Jones-Witten Theory and the Thurston Compactification of Teichmüller Space.

Jørgen Ellegaard Andersen

Thesis submitted Trinity Term, 1992, in support of application to supplicate for the degree of D. Phil.

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Abstract

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In this thesis, we study the Witten invariants of the three-manifolds which arise as mapping cylinders of closed surfaces. We investigate, within the framework of Geometric Quantization, an idea due to Connes which advocates the use of the Thurston compactification of Teichmüller space in studying the Witten invariants of these three-manifolds. Let Σ be a closed surface and G a simple simply connected compact Lie group. Then there is a continuous map F from Teichmüller space to the space of Kähler polarizations on the moduli space $\mathcal{M}(\Sigma)$ of flat connections in some principal G-bundle on Σ. Connes's idea is to give a natural description of the fibers over points of the Thurston boundary of the extension of the holomorphic vector bundle over Teichmüller space one gets by applying Geometric Quantization to this family of Kähler polarizations on the moduli space $\mathcal{M}(\Sigma)$.

We prove that the map F, viewed as a map from Teichmüller space into the space of all polarizations, does not have a continuous extension to the Thurston compactification. However, the study of this problem leads us to the following construction: Given a Strebel point on the Thurston boundary together with some extra data, namely a complete metric on the complement of certain close curves on Σ (determined up to homotopy be the Strebel point), we get a non-negative polarization on the moduli space $\mathcal{M}(\Sigma)$. The real directions of this polarization are completely determined by the Strebel points on the Thurston boundary. Any reducible periodic diffeomorphism of Σ preserves such a polarization on $\mathcal{M}(\Sigma)$. We prove that these non-negative polarizations are reducible and that they are natural limits of Kähler polarizations induced from Teichmüller space. The above mentioned results are proved in chapter 2.

In chapter 3 we study the Geometric Quantization of a smooth symplectic manifold M, with a smooth non-negative reducible polarization. We prove that the cohomology groups with values in the prequantum line bundle are finite-dimensional and that they decompose into a direct sum over the so-called Bohr-Sommerfeld set. Moreover, we prove an index formula which expresses the alternating sum of the dimensions of the cohomology groups as an integral over M of the edge product of the Chern character of the prequantum line

bundle and the $\hat{A}$-roof genus of M. For symplectomorphisms which preserve the polarization and have simple fixed points, we prove a Lefschetz fixed point theorem. A study of linear polarizations on tori gives us an explicit expression for the Witten invariant of mapping cylinders of tori.

We conclude chapter 3 by studying the Bohr-Sommerfeld set for some of the non-negative polarizations constructed in chapter 2 on the moduli space. A simple closed curve on Σ defines a point on the Thurston boundary. A choice of a metric with infinite ends on the complement of that curve gives us a non-negative polarization on $\mathcal{M}(\Sigma)$. We prove that the Bohr-Sommerfeld set of this polarization is precisely the labelling set of the conformal field theory constructed from G.

Acknowledgements

It is with pleasure that I express my deep gratitude to my two supervisors, Prof. Nigel J. Hitchin and Dr. John Roe.

I am very grateful to Prof. Nigel Hitchin for suggesting the problem to me, and for his profound help and many enlightening conversations.

I am deeply indebted to Dr. John Roe for his continuous encouragement, helpful advice and innumerable enlightening discussions.

A further debt of gratitude is due to Prof. Simon Donaldson for insightful exposition. His suggestions during the term he acted as my supervisor were of absolutely crucial importance for the results contained in chapter 2.

Conversations with Prof. Graeme Segal have been of immense help for me, especially with the development of the ideas in chapter 3 of this thesis.

I especially wish to thank my supervisor in Denmark, Lektor H. Pedersen, for his profoundly inspiring attitude and continuous support form the first day I saw him in his office at Odense University.

I have also benefited enormously from conversation with Prof. A Connes, Prof. V. Jones, Prof. E. Witten, Prof. S. Hurder, Dr. F. Kirwan, Dr. R Brussee, Dr. S. Salamon and Dr. N. Woodhouse.

I have been help on innumerable occasions with technical assistance by Guang-Yuan Guo, Alvise Munari, Simon Scott, Oscar Garcia-Prada and Michael Thaddeus.

I would especially like to express my most sincere gratitude to Amit Badiani, Michael Callahan and Mark Lenssen, who help me proof-read this thesis.

I should also like to take this opportunity to thank The Danish Research Council for its financial support.

Finally, I would like to express my deepest gratitude to Annette Øster first of all for her tremendous support and tolerance during the preparation of this thesis. It is with love that I dedicate this work to her.

Contents

0.0 Introduction

Witten has given a path integral definition of new invariants of compact oriented closed 3-manifolds by using a purely topological action, the Chern-Simons functional, on the space of connections on the 3-manifold. He showed that the Jones polynomial could be reinterpreted in terms of expectation values of Wilson lines, i.e. holonomy around embedded knots, in S^3. The adopted path integral approach is however not yet a rigorous mathematical theory. But, as Witten observed, the path integral has some factorization properties when the closed 3-manifold is represented as a union along the boundary of two 3-manifolds with boundary. Using this, Witten related this 2+1 dimensional topological theory to 1+1 dimensional conformal field theory, which in turn he used to prove the relation with the Jones polynomial.

Following Segal's axioms for a conformal field theory, Atiyah then collected the formal properties of the path integral in his axioms for a topological quantum field theory (TQFT). We shall here concentrate on the part of the theory which deals only with 3-manifolds, so not including embedded knots. A 2+1 dimensional TQFT, as Atiyah has described it, is an association of a finite dimensional vector space to closed surfaces and an association of a vector in this vector space to any compact oriented 3-manifold which bounds the surface. These associations are assumed to satisfy certain axioms, from which it follows that one gets a representation of the mapping class group of the surface on the vector space associated to the surface. It will be the character of this representation that will interest us here.

Suppose f is an orientation preserving diffeomorphism of a closed surface Σ. We can then form the mapping cylinder Σ_f of f, and the axioms for a TQFT imply that the Witten invariant $Z(\Sigma_f)$ is given by the character:

$$Z(\Sigma_f) = \operatorname{Tr} Z(f).$$

(Here we have ignored the fact that one needs to choose a 2-framing on Σ_f to define the Witten invariant, and that for this reason one actually only gets a representation of a central extension of the mapping class group. We shall describe this in detail in chapter 1, where we give a precise definition of $Z(\Sigma_f)$.) We thus see that the character contains the information about the Witten invariant for an interesting class of 3-manifolds. We shall investigate an idea of A. Connes, which suggests the use of Thurston's compactification in studying this character.

To describe Connes' idea, we need to consider the actual construction of the vector space $Z(\Sigma)$. We will consider the details of this in chapter 1. $Z(\Sigma)$ is defined to be the covariant constant sections of a bundle $\widehat{H}$ with a flat connection over Teichmüller space T. Moreover the mapping class group acts on $\widehat{H}$ and preserves the connection, so it acts on $Z(\Sigma)$ (we are here ignoring a central extension for simplicity). We observe that if a diffeomorphism f of

Σ preserves a point σ in Teichmüller space, then the action of f on $Z(\Sigma)$ is just the action of f on $\widehat{H}_\sigma$. So for such f, we do not need to know the covariant constant sections of $\widehat{H}$ to describe $\operatorname{Tr} Z(f)$. However, there are many diffeomorphisms which do not preserve any points on Teichmüller space. It is here that Connes has suggested the use of the Thurston compactification $\overline{T}$ of Teichmüller space.

Thurston's compactification of Teichmüller space is topologically a closed ball, and the action of the mapping class group extends continuously to the compactification. By Brouwer's fixed point theorem any diffeomorphism has a fixed point on the compactification. Clearly an extension of $\widehat{H}$ exists as an abstract vector bundle just by affixing a copy of the covariant constant sections of $\widehat{H}$ at each point of the boundary $\partial T = \overline{T} - T$. The mapping class group acts continuously on this extension. Nothing is of course gained by this abstract extension. What we need is a natural description of the fibres of the bundle over the boundary ∂T, so that $Z(f)$ can be described without knowing the covariant constant sections.

We shall here approach this problem within the framework of geometric quantization. The fibres of $\widehat{H}$ over points in Teichmüller space are constructed in the following way: one associates a symplectic manifold $\mathcal{M}(\Sigma)$ to Σ, namely the moduli space of flat connections in some principal G bundle, say P, over Σ. Here G is a Lie group, which we will assume to be compact and simple. There is a natural prequantum line bundle $\mathcal{L}$ over $\mathcal{M}(\Sigma)$ and the mapping class group of Σ acts on $\mathcal{L}$. In order to do the geometric quantization construction one needs a polarization on $\mathcal{M}(\Sigma)$. Each point in Teichmüller space induces a Kähler polarization on $\mathcal{M}(\Sigma)$, and the fibre of $\widehat{H}$ over $\sigma \in T$ is essentially

$$\widehat{H}_\sigma = H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^k),$$

i.e. the geometric quantization of $\mathcal{M}(\Sigma)$ with respect to the Kähler polarization induced from σ (k is here the level, a positive integer.).

If we want to use geometric quantization to construct the fibres over the boundary point, we are naturally led to the following question: is there a continuous map from the Thurston compactification of Teichmüller space into the space of polarizations on $\mathcal{M}(\Sigma)$, which extends the map from Teichmüller space into the space of Kähler polarizations? This question will be analysed in chapter 2. First we shall address the question on the level of the infinite dimensional space of connections. We shall see in section 2.2 that the construction of a complex polarization on the infinite dimensional space of connections in P from a complex structure on the surface generalizes substantially. In fact, any smooth 1-dimensional complex subbundle of the complexified tangent bundle of an open dense subset of the surface induces a polarization on the space of connections. Note that if we have a complex structure on a surface, we have naturally the holomorphic tangent bundle inside the complexified tangent bundle as a smooth 1-dimensional complex subbundle over the whole surface. The points of the Thurston boundary of Teichmüller space are constructed as equivalence classes of measured foliations on the surface. A measured foliation induces a 1-dimensional real subbundle of the tangent bundle on the complement of the singularities of the foliation. By complexifying this 1-dimensional subbundle, we get a natural construction of a real polarization on the space of connections. Using Thurston's classification theorem of diffeomorphisms, we can conclude that any diffeomorphism is homotopic to a diffeomorphism which preserves a non-negative polarization on the space of connections.

The moduli space $\mathcal{M}(\Sigma)$ of gauge equivalence classes of flat connections on the surface Σ, is the Marsden-Weinstein reduction of the space of connections. Whenever one has a Hamiltonian action of a Lie group on a finite dimensional symplectic manifold, then any polarization, which is invariant under the group action, will induce a polarization on the Marsden-Weinstein quotient. We shall see in section 2.3 that polarizations on infinite dimensional Banach manifolds do not in general behave well under Marsden-Weinstein reduction. For Strebel measured foliations we do get a well defined isotropic foliation on the moduli space $\mathcal{M}(\Sigma)$. This isotropic foliation only depends on the equivalence class of the measured foliation, i.e. it is naturally associated to Strebel points of the Thurston boundary. However, this isotropic foliation will in general not be Lagrangian. (For very special foliations, induced from a "pair of pants" decomposition, we do get a polarization on the SU(2) moduli space (see section 2.4).)

Infinitesimally, this isotropic foliation is given by the basic cohomology of the measured foliation. In section 2.4 we describe the global structure of this foliation using gluing techniques.

A measured foliation does not seem to be sufficient data to construct a polarization on the moduli space. Some extra structure on the surface is needed. In section 2.5 we will give a construction of a reducible non-negative polarization $P_{F,\tilde{g}}$ (with singularities) from a measured foliation F and some extra data, namely a metric $\tilde{g}$ on $\tilde{\Sigma} = \Sigma - L_F$. Here L_F is a finite union of non-singular leaves of F, chosen such that any non-singular leaf of F is homotopic to one in L_F, and none of the leaves in L_F is homotopic to any other leaf in L_F. This construction only depends on the equivalence class of F and the conformal class of $\tilde{g}$. The idea here is that the real directions of this non-negative polarization are determined by F, but we need the metric $\tilde{g}$ to determine the complex directions.

In order to see that some of the non-negative polarizations constructed in section 2.5 do lie in the closure of the set of Kähler polarizations, induced from points in Teichmüller space, we shall study some degeneration questions in section 2.6. Suppose F is a Strebel measured foliation on Σ. Then L_F is a union of simple closed curves on a surface. We can then consider a sequence of metrics g_t on Σ, which arises if we fix a metric g on the complement of a tubular neighbourhood of L_F and then stretch the tubular neighbourhood longer and longer. (See section 2.6.2 for a specific desription of g_t.) This gives us a sequence of Kähler polarizations on the moduli space. When the length of the tubular neighbourhoods goes to infinity, we show that the corresponding Kähler polarizations converge pointwise to a non-negative polarization constructed from F and a metric $\tilde{g}$ on $\tilde{\Sigma}$, which is constructed from g simply by attaching standard half-infinite cylinders to the boundaries of $\tilde{\Sigma}$. (Again see section 2.6.2 for precise details.)

The constructions in section 2.5 and the degeneration results in section 2.6 have two important consequences.

The first one is an answer to the question of whether there is a continuous extension to the Thurston compactification of the map that associates Kähler polarizations to points in Teichmüller space. We shall see in section 2.7 that the results of sections 2.5 and 2.6 can be combined with the fact that the action of the mapping class group is minimal on the Thurston boundary to show that there cannot exist a continuous extension to the Thurston compactification of the map which associates Kähler polarizations to points in Teichmüller space.

The second consequence is that for each reducible (in the sense of Thurston) element f of the mapping class group, we have in section 2.5 constructed a reducible non-negative polarization, preserved by f on the moduli space $\mathcal{M}(\Sigma)$. So, although there is no extension of the map from Teichmüller space to the space of non-negative polarizations, we have found an explicit way to construct preserved polarizations for reducible periodic and finite order diffeomorphisms. According to Thurston's classification of diffeomorphisms that only leaves us with the pseudo-Anosov diffeomorphism. The only thing we can say about those, is that they do preserve a polarization on the space of connections.

In order to make use of the constructions in chapter 2, we need to be able to apply geometric quantization to reducible non-negative polarizations. Traditionally, Kähler polarizations have received most attention, since the powerful theory of elliptic operators between Hilbert spaces is applicable in this case, and one gets the machinery of Hodge theory and the index and Lefschetz theorems. In chapter 3 we shall try to initiate the study of index and Lefschetz fixed point theory for more general polarizations. We will see that there is a natural complex with values in the prequantum line bundle $\mathcal{L}$ on a symplectic manifold M with a non-negative polarization P :

$$0 \to C^\infty(\mathcal{L}^k \otimes \delta_P) \overset{\nabla^P}{\to} C^\infty(\overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P) \overset{\nabla^P}{\to} \dots \overset{\nabla^P}{\to} C^\infty(\Lambda^n \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P) \to 0 \qquad (1)$$

which is non-elliptic, if P is not a Kähler polarization (here δ_P is the normal half-form bundle of P). We prove, using a spectral sequence argument, that if the polarization is reducible (meaning the leaf spaces $M/(P+\overline{P})$ and $M/(P\cap\overline{P})$ are Hausdorff manifolds), then the cohomology groups $H^i(M,\nabla^P)$ are finite dimensional. In fact, the spectral sequence gives us some kind of a factorization result, which says that

$$H^i(M,\nabla^P)=0,\ 0\leq i<m,$$

where m is the number of real directions in P, i.e. $\mathrm{rk}(P\cap\overline{P})=m$, and

$$H^i(M,\nabla^P)=\bigoplus_{b\in S_k}H^{i-m}(M_b',\overline{\partial}_b),\ m\leq i\leq n.$$

Here S_k is the Bohr-Sommerfeld subset of the leaf space $M/(P+\overline{P})$, and M_b' is the symplectic reduction at b. (See section 3.2 for further explanation of the notation.)

Now suppose we have a symplectomorphism f with simple fixed points of M, that preserves P, and assume moreover we have a lift f^* of f to $\mathcal{L}$. A local analysis of f combines, by the use of the spectral sequence, with the holomorphic Lefschetz fixed point theorem to prove a Lefschetz fixed point theorem for the complex (1) :

$$\sum_i(-1)^i\,\mathrm{Tr}(f^*:H^i(M,\nabla^P))=\sum_{\{p\in M|f(p)=p\}}\nu_p\sum_i(-1)^i\frac{\mathrm{Tr}(f_p^*:\Lambda^i\overline{P}_p^*\otimes\mathcal{L}_p^k\otimes\delta_{P_p})}{|\,\mathrm{Det}(1-df(p))|},$$

where

$$\nu_p=\mathrm{Sign}\,\mathrm{Det}(1-df_B(p)),$$

and f_B is the induced map on the leaf space M/P.

We can also prove an index theorem for these non-elliptic complexes under a somewhat technical assumption on the non-negative polarization. This assumption is trivialy satisfied for real polarizations and Kähler polarizations. The proof of this index theorem is by a combination of a topological argument and the use of the Hirzebruch-Riemann-Roch formula applied to the Kähler manifolds $M_b',\ b\in S_k$. This index theorem says that

$$\mathrm{Index}(\nabla^P)=(-1)^m\int_M\mathrm{ch}(\mathcal{L}^k)\wedge\widehat{A}(TM).$$

Here $\mathrm{Index}(\nabla^P)$ is the index of the complex (1), i.e. the alternating sum of the dimensions of the cohomology groups $H^i(M,\nabla^P)$.

Nothing is known about the geometric quantization, i.e. the leafwise complex (1) and its cohomology groups, for polarizations which are not reducible. In section 3.5 we study the geometric quantization of a 2n-torus with respect to a linear real polarization, induced from an arbitrary Lagrangian subspace of $\mathbb{R}^{2n}$. The space of distributional sections $\mathcal{H}_P$ of the prequantum line bundle, which are covariant constants along a real linear polarization P is studied. If the polarization P is reducible, then the results of section 3.2-4 shows that $\mathcal{H}_P \cong H^n(M, \nabla^P)$ and

$$\dim \mathcal{H}_P = k^n. \tag{2}$$

For non-reducible P, we explicitly construct an isomorphism from $\mathcal{H}_P$ to $\mathcal{H}_{P'}$, where P' is a reducible real polarization. From this we get that the formula (2) holds for any real linear polarization P. Furthermore, we find an explicit basis for $\mathcal{H}_P$ in terms of certain limits of theta-functions.

Returning to the study of Jones-Witten theory, it is natural to try to apply some of these ideas to the non-negative polarizations constructed on the moduli spaces. Section 3.6 and 3.7 of my thesis are the first steps in this direction.

Section 3.6 is a consideration of the genus one case. In this case, we can in fact carry out the entire programme outlined here for studying the character of the mapping class group. The mapping class group is then $SL(2,\mathbb{Z})$. The upper half plane T is for genus 1 a natural analogue of Teichmüller space. We can compactify T to a closed disc $\overline{T}$ (which parametrizes linear non-negative polarizations on Σ) and the $SL(2,\mathbb{Z})$ action extends to the compactification. Hence, as before, any element of $SL(2,\mathbb{Z})$ has a fixed point on $\overline{T}$. The map that associates to a point on T a complex polarization on the G-moduli space $\mathcal{M}(\Sigma)$, extends to a continuous map from the compactification into the space of non-negative polarizations on $\mathcal{M}(\Sigma)$. We get an explicit description of the fibres of $\widehat{H}$ over the boundary of $\overline{T}$ as the geometric quantization with respect to the corresponding real polarizations on $\mathcal{M}(\Sigma)$. Finally, we use our Lefschetz fixed point theorem to calculate the Witten invariant for the closed 3-manifold Σ_f, where $f \in SL(2,\mathbb{Z})$.

In section 3.7 we conclude chapter 3 by studying the Bohr-Sommerfeld set for the non-negative polarization constructed in section 2.5 on the moduli space $\mathcal{M}(\Sigma)$ of flat connections on a closed surface. The data we need to construct such a non-negative polarization is a finite union of non-homotopic simple closed curves on a closed surface together with a metric with infinite ends on the complement of the curves. In section 3.7 we shall concentrate on the case where we just have one curve γ and a metric $\tilde{g}$ with infinite ends on the complement of that curve. The level k Bohr-Sommerfeld set S_k for the non-negative polarization $P_{[\gamma],\tilde{g}}$ is a finite subset of the space of regular conjugacy classes C_G^{reg} for the Lie group G we are considering. In the case where G is connected, simply connected and semisimple, we prove that S_k is precisely the labeling set of the conformal field theory which Segal constructs from G in [44]. Moreover, the symplectic reduction M'_c of the Bohr-Sommerfeld leaf $c \in S_k$ is precisely the moduli space of flat connections on $\tilde{\Sigma} = \Sigma - \gamma$ with fixed holonomy $c \in S_k$ around the two boundaries of $\tilde{\Sigma}$. We end section 3.7 with a conjectural formula for the Witten invariant for mapping cylinders of Dehn-twists around an arbitrary simple closed curve on a closed surface of any genus.

Section 3.7 is just the initial step in the understanding of the quantization of the moduli spaces with respect to these non-negative polarizations. With the factorization axiom from conformal field theory in mind, there seems to be scope for further interesting study of how the Bohr-Sommerfeld quantization relates to the Kähler quantization of the moduli spaces.

Chapter 1

The geometric approach to Jones-Witten Theory.

1.1 Path Integrals, motivations from QFT and the prequantum line bundle.

In [51] Witten introduced some new invariants for 3-manifolds with an embedded link using ideas from Quantum Field Theory. Witten showed that these invariants gave a generalization of the Jones polynomial for a link embedded in S^3. Here we shall only discuss the case where there are no embedded links. Moreover, we shall not do any calculations with these path integrals, but just use them as motivation for the rigorous construction we shall describe in the following sections. In this section we shall follow paper [39] very closely.

Let Y be an oriented compact 3 manifold. Let $\Sigma = \partial Y$ be the boundary of Y, with the induced orientation. Suppose G is a compact simple and simply connected Lie group. On the Lie algebra g of G there is a bi-invariant inner product $B(\cdot,\cdot)$, which we will normalize such that $B(\alpha,\alpha) = 2$, where α is the highest root of G. Note, that if G is simply laced, then it follows (see [37]) that all the roots have length 2. Let P_Y be a principal G-bundle over Y. We can consider the space $\mathcal{A}_Y$ of connections in P_Y. The group $\mathcal{G}_Y$ of gauge transformations in P_Y acts on $\mathcal{A}_Y$. Since G is assumed to be simply connected, any principal G-bundle P_Y over any compact 3-manifold is trivial. When we choose a trivialization of P_Y, we get an identification $\mathcal{A}_Y \cong \Omega^1(Y, \mathsf{g})$.

Definition 1.1.1 *Define the Chern-Simons functional to be*

$$CS(\tilde{A}) = \frac{1}{4\pi^2}\int_Y B(\tilde{A}\wedge d\tilde{A} + \frac{2}{3}\tilde{A}\wedge[\tilde{A}\wedge\tilde{A}]) \in \mathbb{R}$$

for any $\tilde{A} \in \mathcal{A}_Y$.

We shall now fix a trivialization of $P_Y|_\Sigma$. This means that the Cherns-Simons functional becomes well defined modulo 1 (see [13]), so $\exp(2\pi i CS(\tilde{A})) \in \mathbb{C}$ is well defined for all $\tilde{A} \in \mathcal{A}_Y$. Denote $P = P_Y|_\Sigma$. Let $\mathcal{A}$ be the space of connections of the trivialized bundle P over Σ and $\mathcal{G}$ the group of gauge transformations in P. We have the identifications

$$\mathcal{A} = \Omega^1(\Sigma, \mathrm{g}), \quad \mathcal{G} = \Omega^0(\Sigma, G).$$

Witten now considers the following section s of the trivial bundle $\tilde{\mathcal{L}} = \mathcal{A} \times \mathbb{C}$ over $\mathcal{A}$ given by a path integral:

$$s(A) = \int_{\tilde{A} \in C_A} \exp(2\pi i k CS(\tilde{A})) \mathcal{D}\tilde{A},$$

where $C_A = \{\tilde{A} \in \mathcal{A}_Y | \tilde{A}|_\Sigma = A\}$, $A \in \mathcal{A}$, and $k \in \mathbb{N}$ is the so-called "level" of the theory. Notice that if $\partial Y = \emptyset$, then this is just a complex number, and so this is supposed to be a numerical invariant of Y.

We want to compare $s(A)$ and $s(g^*A)$. Since G is simply connected, there is no obstruction to extending any gauge transformation $g \in \mathcal{G}$, to a gauge transformation $\tilde{g}$ of P_Y. Suppose $\tilde{g} \in \mathcal{G}_Y$ is an extension of g. Let $\tilde{A}_1, \tilde{A}_2 \in C_A$. We then calculate

$$\begin{aligned}
&\exp(2\pi i k(CS(\tilde{g}^*\tilde{A}_1) - CS(\tilde{A}_1)))\exp(-2\pi i k(CS(\tilde{g}^*\tilde{A}_2) - CS(\tilde{A}_2))) \\
&= \exp(2\pi i k(CS(\tilde{g}^*\tilde{B}) - CS(\tilde{B}))),
\end{aligned}$$

where $\tilde{B} = \tilde{A}_1 \bigcup_\Sigma \tilde{A}_2$ is a connection in $P_Y \bigcup_\Sigma P_Y$ over the double $Y \bigcup_\Sigma Y^*$. But $CS(\tilde{g}^*\tilde{B}) - CS(\tilde{B}) \in \mathbb{Z}$ for any gauge transformation $\tilde{g}$ and connection $\tilde{B}$ in a principal bundle over a closed 3-manifold (see [13]). This means we can define a $U(1)$ valued function on the space $\mathcal{A} \times \mathcal{G}$ by

$$\Theta(A, g) = \exp(2\pi i k(CS(\tilde{g}^*\tilde{A}) - CS(\tilde{A}))),$$

where $\tilde{A} \in C_A$ and $\tilde{g} \in \mathcal{G}_Y$ is an extension of g. In fact we can use any extension of A and g in any compatible principal bundle over any 3-manifold that bounds Σ by the same argument as before. We see that Θ is a naturally defined function on $\mathcal{A} \times \mathcal{G}$. One easily verifies that Θ satisfies a cocycle condition:

$$\Theta(A, g)\Theta(g^*A, h) = \Theta(A, gh).$$

Formally, these arguments also show that

$$s(g^*A) = \Theta(A, g)s(A), \tag{1.1}$$

for all $A \in \mathcal{A}$ and $g \in \mathcal{G}$. Before we proceed with a formal discussion of s, let us consider some properties of $\mathcal{A}$ and the cocycle Θ. There is a natural symplectic structure on $\mathcal{A}$ given by

$$\omega(\phi_1, \phi_2) = \int_\Sigma B(\phi_1 \wedge \phi_2), \ \forall \phi_1, \phi_2 \in \Omega^1(\Sigma, \mathrm{ad}\, P).$$

$\mathcal{G}$ preserves the symplectic form, and in fact we can write down a moment map for this action:

$$\mu: \mathcal{A} \to \Omega^0(\Sigma, \operatorname{ad} P)^*,$$

given by

$$\mu(A)(\alpha) = \int_\Sigma B(F_A \wedge \alpha),$$

where $A \in \mathcal{A},\ \alpha \in \Omega^0(\Sigma, \operatorname{ad} P)$, and F_A is the curvature of A.

