Multilevel Monte Carlo Path Simulation

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15th Scottish Computational Mathematics Symposium

SDEs in Finance

In computational finance, stochastic differential equations are used to model the behaviour of

- stocks
- interest rates
- exchange rates
- weather
- electricity/gas demand
- crude oil prices
- . . .

The stochastic term accounts for the uncertainty of unpredictable day-to-day events.

SDEs in Finance

These models are then used to calculate “fair” prices for a huge range of financial options:

- an option to sell a stock portfolio at a specific price in 2 years time
- an option to buy aviation fuel at a specific price in 6 months time
- an option to sell US dollars at a specific exchange rate in 3 years time

In most cases, the buyer of the financial option is trying to reduce their risk.

SDEs in Finance

Examples:

- Geometric Brownian motion (Black-Scholes model for stock prices)

$$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W$$

- Cox-Ingersoll-Ross model (interest rates)

$$\mathrm{d}r = \alpha(b-r)\,\mathrm{d}t + \sigma\,\sqrt{r}\,\mathrm{d}W$$

- Heston stochastic volatility model (stock prices)

$$\begin{aligned} \mathrm{d}S &= r\,S\,\mathrm{d}t + \sqrt{V}\,S\,\mathrm{d}W_1 \\ \mathrm{d}V &= \lambda\,(\sigma^2 - V)\,\mathrm{d}t + \xi\,\sqrt{V}\,\mathrm{d}W_2 \end{aligned}$$

with correlation ρ between $\mathrm{d}W_1$ and $\mathrm{d}W_2$

Generic Problem

Stochastic differential equation with general drift and volatility terms: SDE with general drift and volatility terms:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t)$$

$W(t)$ is a Wiener variable with the properties that for any $q<r<s<t$, $W(t)-W(s)$ is Normally distributed with mean 0 and variance $t-s$, independent of $W(r)-W(q)$.

In many finance applications, we want to compute the expected value of an option dependent on the terminal state

$$P = f(S(T))$$

with a uniform Lipschitz bound,

$$|f(U) - f(V)| \;\leq c\;\|U - V\|\,, \quad \forall\, U, V.$$

Standard MC Approach

Euler discretisation with timestep h:

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n$$

Simplest estimator for expected payoff is an average of N independent path simulations:

$$\widehat{Y} = N^{-1} \sum_{i=1}^{N} f(\widehat{S}^{(i)}_{T/h})$$

Standard MC Approach

Two kinds of errors:

- statistical error, due to finite number of paths

$$V[\widehat{Y}] = N^{-1} V[f(\widehat{S}_{T/h})]$$

  so r.m.s. error = $O(N^{-1/2})$.

- discretisation bias, due to finite number of timesteps
  - weak convergence – $O(h)$ error in expected payoff
  - strong convergence – $O(h^{1/2})$ error in individual path

Standard MC Approach

Mean Square Error is $O\left(N^{-1}+h^2\right)$

- first term comes from variance of estimator
- second term comes from bias due to weak convergence

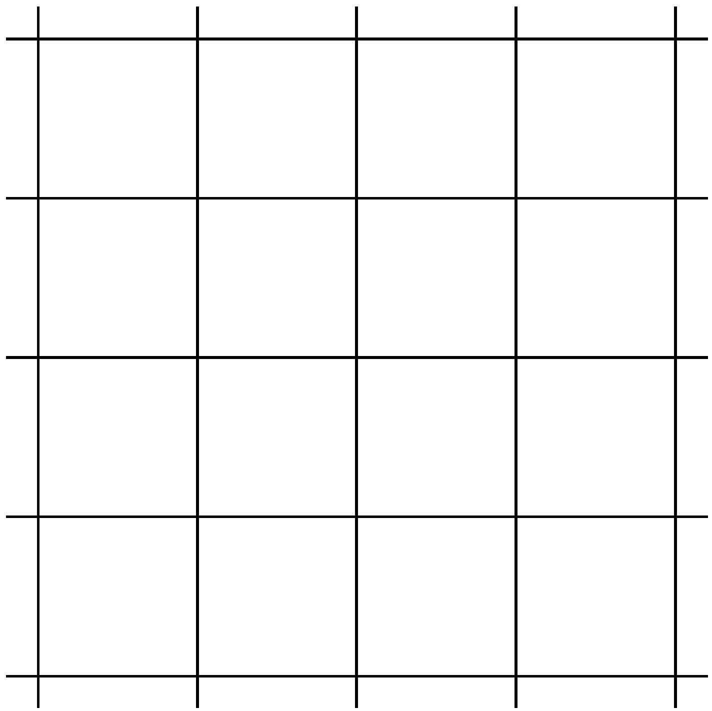
To make this $O(\varepsilon^2)$ requires

$$N=O(\varepsilon^{-2}), \quad h=O(\varepsilon) \quad \Longrightarrow \quad \text{cost}=O(N\,h^{-1})=O(\varepsilon^{-3})$$

Aim is to improve this cost to $O\left(\varepsilon^{-2}(\log\varepsilon)^2\right)$

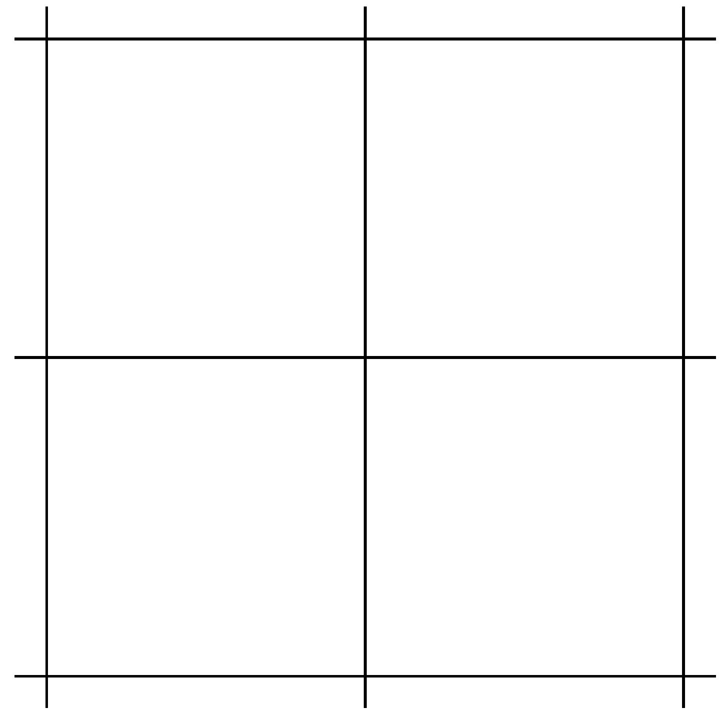
Multigrid

A powerful technique for solving PDE discretisations:



Fine grid

more accurate
more expensive



Coarse grid

less accurate
less expensive

Multigrid

Multigrid combines calculations on a nested sequence of grids to get the accuracy of the finest grid at a much lower computational cost.

