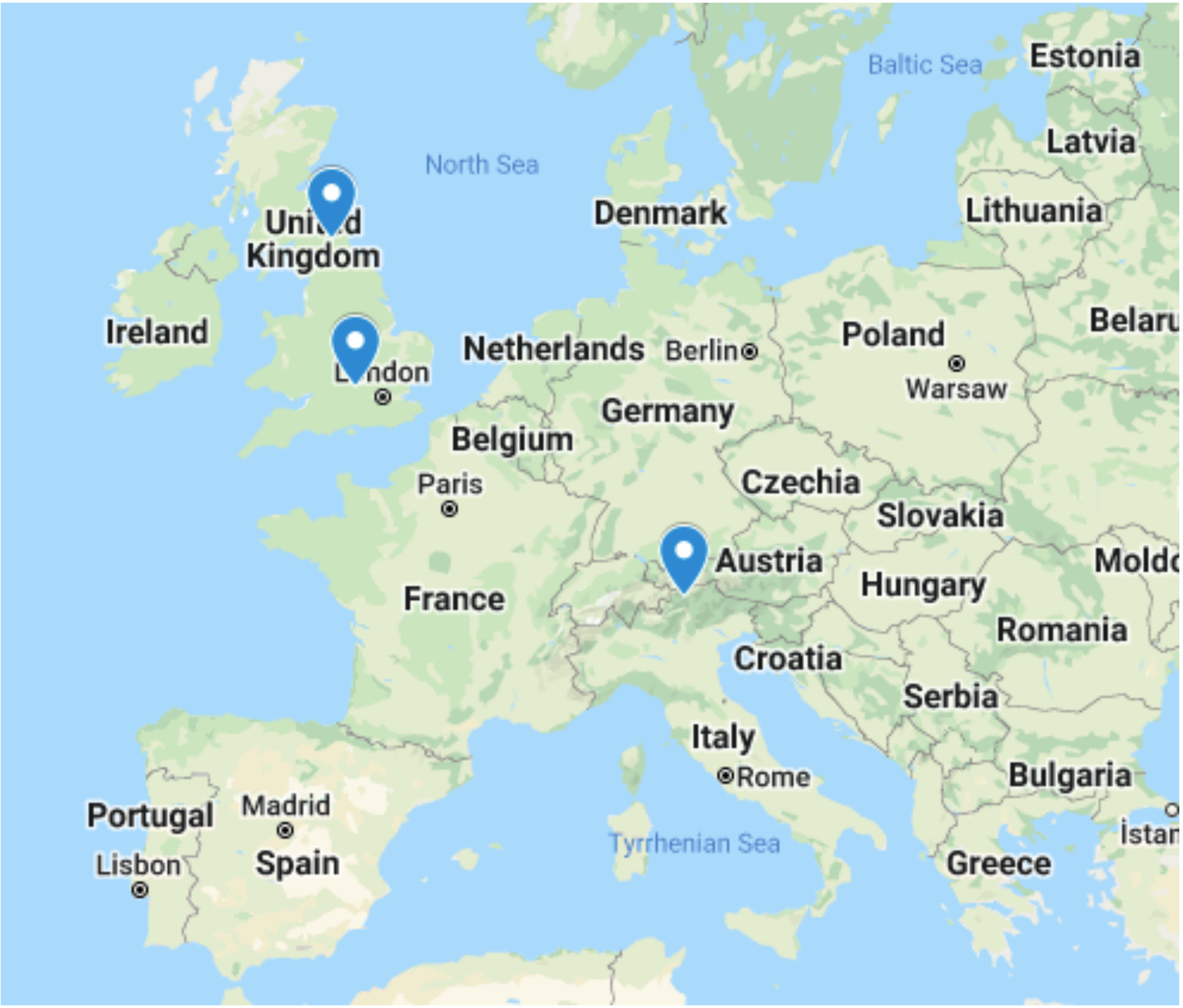
Continuum mechanics

Ian Hewitt, University of Oxford *hewitt@maths.ox.ac.uk*

Background


Baltic Sea
Estonia
Latvia
North Sea
United Kingdom
Denmark
Lithuania
Ireland
Netherlands
Berlin
Poland
Warsaw
London
Germany
Belgium
Paris
Czechia
Slovakia
Austria
France
Hungary
Romania
Croatia
Serbia
Italy
Rome
Bulgaria
Portugal
Madrid
Spain
Lisbon
Tyrrhenian Sea
Greece


Vernagt-Stausee
Lago di Vernago
Seerundweg
Fineilhof
1952
Vernagt
Vernago
Leithof
Kaser
Casera
Schnalser Bach
Bergerstgras
1767
Untergerstgras
1778
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Anderleit alla Costa
Oberhof Maso di Sopra
Schwarzer Adler
Mitterhof
1977
Leltwald
1892
2027
Berglalm
M.ga di Cortiserrade
2214
2302
2125
2401
Gfaltrina
1933
Archeo Parc
Unser Frau in Schnals
Madonna di Senales
Gde. Schnals
Com. Senales
Rainhof
1462
Gfallhof
1840
Auhöfe
Ora
1434
Texel
Sport-zone
1420
Gufigand
Ganda del Covolo
Schmied
Maso del Fabbro
1401
Brugghof
Raindlhöfe
Weghof
Maso Strada
1342
Schnals
Oberpifrol
Piverolo
(Pitair)
Portara
1545
Gorf
Corva
Karthaus
Certosa
1327
Brugghof
1087
Grubhof
la Cave
1283
Schröfwand
Crodarotta
2890
2717
2414
Grafalpe
Alpe della Grova
1960
1892
Schafalpe
B.ta delle Pecore
2737
Klammwald
Parco Naturale
Gruppo di Tessa
Gurschlberg
Monte Corticella
2444
2428
2404
Atzboden
Piano d'Azzo
2494
1993
1978
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M.ga Corticella
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1719
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Rauhstein
Sasso Scabro
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Kleesattel
Sella del Trifoglio
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Gurschlhof
Corticella
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Grubalm
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2270
2348
2151
Gerstgraserboden
Piano di Cortiserrade
Nock Spitze
Cima del Dosso
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Mastaunhof
Maso di Mastaun
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M.ga di Mastaun
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Schnalstal
Val Senales
Mastauntal
Val di Mastaun
Bichele
Mastaunbichl
(Monte Lavez)
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Lafetzalm
Malga Lavez
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Rio di Lavez
Außergufler Wald
Rio di Pinalto
Klosterwald
Albarsch
Ost
Est
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Gerstgraser Spitze
P. di Cortiserrade
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West
Ovest
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2981
Mastaunjoch
Passo di Mastaun
3133
Gamsei
Geigerkofel
2408
Penaudhof
Capanna di Pinalto
1789
2204
Kleine Wiegenspitze
Cima Culla Piccola
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2668
Mastaun Spitze
Mastaun
3200
Mastaunbrunnen
Sorg. di Mastaun
2545
Hohe Wiegenspitze
Cima Culla Alta
2959
2763
2475
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Kreuzspitze
M. Croce
2576
Alpberg
Klosteralm
M.ga del Convento
2152
Schwarze Wand
Croda Nera
2983
Malander
Malandro
3174
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Wiegen Spitze
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Seekar
Quaira del Lago
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Penaudtal
Setznagel
Il Chiodo
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Sardutzspitze
Cima della Tavola
2598
Penauderalm
M.ga di Pinalto
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(2697)
Erdscharte
Forcella Terza
3010
Gamswand
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(2815)
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Saxalbsee
Lago di Sassalbo
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Saxalb Alm
M.ga di Sassalbo
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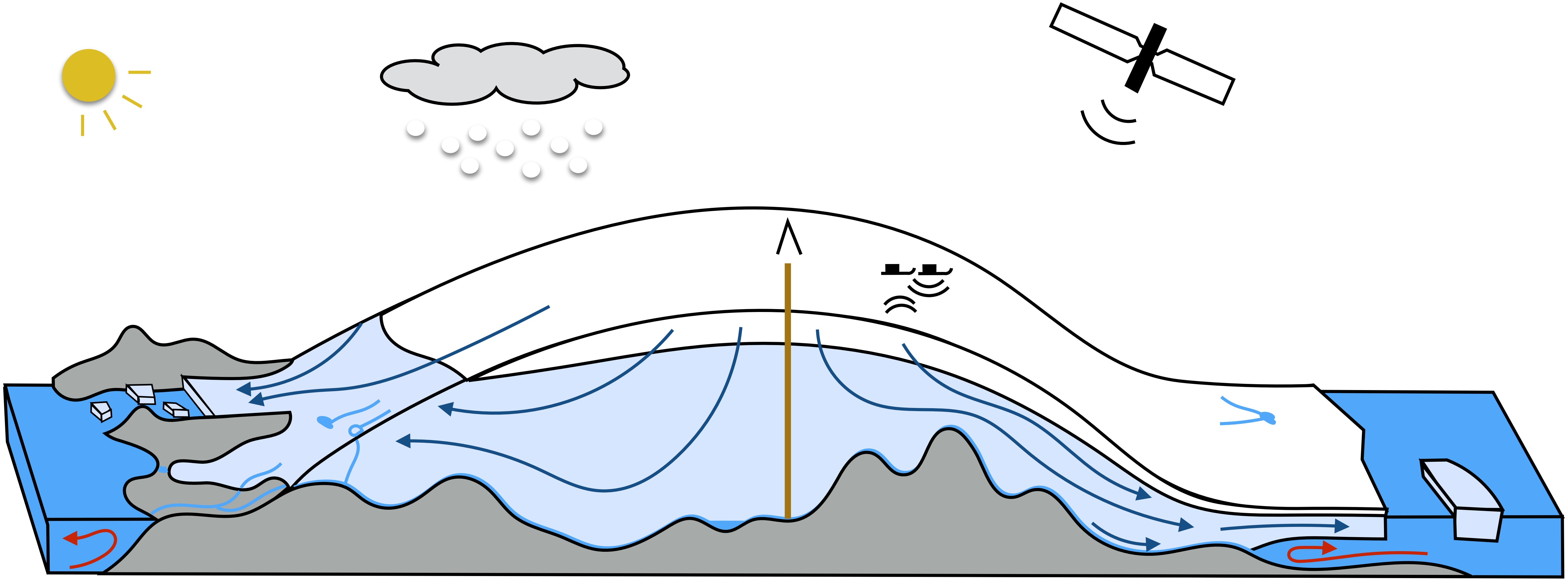
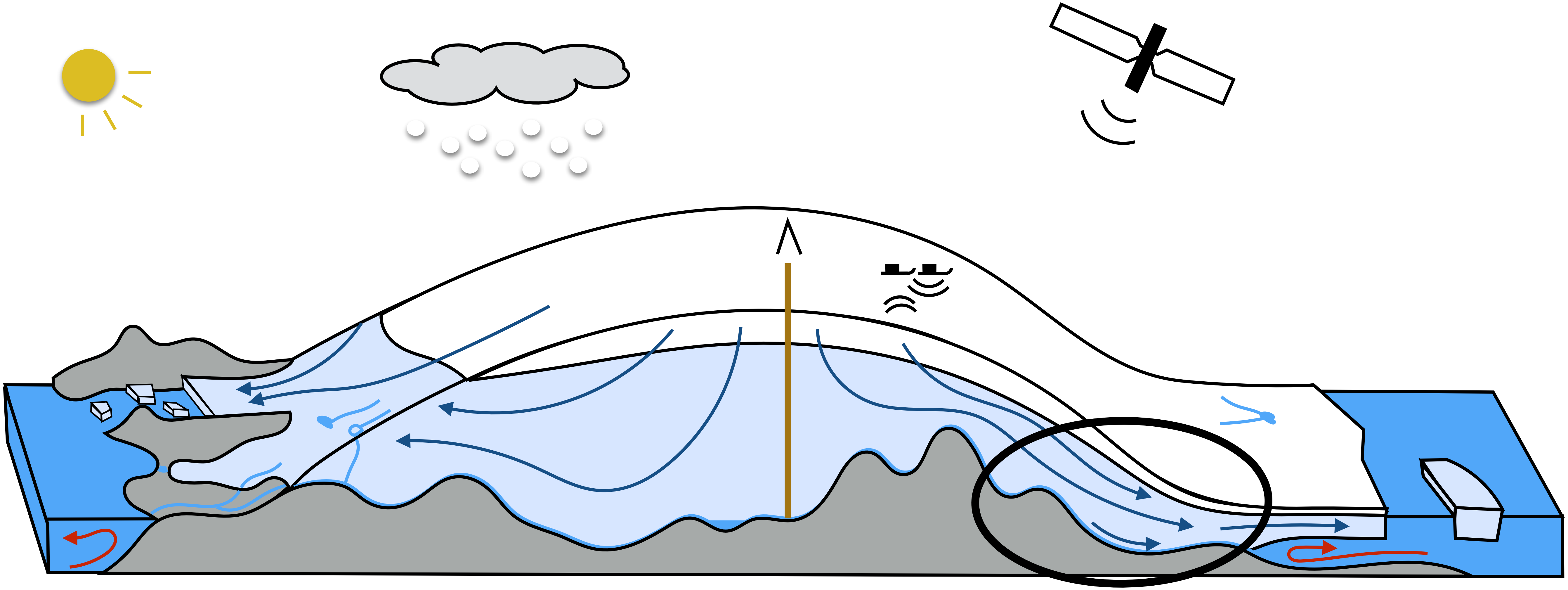
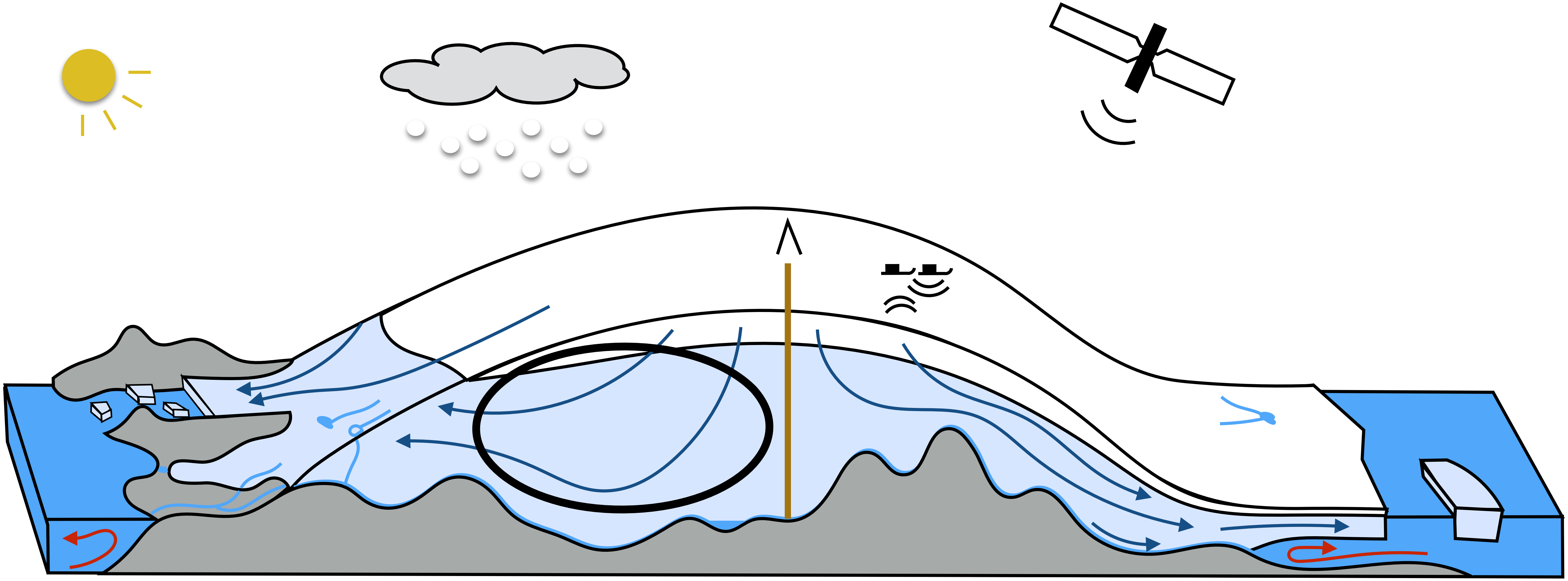
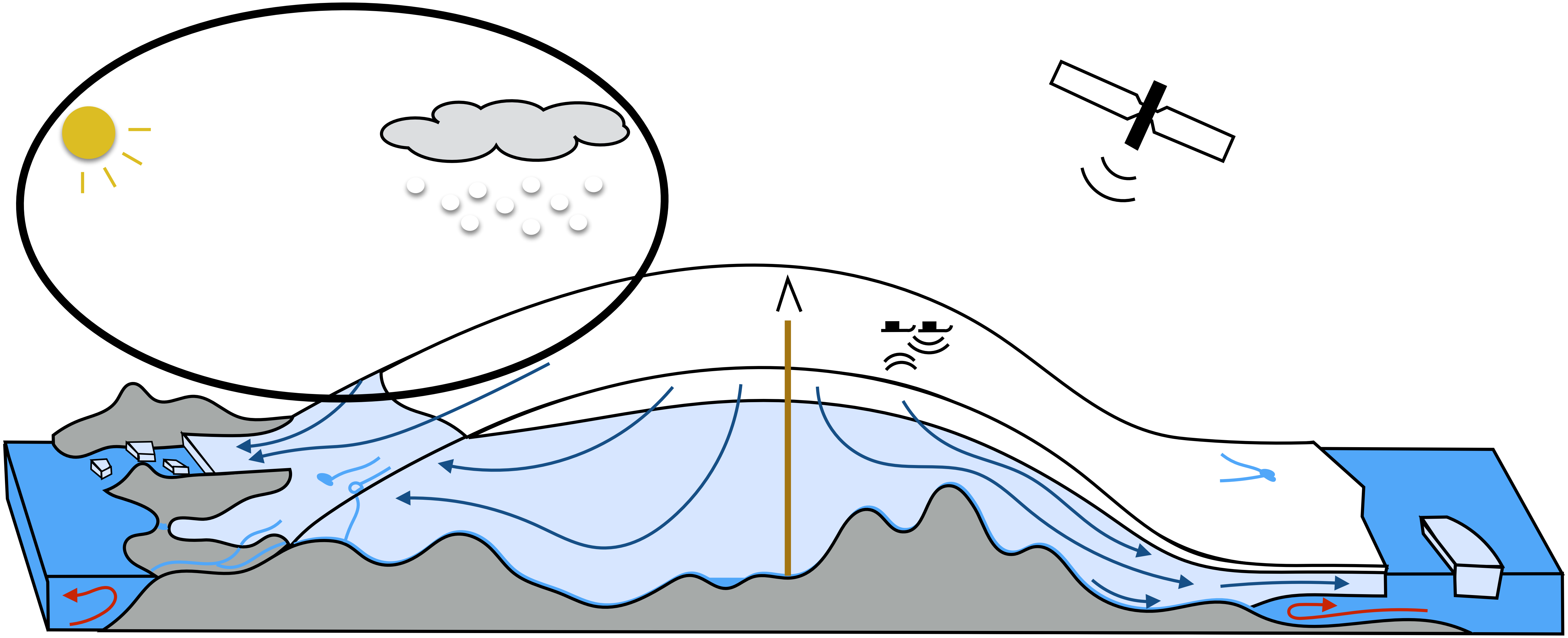
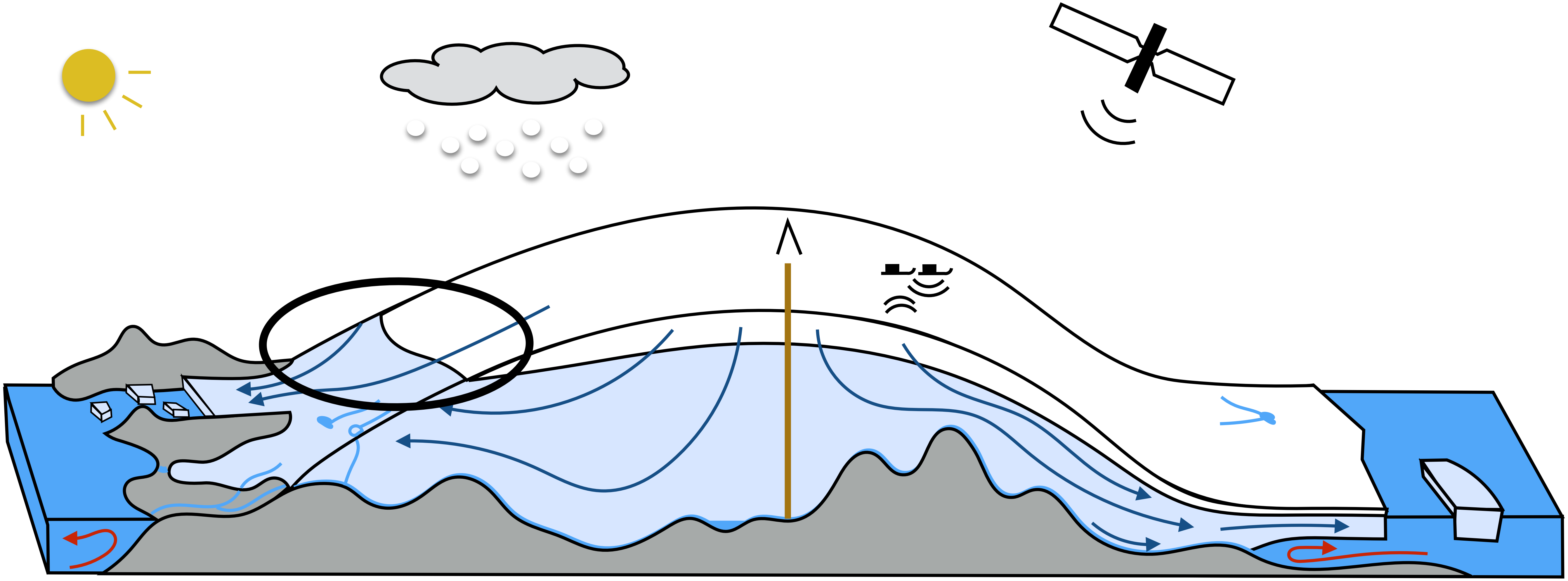
Gamsegg
(Trumser Spitz)
2602
Saxalbe