This is the data one needs to construct the symplectic quotient or the Marsden-Weinstein quotient:

$$\mathcal{M}(\Sigma) = \mu^{-1}(0)/\mathcal{G}.$$

Note that $\mu^{-1}(\Sigma)(0) = \mathcal{A}^F$ is the space of flat connections in P, and so $\mathcal{M}(\Sigma)$ is the moduli space of flat connections in P. Consider the subset $\mathcal{A}^{\mathrm{irr}}$ of $\mathcal{A}^F$ consisting of irreducible flat connections. We note that $\mathcal{G}$ acts on $\mathcal{A}^{\mathrm{irr}}$, and we denote the space of gauge equivalence classes of irreducible flat connections by

$$\mathcal{M}(\Sigma)^{\mathrm{irr}} = \mathcal{A}^{\mathrm{irr}}/\mathcal{G}.$$

One finds that there is naturally induced a manifold structure on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. Moreover $\mathcal{M}(\Sigma)^{\mathrm{irr}}$ is an open dense subset of $\mathcal{M}(\Sigma)$. We shall give more details on the local structure of $\mathcal{M}(\Sigma)$ in section 2.4. For a detailed description of the local struture of $\mathcal{M}(\Sigma)$ we refer to [23].

The symplectic form on $\mathcal{A}$ induces naturally a symplectic structure ω on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. One can identify $\mathcal{M}(\Sigma)$ with the space of representations of $\pi_1(\Sigma)$ into G modulo conjugation, and the induced symplectic structure has a purely algebraic description (see [22]).

On $\mathcal{A}$ there is a 1-form given by

$$\theta_A(\phi) = \int_\Sigma B(A \wedge \phi),$$

where $A \in \mathcal{A}$ and $\phi \in \Omega^1(\Sigma, \operatorname{ad} P)$. Since $d\theta = \omega$, we have that θ is a symplectic potential for ω. Now θ defines a connection in the trivial bundle $\tilde{\mathcal{L}} = \mathcal{A} \times \mathbb{C}$, whose curvature is ω.

The cocycle Θ gives us a lift of the action of $\mathcal{G}$, to the bundle $\tilde{\mathcal{L}}$. Let us now investigate how $\tilde{\mathcal{L}}$ behaves under the Marsden-Weinstein construction. The isotropy group G_A of a connection $A \in \mathcal{A}$ is isomorphic to a subgroup of G, by evaluation in a point on Σ. For the irreducible connections this subgroup must be $Z(G)$, the center of G. Now for $g \in Z(G)$, it is clear that

$$\Theta(A, g) = 1,\ \forall A \in \mathcal{A}.$$

Notice that $\mathcal{G}$ preserves both $\mathcal{A}^F$ and $\mathcal{A}^{\mathrm{irr}}$. From this it follows that we get induced a C^∞ line bundle

$$\mathcal{L} = \mathcal{A}^{\mathrm{irr}} \times_\Theta \mathbb{C}$$

over $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. The total space of $\mathcal{L}$ is defined to be equivalence classes of $\mathcal{A}^{\mathrm{irr}} \times \mathbb{C}$ under the equivalence relation

$$(A, z) \sim (g^* A, \Theta(A, g)z), \ \forall g \in \mathcal{G}.$$

By projecting onto the first factor we get an induced projection onto $\mathcal{M}(\Sigma)^{\mathrm{irr}}$, the smooth part of the moduli space.

The proof in [39], that the connection in $\tilde{\mathcal{L}}$ given by θ is invariant under the action of $\mathcal{G}$ and descends to define a hermitian connection in $\mathcal{L}$, applies without change for the case of the more general groups we are considering here. We can thus formulate a theorem, which we have seen to be a straightforward generalization of Theorem 1 in [39]:

Theorem 1.1.1 *Suppose G is a compact simple and simply connected Lie group. Then there is a naturally defined hermitian C^∞ line bundle $\mathcal{L}$ over $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. It has a natural hermitian connection, whose curvature is ω, ω being the symplectic form on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$.*

Remark. To see that $\mathcal{L}$ extends as a continuous line bundle over $\mathcal{M}(\Sigma)$, we need to show that

$$\Theta(A, g) = 1,$$

for g in the isotropy group G_A and $A \in \mathcal{A}^F$. Let $g \in G_A$. Suppose $\tilde{g} \in \mathcal{G}_Y$ is an extension of g and $\tilde{A} \in \mathcal{A}_Y$ an extension of A. Since $g \in G_A$, we see that $\tilde{g}^* \tilde{A} \bigcup_\Sigma \tilde{A}$ defines a connection on the double of P_Y over the double of Y. Let us now assume there is a smooth curve in G_A which connects g to an element in $Z(G)$. We can then on the double of Y construct a gauge transformation h, such that $h^*(\tilde{g}^* \tilde{A} \bigcup_\Sigma \tilde{A}) = \tilde{A} \bigcup_\Sigma \tilde{A}$. This would implies the desired result. For $SU(2)$ this is clearly the case, since G_A is either $SU(2)$, a copy of the maximal torus, or the centre of $SU(2)$.

We shall now consider how the diffeomorphisms of the surface act on this set-up. Let $\mathrm{Diff}_+(\Sigma)$ be the group of orientation-preserving diffeomorphisms of Σ. $\mathrm{Diff}_+(\Sigma)$ acts on $\mathcal{A}$ and $\mathcal{G}$ by pull back. We claim that there is a natural lift of this action to $\tilde{\mathcal{L}}$. In order to construct this action, we define a $U(1)$ valued function Ψ on $\mathcal{A} \times \mathrm{Diff}_+(\Sigma)$ by

$$\Psi(A, f) = \exp(2\pi i(CS(\widetilde{f^* A}) - CS(\tilde{A}))).$$

Now Ψ is independent of the extensions used, and it is a cocycle. Using Ψ we have a map

$$\Psi(A, f) \colon \tilde{\mathcal{L}}_A \to \tilde{\mathcal{L}}_{f^* A},$$

which gives us a lift of the action of $\mathrm{Diff}_+(\Sigma)$. We note that $\mathrm{Diff}_+(\Sigma)$ preserves both the symplectic form and the symplectic potential, hence the connection in $\tilde{\mathcal{L}}$ is invariant under $\mathrm{Diff}_+(\Sigma)$. One easily verifies that

$$\Theta(f^* A, (g \circ f))\Psi(A, f) = \Psi(g^* A, f)\Theta(A, g).$$

This means that we get an induced action of $\mathrm{Diff}_+(\Sigma)$ on $\mathcal{L}$, which projects to the induced action of $\mathrm{Diff}_+(\Sigma)$ on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. Moreover $\mathrm{Diff}_+(\Sigma)$ preserves the hermitian connection on $\mathcal{L}$.

Let $\mathrm{Diff}_0(\Sigma)$ be the orientation preserving diffeomorphisms homotopic to the identity. Let $f \in \mathrm{Diff}_0(\Sigma)$. By using the homotopy f_t, which connects f to the identity to extend f over a neighbourhood of Σ in a 3-manifold which bounds Σ, we see that

$$\Psi(A, f) = 1.$$

Since $\mathrm{Diff}_0(\Sigma)$ acts trivially on $\mathcal{M}(\Sigma)$, we see that the mapping class group $\Gamma = \mathrm{Diff}_+(\Sigma)/\mathrm{Diff}_0(\Sigma)$ acts on the line bundle $\mathcal{L}$ over $\mathcal{M}(\Sigma)^{\mathrm{irr}}$ and preserves the hermitian connection in $\mathcal{L}$.

In the rest of this section we shall specialize to the group $G = SU(n)$. Suppose we have chosen a conformal structure σ on Σ. We shall now give an alternative description of $\mathcal{M}(\Sigma)$ and $\mathcal{L}$.

Consider the trivial bundle $E = \Sigma \times C^n$, which is the vector bundle associated to P with respect to the defining representation of $SU(n)$. For each $A \in \mathcal{A}$ we get a covariant derivative d_A in E. By using the conformal structure σ on Σ, we get for each A a $\overline{\partial}$-operator $\overline{\partial}_A$ in E, by taking the (0,1)-part of d_A. This gives us an isomorphism between $\mathcal{A}$ and the space $\mathcal{C}_h$ of hermitian holomorphic structures on E with trivial determinant (trivial determinant here means $\Lambda^n E$ is trivial as a holomorphic line bundle). This induces a Kähler structure on $\mathcal{A}$ with Kähler form ω. The complexified gauge group $\mathcal{G}^C = \Omega^0(\Sigma, SL(n, \mathbb{C}))$ acts on $\mathcal{C}_h$. The Narasimhan-Seshadri Theorem (see [36]) says that the algebraic quotient, i.e. the moduli space $\mathcal{M}(\Sigma)_\sigma$ of semi-stable hermitian bundles with trivial determinant, is isomorphic to the moduli space $\mathcal{M}(\Sigma)$ of flat $SU(n)$-connections. The smooth part of $\mathcal{M}(\Sigma)^{\mathrm{irr}}$ is the moduli space $\mathcal{M}(\Sigma)^s_\sigma$ of stable hermitian bundles with trivial determinant. This induces a Kähler structure on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$, and in fact $\mathcal{M}(\Sigma)_\sigma$ is an algebraic variety (see [36]). On the space $\mathcal{C}$ of holomorphic structures on E, Quillen (see [38]) has constructed a holomorphic line bundle $\tilde{\mathcal{L}}_D$, which gets induced a hermitian structure when we choose a metric on Σ in the conformal class σ. Quillen proves that the curvature of the unique connection in $\tilde{\mathcal{L}}_D$, compatible with the holomorphic structure and the hermitian structure, is the Kähler form on $\mathcal{C}$. Now $\Omega^0(\Sigma, U(n))$ acts naturally on $\tilde{\mathcal{L}}_D$ and preserves the connection and the hermitian structure. Hence, if we restrict $\tilde{\mathcal{L}}_D$ to $\mathcal{C}_h$ and pull it back to $\mathcal{A}$, $\mathcal{G}$ acts naturally on $\tilde{\mathcal{L}}_D$ and preserves the connection and the hermitian structure. The fibre of $\tilde{\mathcal{L}}_D$ at $A \in \mathcal{A}$ is

$$\tilde{\mathcal{L}}_D(A) = \det H^1(\Sigma_\sigma, \overline{\partial}_A) \otimes (\det H^0((\Sigma_\sigma, \overline{\partial}_A))^*.$$

As it is explained in [4] p.39, $Z(G) \subset \mathcal{G}$ acts trivially on $\tilde{\mathcal{L}}_D(A)$, so we get induced a holomorphic hermitian line bundle $\mathcal{L}_D$ over $\mathcal{M}(\Sigma)^{\mathrm{irr}}$, whose curvature is ω. In fact (see [4]) $\mathcal{L}_D$ extends to a line bundle over $\mathcal{M}(\Sigma)$. In order to identify $\mathcal{L}$ and $\mathcal{L}_D$ as hermitian line bundles with connection, we just need that $H^1(\mathcal{M}(\Sigma)^{\mathrm{irr}}, \mathbb{R}) = 0$, since they have the same curvature (see [53]). We know that $\mathcal{M}(\Sigma)^{\mathrm{irr}}$ is simply connected (see [5]), so we can state the following straightforward generalization of Theorem 2 in [39].

Theorem 1.1.2 *$\mathcal{L}$ and $\mathcal{L}_D$ are isomorphic as hermitian C^∞ line bundles with connection over $\mathcal{M}(\Sigma)^{\mathrm{irr}}$.*

Let us end this section by returning to the formal discussion of the path integral description of the section s. By choosing a metric ρ on Y, a regularized expression for s can be obtained, and in [39] it is argued that the support of s is concentrated on $\mathcal{A}^F$. But then by equation (1.1) s becomes a section of $\mathcal{L}$ over $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. Furthermore, it is shown in [39] that s is in fact holomorphic with respect to the complex structure induced on $\mathcal{M}(\Sigma)$ by the conformal class of $\rho|_\Sigma$. Hence

$$s_\rho \in H^0(\mathcal{M}(\Sigma)_{[\rho|_\Sigma]}, \mathcal{L}^k).$$

This space of holomorphic sections of $\mathcal{L}^k$ is a finite dimensional complex vector space.

We shall not here discuss how s_ρ depends on the metric ρ, but just refer to Witten's paper [51]. In fact Witten argues in the paper [51] that the dependence on the metric is not essential, and that the invariant for a 3-manifold Y only depends on a framing of Y. This assumes that we have some way to identify the spaces $H^0(\mathcal{M}(\Sigma)_{[\rho|_\Sigma]}, \mathcal{L}^k)$ for varying $[\rho|_\Sigma]$. We shall return to this problem in section 1.3, where we will also discuss the framing dependence. In the next section the framing issue will be suppressed. When we do this, the discussion in this section then motivates that the invariant of an oriented compact 3-manifold should be a vector in a finite dimensional vector space, canonically associated to the boundary $\partial Y = \Sigma$.

1.2 Atiyah's axioms for a TQFT.

The use of the Path Integrals in the previous section is not rigorous nor well defined mathematically yet. One can however try to axiomatize the theory by extracting formal properties of the Path Integral. This has been done by Atiyah (see [3] for a complete discussion). Let us restate the axioms for a (2+1)-dimensional Topological Quantum Field Theory (TQFT), and derive the formula we shall be interested in here.

We require the following assignment in a (2+1)-dimensional TQFT:

- We have a functor Z from the category of closed oriented surfaces to the category of finite dimensional complex vector spaces.

- To each compact oriented 3-dimensional manifold Y there is associated a vector $Z(Y) \in Z(\partial Y)$.

Furthermore $Z(Y)$ is assumed to behave functorially with respect to orientation preserving diffeomorphisms, i.e. if $f\colon Y \to X$ is an orientation preserving diffeomorphism then

$$Z(f|_{\partial Y})(Z(Y)) = Z(X),$$

where

$$Z(f|_{\partial Y})\colon Z(\partial Y) \to Z(\partial X)$$

is the induced map of $f|_{\partial Y}\colon \partial Y \to \partial X$.

The assignment is assumed to satisfy the following axioms:

Axiom 1) (Involutory)

$$Z(\Sigma^*) = Z(\Sigma)^*,$$

where Σ^* denotes Σ with the opposite orientation.

Axiom 2) (Multiplicative)

$$Z(\Sigma_1 \coprod \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2),$$

where $\Sigma_1 \coprod \Sigma_2$ denotes the disjoint union.

Suppose Y is a compact oriented 3-manifold with boundary $\partial Y = \Sigma_1^* \coprod \Sigma_2$. From axiom 1) and 2) we see that

$$Z(Y) \in \mathrm{Hom}(Z(\Sigma_1), Z(\Sigma_2)).$$

Using this interpretation of $Z(Y)$, we formulate the third axiom:

Axiom 3) (Associativity)

$$Z(Y) = Z(Y_1) \circ Z(Y_2),$$

where $Y = Y_1 \bigcup_{\Sigma_2} Y_2$ and $\partial Y_1 = \Sigma_1^* \coprod \Sigma_2,\ \partial Y_2 = \Sigma_2^* \coprod \Sigma_3$.

From axioms 1), 2) and 3) it follows that $Z(\emptyset) = 0$ or $\mathbb{C}$ and that $Z(\Sigma \times [0,1]) \in \mathrm{Hom}(Z(\Sigma))$ is an idempotent. We shall therefore add the following two non-triviality axioms.

Axiom 4)

$$Z(\emptyset) = \mathbb{C}.$$

Axiom 5)

$$Z(\Sigma \times [0,1]) = I_{Z(\Sigma)}.$$

We note that for a closed 3-manifold Y, $Z(Y)$ is a numerical invariant of Y.

One easily derives from the axioms that $Z(f) = Z(g)$ if $f, g \in \mathrm{Diff}_+(\Sigma)$ are homotopic, so we get a representation of the mapping class group Γ. Suppose $f \in \mathrm{Diff}_+(\Sigma)$. We can then form a closed 3-manifold

$$\Sigma_f = \Sigma \times [0,1]/(x,0) \sim (f(x),1).$$

Note that the diffeomorphism type Σ_f only depends on the class of f in the mapping class group Γ. The axioms imply that the numerical invariant for the closed 3-manifold Σ_f is given by

$$Z(\Sigma_f) = \mathrm{Tr}\, Z(f).$$

If we know the representations of $\mathrm{Diff}_+(\Sigma)$ on $Z(\Sigma)$, this formula clearly gives us a complete understanding of $Z(\Sigma_f)$. We shall here discuss ideas which aim at giving a geometric expression for $Z(\Sigma_f)$. These ideas will be presented in section 1.4, when we have reviewed the geometric construction of $Z(\Sigma)$ for closed surfaces Σ in the next section.

1.3 On construction of the functor Z.

The formal discussion in section 1.1 suggests that $Z(\Sigma)$ should be constructed by applying geometric quantization to the moduli space of flat connections over Σ. This has been done and worked out in [8]. For a purely mathematical approach using complex/algebraic geometry see [29]. For a construction of a theory, which also deals with surfaces with boundaries, see [44]. We refer to these papers for a detailed discussion. Here we shall review the construction with emphasis on the aspects which are of particular interest to our discussion.

We shall assume the following set-up. Σ is a closed surface of genus $g > 1$. Let G be a simple, compact and simply connected Lie group, P a principal G-bundle over Σ. $\mathcal{A}$ is the space of connections in P, and $\mathcal{G}$ is the group of gauge transformations. A bi-invariant inner product on the Lie-algebra of G is chosen as specified in the beginning of section 1.1. This induces, as we saw in section 1.1, a symplectic structure ω on $\mathcal{A}$ and the Marsden-Weinstein quotient with respect to the action of $\mathcal{G}$, is the moduli space $\mathcal{M}(\Sigma)$ of flat G-connections. Now ω induces a symplectic structure on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. Theorem 1 tells us there is a natural prequantum line bundle over $\mathcal{M}(\Sigma)^{\mathrm{irr}}$, when a trivialization of P is chosen. Moreover, there is a natural action of the mapping class group Γ induced on $\mathcal{L}$. Different trivializations of P give isomorphic $\mathcal{L}$ and isomorphic Γ actions.

In order to apply the construction of geometric quantization to $\mathcal{M}(\Sigma)^{\mathrm{irr}}$ (see [53]), we need a polarization. Let us here digress to review the definition of a polarization on a general symplectic manifold (M, ω). We shall consider a very general definition of polarizations, which will include singular polarizations. The reason being, as we shall see in chapter 2, that polarizations with singularities arise naturally on the moduli spaces.

Let us first briefly review the complex Lagrangian Grassmannian of a symplectic vector space. Let (V, ω) be a real $2n$-dimensional symplectic vector space. Extend ω by complex linear to $V \otimes \mathbb{C}$. Consider the complex Lagrangian Grassmannian $L(V \otimes \mathbb{C})$, the space of non-negative Lagrangian subspaces of $V \otimes \mathbb{C}$. Recall, that a subspace W is non-negative if

$$-i\omega(v, \overline{v}) \geq 0 \;\forall v \in W.$$

Note that non-negativity is part of our definition of a Lagrangian subspace, since we will only be interested in such subspaces. There is a function

$$\rho : L(V \otimes \mathbb{C}) \to \{0, \ldots, n\},$$

which counts the number of real directions in a subspace , i.e.

$$\rho(W) = \dim(W \cap \overline{W} \cap V),\; W \in L(V \otimes \mathbb{C}).$$

We use the following notation:

$$L_i(V \otimes \mathbb{C}) = \rho^{-1}(i).$$

General theory (see [53]) shows that $L(V \otimes \mathbb{C})$ is topologically a closed ball. The interior of $L(V \otimes \mathbb{C})$ is $L_0(V \otimes \mathbb{C})$ and

$$\overline{L_k(V \otimes \mathbb{C})} = \bigcup_{m=k}^{n} L_m(V \otimes \mathbb{C}).$$

Suppose now (M, ω) is a symplectic manifold. Denote by $L(M)$ the topological fibre bundle over M, whose fibre over $p \in M$ is $L_p(M) = L(T_pM \otimes \mathbb{C})$. Now assume $P : M \to L(M)$ is a section of this fibration. Define

$$U_l(P) = \{p \in M | \rho(P(p)) = l\}.$$

These sets give a partition of M for each section P:

$$M = \bigcup_{l=0}^{n} U_l(P).$$

We now make the very general definition of a singular non-negative polarization.

Definition 1.3.1 *A singular non-negative polarization of a symplectic manifold (M, ω) is a section $P : M \to L(M)$ with the following properties:*

1. *There is a $\rho(P) \in \{0, \ldots, n\}$ such that $U_{\rho(P)}(P)$ is open and dense in M.*

2. *$P|_{U_{\rho(P)}(P)}$ is a smooth subbundle of $TM \otimes \mathbb{C}|_{U_{\rho(P)}(P)}$.*

3. *$P|_{U_{\rho(P)}(P)}$ is integrable, meaning that around each point $p \in U_{\rho(P)}(P)$ there are m smooth complex functions, whose Hamiltonian vector fields trivialize P in a neighbourhood of p.*

We say that a polarization is smooth if $U_{\rho(P)}(P) = M$.

Let $P(M)$ be the space of singular non-negative polarizations on M. We will consider the pointwise convergence topology on $P(M)$, i.e. $P_n \to P$ if $P_n(p) \to P(p)$ $\forall p \in M$. Let us denote the space of compatible Kähler structures on (M, ω) by $P_0^s(M)$. Notice that there is a one-to-one correspondence between compatible Kähler structures $J \in P_0^s(M)$ and polarizations on (M, ω), for which $U_0 = M$. (The correspondence being: take the i-eigenspace of $J_p\colon T_pM \otimes \mathbb{C} \to T_pM \otimes \mathbb{C}$, and in fact that space determines J_p uniquely.) We will consider $P_0^s(M)$ as a subspace of $P(M)$.

Let us now return to the situation where $M = \mathcal{M}(\Sigma)^{\mathrm{irr}}$. We indicated in section 1.1 how a conformal or complex structure on Σ induces a Kähler structure on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. We shall now investigate this in detail. Consider the associated adjoint bundle $\operatorname{ad} P$. A connection A induces a covariant derivative d_A in $\operatorname{ad} P$. Now suppose A is a flat connection. We then get a complex:

$$0 \to \Omega^0(\Sigma, \operatorname{ad} P) \stackrel{d_A}{\to} \Omega^1(\Sigma, \operatorname{ad} P) \stackrel{d_A}{\to} \Omega^2(\Sigma, \operatorname{ad} P) \to 0. \tag{1.2}$$

The linearization of the equation $F_A = 0$ is

$$d_A \varphi = 0$$

for $\varphi \in T_A\mathcal{A} = \Omega^1(\Sigma, \operatorname{ad} P)$, so we see

$$T_A\mathcal{A}^F = \ker d_A.$$

By considering the action of the Lie-algebra of $\mathcal{G}$

$$\operatorname{Lie}(\mathcal{G}) = \Omega^0(\Sigma, \operatorname{ad} P),$$

one easily sees that the tangent space to the orbit $\mathcal{G}_A$ through A is

$$T_A\mathcal{G}_A = \operatorname{Im} d_A.$$

When A is a flat irreducible connection the isotropy group of A is trivial, and we get an identification of the first cohomology group of the complex (1.2) and the tangent space to $\mathcal{M}(\Sigma)^{\mathrm{irr}}$ at $[A]$:

$$H^1(\Sigma, d_A) \cong T_{[A]}\mathcal{M}(\Sigma)^{\mathrm{irr}}.$$

Moreover we have, since A is irreducible, that

$$H^0(\Sigma, d_A) = H^2(\Sigma, d_A) = 0,$$

by Poincaré duality (see section 2.3). By applying the index theorem we see that

$$\dim \mathcal{M}(\Sigma)^{\mathrm{irr}} = 2(g-1)\dim G.$$

Let σ be a complex structure on Σ. We can then use Hodge theory to get harmonic representatives for cohomology classes in $H^1(\Sigma, d_A)$. Moreover we get a decomposition

$$H^1(\Sigma, d_A) \otimes \mathbb{C} = H^{1,0}(\Sigma_\sigma, d_A) \oplus H^{0,1}(\Sigma_\sigma, d_A).$$

The gauge group preserves this decomposition and so $H^{1,0}(\Sigma_\sigma, d_A)$ induces a well defined positive Lagrangian subspace of $T_{[A]}\mathcal{M}(\Sigma)^{\mathrm{irr}} \otimes \mathbb{C}$. From the construction of the smooth structure of $\mathcal{M}(\Sigma)^{\mathrm{irr}}$, using Fredholm theory and the implicit function theorem for Banach spaces, one shows that this is a smooth integrable distribution and therefore defines a Kähler structure on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$.

The group $\mathrm{Diff}_+(\Sigma)$ acts on the space $\mathcal{C}_\Sigma$ of complex structures on Σ. Let $f \in \mathrm{Diff}_+(\Sigma)$. If σ is a complex structure on Σ, then σ and $f^*\sigma$ will induce two Kähler structures on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. Now if $f \in \mathrm{Diff}_0(\Sigma)$, then the induced map on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$ is the identity, so we see that σ and $f^*\sigma$ induce the same Kähler structure on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. Since the quotient space of the space of complex structures modulo $\mathrm{Diff}_0(\Sigma)$ is Teichmüller space T (see [17]), we get a continuous equivariant map

$$F: T \to P_0(\mathcal{M}(\Sigma)^{\mathrm{irr}}). \tag{1.3}$$

In the rest of this section, we specialize to the case $G = SU(n)$. The Kähler structure $F(\sigma)$ on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$ is the same as the one induced from the stable bundle description of $\mathcal{M}(\Sigma)^{\mathrm{irr}}$, which we saw in section 1.1. Using the semi-stable bundle description of $\mathcal{M}(\Sigma)$, one in fact sees that $\mathcal{M}(\Sigma)$ is a complex projective variety (see [36]). From now on we will use the notation $\mathcal{M}(\Sigma)_\sigma$ for $\mathcal{M}(\Sigma)$ with the induced Kähler structure $F(\sigma)$. In the light of the discussion in section 1.1, we are interested in whether there is a natural way to identify the spaces $H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^k)$ of holomorphic sections of $\mathcal{L}^k$ with respect to the Kähler structure $F(\sigma)$, when σ varies in Teichmüller space. Recall that the mapping class group Γ acts on $\mathcal{L}$ and hence induces an action on the collection of the vector spaces $H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^k)$, $\sigma \in T$, covering the action of Γ on T. We would like the identification of these spaces to be compatible with the mapping class group action.