We will use a similar idea to achieve variance reduction in Monte Carlo path calculations, combining simulations with different numbers of timesteps – same accuracy as finest calculations, but at a much lower computational cost.

Other Research

- In Dec. 2005, Ahmed Kebaier published an article in *Annals of Applied Probability* describing a two-level method which reduces the cost to $O\left(\varepsilon^{-2.5}\right)$.
- Also in Dec. 2005, Adam Speight wrote a working paper describing a similar multilevel use of control variates, but without an analysis of its complexity.
- There are also close similarities to a multilevel technique developed by Stefan Heinrich for parametric integration (*Journal of Complexity*, 1998)

Multilevel MC Approach

Consider multiple sets of simulations with different timesteps $h_l = 2^{-l}\, T,\ l = 0, 1, \ldots, L$, and payoff $\widehat{P}_l$

$$E[\widehat{P}_L] = E[\widehat{P}_0] + \sum_{l=1}^{L} E[\widehat{P}_l - \widehat{P}_{l-1}]$$

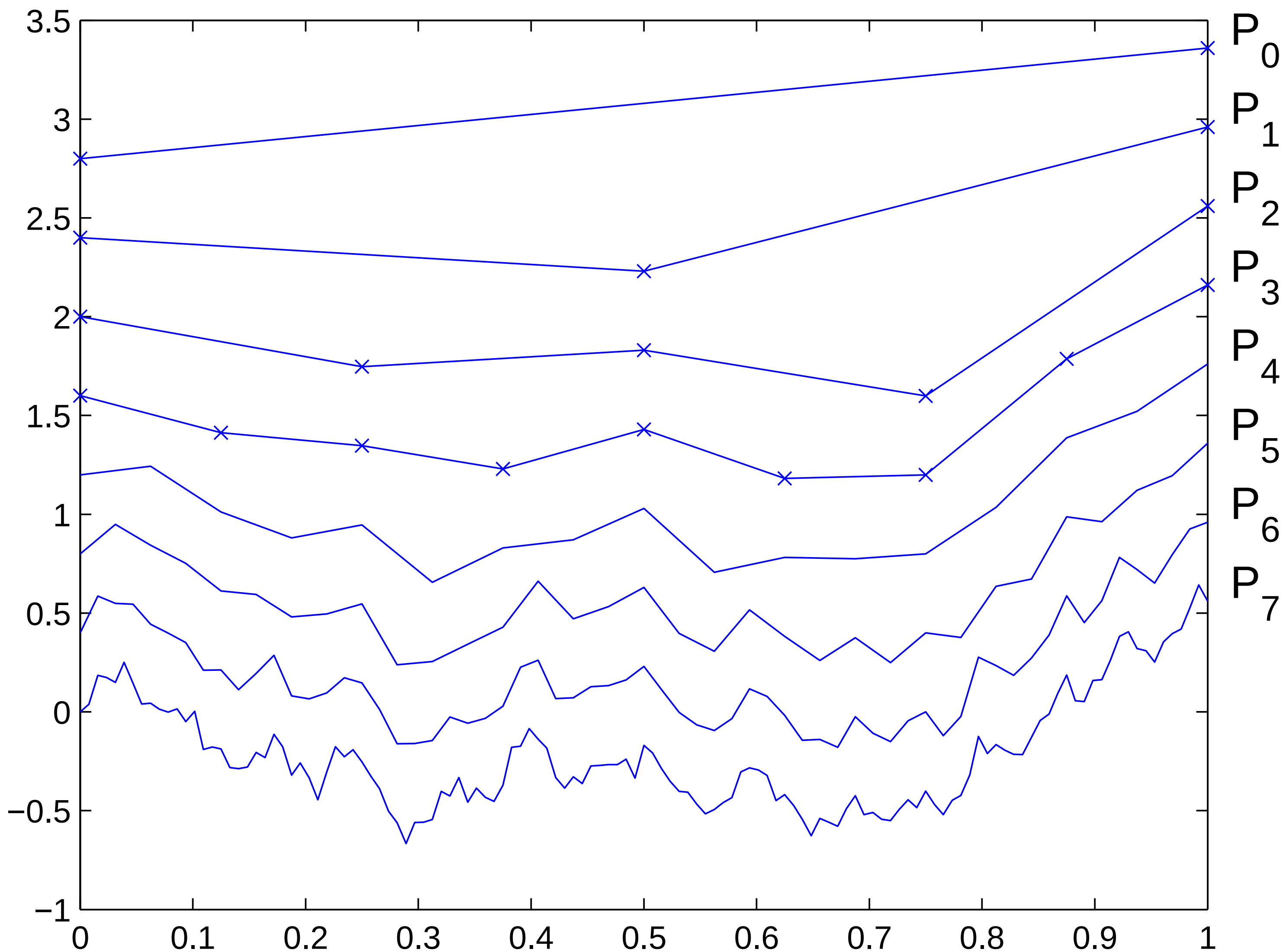
Expected value is same – aim is to reduce variance of estimator for a fixed computational cost.

