Multilevel Monte Carlo Path Simulation

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SDEs in Finance

In computational finance, stochastic differential equations are used to model the behaviour of

- stocks
- interest rates
- exchange rates
- weather
- electricity/gas demand
- crude oil prices
- . . .

The stochastic term accounts for the uncertainty of unpredictable day-to-day events.

SDEs in Finance

These models are then used to calculate “fair” prices for a huge range of financial options:

- an option to sell a stock portfolio at a specific price in 2 years time
- an option to buy aviation fuel at a specific price in 6 months time
- an option to sell US dollars at a specific exchange rate in 3 years time

In most cases, the buyer of the financial option is trying to reduce their risk.

SDEs in Finance

Examples:

- Geometric Brownian motion (Black-Scholes model for stock prices)

$$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W$$

- Cox-Ingersoll-Ross model (interest rates)

$$\mathrm{d}r = \alpha(b-r)\,\mathrm{d}t + \sigma\,\sqrt{r}\,\mathrm{d}W$$

- Heston stochastic volatility model (stock prices)

$$\begin{aligned}\mathrm{d}S &= r\,S\,\mathrm{d}t + \sqrt{V}\,S\,\mathrm{d}W_1 \\ \mathrm{d}V &= \lambda\,(\sigma^2 - V)\,\mathrm{d}t + \xi\,\sqrt{V}\,\mathrm{d}W_2\end{aligned}$$

with correlation ρ between $\mathrm{d}W_1$ and $\mathrm{d}W_2$

Generic Problem

Stochastic differential equation with general drift and volatility terms: SDE with general drift and volatility terms:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t)$$

$W(t)$ is a Wiener variable with the properties that for any $q<r<s<t$, $W(t)-W(s)$ is Normally distributed with mean 0 and variance $t-s$, independent of $W(r)-W(q)$.

In many finance applications, we want to compute the expected value of an option dependent on the terminal state

$$P = f(S(T))$$

with a uniform Lipschitz bound,

$$|f(U) - f(V)| \ \leq c\ \|U - V\|\,, \quad \forall\ U,V.$$

Standard MC Approach

Euler discretisation with timestep h:

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n$$

Simplest estimator for expected payoff is an average of N independent path simulations:

$$\widehat{Y} = N^{-1} \sum_{i=1}^{N} f(\widehat{S}^{(i)}_{T/h})$$

Standard MC Approach

Two kinds of errors:

- statistical error, due to finite number of paths

$$V[\widehat{Y}] = N^{-1} V[f(\widehat{S}_{T/h})]$$

  so r.m.s. error = $O(N^{-1/2})$.

- discretisation bias, due to finite number of timesteps
  - weak convergence – $O(h)$ error in expected payoff
  - strong convergence – $O(h^{1/2})$ error in individual path

Standard MC Approach

Mean Square Error is $O\left(N^{-1} + h^2\right)$

- first term comes from variance of estimator
- second term comes from bias due to weak convergence

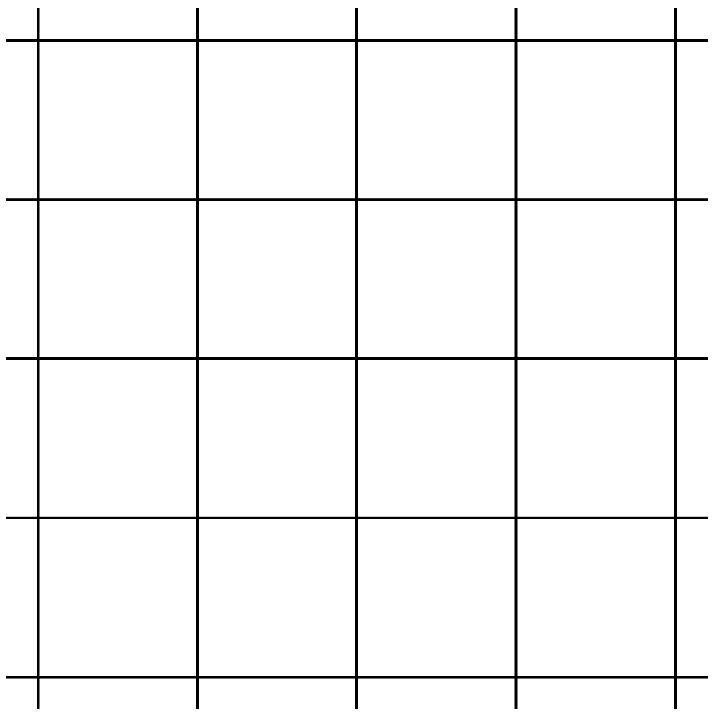
To make this $O(\varepsilon^2)$ requires

$$N = O(\varepsilon^{-2}), \quad h = O(\varepsilon) \quad \Longrightarrow \quad \text{cost} = O(N\,h^{-1}) = O(\varepsilon^{-3})$$

Aim is to improve this cost to $O\left(\varepsilon^{-2}(\log \varepsilon)^2\right)$