This has been worked out in detail in [29], [8] and [44] and we will just state the theorem here.

Theorem 1.3.1 *(Axelrod, Della Pietra and Witten; Hitchin; Segal) The spaces $H^0(M_\sigma, \mathcal{L}^k)$, $\sigma \in T$, form a holomorphic vector bundle H over Teichmüller space T. There is a natural projectively flat mapping class group invariant connection compatible with the holomorphic structure in the bundle H.*

This gives us a canonical identification of the projectivized spaces $P(H^0(\mathcal{M}_\sigma, \mathcal{L}^k))$, since Teichmüller space is contractible.

Let us now discuss the curvature of the connection in H. Over the space of complex structures $\mathcal{C}_\Sigma$ on Σ we have the Quillen determinant line bundle, which to each complex structure σ associates the canonical determinant line of the Fredholm operator

$$\overline{\partial}_\sigma: \Omega^0(\Sigma) \to \Omega^{0,1}_\sigma(\Sigma).$$

Now $\mathcal{C}_\Sigma$ is a principal $\mathrm{Diff}_0(\Sigma)$-bundle over Teichmüller space T (see [17]) and so we get an induced Quillen determinant line bundle $\mathcal{L}_\Sigma$ over Teichmüller space. There is a canonical mapping class group invariant connection in $\mathcal{L}_\Sigma$. Moreover Segal explains in [44] how the curvature of H and $\mathcal{L}_\Sigma$ are related:

$$F_{\nabla_H} = -\frac{1}{2} c F_{\nabla_{\mathcal{L}_\Sigma}},$$

where $c \in \mathbb{R}$ is called the *central charge* of H. The central charge depends on the group G and level k in the following way

$$c = \frac{|G| k}{k+h}.$$

Here $|G| = \dim G$ and h is the dual Coxeter number, i.e. the quadratic Casimir of the adjoint representation of G, e.g. for $SU(2)$ we have that $h = 2$ and $|G| = 3$.

Hence, if we can construct $\mathcal{L}_\Sigma^{\frac{1}{2}c}$, we can then modify the bundle H by tensoring on this fractional power, to get a flat bundle over T. Before we embark on such a construction, we note that it will not be possible to do this in a mapping class group invariant way, since such a line bundle would descend to a line bundle over the moduli space of Riemann surfaces, $\mathcal{M}_\Sigma$, with a fractional first Chern class, i.e. not in the integer cohomology of $\mathcal{M}_\Sigma$, which is a contradiction. Over Teichmüller space, however, $\mathcal{L}_\Sigma$ is trivial and so any fractional power of $\mathcal{L}_\Sigma$ exists. Let $\mathcal{L}_\Sigma^* = \mathcal{L}_\Sigma - 0$. Since T is contractible, $\mathcal{L}_\Sigma^*$ is homotopy equivalent to S^1. Choose a universal covering $\tilde{\mathcal{L}}_\Sigma^*$ of $\mathcal{L}_\Sigma^*$. $\mathbb{C}$ acts on $\tilde{\mathcal{L}}_\Sigma^*$ covering (by the usual covering map $\exp(2\pi i \cdot) \colon \mathbb{C} \to \mathbb{C}^*$) the $\mathbb{C}^*$ action of $\mathcal{L}_\Sigma^*$. We then define

$$\mathcal{L}_\Sigma^{\frac{1}{2}c} = \tilde{\mathcal{L}}_\Sigma^* \times_{\mathbb{C}} \mathbb{C},$$

where $\mathbb{C}$ now acts on $\mathbb{C}$ by

$$z(\lambda) = \exp(2\pi i (\frac{1}{2}c) z)\lambda, \;\; z \in \mathbb{C}, \;\; \lambda \in \mathbb{C}.$$

Different choices of the universal covering $\tilde{\mathcal{L}}_\Sigma^*$ are of cause isomorphic via some covering transformation, but not canonically so. Hence the choice of $\tilde{\mathcal{L}}_\Sigma^*$ defines for us a central extension $\tilde{\Gamma}$ of the mapping class group Γ:

$$0 \to \mathbb{Z} \to \tilde{\Gamma} \to \Gamma \to 0.$$

Segal explains in [44] that this is the universal central extension of Γ. By construction, $\tilde{\Gamma}$ acts on $\mathcal{L}_\Sigma^{\frac{1}{2}c}$ and preserves the induced connection. The bundle

$$\widehat{H} = H \otimes \mathcal{L}_\Sigma^{\frac{1}{2}c}$$

has a natural flat connection, so the fibres of this bundle are canonically identified by the connection. We can now define $Z(\Sigma)$:

Definition 1.3.2 *The vector space $Z(\Sigma)$ associated to a closed surface is the space of covariant constant sections of the flat bundle $\widehat{H}$.*

From the discussion above we see that the universal central extension $\tilde{\Gamma}$ of the mapping class group acts naturally on $Z(\Sigma)$. This is what we would have required from the functor, if the framing dependence had been considered in section 1.2. Let us now see how the framings of 3-manifolds and this central extension $\tilde{\Gamma}$ relate.

Atiyah explains in his paper [2] how one constructs a central extension $\widehat{\Gamma}$ of the mapping class group Γ by using 2-framings of the closed 3-manifolds Σ_f, described in section 1.2. Atiyah also describes how there is a canonical choice of a 2-framing for any oriented closed 3-manifold, hence he constructs a natural section

$$s\colon \Gamma \to \widehat{\Gamma},$$

of this central extension. So if we want to "compute" the invariant $Z(\Sigma_f)$, for a $f \in \Gamma$, it should be with respect to the 2-framed manifold $s(f)$. Hence, in order to write down a formula for $Z(\Sigma_f)$, using the action of $\tilde{\Gamma}$ on $Z(\Sigma)$, we need to relate $\tilde{\Gamma}$ and $\widehat{\Gamma}$. From proposition 2 in [2] we see that there is an inclusion $\tilde{\Gamma} \to \widehat{\Gamma}$, and the corresponding map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by 12. Let $f \in \Gamma$ and consider $s(f) \in \widehat{\Gamma}$. If $s(f) \notin \tilde{\Gamma}$, we can try to alter the framing so we end up with an element in $\tilde{\Gamma}$, but we must correct the formula for the framing change. $12 \cdot s(f)$ will definitely be in $\tilde{\Gamma}$. Let us denote the difference in framing between $12 \cdot s(f)$ and $s(f)$ by $n(f) \in \mathbb{Z}$. When we change the framing of a closed 3-manifold Y by 1 unit, the Witten invariant of Y multiplies by a fixed factor, namely $\exp(2\pi i \frac{c}{24})$, where c is the central charge (see [51]). We are now ready to write down what for us will be the definition of the Witten invariant of the manifolds we are particularly interested in here.

Definition 1.3.3 *Let $f \in \Gamma$. We then define the* Witten Invariant *for the closed 3 manifold Σ_f to be*

$$Z(\Sigma_f) = \exp(-2\pi i \frac{c \cdot n(f)}{24}) \cdot \mathrm{Tr}(12 \cdot s(f))$$

Here Tr *is the character of the action of $\tilde{\Gamma}$ on $Z(\Sigma)$, the covariant constant sections of the flat bundle $\widehat{H}$.*

Note that $\mathrm{Tr}(\tilde{f} + 1) = \exp(2\pi i \frac{c}{2}) \mathrm{Tr}(\tilde{f})$, $\tilde{f} \in \tilde{\Gamma}$, which is the multiplicative factor corresponding to the 12 unit framing change.

In the next section we shall discuss ideas of how to make this formula more explicit.

1.4 Can we use the Thurston boundary in connection with Jones-Witten theory?

Let us start this section by giving a simple description of the fibres of the Quillen determinant line bundle over Teichmüller space.

The construction of Kähler polarizations on the moduli space of flat connections from points in Teichmüller space quite clearly also applies in the abelian case, where $G = U(1)$. In fact in the abelian case the Kähler structure on the Jacobian is linear, and we just get an equivariant smooth map

$$F: T \to L_0(H^1(\Sigma, \mathbb{C})),$$

where

$$F(\sigma) = H^{1,0}(\Sigma_\sigma).$$

Over $L_0(H^1(\Sigma, \mathbb{C}))$, we have a canonical holomorphic hermitian line bundle, Det, whose fibre over $H \in L_0(H^1(\Sigma, \mathbb{C}))$ is the line

$$\mathrm{Det}_H = (\det H)^*.$$

The fibre of the Quillen determinant line bundle over T is the fibre pullback of Det by F,

$$\mathcal{L}_\Sigma \cong F^*(\mathrm{Det}).$$

Note that this is only an isomorphism of C^∞ line bundles. Let us now investigate the formula for the Witten invariant in definition 1.3.3, when we have a diffeomorphism $f \in \Gamma$ which has a fixed point $\sigma \in T$. Now the action of $12 \cdot s(f)$ on $Z(\Sigma)$ is of course completely determined by its action on $\widehat{H}_\sigma$. Let us define $\alpha(f) \in \mathbb{Z}$ by requiring that

$$\begin{aligned} &\exp(-2\pi i \frac{c \cdot n(f)}{24}) \cdot \mathrm{Tr}(12 \cdot s(f)_\sigma : \mathcal{L}_\Sigma^{\frac{1}{2}c}(\sigma) \to \mathcal{L}_\Sigma^{\frac{1}{2}c}(\sigma)) \\ = \quad &\exp(2\pi i \frac{c \cdot \alpha(f)}{24}) \cdot \det(f)^{\frac{1}{2}c}, \end{aligned}$$

where $\det(f)$ is the complex number

$$\det(f) = \mathrm{Tr}(\Lambda^g(f) \colon \Lambda^g(H^{0,1}(\Sigma_\sigma)) \to \Lambda^g(H^{0,1}(\Sigma_\sigma)).$$

So we get the expression

$$Z(\Sigma_f) = \exp(2\pi i \frac{c \cdot \alpha(f)}{24}) \cdot \det(f)^{\frac{1}{2}c} \cdot \mathrm{Tr}(f^*).$$

Here $\mathrm{Tr}(f^*)$ denotes the trace of $f^* \colon H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^k) \to H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^k)$. Note that since $f^*\sigma = \sigma$, the induced map $f \colon \mathcal{M}(\Sigma)_\sigma \to \mathcal{M}(\Sigma)_\sigma$ is holomorphic, and so the lifted action f^* of f to $\mathcal{L}$ is a geometric endomorphism of the $\overline{\partial}$-complex in $\mathcal{L}$.

If $\mathcal{M}(\Sigma)_\sigma$ was a smooth manifold, one would proceed as follows: from the Kodaira-Nakano vanishing theorem it would follow that

$$H^i(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^k) = 0, \; i > 0.$$

Thus, if we made the extra assumption that f had simple fixed points on $\mathcal{M}(\Sigma)_\sigma$, we could apply the holomorphic Lefschetz fixed point theorem to compute $\mathrm{Tr}(f^*)$. This would enable us to give a purely geometric expression of $Z(\Sigma_f)$ for such f, which would involve a sum over the gauge equivalence classes of flat connections on Σ_f. It seems an interesting project for further research to understand to what extend this discussion in fact could be made rigorous and precise for the singular moduli spaces $\mathcal{M}(\Sigma)_\sigma$. Here we shall just remark that f's action on the covairant constant sections of $\widehat{H}$ is of cause determined by f's action on $\widehat{H}_\sigma$. However, there are many diffeomorphisms $f \in \Gamma$, which do not have any fixed points on T (see section 2.1.1).

This leads us quite naturally to the following idea, which is due to Connes: consider the Thurston compactification $\overline{T}$ of Teichmüller space T. $\overline{T}$ has two very nice properties (see section 2.1.1). $\overline{T}$ is homeomorphic to a closed ball and the action of the mapping class group Γ extends continuously to $\overline{T}$. This means by Brouwer's fixed point theorem that any $f \in \Gamma$ has a fixed point on $\overline{T}$. Hence, if there is a natural extension of the bundle $\widehat{H}$ to $\overline{T}$, we would be able to repeat the previous discussion, i.e. $Z(\Sigma_f)$ is determind by f's action on the fibre of $\widehat{H}$ over a fixed point of f on $\overline{T}$.

The way we shall try to approach this idea here is within the framework of geometric quantization. The reason for this being that $\widehat{H}$ is essentially constructed by quantizing a fixed symplectic manifold, namely the moduli space of flat connection on Σ with a fixed prequantum line bundle, with respect to varying polarizations, parametrized by Teichmüller space. Now points on the Thurston boundary are geometrically realized as equivalence classes of certain structures (see section 2.1.1) on the surface Σ. It therefore seems natural to keep the moduli space and the prequantum line bundle fixed, and investigate whether there is a natural extended family of polarizations parametrized by the Thurston compactification of Teichmüller space. To phrase this precisely (recall (1.3)): Does there exist a continuous extension of the map

$$F: T \to P(\mathcal{M}(\Sigma)^{\mathrm{irr}}),$$

to the Thurston compactification?

We shall study this problem in detail in chapter 2.

Chapter 2

The Thurston compactification and polarizations on moduli spaces.

2.1 The Thurston boundary of Teichmüller space.

2.1.1 Thurston's construction of the boundary of Teichmüller space.

We shall in this section briefly review how the Thurston compactification of Teichmüller space is constructed using measured foliations. We shall follow the lines and notation of [19]. The definition of a measured foliation is adapted from [31]. For proof of the theorems stated in this section we refer to [19].

Let Σ be a closed oriented surface of genus $g \geq 2$. By a measured foliation on the surface Σ, we shall mean a 1-dimensional foliation on Σ with certain allowed singularities and a holonomy invariant transverse measure. More precisely, we define, following Hubbard and Masur [31] :

Definition 2.1.1 *A measured foliation F on Σ with singularities of order $k_1, \ldots, k_n$ at $x_1, \ldots, x_n$ is given by an open cover $\{U_i \mid i \in I\}$ of $\Sigma' = \Sigma - \{x_1, \ldots, x_n\}$, and everywhere nonzero closed 1-forms $\varphi_i \in C^\infty(T^*U_i)$ for all $i \in I$, such that*

a) $\varphi_i = \pm\varphi_j$ *on* $U_i \cap U_j$.

b) *Around x_j there is an open coordinate neighbourhood $z\colon V_j \to \mathbb{C}$, such that $\varphi_i = \operatorname{Im}(\sqrt{z^{k_j}}dz)$ for some branch of the square root on $V_j \cap U_i$.*

The singularities will also be called critical points.

Let F be a measured foliation. F then defines a 1-dimensional subbundle D_F of $T\Sigma'$, given by

$$D_F = \{v \in T\Sigma' | \varphi_i(v) = 0\}.$$

Now D_F defines a smooth foliation on Σ', which on Σ have a finite number of singularities, modeled on the standard examples

$$D_k = \{v \in T\mathbb{C} | \operatorname{Im}(\sqrt{z^k}dz)(v) = 0\}$$

in the complex plane.

The densities $|\varphi_i|$ define a transverse measure, that is, for any 1-dimensional immersed submanifold L of Σ, which is transversal to F (meaning TL transverse to D_F), we get induced a 1-density on L. (See e.g. [12] for the exact definition of a transverse measure.) The fact that the φ_i's are closed, implies that this transverse measure is holonomy invariant. Hence the notation "F is a measured foliation".

Two measured foliations $F = \{U_i, \varphi_i\}$ and $F' = \{V_i, \psi_i\}$ with the same singularities, are considered to be equal if

$$\varphi_i = \pm\psi_j \text{ on } U_i \cap V_j.$$

A holomorphic quadratic differential $\Phi \in H^0(\Sigma, K^2) - \{0\}$ on a Riemann surface Σ induces a measured foliation

$$F = \operatorname{Re}\sqrt{\Phi}$$

on Σ, which is called the vertical measured foliation associated to Φ . $\operatorname{Im}\sqrt{\Phi}$ is called the horizontal measured foliation associated to Φ. As we shall see later, up to a certain equivalence, any measured foliation arises in this way. See Theorem 2.1.6.

Let F be a measured foliation. Consider F', the restriction of F to Σ'. F' is by definition non-singular and its underlying foliation D_F is a smooth distribution. D_F is clearly integrable. The *leaves* of F are the maximal integrable submanifolds of F'. If a leaf converges to a critical point, we will consider the critical point to be contained in the leaf. A leaf of F which contains a critical point, is called a *critical leaf*. The union of the compact critical leaves is called the *critical graph*. We shall use the terminology *critical segment* for a critical leaf which connects two different critical points. A measured foliation whose non critical leaves are closed, is called a *Strebel measured foliation.*

We shall now define an equivalence relation on measured foliations. We consider the set S of nontrivial homotopy classes of simple closed curves. Let F be a measured foliation on Σ given by $\{U_i, \varphi_i\}$. For a simple closed C^1 curve γ we define

$$l_F(\gamma) = \int_0^1 | \varphi_i(\gamma'(t)) | \, dt.$$

For $c \in S$ we let

$$F(c) = \inf_{\gamma \in c} l_F(\gamma).$$

Let $\mathcal{F}_\Sigma$ be the set of measured foliations on Σ. We then have a map $I: \mathcal{F}_\Sigma \to \mathbb{R}^S$ given by

$$I(F)(c) = F(c) \; \forall c \in S.$$

We call two measured foliations *equivalent*, if they have the same image in $\mathbb{R}^S$. Thurston has given a more geometric description of this equivalence, which uses the following two equivalences :

Definition 2.1.2 *Two measured foliations $F = \{U_i, \varphi_i\}$ and $F' = \{V_j, \psi_j\}$ are* homotopic, *if there exists a diffeomorphism f of Σ homotopic to the identity, such that*

$$f_*F = \{f(U_i), (f^{-1})^*\varphi_i\}$$

is equal to F'.

Definition 2.1.3 *Two measured foliations $F = \{U_i, \varphi_i\}$ and $F' = \{V_j, \psi_j\}$ are Whitehead equivalent, if there exists a map $f\colon \Sigma \to \Sigma$ homotopic to the identity, satisfying the following condition :*

1. *There exists a critical segment γ of F, such that $f|_{\Sigma-\gamma}$ is a diffeomorphism of $\Sigma - \gamma$ onto $\Sigma - \{x\}$, x being a critical point of F'.*
2. *$f_*F = F'$.*

With these two definitions at hand we can formulate one of Thurston's key results in describing the boundary of Teichmüller space as equivalence classes of measured foliations.

Theorem 2.1.1 (Thurston) *Two measured foliations F and F' are equivalent, i.e. $I(F) = I(F')$, if, and only if, F and F' are equivalent under a combination of homotopy equivalences and Whitehead equivalences.*

We will denote the space of equivalence classes of measured foliations $\mathcal{MF}_\Sigma$. It is shown in [19], following Thurston, that $\mathcal{MF}_\Sigma$ is homeomorphic to $S^{6g-7} \times \mathbb{R}_+$. Notice the map $I_* : \mathcal{MF}_\Sigma \to \mathbb{R}^S$, induced by the map $I\colon \mathcal{F}_\Sigma \to \mathbb{R}^S$, is an equivalence with respect to the action of the mapping class group $\Gamma = \pi_0(\mathrm{Diff}_+(\Sigma))$. Let $\mathcal{PF}_\Sigma$ denote the projectification of $\mathcal{MF}_\Sigma$, i.e. we identify rays in $\mathcal{MF}_\Sigma$ to points in $\mathcal{PF}_\Sigma$. Let us here review some of the constructions in [19], which we will use in this chapter.

There is a natural map (see proposition 16 p. 52 in [19])

$$i_* : \mathbb{R}_+ \times S \to \mathbb{R}_+^S - \{0\}$$

by mapping $(\lambda, c) \in \mathbb{R}_+ \times S$ to the function

$$i_*(\lambda, c)(d) = \lambda i(c, d),\ \forall d \in S.$$

Here $i(c, d)$ is the geometric intersection number of c and d, i.e. the minimal number of intersection points of a representative of c with a representative of d. If we compose i_* with the projection $\pi : \mathbb{R}_+^S - \{0\} \to P(\mathbb{R}_+^S)$, we get in fact an inclusion (see proposition 16 p. 52 in [19])

$$\pi i_* : S \to P(\mathbb{R}_+^S).$$

Let S' be the space of isotopy classes of 1-dimensional submanifolds, such that none of the components are null homotopic. Hence, the elements of S' are just finite unions of elements of S, and clearly $S \subset S'$. Likewise, one gets a map

$$\pi i_* : S' \to P(\mathbb{R}_+^S).$$

Theorem 4 p. 59 in [19] says that

$$\overline{\pi i_*(S')} = \overline{\pi i_*(S)},$$

and proposition 1 on p. 58 in [19] tells us that this set is compact.

A construction of an inclusion

$$\mathbb{R}_+ \times S \hookrightarrow \mathcal{MF}_\Sigma,$$

is described in exposé 5, paragraph III in [19]. We refer to that paper for the details of the construction. The idea in the construction is as follows:

Let $(\lambda, c) \in \mathbb{R}_+ \times S$. Let γ be a representative of c. Choose a cylindrical neighbourhood C of γ and a foliation F_C with closed leaves on C, such that γ and the two components of ∂C are leaves of F_C. One easily constructs a holonomy invariant measure μ_C for F_C, which gives the cylinder the length λ. Let B be a 1-skeleton of $\Sigma - C$. Now let j be a piecewise smooth map from C to Σ, which is the identity in a small neighbourhood of γ, maps ∂C onto B and is a diffeomorphism of $C - \partial C$ onto $\Sigma - B$. We then define

$$F_{(\lambda,c)} = j_* F_C.$$

It turns out, that $[F_{(\lambda,c)}] \in \mathcal{MF}$ only depends on $(\lambda, c) \in \mathbb{R}_+ \times S$, and that this construction gives us an inclusion. This inclusion is constructed such that

$$i_*(\lambda, c) = I_*(F_{(\lambda,c)})$$

in $P(\mathbb{R}_+^S)$. A result we shall use several times through out this chapter is proposition VI.2 on p. 117 in [19], which says that

$$\overline{\pi i_*(S')} = \overline{\pi i_*(S)} = \pi I_*(\mathcal{MF}_\Sigma) \cong \mathcal{PF}_\Sigma.$$

We shall now give a description of Teichmüller space T. Let $\mathcal{H}$ be the space of metrics on Σ with constant negative curvature -1. The group $\mathrm{Diff}_0(\Sigma)$ of diffeomorphisms homotopic to the identity, considered with the C^∞-topology, acts freely on $\mathcal{H}$ and we define Teichmüller space T of Σ to be the orbit space of that action. It is well known that T is homeomorphic to $\mathbb{R}^{6g-6}$. Given a metric $\rho \in \mathcal{H}$ we define the length of a curve γ by

$$l_\rho(\gamma) = \int_0^1 \mid \gamma'(t) \mid_\rho dt.$$

For $c \in S$ we let

$$\rho(c) = \inf_{\gamma \in c} l_\rho(\gamma).$$

Hence we have a map $I: \mathcal{H} \to \mathbb{R}^S$ given by

$$I(\rho)(c) = \rho(c) \ \forall c \in S.$$

Since I is invariant under the action of $\mathrm{Diff}_0(\Sigma)$, the map I will induce a map $I_*: T \to \mathbb{R}^S$. The following theorem by Thurston realizes Teichmüller space as a subset of $\mathbb{R}^S$:

Theorem 2.1.2 (Thurston) *The map $I_* : T \to \mathbb{R}^S$ is a homeomorphism on its image.*

Clearly $I_*: \mathrm{T} \to \mathbb{R}^S$ is an equivalence with respect to the action of Γ.

We are now in a position to describe the Thurston boundary of Teichmüller space. We note that $I_*(\mathrm{T}) \cap (I_*(\mathcal{MF}_\Sigma) \cup 0) = \emptyset$.

Definition 2.1.4 (The Thurston compactification) *The Thurston compactification of Teichmüller space is defined to be*

$$\overline{T} = T \cup \mathcal{PF}_\Sigma.$$

We give $\overline{T}$ a topology the following way : Consider $I_*(T) \cup I_*(\mathcal{MF}_\Sigma)$ with the induced topology from $\mathbb{R}^S$. Identify each ray in $I_*(\mathcal{MF}_\Sigma)$ to a point and take the quotient topology.

Note that the action of Γ by construction extends to $\overline{\mathrm{T}}$. It is also shown in [19] that $\overline{\mathrm{T}}$ is homeomorphic to the closed ball B^{6g-6}. Using this compactification, Thurston was able to make a classification of the diffeomorphisms of closed surfaces:

Theorem 2.1.3 (Thurston) *Any diffeomorphism of Σ is homotopic to a diffeomorphism f, which possesses exactly one of the three properties :*

a) *f fixes a point in T, and is of finite order. In this case, we say that f is* periodic.

b) *f is reducible, in the sense that it preserves a non-empty system of disjoint simple closed non-trivial curves on Σ.*

c) *f is a pseudo-Anosov diffeomorphism : There exists $\lambda > 1$ and two transverse measured foliations F^s and F^u such that*

$$f_*(F^s) = \lambda^{-1} F^s,\ f_*(F^u) = \lambda F^u.$$

Here one should note that a "pseudo-Anosov diffeomorphism" is a diffeomorphism away from $\mathrm{sing}(F^s)$, but only C^1 at the finite subset $\mathrm{sing}(F^s)$. (See pp. 174-75 in [19].) The theory for surfaces with boundaries has also been worked out by Thurston. In order to formulate Thurston's results on diffeomorphism of compact surfaces, we recall the definition of a periodic and a pseudo-Anosov diffeomorphism for compact surfaces from [19].

Definition 2.1.5 *A periodic diffeomorphism of a compact surface is a diffeomorphism which preserves a hyperbolic metric on the surface.*

Definition 2.1.6 *A pseudo-Anosov diffeomorphism f of a compact surface Σ is a diffeomorphism for which there exist two measured foliations F and F' and a $\lambda > 1$ with the following properties:*

1) $f_*(F) = \frac{1}{\lambda} F$,

2) $f_*(F') = \lambda F'$,

3) *F and F' are transverse at any interior point of Σ*

4) *The components of $\partial\Sigma$ are unions of leaves of F and of F' and they contain singularities from both foliations. Moreover, f is the identity on the boundary.*

With these definitions at hand, we can now formulate Thurstons main theorem on diffeomorphisms of compact surfaces.

Theorem 2.1.4 (Thurston) *Let f be a diffeomorphism of a compact surface Σ. Then there exist compact surfaces $\Sigma_1, \ldots, \Sigma_k$ with the following properties:*

1) $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_k$.