Key point: approximate $E[\widehat{P}_l - \widehat{P}_{l-1}]$ using N_l simulations with $\widehat{P}_l$ and $\widehat{P}_{l-1}$ obtained using <u>same</u> Brownian path.

$$\widehat{Y}_l = N_l^{-1} \sum_{i=1}^{N_l} \left(\widehat{P}_l^{(i)} - \widehat{P}_{l-1}^{(i)} \right)$$

Multilevel MC Approach

Discrete Brownian path at different levels



Multilevel MC Approach

- each level adds more detail to Brownian path
- $E[\widehat{P}_l - \widehat{P}_{l-1}]$ reflects impact of that extra detail on the payoff
- different timescales handled by different levels
  – similar to different wavelengths being handled by different grids in multigrid

Multilevel MC Approach

Using independent paths for each level, the variance of the combined estimator is

$$V\left[\sum_{l=0}^{L} \widehat{Y}_l\right] = \sum_{l=0}^{L} N_l^{-1} V_l, \qquad V_l \equiv V[\widehat{P}_l - \widehat{P}_{l-1}],$$

and the computational cost is proportional to $\displaystyle\sum_{l=0}^{L} N_l\, h_l^{-1}$.

Hence, the variance is minimised for a fixed computational cost by choosing N_l to be proportional to $\sqrt{V_l\, h_l}$.

The constant of proportionality can be chosen so that the combined variance is $O(\varepsilon^2)$.

Multilevel MC Approach

For the Euler discretisation and the Lipschitz payoff function

$$V[\widehat{P}_l - P] = O(h_l) \quad \Longrightarrow \quad V[\widehat{P}_l - \widehat{P}_{l-1}] = O(h_l)$$

and the optimal N_l is asymptotically proportional to h_l.

To make the combined variance $O(\varepsilon^2)$ requires

$$N_l = O(\varepsilon^{-2} L\, h_l).$$

To make the bias $O(\varepsilon)$ requires

$$L = \log_2 \varepsilon^{-1} + O(1) \quad \Longrightarrow \quad h_L = O(\varepsilon).$$

Hence, we obtain an $O(\varepsilon^2)$ MSE for a computational cost which is $O(\varepsilon^{-2} L^2) = O(\varepsilon^{-2} (\log \varepsilon)^2)$.

Multilevel MC Approach

Theorem: *Let P be a functional of the solution of a stochastic o.d.e., and $\widehat{P}_l$ the discrete approximation using a timestep $h_l = M^{-l}\,T$.*

If *there exist independent estimators $\widehat{Y}_l$ based on N_l Monte Carlo samples, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that*

i) $E[\widehat{P}_l - P] \leq c_1\, h_l^\alpha$

ii) $E[\widehat{Y}_l] = \begin{cases} E[\widehat{P}_0], & l=0 \\ E[\widehat{P}_l - \widehat{P}_{l-1}], & l>0 \end{cases}$

iii) $V[\widehat{Y}_l] \leq c_2\, N_l^{-1} h_l^\beta$

iv) C_l*, the computational complexity of* $\widehat{Y}_l$*, is bounded by*

$$C_l \leq c_3\, N_l\, h_l^{-1}$$

Multilevel MC Approach

then *there exists a positive constant* c_4 *such that for any* $\varepsilon < e^{-1}$ *there are values* L *and* N_l *for which the multi-level estimator*

$$\widehat{Y} = \sum_{l=0}^{L} \widehat{Y}_l,$$

has Mean Square Error $MSE \equiv E\left[\left(\widehat{Y} - E[P]\right)^2\right] < \varepsilon^2$

with a computational complexity C *with bound*

$$C \leq \begin{cases} c_4\, \varepsilon^{-2}, & \beta > 1, \\ c_4\, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_4\, \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Results

Geometric Brownian motion:

$$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W, \qquad 0 < t < 1,$$

$S(0)\!=\!1$, $r\!=\!0.05$, $\sigma\!=\!0.2$

Heston model:

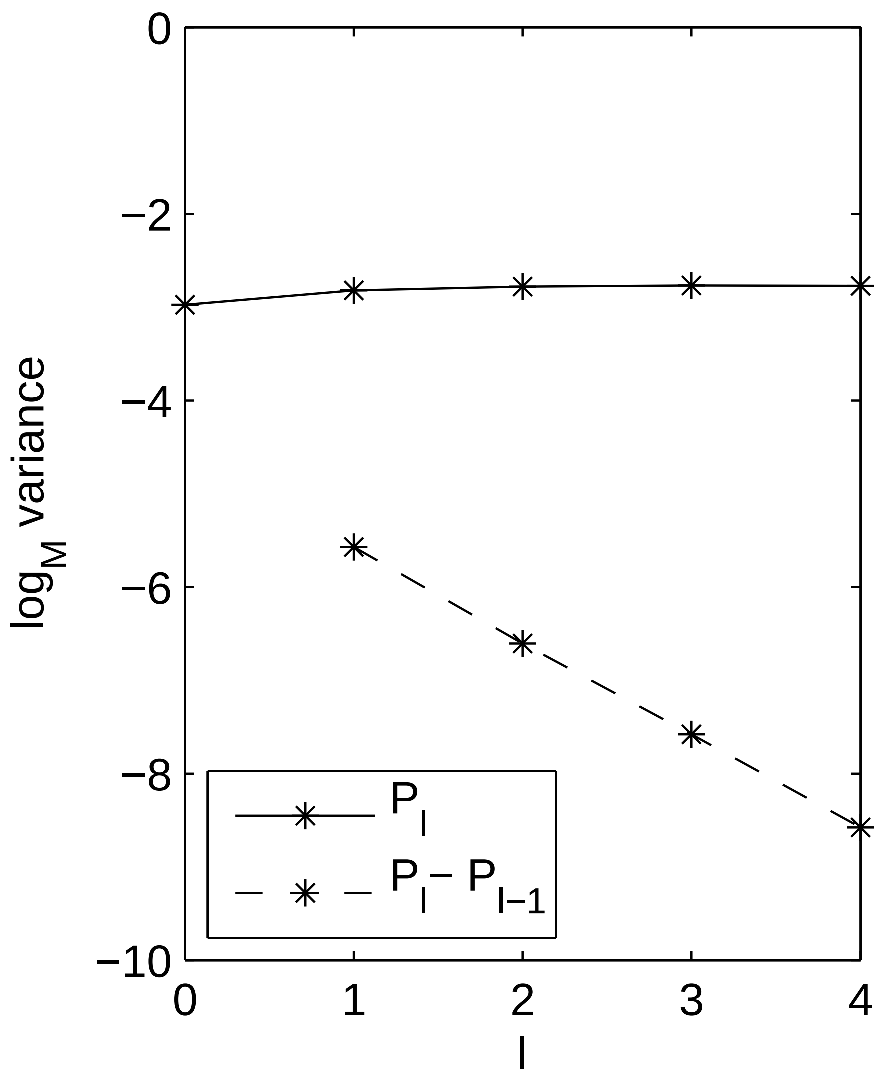
$$\begin{aligned} \mathrm{d}S &= r\,S\,\mathrm{d}t + \sqrt{V}\,S\,\mathrm{d}W_1, \qquad 0 < t < 1 \\ \mathrm{d}V &= \lambda\,(\sigma^2 - V)\,\mathrm{d}t + \xi\,\sqrt{V}\,\mathrm{d}W_2, \end{aligned}$$