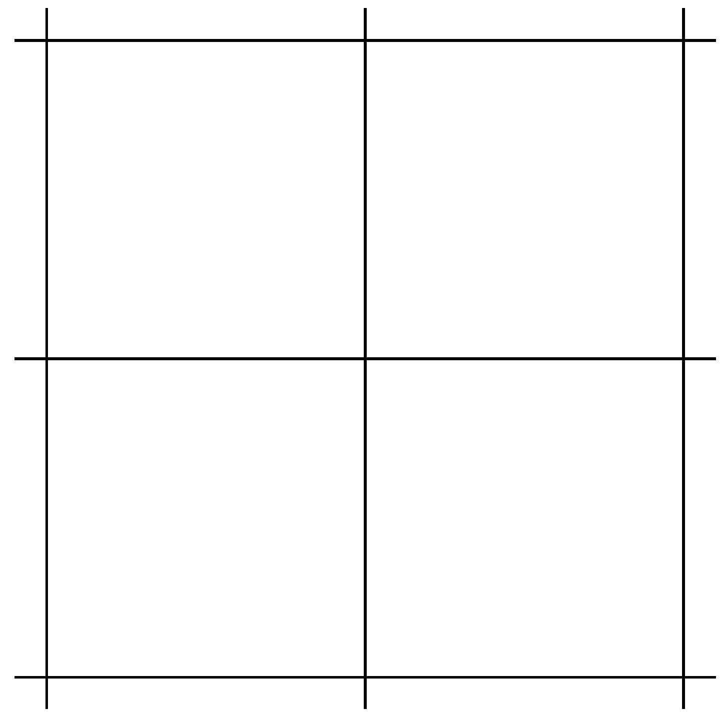
Multigrid

A powerful technique for solving PDE discretisations:



Fine grid

more accurate
more expensive



Coarse grid

less accurate
less expensive

Multigrid

Multigrid combines calculations on a nested sequence of grids to get the accuracy of the finest grid at a much lower computational cost.

We will use a similar idea to achieve variance reduction in Monte Carlo path calculations, combining simulations with different numbers of timesteps – same accuracy as finest calculations, but at a much lower computational cost.

Other Research

- In Dec. 2005, Ahmed Kebaier published an article in *Annals of Applied Probability* describing a two-level method which reduces the cost to $O\left(\varepsilon^{-2.5}\right)$.
- Also in Dec. 2005, Adam Speight wrote a working paper describing a similar multilevel use of control variates, but without an analysis of its complexity.
- There are also close similarities to a multilevel technique developed by Stefan Heinrich for parametric integration (*Journal of Complexity*, 1998)

Multilevel MC Approach

Consider multiple sets of simulations with different timesteps $h_l = 2^{-l}\,T,\ l = 0, 1, \ldots, L$, and payoff $\widehat{P}_l$

$$E[\widehat{P}_L] = E[\widehat{P}_0] + \sum_{l=1}^{L} E[\widehat{P}_l - \widehat{P}_{l-1}]$$

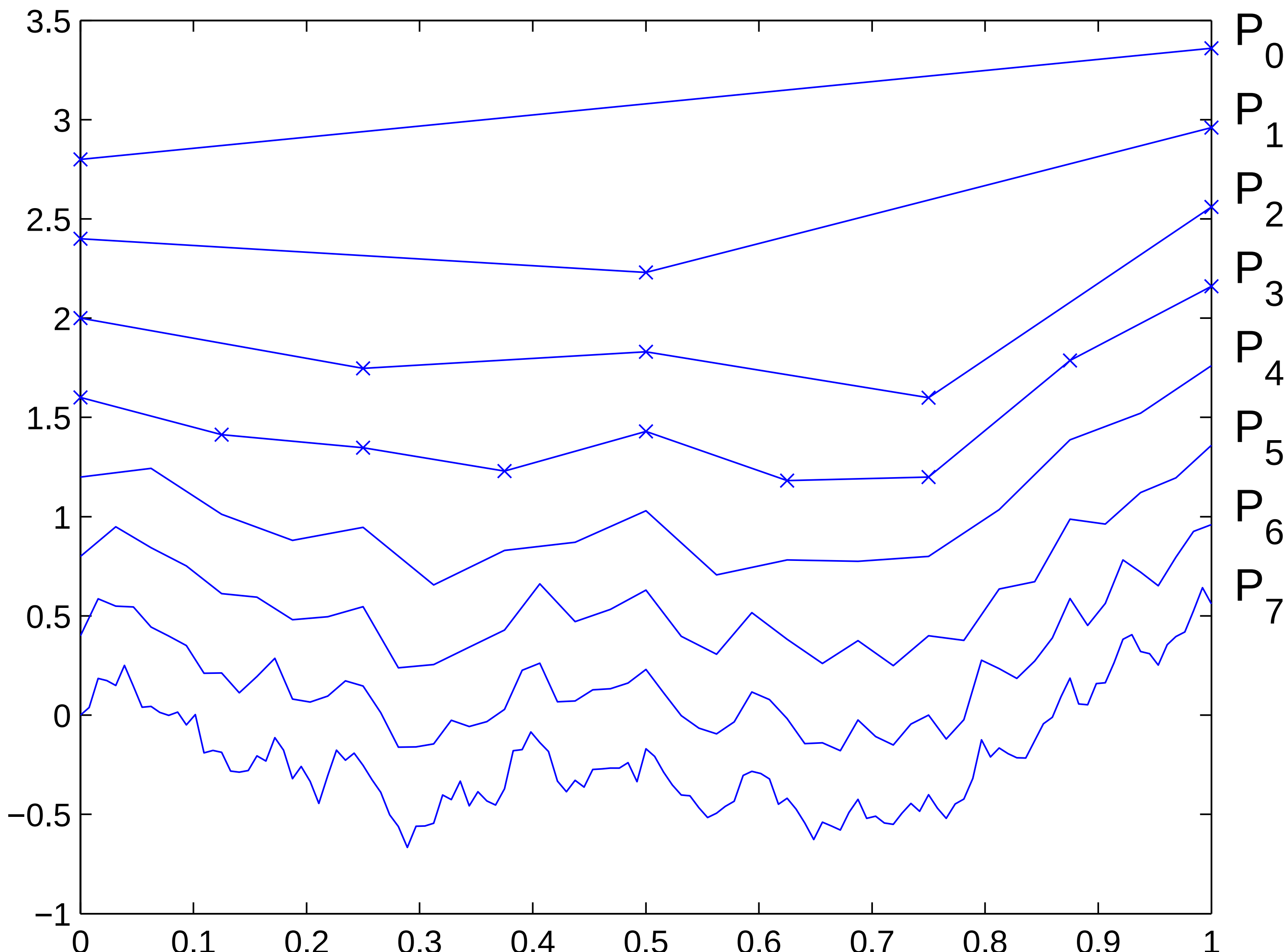
Expected value is same – aim is to reduce variance of estimator for a fixed computational cost.

