Numerical analysis of the multilevel Milstein discretisation

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Multilevel Monte Carlo

Given a scalar SDE driven by a Brownian diffusion

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t), \qquad 0<t<T$$

to estimate $\mathbb{E}[P]$ where the path-dependent payoff P can be approximated by $\widehat{P}_\ell$ using 2^ℓ uniform timesteps, we use

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}].$$

$\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ is estimated using N_ℓ simulations with same $W(t)$ for both $\widehat{P}_\ell$ and $\widehat{P}_{\ell-1}$,

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(\widehat{P}_\ell^{(i)} - \widehat{P}_{\ell-1}^{(i)} \right)$$

MLMC Theorem

Theorem: *Let P be a functional of the solution of an SDE, and $\widehat{P}_\ell$ the discrete approximation using a timestep $h_\ell = 2^{-\ell}\, T$.*

If *there exist independent estimators $\widehat{Y}_\ell$ based on N_ℓ Monte Carlo samples, with computational complexity (cost) C_ℓ, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that*

i) $\left| \mathbb{E}[\widehat{P}_\ell - P] \right| \leq c_1\, h_\ell^\alpha$

ii) $\mathbb{E}[\widehat{Y}_\ell] = \begin{cases} \mathbb{E}[\widehat{P}_0], & \ell = 0 \\ \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & \ell > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_\ell] \leq c_2\, N_\ell^{-1} h_\ell^\beta$

iv) $C_\ell \leq c_3\, N_\ell\, h_\ell^{-1}$

MLMC Theorem

then *there exists a positive constant* c_4 *such that for any* $\varepsilon < e^{-1}$ *there are values* L *and* N_ℓ *for which the multilevel estimator*

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell,$$

has Mean Square Error $MSE \equiv \mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$

with a computational complexity C *with bound*

$$C \leq \begin{cases} c_4\,\varepsilon^{-2}, & \beta > 1, \\ c_4\,\varepsilon^{-2}(\log\varepsilon)^2, & \beta = 1, \\ c_4\,\varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Numerical Analysis

If P is a Lipschitz function of $S(T)$, value of underlying path simulation at a fixed time, the strong convergence property

$$\left(\mathbb{E}\left[(\widehat{S}_N - S(T))^2\right]\right)^{1/2} = O(h^\gamma)$$

implies that $\mathbb{V}[\widehat{P}_\ell - P] = O(h_\ell^{2\gamma})$ and hence

$$V_\ell \;\equiv\; \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}] = O(h_\ell^{2\gamma}).$$

Therefore $\beta = 1$ for Euler-Maruyama discretisation, and $\beta = 2$ for the Milstein discretisation.

However, in general, good strong convergence is neither necessary nor sufficient for good convergence for V_ℓ.

Numerics and Analysis

option	Euler numerics	Euler analysis	Milstein numerics	Milstein analysis
Lipschitz	$O(h)$	$O(h)$	$O(h^2)$	$O(h^2)$
Asian	$O(h)$	$O(h)$	$O(h^2)$	$O(h^2)$
lookback	$O(h)$	$O(h)$	$O(h^2)$	$o(h^{2-\delta})$
barrier	$O(h^{1/2})$	$o(h^{1/2-\delta})$	$O(h^{3/2})$	$o(h^{3/2-\delta})$
digital	$O(h^{1/2})$	$O(h^{1/2}\log h)$	$O(h^{3/2})$	$o(h^{3/2-\delta})$

Table: V_ℓ convergence observed numerically (for GBM) and proved analytically (for more general SDEs) for both the Euler and Milstein discretisations. δ can be any strictly positive constant.

Numerical Analysis

Analysis for Euler discretisations:

- lookback and barrier options: Giles, Higham & Mao (*Finance & Stochastics, 2009*)
  - lookback analysis follows from strong convergence
  - barrier analysis shows dominant contribution comes from paths which are near the barrier; uses asymptotic analysis, first proving that “extreme” paths have negligible contribution
  - similar analysis for digital options gives $O(h^{1/2-\delta})$ bound instead of $O(h^{1/2}\log h)$
- digital options: Avikainen (*Finance & Stochastics, 2009*)
  - method of analysis is quite different

Numerical Analysis

Analysis for Milstein discretisations:

- builds on approach in paper with Higham and Mao
- key idea is to use boundedness of all moments to bound the contribution to V_ℓ from “extreme” paths (e.g. for which $\max_n |\Delta W_n| > h^{1/2-\delta}$ for some $\delta > 0$)
- uses asymptotic analysis to bound the contribution from paths which are not “extreme”