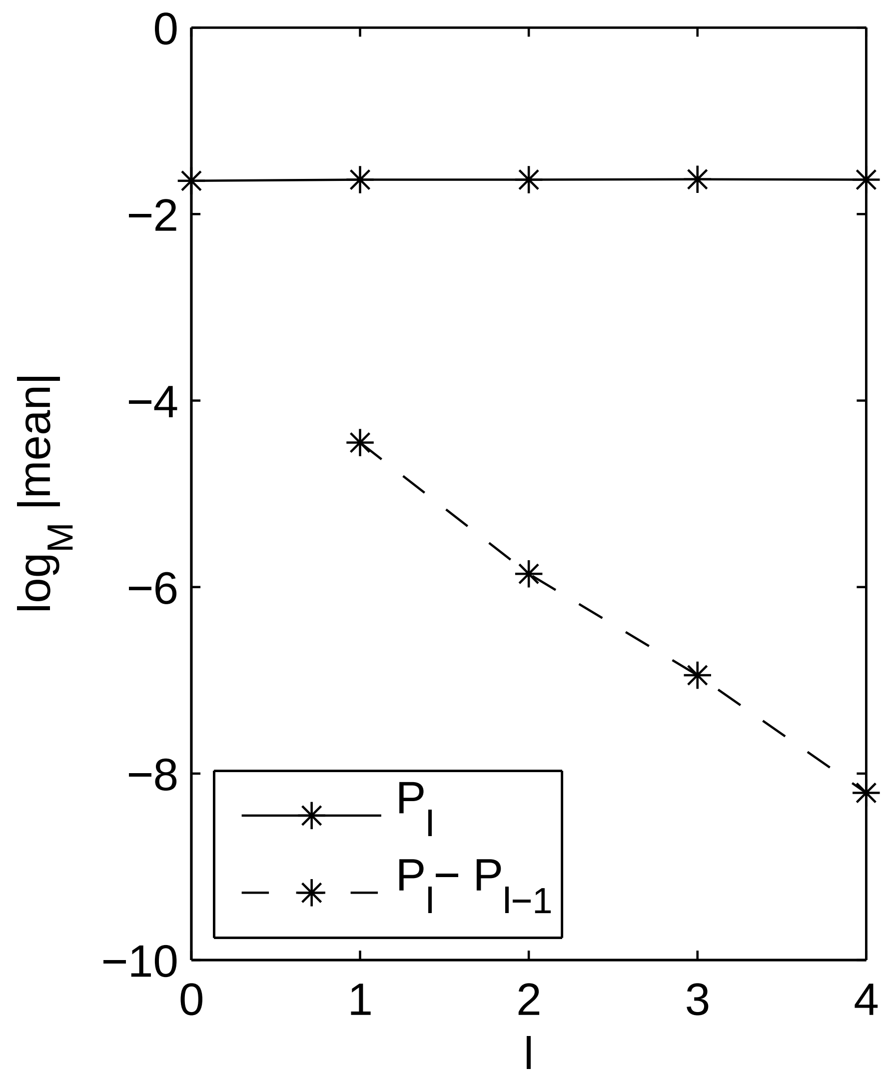
$S(0)\!=\!1$, $V(0)\!=\!0.04$, $r\!=\!0.05$, $\sigma\!=\!0.2$, $\lambda\!=\!5$, $\xi\!=\!0.25$, $\rho\!=\!-0.5$

All calculations use $M\!=\!4$, more efficient than $M\!=\!2$.