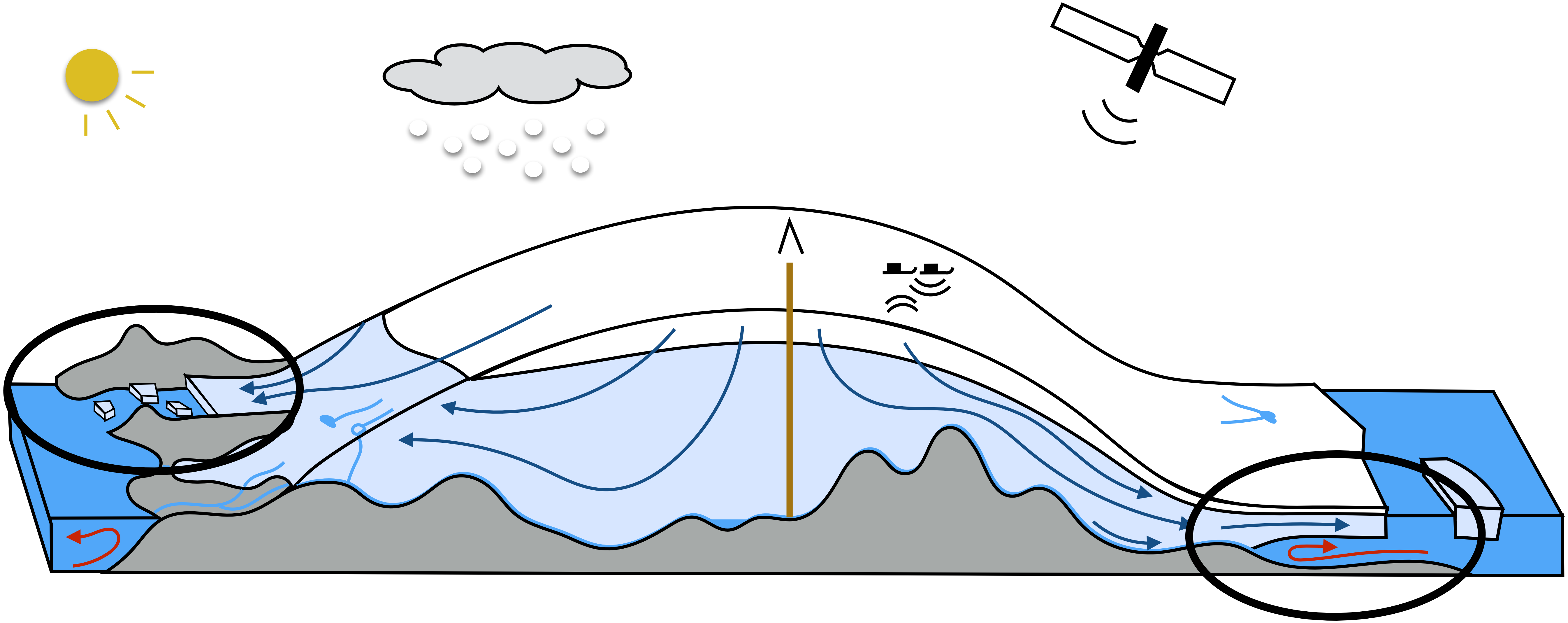


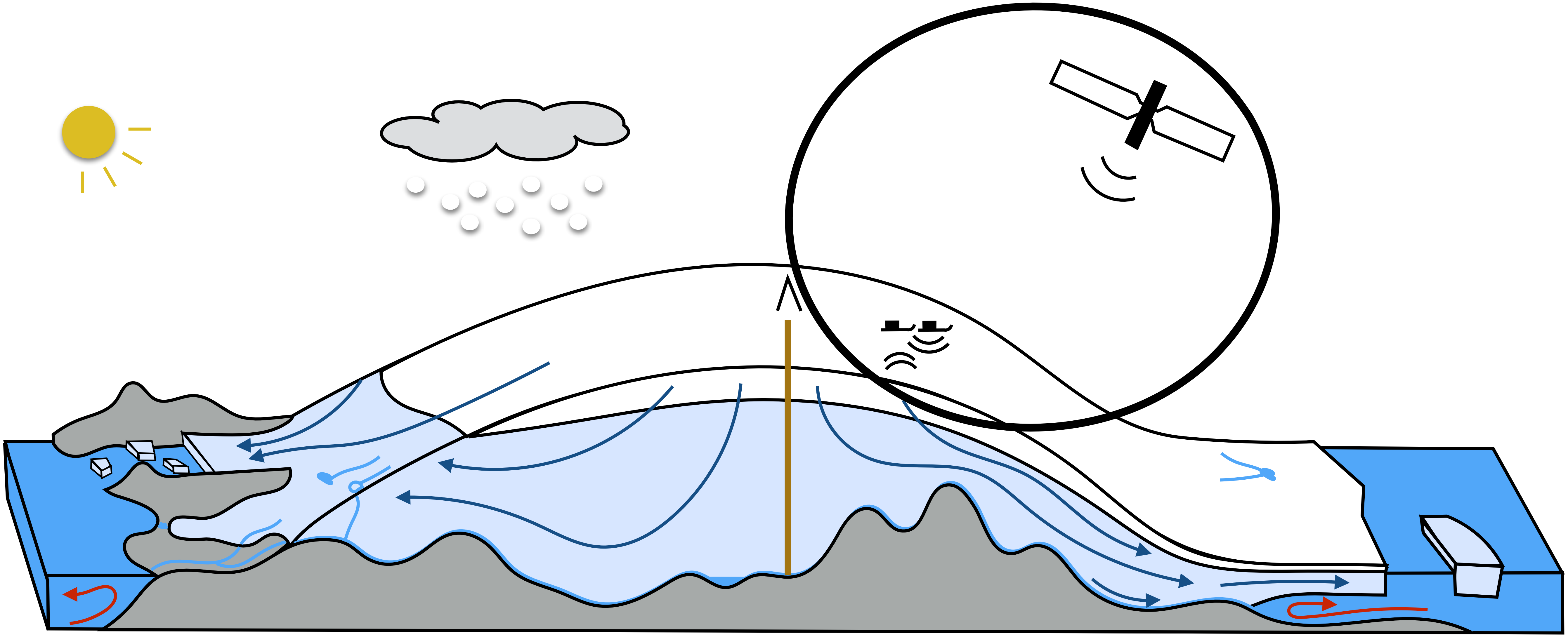


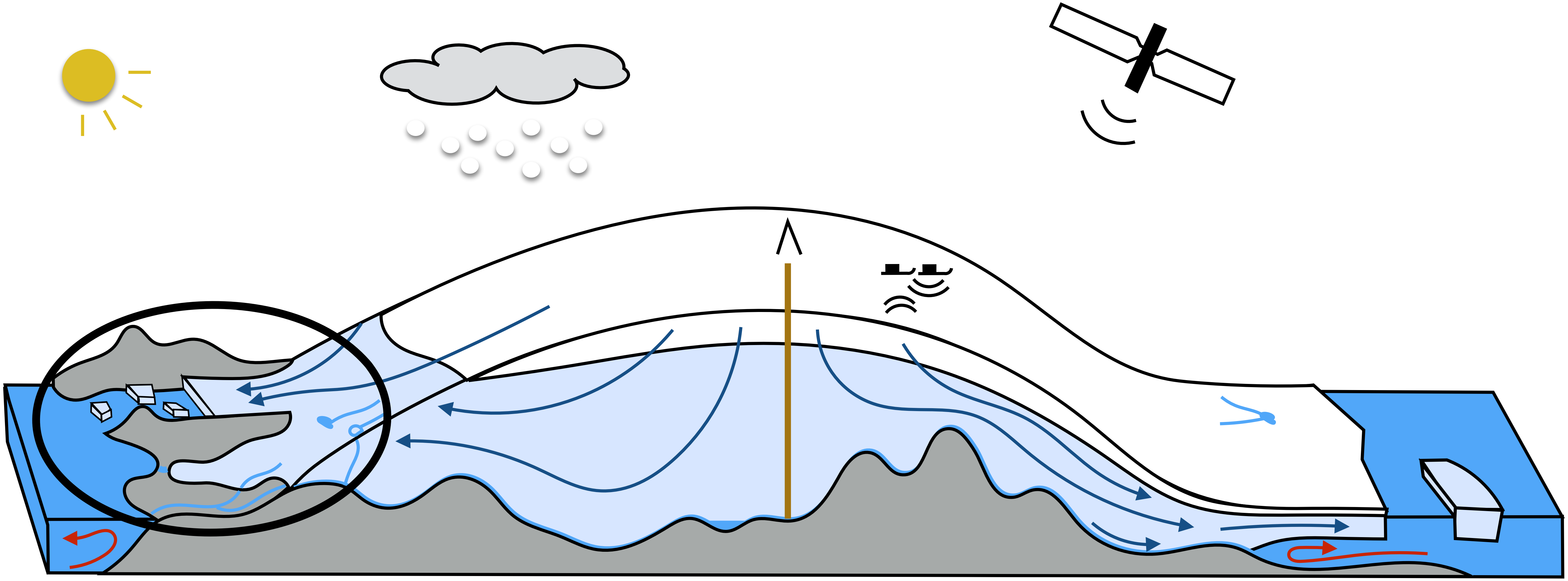


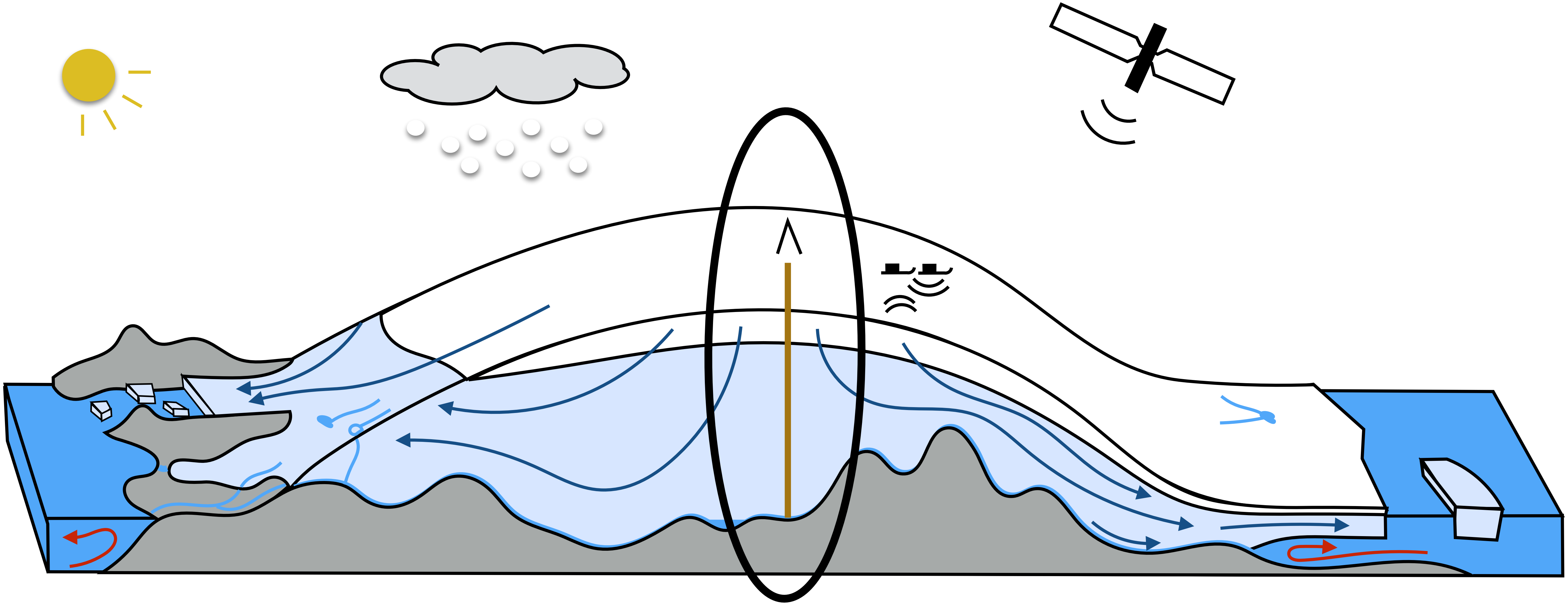










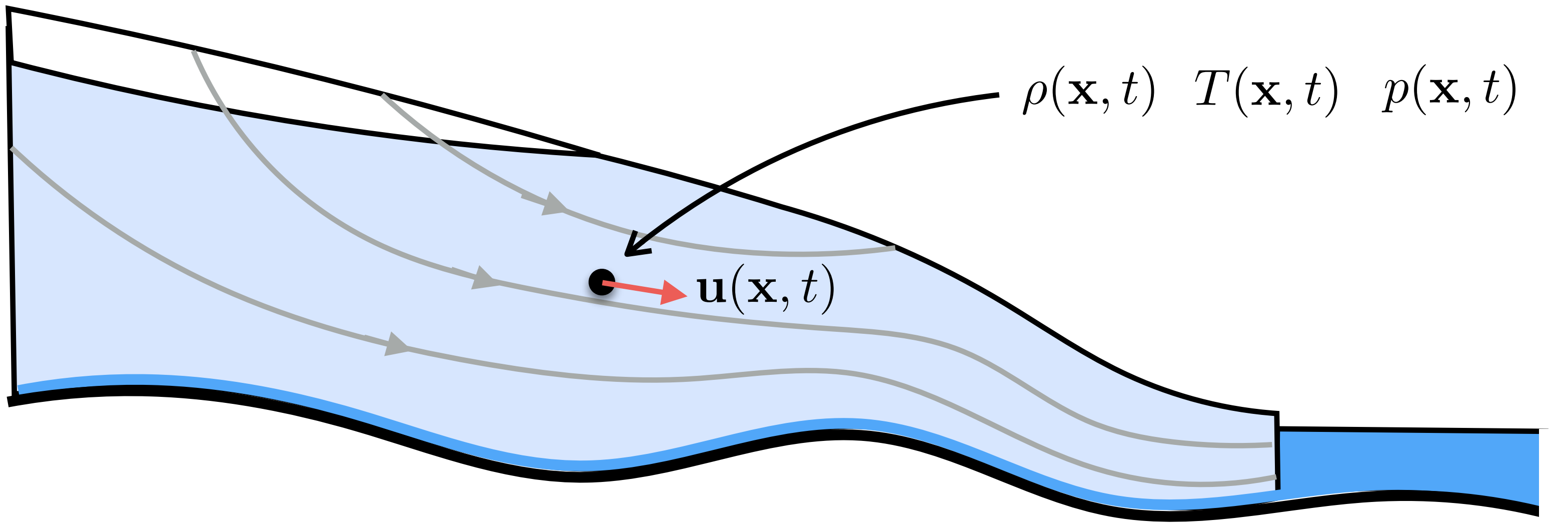


Continuum mechanics

Continuum mechanics

A **continuum approximation** treats a material as having a continuous distribution of mass. It applies on scales much larger than inter-molecular distances.

Each 'point' of the continuum can be ascribed properties, such as **density, temperature, velocity, pressure,** etc.



Continuum mechanics provides a mathematical framework to describe how these properties vary in space and time.

Continuum mechanics can be used to describe both 'fluids' and 'solids' - we focus on fluids.

Kinematics

- Coordinate systems / derivatives
- Strain rate

Dynamics

- Stress tensor
- Constitutive laws

Conservation laws

- Conservation of mass
- Conservation of momentum
- Navier-Stokes equations
- Conservation of energy

Boundary conditions

Depth-integrated models

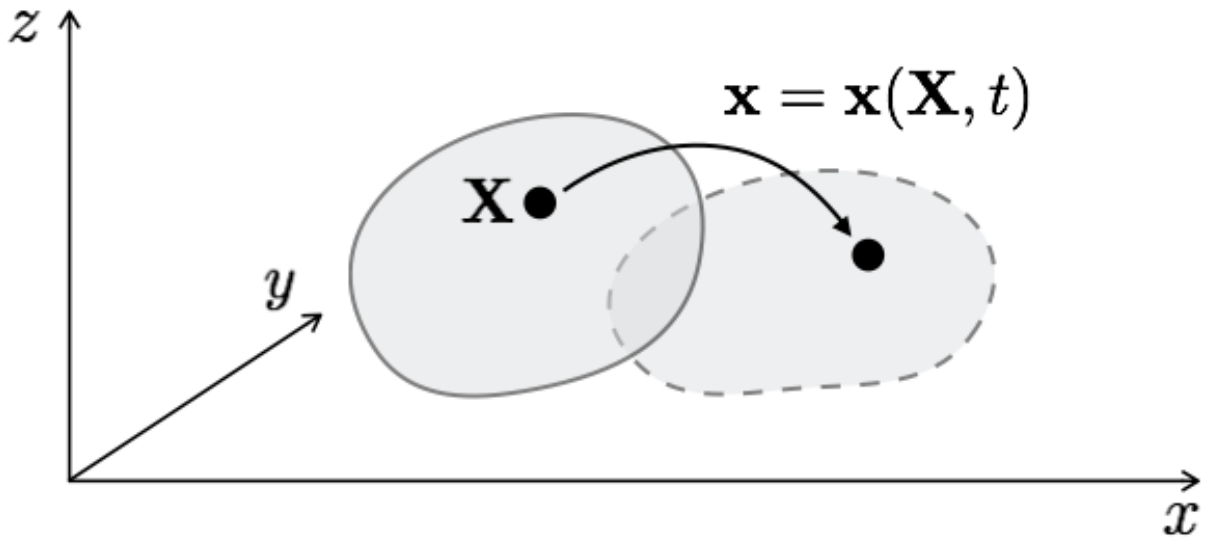
Kinematics

Coordinate systems

Eulerian description $(\mathbf{x}, t)$

$\mathbf{x}$ Spatial coordinates, fixed in space

$$\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$$



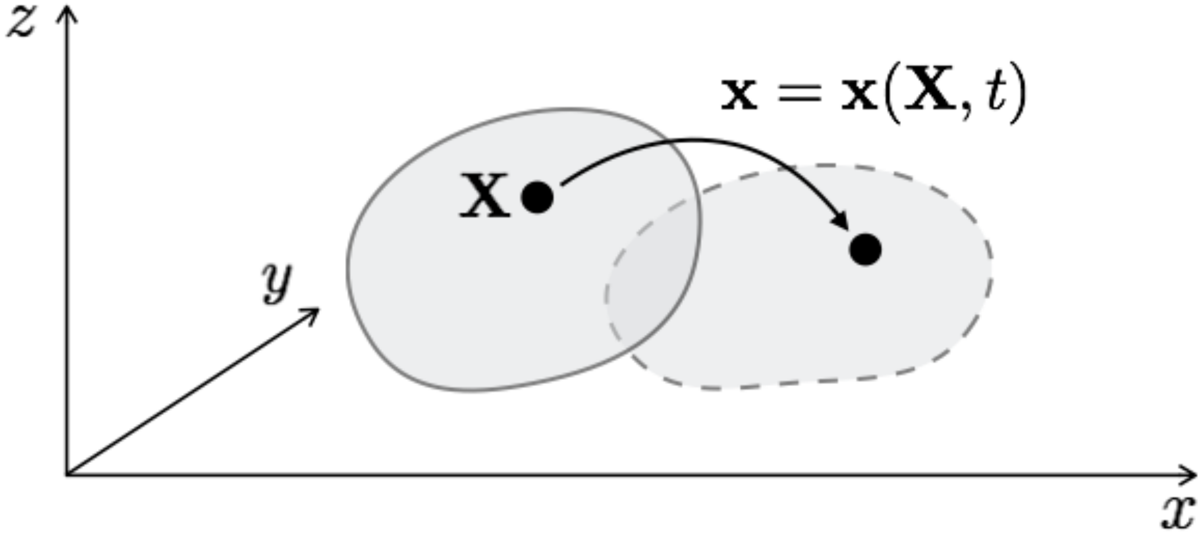
Lagrangian description $(\mathbf{X}, t)$

$\mathbf{X}$ Spatial coordinates, fixed in material

We usually choose these as the coordinates of a reference configuration at $t = 0$

Coordinate systems

Eulerian description $(\mathbf{x}, t)$



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Lagrangian description $(\mathbf{X}, t)$

$\mathbf{X}$ Spatial coordinates, fixed in material

We usually choose these as the coordinates of a reference configuration at $t = 0$

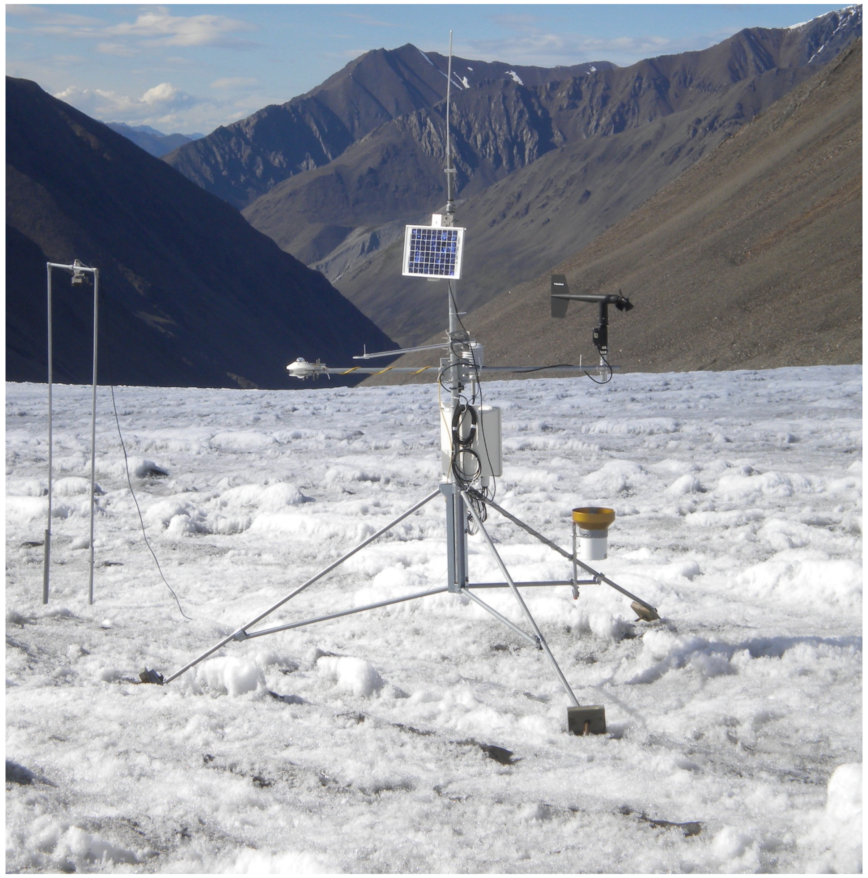
velocity $\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$

