Continuum Mechanics Exercises

1. **Material derivatives** Imagine you had carried a thermometer with you to Karthaus, and attached it to the outside of every bus/train/plane/*etc.* that you used. Sketch the temperature it would have measured as a function of time. Label the periods when the material derivative DT/Dt (as measured by the thermometer) was less than than the local derivative $\partial T/\partial t$.

2. **Strain rates** For each of the following velocity fields $\mathbf{u}(x, y, z)$, find all non-zero components of the strain-rate tensor $\dot{\varepsilon}_{ij}$ and confirm that the velocity field is incompressible (α, λ, H and U are positive constants). Where on an ice sheet or glacier might you see each type of velocity field?

$$\text{(i) } \mathbf{u} = \begin{pmatrix} U + \alpha y \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(ii) } \mathbf{u} = \lambda \begin{pmatrix} H^4 - (H - z)^4 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(iii) } \mathbf{u} = \begin{pmatrix} U + \alpha x \\ 0 \\ -\alpha z \end{pmatrix}.$$

[Recall that $\dot{\varepsilon}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ and $\nabla \cdot \mathbf{u} = \dot{\varepsilon}_{ii} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$.]

3. **Shallow ice approximation** The approximate form of the momentum equations and Glen's flow law for the shallow ice approximation in two dimensions is

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z}, \qquad 0 = -\frac{\partial p}{\partial z} - \rho g, \qquad \frac{\partial u}{\partial z} = 2A\tau^n,$$

where $\tau = \tau_{xz}$ is the shear stress, p is the pressure, ρ is the density, g is gravity, and A is the flow-law coefficient, assumed constant.

   (i) Consider an ice sheet on a flat bed at $z = 0$, with surface at $z = h(x)$. What are the appropriate boundary conditions for p and τ at the surface? Calculate p and show that

   $$\tau(x, z) = -\rho g \frac{\partial h}{\partial x} (h - z).$$

   (ii) Show that the velocity profile assuming no slip at the bed is

   $$u(x, z) = \frac{2A}{n + 1} \left(-\rho g \frac{\partial h}{\partial x} \right)^n \left[h^{n+1} - (h - z)^{n+1} \right].$$

   (iii) Suppose the ice thickness is given by $h(x) = H \left(1 - x^2/L^2 \right)$ where H and L are constant. Draw a sketch of the ice sheet, and label the point with (a) the largest shear stress, and (b) the largest velocity.
   [You should be able to roughly locate these with physical reasoning, but may need to use some calculus to find the precise locations. You can save yourself some algebra by first deciding on the relevant value of z and then maximising over x.]

4. **Shallow ice approximation** (Harder extension) Use the two-dimensional continuity equation $\nabla \cdot \mathbf{u} = 0$, together with the kinematic conditions at $z = 0$ and $z = h(x, t)$, to derive the depth-integrated mass conservation equation

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a,$$

where $q = \int_0^h u \, dz$ is the ice flux, and a is the net accumulation.

Draw a sketch of $q(x)$ for the ice sheet in Q3. Assuming this is a steady state, also draw a sketch of $a(x)$ and show that the equilibrium line altitude (the surface height at which $a = 0$) is $\frac{2n+4}{3n+4} H$.