2) *For $i \neq j$, we have that $\Sigma_i \cap \Sigma_j = \partial\Sigma_i \cap \partial\Sigma_j$ is a union of simple closed curves $C_1, \ldots C_m$, such that for $k \neq l$, C_l is not homotopic to C_k on Σ.*

3) *f is homotopic to a diffeomorphism f' such that $f'(\Sigma_i) = \Sigma_i$ for all i.*

4) *$f'|_{\Sigma_i}$ is homotopic in $\mathrm{Diff}(\Sigma_i)$ to a periodic diffeomorphism or a pseudo–Anosov diffeomorphism.*

We shall say that a diffeomorphism is *reducible periodic*, if the $f'|_{\Sigma_i}$'s are homotopic to periodic diffeomorphisms, and that a diffeomorphism is *reducible pseudo-Anosov*, if the $f'|_{\Sigma_i}$'s are homotopic to pseudo-Anosov diffeomorphisms.

We shall make use of Thuston's work on diffeomorphisms in sections 2.3 and 2.5

2.1.2 The Thurston boundary and holomorphic quadratic differentials.

The questions we shall address in section 2.6 will be concerned with the degeneration of the Kähler structure on moduli space, when we degenerate the conformal structure on the surface to a boundary point in the Thurston compactification. That means we have to translate the geometric data given in terms of length of geodesics into information about the holomorphic differentials on the underlying Riemann Surface. As a tool in this translation process, we shall appeal to an alternative and more analytic description of the Thurston boundary by Wolf [52]. For the convenience of the reader, we shall give a short review of the results we will need in section 2.6.1.

Let Σ be a closed oriented compact surface of genus $g \geq 2$. Suppose $\sigma, \rho \in \mathcal{H}$ are metrics of constant curvature -1 on Σ. Consider the quadratic differential $\Phi(\rho) = \rho^{2,0}$, the (2,0)-part of ρ with respect to the complex structure $[\sigma]$, induced by σ. If we assume the identity map $\mathrm{id} : (\Sigma, \sigma) \to (\Sigma, \rho)$ is harmonic, it follows (see [43]), that $\Phi(\rho)$ is holomorphic with respect to $[\sigma]$, i.e. $\Phi(\rho) \in H^0(\Sigma_\sigma, K^2)$. From work of Eells and Sampson [18] it follows that there exists a metric ρ over each point in Teichmüller space such that the identity map is harmonic. A result by Hartman [28] implies that ρ is unique. Hence there is an induced map $\Phi : \mathrm{T} \to H^0(\Sigma_\sigma, K^2)$. Using Higgs bundle techniques, Hitchin [30] showed that Φ is a homeomorphism. Wolf [52] also proves that Φ is a homeomorphism, using completely different techniques.

On $H^0(\Sigma_\sigma, K^2)$ we have the norm $\|\cdot\|$ (see [47]) given by:

$$\|\Phi\| = \int_\Sigma |\ \Phi\ |\ dz d\overline{z}.$$

Following Wolf's notation we define

$$\begin{aligned}
BQD(\sigma) &= \{\Phi \in QD(\sigma) \mid \|\Phi\| < 1\} \\
SQD(\sigma) &= \{\Phi \in QD(\sigma) \mid \|\Phi\| = 1\} \\
\overline{BQD(\sigma)} &= BQD(\sigma) \cup SQD(\sigma).
\end{aligned}$$

Consider the map

$$\tilde{\Phi} : T \to BQD(\sigma),$$

given by

$$\tilde{\Phi}(\rho) = \frac{4\Phi(\rho)}{1 + 4\|\Phi(\rho)\|}.$$

Now $\tilde{\Phi}$ is a homeomorphism of T onto $BQD(\sigma)$. Now define

$$\overline{T^h(\sigma)} = \overline{BQD(\sigma)}.$$

Wolf's main theorem [52] is

Theorem 2.1.5 (Wolf) *The map $\tilde{\Phi}$ extends to a homeomorphism*

$$\tilde{\Phi} : \overline{T^h(\sigma)} \to \overline{T}.$$

From this theorem it follows that $\overline{T^h(\sigma)} = \overline{T}$ is independent of the choice of base point σ, and the action of the mapping class group extends to $\overline{T^h(\sigma)}$.

Wolf's proof uses the fundamental work of Hubbard and Masur [31] on the relation between holomorphic quadratic differentials and measured foliations :

Theorem 2.1.6 (Hubbard & Masur) *Let F be a measured foliation on Σ and σ a complex structure on Σ. Then there exists a unique quadratic differential holomorphic with respect to σ, such that the vertical measured foliation is equivalent to F.*

We shall not use this theorem here. However, we shall use one of the key analytic facts in the proof of theorem 2.1.5, which will be of crucial importance to us :

Proposition 2.1.1 (Wolf) *Let $\Phi_0 \in SQD(\sigma)$. Let $\{\rho_t\} = \Phi^{-1}\{t\Phi_0 \mid t > 0\}$ be the family of hyperbolic metrics, determined by $t\Phi_0$. Let $\nu(t) \in$ (Beltσ) be the family of Beltrami differentials given in local conformal coordinates $\rho_t = \rho_t dw(t)\overline{dw(t)}$, $\sigma = \sigma dz d\overline{z}$ by*

$$\nu(t) = \frac{w(t)_{\overline{z}}}{w(t)_z}.$$

Then, if $\Phi_0(p) \neq 0$, $|\nu(t)(p)|^2 \uparrow 1$.

We end this section by the following observation from Wolf's proof of the theorem 2.1.5: For any $\Phi_0 \in SQD(\sigma)$, $\tilde{\Phi}^{-1}(\Phi_0) \in \partial T$ is represented by the vertical measured foliation $\mathrm{Im}\sqrt{\Phi_0}$ associated to Φ_0.

2.2 Construction of polarizations on the space of connections.

We shall see in this section that 1-dimensional singular distributions of a very general kind on a surface induce polarizations on the space of connections in some principal bundle over the surface. In particular a complex structure or a measured foliation on the surface are examples of such structures. These polarizations will be gauge invariant. As mentioned in Chapter 1, the moduli space of flat connections is the Marsden-Weinstein quotient of the space of connections. Whenever one has a polarization on some finite dimensional symplectic manifold, which is invariant under some hamiltonian action, then that automatically induces a polarization on the Marsden-Weinstein quotient. We shall here repeat the argument for the reduction construction in the infinite dimensional case at hand, and see where it fails in general.

In fact in the next section, we shall see that measured foliations on a closed surface of genus $g > 1$ induce polarizations on the space of connections $\mathcal{A}$, which do not reduce to polarizations on the moduli space under the Marsden-Weinstein quotient construction.

Let Σ be a compact closed surface of genus g. Over Σ, we will consider a principal G-bundle, where G is a compact Lie-group. Assume moreover, we have chosen a bi-invariant inner product $B(\cdot,\cdot)$ on the Lie-algebra $\mathfrak{g}$ of G. Let $\mathcal{A}$ be the space of connections on P. $\mathcal{A}$ is an affine space modelled on $\Omega^1(\Sigma, \operatorname{ad} P)$, so there is a natural identification

$$T_A\mathcal{A} = \Omega^1(\Sigma, \operatorname{ad} P).$$

The space of bundle automorphisms $\mathcal{G}$ of P covering the identity on Σ, also called the gauge group, acts on $\mathcal{A}$. To make the following discussion precise, we shall, following section 14 of [5] consider the Sobolev completions of these spaces.

Let $k \geq 2$. Suppose we have chosen a metric on Σ. Consider the L^2_{k-1}-completion $\mathcal{A}_{k-1}$ of the space of connections $\mathcal{A}$. The L^2_k-completion $\mathcal{G}_k$ of $\mathcal{G}$ will be a Banach group acting smoothly on the Banach manifold $\mathcal{A}_{k-1}$. We will denote the L^2_k-completion of $\Omega^i(\Sigma, \operatorname{ad} P)$ by $\Omega^i_k(\Sigma, \operatorname{ad} P)$.

On $\Omega^1_{k-1}(\Sigma, \operatorname{ad} P)$, we have a natural weak symplectic structure

$$\omega(\varphi_1, \varphi_2) = \int_\Sigma B(\varphi_1 \wedge \varphi_2),$$

which gives $\mathcal{A}_{k-1}$ the structure of a weak symplectic Banach manifold. Recall that a weak symplectic Banach space (V, ω) is a Banach space V with a skew-symmetric bounded bilinear form ω, which induces an injective map $\tilde{\omega} : V \to V^*$. Note that the weak symplectic structure on $\mathcal{A}_{k-1}$ is invariant under the $\mathcal{G}_k$ action.

Consider now the following kind of singular distributions on a surface.

Definition 2.2.1 *A generalized distribution on Σ is an open dense subset $\Sigma' \subset \Sigma$ and a 1-dimensional smooth subbundle $F \subset T\Sigma' \otimes \mathbb{C}$.*

We shall see that this general structure on Σ induces a polarization on $\mathcal{A}_{k-1}$, which is $\mathcal{G}_k$-invariant. Let F be a generalized distribution, and consider

$$P_F = \{\varphi \in \Omega^1_{k-1}(\Sigma, \operatorname{ad} P \otimes \mathbb{C}) | i_X \varphi = 0, \forall X \in C^\infty(\Sigma', F)\}.$$

Lemma 2.2.1 *P_F is both isotropic and coisotropic.*

Proof. Suppose that $\varphi_1, \varphi_2 \in P_F$. We then have that

$$i_X(\varphi_1 \wedge \varphi_2) = i_X \varphi_1 \varphi_2 - \varphi_1 i_X \varphi_2 = 0 \ \forall X \in C^\infty(\Sigma', F).$$

So we conclude that $\varphi_1 \wedge \varphi_2$ is zero on Σ', hence

$$\omega(\varphi_1, \varphi_2) = 0.$$

This shows that P_F is isotropic. Now suppose that $\varphi \in \Omega^1_{k-1}(\Sigma, \operatorname{ad} P)$, such that

$$\omega(\varphi, \psi) = 0 \ \forall \psi \in P_F.$$

On Σ' we can locally find coordinates (x, y), such that $\frac{\partial}{\partial y} \in C^\infty(F)$. Assume that

$$\varphi = a \ dx + b \ dy$$

in the local coordinates U on Σ'. We then get that

$$\omega(\varphi, (f dx)) = \int_U -B(f, b) dx dy = 0$$

for any $f \in C^\infty_c(U, \mathfrak{g} \otimes \mathbb{C})$. Hence we conclude that b is zero on U. This implies that

$$\varphi \in P_F.$$

$\diamond$

For a weak symplectic Banach space V, one defines an isotropic subspace W to be Lagrangian if it is maximal. Since an isotropic subspace, which is also coisotropic must be maximal, lemma 2.2.1 implies that P_F is in fact Lagrangian, and hence defines a polarization on $\mathcal{A}_{k-1}$, which is easily seen to be $\mathcal{G}_k$ invariant.

Remark. From Thurston's classification of diffeomorphisms, we get that any diffeomorphism f of a surface is homotopic to a diffeomorphism $\tilde{f}$ which preserves a generalized distribution F on Σ, i.e. $\tilde{f}^* F = F$. We see this in the following way. Theorem 2.1.4 tells us that we can on the compliment, say Σ', of a number of preserved closed curves, change f by a homotopy, to get a new diffeomorphism of Σ', which preserved a generalized distribution on Σ'. The problem here is that the homotopy is free on the boundary of Σ'. (E.g. a Dehn twist around a simple closed curve is homotopic to the identity on the compliment of that curve.) However, by first changing f by a homotopy on Σ, we can assume that f is the identity in a small cylindrical neighbourhood of the preserved curves. Say f preserves the curves $\gamma_1, \dots, \gamma_n$ (coming from the decomposition of Σ in theorem 2.1.4), and we assume that we have

$$f|_{C_{\gamma_i}} = \mathrm{Id},$$

where $C_{\gamma_i} \cong S^1 \times I$ is a cylindrical neighbourhood of γ_i. Then let $\tilde{\Sigma} = \Sigma - \cup_i C_{\gamma_i}$. By theorem 2.1.4, we get a homotopy $f_t : \tilde{\Sigma} \to \tilde{\Sigma}$, such that f_1 preserves a complex structure or a foliation on $\tilde{\Sigma}$. Since $f_t(\alpha) = \alpha$ for each boundary component of $\partial\tilde{\Sigma}$, we can easily extend f_t to be a diffeomorphism $\tilde{f}_t$ of Σ, by just "undoing" the twist of $f_t|_{\partial\tilde{\Sigma}}$ over C. Moreover, we can do this, such that $\tilde{f}_t$ preserves the natural foliation by the circles $S^1 \times \{s\}$ on C_{γ_i}. From this, it is now clear that $\tilde{f}_1$ preserves a generalized distribution on Σ, and $\tilde{f}_1$ is homotopic to f. $\tilde{f}$ will therefore preserve P_F on $\mathcal{A}_{k-1}$.

In the pseuso-Anosov case $\tilde{f}_1$ is only C^1 at the singularities of F, so we have to be more careful. However, by studying $\tilde{f}_1$ locally around the singularities of F, we see that $\tilde{f}_1$ does act on $\mathcal{A}_1$ and preserves the polarization P_F on $\mathcal{A}_1$.

Let us now recall the Marsden-Weinstein quotient construction of the space of connections. The Lie-algebra of the gauge group $\mathcal{G}_k$ is

$$L\mathcal{G}_k = \Omega^0_k(\Sigma, \mathrm{ad}\, P).$$

We shall write $\Omega^0_{-k}(\Sigma, \mathrm{ad}\, P)$ for the dual of $\Omega^0_k(\Sigma, \mathrm{ad}\, P)$. We have a smooth map

$$F : \mathcal{A}_{k-1} \to \Omega^2_{k-2}(\Sigma, \mathrm{ad}\, P),$$

which is the curvature

$$F(A) = dA + \frac{1}{2}[A, A].$$

One easily sees that the map

$$\mu : \mathcal{A}_{k-1} \to (L\mathcal{G}_k)^*,$$

given by

$$\mu(A)(\xi) = \int_\Sigma B(\xi, F(A)),$$

is the moment map for the $\mathcal{G}_k$ action on $\mathcal{A}_{k-1}$. By lemma 14.8, 14.9 and 14.11 in [5], it is clear that the Marsden-Weinstein quotient of the Banach manifold $\mathcal{A}_k$, by the Banach

Lie-group $\mathcal{G}_k$, is diffeomorphic to the moduli space $\mathcal{M}(\Sigma)$ of gauge equivalence classes of smooth connections in P, i.e.

$$\mathcal{M}(\Sigma) \cong \mathcal{A}^F_{k-1}/\mathcal{G}_k.$$

Here $\mathcal{A}^F_{k-1} = \mu^{-1}(0)$ is the space of flat L^2_{k-1}-connections. Let $A \in \mathcal{A}^F_{k-1}$. We then identify the derivative at A of F

$$D_A F : T_A\mathcal{A}_{k-1} \to \Omega^2_{k-2}(\Sigma, \mathrm{ad}\, P),$$

with

$$d_A : \Omega^1_{k-1}(\Sigma, \mathrm{ad}\, P) \to \Omega^2_{k-2}(\Sigma, \mathrm{ad}\, P).$$

We get from this that

$$T_A\mathcal{A}^F_{k-1} = \ker d_A.$$

Suppose, that the centralizer of A in $\mathcal{G}_k$ is a finite subgroup of G. We can then identify

$$T_A(\mathcal{G}_k(A)) = \mathrm{Im}\, d_A.$$

Now $\mathrm{Im}\, d_A$ is closed, and standard Hodge theory shows that

$$T_{[A]}\mathcal{M}(\Sigma) \cong \frac{\ker d_A}{\mathrm{Im}\, d_A},$$

and

$$H^1(\Sigma, d_A) \cong \frac{\ker d_A}{\mathrm{Im}\, d_A}.$$

Here $H^1(\Sigma, d_A)$ is the first cohomology group of the complex

$$0 \to \Omega^0(\Sigma, \mathrm{ad}\, P) \overset{d_A}{\to} \Omega^1(\Sigma, \mathrm{ad}\, P) \overset{d_A}{\to} \Omega^2(\Sigma, \mathrm{ad}\, P) \to 0.$$

Let us now emphasize the following fact, by recalling the proof.

Claim 2.2.1

$$(\ker d_A)^0 = \mathrm{Im}\, d_A.$$

Here $(\ker d_A)^0$ is the symplectic annihilator of $\ker d_A$ in $\Omega^1_{k-1}(\Sigma, \mathrm{ad}\, P)$.

Proof. It is clear that $\mathrm{Im}\, d_A \subset (\ker d_A)^0$. Suppose $\varphi \in \Omega^1_{k-1}(\Sigma, \mathrm{ad}\, P)$ such that

$$\omega(\varphi, \psi) = 0 \ \forall \psi \in \ker d_A.$$

We have that

$$\Omega^1_{k-1}(\Sigma, \mathrm{ad}\, P) = \ker d_A^* \oplus \mathrm{Im}\, d_A.$$

Using this decomposition, we can write

$$\varphi = \varphi_1 + d_A f,$$

where $\varphi_1 \in \ker d_A^*$, and $f \in \Omega_k^0(\Sigma, \operatorname{ad} P)$. Let $*$ be the $*$-operator, associated to the metric we have chosen on Σ. Then $*\varphi_1 \in \ker d_A$, and we get

$$|\varphi_1|_{L^2}^2 = \omega(*\varphi_1, \varphi_1) = \omega(*\varphi_1, \varphi) = 0.$$

So we see $\varphi \in \operatorname{Im} d_A$.

$\diamond$

This clearly shows that the symplectic structure ω induces a symplectic structure on

$$H^1(\Sigma, d_A) = \frac{\ker d_A}{\operatorname{Im} d_A}.$$

In analogy with the finite dimensional Marsden-Weinstein reduction of invariant polarizations, we now consider

$$\begin{aligned} H_F^1(\Sigma, d_A) &= \frac{P_F \cap (T_A \mathcal{A}_{k-1}^F \otimes \mathbb{C})}{T_A \mathcal{G}_k \otimes \mathbb{C}} \\ &= \frac{P_F \cap (\ker d_A \otimes \mathbb{C})}{\operatorname{Im} d_A \otimes \mathbb{C}} \subset H^1(\Sigma, d_A) \otimes \mathbb{C}. \end{aligned}$$

In the rest of this chapter, we shall use the notation $H^1(\Sigma, d_A)$ for both $H^1(\Sigma, d_A)$ and $H^1(\Sigma, d_A) \otimes \mathbb{C}$. It should be clear from the context what is meant.

Proposition 2.2.1 *$H_F^1(\Sigma, d_A)$ is Lagrangian if*

$$(P_F + \operatorname{Im} d_A) \cap \ker d_A = (\overline{P_F + \operatorname{Im} d_A}) \cap \ker d_A.$$

Proof. It is clear that $H_F^1(\Sigma, d_A)$ is isotropic. We need to show that $(H_F^1(\Sigma, d_A))^0 \subset H_F^1(\Sigma, d_A)$. Let $a \in (H_F^1(\Sigma, d_A))^0$. Then there is a $\varphi \in \ker d_A$, such that $[\varphi] = a$, and

$$\omega(\varphi, \psi) = 0 \ \forall \psi \in P_F \cap \ker d_A.$$

We see that $\varphi \in (P_F \cap \ker d_A)^0$. General theory gives

$$(P_F \cap \ker d_A)^0 = \overline{P_F^0 + (\ker d_A)^0}.$$

So we see that

$$\varphi \in (\overline{P_F + \operatorname{Im} d_A}) \cap \ker d_A.$$

Now by the assumption, $(\overline{P_F + \operatorname{Im} d_A}) \cap \ker d_A = (P_F + \operatorname{Im} d_A) \cap \ker d_A$, and so we see that

$$[\varphi] \in H_F^1(\Sigma, d_A).$$

We conclude that $H_F^1(\Sigma, d_A)$ is Lagrangian.

$\diamond$

From the proof, we see in general that

$$(H^1_F(\Sigma, d_A))^0 = \frac{(\overline{P_F + \operatorname{Im} d_A}) \cap \ker d_A}{\operatorname{Im} d_A}.$$

Note, that in a finite dimensional vector space the sum of two closed subspaces is of course closed, since all subspaces are closed, and so the previous proof shows that invariant polarizations on finite dimensional symplectic manifolds behave well under the Marsden-Weinstein quotient construction.

We will however see in the next section, that as soon as we allow F to have singularities, we will not have that $H^1_F(\Sigma, d_A)$ is Lagrangian. In fact, the real distribution associated to a Strebel measured foliation has the mildest possible singularities, but $H^1_F(\Sigma, d_A)$ will in general not be Lagrangian.

2.3 The basic cohomology of a measured foliation.

Let F be a measured foliation on a closed surface Σ of genus g. Let P be a principal G-bundle over Σ, and A a flat connection in P. The connection A then induces a covariant derivative in the adjoint bundle $\operatorname{ad} P$. We shall also use the notation F for the underlying foliation associated to the measured foliation F. In general, when one has a foliation on a manifold, one gets a filtration of the de Rham complex. This filtration gives rise to a spectral sequence, which has been studied a lot for smooth foliations on smooth manifolds. (See [33] and [34]. See also the bibliography of [48].) For a 1-dimensional foliation on a 2-dimensional manifold, there are only 4 non-zero terms in the spectral sequence, and it collapses at the E_2-term. Moreover, one easily sees in this special case, that all terms in the spectral sequence can be explicitly identified in terms of the cohomology of the 2-manifold and what is called the "basic cohomology" of the foliation. We shall, therefore, here only consider the basic cohomology. Adapting the definition to forms with values in a flat bundle, we shall follow [48] in defining the basic complex of F with values in $\operatorname{ad} P$:

$$0 \to \Omega^0_F(\Sigma, \operatorname{ad} P) \xrightarrow{d_A} \Omega^1_F(\Sigma, \operatorname{ad} P) \to 0, \tag{2.1}$$

where

$$\Omega^i_F(\Sigma, \operatorname{ad} P) = \{\varphi \in \Omega^i(\Sigma, \operatorname{ad} P) \mid i_X\varphi = 0, i_X d_A\varphi = 0 \ \forall X \in C^\infty(F)\}.$$

The cohomology of the complex (2.1) is called the basic cohomology $H^i_F(\Sigma, d_A)$. A moment's thought shows that

$$H^0_F(\Sigma, d_A) = H^0(\Sigma, d_A),$$

and that we have a natural inclusion

$$H^1_F(\Sigma, d_A) \subset H^1(\Sigma, d_A).$$

These are general facts about the two first cohomology groups of the basic complex of any foliation.

Recall that $H^1(\Sigma, d_A)$ carries a symplectic structure

$$\omega(\varphi_1, \varphi_2) = \int_\Sigma B(\varphi_1 \wedge \varphi_2),$$

whenever we have chosen an invariant inner product $B(\cdot,\cdot)$ on $\mathfrak{g}$. Note that $H^1_F(\Sigma, d_A)$ is an isotropic subspace of $H^1(\Sigma, d_A)$, for let φ_1, φ_2 be basic forms representing two classes in $H^1_F(\Sigma, d_A)$. We then see

$$i_X(\varphi_1 \wedge \varphi_2) = i_X\varphi_1\varphi_2 - \varphi_1 i_X\varphi_2 = 0$$

for any $X \in C^\infty(F)$, hence

$$\omega(\varphi_1, \varphi_2) = 0.$$

In order to get some understanding of the dimension of the space $H^1_F(\Sigma, d_A)$, we shall give an alternative definition for a Strebel measured foliation.

We can consider homology with coefficients in the flat bundle $(\mathrm{ad}\, P, d_A)$. Let us denote the first homology group with coefficients in $(\mathrm{ad}\, P, d_A)$ by $H_1(\Sigma, d_A)$. For details and definitions we refer to [10], [14], [21] and [46]. As it is explained in [21], a smooth map $\alpha : S^1 \to \Sigma$ and a covariant constant section ξ of $\mathrm{ad}\, P|_{\alpha(S^1)}$ define an element of $H_1(\Sigma, d_A)$, which we denote $[\alpha \otimes \xi]$. Moreover, such elements span $H_1(\Sigma, d_A)$. Suppose $\alpha : S^1 \to \Sigma$ and $\beta : S^1 \to \Sigma$ are smooth maps, which are homotopic, i.e. there exists a smooth map

$$F : S^1 \times I \to \Sigma$$

such that

$$F|_{S^1 \times \{0\}} = \alpha,\ F|_{S^1 \times \{1\}} = \beta.$$

Suppose $\xi_\alpha \in H^0(\alpha(S^1), d_A)$ and $\xi_\beta \in H^0(\beta(S^1), d_A)$. Modulo a gauge transformation, we can assume that say

$$F^*A = A_1 d\theta,\ A_1 \in \mathfrak{g},$$

where (θ, r) are the standard coordinates on $S^1 \times I$. Using F, one identifies $H^0(F(S^1 \times \{s\}), d_A)$ with $N_A = \{\eta \in \mathfrak{g} | [\eta, A] = 0\}$, and we see that

$$[\alpha \otimes \xi_\alpha] = [\beta \otimes \xi_\beta]$$

in $H_1(\Sigma, d_A)$, if

$$\xi_\alpha = \xi_\beta$$

as elements of N_A.

Now the flat bundle (P, A) gives us a representation $\phi: \pi_1(\Sigma) \to G$, hence by composing with the adjoint representation of G, we get a $\pi_1(\Sigma)$-module structure on $\mathfrak{g}$. In the following discussion we shall often use the identifications (see [21])

$$H^1(\Sigma, d_A) \cong H^1(\pi_1(\Sigma), \mathfrak{g}_{Ad\phi})$$

and

$$H_1(\Sigma, d_A) \cong H_1(\pi_1(\Sigma), \mathfrak{g}_{Ad\phi}).$$

It is explained in [21] how the inner product B induces a natural perfect pairing

$$B_* : H^1(\Sigma, d_A) \times H_1(\Sigma, d_A) \to \mathbb{R}.$$

Suppose we have a d_A-closed $\mathrm{ad}\, P$ valued 1-form φ representing an element in $H^1(\Sigma, d_A)$. Let $\alpha : S^1 \to \Sigma$ be a smooth map and ξ a covariant constant section of $\mathrm{ad}\, P$ over $\alpha(S^1)$. We then have the following explicit description of the pairing

$$B_*([\varphi], [\alpha \otimes \xi]) = B(\int_\alpha \varphi, \xi(\alpha(0))),$$

where

$$\int_\alpha \varphi = \int_0^1 P_t \varphi(\alpha'(t)) dt,$$

and

$$P_t : \mathrm{ad}\, P_{\alpha(t)} \to \mathrm{ad}\, P_{\alpha(0)},$$

is parallel transport in $\mathrm{ad}\, P$ along α.