Material paths $\mathbf{x}(\mathbf{X}, t)$ are governed by $\dfrac{D\mathbf{x}}{Dt} = \mathbf{u}$ $\quad \mathbf{x}|_{t=0} = \mathbf{X}$

where $\dfrac{D}{Dt}$ is the time rate of change for fixed $\mathbf{X}$ (i.e. the derivative 'following the fluid')

Coordinate systems

A stake drilled into the ice tracks the ice motion in a Lagrangian system.



A weather station on the ice surface measures atmospheric properties in a (roughly) Eulerian framework.

Fluid **models** are usually written in an Eulerian coordinate system.

Material derivative

Given some function of Eulerian coordinates (e.g. temperature) $T = f(\mathbf{x}, t)$

we can calculate the **material derivative** using the chain rule (recall $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$)

$$\boxed{\frac{DT}{Dt} = \underset{\substack{\uparrow \\ \text{local term}}}{\frac{\partial T}{\partial t}} + \underset{\substack{\uparrow \\ \text{advective term}}}{\mathbf{u} \cdot \nabla T}}$$

$\partial T/\partial t$	rate of change with respect to time at fixed $\mathbf{x}$
∇T	rate of change with respect to $\mathbf{x}$
$\mathbf{u}$	rate of change of $\mathbf{x}$ with respect to time at fixed $\mathbf{X}$

The material derivative is also called the 'convective' derivative or 'total' derivative.

In components, $$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

We use the **summation convention** (repeated indices imply a sum): $$u_i \frac{\partial T}{\partial x_i} = \sum_{i=1}^{3} u_i \frac{\partial T}{\partial x_i}$$

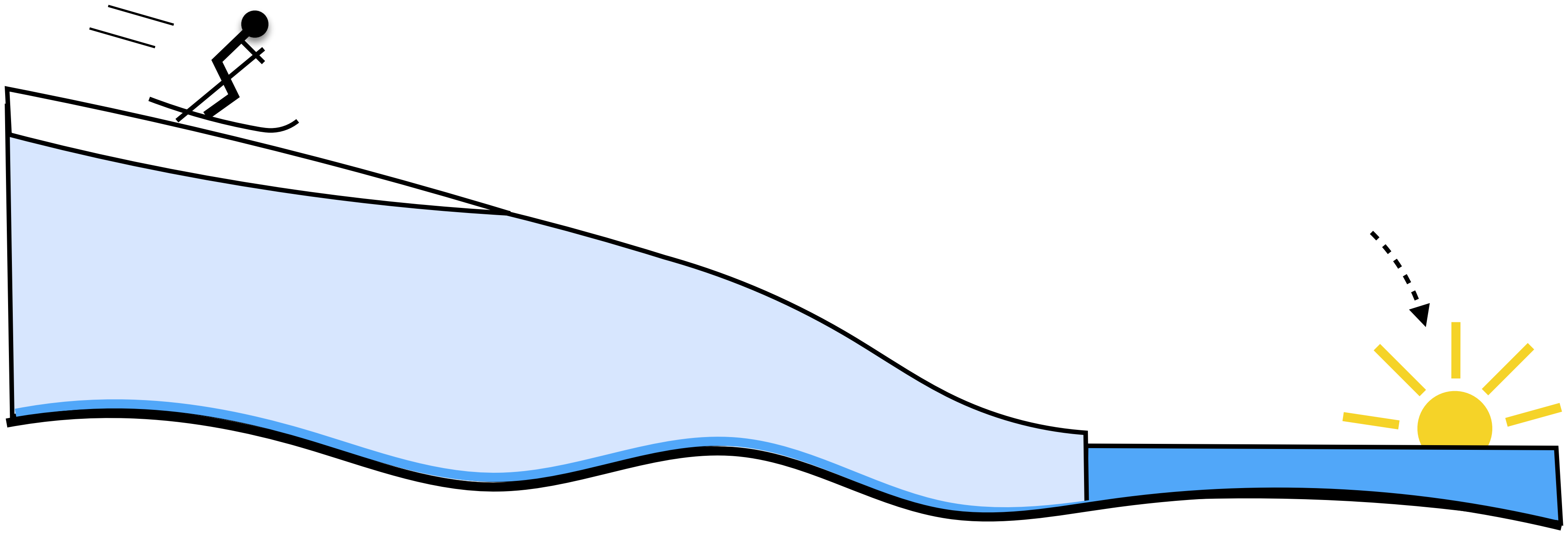
Material derivative

Example

The rate of change of temperature as measured by a skier has components due to:

- the temperature decreasing through the evening
- the temperature increasing as they travel downhill

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T$$

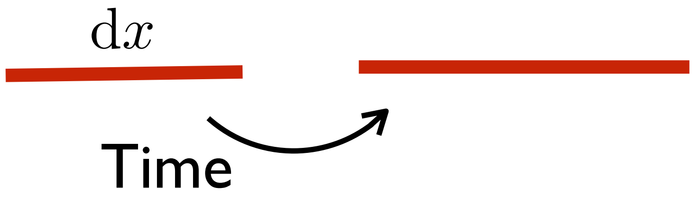


Strain rate

Strain is a measure of deformation. The strain rate is a measure of how fast strain is changing.