Results

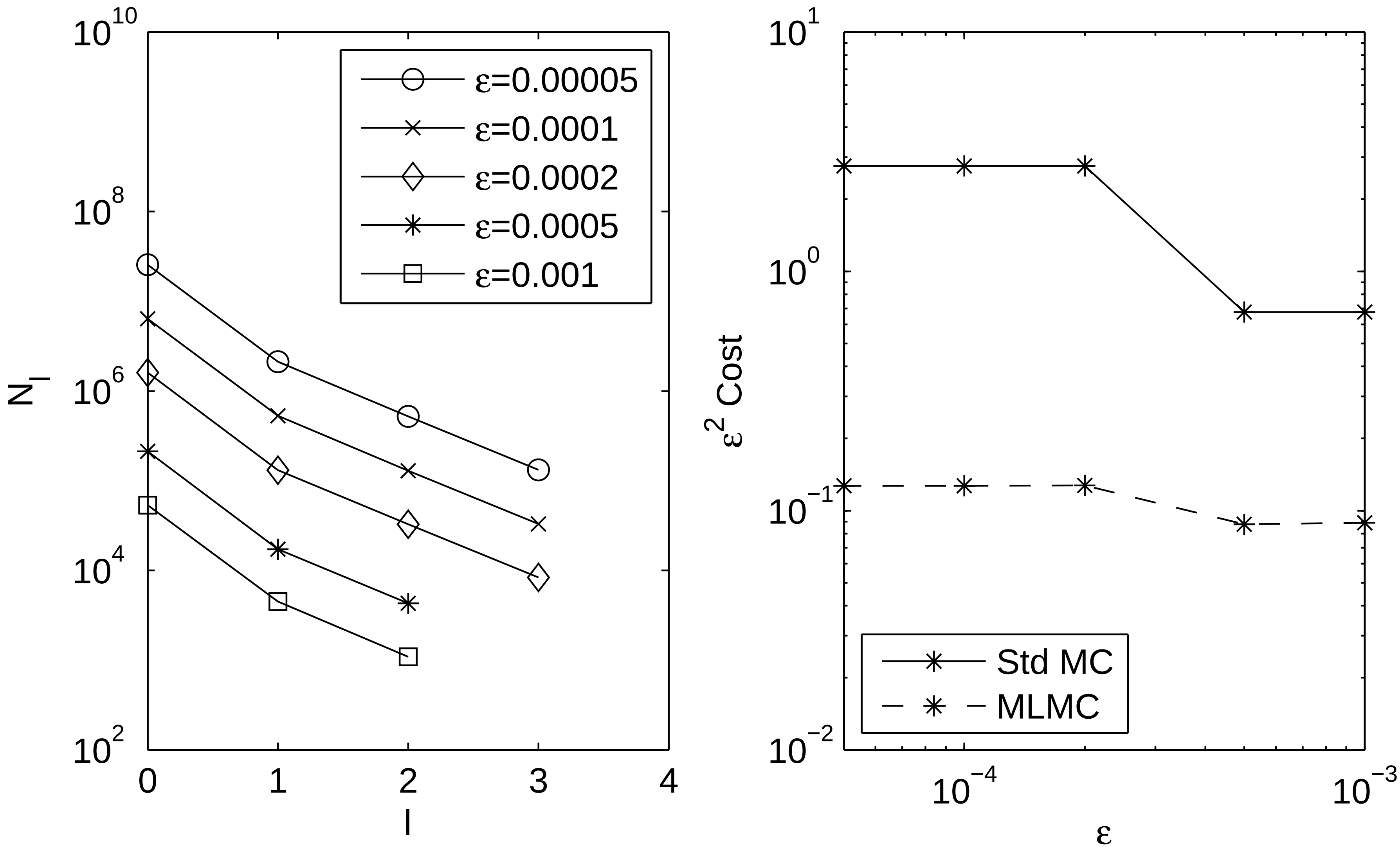
GBM: European call, $\max(S(1)-1,0)$





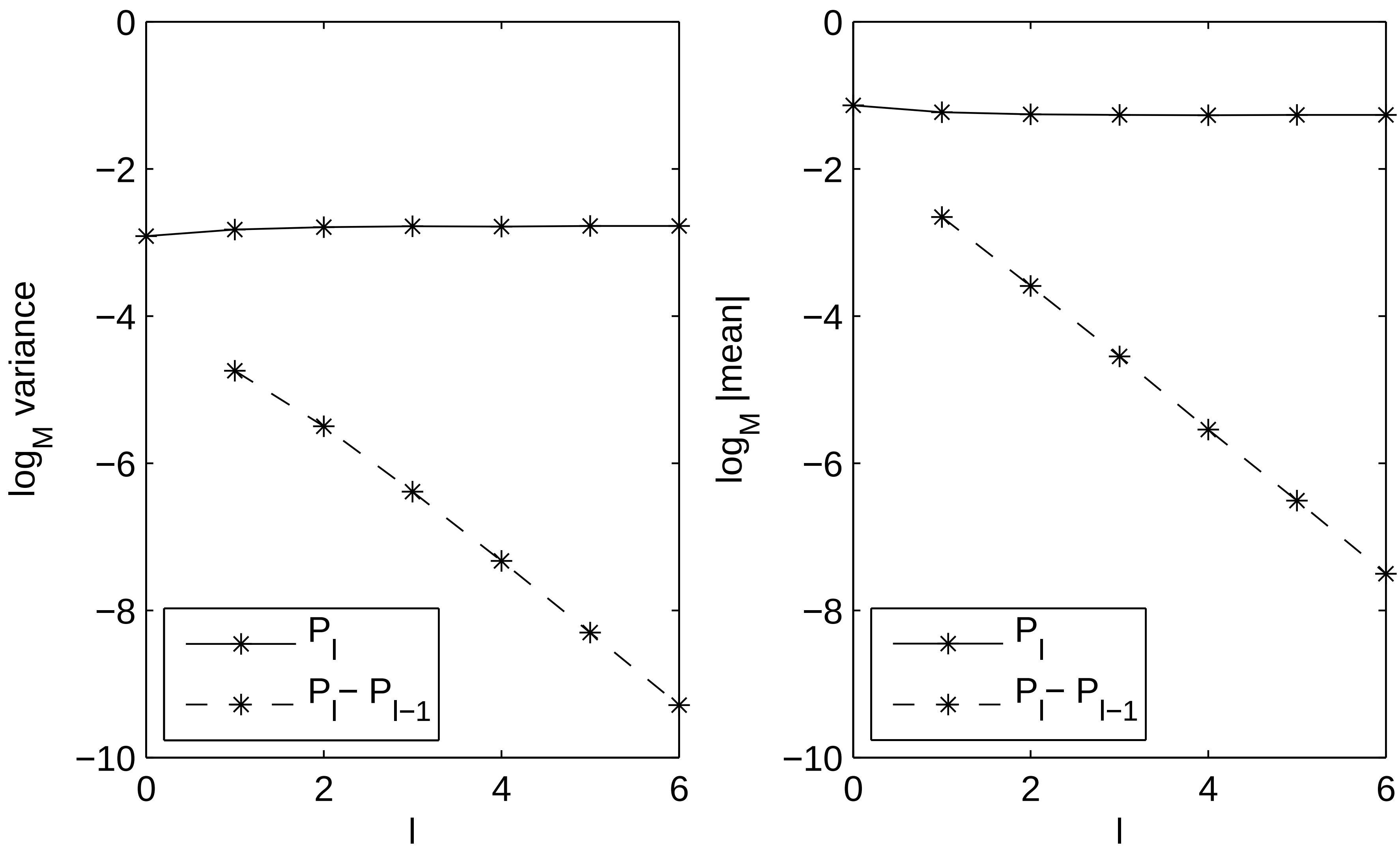
Results

GBM: European call, $\max(S(1)-1, 0)$