We also have Poincaré duality denoted

$$\cap[\Sigma] : H^1(\Sigma, d_A) \to H_1(\Sigma, d_A),$$

which is defined by

$$B_*([\varphi] \cap [\Sigma], [\alpha \otimes \xi]) = B_*([\varphi], [\alpha \otimes \xi]).$$

The pairing on the left is the intersection product (see p.276 of [21])

$$B_* : H_1(\Sigma, d_A) \times H_1(\Sigma, d_A) \to \mathbb{R}.$$

Let F be a Strebel measured foliation. We shall now define what we call a cylindrical leaf choice.

Definition 2.3.1 *A cylindrical leaf choice for F is a set L_F, consisting of smooth maps $\alpha_i : S^1 \to \Sigma$, $i = 1, \dots, m$ with the following properties*

1) *$\alpha_i(S^1)$ is a non-singular leaf of F.*

2) *α_i is not homotopic to α_j, $i \neq j$.*

3) *Any non-singular leaf of F is homotopic to α_i for some $i \in \{1, \dots, m\}$.*

For a Strebel measured foliation, the compliment of the critical graph is a disjoint union of a finite number of foliated cylinders. Hence, one gets a cylindrical leaf choice L_F for F, by chosing a parametrisation of precisely one leaf from each cylinder. Let L_F be a cylindrical leaf choice for F. We can then define the basic homology of F to be

$$H_1^F(\Sigma, d_A) = \mathrm{Span}\{[\alpha \otimes \xi] | \alpha \in L_F,\ \xi \in H^0(\alpha, d_A)\} \subset H_1(\Sigma, d_A).$$

By the remarks made earlier about $H_1(\Sigma, d_A)$, we see that $H_1^F(\Sigma, d_A)$ does not depend on L_F and in fact only depends on the equivalence class of F, i.e. $H_1^F(\Sigma, d_A)$ is naturally associated to Strebel points of the Thurston boundary of Teichmüller space.

The following theorem tells us that the basic cohomology and the basic homology are identified under Poincaré duality.

Theorem 2.3.1 *Let Σ be a closed surface and F a Strebel measured foliation on Σ. Poincaré duality maps*

$$(H_F^1) \cap [\Sigma] = H_1^F.$$

Note that a corollary of this theorem is that the basic cohomology is a subspace of $H^1(\Sigma, d_A)$ naturally associated to the Strebel points of the Thurston boundary.

Before we can prove this theorem, we need some facts about the basic forms of a Strebel measured foliation. We give these in Lemma 2.3.1 – 2.3.4.

Lemma 2.3.1 *Let α be a simple closed curve on Σ and η a covariant constant section of* $\mathrm{ad}\, P$ *over α. The Poincaré dual of*

$$[\alpha \otimes \eta] \in H_1(\Sigma, d_A)$$

can be represented by $\varphi \in \Omega^1(\Sigma, \mathrm{ad}\, P)$, $d_A\varphi = 0$ with compact support in any neighbourhood of α.

Proof. Suppose $\tilde{\varphi} \in \Omega^1(\Sigma, \mathrm{ad}\, P), d_A\tilde{\varphi} = 0$ is a representative of the Poincaré dual of $[\alpha \otimes \eta]$. Denote $\Sigma' = \Sigma - \alpha$. Let us denote the inclusion $i : \Sigma' \to \Sigma$. Let β be a simple closed curve in Σ' and ξ a covariant constant section (with respect to d_{i^*A}) of $i^*(\mathrm{ad}\, P)$ over β. We then get

$$B_*(i^*[\tilde{\varphi}], [\beta \otimes \xi]) = B_*([\tilde{\varphi}], [\beta \otimes \xi]) = 0.$$

So we conclude that for $[i^*\tilde{\varphi}] \in H^1(\Sigma', d_A)$, we have

$$B_*([i^*\tilde{\varphi}], a) = 0 \ \forall a \in H_1(\Sigma', d_{i^*A}).$$

The de Rham theorem for twisted coefficient systems then implies that there exists $f \in \Omega^0(\Sigma', \mathrm{ad}\, P)$, such that

$$\psi = d_{i^*A} f.$$

Let U be a neighbourhood of α, and choose $\rho \in C^\infty(\Sigma)$ with support in U, and such that ρ is 1 near α. Consider

$$\varphi = \tilde{\varphi} - d_A((1 - \rho)f).$$

Then $[\varphi]$ is the Poincaré dual of $[\alpha \otimes \eta]$, and φ has compact support in U.

◇

Lemma 2.3.2 *Let F be a Strebel measured foliation on Σ, α be a non-singular leaf of F, and η be a covariant constant section of $\operatorname{ad} P|_\alpha$. Then the Poincaré dual of $[\alpha \otimes \eta] \in H_1(\Sigma, d_A)$ is contained in the basic cohomology, i.e.*

$$[\alpha \otimes \eta](\cap[\Sigma])^{-1} \in H^1_F(\Sigma, d_A).$$

Proof. By lemma 2.3.1 we know that $[\alpha \otimes \eta](\cap[\Sigma])^{-1}$ can be represented by $\varphi \in \Omega^1(\Sigma, \operatorname{ad} P)$, $d_A\varphi = 0$, with compact support in a cylinder neighbourhood C of α, not intersecting the critical graph of F. If we restrict φ to C, we know by the de Rham theorem that there exists an $f \in \Omega^0(C, \operatorname{ad} P)$, such that $d_A f = \varphi|_C$. Let $\rho \in \Omega^0_F(\Sigma)$ be such that ρ is 1 on a neighbourhood of the support of φ and with compact support in C. Consider

$$\tilde{\varphi} = \varphi - d_A(\rho f).$$

In a neighbourhood of the support of φ, $\tilde{\varphi}$ is zero, and on the complement of the support of φ we have

$$\tilde{\varphi} = -d_A(\rho f) = -f d\rho - \rho d_A f = -f d\rho.$$

So we conclude that

$$i_X\tilde{\varphi} = 0 \ \forall X \in C^\infty(F),$$

hence $[\tilde{\varphi}] \in H^1_F(\Sigma, d_A)$.

◇

Lemma 2.3.3 *Suppose F is a Strebel measured foliation on Σ. For any basic form $\varphi \in \Omega^1_F(\Sigma, \operatorname{ad} P)$, we can find a basic form $\tilde{\varphi} \in \Omega^1_F(\Sigma, \operatorname{ad} P)$, such that $\tilde{\varphi}$ represents the same cohomology class as φ, and the support of $\tilde{\varphi}$ is disjoint from the critical graph of F.*

Proof. Let $C_1, \dots, C_n$ be the connected components of the complements of the critical graph. Each C_i is a foliated cylinder $S^1 \times I$. Let α_i be a leaf in C_i. Now let ρ be a smooth basic function with support in $\cup_i C_i$ and 1 in a neighbourhood of α_i.

Suppose that $\varphi \in \Omega^1_F(\Sigma, \operatorname{ad} P)$. Let $\Sigma' = \Sigma - \cup_i \alpha_i$, and consider $\psi = \varphi|_{\Sigma'}$. Suppose $[\beta \otimes \xi] \in H_1(\Sigma', d_A)$. Since Σ' retracts onto the critical graph of F, we can represent β by a curve, which is contained in the critical graph of F. So when we evaluate $[\psi]$ against $[\beta \otimes \xi]$, we get zero, since ψ is basic. By the de Rham theorem, we conclude there exists $f \in \Omega^1(\Sigma', \operatorname{ad} P)$, such that

$$d_A f = \psi \text{ on } \Sigma'.$$

Consider

$$\tilde{\varphi} = \varphi - d_A((1-\rho)f).$$

Clearly $\tilde{\varphi}$ is basic, and the support of $\tilde{\varphi}$ is contained in $\cup_i C_i$. This concludes the proof, since $[\varphi] = [\tilde{\varphi}]$.

◇

Lemma 2.3.4 *Let F be a Strebel measured foliation of a closed surface Σ. Suppose $\varphi \in \Omega^1_F(\Sigma, \mathrm{ad}\, P)$ is a basic form with support in a cylinder neighbourhood $C = S^1 \times I$ of a non-critical leaf α. Then there exists a covariant constant section ζ of $\mathrm{ad}\, P$ over α, such that*

$$[\varphi] \cap [\Sigma] = [\alpha \otimes \zeta].$$

Proof. We can assume we have coordinates $(\theta, r) : C \to S^1 \times I$, such that $r^{-1}(k),\ k \in I$ are the leaves of F on C, and $\alpha = r^{-1}(\frac{1}{2})$. Let $p = (\theta, r)^{-1}(0, \frac{1}{2})$. Suppose $\beta \in \pi_1(p, \Sigma)$. If $i(\beta, \alpha) = 0$, then $B_*([\varphi], [\alpha \otimes \xi]) = 0$ for any $\xi \in H^0(\beta, d_A)$, so assume $i(\beta, \alpha) > 0$. We can then choose curves $\beta_1, \ldots, \beta_i$, $i = i(\beta, \alpha)$, such that $\beta_j \cap C = \{0\} \times I$, and β_j begins and ends at p. Furthermore

$$\beta = [\beta_1 \cdot \ldots \cdot \beta_i].$$

On C we have, say $\varphi = \varphi_1 dr$, so

$$\int_{\beta_j} \varphi = \int_{\frac{1}{2}}^{1} P_s \varphi_1(s) ds + \int_0^{\frac{1}{2}} Ad\phi(\beta_j) \tilde{P}_s \varphi_1(s) ds,$$

where

$$P_s \colon \mathrm{ad}\, P_{(0,s)} \to \mathrm{ad}\, P_{(0,\frac{1}{2})},\ \ s \in [\frac{1}{2}, 1],$$

and

$$\tilde{P}_s : \mathrm{ad}\, P_{(0,s)} \to \mathrm{ad}\, P_{(0,\frac{1}{2})},\ \ s \in [0, \frac{1}{2}],$$

are parallel transports in $\mathrm{ad}\, P$ along $\{0\} \times I$.

Suppose $[\beta \otimes \xi] \in H_1(\pi_1(p, \Sigma), \mathfrak{g}_{\mathrm{Ad}\,\phi})$. We then have that

$$\mathrm{Ad}\,\phi(\beta_1) \circ \cdots \circ \mathrm{Ad}\,\phi(\beta_i)\xi = \xi.$$

Using this, we calculate

$$B(\int_\beta \varphi, \xi)$$

$$= \quad B\left(\sum_{j=1}^{i} \mathrm{Ad}\,\phi(\beta_1) \circ \cdots \circ \mathrm{Ad}\,\phi(\beta_{j-1}) \right.$$

$$
\begin{aligned}
& \left(\int_{\frac{1}{2}}^{1} P_s\varphi_1(s)ds + \int_0^{\frac{1}{2}} \operatorname{Ad}\phi(\beta_j)\tilde{P}_s\varphi_1(s)ds), \xi \right) \\
= \; & B\left(\int_{\frac{1}{2}}^{1} P_s\varphi_1(s)ds, \sum_{j=1}^{i} \operatorname{Ad}\phi(\beta_{j-1}^{-1}) \circ \cdots \circ \operatorname{Ad}\phi(\beta_1^{-1})\xi \right) \\
& + B\left(\int_0^{\frac{1}{2}} \tilde{P}_s\varphi_1(s)ds, \sum_{j=1}^{i} \operatorname{Ad}\phi(\beta_j^{-1}) \circ \cdots \circ \operatorname{Ad}\phi(\beta_1^{-1})\xi \right) \\
= \; & B\left(\int_{\frac{1}{2}}^{1} P_s\varphi_1(s)ds + \int_0^{\frac{1}{2}} \tilde{P}_s\varphi_1(s)ds, \sum_{j=1}^{i} \operatorname{Ad}\phi(\beta_j^{-1}) \circ \cdots \circ \operatorname{Ad}\phi(\beta_1^{-1})\xi \right)
\end{aligned}
$$

Let

$$\zeta = \int_{\frac{1}{2}}^{1} P_s\varphi_1(s)ds + \int_0^{\frac{1}{2}} \tilde{P}_s\varphi_1(s)ds.$$

We shall now establish

$$\operatorname{Ad}\phi(\alpha)\varphi_1(s) = \varphi_1(s), \;\; \forall s \in [0,1], \tag{2.2}$$

by showing that $i_{\frac{\partial}{\partial r}}\varphi$ is covariant constant on $S^1 \times \{s\}$.

Since

$$\varphi = \varphi_1 dr,$$

and say

$$A = A_1 d\theta + A_2 dr,$$

we get from the d_A-closedness of φ that

$$\frac{\partial\varphi_1}{\partial\theta} + [A_1, \varphi_1] = 0.$$

Using this, we calculate

$$i_{\frac{\partial}{\partial\theta}}(d_A(i_{\frac{\partial}{\partial r}}\varphi)) = i_{\frac{\partial}{\partial\theta}}(\frac{\partial\varphi_1}{\partial\theta}d\theta + \frac{\partial\varphi_1}{\partial r}dr + [A_1,\varphi_1]d\theta + [A_2,\varphi_1]dr) = 0.$$

We conclude that $i_{\frac{\partial}{\partial r}}\varphi$ is covariant constant. From (2.2) we can deduce that

$$
\begin{aligned}
\operatorname{Ad}\phi(\alpha)\zeta &= \int_{\frac{1}{2}}^{1} \operatorname{Ad}\phi(\alpha)P_s\varphi_1(s)ds + \int_0^{\frac{1}{2}} \operatorname{Ad}\phi(\alpha)\tilde{P}_s\varphi_1(s)ds \\
&= \int_{\frac{1}{2}}^{1} P_s \operatorname{Ad}\phi(\alpha)\varphi_1(s)ds + \int_0^{\frac{1}{2}} \tilde{P}_s \operatorname{Ad}\phi(\alpha)\varphi_1(s)ds \\
&= \zeta.
\end{aligned}
$$

Hence we see $\zeta \in \mathfrak{g}^{\operatorname{Ad}\phi(\alpha)}$, so $[\alpha \otimes \zeta]$ gives us a well defined element of $H_1(\pi_1(p,\Sigma), \mathfrak{g}_{\operatorname{Ad}\phi})$. From the above calculations we have that

$$B_*([\alpha\otimes\zeta],[\beta\otimes\xi]) = B(\int_\beta \varphi, \xi),$$

and if $i(\beta, \alpha) = 0$, we have that

$$B_*([\alpha \otimes \zeta], [\beta \otimes \xi]) = 0.$$

Hence we conclude that

$$[\varphi] \cap [\Sigma] = [\alpha \otimes \zeta].$$

$\diamond$

We are now in a position to prove theorem 2.3.1.

Proof of theorem 2.3.1. By lemma 2.3.2 we have

$$(H_1^F(\Sigma, d_A))(\cap[\Sigma])^{-1} \subset H_F^1(\Sigma, d_A).$$

The other inclusion follows from lemma 2.3.3 and 2.3.4. Hence

$$H_F^1(\Sigma, d_A) \cap [\Sigma] = H_1^F(\Sigma, d_A).$$

$\diamond$

We shall use the above results on the basic forms and the basic cohomology in the subsequent sections of this chapter. For now, let us just use theorem 2.3.1 to get some upper bounds on $\dim H_F^1(\Sigma, d_A)$. We shall assume for the rest of the section that $g \geq 2$. Let G be a semisimple and compact group. Let P be a principal G-bundle and A an irreducible flat connection in P. Denote by ϕ_A a representative for the conjugacy class of representations of $\pi_1(\Sigma)$ into G given by A. We have, since A is irreducible (see for example [29]), that

$$\dim H^1(\Sigma, d_A) = 2(g-1)\dim G. \tag{2.3}$$

Let L_F be a cylindrical leaf choice of F.

From the definition of $H_1^F(\Sigma, d_A)$, we have

$$\dim H_1^F(\Sigma, d_A) \leq \sum_{\alpha \in L_F} \dim \mathrm{g}^{\mathrm{Ad}\,\phi_A(\alpha)}. \tag{2.4}$$

So if $H_F^1(\Sigma, d_A)$ was Lagrangian, we would have that

$$(g-1)\dim G \leq \sum_{\alpha \in L_F} \dim \mathrm{g}^{\mathrm{Ad}\,\phi_A(\alpha)}.$$

Since we clearly have that

$$\dim \mathrm{g}^{\mathrm{Ad}\,\phi_A(\alpha)} \leq \dim G,$$

for any connection A, we see that if $\#L_F < g-1$, then $H^1_F(\Sigma, d_A)$ will not be Lagrangian. Typically, one would expect that

$$\dim \mathsf{g}^{\operatorname{Ad}\phi_A(\alpha)} = \operatorname{Rank} G$$

for a generic flat connection A. Moreover, we always have that $\#L_F \leq 3g-3$, since $3g-3$ is the maximal number of non-intersecting, homotopicly non-trivial, pairwise non-homotopic curves on a genus g surface. In which case, $H^1_F(\Sigma, d_A)$ Lagrangain implies

$$(g-1)\dim G \leq 3(g-1)\operatorname{Rank} G.$$

So $\dim G \leq 3 \operatorname{Rank} G$, which is impossible for a simple group G, unless $G = SU(2)$ or $G = SO(3)$.

In section 2.4, we shall see that there are some measured foliations F, for which $H^1_F(\Sigma, d_A)$ is Lagrangian for generic flat $SU(2)$-connections A.

If the measured foliation F has a leaf which is dense, then we get the following bound on the dimension

$$\dim H^1_F(\Sigma, d_A) \leq \dim G,$$

since any basic form is determined by its restiction to this dense leaf. Note that any measured foliation F preserved by a pseudo-Anosov diffeomorphism has all leaves dense (see [19] lemma 6 p. 165). Note that lemma 2.3.3 is also valid for any basic 1-form φ in $\Omega^1_k(\Sigma, \operatorname{ad} P)$, when $k \geq 2$. This implies that we can identify "$H^1_F(\Sigma, d_A)$" from the previous section with the basic cohomology defined in this section. So in reference to the previous section and the dimension estimates on the basic cohomology of this section, we conclude that the polarizations we constructed in section 2.2 do not behave well under Marsden-Weinstein quotient construction of the space of connections.

Let us end this section with a discussion of the basic cohomology in the abelian case, $G = U(1)$. We are then considering

$$H^1_F = \{\varphi \in \Omega^1(\Sigma) \mid i_X\varphi = 0, d\varphi = 0,\ \forall X \in C^\infty(F)\} \subset H^1(\Sigma),$$

and for a Strebel measured foliation, we have that

$$H_1^F = \operatorname{Span} L_F \subset H_1(\Sigma),$$

where L_F is a cylindrical leaf choice of F. It is easy to read off the dimension of H_1^F, and therefore by theorem 2.3.1 the dimension of H^1_F. Suppose $\alpha_1, \ldots, \alpha_g$ are g non-intersecting curves on a surface Σ of genus g, such that $\alpha_1, \ldots, \alpha_g$ represent g linearly independent elements of homology. These curves then define a point on the Thurston boundary of Teichmüller space. The basic cohomology of this equivalence class of measured foliations is then Lagrangian, since the dimension of H_1^F is g. Conversely, suppose we have a Strebel measured foliation F, such that the basic cohomology is Lagrangian. Then by the definition of H_1^F, we can find g leaves of F, which represent independent cohomology classes. This gives us a description of the Strebel points with Lagrangian basic cohomology H_F^1 of the Thurston boundary. Clearly, the set of such points is preserved by the mapping class group, and hence forms a dense subset of the Thurston boundary.

On the other hand, L_F consists of homologically trivial curves if, and only if $H_F^1 = \{0\}$. The set of Strebel points on the Thurston boundary with trivial basic cohomology is non-empty and invariant under the mapping class group, and hence also dense.

2.4 Isotropic foliations on the moduli spaces.

Let P be a principal G bundle over a closed surface Σ. Assume that G is a connected compact simple Lie group. As we saw in section 2.2, $H^1(\Sigma, d_A)$ is generically identified with the tangent space to the moduli space $\mathcal{M}(\Sigma)$ of flat connections in P. Given a Strebel measured foliation F on Σ, we get an isotropic subspace $H^1_F(\Sigma, d_A)$, which induces an isotropic singular distribution on the moduli space $\mathcal{M}(\Sigma)$. We shall now give an alternative description of this distribution for Strebel measured foliations, using the gluing techniques described in [15]. The description will show that this distribution induces a singular isotropic foliation with compact leaves on $\mathcal{M}(\Sigma)$.

So, let us assume that F is a Strebel measured foliation. Let L_F be a cylindrical leaf choice for F. Now given such a system L_F of non-homotopic simple closed curves, we shall, following [15], describe a singular foliation structure on $\mathcal{M}(\Sigma)$.

Let $\gamma \in L_F$. Since G is connected, the restriction P_γ of P to γ is trivial. A connection A in P induces a connection in P_γ. Now principal bundles with connections over a circle is classified up to gauge equivalence by the holonomy, which is naturally an element of C_G, the space of conjugacy classes of the structure group G. From this we get a map

$$h : \mathcal{M}(\Sigma) \to \times_m C_G,$$

where $m = \#L_F$, simply by mapping $[A] \in \mathcal{M}(\Sigma)$ to the holonomies of A around the curves in L_F. Let $c \in \times_m C_G$, and denote

$$\mathcal{M}(\Sigma)_c = h^{-1}(c).$$

Let $\tilde{\Sigma}$ be the surface with boundary one obtains by doubling all the curves in L_F. That is we have a map $p : \tilde{\Sigma} \to \Sigma$ such that

$$p|_{\tilde{\Sigma}-\partial\tilde{\Sigma}} : \tilde{\Sigma} - \partial\tilde{\Sigma} \to \Sigma - \bigcup_{\gamma\in L_F} \gamma$$

is a diffeomorphism and $p^{-1}(\gamma)$, $\gamma \in L_F$ is a union of two disjoint boundary components γ' and γ'' of $\tilde{\Sigma}$. Consider $\mathcal{M}(\tilde{\Sigma})$, the moduli space of flat connections in P over $\tilde{\Sigma}$. On $\mathcal{M}(\tilde{\Sigma})$ we likewise have a map

$$h : \mathcal{M}(\tilde{\Sigma}) \to \times_m C_G \;\times\; \times_m C_G,$$

by mapping $[A] \in \mathcal{M}(\Sigma)$ to the holonomies of A around the curves γ' for $\gamma \in L_F$ followed by the holonomies af A around the curves γ'' for $\gamma \in L_F$. We denote

$$\mathcal{M}(\tilde{\Sigma})^{c_1}_{c_2} = h^{-1}(c_1, c_2).$$

There clearly is a projection

$$\pi_c : \mathcal{M}(\Sigma)_c \to \mathcal{M}(\tilde{\Sigma})^c_c.$$

Suppose $\tilde{A}$ is a flat connection in P over $\tilde{\Sigma}$, such that $[\tilde{A}] \in \mathcal{M}(\tilde{\Sigma})_{c1}^{c_2}$. Let $Z(\tilde{A})$ be the set of bundle isomorphisms from $\times_{L_F}(P_{\gamma'}, \tilde{A}|_{\gamma'})$ to $\times_{L_F}(P_{\gamma''}, \tilde{A}|_{\gamma''})$. Let $\Gamma_{\tilde{A}}$ be the automorphism group of $\tilde{A}$ over $\tilde{\Sigma}$. Then $\Gamma_{\tilde{A}}$ acts on $Z(\tilde{A})$ in the natural way. A gluing parameter $\rho \in Z(\tilde{A})$ defines a flat connection $\tilde{A}^{\#(\rho)}$ on Σ. It is easily seen that two flat connections $\tilde{A}^{\#(\rho)}$ and $\tilde{A}^{\#(\rho')}$, that arise this way, are isomorphic, if and only if ρ and ρ' is in the same $\Gamma_{\tilde{A}}$ orbit of $Z(\tilde{A})$.

It is clear that any $[A] \in \pi_c^{-1}([\tilde{A}])$ will be of this form, hence we see

$$\pi_c^{-1}([\tilde{A}]) \cong \frac{Z(\tilde{A})}{\Gamma_{\tilde{A}}}.$$

We shall now identify the tangent space to the fibres of π_c with the basic cohomology of the foliation F. In order to do this we first need to describe the precise relation between the cohomology $H^1(\Sigma, d_A)$ and the local structure of $\mathcal{M}(\Sigma)$ around $[A]$.

There is a natural pairing

$$[\cdot,\cdot] : H^1(\Sigma, d_A) \times H^1(\Sigma, d_A) \to H^2(\Sigma, d_A),$$

which is a combination of the wedge product and commutator in the Lie-algebra. We have the following theorem. (See [23].)

Theorem 2.4.1 (Goldman and Millson) *Let A be a flat connection. Then there is a neighbourhood of $[A]$ in the moduli space $\mathcal{M}(\Sigma)$, which is diffeomorphic to*

$$\{\varphi \in H^1(\Sigma, d_A) | [\varphi, \varphi] = 0\}/Z(A),$$

where $Z(A)$ is the stabilizer of A.

Note that if A is irreducible, then $Z(A) = Z(G)$, and $H^0(\Sigma, d_A) = \{0\}$. By Poincaré duality, we get that $H^2(\Sigma, d_A) = \{0\}$. Hence we see that $\mathcal{M}^{\mathrm{irr}}(\Sigma)$, the moduli space of irreducible connections, is smooth. Now $\mathcal{M}^{\mathrm{irr}}(\Sigma)$ is in fact open and dense in $\mathcal{M}(\Sigma)$, when the genus g of the surface is greater than or equal to 2. We therefore get that

$$T_{[A]}\mathcal{M}(\Sigma) \cong H^1(\Sigma, d_A),$$

for all irreducible connections.

Recalling the construction of $H_F^1(\Sigma, d_A) \subset H^1(\Sigma, d_A)$, one sees that

$$H_F^1(\Sigma, d_A) \subset \{\varphi \in H^1(\Sigma, d_A) | [\varphi, \varphi] = 0\},$$

and $Z(A)$ acts on $H_F^1(\Sigma, d_A)$, so we have

$$H_F^1(\Sigma, d_A)/Z(A) \subset \{\varphi \in H^1(\Sigma, d_A) | [\varphi, \varphi] = 0\}/Z(A).$$

Lemma 2.4.1 *Let A be a flat connection on Σ and denote $\tilde{A} = A|_{\tilde{\Sigma}}$. Let g_s be a smooth curve through the identity of $Z(\tilde{A})$. We then get a curve $g_s^* A$ of flat connections. The tangent vector at A to this curve is a d_A closed 1-form, which represents a basic cohomology class. Moreover, any basic cohomology class can be represented this way.*

Note, that if A is an irreducible connection and $[A]$ is a smooth point of the fibre of π_c through $[A]$, then this lemma is just saying that the tangent space to the fibre is the basic cohomology of the foliation F.