One dimension

Consider the rate of change of length of a small fluid element



$$\frac{D}{Dt}(\mathrm{d}x) = \mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x \quad \Rightarrow \quad \frac{1}{\mathrm{d}x}\frac{D}{Dt}(\mathrm{d}x) = \frac{\partial u}{\partial x}$$

stretching rate

Strain rate

Strain is a measure of deformation. The strain rate is a measure of how fast strain is changing.

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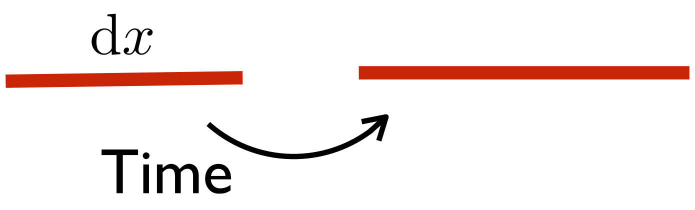

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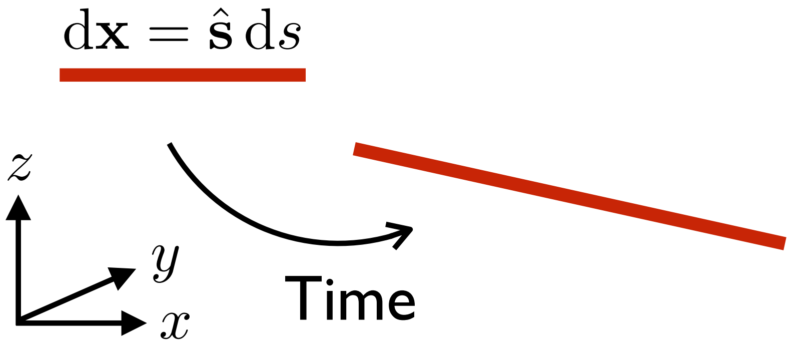
stretching rate

Three dimensions

The strain rate is now described by a rank-two tensor (a matrix)



$$\frac{1}{\mathrm{d}s}\frac{D}{Dt}(\mathrm{d}s) = \frac{1}{2}\hat{\mathbf{s}}^T(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\hat{\mathbf{s}} \;=\; \hat{s}_i\dot{\varepsilon}_{ij}\hat{s}_j$$

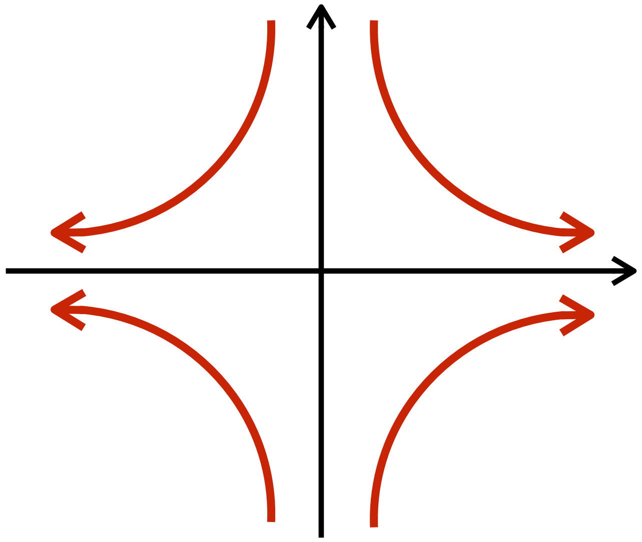
where the **strain rate tensor** is

$$\boxed{\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

Strain rate

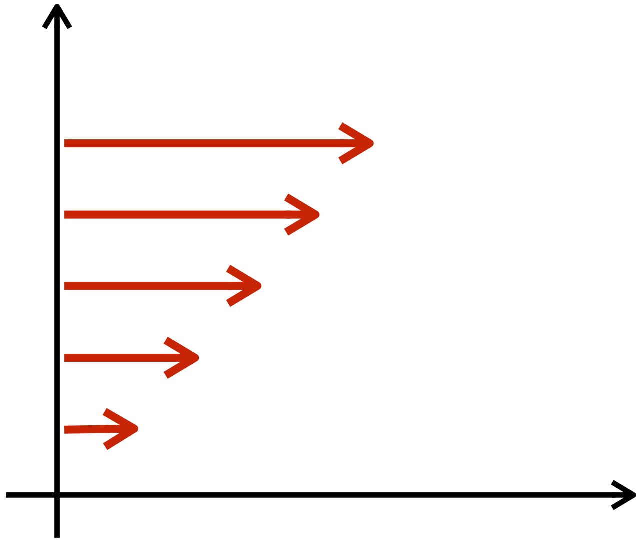
Examples

$$\mathbf{u} = \begin{pmatrix} x \\ 0 \\ -z \end{pmatrix}$$



$$\dot{\varepsilon}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

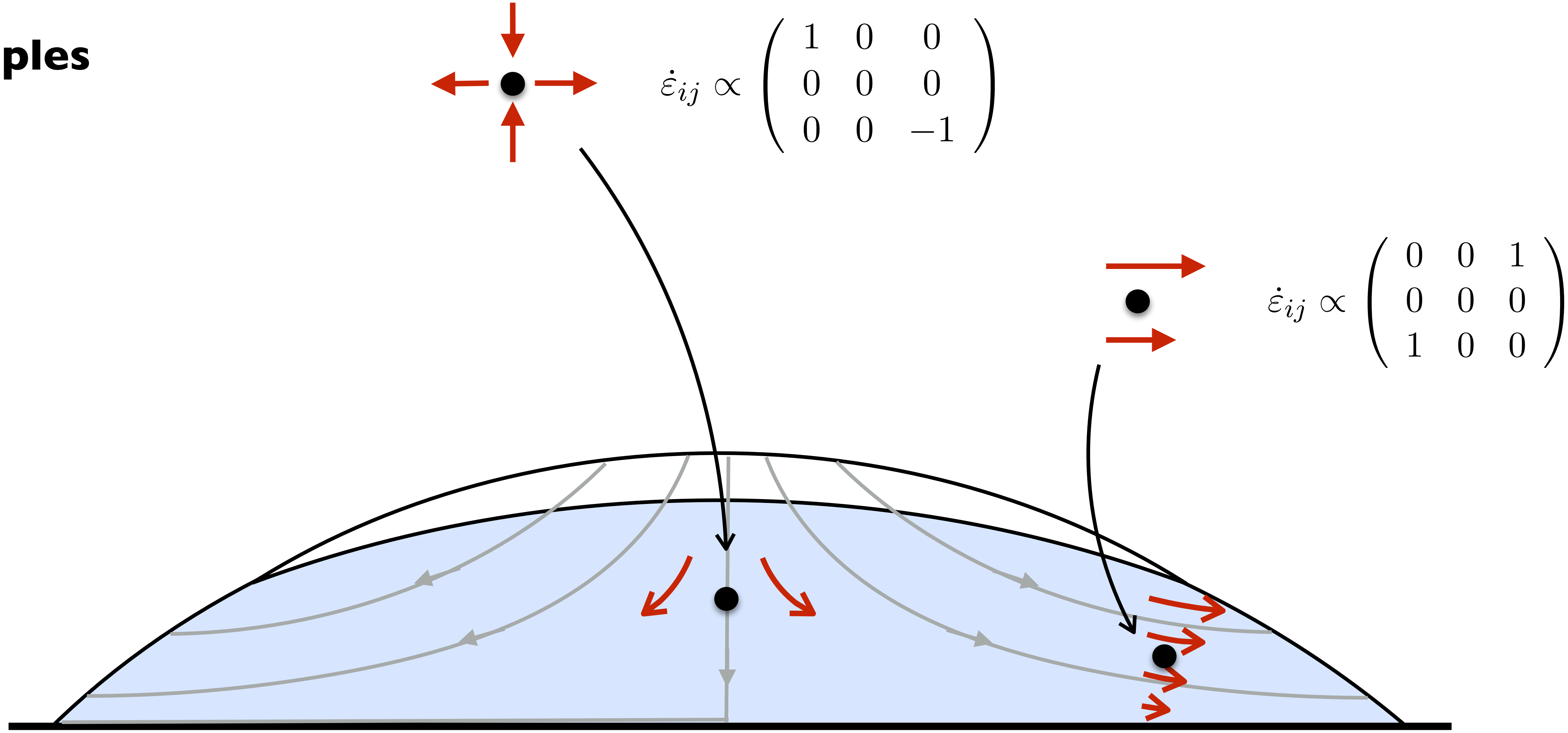
$$\mathbf{u} = \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}$$



$$\dot{\varepsilon}_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

Strain rate

Examples



$$\dot{\varepsilon}_{ij} \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

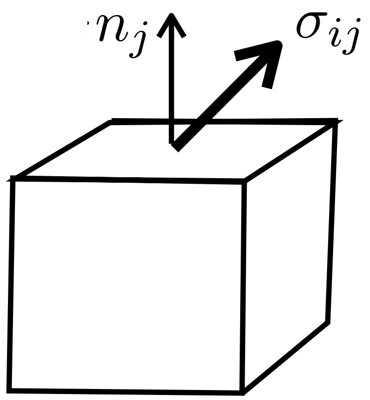
$$\dot{\varepsilon}_{ij} \propto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Dynamics

Stress tensor

Stress is force per unit area. The stress state is described by a rank-two tensor.

At each point in the material, consider a small cube.



We define the **Cauchy stress tensor** $\boldsymbol{\sigma} = \sigma_{ij}$ as the force per unit area in the i direction on the face with normal in the j direction.

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

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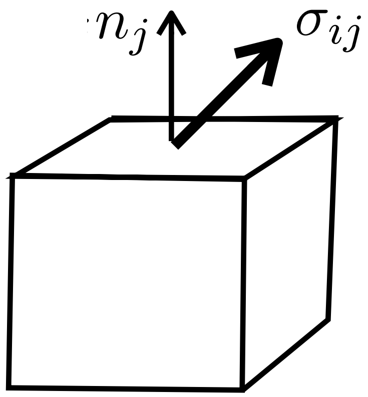

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We define the **pressure** by $p = -\frac{1}{3}\sigma_{ii}$

and the **deviatoric stress tensor** $\boldsymbol{\tau}$ by $\boldsymbol{\sigma} = -p\boldsymbol{\delta} + \boldsymbol{\tau}$ or $\boxed{\sigma_{ij} = -p\delta_{ij} + \tau_{ij}}$

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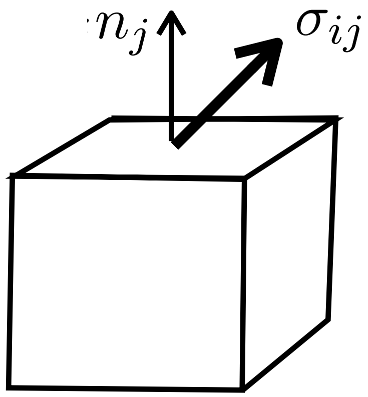

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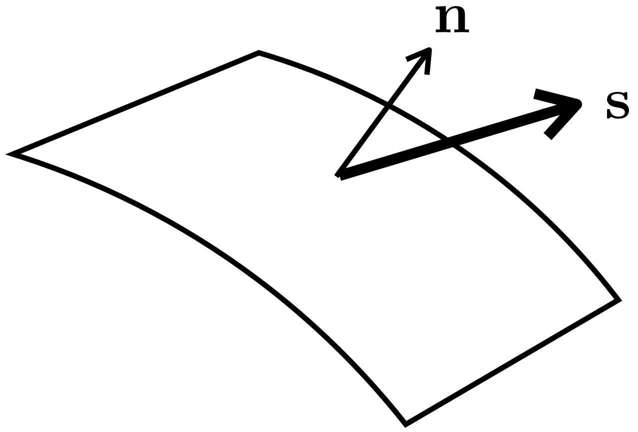
The stress acting on a general surface with unit normal $\mathbf{n}$ is

$\mathbf{s} = \boldsymbol{\sigma} \cdot \mathbf{n}$ or, in index notation, $s_i = \sigma_{ij} n_j$



Constitutive law

The constitutive law describes a relationship between stress and strain rates - it characterises the **rheology** of the material

For a Newtonian fluid (e.g. water)

$$\boxed{\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}}$$

η is the viscosity

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For ice, it is common to use Glen's flow law

$$\boxed{\dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}} \qquad \tau = \sqrt{\tfrac{1}{2}\tau_{ij}\tau_{ij}} \qquad n \approx 3 \qquad A \approx 2.4 \times 10^{-24}\ \mathrm{Pa}^{-3}\ \mathrm{s}^{-1}\ \text{at}\ 0^\circ\ \mathrm{C}$$

This can be written in the form of a Newtonian fluid but with an effective (non-constant) viscosity

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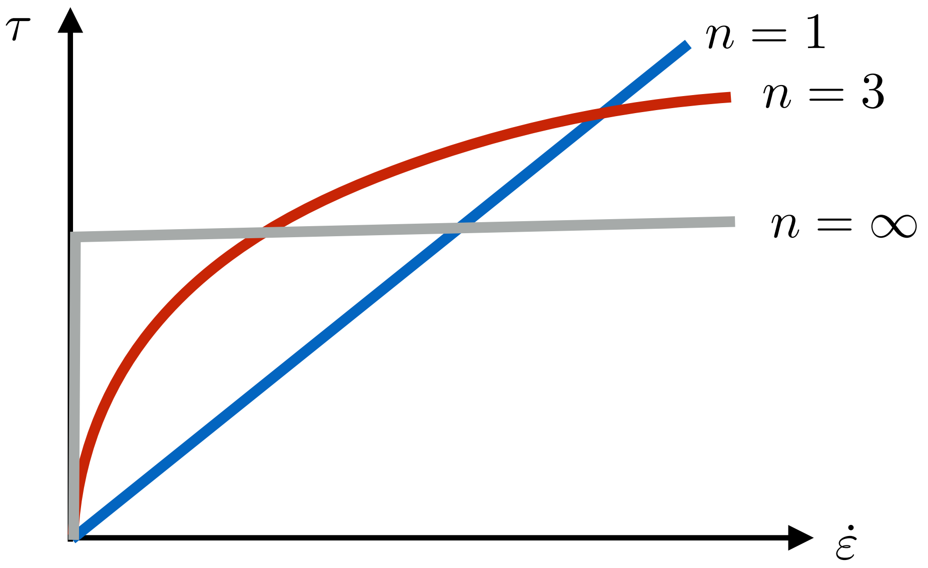
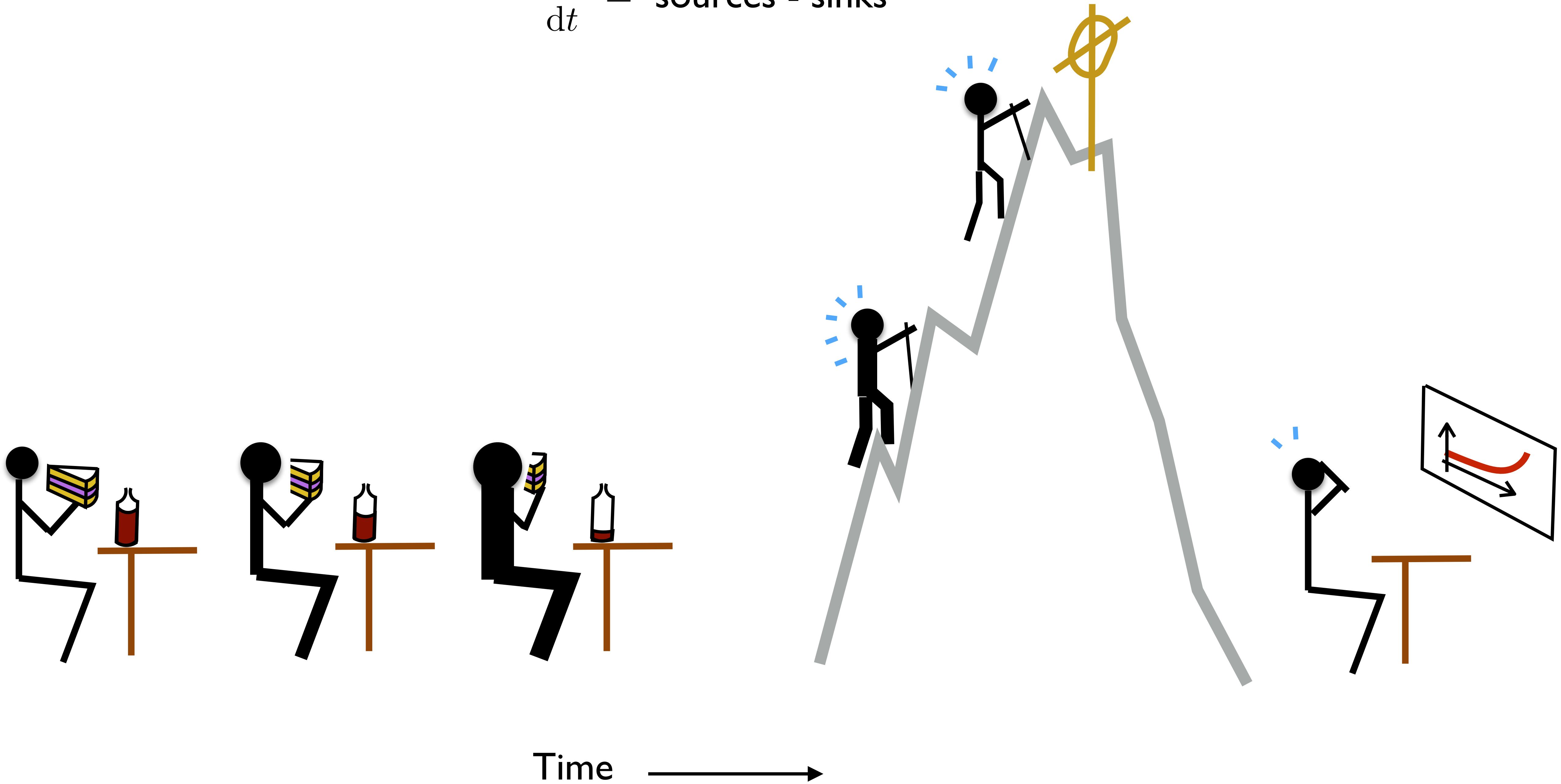