Milstein Scheme

MLMC Theorem allows different approximations on the coarse and fine levels:

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{n=1}^{N_\ell} \left(\widehat{P}_\ell^f(\omega^{(n)}) - \widehat{P}_{\ell-1}^c(\omega^{(n)}) \right)$$

The telescoping sum still works provided

$$\mathbb{E}\left[\widehat{P}_\ell^f\right] \;=\; \mathbb{E}\left[\widehat{P}_\ell^c\right].$$

The key is to exploit this freedom to reduce the variance

$$\mathbb{V}\left[\widehat{P}_\ell^f - \widehat{P}_{\ell-1}^c\right].$$

Milstein Scheme

Fine path Brownian interpolation: within each timestep, model the behaviour as simple Brownian motion (constant drift and volatility) conditional on two end-points

$$\begin{aligned}\widehat{S}^f(t) &= \widehat{S}^f_n + \lambda(t)(\widehat{S}^f_{n+1} - \widehat{S}^f_n) \\ &\quad + b_n \Big(W(t) - W_n - \lambda(t)(W_{n+1} - W_n) \Big),\end{aligned}$$

where $\lambda(t) = \dfrac{t - t_n}{t_{n+1} - t_n}$.

There then exist analytic results for the distribution of the min/max/average over each timestep, and probability of crossing a barrier.

Milstein Scheme

Coarse path Brownian interpolation: exactly the same, but with double the timestep, so for even n

$$\begin{aligned}\widehat{S}^c(t) &= \widehat{S}^c_n + \lambda(t)(\widehat{S}^c_{n+2} - \widehat{S}^c_n) \\ &\quad + b_n \Big(W(t) - W_n - \lambda(t)(W_{n+2} - W_n) \Big),\end{aligned}$$

where $\lambda(t) = \dfrac{t - t_n}{t_{n+2} - t_n}$. Hence, in particular,

$$\begin{aligned}\widehat{S}^c_{n+1} \equiv \widehat{S}^c(t_{n+1}) &= \tfrac{1}{2}(\widehat{S}^c_n + \widehat{S}^c_{n+2}) \\ &\quad + b_n \Big(W_{n+1} - \tfrac{1}{2}(W_n + W_{n+2}) \Big),\end{aligned}$$

Milstein Scheme

Theorem: Under standard conditions,

- $$\mathbb{E}\left[\sup_{[0,T]} \left|\widehat{S}(t) - S(t)\right|^m\right] = O((h \log h)^m),$$

- $$\sup_{[0,T]} \mathbb{E}\left[\left|\widehat{S}(t) - S(t)\right|^m\right] = O(h^m),$$

- $$\mathbb{E}\left[\left(\int_0^T \widehat{S}(t) - S(t)\ \mathrm{d}t\right)^2\right] = O(h^2).$$

Milstein Scheme

The variance convergence for the Asian option comes directly from this.

Will now outline the analysis for the lookback option – the barrier is similar but more complicated.

The digital option is based on a Brownian extrapolation from one timestep before the end – the analysis is similar.

The analysis for the lookback, barrier and digital options uses the idea of "extreme" paths which are highly improbable – the variance comes mainly from non-extreme paths for which one can use asymptotic analysis.

Extreme Paths

Lemma: If X_ℓ is a random variable on level l, and $\mathbb{E}[\,|X_\ell|^m] \leq C_m$ is uniformly bounded, then, for any $\delta > 0$,

$$\mathbb{P}[\,|X_\ell| > h_\ell^{-\delta}] = o(h_\ell^p), \qquad \forall p > 0.$$

Proof: Markov inequality $\mathbb{P}[\,|X_\ell|^m > h_\ell^{-m\delta}] \ < \ h_\ell^{-m\delta}\,\mathbb{E}[\,|X_\ell|^m]$.

Lemma: If Y_ℓ is a random variable on level ℓ, $\mathbb{E}[\,Y_\ell^2]$ is uniformly bounded, and the indicator function $\mathbf{1}_{E_\ell}$ satisfies $\mathbb{E}[\mathbf{1}_{E_\ell}] = o(h_\ell^p),\ \forall p > 0$ then

$$\mathbb{E}[\,|Y_\ell|\,\mathbf{1}_{E_\ell}] = o(h_\ell^p), \qquad \forall p > 0.$$

Proof: Hölder inequality $\mathbb{E}[\,|Y_\ell|\,\mathbf{1}_{E_\ell}] \leq \sqrt{\mathbb{E}[\,Y_\ell^2]\,\mathbb{E}[\mathbf{1}_{E_\ell}]}$