Key point: approximate $E[\widehat{P}_l - \widehat{P}_{l-1}]$ using N_l simulations with $\widehat{P}_l$ and $\widehat{P}_{l-1}$ obtained using <u>same</u> Brownian path.

$$\widehat{Y}_l = N_l^{-1} \sum_{i=1}^{N_l} \left(\widehat{P}_l^{(i)} - \widehat{P}_{l-1}^{(i)} \right)$$

Multilevel MC Approach

Discrete Brownian path at different levels



Multilevel MC Approach

- each level adds more detail to Brownian path
- $E[\widehat{P}_l - \widehat{P}_{l-1}]$ reflects impact of that extra detail on the payoff
- different timescales handled by different levels
– similar to different wavelengths being handled by different grids in multigrid

Multilevel MC Approach

Using independent paths for each level, the variance of the combined estimator is

$$V\left[\sum_{l=0}^{L} \widehat{Y}_l\right] = \sum_{l=0}^{L} N_l^{-1} V_l, \qquad V_l \equiv V[\widehat{P}_l - \widehat{P}_{l-1}],$$

and the computational cost is proportional to $\displaystyle\sum_{l=0}^{L} N_l\, h_l^{-1}$.

Hence, the variance is minimised for a fixed computational cost by choosing N_l to be proportional to $\sqrt{V_l\, h_l}$.

The constant of proportionality can be chosen so that the combined variance is $O(\varepsilon^2)$.

Multilevel MC Approach

For the Euler discretisation and the Lipschitz payoff function

$$V[\widehat{P}_l - P] = O(h_l) \quad \Longrightarrow \quad V[\widehat{P}_l - \widehat{P}_{l-1}] = O(h_l)$$

and the optimal N_l is asymptotically proportional to h_l.

To make the combined variance $O(\varepsilon^2)$ requires

$$N_l = O(\varepsilon^{-2} L\, h_l).$$

To make the bias $O(\varepsilon)$ requires

$$L = \log_2 \varepsilon^{-1} + O(1) \quad \Longrightarrow \quad h_L = O(\varepsilon).$$

Hence, we obtain an $O(\varepsilon^2)$ MSE for a computational cost which is $O(\varepsilon^{-2} L^2) = O(\varepsilon^{-2} (\log \varepsilon)^2)$.

Multilevel MC Approach

Theorem: *Let P be a functional of the solution of a stochastic o.d.e., and $\widehat{P}_l$ the discrete approximation using a timestep $h_l = M^{-l}\, T$.*

If *there exist independent estimators $\widehat{Y}_l$ based on N_l Monte Carlo samples, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that*

i) $E[\widehat{P}_l - P] \leq c_1\, h_l^\alpha$

ii) $E[\widehat{Y}_l] = \begin{cases} E[\widehat{P}_0], & l = 0 \\ E[\widehat{P}_l - \widehat{P}_{l-1}], & l > 0 \end{cases}$

iii) $V[\widehat{Y}_l] \leq c_2\, N_l^{-1} h_l^\beta$

iv) C_l*, the computational complexity of* $\widehat{Y}_l$*, is bounded by*

$$C_l \leq c_3\, N_l\, h_l^{-1}$$

Multilevel MC Approach

then *there exists a positive constant* c_4 *such that for any* $\varepsilon < e^{-1}$ *there are values* L *and* N_l *for which the multi-level estimator*

$$\widehat{Y} = \sum_{l=0}^{L} \widehat{Y}_l,$$

has Mean Square Error $MSE \equiv E\left[\left(\widehat{Y} - E[P]\right)^2\right] < \varepsilon^2$

with a computational complexity C *with bound*

$$C \leq \begin{cases} c_4\, \varepsilon^{-2}, & \beta > 1, \\ c_4\, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_4\, \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Results

Geometric Brownian motion:

$$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W, \qquad 0 < t < 1,$$

$S(0)\!=\!1$, $r\!=\!0.05$, $\sigma\!=\!0.2$

Heston model:

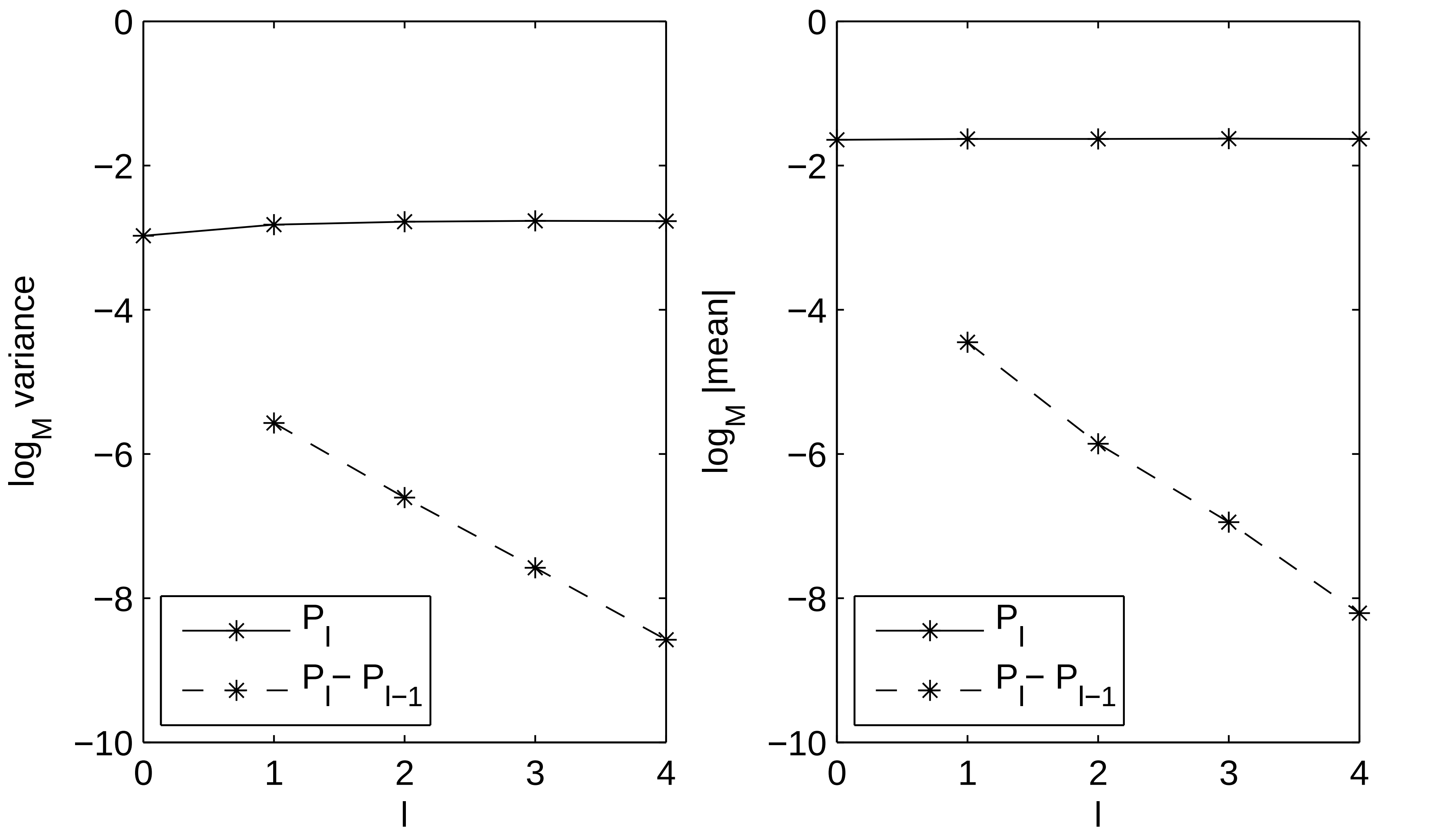
$$\begin{aligned} \mathrm{d}S &= r\,S\,\mathrm{d}t + \sqrt{V}\,S\,\mathrm{d}W_1, \qquad 0 < t < 1 \\ \mathrm{d}V &= \lambda\,(\sigma^2 - V)\,\mathrm{d}t + \xi\,\sqrt{V}\,\mathrm{d}W_2, \end{aligned}$$

$S(0)\!=\!1$, $V(0)\!=\!0.04$, $r\!=\!0.05$, $\sigma\!=\!0.2$, $\lambda\!=\!5$, $\xi\!=\!0.25$, $\rho\!=\!-0.5$

All calculations use $M\!=\!4$, more efficient than $M\!=\!2$.