Conservation laws

Conservation of mass

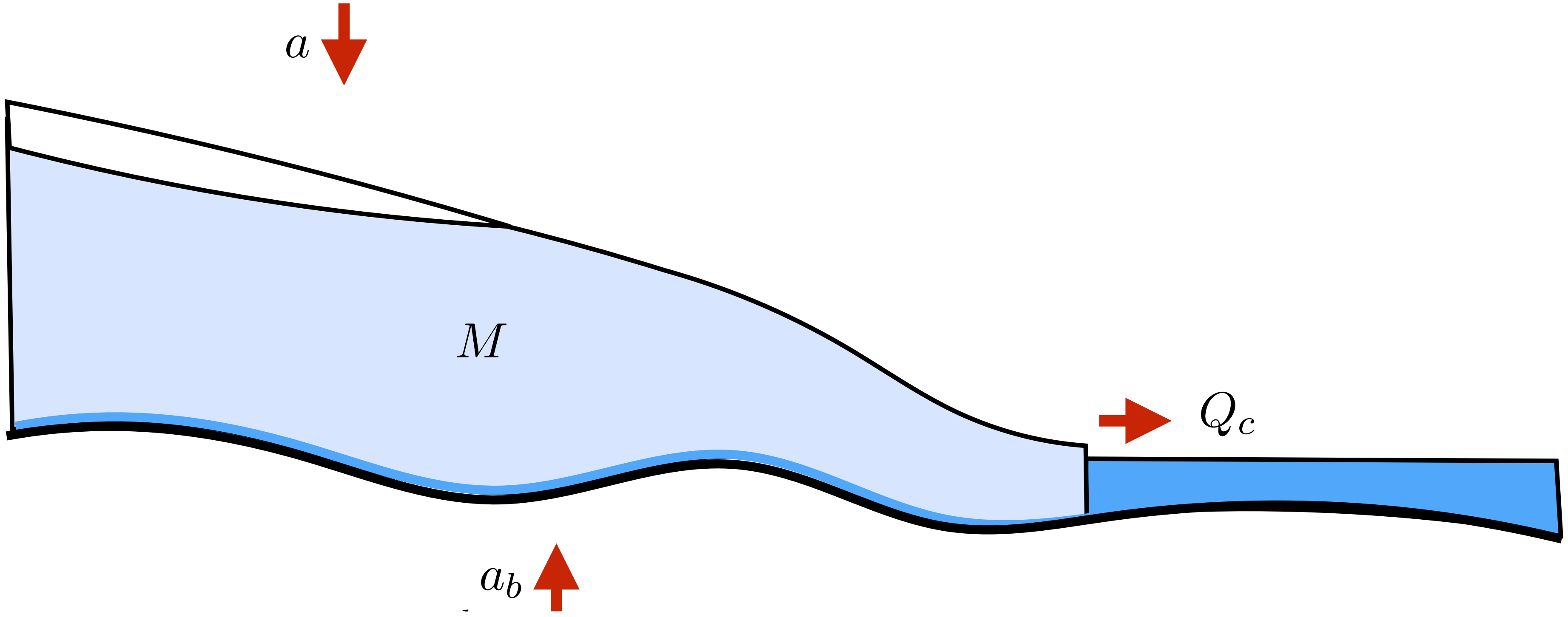
A major concern at Karthaus …

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \text{sources - sinks}$$



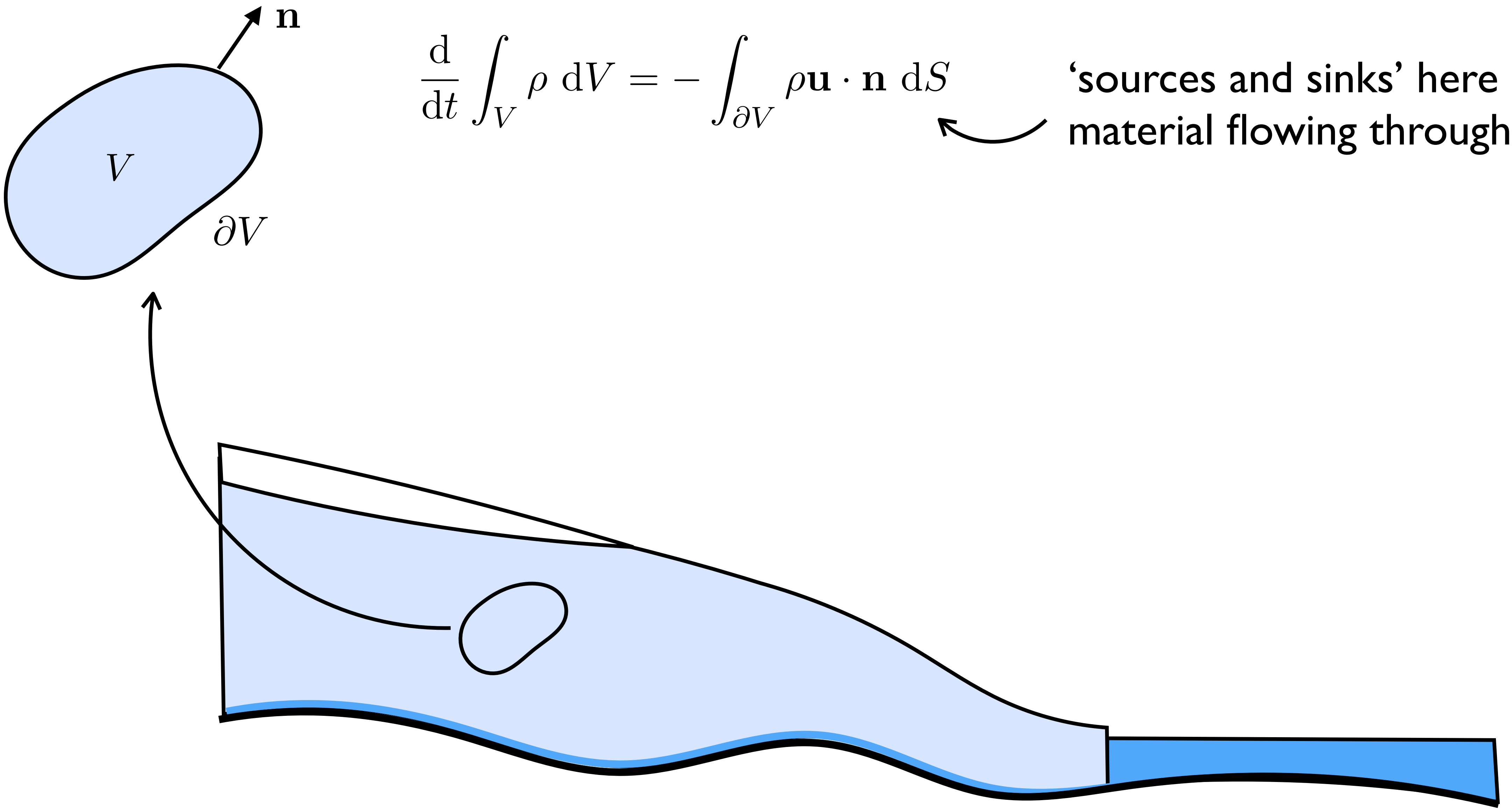
Conservation of mass

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{surface} a \ \mathrm{d}S + \int_{bed} a_b \ \mathrm{d}S - Q_c$$



Conservation of mass

Conservation of mass applies to each arbitrary (Eulerian) volume V in the ice.



$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \; \mathrm{d}V = -\int_{\partial V} \rho \mathbf{u} \cdot \mathbf{n} \; \mathrm{d}S$$

‘sources and sinks’ here are due to material flowing through the boundary

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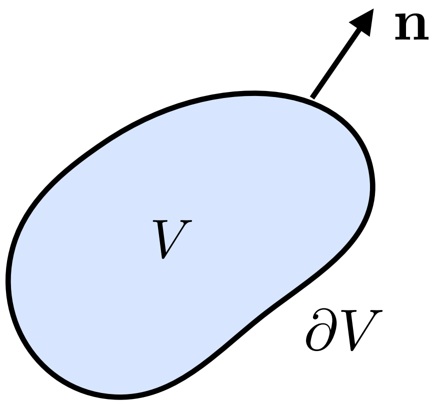


$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \,\mathrm{d}V = -\int_{\partial V} \rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S$$

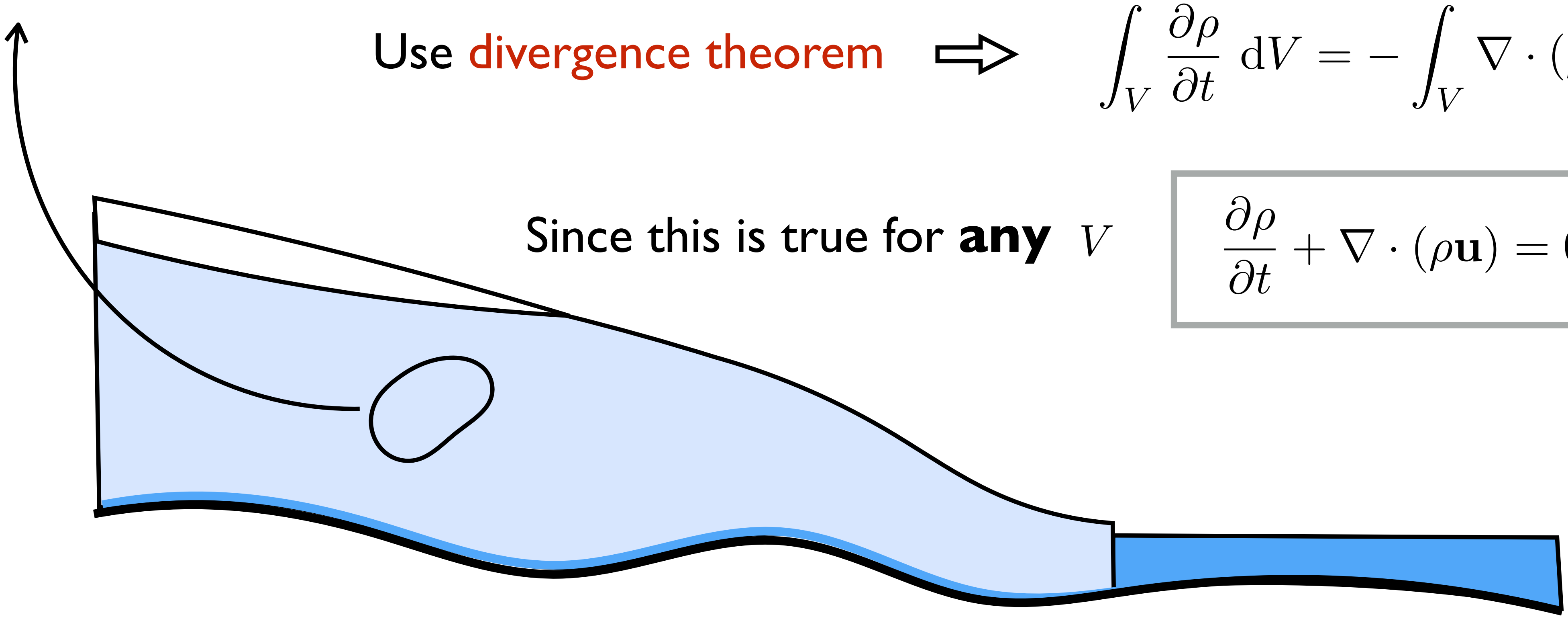
'sources and sinks' here are due to material flowing through the boundary

Use divergence theorem $\Rightarrow$

$$\int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V = -\int_V \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V$$

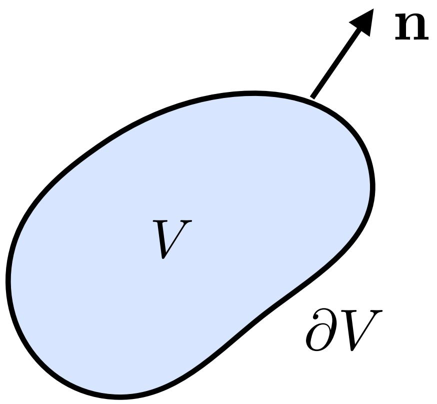
Since this is true for **any** V

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0}$$



Conservation of mass

Conservation of mass applies to each arbitrary (Eulerian) volume V in the ice.