Extreme Paths

Theorem: For any $\gamma > 0$, the probability that $W(t)$, its increments ΔW_n and the corresponding SDE solution $S(t)$ and approximations $\widehat{S}_n^f$ and $\widehat{S}_n^c$ satisfy any of the following "extreme" conditions

$$
\begin{aligned}
\max_n \left(\max(|S(nh)|,\ |\widehat{S}_n^f|,\ |\widehat{S}_n^c|) \right) &> h^{-\gamma} \\
\max_n \left(\max(|S(nh) - \widehat{S}_n^c|,\ |S(nh) - \widehat{S}_n^f|,\ |\widehat{S}_n^f - \widehat{S}_n^c|) \right) &> h^{1-\gamma} \\
\max_n |\Delta W_n| &> h^{1/2-\gamma}
\end{aligned}
$$

is $o(h^p)$ for all $p > 0$.

Non-extreme paths

Furthermore, there exist constants c_1, c_2, c_3, c_4 such that if none of these conditions is satisfied, and $\gamma < \frac{1}{2}$, then

$$
\begin{aligned}
\max_n |\widehat{S}^f_n - \widehat{S}^f_{n-1}| &\leq c_1\, h^{1/2-2\gamma} \\
\max_n |b^f_n - b^f_{n-1}| &\leq c_2\, h^{1/2-2\gamma} \\
\max_n \left(|b^f_n| + |b^c_n|\right) &\leq c_3\, h^{-\gamma} \\
\max_n |b^f_n - b^c_n| &\leq c_4\, h^{1/2-2\gamma}
\end{aligned}
$$

where b^c_n is defined to equal b^c_{n-1} if n is odd.

Lookback Option

Lookback options are a Lipschitz function of the minimum over the whole simulation path.

For the fine path, the minimum over one timestep is

$$\widehat{S}^f_{n,min} = \frac{1}{2}\left(\widehat{S}^f_n + \widehat{S}^f_{n+1} - \sqrt{\left(\widehat{S}^f_{n+1} - \widehat{S}^f_n\right)^2 - 2\left(b^f_n\right)^2 h_\ell \, \log U_n} \right)$$

where U_m is a $(0,1]$ uniform random variable.

For the coarse path, define $\widehat{S}^c_n$ for odd n using conditional Brownian interpolation, then use the same expression for the minimum with same U_n – this doesn't change the distribution of the computed minimum over the coarse timestep, so the telescoping sum is OK.

Lookback Option

$\mathbb{E}[(\widehat{P}_\ell - \widehat{P}_{\ell-1})^4]$ is bounded and therefore extreme paths have negligible contribution to $\mathbb{E}[(\widehat{P}_\ell - \widehat{P}_{\ell-1})^2]$.

For non-extreme paths, tedious asymptotic analysis gives

$$\begin{aligned} \left| \widehat{S}^f_{min} - \widehat{S}^c_{min} \right| &\leq \max_n \left| \widehat{S}^f_{n,min} - \widehat{S}^c_{n,min} \right| \\ &= o(h_\ell^{1-5\gamma/2}) \end{aligned}$$

and hence $\mathbb{E}[(\widehat{P}_\ell - \widehat{P}_{\ell-1})^2] = o(h_\ell^{2-\delta})$ for any $\delta > 0$.

Barrier and Digital Options

For barrier options, split paths into 3 subsets:

- extreme paths
- paths with a minimum within $O(h^{1/2-\gamma})$ of the barrier
- rest

– dominant contribution comes from the second subset.

For digital options, again split paths into 3 subsets:

- extreme paths
- paths with final $S(T)$ within $O(h^{1/2-\gamma})$ of the strike
- rest

– dominant contribution again from the second subset.

Other numerical analysis

- multi-dimensional Milstein – next talk by Lukas Szpruch
- multilevel scalar finite rate jump-diffusion, including path-dependent Poisson rate – Yuan Xia
  - algorithm and numerical results presented at MCQMC'10
  - numerical analysis completed, paper almost ready
- multilevel Greeks – Sylvestre Burgos
  - algorithm and numerical results presented at MCQMC'10
  - numerical analysis in progress

Conclusions

- numerical analysis of multilevel variance achieves bounds which match numerical experiments for Milstein discretisation of scalar SDEs and all common payoffs
- Brownian interpolation is key to obtaining rapid convergence of the multilevel variance for complex payoffs
- excluding the significance of “extreme” paths and using asymptotic analysis for the rest is a non-standard approach to numerical analysis, but seems quite flexible

Multilevel papers are available from:
`people.maths.ox.ac.uk/gilesm/mlmc.html`

Paper on this work should be on ArXiv soon (finally!)