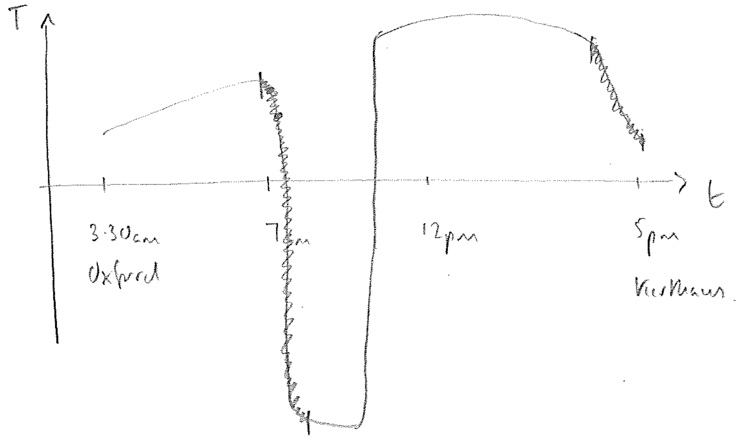
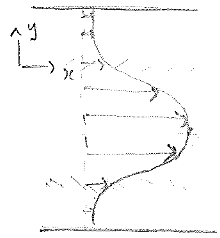
1.



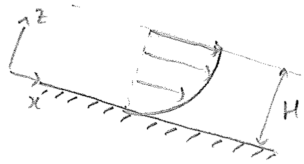
Shaded periods have $\frac{DT}{Dt} < \frac{\partial T}{\partial t}$

(ie advective component $\underline{u} \cdot \nabla T < 0$; generally when travelling uphill)

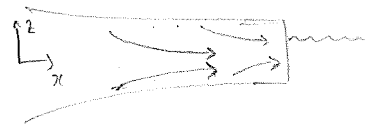
2. (i) $\underline{u} = \begin{pmatrix} \alpha y \\ 0 \\ 0 \end{pmatrix}$ $\quad \dot{\varepsilon}_{ij} = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \nabla \cdot \underline{u} = 0.$ eg. ice-stream margin



(ii) $\underline{u} = \lambda \begin{pmatrix} H^4 - (H-z)^4 \\ 0 \\ 0 \end{pmatrix}$ $\quad \dot{\varepsilon}_{ij} = \begin{pmatrix} 0 & 0 & \lambda(H-z)^3 \\ 0 & 0 & 0 \\ \lambda(H-z)^3 & 0 & 0 \end{pmatrix}$ $\quad \nabla \cdot \underline{u} = 0.$ Shallow ice approx.
eg. valley glacier



(iii) $\underline{u} = \begin{pmatrix} \alpha x \\ 0 \\ -\alpha z \end{pmatrix}$ $\quad \dot{\varepsilon}_{ij} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha \end{pmatrix}$ $\quad \nabla \cdot \underline{u} = \alpha - \alpha = 0$ eg. ice shelf. (stretching flow)



3. (i) $\boxed{p = \tau = 0 \quad \text{at } z = h}$ (atmospheric pressure taken as zero)

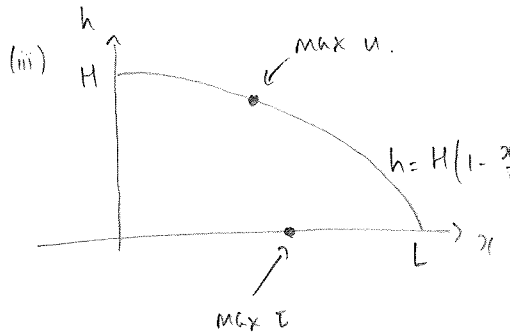
$$0 = -\frac{\partial p}{\partial z} - \rho g \Rightarrow \boxed{p = \rho g (h - z)}$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z} \Rightarrow \frac{\partial \tau}{\partial z} = \rho g \frac{\partial h}{\partial x} \Rightarrow \boxed{\tau = \left(-\rho g \frac{\partial h}{\partial x}\right)(h - z)}$$

(ii) $$\frac{\partial u}{\partial z} = 2A\tau^n = 2A\left(-\rho g \frac{\partial h}{\partial x}\right)^n (h-z)^n \Rightarrow \boxed{u = \frac{2A}{n+1}\left(-\rho g \frac{\partial h}{\partial x}\right)^n \left[h^{n+1} - (h-z)^{n+1}\right]}$$

(constant chosen so that $u = 0$ at $z = 0$.)