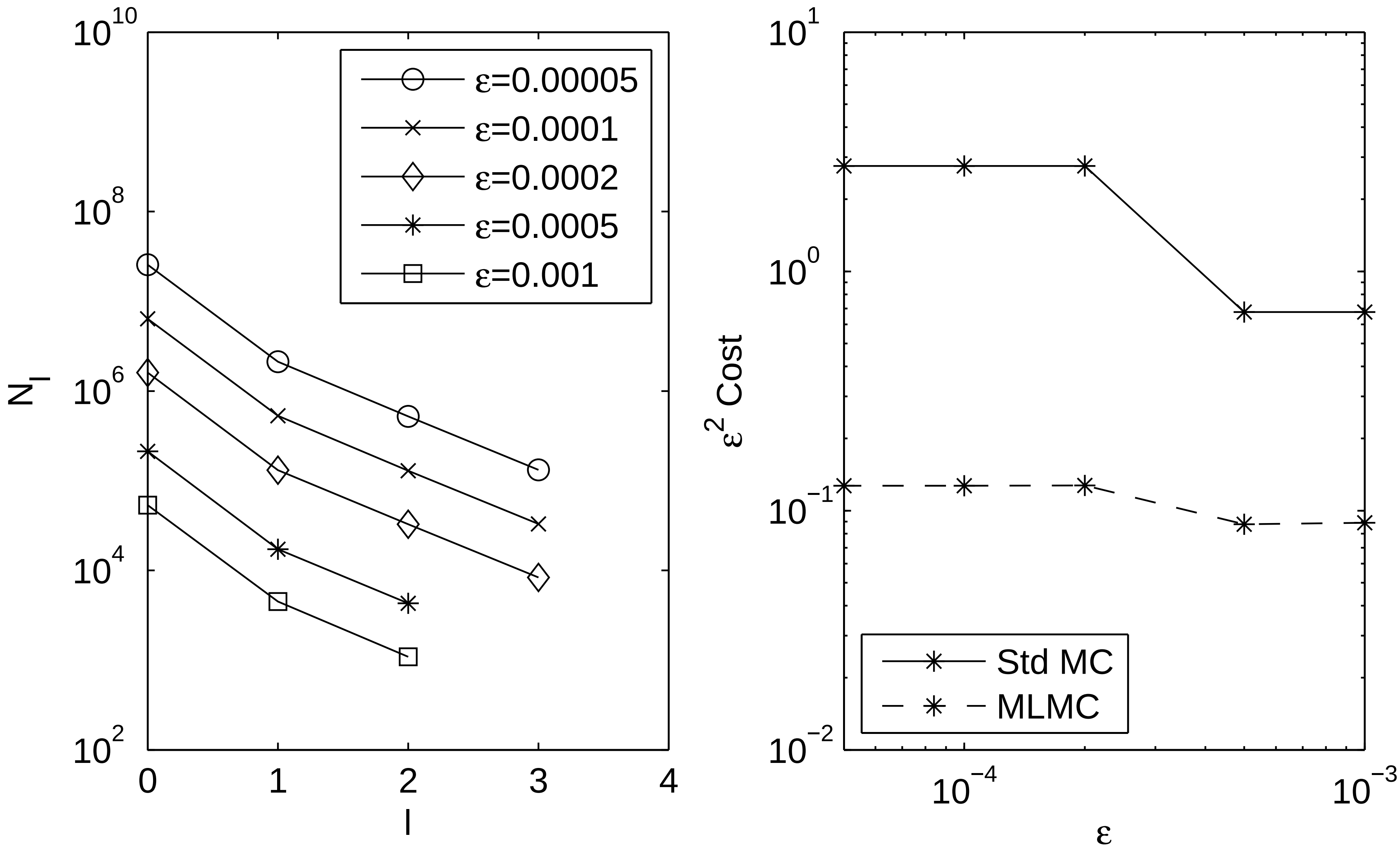
Results

GBM: European call, $\max(S(1)-1, 0)$



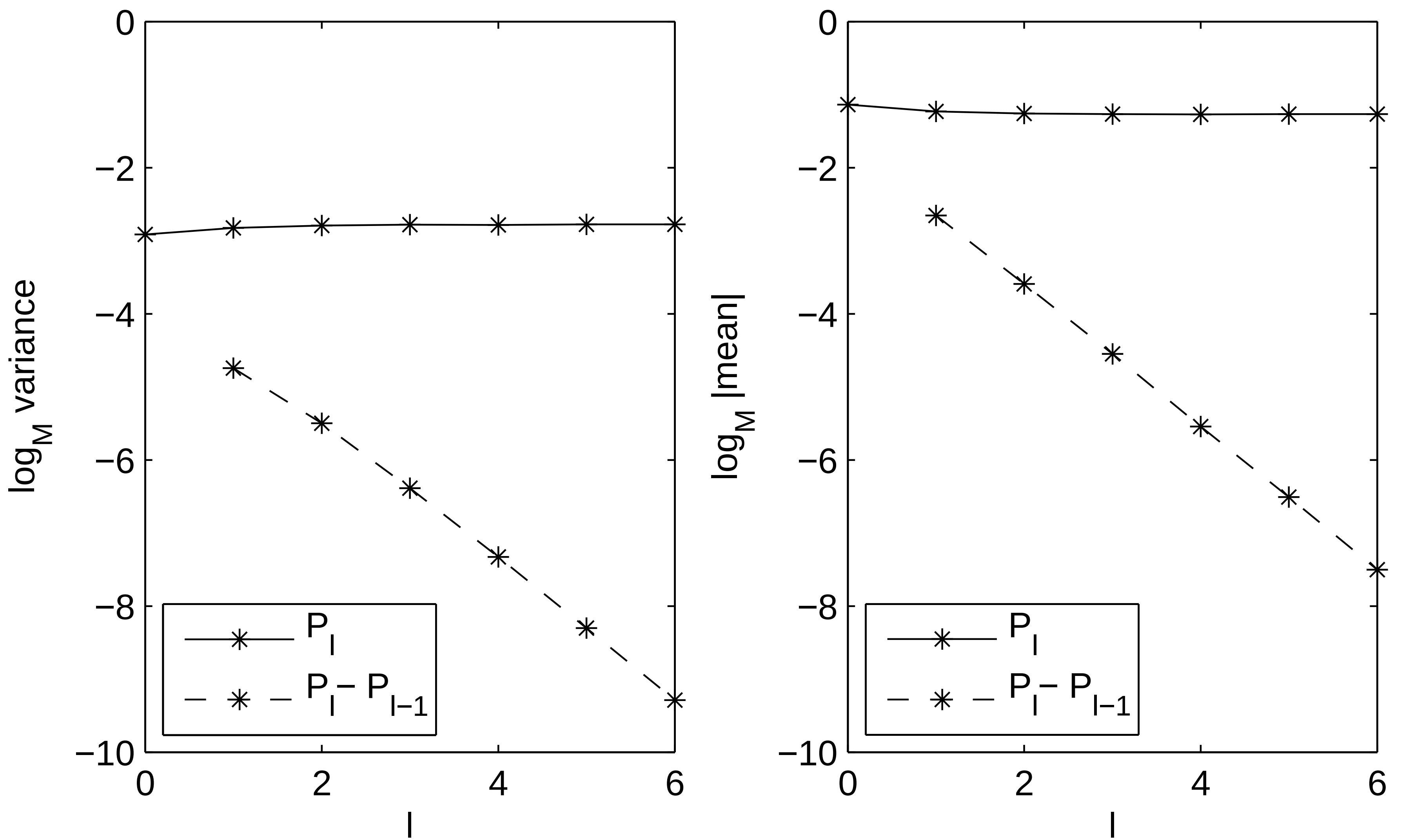
Results

GBM: European call, $\max(S(1)-1, 0)$



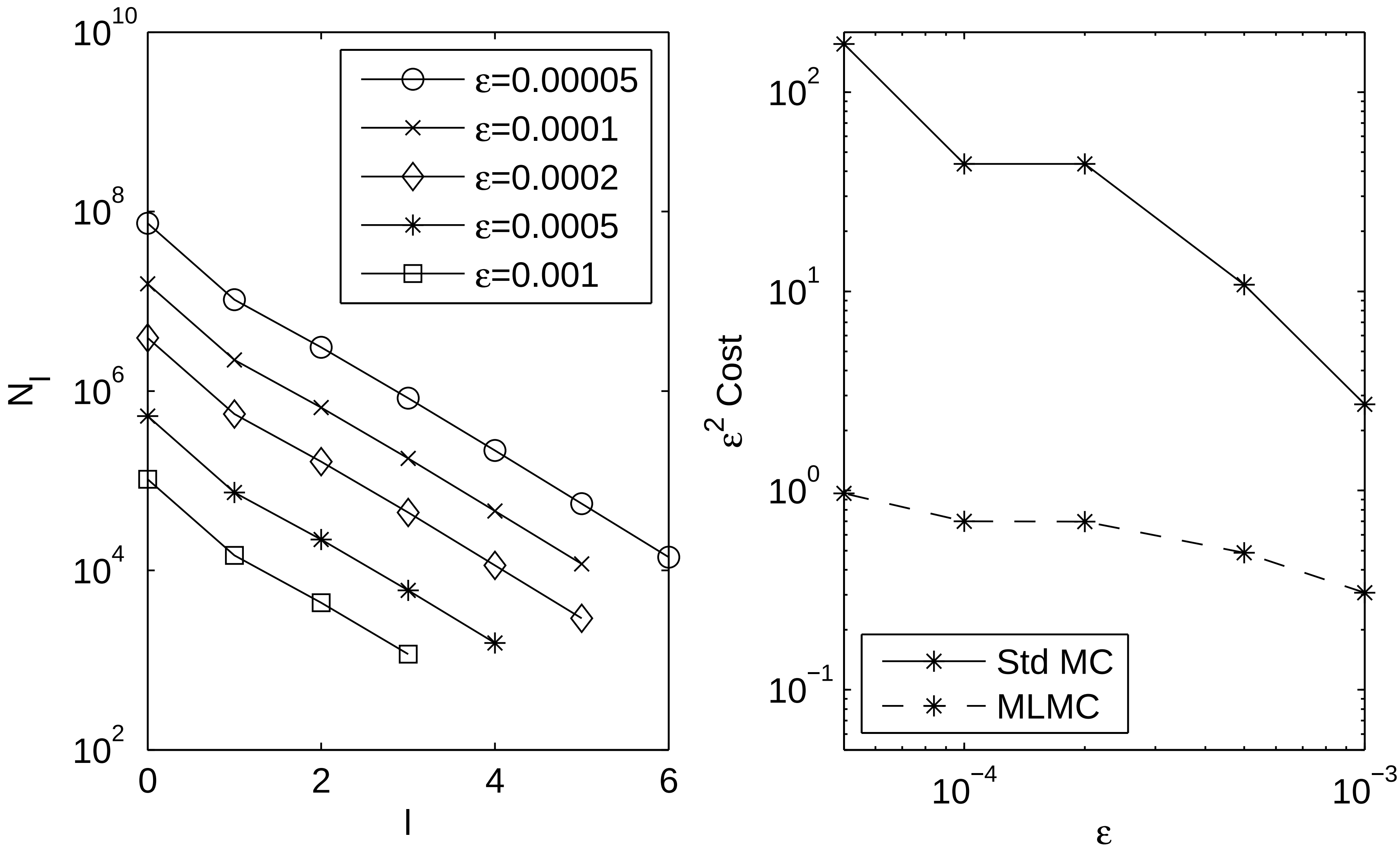
Results

GBM: lookback option, $S(1) - \min_{0<t<1} S(t)$



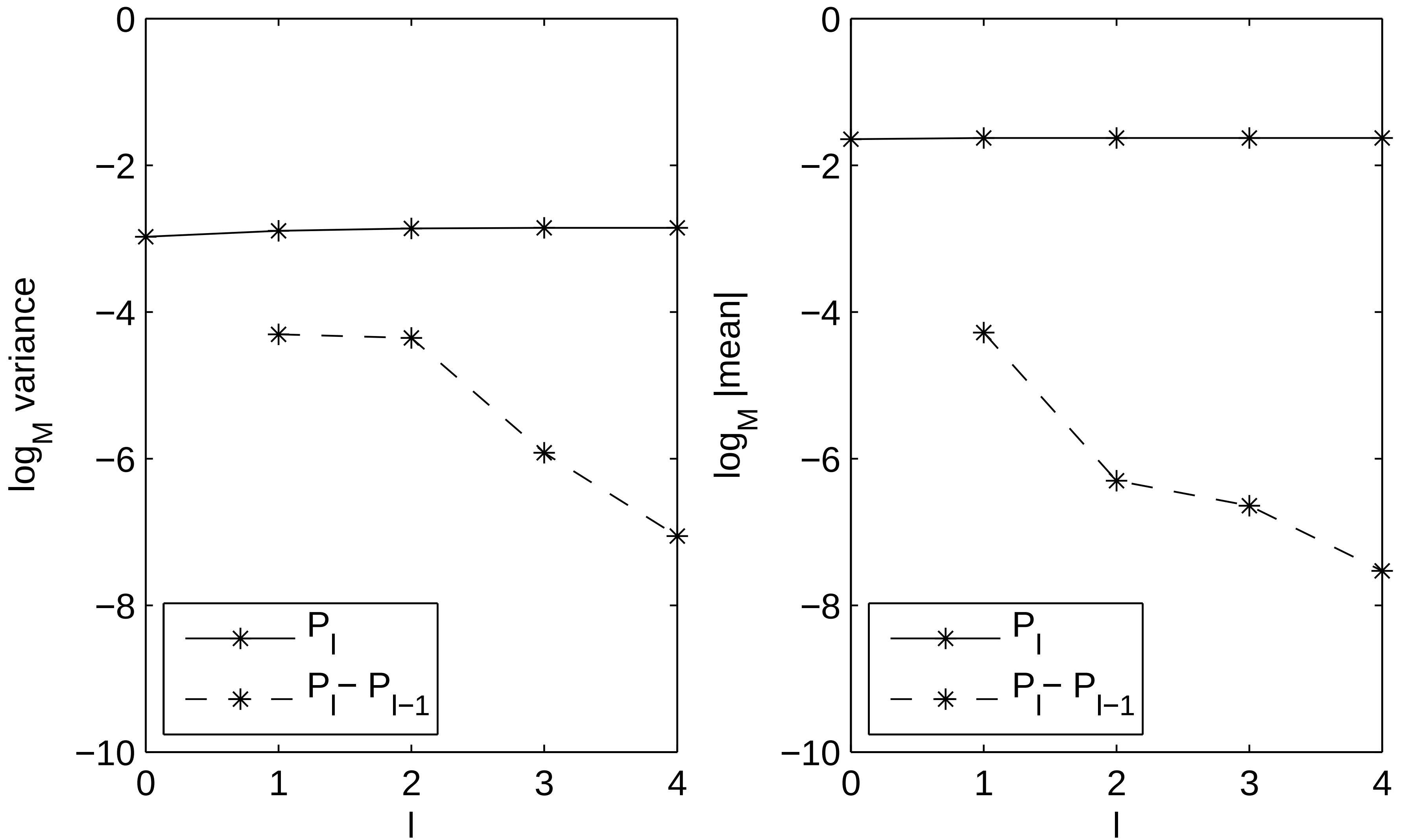
Results

GBM: lookback option, $S(1) - \min_{0<t<1} S(t)$



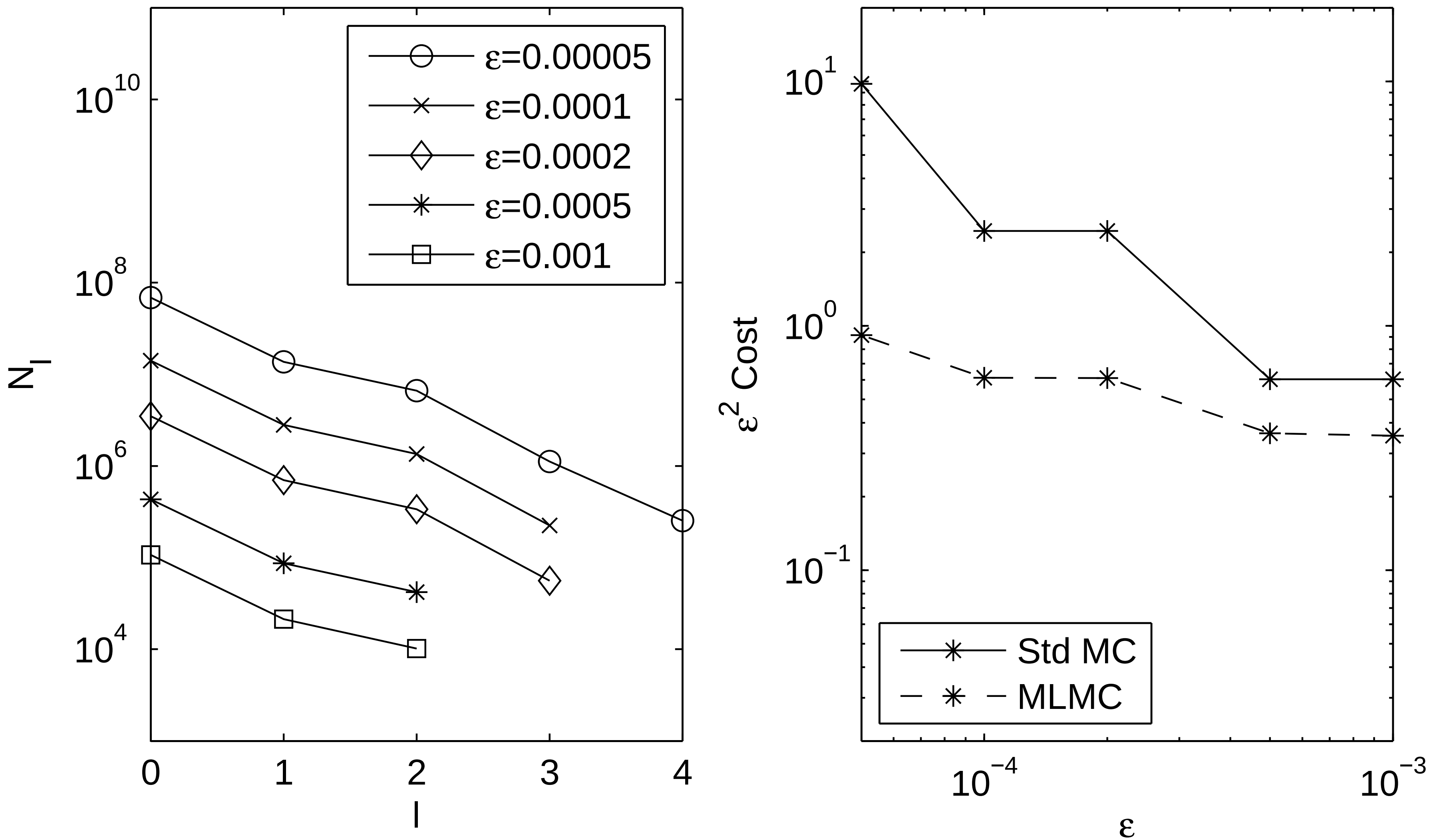
Results

Heston model: European call



Results

Heston model: European call



Conclusions

Results so far:

- improved order of complexity
- easy to implement
- significant benefits for model problems

However:

- lots of scope for further improvement
- need to test ideas on “real” finance applications

Future Work

- Milstein method for scalar SDEs
- extension to multi-dimensional SDEs
- use of quasi-Monte Carlo methods
- Greeks and calibration
- numerical analysis

Milstein Scheme

Generic scalar SDE:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t), \quad 0<t<T.$$