$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \, \mathrm{d}V = -\int_{\partial V} \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S$$

‘sources and sinks’ here are due to material flowing through the boundary

Use divergence theorem $\Rightarrow$ $\displaystyle\int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V = -\int_V \nabla \cdot (\rho \mathbf{u}) \, \mathrm{d}V$

Since this is true for **any** V

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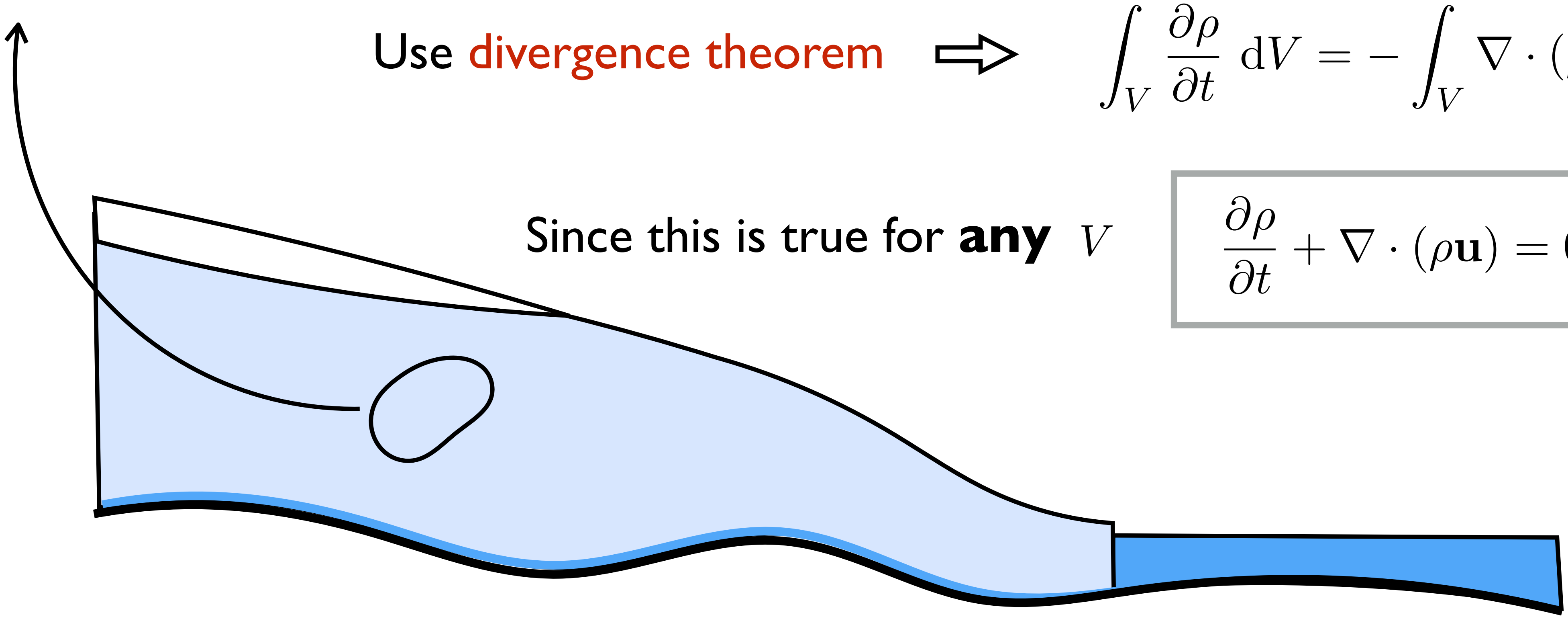
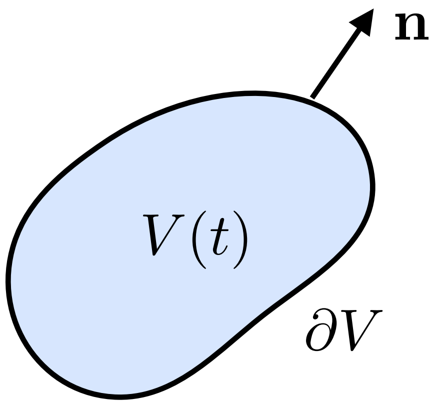


If the material is **incompressible**, $\dfrac{D\rho}{Dt} = 0$, we obtain $\boxed{\nabla \cdot \mathbf{u} = 0}$

Conservation of mass

An **alternative** derivation is to consider arbitrary material (Lagrangian) volumes $V(t)$

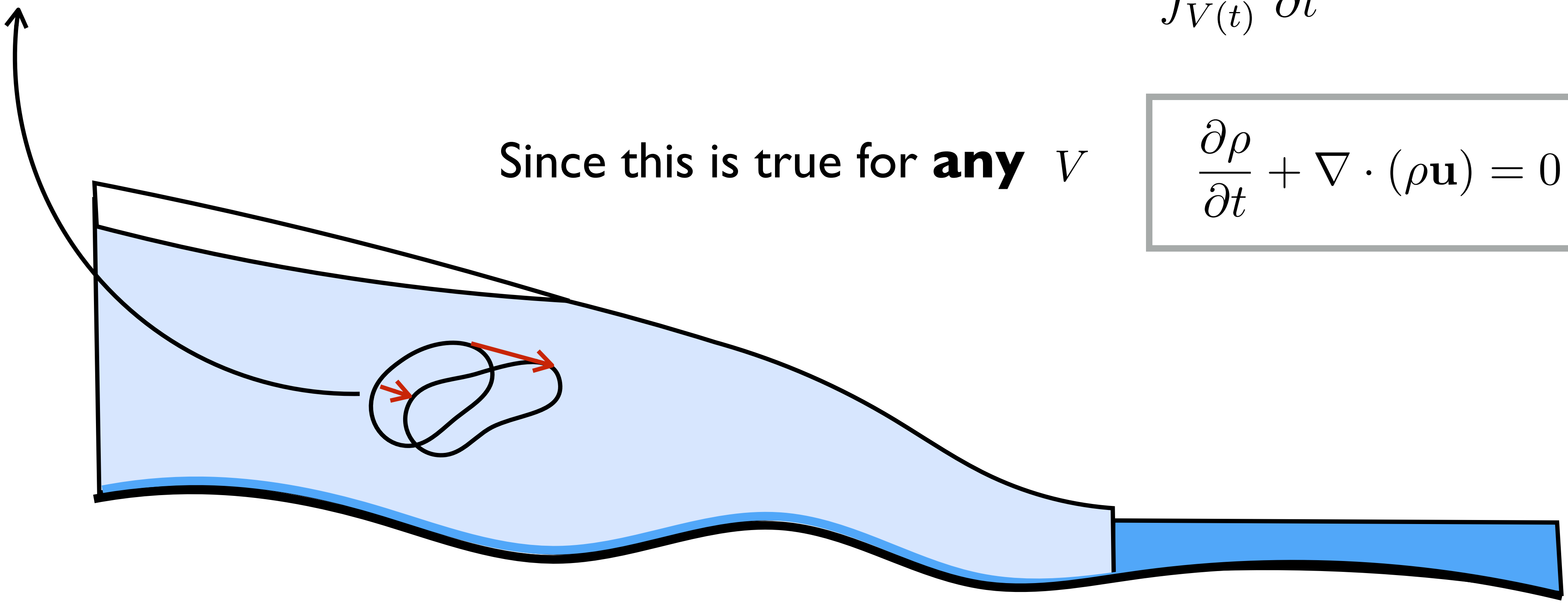


$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho \, \mathrm{d}V = 0$$

since the boundary is now a material surface, no mass crosses it.

Use Reynolds transport theorem $\Rightarrow$ $\displaystyle\int_{V(t)} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \, \mathrm{d}V = 0$

Since this is true for **any** V $\quad \boxed{\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$

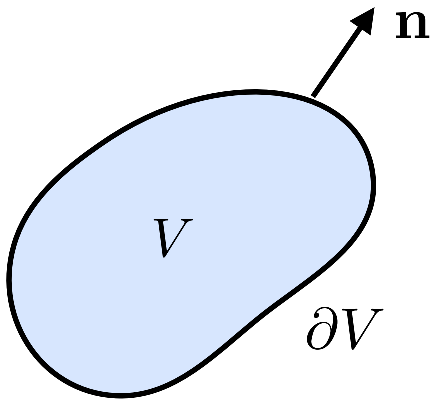


Conservation of momentum

We apply a similar argument to conserve **momentum** for each volume V

Momentum conservation is equivalent to **Newton's second law**:

Rate of change of momentum is equal to the forces acting



$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i \ \mathrm{d}V = -\int_{\partial V} \underset{\substack{\uparrow \\ \text{flux of momentum} \\ \text{through boundary}}}{\rho u_i u_j n_j} \ \mathrm{d}S + \int_{\partial V} \underset{\substack{\uparrow \\ \text{surface forces}}}{\sigma_{ij} n_j} \ \mathrm{d}S + \int_V \underset{\substack{\uparrow \\ \text{body force} \\ \text{(gravity)}}}{\rho g_i} \ \mathrm{d}V$$

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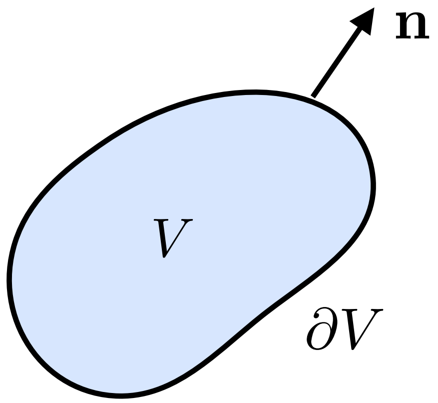

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Apply divergence theorem

$$\int_V \frac{\partial}{\partial t}(\rho u_i) \ \mathrm{d}V = \int_V -\frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i \ \mathrm{d}V$$

Use that volume is arbitrary

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i$$

Use conservation of mass

$$\boxed{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{g}}$$

(in vector form)

Navier-Stokes equations

We have derived mass and momentum equations for an incompressible fluid

$$\nabla \cdot \mathbf{u} = 0 \qquad\qquad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g}$$

Navier-Stokes equations

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Combining with the Newtonian rheology $\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}$ gives the **Navier-Stokes equations**

$$\boxed{\nabla \cdot \mathbf{u} = 0 \qquad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \eta \nabla^2 \mathbf{u} + \rho \mathbf{g}}$$

constant viscosity is used here

this term is non linear!

Reynolds number

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}$$

Estimate the size of terms in the momentum equation for an ice sheet

$$\mathbf{u} \sim U \approx 100 \text{ m y}^{-1} \qquad \mathbf{x} \sim L \approx 1000 \text{ m} \qquad \mathbf{g} \sim g \approx 9.8 \text{ m s}^{-2} \qquad \boldsymbol{\sigma} \sim \rho g z$$

$$\Rightarrow \quad \mathbf{u} \cdot \nabla \mathbf{u} \sim 10^{-14} \text{ m s}^{-2}$$

The inertial terms on the left are much much smaller than those on the right.

More generally, the relative size of these terms is measured by the Reynolds number

$$\boxed{Re = \frac{\rho U L}{\eta}}$$

- this is a measure of how ‘fast’ the flow is.

Reynolds number

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For small Reynolds number (‘slow flow’) we have the **Stokes equations**

$$\boxed{\nabla \cdot \mathbf{u} = 0 \qquad \mathbf{0} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} \qquad \dot{\varepsilon}_{ij} = A(T) \tau^{n-1} \tau_{ij}}$$

High Reynolds number flows

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}$$

For flows with high Reynolds number (e.g. most atmosphere and ocean processes) we can usually ignore the viscous terms.

However, such flows are often **turbulent**, and there are Reynolds stresses (due to fluctuations in the velocity field) that have to be parameterised

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \cdot \langle \mathbf{u}' \mathbf{u}' \rangle + \mathbf{g}$$

↖ Reynolds stresses

When inertia is important we may also have to worry about the effects of Earth's **rotation**

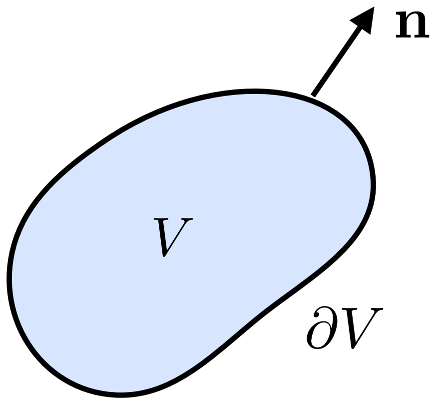
$\frac{D\mathbf{u}}{Dt}$ becomes $\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \wedge \mathbf{u} + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{x})$

Conservation of energy

The same methods work to derive an **energy** equation.

Rate of change of energy is equal to the work done by forces and net conductive heat transfer

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(e+\tfrac{1}{2}|\mathbf{u}|^2)\ \mathrm{d}V = -\underset{\substack{\uparrow \\ \text{flux of energy} \\ \text{through boundary}}}{\int_{\partial V} \rho(e+\tfrac{1}{2}|\mathbf{u}|^2)\mathbf{u}\cdot\mathbf{n}\ \mathrm{d}S}+\underset{\substack{\uparrow \\ \text{conductive} \\ \text{transfer}}}{\int_{\partial V} k\nabla T\cdot\mathbf{n}\ \mathrm{d}S}+\underset{\substack{\uparrow \\ \text{work done against} \\ \text{surface forces}}}{\int_{\partial V} \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\ \mathrm{d}S}+\underset{\substack{\uparrow \\ \text{work done} \\ \text{against gravity}}}{\int_V \rho\mathbf{u}\cdot\mathbf{g}\ \mathrm{d}V}$$

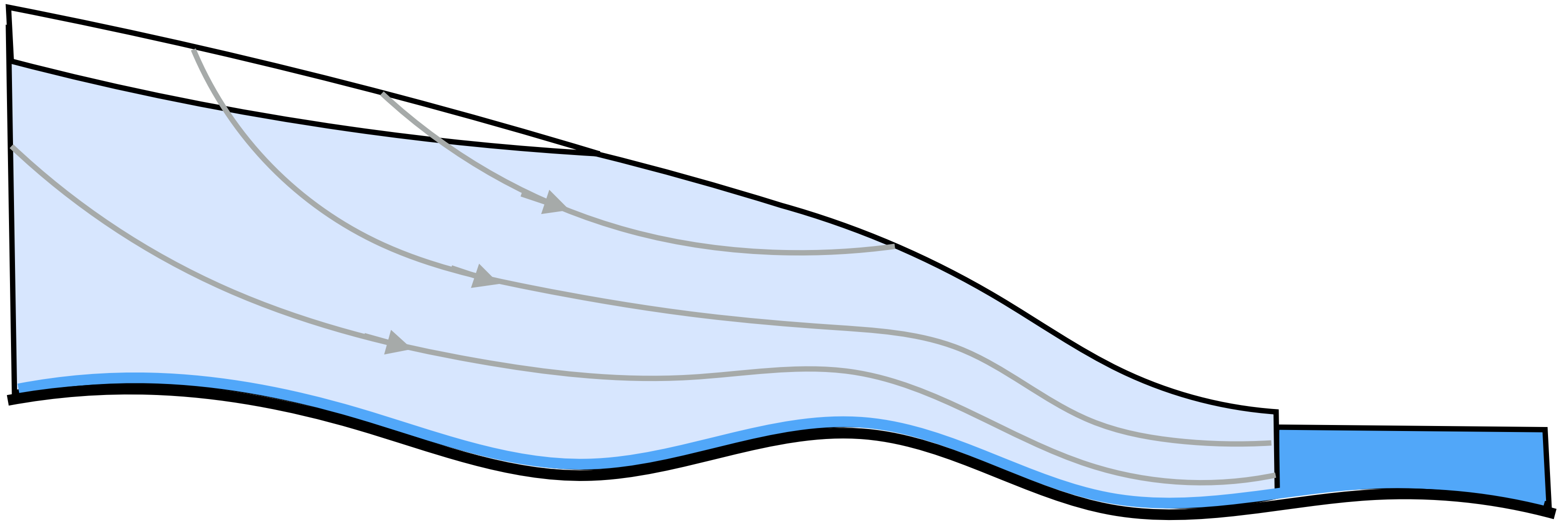


Applying the usual arguments …

$$\boxed{\rho c_p\left(\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T\right)=\nabla\cdot(k\nabla T)+\tau_{ij}\dot{\varepsilon}_{ij}}$$

$$\frac{De}{Dt}=c_p\frac{DT}{Dt}$$

Boundary conditions



Kinematic boundary conditions

At a **rigid boundary** (e.g. the glacier bed* in absence of melting/freezing), we must usually have no normal flow

$$\mathbf{u} \cdot \mathbf{n} = 0$$

For a viscous fluid we also usually have no slip

$$\mathbf{u}_b = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n} = 0$$

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However, glaciers often slide at the base, for which we adopt a **sliding law** relating basal speed and basal shear stress $\boldsymbol{\tau}_b = \boldsymbol{\sigma} \cdot \mathbf{n} - (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n})\mathbf{n}$

$$\boldsymbol{\tau}_b = f(|\mathbf{u}_b|)\frac{\mathbf{u}_b}{|\mathbf{u}_b|}$$

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At a **material boundary** (e.g. the glacier surface in absence of accumulation or melting) we insist particles must stay on the surface

$$\frac{D}{Dt}(z - s(x,y,t)) = 0$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w$$

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$$\frac{D}{Dt}(z-s(x,y,t))=0$$

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}=w$$

If there is mass transfer (e.g. accumulation) at a boundary, conservation of material demands a modification of the kinematic condition

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}=w+a$$

Dynamic boundary conditions

At free boundaries we apply atmospheric stress conditions

$$\boldsymbol{\sigma} \cdot \mathbf{n} = -p_a \mathbf{n}$$

(atmospheric pressure is often chosen as the gauge pressure and set to zero)