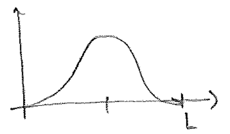
(iii)



$$h = H\left(1 - \frac{x^2}{L^2}\right) \Rightarrow \frac{\partial h}{\partial x} = -\frac{2Hx}{L^2}$$

- u increases with z, so the largest velocity must be at the surface $z = h$, where

$$u = \frac{2A}{n+1}\left(-\rho g \frac{\partial h}{\partial x}\right)^n h^{n+1} \propto x^n \left(1 - \frac{x^2}{L^2}\right)^{n+1}$$



(product rule)

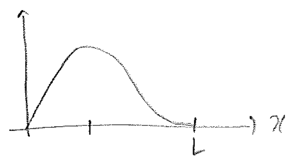
Maximum where $\frac{\partial}{\partial x} = 0$ i.e. $n x^{n-1}\left(1 - \frac{x^2}{L^2}\right)^{n+1} + x^n (n+1)\left(1 - \frac{x^2}{L^2}\right)^n \left(-\frac{2x}{L^2}\right) = 0$

$$\Rightarrow n\left(1 - \frac{x^2}{L^2}\right) = 2(n+1)\frac{x^2}{L^2} \quad \text{(cancelling factors)}$$

$$\Rightarrow \left(\frac{x}{L}\right)^2 = \frac{n}{3n+2} = \frac{3}{11} \text{ for } n = 3. \quad \text{so } \boxed{x = \left(\frac{n}{3n+2}\right)^{1/2} L}$$

- τ is largest at $z = 0$ for each x, so largest value must be at $z = 0$, where

$$\tau = -\rho g h \frac{\partial h}{\partial x} \propto x\left(1 - \frac{x^2}{L^2}\right)$$



Maximum where $\frac{\partial}{\partial x} = 0$ i.e. $1 - \frac{3x^2}{L^2} = 0 \Rightarrow \boxed{x = \left(\frac{1}{3}\right)^{1/2} L.}$

(Note the maximum τ occurs at a larger value of x than the maximum u)

4. $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$ ($\nabla \cdot \underline{u} = 0$ in 2-d) with $w = 0$ at $z = 0$

$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = w + a$ at $z = h(x,t)$

↑ accumulation

Integrate from 0 to h; $\underbrace{\int_0^h \frac{\partial u}{\partial x}\, dz}_{= \frac{\partial}{\partial x}\Big(\underbrace{\int_0^h u\, dz}_{q}\Big) - \cancel{u|_h \frac{\partial h}{\partial x}}} = -[w]_0^h = -\frac{\partial h}{\partial t} - \cancel{u|_h \frac{\partial h}{\partial x}} + a$.

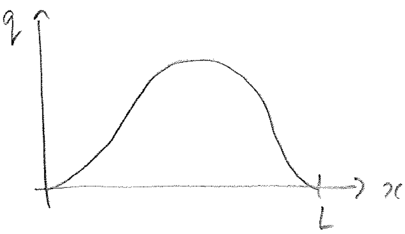
$$\Rightarrow \boxed{\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a}$$

From Q3, $q = \int_0^h u\, dz = \frac{2A\left(-\rho g \frac{\partial h}{\partial x}\right)^n}{n+1} \underbrace{\int_0^h h^{n+1} - (h-z)^{n+1}\, dz}$

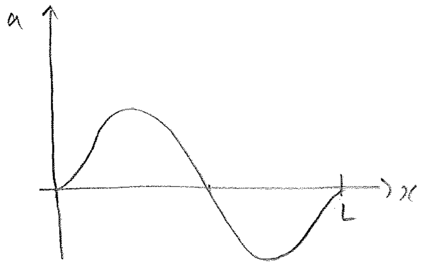
$$\left[-h^{n+1}(h-z) + \frac{1}{n+2}(h-z)^{n+2}\right]_0^h$$

$$= h^{n+2} - \frac{1}{n+2}h^{n+2} = \frac{n+1}{n+2}h^{n+2}$$

so $\boxed{q = \frac{2A}{n+2}\left(-\rho g \frac{\partial h}{\partial x}\right)^n h^{n+2}}$



In steady state $a = \frac{\partial q}{\partial x}$



$q \propto x^n\left(1 - \frac{x^2}{L^2}\right)^{n+2} \Rightarrow a = \frac{\partial q}{\partial x} \propto n x^{n-1}\left(1 - \frac{x^2}{L^2}\right)^{n+2} + (n+2)x^n\left(1 - \frac{x^2}{L^2}\right)^{n+1}\left(-\frac{2x}{L^2}\right)$

$$= x^{n-1}\left(1 - \frac{x^2}{L^2}\right)^{n+1}\left[n\left(1 - \frac{x^2}{L^2}\right) - 2(n+2)\frac{x^2}{L^2}\right]$$

$a = 0$ at $\boxed{\frac{x^2}{L^2} = \frac{n}{3n+4}}$, at which point $\boxed{h = H\left(1 - \frac{n}{3n+4}\right) = \frac{2n+4}{3n+4}H}$