Milstein scheme:

$$\widehat{S}_{n+1} = \widehat{S}_n + a\,h + b\,\Delta W_n + \tfrac{1}{2}\,b'\,b\left((\Delta W_n)^2 - h\right).$$

Milstein Scheme

In scalar case:

- $O(h)$ strong convergence
- $O(\varepsilon^{-2})$ complexity for Lipschitz payoffs – trivial
- $O(\varepsilon^{-2})$ complexity for Asian, lookback, barrier and digital options using carefully constructed estimators, based on Brownian interpolation
- key idea: within each timestep, model the behaviour as simple Brownian motion conditional on the two end-points – analytic results exist for distribution of min/max/average

Milstein Scheme

Generic vector SDE:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t), \quad 0<t<T,$$

with correlation matrix $\Omega(S,t)$ between elements of $\mathrm{d}W(t)$.

Milstein scheme:

$$\begin{aligned} \widehat{S}_{i,n+1} &= \widehat{S}_{i,n} + a_i\,h + b_{ij}\,\Delta W_{j,n} \\ &\quad + \tfrac{1}{2}\,\frac{\partial b_{ij}}{\partial S_l}\,b_{lk}\left(\Delta W_{j,n}\,\Delta W_{k,n} - h\,\Omega_{jk} - A_{jk,n}\right) \end{aligned}$$