Proof. Let $(\theta_\gamma, r_\gamma) : U_\gamma \to S^1 \times [0,1]$ be coordinates near γ compatible with F, i.e. the leaves of F are given by $r_\gamma^{-1}(k)$, $k \in [0,1]$. (Assume that $\gamma = r^{-1}(\frac{1}{2})$.) We can suppose that A is of the form $A_\gamma d\theta_\gamma$, where $A_\gamma \in \mathfrak{g}$, on U_γ. The holonomy of A around γ is represented by $\exp A_\gamma \in G$. We can identify

$$Z(\tilde{A}) = \times_{\gamma \in L_F} Z(\exp A_\gamma).$$

Now let $\xi = (\xi_\gamma) \in L(Z(\tilde{A}))$. Consider a smooth function ρ on $[\frac{1}{2}, 1]$, which is constant 1 in a neighbourhood of $\frac{1}{2}$ in $[\frac{1}{2}, 1]$ and constant 0 in a neighbourhood of 1 in $[\frac{1}{2}, 1]$. Then define the following "broken" gauge transformation of P

$$g_t(\theta_\gamma, r_\gamma) = \begin{cases} e & r_\gamma \in [0, \frac{1}{2}) \\ \exp(t\rho(r_\gamma)\xi_\gamma) & r_\gamma \in [\frac{1}{2}, 1] \end{cases} .$$

Since $g_t^*(\frac{1}{2})(A|_{L_F}) = A|_{L_F}$, we see that $g_t^*(A)$ gives a smooth curve of flat connections on Σ. Explicitly we get

$$g_t^*(A)(\theta_\gamma, r_\gamma) = \begin{cases} A_\gamma d\theta_\gamma & r_\gamma \in [0, \frac{1}{2}) \\ A_\gamma d\theta_\gamma + t\rho'(r_\gamma)\xi_\gamma dr_\gamma & r_\gamma \in [\frac{1}{2}, 1] \end{cases} .$$

Hence the tangent vector to this curve is represented by the d_A-closed form

$$\varphi = \sum_\gamma \rho'(r_\gamma)\xi_\gamma dr_\gamma, \tag{2.5}$$

which is a basic 1-form, i.e.

$$[\varphi] \in H_F^1(\Sigma, d_A).$$

Now suppose that φ is a d_A-closed form, which represent a class in $H_F^1(\Sigma, d_A)$. By the study of the basic cohomology in section 2.3, we can assume that φ is supported in $\cup_{\gamma \in L_F} r_\gamma^{-1}((\frac{1}{2}, 1))$. In fact we can assume that φ is of the form (2.5) , where $\xi_\gamma \in L(Z(\exp A_\gamma))$ from the proof of lemma 2.3.2. This concludes the proof.

$\diamond$

This lemma gives an explicit description of the leaves of the isotropic foliation which a Strebel measured foliation induces on the moduli space $\mathcal{M}(\Sigma)$.

Consider the open subset C_G^{reg} of C_G consisting of regular conjugacy classes, and define the following subset of $\mathcal{M}(\Sigma)$

$$\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}} = \bigcup_{c\in C_G^{reg}} \pi_c^{-1}(\mathcal{M}(\tilde{\Sigma})_c^c(\mathrm{irr})),$$

where $\mathcal{M}(\tilde{\Sigma})_c^c(\mathrm{irr})$ denotes the subset of $\mathcal{M}(\tilde{\Sigma})_c^c$, consisting of irreducible connections.

Let

$$\mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots} = h^{-1}(\triangle),$$

where

$$\triangle = \{(c_1, c_2) \in \times_m C_G \ \times \ \times_m C_G | c_1 = c_2\}.$$

We denote $\mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots}(\mathrm{irr})$ the subset of irreducible connections in $\mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots}$. We have the restriction map

$$r : \mathcal{M}(\Sigma) \to \mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots}.$$

We then see that

$$\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}} = r^{-1}(\mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots}(\mathrm{irr})) \cap h^{-1}(\times_m C_G^{\mathrm{reg}}).$$

Note that $\mathcal{M}(\Sigma)$, $\mathcal{M}(\tilde{\Sigma})$, $\mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots}$ and $\times_m C_G$ are real algebraic varieties and that h and r are real algebraic maps (see e.g. [22]). We have that $\mathcal{M}(\Sigma)_{\cdots}^{\cdots}(\mathrm{irr})$ is Zariski open in $\mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots}$ and $\times_m C_G^{\mathrm{reg}}$ is Zariski open in $\times_m C_G$. Thus we get that $\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$ is Zariski open in $\mathcal{M}(\Sigma)$. So whenever $\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$ is non-empty, it will automaticly be open and dense in the topology induced from the space of flat connections on $\mathcal{M}(\Sigma)$.

Denote by $\tilde{\Sigma}'$, the closed surface, one obtains from $\tilde{\Sigma}$ by sawing on discs on each boundary component of $\tilde{\Sigma}$. We have the following existence result.

Lemma 2.4.2 *Suppose that the genus of the connected components of $\tilde{\Sigma}'$ all are greater than or equal to two. It then follows that $\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$ is an open dense subset of $\mathcal{M}(\Sigma)$.*

Proof. By the above discussion, we just need to find $a_1, a_2 \in \mathcal{M}(\Sigma)$ such that $r(a_1) \in \mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots}(\mathrm{irr})$ and $h(a_2) \in \times_m C_G^{\mathrm{reg}}$.

In order to find $a_1 \in \mathcal{M}(\Sigma)$, we note that

$$\mathcal{M}(\tilde{\Sigma})_{[1],\ldots,[1]}^{[1],\ldots,[1]} = \mathcal{M}(\tilde{\Sigma}'),$$

and so

$$\mathcal{M}(\tilde{\Sigma})_{[1],\ldots,[1]}^{[1],\ldots,[1]}(\mathrm{irr}) = \mathcal{M}(\tilde{\Sigma}')(\mathrm{irr}).$$

Now we know that $\mathcal{M}(\tilde{\Sigma}')(\mathrm{irr}) \neq \emptyset$, since the connected components of $\tilde{\Sigma}'$ are assume to have genus greater than or equal to two. (See e.g. proposition 7.7 in [40].) So we see that there is an $a_1 \in \mathcal{M}(\Sigma)$ such that

$$r(a_1) \in \mathcal{M}(\tilde{\Sigma})_{\cdots}^{\cdots}(\mathrm{irr}).$$

Suppose $c \in \times_m C_G^{\mathrm{reg}}$. Because G is assumed to be simple, we have that the image of the map

$$[\cdot,\cdot] : G \times G \to G$$

contains an open neighbourhood of the identity. From this it follows, that if we choose $c \in \times_m C_G^{\mathrm{reg}}$ sufficiently close to $([1], \ldots, [1]) \in \times_M C_G$, we can, since the genus of each of the connected components of $\tilde{\Sigma}'$ is positive, find an $a_2 \in \mathcal{M}(\Sigma)$ such that $h(a_2) = c$ by gluing connections with appropriate holonomies on each of the components of $\tilde{\Sigma}$ together to a connection representing a_2 on Σ.

$\diamond$

One easily sees that $\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$ is always non-empty for the group $SU(2)$, since any to elements in $SU(2)$ which are not in the same maximal torus generate $SU(2)$. (Note that non of the curves in L_F are homotopic, so we cannot have that a component of $\tilde{\Sigma}$ is a cylinder).

We shall for the rest of chapter 2 and in section 3.7 restrict attention to the case, where $\mathcal{M}(\Sigma)_{LF,\mathrm{reg}}$ is non-empty.

Note that $\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}} \subset \mathcal{M}(\Sigma)^{\mathrm{irr}}$.

Example: Let P be a $SU(2)$-bundle over Σ and $\mathcal{M}(\Sigma)$ the moduli space of flat connections in P. (The case of flat SO(3) connections in a nontrivial SO(3)-bundle is examined in detail in [15].) Let F be a Strebel measured foliation, which is equivalent to $[\gamma]$, where γ is a nonseparating simple closed curve on Σ. One can identify the space $C_{SU(2)}$ of conjugacy classes of $SU(2)$ with the unit interval, by mapping

$$\left[\begin{pmatrix} e^{\pi i t} & 0 \\ 0 & e^{-\pi i t} \end{pmatrix}\right] \in C_{SU(2)}$$

to $t \in [0,1]$. We therefore get a map

$$h : \mathcal{M}(\Sigma) \to [0,1],$$

and the projections

$$\pi_t : \mathcal{M}(\Sigma)_t \to \mathcal{M}(\tilde{\Sigma})_t^t.$$

Let us now study the fibres $Z(\tilde{A})/\Gamma_{\tilde{A}}$ for $[\tilde{A}] \in \mathcal{M}(\tilde{\Sigma})^t_t$. Since $\Gamma_{\tilde{A}}$ is the centralizer of the holonomy group of $\tilde{A}$, we see that if $\Gamma_{\tilde{A}} = SU(2)$, then $t = \pm 1$.

Case $t = \pm 1$: We then see that $Z(\tilde{A}) = SU(2)$. $\mathcal{M}(\tilde{\Sigma})^t_t$ is stratified by the following sets.

$$\mathcal{M}(\tilde{\Sigma})^t_t = \mathcal{M}(\tilde{\Sigma})^{\mathrm{irr}}_{t,t} \cup \mathcal{M}(\tilde{\Sigma})^{\mathrm{ab}}_{t,t} \cup \mathcal{M}(\tilde{\Sigma})^{\mathrm{sing}}_{t,t},$$

where

$$\mathcal{M}(\tilde{\Sigma})^{\mathrm{irr}}_{t,t} = \{[\tilde{A}] \in \mathcal{M}(\tilde{\Sigma})^t_t | \Gamma_{\tilde{A}} = Z(SU(2))\},$$

$$\mathcal{M}(\tilde{\Sigma})^{\mathrm{ab}}_{t,t} = \{[\tilde{A}] \in \mathcal{M}(\tilde{\Sigma})^t_t | \Gamma_{\tilde{A}} = U(1)\},$$

$$\mathcal{M}(\tilde{\Sigma})^{\mathrm{sing}}_{t,t} = \{[\tilde{A}] \in \mathcal{M}(\tilde{\Sigma})^t_t | \Gamma_{\tilde{A}} = SU(2)\}.$$

Now over the irreducible part $\mathcal{M}(\tilde{\Sigma})^{\mathrm{irr}}_{t,t}$, we have $\Gamma_{\tilde{A}} = Z(SU(2))$, so $\pi_t^{-1}(\mathcal{M}(\tilde{\Sigma})^{\mathrm{irr}}_{t,t})$ is simply a $SU(2)$ fibre bundle over $\mathcal{M}(\tilde{\Sigma})^{\mathrm{irr}}_{t,t}$. Over $\mathcal{M}(\tilde{\Sigma})^{\mathrm{ab}}_{t,t}$, we see that $\pi^{-1}(\mathcal{M}(\tilde{\Sigma})^{\mathrm{ab}}_{t,t})$ is a $SU(2)/U(1) = S^2$ bundle. $\mathcal{M}(\tilde{\Sigma})^{\mathrm{sing}}_{t,t}$ is a discrete set, and the fibres of π_t are $C_{SU(2)} \cong [0,1]$.

Case $t \in (0,1)$: Here we have that $Z(\tilde{A}) = U(1)$. $\mathcal{M}(\tilde{\Sigma})^t_t$ is now stratified by

$$\mathcal{M}(\tilde{\Sigma})^t_t = \mathcal{M}(\tilde{\Sigma})^{\mathrm{irr}}_{t,t} \cup \mathcal{M}(\tilde{\Sigma})^{\mathrm{ab}}_{t,t},$$

since $\mathcal{M}(\tilde{\Sigma})^{\mathrm{sing}}_{t,t}$ is empty. Over $\mathcal{M}(\tilde{\Sigma})^{\mathrm{ab}}_{t,t}$, $\Gamma_{\tilde{A}} = U(1)$ acts trivially on $Z(\tilde{A})$, and so we see that $\pi_t^{-1}(\mathcal{M}(\tilde{\Sigma})^t_t)$ is a U(1) bundle over $\mathcal{M}(\tilde{\Sigma})^t_t$.

In the next section, we shall see how a metric with infinite ends on $\tilde{\Sigma}$ induces a complex structure on $\mathcal{M}(\tilde{\Sigma})^c_c$, and so, combined with this isotropic distribution, induces a singular non-negative polarization on $\mathcal{M}(\Sigma)$.

Let us end this section, by considering a degenerate case of the situation considered above. For very special Strebel measured foliations, we do get a singular polarization on the $SU(2)$-moduli space. Suppose Σ is a closed surface of genus g. Let $D = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be $3g-3$ simple curves on Σ, which are pairwise disjoint, pairwise not homotopic and not null homotopic. That is the maximal number of such curves, and the connected components of the complement of the union of the curves must be homeomorphic to S^2 with 3 disjoint discs removed. We thus see that D is a "pair of pants" decomposition of Σ.

As it was explained in section 2.1.1 in the remarks after Theorem 2.1.1, D is an element of S' and so defines a point $[D]$ on the Thurston boundary of Teichmüller space. Let F be a measured foliation representing the point $[D]$. Now let P be a principal $SU(2)$-bundle over Σ. If A is a flat connection in P, then associated to $[D]$ we have the basic cohomology $H^1_F(\Sigma, d_A) \subset H^1(\Sigma, d_A)$. We want to give an alternative description of $H^1_F(\Sigma, d_A)$. Let $\mathcal{M}(\Sigma)$ be the moduli space of flat $SU(2)$-connections in P:

$$\mathcal{M}(\Sigma) = \mathrm{hom}(\pi_1(\Sigma), \mathrm{SU}(2))/\mathrm{SU}(2).$$

Following Goldman (See [21]), we define $f_i \in C^\infty(\mathcal{M}(\Sigma))$, $i = 1, \ldots, 3g-3$ by

$$f_i([\phi]) = f(\phi(\alpha_i)),\ [\phi] \in \mathcal{M}(\Sigma),$$

where $f\colon SU(2) \to \mathbb{R}$ is given by

$$f(C) = 2\,\mathrm{Re}\,\mathrm{Tr}(C),\ \forall C \in SU(2).$$

By using f to identify $C_{SU(2)}$ with $[-1,1]$, we see that

$$h = (f_1, \ldots, f_{3g-3}) : \mathcal{M}(\Sigma) \to [-1,1]^{3g-3}.$$

From lemma 2.4.1, it follows that the leaves of h are isotropic. For each pair of pants with boundary $\gamma_i \cup \gamma_j \cup \gamma_k$, we have that the image of (f_i, f_j, f_k) is a tetrahedron inside $[-1,1]^3$, and so the interior of the image of h is an open subset of $[-1,1]^{3g-3}$. From this, one concludes that the fibres of h over the interior of $\mathrm{Im}\,h$ are Lagragian. Hence, we see that the basic cohomology of $[D]$ induced a real polarization on the $SU(2)$-moduli space.

This real polarization was first constructed by J. Weitsman in [49]. We have here given an alternative proof, that the fibres of h are isotropic.

2.5 Construction of non-negative polarizations on the moduli spaces.

Let Σ be a closed surface. Suppose F is a Strebel measured foliation on Σ. Let L_F be a cylindrical leaf choice. Let $\tilde{g}$ be a metric on $\tilde{\Sigma} = \Sigma - L_F$. Let U_F be a closed neighbourhood of L_F, and set $\tilde{U_F} = U_F - L_F$. We shall require each of the connected components of $\tilde{U_F}$ to be conformally equivalent to the standard semi-infinite straight cylinder $S^1 \times [0, \infty)$. We shall say such a metric has "infinite ends". In fact throughout this section, we will assume that our metric $\tilde{g}$ is isometric to the standard semi-infinite cylinder over the ends of $\tilde{\Sigma}$. The non-negative polarization we shall construct from such a metric only depends on the underlying conformal structure of $\tilde{g}$. However, for proving that this polarization is integrable, we use this special metric and the associated weighted Sobolev spaces. Let G be a compact simply connected Lie-group and P a principal G-bundle over Σ. We shall choose a G-bi-invariant inner product B on the Lie-algebra of G. Suppose that A is a flat connection in P. This then induces a covariant derivative d_A in $\operatorname{ad} P$. Inside the first cohomology group with coefficients in $\operatorname{ad} P$, $H^1(\Sigma, d_A)$, we have the basic cohomology $H^1_F(\Sigma, d_A)$. The basic cohomology is an isotropic subspace with respect to the natural symplectic structure ω on $H^1(\Sigma, d_A)$. Let us denote the quotient space

$$V(A) = \frac{H^1_F(\Sigma, d_A)^0}{H^1_F(\Sigma, d_A)}.$$

Since $H^1_F(\Sigma, d_A)$ is isotropic, ω induces a symplectic structure on $V(A)$.

We shall see that $\tilde{g}$ induces a complex structure compatible with the symplectic structure on $V(A)$.

Before we can describe the complex structure, we need to study the covariant derivative over the semi-infinite ends of $\tilde{\Sigma}$.

So, for a moment, we shall just consider the semi-infinite cylinder $C = S^1 \times [0, \infty)$. Over C, we shall consider a principal G-bundle P with a connection A in P over C. We have a projection $\pi : C \to S^1$, and we will assume that a trivialization $P \cong \pi^*(P|_{S^1})$ has been chosen. Moreover, we assume that $A = \pi^*(A|_{S^1})$. In coordinates (θ, r) on the cylinder, the $*$-operator associated to the standard metric is determined by

$$*d\theta = dr, \quad * \, dr = -d\theta.$$

We define d_A^* by

$$d_A^* = - * d_A * .$$

We shall now consider the operator

$$d_A + d_A^* : \Omega^0(C, \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^2(C, \operatorname{ad} P \otimes \mathbb{C}) \to \Omega^1(\Sigma, \operatorname{ad} P \otimes \mathbb{C}).$$

Let $f \in \Omega^0(C, \mathrm{ad}\, P \otimes \mathbb{C})$. We then get

$$d_A f = \frac{\partial f}{\partial r} dr + (\frac{\partial f}{\partial \theta} + [A, f]) d\theta.$$

If $\omega d\theta \wedge dr \in \Omega^2(C, \mathrm{ad}\, P \otimes \mathbb{C})$, we then get

$$d_A^*(\omega d\theta \wedge dr) = -\frac{\partial \omega}{\partial r} d\theta + (\frac{\partial \omega}{\partial \theta} + [A, \omega]) dr.$$

We thus get the following operator

$$\begin{aligned}(d_A + d_A^*)(f, \omega d\theta \wedge dr) & \\ &= (\frac{\partial f}{\partial r} + (\frac{\partial \omega}{\partial \theta} + [A, \omega]))dr \\ &\quad -(\frac{\partial \omega}{\partial r} - (\frac{\partial f}{\partial \theta} + [A, f]))d\theta.\end{aligned}$$

Consider the isomorphism

$$\Omega^1(C, \mathrm{ad}\, P \otimes \mathbb{C}) \to \Omega^0(C, \mathrm{ad}\, P \otimes \mathbb{C}) \oplus \Omega^0(C, \mathrm{ad}\, P \otimes \mathbb{C}),$$

given by

$$\varphi_1 dr + \varphi_2 d\theta \mapsto (\varphi_1, \varphi_2),$$

and the isomorphism

$$\Omega^0(C, \mathrm{ad}\, P \otimes \mathbb{C}) \to \Omega^2(C, \mathrm{ad}\, P \otimes \mathbb{C})$$

given by

$$\omega \mapsto \omega d\theta \wedge dr.$$

By using these two isomorphisms we get that

$$d_A + d_A^* = \frac{\partial}{\partial r} + \begin{pmatrix} 0 & -(\frac{\partial}{\partial \theta} + [A, \cdot]) \\ (\frac{\partial}{\partial \theta} + [A, \cdot]) & 0 \end{pmatrix}.$$

Now

$$D_A = \begin{pmatrix} 0 & -(\frac{\partial}{\partial \theta} + [A, \cdot]) \\ (\frac{\partial}{\partial \theta} + [A, \cdot]) & 0 \end{pmatrix}$$

is a self-adjoint first order elliptic operator over the compact manifold S^1. From general theory (see e.g. lemma 1.6.3 p. 43 in [20]) we know there exists a complete orthonormal basis $\{\varphi_{\lambda_\lambda} | \lambda \in \Lambda\}$ for $L_2(S^1, (\mathrm{ad}\, P \oplus \mathrm{ad}\, P) \otimes \mathbb{C})$ of eigenvectors for D_A indexed by the set Λ of real eigenvalues of D_A. We shall use the following notation $\Lambda_+ = \{\lambda \in \Lambda | \lambda > 0\}$ and $\Lambda_- = \{\lambda \in \Lambda | \lambda < 0\}$. The eigenvectors φ_λ, $\lambda \in \Lambda$ are smooth. The absolute values of the non-zero eigenvalues are bounded below by a positive constant, and they have ∞ as the only accumulation point. The analysis of [6] now applies, when we work with the weighted

Sobolev spaces (which we shall define shortly). We will here repeat the argument, with emphasis on the changes that are needed.

Suppose $\psi \in \Omega^1(C, \operatorname{ad} P \otimes \mathbb{C})$. We then want to solve the equation

$$(d_A + d_A^*)\phi = \psi, \tag{2.6}$$

where $\phi \in \Omega^0(C, \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^2(C, \operatorname{ad} P \otimes \mathbb{C})$. We write

$$\phi = \sum_{\lambda \in \Lambda} f_\lambda \varphi_\lambda, \ \psi = \sum_{\lambda \in \Lambda} g_\lambda \varphi_\lambda.$$

Now the equation (2.6) becomes

$$f'_\lambda + \lambda f_\lambda = g_\lambda.$$

We can explicitly write down the following solution

$$f_\lambda(u) = \int_0^u e^{\lambda(v-u)} g_\lambda(v) dv \ \ \lambda \in \Lambda - \Lambda_- \tag{2.7}$$

$$- \int_u^\infty e^{\lambda(v-u)} g_\lambda(v) dv \ \ \lambda \in \Lambda_-. \tag{2.8}$$

Define a linear map

$$Q : \Omega_c^1(C, \operatorname{ad} P \otimes \mathbb{C}) \to \Omega^0(C, \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^2(C, \operatorname{ad} P \otimes \mathbb{C})$$

by

$$Q(\psi) = \sum_{\lambda \in \Lambda} f_\lambda \phi_\lambda$$

where f_λ is define by (2.7) or (2.8). Then we get formally that

$$(d_A + d_A^*) Q\psi = \psi \ \forall \psi \in \Omega_c^1(C, \operatorname{ad} P \otimes \mathbb{C}),$$

and

$$Q(d_A + d_A^*)\phi = \phi,$$

for all $\phi = \sum_{\lambda \in \Lambda} f_\lambda \varphi_\lambda$ with the following properties

$$f_\lambda(0) = 0 \ \forall \lambda \in \Lambda - \Lambda_-,$$

f_0 is smooth and constant in a neighbourhood of ∞, and $f_\lambda \in C_c^\infty([0, \infty))$ for $\lambda \in \Lambda_-$.

We shall now introduce certain weighted Sobolev spaces, which will allow us to show that Q in fact define a bounded operator between some of these weighted spaces, for particular choice of the weights. Let $\epsilon \in \mathbb{R}$. We consider the following norms:

$$|\psi|^2_{\epsilon,k} = \int_C \sum_{0 \le l \le k} |\nabla^l (e^{\epsilon u} \psi(u, \theta))|^2 du d\theta,$$

for $\psi \in \Omega^i_c(C, \mathrm{ad}\, P \otimes \mathbb{C})$. We shall denote the completion of $\Omega^i_c(C, \mathrm{ad}\, P \otimes \mathbb{C})$ in the norm $|\cdot|_{\epsilon,k}$ by $\Omega^i_{\epsilon,k}(C, \mathrm{ad}\, P \otimes \mathbb{C})$. Denote by N_A the kernel

$$N_A = \left\{ \varphi \in \Omega^0(S^1, \mathrm{ad}\, P \otimes \mathbb{C}) \,\middle|\, \frac{\partial \varphi}{\partial \theta} + [A, \varphi] = 0 \right\}. \tag{2.9}$$

Fix a function ρ with support in $(0, \infty)$ which is constant one near ∞. We define subspaces $\Omega^i_{N_A}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \subset \Omega^i(C, \mathrm{ad}\, P \otimes \mathbb{C})$ $(i = 0, 2)$, which consist of the $\phi \in \Omega^i(C, \mathrm{ad}\, P \otimes \mathbb{C})$ for which there exists $\phi_\infty \in N_A$, such that

$$\phi - \rho\phi_\infty \in \Omega^i_c(C, \mathrm{ad}\, P \otimes \mathbb{C}).$$

We define $\Omega^1_{N_A}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \subset \Omega^1(C, \mathrm{ad}\, P \otimes \mathbb{C})$ to be the $\phi \in \Omega^1(C, \mathrm{ad}\, P \otimes \mathbb{C})$ for which there exists $\varphi_\infty \in N_A \oplus N_A$ such that

$$\varphi - \rho\varphi_\infty \in \Omega^1_c(C, \mathrm{ad}\, P \otimes \mathbb{C}).$$

On $\Omega^i_{N_A}(C, \mathrm{ad}\, P \otimes \mathbb{C})$ we consider the norm

$$|\phi|^2_{\epsilon,k,\infty} = \int_C \sum_{0 \le l \le k} |\nabla^l (e^{\epsilon u}(\phi(u,\theta) - \rho\phi_\infty(\theta)))|^2 du d\theta + \int_{S^1} |\phi_\infty(\theta)|^2 d\theta.$$

The completion of $\Omega^i_{N_A}(C, \mathrm{ad}\, P \otimes \mathbb{C})$ in the norm $|\cdot|_{\epsilon,k,\infty}$ is denoted $\Omega^i_{\epsilon,k,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C})$.