This is often broken into normal and shear components

$$\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} = -p_a$$

$$\boldsymbol{\tau}_s = \boldsymbol{\sigma} \cdot \mathbf{n} - (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n})\mathbf{n} = \mathbf{0}$$

Stokes equations + boundary conditions

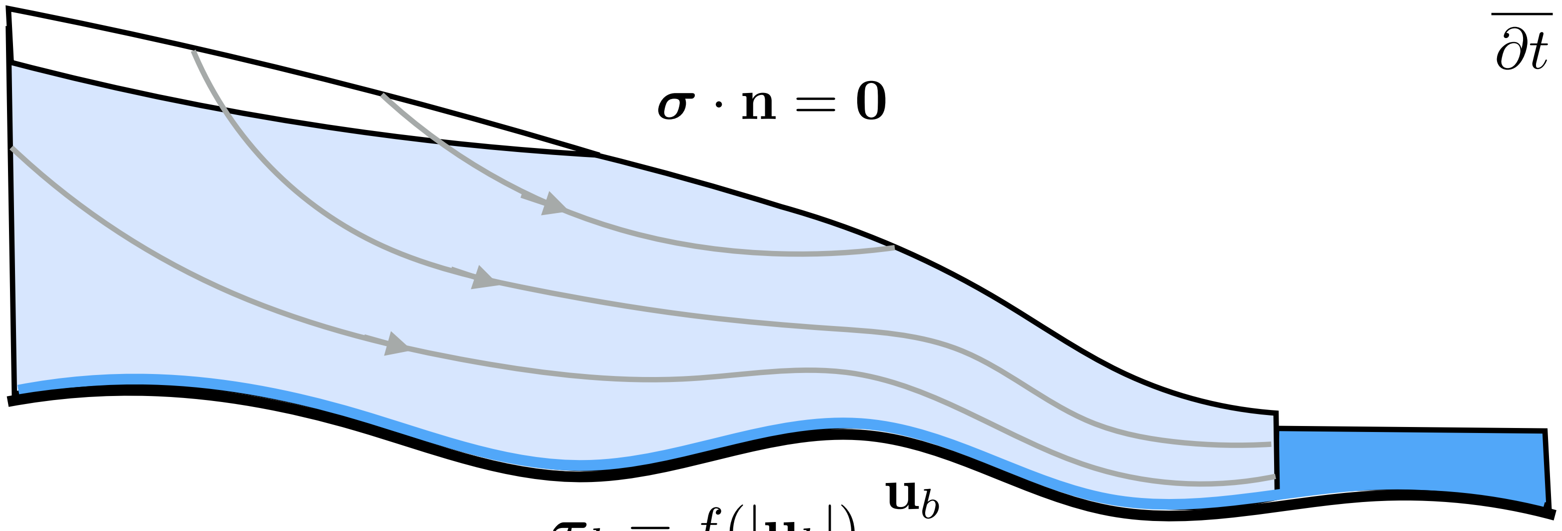
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

$$\dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \rho g$$



$$z = s(x, y, t)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$$

$$z = b(x, y)$$

$$\boldsymbol{\tau}_b = f(|\mathbf{u}_b|)\frac{\mathbf{u}_b}{|\mathbf{u}_b|}$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w + a$$

$$u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} = w$$

Shallow approximation (lubrication theory, 2D)

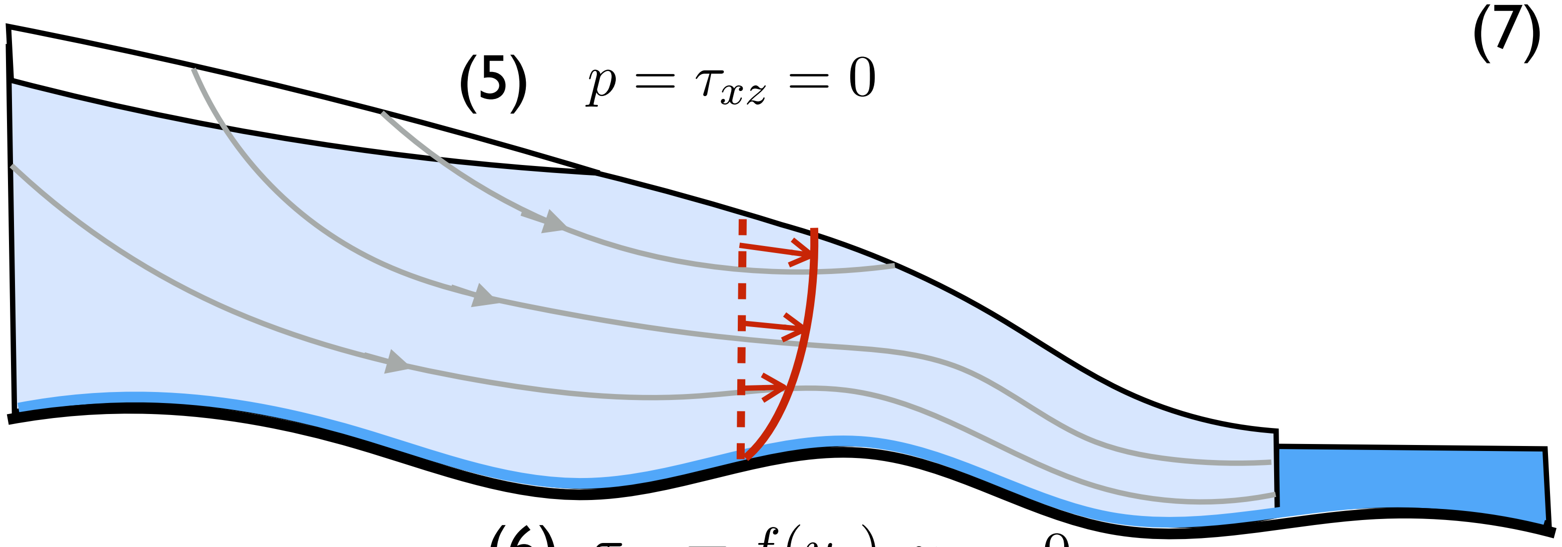
(1) $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

(2) $$0 = -\frac{\partial p}{\partial x} + \cancel{\frac{\partial \tau_{xx}}{\partial x}} + \frac{\partial \tau_{xz}}{\partial z}$$

$$\frac{1}{2}\frac{\partial u}{\partial z} = A|\tau_{xz}|^{n-1}\tau_{xz}$$

(4) ~~$\dot{\epsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$~~

(3) $$0 = -\frac{\partial p}{\partial z} + \cancel{\frac{\partial \tau_{zx}}{\partial x}} + \cancel{\frac{\partial \tau_{zz}}{\partial z}} - \rho g$$



$z = s(x,t)$

$z = b(x)$

(5) $p = \tau_{xz} = 0$

(6) ~~$\tau_{xz} = f(u_b)$~~ $u_b = 0$

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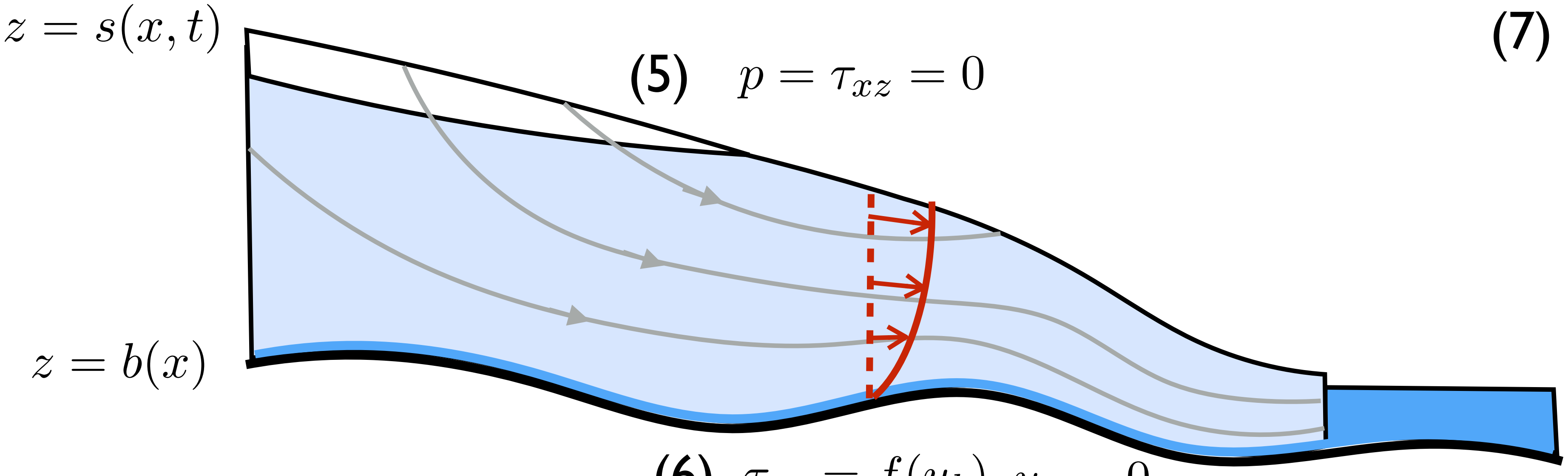

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Integrate (1) with (7) and (8)

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a}$$

$h = s - b$ $\qquad$ $q = \int_b^s u \,\mathrm{d}z$

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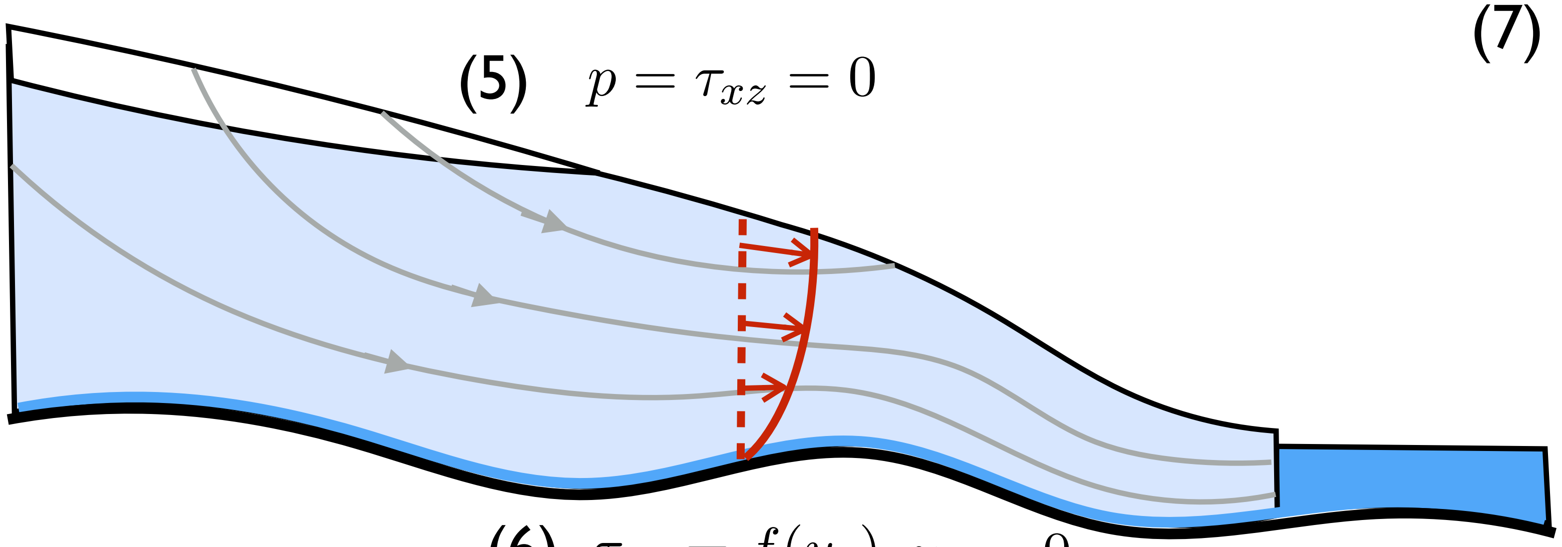

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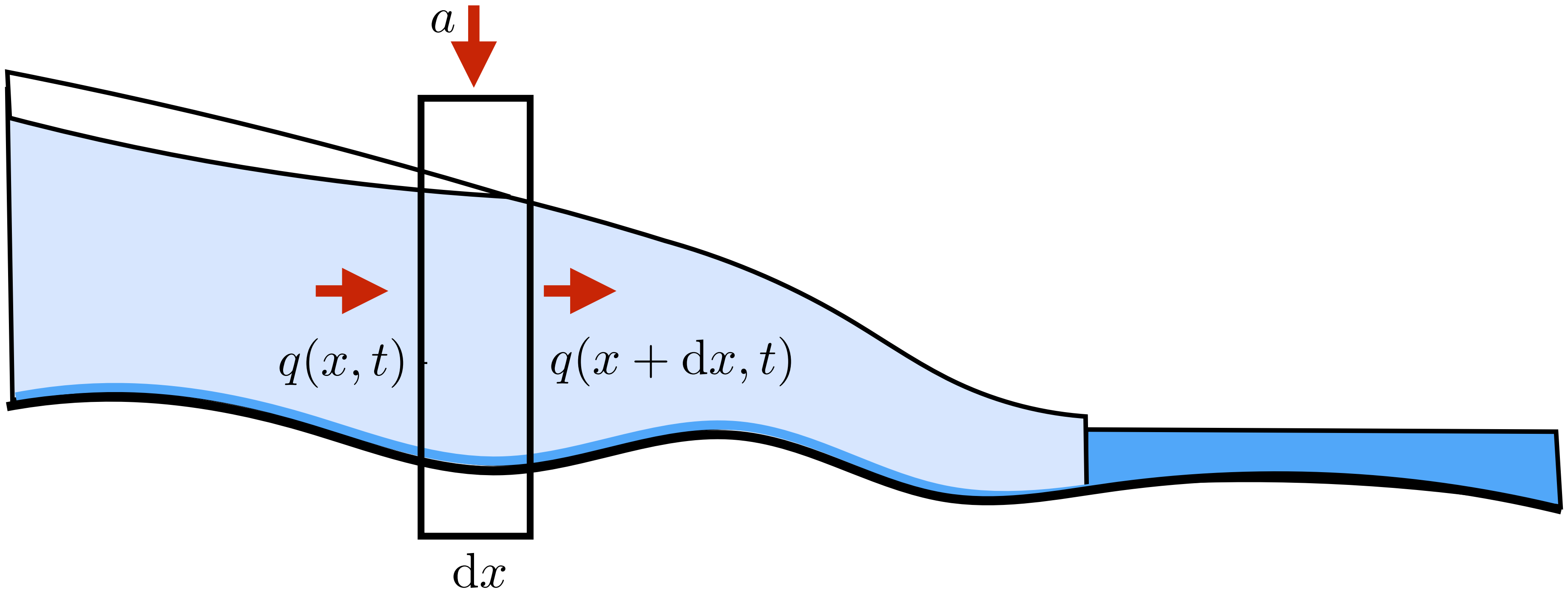
Integrate (2)-(4) with (5) and (6)

$$\boxed{q = -\frac{2A(\rho g)^n}{n+2}h^{n+2}\left|\frac{\partial s}{\partial x}\right|^{n-1}\frac{\partial s}{\partial x}}$$

Depth-integrated equations directly



Depth-integrated mass conservation $$\frac{\partial}{\partial t}(h\ \mathrm{d}x) = q(x,t) - q(x+\mathrm{d}x,t) + a\ \mathrm{d}x$$

Rearrange $$\frac{\partial h}{\partial t} + \frac{q(x+\mathrm{d}x,t) - q(x,t)}{\mathrm{d}x} = a$$

Take limit $\mathrm{d}x \to 0$ $$\boxed{\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a}$$