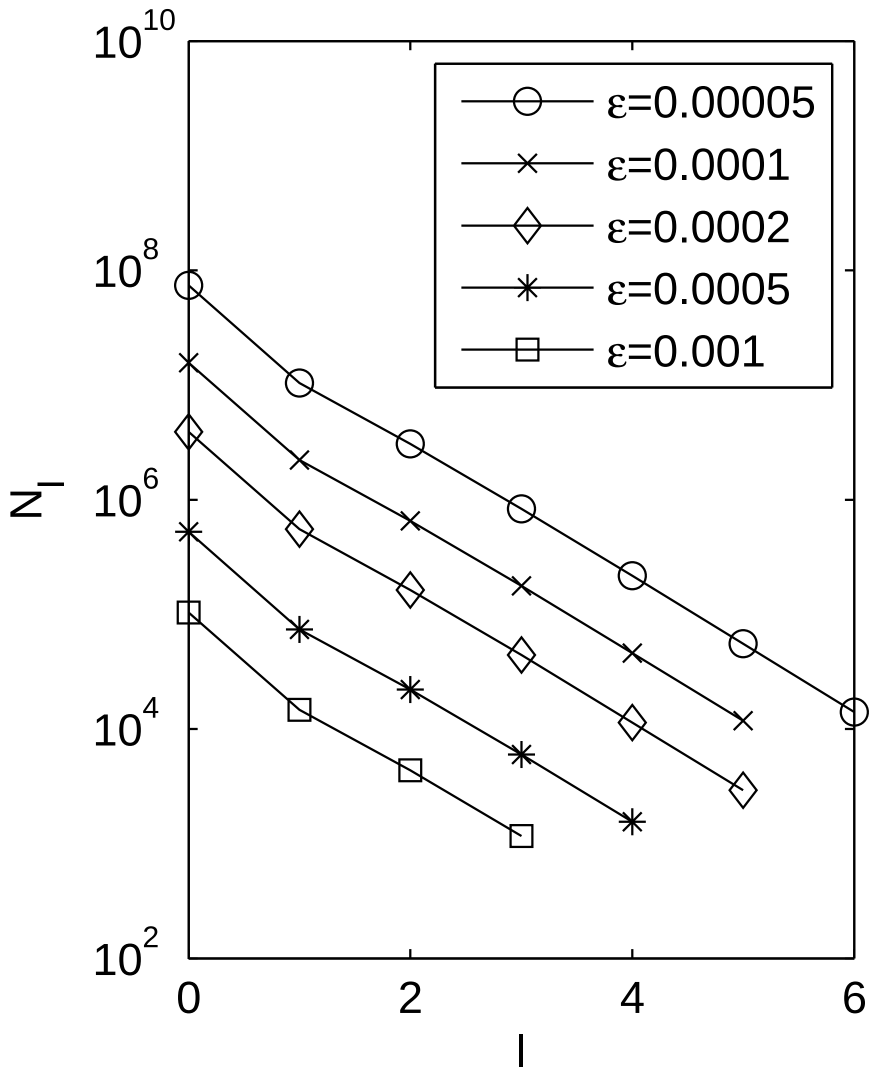
Results

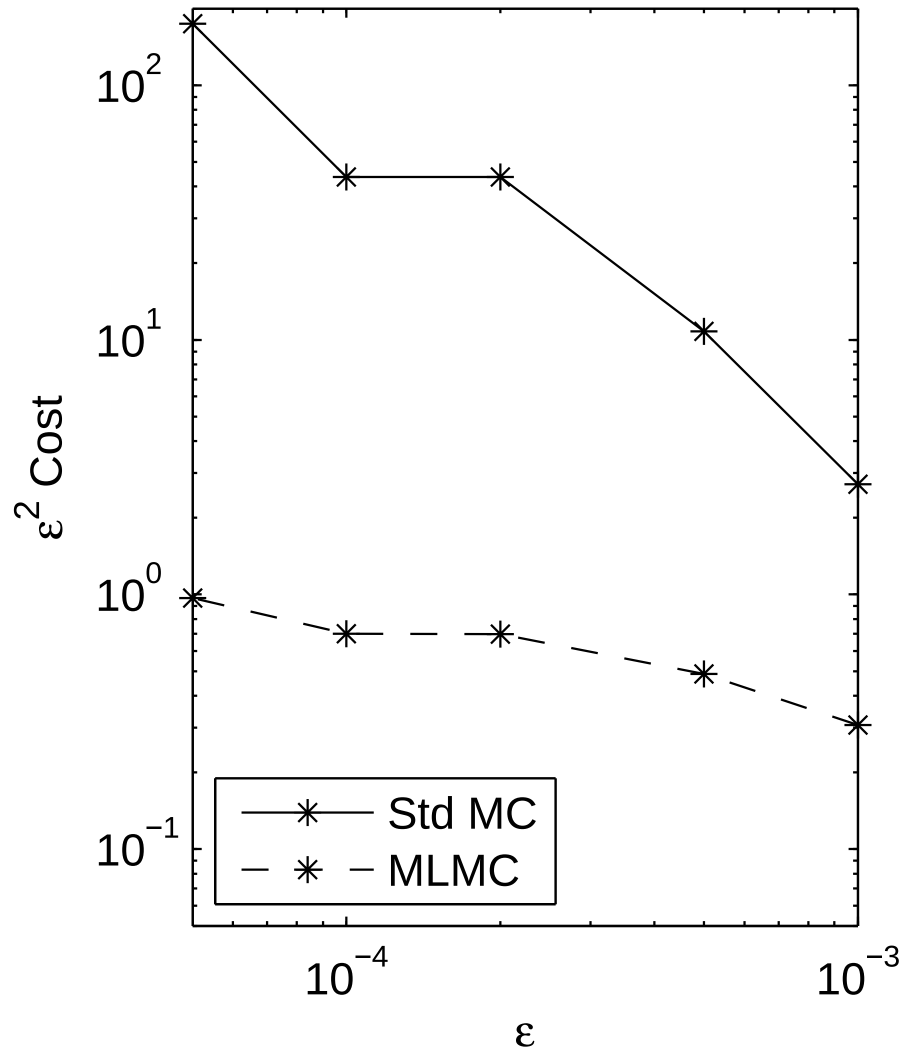
GBM: lookback option, $S(1) - \min_{0<t<1} S(t)$