In order to establish some norm estimates, we rewrite (2.7) and (2.8) in the form

$$\begin{aligned} e^{\epsilon u} f_\lambda(u) \quad &= \quad \int_0^u e^{(\lambda-\epsilon)(v-u)} g_\lambda(v) e^{\epsilon v} dv \;\; \lambda \in \Lambda_+ \\ &\quad - \int_u^\infty e^{(\lambda-\epsilon)(v-u)} g_\lambda(v) e^{\epsilon v} dv \;\; \lambda \in \Lambda_-, \end{aligned}$$

and

$$e^{\epsilon u}(f_0(u) - f_0(\infty)) = -\int_u^\infty e^{-\epsilon(v-u)} g_0(v) e^{\epsilon v} dv.$$

If we choose an ϵ such that $0 < \epsilon < \min_{\lambda \in \Lambda_+} \lambda$, we get from the analysis in [6] on pp. 49-50, that

$$|\lambda - \epsilon|^p \parallel \frac{\partial^q (e^{\epsilon u} f_\lambda(u))}{\partial u^q} \parallel \le 3|\lambda-\epsilon|^{p+q+1} \parallel g_\lambda(u) e^{\epsilon u} \parallel + \sum_{r=1}^{q-1} |\lambda-\epsilon|^{p+q-r-1} \parallel \frac{\partial}{\partial u^r}(g_\lambda(u) e^{\epsilon u}) \parallel \;\; \lambda \ne 0,$$

and

$$|\epsilon|^p \parallel \frac{\partial^q (e^{\epsilon u}(f_0(u) - f_0(\infty)))}{\partial u^q} \parallel \le 3|\epsilon|^{p+q+1} \parallel g_0(u) e^{\epsilon u} \parallel + \sum_{r=1}^{q-1} |\epsilon|^{p+q-r-1} \parallel \frac{\partial}{\partial u^r}(g_0(u) e^{\epsilon u}) \parallel .$$

Note that

$$|f_0(\infty)|^2 = \left| \int_0^\infty g_0(v) dv \right| \le |\psi|^2_{\epsilon,k}.$$

From these estimates, it follows that

$$|Q\psi|_{\epsilon,k+1,\infty} \leq C|\psi|_{\epsilon,k}.$$

Hence we see that

$$Q : \Omega^1_{\epsilon,k}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \to \Omega^0_{\epsilon,k+1,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k+1,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C})$$

is continuous. Moreover, we get that

$$(d_A + d_A^*)Q\psi = \psi \;\forall \psi \in \Omega^1_{\epsilon,k}(C, \mathrm{ad}\, P \otimes \mathbb{C}), \tag{2.10}$$

and

$$Q(d_A + d_A^*)\phi = \phi,$$

for all $\phi \in \Omega^0_{\epsilon,k+1,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k+1,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C})$ such that

$$\phi|_{S^1\times\{0\}} = 0.$$

Similarly we can consider

$$d_A^* + d_A : \Omega^1_{\epsilon,k,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \to \Omega^0_{\epsilon,k-1}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k-1}(C, \mathrm{ad}\, P \otimes \mathbb{C})$$

and the same analysis shows that there exists

$$\tilde{Q} : \Omega^0_{\epsilon,k-1}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k-1}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \to \Omega^1_{\epsilon,k,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C})$$

such that

$$(d_A^* + d_A)\tilde{Q}\phi = \phi \;\forall \phi \in \Omega^0_{\epsilon,k+1,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k+1,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C}), \tag{2.11}$$

and

$$Q(d_A^* + d_A)\psi = \psi,$$

for all $\psi \in \Omega^1_{\epsilon,k}(C, \mathrm{ad}\, P \otimes \mathbb{C})$ such that

$$\psi|_{S^1\times\{0\}} = 0.$$

Let us now return to the situation considered in the beginning of this section. The aim is to define weighted Sobolev spaces on the surface $\tilde{\Sigma}$, which locally on the ends of $\tilde{\Sigma}$ are equivalent to the ones defined on the semi-infinite cylinders C. In order to do this, we need to fix trivializations of P over the ends of $\tilde{\Sigma}$, i.e. over the ends $S^1 \times [0, \infty)$, we have a projection π onto the S^1-component, and we assume that a trivialisation $P \cong \pi^*(P|_{S^1})$ has been chosen. Moreover, we assume that $A = \pi^*(A|_{S^1})$. If we start with a connection A in

P over Σ we can always find a trivialisation of P near L_F such that the restriction of A to $\tilde{\Sigma}$ is of the required form on the ends.

Now for each connected component $\tilde{\Sigma}_i$ of $\tilde{\Sigma}$, choose a point $p_i \in \tilde{\Sigma}_i$, and let d be the function on $\tilde{\Sigma} = \cup_i \tilde{\Sigma}_i$, which measures the distance from any point $p \in \tilde{\Sigma}_i$ to $p_i \in \tilde{\Sigma}_i$. Fix a function ρ on $\tilde{\Sigma}$ with support on the cylinders and such that ρ is 1 near the ends of the cylinders. Consider the spaces

$$\begin{aligned}\Omega^i_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \;=\; & \Big\{ f \in \Omega^i_{L^2_{loc}}(\Sigma, \operatorname{ad} P \otimes \mathbb{C}) \Big| \\ & \exists f_\infty \in H^0(L_F, d_A|_{L_F}) : \sum_{0 \le l \le k} \int_{\tilde{\Sigma}} |\nabla^l (e^{\epsilon d}(f - \rho f_\infty))|^2 + \int_{\partial \tilde{\Sigma}} |f_\infty|^2 < \infty \Big\}\end{aligned}$$

for $i = 0, 2$,

$$\begin{aligned}\Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \;=\; & \Big\{ f \in \Omega^i_{L^2_{loc}}(\Sigma, \operatorname{ad} P \otimes \mathbb{C}) \Big| \;\; \exists f^1_\infty, f^2_\infty \in H^0(L_F, d_A|_{L_F}) : \\ & \sum_{0 \le l \le k} \int_{\tilde{\Sigma}} |\nabla^l (e^{\epsilon d}(f - \rho(f^1_\infty \oplus f^2_\infty)))|^2 + \int_{\partial \tilde{\Sigma}} |f^1_\infty \oplus f^2_\infty|^2 < \infty \Big\}\end{aligned}$$

and

$$\Omega^i_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) = \left\{ \varphi \in \Omega^i_{L^2_{loc}}(\Sigma, \operatorname{ad} P \otimes \mathbb{C}) \;\middle|\; \sum_{0 \le l \le k} \int_{\tilde{\Sigma}} |e^{\epsilon d} \nabla^l \varphi|^2 < \infty \right\}.$$

The flat connection A in P over Σ gives us a covariant derivative in $\operatorname{ad} P \otimes \mathbb{C}$ over $\tilde{\Sigma}$, and we get a complex

$$0 \to \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \xrightarrow{d_A} \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \xrightarrow{d_A} \Omega^2_{\epsilon,k-1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \to 0.$$

We shall denote the first cohomology group of this complex by

$$H^1_{\epsilon,k}(\tilde{\Sigma}, d_A) = \frac{\ker d_A}{\operatorname{Im} d_A}.$$

By using the Hodge $*$-operator associated to the metric $\tilde{g}$ on $\tilde{\Sigma}$ we can consider

$$d_A^* = - * d_A *$$

on $\Omega^i(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$. Now consider the operator

$$d_A + d_A^* : \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \to \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}).$$

By using the operator Q on each of the cylinders and a partition of unity argument, we can now construct a continuous operator

$$P : \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \to \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}),$$

provided ϵ is positive and sufficiently small. By the construction of P we have that the operators

$$P(d_A+d_A^*)-I:\Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma},\operatorname{ad}P\otimes\mathbb{C})\oplus\Omega^2_{\epsilon,k+1,\infty}(\tilde{\Sigma},\operatorname{ad}P\otimes\mathbb{C})\hookleftarrow$$

and

$$(d_A+d_A^*)P-I:\Omega^1_{\epsilon,k}(\tilde{\Sigma},\operatorname{ad}P\otimes\mathbb{C})\hookleftarrow$$

have smooth kernels with compact support,. Hence they are compact operators on the two Hilbert spaces. Hence we conclude that $d_A+d_A^*$ is a Fredholm operator.

By the same construction, one finds that

$$d_A^*+d_A:\Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma},\operatorname{ad}P\otimes\mathbb{C})\to\Omega^0_{\epsilon,k-1}(\tilde{\Sigma},\operatorname{ad}P\otimes\mathbb{C})\oplus\Omega^2_{\epsilon,k-1}(\tilde{\Sigma},\operatorname{ad}P\otimes\mathbb{C})$$

is Fredholm for ϵ sufficiently small and positive.

Over the cylinder C, we can rather explicitly describe the elements of $\ker(d_A^*+d_A)$. Suppose $\varphi\in\Omega^1_{\epsilon,k}(\tilde{\Sigma},\operatorname{ad}P\otimes\mathbb{C})$, and say

$$\varphi=\sum_{\lambda\in\Lambda}h_\lambda\varphi_\lambda.$$

The equation

$$(d_A^*+d_A)\varphi=0$$

then becomes

$$h_\lambda^{'}-\lambda h_\lambda=0\ \forall\lambda\in\Lambda.$$

So from this we see that

$$h_\lambda=\begin{cases}c_\lambda e^{-\lambda r} & \lambda\in\Lambda_+\\ 0 & \lambda=0\\ c_\lambda e^{\lambda r} & \lambda\in\Lambda_-\end{cases},$$

hence

$$\varphi=\sum_{\lambda\in\Lambda_-}c_\lambda e^{\lambda r}\varphi_\lambda+\sum_{\lambda\in\Lambda_+}c_\lambda e^{-\lambda r}\varphi_\lambda.$$

Note that the L^2-kernel of $d_A^*+d_A$ is the same as the $\Omega^1_{\epsilon,k}$-kernel of $d_A^*+d_A$, provided ϵ is sufficiently small and positive. Moreover, the extended L^2-kernel of $d_A^*+d_A$ is the same as the $\Omega^1_{\epsilon,k,\infty}$-kernel of $d_A^*+d_A$ again provided ϵ is sufficiently small and positive. We shall use this explicit expression for elements of $\ker(d_A^*+d_A)$ in the proof of lemma 2.5.3. The following lemmas will help us establish a Hodge theory for these waited Sobolev spaces.

Lemma 2.5.1 *Any element of $H^1_{\epsilon,k}(\tilde{\Sigma},d_A)$ can be represented by a d_A−closed form on $\tilde{\Sigma}$ with compact support.*

Proof. We consider the self-adjoint first order elliptic operator $E_A = i(\frac{\partial}{\partial\theta} + [A, \cdot])$ on the compact manifold S^1. From general theory we know there exists a complete orthonormal basis $\{\phi_\gamma | \gamma \in \Gamma\}$ for $L_2(S^1, \mathrm{ad}\, P \otimes \mathbb{C})$ of eigenvectors for E_A index by the set Γ of real eigenvalues of E_A. Now let $\varphi \in \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \mathrm{ad}\, P \otimes \mathbb{C})$ be a d_A-closed form representing an element of $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$. Let $\psi = \varphi|_C$. We then have in the coordinates (θ, r) on C, say

$$\psi = \sum_{\gamma\in\Gamma} h_\gamma(r)\phi_\gamma d\theta + \sum_{\gamma\in\Gamma} g_\gamma(r)\phi_\gamma dr.$$

Now compute that

$$d_A\psi = \sum_{\gamma\in\Gamma}(h'_\gamma(r) - g_\gamma(r)i\gamma)\phi_\gamma dr \wedge d\theta = 0,$$

hence we have

$$h'_\gamma(r) = g_\gamma(r)i\gamma.$$

Suppose we have that

$$f = \sum_{\gamma\in\Gamma} f_\gamma(r)\phi_\gamma.$$

Then

$$d_A f = \sum_{\gamma\in\Gamma} f'_\gamma(r)\phi_\gamma dr + \sum_{\gamma\in\Gamma} f_\gamma(r)i\gamma\phi_\gamma d\theta.$$

Now, since $\psi \in \Omega^1_{\epsilon,k}(C, \mathrm{ad}\, P \otimes \mathbb{C})$, we conclude that $h_o(r) = 0$. Define

$$f = \sum_{\gamma\in\Gamma-\{0\}} \frac{1}{i\gamma} h_\gamma(r)\phi_\gamma + \int_0^r g_0(s)ds\phi_0.$$

We then have that $d_A f = \psi$. Moreover, since $\psi \in \Omega^1_{\epsilon,k}(C, \mathrm{ad}\, P \otimes \mathbb{C})$ we can conclude that $f \in \Omega^0_{\epsilon,k+1,\infty}(C, \mathrm{ad}\, P \otimes \mathbb{C})$. Now let ρ be a function on $[0, \infty)$ which is zero in a neighbourhood of 0 and one in a neighbourhood of ∞. Then $\varphi' = \varphi - d_A(\rho f)$ has compact support and

$$[\varphi'] = [\varphi] \in H^1_{\epsilon,k}(\tilde{\Sigma}, d_A).$$

$\diamond$

Let us now consider

$$\mathrm{coker}(d_A^* + d_A) = \frac{\Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \mathrm{ad}\, P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \mathrm{ad}\, P \otimes \mathbb{C})}{\mathrm{Im}(d_A^* + d_A : \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \mathrm{ad}\, P \otimes \mathbb{C}) \to \cdot)}$$

Lemma 2.5.2 *Any element of* $\mathrm{coker}(d_A^* + d_A)$ *can be represented by an element of* $\Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \mathrm{ad}\, P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \mathrm{ad}\, P \otimes \mathbb{C})$ *with compact support.*

Proof. Let $\varphi \in \Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$ represent an element of $\operatorname{coker}(d_A^* + d_A)$. Let $\phi = \varphi|_C$ (C being a tubular neighbourhood of the boundary of $\tilde{\Sigma}$). We then have by (2.11) that

$$(d_A^* + d_A)\tilde{Q}\phi = \phi.$$

Now let ρ be a function on $\tilde{\Sigma}$, which is constant 1 near the boundary of $\tilde{\Sigma}$ and with support in C. Then $\rho\tilde{Q}\phi \in \Omega^0_{\epsilon,k+1,\infty}(C, \operatorname{ad} P \otimes \mathbb{C})$, and

$$\tilde{\varphi} = \varphi - (d_A^* + d_A)\rho\tilde{Q}\phi$$

has compact support on $\tilde{\Sigma}$.

$\diamond$

Let us now establish a Hodge-theory for these weighted spaces.

Lemma 2.5.3 *We have a natural isomorphism*

$$H^1_{\epsilon,k}(\tilde{\Sigma}, d_A) \cong \ker(d_A^* + d_A : \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \to \cdot).$$

Proof. Since we have that

$$\operatorname{Im}(d_A) \perp \ker(d_A^*)$$

in the continuous L^2-inner product on $\Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P)$, we see that there is an injective map

$$i : \ker(d_A^* + d_A) \to H^1_{\epsilon,k}(\tilde{\Sigma}, d_A).$$

Suppose $[\varphi] \in H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$. We know by lemma 2.5.1 that we can choose $\varphi \in \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$ d_A-closed and with compact support. By general theory (see [41] p. 169 or [6] proof of proposition (4.9) and comment (3) p. 408 in [7]) there exists $\phi \in \ker(d_A^* + d_A)$ and a distributional section f of $\operatorname{ad} P \otimes \mathbb{C}$ over $\tilde{\Sigma}$, such that

$$\varphi - \phi = d_A f.$$

Now on the complement of the support of φ, we have that $\phi = d_A f$. By the explicit expressions over the ends of elements in $\ker(d_A^* + d_A)$, we see that f is smooth on the compliment of the support of φ, and that f has finite $|\cdot|_{\epsilon,k-1,\infty}$-norm over the ends of the surface.

Now let ρ be a function with compact support on $\tilde{\Sigma}$, such that ρ is constant 1 on the support of φ. We then get that

$$\rho\varphi - \rho\phi + d(\rho)f = d_A(\rho f) = (d_A + d_A^*)(\rho f, 0).$$

By elliptic regularity of $d_A + d_A^*$ on the double of $\tilde{\Sigma}$ we see that ρf is smooth. The two observations about f then imply that $f \in \Omega^0_{\epsilon,k-1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$, and so

$$[\varphi] = [\phi] \in H^1_{\epsilon,k}(\tilde{\Sigma}, d_A).$$

Thus i is surjective.

$\diamond$

Lemma 2.5.4 *We have a natural isomorphism*

$$\operatorname{coker}(d_A^* + d_A) \cong \ker(d_A + d_A^* : \Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}) \to \cdot).$$

Proof. The proof is completely analoguous to the proof of lemma 2.5.3.

$\diamond$

With these analytical tools at hand, we can now describe how $\tilde{g}$ induces a complex structure on $V(A)$.

The $*$-operator on $(\tilde{\Sigma}, \tilde{g})$ induces a complex structure on $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$ by the above isomorphism. Note that there is a natural symplectic structure on $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$ induced from the symplectic structure on $\Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$. We shall now see that we can identify $V(A)$ and $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$ as symplectic vector spaces. The complex structure on $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$ hence induces a non-negative Lagrangian subspace of $H^1(\Sigma, d_A)$, by pulling the i-eigenvalues of $*$ on $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$ back to $H^1(\Sigma, d_A)$. We will denote this Lagrangian subspace by

$$P_{F,\tilde{g}}(A) \subset H^1(\Sigma, d_A).$$

Let us now prove the following lemma, which allows us to define $P_{F,\tilde{g}}(A)$.

Lemma 2.5.5 *We have an isomorphism*

$$V(A) \cong H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$$

which identifies $V(A)$ and $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$ as symplectic vector spaces.

Proof. Suppose $\psi \in \Omega^1(\Sigma, \operatorname{ad} P \otimes \mathbb{C})$ is d_A-closed, such that

$$\int_\Sigma B(\psi \wedge \varphi) = 0$$

for all basic 1-forms φ. By the argument in the proof of lemma 2.3.1, we get that if we restrict ψ to a tubular neighbourhood $\tilde{C}$ of L_F, we can find a section $f \in \Omega^0(\tilde{C}, \operatorname{ad} P \otimes \mathbb{C})$ such that

$$d_A f = \psi|_{\tilde{C}}.$$

Let ρ be a bump-function with compact support in $\tilde{C}$ and constant 1 near L_F. Consider

$$\psi' = \psi - d_A(\rho f).$$

It is clear that $\psi' \in \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$, hence ψ' defines a class in $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$. This gives us a well defined map

$$r : H^1_F(\Sigma, d_A)^0 \to H^1_{\epsilon,k}(\tilde{\Sigma}, d_A),$$

which is independent of the choice of ρ.

Suppose $\tilde{\varphi} \in \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$ is d_A-closed. We then know that there exists $f \in \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$ such that

$$\psi = \tilde{\varphi} - d_A f$$

has compact support on $\tilde{\Sigma}$. Hence we see that $[\psi] \in (H^1_F(\Sigma, d_A))^0$, and

$$r([\psi]) = [\tilde{\varphi}].$$

We conclude that r is surjective.

Recall that any element $[\varphi]$ of $H^1_F(\Sigma, d_A)$ can be represented by a d_A-closed 1-form φ, which is given by the expression (2.5). Recall that ρ is defined to be zero on $[0, \frac{1}{2})$ and constant one in a neighbourhood of $\frac{1}{2}$ in $[\frac{1}{2}, 1]$ and constant zero in a neighbourhood of 1 in $[\frac{1}{2}, 1]$ so $\rho(r_\gamma)$ is not continuous on Σ, but we do have that

$$\sum_\gamma \rho(r_\gamma)\xi_\gamma \in \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C}),$$

and

$$\varphi|_{\tilde{\Sigma}} = d_A(\sum_\gamma \rho(r_\gamma)\xi_\gamma).$$

Hence we see that

$$H^1_F(\Sigma, d_A) \subset \ker r.$$

Now suppose $\varphi \in \Omega^1(\Sigma, \operatorname{ad} P \otimes \mathbb{C})$ is a d_A-closed from which represents an element $[\varphi] \in H^1_F(\Sigma, d_A)^0$, such that $r([\varphi]) = 0$. Let $g \in \Omega^0(\tilde{C}, \operatorname{ad} P)$ be such that

$$d_A g = \varphi|_{\tilde{C}}.$$

Let now ρ be a bump-function with compact support in $\tilde{C}$ and constant 1 near L_F. Set

$$\varphi' = \varphi - d_A(\rho g).$$

By the assumption, we have that $[\varphi'] = 0$ in $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$. Hence there exists $f \in \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$ such that

$$d_A f = \varphi'|_{\tilde{\Sigma}}.$$

Then
$$\varphi'' = \varphi' - d_A((1-\rho)f) = \varphi - d_A((1-\rho)f + \rho g)$$
has compact support in $\tilde{C}$. From the Proof of Lemma 2.3.2, we conclude that
$$[\varphi] = [\varphi''] \in H^1_F(\Sigma, d_A).$$
This shows that r induces an isomorphism from $V(A)$ to $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$. It is clear that r is compatible with the two symplectic forms.

$\diamond$

We shall now see that $P_{F,\tilde{g}}(A)$ defines a singular non-negative polarization on $\mathcal{M}(\Sigma)$. Moreover we have that
$$\rho(P_{F,\tilde{g}}) = \# L_F \operatorname{Rank} G,$$
since the real directions are given by the real foliation on $\mathcal{M}(\Sigma)$ described in section 2.4. By the results of section 2.4, we just need to show that $P_{F,\tilde{g}}$ is integrable. We shall establish that $P_{F,\tilde{g}}$ is integrable on the open dense subset $\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$ defined in section 2.4.

Theorem 2.5.1 *$P_{F,\tilde{g}}$ is integrable on the open dense subset $\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$.*

Proof. Let A be a flat connection in P over Σ. Assume that $[A] \in \mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$. We have the identification
$$T_{[A]}\mathcal{M}(\Sigma) \cong H^1(\Sigma, d_A).$$
By the definition of $P_{F,\tilde{g}}$, we have that the basic cohomology is the real directions of $P_{F,\tilde{g}}(A)$
$$P_{F,\tilde{g}}(A) \cap \overline{P_{F,\tilde{g}}(A)} = H^1_F(\Sigma, d_A) \subset H^1(\Sigma, d_A).$$
Choose a conformal structure on Σ. (Note that this choice is only an auxiliary tool in showing that $P_{F,\tilde{g}}$ is integrable.) We then have that $*(H^1_F(\Sigma, d_A))$ will be a complement to $H^1_F(\Sigma, d_A)^0$, i.e.
$$H^1_F(\Sigma, d_A)^0 \oplus *H^1_F(\Sigma, d_A) = H^1(\Sigma, d_A).$$
By using the Coulomb gauge fixing condition, a neighbourhood W of zero in $*H^1_F(\Sigma, d_A)$ will parametrize a set $\{\tilde{A}_t | t \in W\}$ of flat connections near A which will project to a transversal of the foliation $H^1_F(\Sigma, d_A)^0 \subset H^1(\Sigma, d_A)$ near $[A]$ in $\mathcal{M}(\Sigma)$. Let $\tilde{C}$ be a cylindrical neighbourhood of L_F. Let $\tilde{C}_\gamma$ be the connected component of $\tilde{C}$, containing $\gamma \in L_F$.

Let us for a moment digress to study the space of flat connections $\mathcal{A}^F(C)$ in a trivial principal G-bundle P on a cylinder $C = S^1 \times I$. We have a map
$$h_c : \mathcal{A}^F(C) \to C_G,$$

which asssigns to a connection its holonomy around S^1. The moduli space is naturally identified with the space of conjugacy classes:

$$\mathcal{M}(C) = C_G.$$

Let (θ, r) be the standard coordinates on C. Choose a maximal torus T of G. Let $\mathfrak{t}$ be the Lie-algebra of T considered as a Lie-subalgebra of $\mathfrak{g}$, the Lie-algebra of G. Let U be a fundamental domain for the extended Weyl group's action on $\mathfrak{t}$. We can then identify

$$C_G = \overline{U}.$$

Let $\xi \in \overline{U}$. We then get a flat connection

$$A_\xi = \xi d\theta.$$

The space $\mathcal{A}_\xi$ of flat connections which are gauge equivalent to A_ξ, is identified with

$$\mathcal{A}_\xi = \mathcal{G}(C)/Z(\exp \xi).$$

where $\mathcal{G}(C)$ is the gauge group. For $\xi \in U$, we can identify $Z(\exp \xi)$ with T. Then $\mathcal{G}(C) \times U$ is a principal T-bundle over $h_C^{-1}(U) \subset \mathcal{A}^F(C)$.

Let us now return to the study of the polarization $P_{F,\tilde{g}}$ on $\mathcal{M}(\Sigma)$. By restricting the smooth family $\{\tilde{A}_t | t \in W\}$ of flat connections to $\tilde{C}_\gamma$, we get a smooth family of connections in $h_{\tilde{C}_\gamma}^{-1}(U)$. Since $\mathcal{G}(\tilde{C}_\gamma) \times U$ is a principal T-bundle over $h_{\tilde{C}_\gamma}^{-1}(U)$, we can choose a lift $\{g_t^\gamma | t \in W\} \subset \mathcal{G}(\tilde{C}_\gamma)$ of $\{A_t|_{\tilde{C}_\gamma} | t \in W\}$. We now extend the lifts $\{g_t^\gamma | t \in W\}$, $\gamma \in L_F$, to a smooth family of gauge transformations $\{g_t | t \in W\}$ in P (we are using here the fact that G is simply connected). Let $A_t = g_t^*(\tilde{A}_t)$, $t \in W$. By the construction of g_t, we have that

$$A_t|_{\tilde{C}_\gamma} = g_t^*(\tilde{A}_t)|_{\tilde{C}_\gamma} = \xi(t) d\theta,$$

where $\xi(t) \in U$ for all $t \in W$. We shall now consider the smooth family of connections $\{A_t | t \in W\}$, which also projects to a smooth transversal of the foliation $H_F^1(\Sigma, d_A)^0 \subset H^1(\Sigma, d_A)$ through $[A]$ in $\mathcal{M}(\Sigma)$.

By chosing a basis $[\varphi_1], \ldots, [\varphi_m]$ of $H_F^1(\Sigma, d_A)$ such that φ_i is of the form (2.5), we get that $*W \times W$ parametrizes a smooth family $A(s,t)$, $(s,t) \in *W \times W$ near A, by letting

$$A(s,t) = A_t + \sum_{i=1}^m s_i \varphi_i.$$

Note that the flat connections $A(s,t)$, $(s,t) \in *W \times W$ is of a form for which the analysis in the beginning of this section applies. Using the definition 2.9, we see that

$$N_{A(0,0)} = N_{A(s,t)} = N_{A(s,t)+\xi}$$

for any $\xi \in \mathfrak{t}$, hence we can use $A(0,0)$ to define $\Omega^i_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P)$.