Rapid sliding (membrane theory)

$$u(x,z,t) \approx u_b(x,t)$$

(1) $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

(2) $$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

(4) $$\frac{\partial u}{\partial x} = A|\tau_{xx}|^{n-1}\tau_{xx}$$

~~$\dot{\epsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$~~

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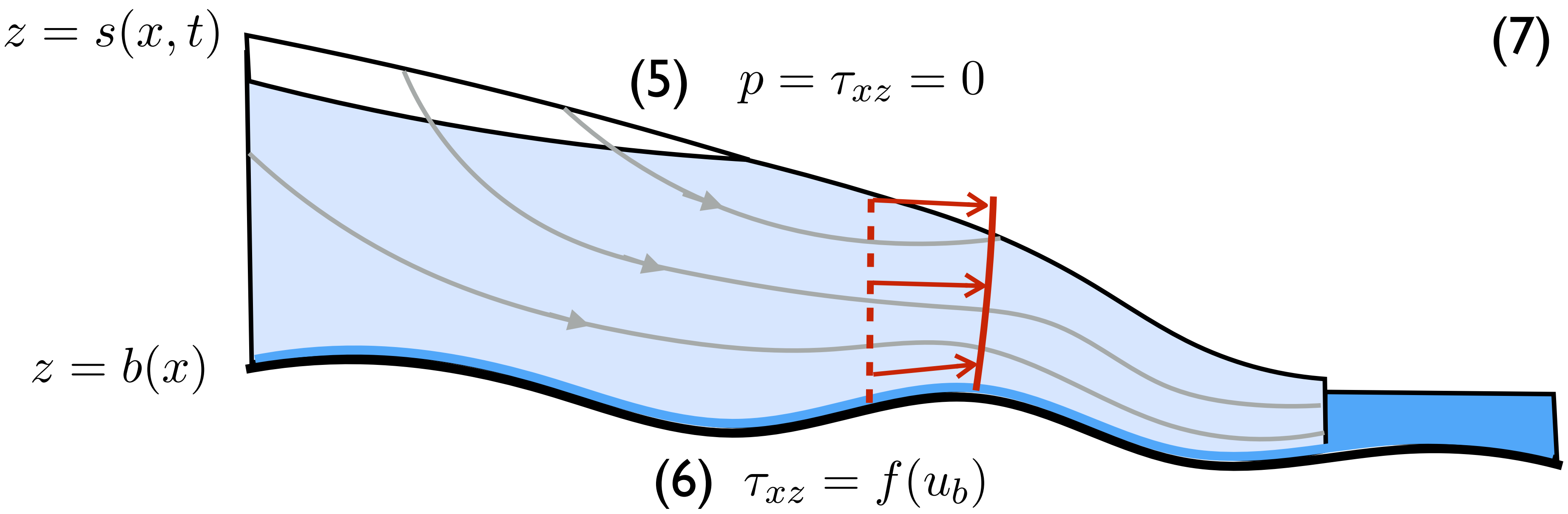

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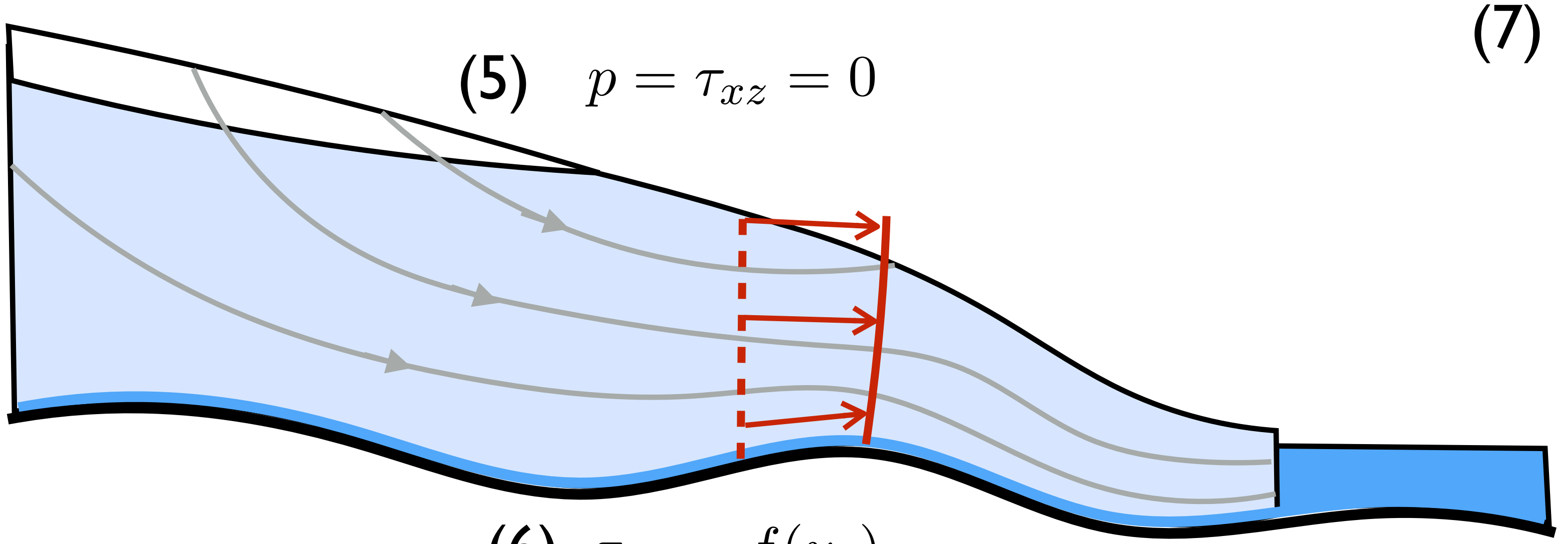

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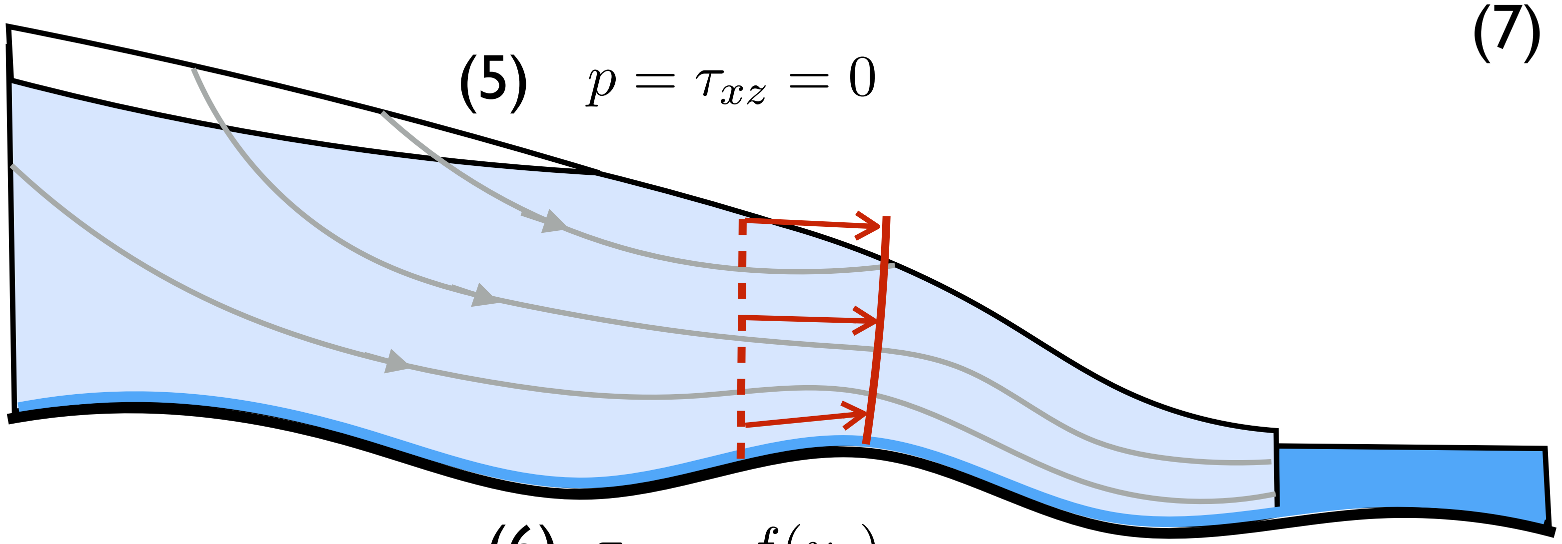

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$h = s - b$ $q = hu$

Integrate (2)-(4) with (5) and (6)

$$\boxed{0 = -\rho g h\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left(2hA^{-1/n}\left|\frac{\partial u}{\partial x}\right|^{1/n-1}\frac{\partial u}{\partial x}\right) - f(u)}$$

Summary

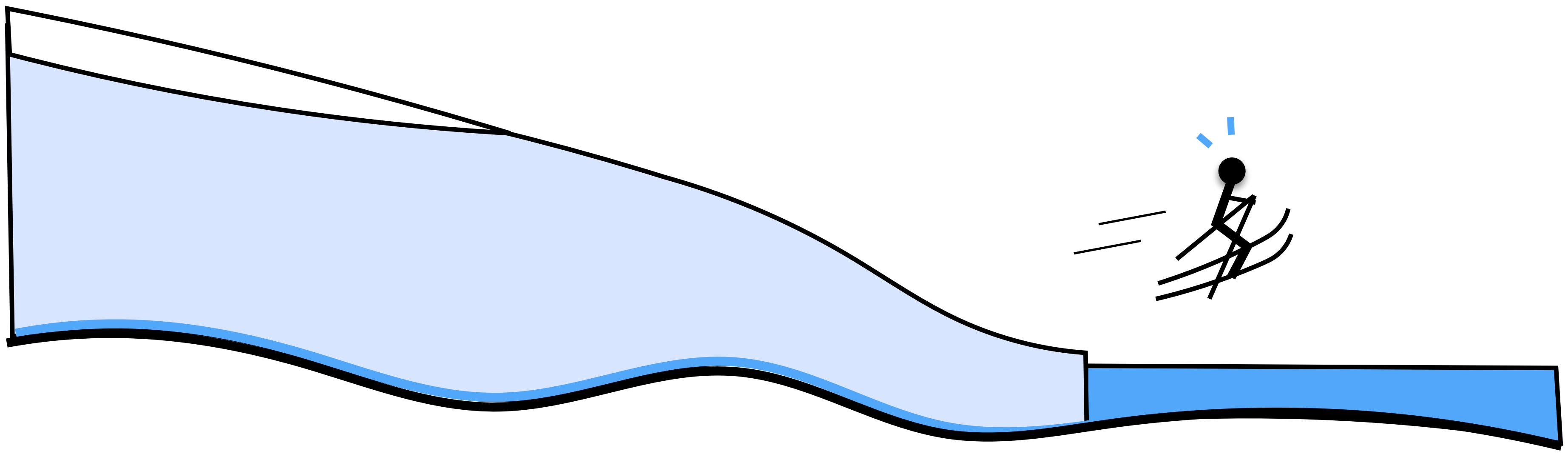
Continuum variables can be described in terms of **Eulerian** or **Lagrangian** coordinates.

The material derivative is the derivative following fluid particles.

Stress and strain rate tensors describe the forces and the rates of deformations in the material.

The principles of mass and momentum conservation lead to coupled PDEs for velocity, pressure and deviatoric stress. Together with a constitutive law these lead to the **Navier-Stokes** or **Stokes** equations.

Various **boundary conditions** are applicable for different types of bounding surfaces.



Continuum mechanics summary

Continuum mechanics

- Treats fluid as a continuous distribution, with properties (e.g. temperature) assigned to each point.

Coordinate systems

- Eulerian $\mathbf{x}$ (coordinate is fixed in space, material points moves through the coordinates)

$T = f(\mathbf{x}, t)$ e.g. satellite footprint

- Lagrangian $\mathbf{X}$ (coordinate moves with material, material points defined by initial position)

$T = f(\mathbf{x}(\mathbf{X}, t), t) = F(\mathbf{X}, t)$ $\quad \dfrac{D\mathbf{x}}{Dt} = \mathbf{u}$ $\quad \mathbf{x}|_{t=0} = \mathbf{X}$ e.g. GPS stake

Rates of change

- Eulerian derivative $\dfrac{\partial T}{\partial t} = \left.\dfrac{\partial f}{\partial t}\right|_{\mathbf{x}\ \text{fixed}}$

- Material derivative $\dfrac{DT}{Dt} = \left.\dfrac{\partial F}{\partial t}\right|_{\mathbf{X}\ \text{fixed}}$ (rate of change ‘following the fluid’)

- using chain rule $\boxed{\dfrac{DT}{Dt} = \dfrac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T}$ (also called ‘convective’ or ‘total’ derivative)

Notation

- Eulerian coordinates $\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$
- Velocities $\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$
- Summation convention $\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} \left(= \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \right)$ (repeated indices are summed over)
- Kronecker delta $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Strain rate tensor

$$\boxed{\dot{\varepsilon}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}$$

- Describes local rates of deformation

Stress tensor

$$\boxed{\sigma_{ij} = -p\delta_{ij} + \tau_{ij}}$$

- Describes force per unit area in i direction on a surface with normal in j direction $s_i = \sigma_{ij} n_j$
- Pressure $p = -\frac{1}{3}\sigma_{ii}$
- Deviatoric stress tensor τ_{ij}

Constitutive law

- Relates stress to strain rate - general form $\tau_{ij} = c_{ijkl}\dot{\varepsilon}_{kl}$
- Newtonian fluid $\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}$ η viscosity
- Glen's law for ice $\boxed{\dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}}$ $\tau = \sqrt{\tfrac{1}{2}\tau_{ij}\tau_{ij}}$ $n \approx 3$ $A \approx 2.4 \times 10^{-24}\ \mathrm{Pa}^{-3}\ \mathrm{s}^{-1}$ **at** $0^\circ\ \mathrm{C}$

Conservation of mass

- Consider rate of change of mass in a control volume $\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\ \mathrm{d}V = -\int_{\partial V} \rho u_j n_j\ \mathrm{d}S$
- using divergence theorem, and since volume is arbitrary, $\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0}$
- If incompressible $\frac{D\rho}{Dt} = 0$, then $\boxed{\nabla\cdot\mathbf{u} = 0}$

Conservation of momentum

- Apply Newton’s second law to a control volume

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i \ \mathrm{d}V = -\int_{\partial V} \rho u_i u_j n_j \ \mathrm{d}S + \int_{\partial V} \sigma_{ij} n_j \ \mathrm{d}S + \int_V \rho g_i \ \mathrm{d}V$$

- manipulating, and since volume is arbitrary, $\boxed{\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}}$

- e.g. for incompressible Newtonian fluid, these are the Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0 \qquad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \eta \nabla^2 \mathbf{u} + \rho \mathbf{g}$$

Reynolds number

$$Re = \frac{\rho U L}{\eta}$$

- measures the importance of the inertia terms (LHS of momentum eqn).

- typically very small for ice flow, so approximate as ‘Stokes flow’, $\boxed{\mathbf{0} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g}}$

Boundary conditions (general)

- rigid boundaries $\mathbf{u} \cdot \mathbf{n} = 0$ (no normal flow)

  $\mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n} = 0$ (no slip)

- free surfaces $z = f(x, y, t)$

  $w = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$ (kinematic condition)

  $\sigma_{ij} n_j = -p_a \delta_{ij}$ (stress continuity)

Glacier boundary conditions

- glacier surface $z = s(x, y, t)$

  $\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} - w = a$

  $\sigma_{ij} n_j = -p_a \delta_{ij}$

  a accumulation/ablation

  p_a atmospheric pressure

- glacier bed $z = b(x, y)$

  $u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} - w = a_b \ (\approx 0 \ )$

  a_b basal accumulation

  $\boldsymbol{\tau}_b = f(|\mathbf{u}_b|)\frac{\mathbf{u}_b}{|\mathbf{u}_b|}$

  friction law to account for slip

  $\boldsymbol{\tau}_b = \sigma_{ij} n_j - (n_k \sigma_{kj} n_j) n_i$

  $\mathbf{u}_b = u_i - (u_k n_k) n_i$

Shallow-layer models

- Exploit small aspect ratio to reduce complexity of the model.

- Derive by systematically approximating and integrating the governing equations, or by using conservation laws for depth averaged quantities.

- Depth-averaged conservation of mass

$$\boxed{\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{q} = a}$$

$h = s - b$ ice depth

$\mathbf{q} = h\overline{\mathbf{u}} = \int_b^s \mathbf{u}\, \mathrm{d}x$ ice flux

- Depth-averaged conservation of momentum

$$\boxed{0 = -\rho g h \nabla s - \boldsymbol{\tau}_b + \nabla \cdot (h \overline{\boldsymbol{\tau}})}$$

$\overline{\boldsymbol{\tau}} = \begin{pmatrix} 2\overline{\tau}_{xx} + \overline{\tau}_{yy} & \overline{\tau}_{xy} \\ \overline{\tau}_{xy} & \overline{\tau}_{xx} + 2\overline{\tau}_{yy} \end{pmatrix}$ membrane stress