Results

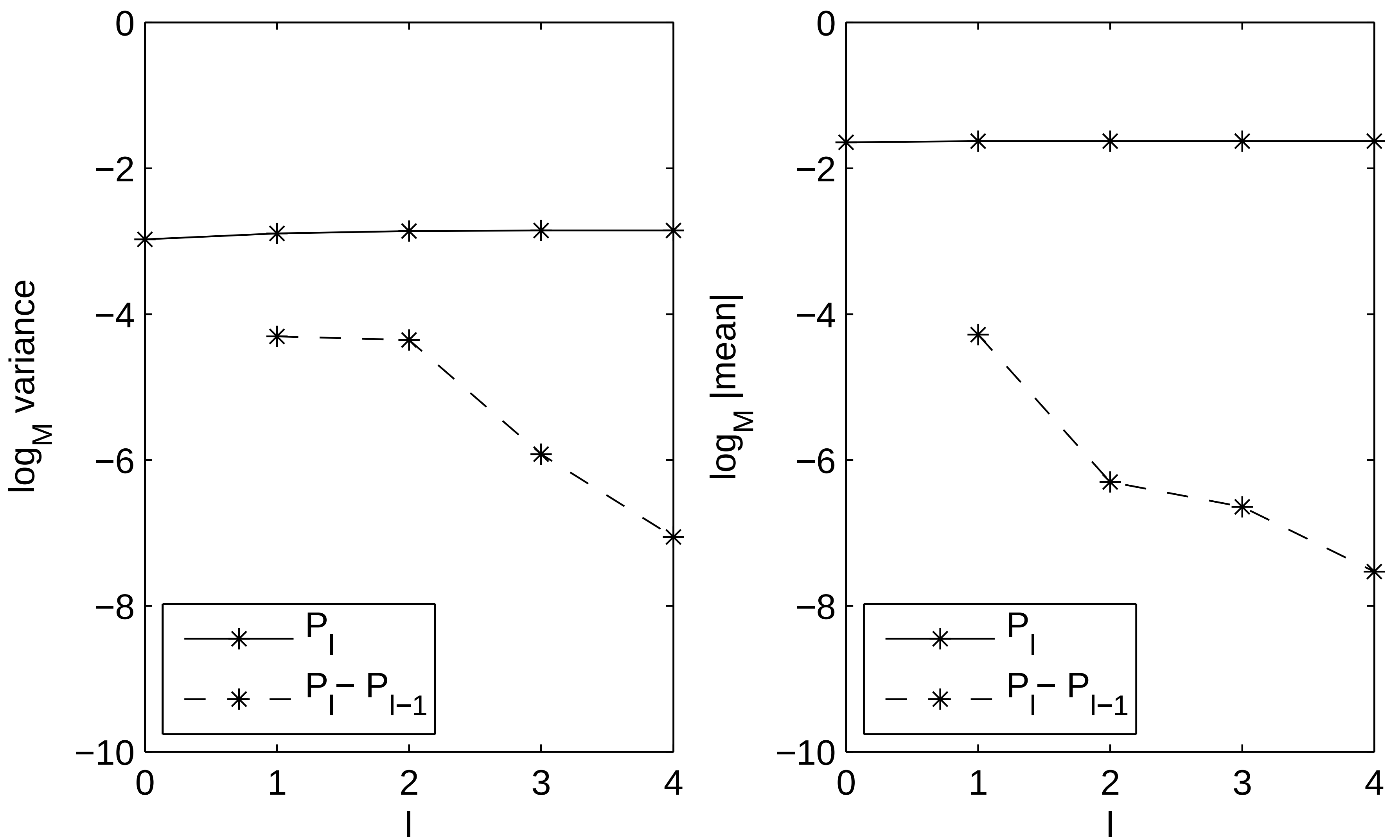
GBM: lookback option, $S(1) - \min_{0<t<1} S(t)$