with implied summation, and Lévy areas defined as

$$A_{jk,n} = \int_{t_n}^{t_{n+1}} (W_j(t) - W_j(t_n))\,\mathrm{d}W_k - (W_k(t) - W_k(t_n))\,\mathrm{d}W_j.$$

Milstein Scheme

In vector case:

- $O(h)$ strong convergence if Lévy areas are simulated correctly – expensive
- $O(h^{1/2})$ strong convergence in general if Lévy areas are omitted, except if a certain commutativity condition is satisfied (useful for a number of real cases)
- Lipschitz payoffs can be handled well using antithetic variables
- Other cases may require approximate simulation of Lévy areas

Quasi-Monte Carlo

Quasi-Monte Carlo methods can offer greatly improved convergence with respect to the number of samples N:

- in the best case, $O(N^{-1+\delta})$ error for arbitrary $\delta > 0$, instead of $O(N^{-1/2})$
- depends on knowledge/identification of “important dimensions” in an application
  - Brownian Bridge
  - Principal Component Analysis
- confidence intervals can be obtained by using randomized QMC
- working with Sloan, Kuo and Waterhouse, will try both rank-1 lattice rules and Sobol sequences

Numerical Analysis

In the simplest case (Euler discretisation, Lipschitz payoff) analysis of the multilevel method is an immediate consequence of existing literature – in other cases, it is not

Currently developing methods based on non-rigorous asymptotic analysis (looking at leading order errors)

Starting collaboration with Des Higham and Xuerong Mao to analyse these methods properly

Final words

- interesting new approach to Monte Carlo path simulation
- preliminary results look very encouraging
- lots more research to be done
- enjoying new collaborations to develop the methods and their numerical analysis

Working Paper

M.B. Giles, "Multi-level Monte Carlo path simulation"
Oxford University Computing Laboratory
Numerical Analysis Report NA-06/03

`www.comlab.ox.ac.uk/mike.giles/finance.html`

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