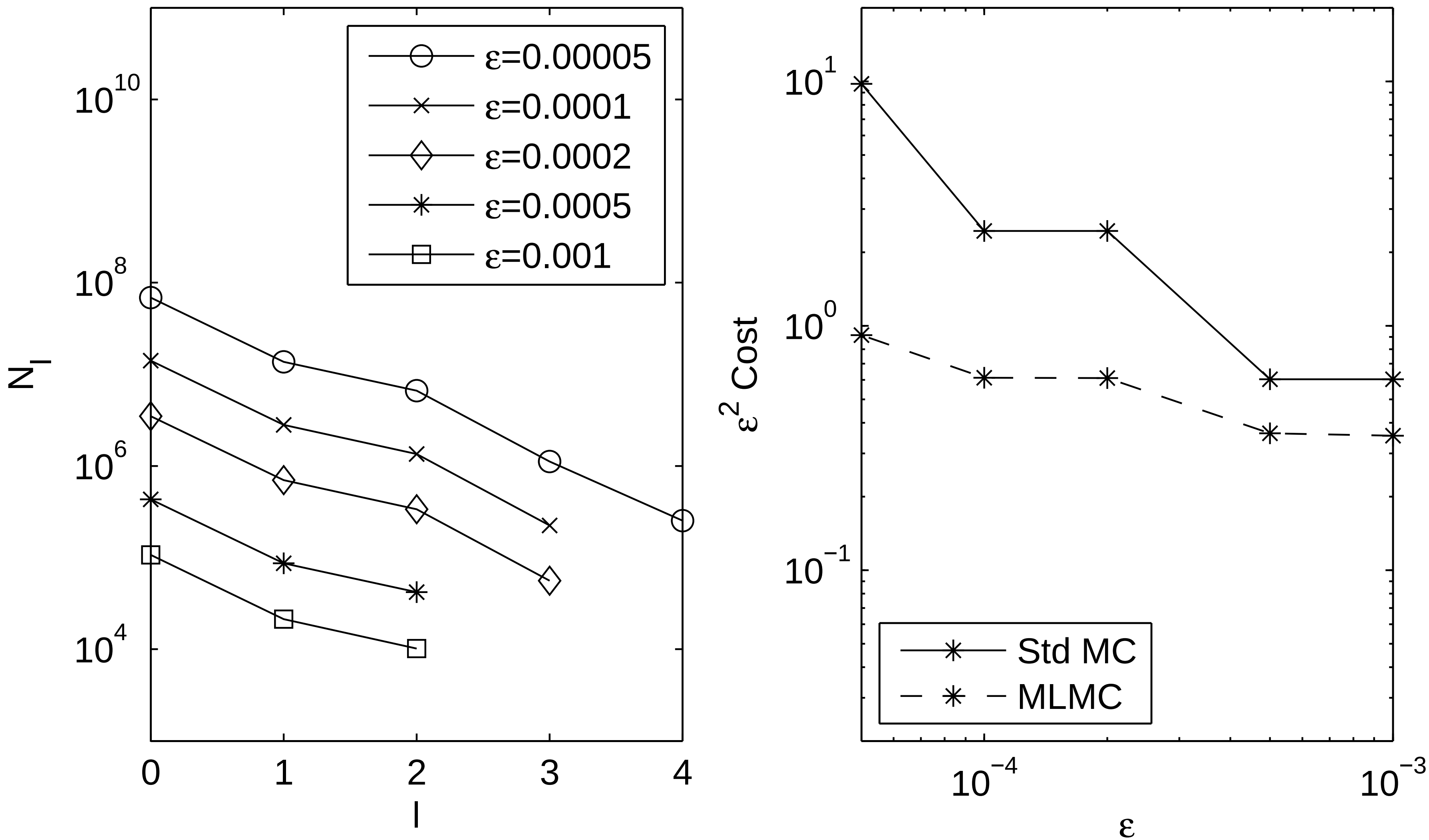
Results

Heston model: European call



Results

Heston model: European call



Conclusions

Results so far:

- improved order of complexity
- easy to implement
- significant benefits for model problems

Current research:

- use of Milstein method (and antithetic variables in multi-dimensional case) to reduce complexity to $O(\varepsilon^{-2})$
- adaptive sampling to treat discontinuous payoffs and pathwise derivatives for Greeks
- use of quasi-Monte Carlo methods, to reduce complexity towards $O(\varepsilon^{-1})$

Working Paper

M.B. Giles, "Multi-level Monte Carlo path simulation"
Oxford University Computing Laboratory
Numerical Analysis Report NA-06/03

`www.comlab.ox.ac.uk/mike.giles/finance.html`

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Acknowledgements:

- Paul Glasserman and Mark Broadie for early feedback
- Microsoft for current research funding

Milstein Scheme

Generic SDE:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t), \quad 0<t<T,$$

with correlation matrix $\Omega(S,t)$ between elements of $\mathrm{d}W(t)$.

Simplest Milstein scheme sets Lévy areas to zero to give

$$\widehat{S}_{i,n+1} = \widehat{S}_{i,n} + a_i\,h + b_{ij}\,\Delta W_{j,n} + \tfrac{1}{2}\,\frac{\partial b_{ij}}{\partial S_l}\,b_{lk}\left(\Delta W_{j,n}\,\Delta W_{k,n} - h\,\Omega_{jk}\right)$$

using implied summation convention.

Milstein Scheme

In scalar case:

- $O(h)$ strong convergence
- $O(\varepsilon^{-2})$ complexity for Lipschitz payoffs
- $O(\varepsilon^{-2})$ complexity for lookback, barrier and digital options using carefully constructed estimators

In multi-dimensional case:

- still only $O(h^{1/2})$ strong convergence
- but $\widehat{S}_n - E[S \,|\, W_n] = O(h)$

Milstein Scheme

If a coarse path with timestep $2h$ is constructed using

$$\Delta W^c_n = \sqrt{2h}\, Y_n$$

where the Y_n are $N(0,1)$ random variables, and the fine path uses a Brownian Bridge construction with

$$\Delta W^f_n = \tfrac{1}{2}\sqrt{2h}\,(Y_n\!+\!Z_n), \quad \Delta W^f_{n+\frac{1}{2}} = \tfrac{1}{2}\sqrt{2h}\,(Y_n\!-\!Z_n).$$

where the Z_n are also $N(0,1)$ random variables, then perturbation analysis shows that the $O(h^{1/2})$ difference between the two paths comes from a sum of terms proportional to

$$Y_{j,n}Z_{k,n} - Y_{k,n}Z_{j,n}.$$

Milstein Scheme

Using the idea of antithetic variables, we use the estimator

$$\widehat{Y}_l = N_l^{-1} \sum_{i=1}^{N_l} \left(\tfrac{1}{2} \left(\widehat{P}_l^{(i)} + \widehat{P}_l^{(i)*} \right) - \widehat{P}_{l-1}^{(i)} \right),$$

where $\widehat{P}_l^{(i)*}$ is based on the same coarse path Y_n, but with Z_n replaced by $-Z_n$, which leads to the cancellation of the leading order error proportional to Z_n.

- $V[\widehat{Y}_l] = O(h^2)$ for smooth payoffs, $O(h^{3/2})$ for Lipschitz
- in both cases, gives $O(\varepsilon^{-2})$ complexity for $O(\varepsilon)$ accuracy

Adaptive sampling

With digital options, the problem is that small path changes can lead to an $O(1)$ change in the payoff

For the Euler discretisation, $O(h^{1/2})$ strong convergence $\Longrightarrow$ $O(h^{1/2})$ paths have an $O(1)$ value for $\widehat{P}_l - \widehat{P}_{l-1}$

Hence,

$$V_l = O(h^{1/2}).$$

For improved results, need more samples of paths near payoff discontinuities.

Adaptive sampling

Two ideas for adaptive sampling are both based on Brownian Bridge constructions, using coarse timestep realisations to decide which paths are important

- idea 1: start with relatively few paths, and sub-divide those which look interesting (splitting)
- idea 2: start with lots of paths, and prune those which are unimportant (Russian roulette)
- use path weights to ensure estimator remains unbiased
- initial results (combining 2 ideas to keep a fixed number of paths) look good for a digital option, and it should also handle barrier options

Quasi-Monte Carlo

Quasi-Monte Carlo methods can offer greatly improved convergence with respect to the number of samples N:

- in the best case, $O(N^{-1+\delta})$ error for arbitrary $\delta > 0$, instead of $O(N^{-1/2})$
- depends on knowledge/identification of “important dimensions” in an application
  - Brownian Bridge
  - Principal Component Analysis
- confidence intervals can be obtained by using randomized QMC
- working with Sloan, Kuo and Waterhouse, will try both rank-1 lattice rules and Sobol sequences