Consider now the following map

$$\tilde{F} : *W \times W \times \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P) \to \Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P)$$

given by

$$\tilde{F}(s,t,a) = -i * F(A(s,t) + *a) \oplus F(A(s,t) + a).$$

The derivative of this map in the last variable at $a \in \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P)$ is

$$D_a \tilde{F}_{(s,t,a)} = -i * d_{A(s,t)+a} * + d_{A(s,t)+a}.$$

We note that

$$D_a \tilde{F}_{(s,t,a)}(*\varphi) = i * D_a \tilde{F}_{(s,t,a)}(\varphi).$$

Hence $D_a\tilde{F}$ is complex linear with respect to the complex strutures $*$ on $\Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P)$ and $i*$ on $\Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P)$. Moreover, $D_a\tilde{F}_{(s,t,0)} = -i * d_{A(s,t)} * + d_{A(s,t)}$ is Fredholm. Since $[A] \in \mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$ we conclude that

$$\ker(d_{A(s,t)} : \Omega^0_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P) \to \Omega^1_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P)) = 0,$$

for small s,t. Using the $*$-operator, we conclude that

$$\ker(d_{A(s,t)} + d^*_{A(s,t)} : \Omega^0_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P) \oplus \Omega^2_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P) \to \Omega^1_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P)) = 0$$

for small s,t. By Lemma 2.5.4, we conclude that cokernel of $D_aF_A(s,t,0)$ is zero for small s,t. By (3.25) in [6] p.60 we conclude that we have an exact sequense :

$$0 \to \ker(d^*_{A(s,t)} + d_{A(s,t)} : \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P)) \to \ker(d^*_{A(s,t)} + d_{A(s,t)} : \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P)) \to N_A \oplus N_A \to 0.$$

So if we let

$$H = \ker(d^*_{A(s,t)} + d_{A(s,t)} : \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P) \to \cdot)^\perp \subset \Omega_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P),$$

we get a decomposistion

$$\Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P) = \ker(d^*_{A(s,t)} + d_{A(s,t)} : \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P) \to \cdot) \oplus H.$$

We then have that

$$D_a \tilde{F}_{(0,0,0)} : H \to \Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{ad} P)$$

is an isomorphism. From the implicit function theorem, we then get a smooth map

$$h : *W \times W \times N \to H,$$

where N is a neighbourhood of zero in $\ker(d^*_{A(0,0)} + d_{A(0,0)})$, such that

$$\tilde{F}(s,t,a+h(s,t,a)) = 0 \ \forall (s,t,a) \in *W \times W \times N.$$

Moreover, the map

$$h_{s,t} : N \to \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{ad} P),$$

given by

$$h_{s,t}(a) = h(s,t,a) \ \forall a \in N,$$

will be holomorphic. Now consider

$$N' = N \cap \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{ad} P).$$

We then get a map

$$h : *W \times W \times N' \to \mathcal{A}^F_k(\Sigma),$$

and therefore a map

$$h : *W \times W \times N' \to \mathcal{M}(\Sigma),$$

whose derivative is complex linear in the last variable with respect to $P_{F,\tilde{g}}$. By lemma 2.5.5, we see that h defines a local diffeomorphism, and so if we decompose h^{-1} relative to the decomposition

$$H^1(\Sigma, d_A) \cong *H^1_F(\Sigma, d_A) \oplus H^1_F(\Sigma, d_A) \oplus V(A),$$

to get say

$$h^{-1} = (x_1, \ldots, x_m, y_1, \ldots, y_m, z_{m+1}, \ldots, z_{n-m}),$$

then the Hamiltonian vectorfields of (y,z) trivialize $P_{F,\tilde{g}}$ locally. Hence we conclude that $P_{F,\tilde{g}}$ is integrable on the set $\mathcal{M}(\Sigma)_{L_F,\mathrm{reg}}$.

$\diamond$

If f is an element of the mapping class group, which is reducible periodic (see section 2.1.1), then there will be a Strebel measured foliation F which represents a fixed point for f on the Thurston boundary. It is then possible to choose L_F and $\tilde{g}$ such that they are preserved by f, since we have assumed f is reducible periodic. That means $P_{F,\tilde{g}}$ will be preserved by f. E.g. if f is a product of Dehn-twists around a union L of simple closed curves which do not intersect we can then choose any metric with infinite ends, say $\tilde{g}$, on $\tilde{\Sigma} = \Sigma - L$, and $P_{[L],\tilde{g}}$ will be preserved by f.

2.6 Degeneration of induced Kähler structures on moduli space.

2.6.1 Degeneration of holomorphic forms along "straight rays" in Teichmüller space.

Let P be a principal G-bundle over Σ. Let A be a flat connection in P. Suppose F is a measured foliation on Σ, for which there exists a complex structure ρ_0 on Σ and a holomorphic quadratic differential q, such that the vertical measured foliation associated to q is F. Using the notation from section 2.1 we have the global coordinates

$$\Phi: T \to QD(\rho_0),$$

and by Wolf's work, the ray

$$\rho_t = \Phi^{-1}(tq),\ t \in \overline{\mathbb{R}}_+,$$

will converge to $[F]$ as $t \to \infty$. Let $\{t_n\}$ be a sequence of positive real numbers such that

$$\tau_n = [\rho_{t_n}] \to [F]\ as\ n \to \infty,$$

and

$$H_n = H^{1,0}(\Sigma_{\tau_n}, d_A) \in L(H^1(\Sigma, d_A))$$

converges to say $H \in L(H^1(\Sigma, d_A))$.

Theorem 2.6.1 *Let $H^1_F(\Sigma, d_A)$ be the basic cohomology of F. We then have that*

$$H^1_F(\Sigma, d_A) \subset (H \cap \overline{H}).$$

Proof. From the assumption that H_n converges to H in $L(H^1(\Sigma, d_A))$, we conclude that we can find a basis of holomorphic forms $(\alpha_n^1, \ldots, \alpha_n^m)$ of $H^{1,0}(\Sigma_{\tau_n}, d_A)$ for each n and a basis of cohomology classes $(\alpha^1, \ldots, \alpha^m)$ of H, such that α_n^i as elements of $H^1(\Sigma, d_A)$ converges to α^i as $n \to \infty$. We notice that $H^1_F(\Sigma, d_A)$ is real, so we just need to show that

$$H^1_F(\Sigma, d_A) \subset H,$$

or equivalently

$$H \subset (H^1_F(\Sigma, d_A))^0,$$

since H is Lagrangian. That means we just have to prove

$$\int_\Sigma B(\alpha^i \wedge \varphi) = 0,\ i = 1, \ldots, m$$

for all basic cohomology classes $\varphi \in H^1_F(\Sigma, d_A)$. Hence it is enough to prove that

$$\int_\Sigma B(\alpha_n^i \wedge \varphi) \to 0 \text{ for } n \to \infty, \ i = 1, \dots, m,$$

for all basic forms φ. In local coordinates we can write

$$\alpha_n^i = f_n^i (dz + \nu_n d\overline{z}).$$

So

$$\int_\Sigma B(\alpha_n^i \wedge \overline{\alpha}_n^i) = \int \mid f_n^i \mid^2 (1- \mid \nu_n \mid^2) dz \wedge d\overline{z}.$$

Now

$$\int_\Sigma B(\alpha_n^i \wedge \overline{\alpha}_n^i) \to \int_\Sigma B(\alpha^i \wedge \overline{\alpha}^i) \ for \ n \to \infty, \ i = 1, \dots, m.$$

This means there exists a constant C, independent of the coordinates, and n, such that

$$\left| \int \mid f_n^i \mid^2 (1- \mid \nu_n \mid^2) dz \wedge d\overline{z} \right| \leq C.$$

We shall now consider special local coordinates, for which we can say more about ν_n. Suppose $p \in \Sigma$, such that $q(p) \neq 0$. We can then find holomorphic coordinates $z = x + iy$ in a neighbourhood U_p of p, with $q = dz^2$. Using formula (IX) on page 453 in [52], we have

$$\nu_n(p') \in \mathbb{R}_+, \text{ for } p' \in U_p.$$

From Wolf's proposition 2.1.1 we conclude that

$$\nu_n(p') \to 1 \text{ as } n \to \infty \text{ for } p' \in U_p.$$

We note that the vertical foliation of q, i.e. F, is given by $dz + d\overline{z}$ on U_p. If p is such that $q(p) = 0$ then we can find holomorphic coordinates ζ in a neighbourhood U_p of p, such that $q = \zeta^k d\zeta^2$ for some $k \in N$. F divides U_p into $k+2$ wedges U_p^j. In each U_p^j we can choose a holomorphic branch of the square root and define a holomorphic coordinate z on U_p^j by

$$z = \frac{2}{k+2} \zeta^{\frac{k+2}{2}}$$

(see [47]). We then have that $q = dz^2$ on U_p^j, so again

$$\nu_n(p') \to 1 \text{ as } n \to \infty \text{ for } p' \in U_p^j,$$

and the vertical foliation of F is given by $dz + d\overline{z}$ on U_p^j.

We conclude that we can cover an open dense subset of Σ with a finite number of such special local coordinates. In these special coordinates a basic 1-form can be written as

$$\varphi = g(dz + d\overline{z}),$$

so

$$B(\alpha_n^i \wedge \varphi) = B(f_n^i, g(1-\nu_n))dz \wedge d\overline{z}.$$

We calculate

$$\begin{aligned}
\left| \int_\Sigma B(\alpha_n^i \wedge \varphi) \right| &\leq \left| \int_\Sigma B(f_n^i, g(1-\nu_n)) \right| \\
&\leq \left(\int \mid f_n^i \mid^2 \mid 1-\nu_n \mid \right)^{\frac{1}{2}} \left(\int \mid g \mid^2 \mid 1-\nu_n \mid \right)^{\frac{1}{2}} \\
&\leq C \left(\int \mid g \mid^2 \mid 1-\nu_n \mid \right)^{\frac{1}{2}}.
\end{aligned}$$

We are here using that $0 < \nu_n < 1$, so

$$\mid 1-\nu_n \mid \leq 1 - \mid \nu_n \mid^2 .$$

By Lebesgue's dominated convergence theorem we conclude that

$$\int_\Sigma B(\alpha_n^i \wedge \varphi) \to 0 \text{ as } n \to \infty, \text{ for } i = 1, \ldots, m.$$

$\diamond$

Remark that if $H^1_F(\Sigma, d_A)$ is Lagrangian then it follows from 2.6.1 that

$$H^{1,0}(\Sigma_{\Phi^{-1}(t\rho_0)}, d_A) \to H^1_F(\Sigma, d_A) \text{ as } t \to \infty.$$

Consider the real polarization on the SU(2)-moduli space $\mathcal{M}(\Sigma)$ arising from a "pair of pants" decomposition D as described in section 2.4. As we have seen before, D defines a point $[D]$ on the Thurston boundary. This means there exists a $q_D \in QD(\rho_0)$ such that

$$\Phi^{-1}(tq_D) \to [D], \text{ as } t \to \infty.$$

From theorem 2.6.1 and the discussion ending section 2.4, it now follows that the sequence of Kähler polarisations $F(\Phi^{-1}(tq_D))$ converges to the real polarization, given by the basic cohomology associated to $[D]$.

2.6.2 Stretching necks around simple closed curves on a surface.

Let Σ be a closed surface, and let P be a principal G-bundle over Σ. Let A be a flat connection in P. Suppose we have a Strebel measured foliation F on Σ. Let L_F be a non-singular leaf choice of F. Let $\tilde{C}'$ be an open tubular neighbourhood of L_F.

We denote the complement of $\tilde{C}'$ by

$$\tilde{\Sigma}' = \Sigma - \tilde{C}'.$$

For each $\gamma \in L_F$, denote by $\tilde{C}'_\gamma$ the connected component of $\tilde{C}'$ which contains γ. Denote the two components of $\tilde{C}'_\gamma - \gamma$ by $\tilde{C}^1_\gamma$ and $\tilde{C}^2_\gamma$. That gives us the disjoint union

$$\tilde{C}'_\gamma = \tilde{C}^1_\gamma \cup \gamma \cup \tilde{C}^2_\gamma.$$

Let $\tilde{g}'$ be a metric on $\tilde{\Sigma}'$ with the following property. There exist coordinates $\underline{x} : U^i_\gamma \to S^1 \times (-\delta, \infty)$ in a neighbourhood U^i_γ of each $\tilde{C}^i_\gamma$ $(i = 1, 2)$ which give an isometry of $U^i_\gamma - \tilde{C}^i_\gamma$ to $S^1 \times (-\delta, 0]$ with the standard metric. Note that given any metric on $\tilde{\Sigma}'$, we can, by making a conformal change of that metric, obtain a metric with this property.

By pulling back the standard metrics on $S^1 \times [0, \infty)$ to each $\tilde{C}^i_\gamma$, we construct a metric $\tilde{g}$ on $\tilde{\Sigma} = \Sigma - L_F$, which extends $\tilde{g}'$ on $\tilde{\Sigma}'$. Note that $\tilde{g}$ has infinite ends.

We shall now consider sequences of metrics $\{g_t\}$ on Σ, such that g_t equals $\tilde{g}'$ on the complements of $\tilde{C}'$ and which make the length of $\tilde{C}'$ go to infinity as t goes to infinity. For each $\gamma \in L_F$, choose a closed tubular neighbourhood $C_\gamma \subset \Sigma$ of γ such that

$$C_\gamma \subset \tilde{C}'_\gamma.$$

Suppose that on C_γ we have coordinates $(\theta, r) : C_\gamma \to S^1 \times [-1, 1]$ such that $(\theta, r)(\gamma) = S^1 \times \{0\}$. Now consider a sequence of metrics $\{g_t\}$ which satisfies the following two conditions.

First, the sequence $\{g_t\}$ is constant on the complement of $\tilde{C}'$, i.e.,

$$g_t|_{\tilde{\Sigma}'} = \tilde{g}'.$$

Second, in the coordinates (θ, r) on C_γ we have

$$g_t|_{C_\gamma} = \begin{pmatrix} L_\gamma(t)^{-1} & 0 \\ 0 & L_\gamma(t) \end{pmatrix},$$

where

$$L_\gamma : [1, \infty) \to \mathbb{R}_+,$$

such that

$$L_\gamma(t) \to \infty \text{ as } t \to \infty.$$

It is not hard to see that given any metric $\tilde{g}'$ on $\tilde{\Sigma}'$ one can construct such a sequence. Moreover, for any such sequence, the length of C_γ, and therefore the length of $\tilde{C}'_\gamma$, tends to infinity as t goes to infinity.

Associated to the sequence of metrics, we have a sequence of complex structures J_t, which on C_γ is given by

$$J_t|_{C_\gamma} = \begin{pmatrix} 0 & -L_\gamma(t) \\ L_\gamma(t)^{-1} & 0 \end{pmatrix}.$$

Now J_t induces a sequence of complex Lagrangian subspaces $P_t(A)$ of $H^1(\Sigma, d_A)$. By taking a subsequence, we can assume that $P_t(A)$ converges to say $P_\infty(A)$ as Lagrangian subspaces of $H^1(\Sigma, d_A)$. This means that we can find a basis for $P_t(A)$ consisting of holomorphic 1-forms $(\alpha_t^1, \ldots, \alpha_t^m)$, and a basis for $P_\infty(A)$ consisting of cohomology classes, say $(\alpha^1, \ldots, \alpha^m)$, such that

$$\alpha_t^i \to \alpha^i$$

as elements of $H^1(\Sigma, d_A)$.

By changing A by a gauge transformation, we can assume that $A = A_\gamma d\theta$ for some $A_\gamma \in \mathfrak{g}$ on each $\tilde{C}'_\gamma$.

Recall from section 2.5 that from the data $(F, \tilde{g})$ and such a connection, we constructed a non-negative Lagrangian subspace $P_{F,\tilde{g}}(A)$ of $H^1(\Sigma, d_A)$. We now claim that

Theorem 2.6.2 *We can identify*

$$P_\infty(A) = P_{F,\tilde{g}}(A).$$

Proof. We just need to prove that

$$\omega(a, b) = 0 \ \forall a \in P_\infty(A), \ b \in P_{F,\tilde{g}}(A),$$

hence it is enough to prove that

$$\lim_{t\to\infty} \omega(\alpha_t^i, b) = 0 \ \forall b \in P_{F,\tilde{g}}(A). \tag{2.12}$$

We shall first prove that $P_\infty(A) \subset H^1_F(\Sigma, d_A)^0$.

Let φ be a d_A-closed basic 1-form on Σ with compact support in C. (By the proof of 2.3.3 in section 2.3, we just need to consider such basic 1-forms.) In the local coordinates (θ, r) on C_γ, we have that

$$J_t(\frac{\partial}{\partial\theta}) = L_\gamma(t)^{-1}\frac{\partial}{\partial r}, \ J_t(\frac{\partial}{\partial r}) = L_\gamma(t)\frac{\partial}{\partial\theta}.$$

So

$$J_t(d\theta) = -L_\gamma(t)dr, \ J_t(dr) = L_\gamma(t)^{-1}d\theta.$$

Now suppose

$$\alpha_t^i = \alpha_t^i(dr + iJ_t(dr)) = \alpha_t^i(dr + iL_\gamma(t)^{-1}d\theta).$$

Then

$$\int_{C_\gamma} B(\alpha_t^i \wedge \overline{\alpha}_t^i) = i\int_{C_\gamma} \frac{2}{L_\gamma(t)}|\alpha_t^i|^2 d\theta \wedge dr.$$

Since

$$\omega(\alpha_t^i, \overline{\alpha}_t^i) \to \omega(\alpha^i, \overline{\alpha}^i)$$

we conclude that there is a constant K such that

$$\left|\int_C B(\alpha_t^i \wedge \overline{\alpha}_t^i)\right| \leq K$$

for large enough t. In the coordinates we write

$$\varphi = \varphi dr.$$

So we get that

$$\begin{aligned}
&|\omega(\alpha_t^i, \varphi)| \\
&\quad = \left|\int_C B(\alpha_t^i \wedge \varphi)\right| \\
&\quad \leq \sum_\gamma \int_{C_\gamma} |\alpha_t^i||\varphi| \frac{1}{L_\gamma(t)} d\theta dr \\
&\quad \leq \sum_\gamma \left(\frac{1}{L_\gamma(t)} \int_{C_\gamma} |\alpha_t^i|^2 d\theta dr\right)^{\frac{1}{2}} \left(\frac{1}{L_\gamma(t)} \int_{C_\gamma} |\varphi|^2 d\theta \wedge dr\right)^{\frac{1}{2}}.
\end{aligned}$$

From this we conclude that

$$\omega(\alpha_i, \varphi) = 0,$$

which proves that $P_\infty(A) \subset H^1_F(\Sigma, d_A)^0$.

Suppose now $\varphi \in \ker(d_A^* + d_A)$ is an i-eigenvector for $*$ on $\tilde{\Sigma}$. Here $*$ is the Hodge $*$-operator, associated to $\tilde{g}$ on $\tilde{\Sigma}$. We know there exists $f \in \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$ such that

$$\psi = \varphi - d_A f$$

has compact support on $\tilde{\Sigma} - C$. From the construction of $P_{F,\tilde{g}}(A)$, we just need to show that

$$\omega(\alpha^i, \psi) = 0$$

in order to prove (2.12).

Let $\tilde{C}$ be a tubular neighbourhood of L_F inside C. We now want to find diffeomorphisms $F_t : \Sigma \to \Sigma$ with the following properties.

1. $F_t|_{L_F} = \mathrm{Id}$.

2. $F_t|_{\Sigma - \tilde{C}'} = \mathrm{Id}$.

3. $(F_t)_*(\tilde{g})|_{\Sigma - \tilde{C}}$ is conformally equivalent to $g_t|_{\Sigma - \tilde{C}}$.

4. F_t is homotopic to the identity.

Since g_t gives the connected components of $\tilde{C}' - \tilde{C}$, a finite modulus and $\tilde{g}|_{\tilde{C}'-L_F}$ are conformally equivalent to infinite cylinders, such a diffeomorphism exists . Note that we can choose F_t, such that it preserves the connection A.

Now let ρ be a function which is constant 1 on $\tilde{C}$ and with compact support inside C. Set

$$\psi_t = F_t^*\varphi - d_A(\rho F_t^* f).$$

Note that ψ_t is supported on the complement of $\tilde{C}$. We calculate that

$$\begin{aligned} &\psi_t - F_t^*\psi \\ &\quad = F_t^*\varphi - d_A(\rho F_t^* f) - (F_t^*\varphi - d_A(F_t^*(\rho)F_t^*(f))) \\ &\quad = d_A((F_t^*(\rho) - \rho)F_t^* f). \end{aligned}$$

Since $(F_t^*(\rho) - \rho)F_t^* f \in \Omega^0(\Sigma, \operatorname{ad} P \otimes \mathbb{C})$, we have that $[\psi_t] = [F_t^*\psi] = [\psi]$ in $H^1(\Sigma, d_A)$. We have here used that F_t^* acts trivially on $H^1(\Sigma, d_A)$, since it is homotopic to the identity.

This means we just need to show that

$$\lim_{t\to\infty} \omega(\alpha_t^i, \psi_t) = 0.$$

By the construction of F_t, we see that both α_t^i and $F_t^*\varphi$ are $(0,1)$-forms with respect to g_t on $\Sigma - \tilde{C}$, hence their wedge product vanishes on $\Sigma - \tilde{C}$. Since

$$\rho F_t^*\varphi = \rho F_t^* d_A f = \rho d_A F_t^* f,$$

we also see that $\rho d_A F_t^* f$ is a $(0,1)$-form with respect to g_t on $\Sigma - \tilde{C}$. From this we conclude that

$$\alpha_t^i \wedge \psi_t = \alpha_t^i \wedge d(\rho) F_t^* f.$$

We then get the following calculation.

$$\begin{aligned} &|\omega(\alpha_t^i, \psi_t)| \\ &\quad = \left| \int_{C-\tilde{C}} B(\alpha_t^i \wedge d\rho F_t^* f) \right| \\ &\quad \leq \sum_\gamma \int_{C_\gamma - \tilde{C}_\gamma} |\alpha_t^i||\rho'(r)||F_t^* f| \frac{1}{L_\gamma(t)} d\theta \wedge dr \\ &\quad \leq \sum_\gamma \left(\int_{C_\gamma - \tilde{C}_\gamma} |\alpha_t^i|^2 \frac{1}{L_\gamma(t)} d\theta \wedge dr \right)^{\frac{1}{2}} \left(\int_{C_\gamma - \tilde{C}_\gamma} |\rho'(r)|^2 |F_t^* f|^2 \frac{1}{L_\gamma(t)} d\theta \wedge dr \right)^{\frac{1}{2}}. \end{aligned}$$

Since we have that $f \in \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{ad} P \otimes \mathbb{C})$, we can conclude that f is bounded over $\tilde{\Sigma}$, hence there exists a constant K' such that

$$\left| \int_{C_\gamma - \tilde{C}_\gamma} |\rho'(r)|^2 |F_t^* f|^2 d\theta \wedge dr \right| \leq K'.$$

But then

$$|\omega(\alpha_t^i, \psi_t)| \leq \sum_\gamma \frac{1}{L_\gamma(t)^{\frac{1}{2}}} K \cdot K'.$$

Hence we see that

$$\lim_{t \to \infty} \omega(\alpha_t^i, \psi_t) = 0,$$

which proves (2.12) and thus concludes the proof.

$\diamond$

The sequence $[g_t]$ in Teichmüller space induced a sequence $F([g_t])$ of Kähler polarizations on the moduli space $\mathcal{M}(\Sigma)$. From this theorem we see that

$$F([g_t]) \to P_{F,\tilde{g}}$$

in the topology we have defined on $P(\mathcal{M}(\Sigma)^{\mathrm{irr}})$ (See the discussion after definition 1.3.1).

2.7 On the association of polarizations to points in the Thurston compactification of Teichmüller space.

We shall start this section with some general remarks on the space of polarizations of a symplectic manifold. Let (M, ω) be a symplectic manifold. Recall the definition of the space of polarizations $P(M)$ in section 1.3. We have a map

$$\rho: P(M) \to \{0, \ldots, n\},$$

and we define

$$P_k(M) = \rho^{-1}(k).$$

$P_0(M)$ is the space of Kähler structures on M, and $P_n(M)$ is the space of real polarizations on M. Suppose we have a sequence of polarizations $(P_n) \subset P(M)$, which converges to say $P \in P_k(M)$. Let

$$U_n = \{p \in M | \operatorname{Rank}(P_n(p) \cap \overline{P_n(p)} \cap TM_p) = \rho(P_n)\},$$

and

$$U = \{p \in M | \operatorname{Rank}(P(p) \cap \overline{P(p)} \cap TM_p) = \rho(P)\}.$$

Now, U_n and U are open and dense subsets of M, by the definition of $P(M)$. That implies $\cap U_n \cap U$ is not empty. Let $p \in \cap U_n \cap U$. Since $P_n(p) \to P(p)$, we conclude that

$$\dim(P(p) \cap \overline{P(p)} \cap TM_p) \geq \rho(P_n)$$

for n large. Hence $\rho(P) \geq k$. This shows that

$$\overline{P_k(M)} \subset \bigcup_{i=k}^{n} P_i(M).$$

In particular $P_n(M)$ is closed.

Let us now return to the situation where M is the moduli space $\mathcal{M}(\Sigma)^{\text{irr}}$. As explained in section 1.4, we have a continuous equivariant map

$$F: T \to P_0(\mathcal{M}(\Sigma)^{\text{irr}}).$$

We are now in a position to answer the extension question, which was raised in section 1.4.

Theorem 2.7.1 *There is no continuous extension of*

$$F : T \to P(\mathcal{M}(\Sigma)^{\text{irr}})$$

to the Thurston compactification of Teichmüller space.

Proof. Let γ_1 and γ_2 be two simple closed curves on Σ. We will require γ_i to be homologically non-trivial and γ_1 not homotopic to γ_2 (Choose e.g. two non-intersecting curves from a standard set of generators of $\pi_1(\Sigma)$).

As explained in section 2.1.1, $[\gamma_1]$ and $[\gamma_1 \cup \gamma_2]$ determine two elements of the Thurston boundary, and we can represent them by two measured foliations, respectively F_1 and F_2. Now choose metrics $\tilde{g}_1$ on $\tilde{\Sigma}_1 = \Sigma - \gamma_1$ and $\tilde{g}_2$ on $\tilde{\Sigma}_2 = \Sigma - (\gamma_1 \cup \gamma_2)$. The metrics should be chosen such that $\tilde{g}_1$ is conformally equivalent to a half infinite cylinder on each side of γ_1, and $\tilde{g}_2$ is conformally equivalent to a half infinite cylinder on each side of both γ_1 and γ_2.

As it was explained in section 2.5, we get two non-negative polarizations $P_{F_1,\tilde{g}_1}$ and $P_{F_2,\tilde{g}_2}$ induced on $\mathcal{M}(\Sigma)^{\mathrm{irr}}$. Moreover we saw that

$$\rho(P_{F_1,\tilde{g}_1}) = \operatorname{Rank} G$$

and

$$\rho(P_{F_2,\tilde{g}_2}) = 2 \operatorname{Rank} G.$$

From section 2.6.2 we know there are sequences (τ_n^i) for $i = 1, 2$ in Teichmüller space, such that

$$F(\tau_n^i) \to P_{F_i,\tilde{g}_i}$$

in the topology on $P(\mathcal{M}(\Sigma)^{\mathrm{irr}})$. Hence

$$P_{F_i,\tilde{g}_i} \in \overline{\operatorname{Im} F} - \operatorname{Im} F.$$

Now assume we can find a continuous extension

$$F : \overline{T} \to P(\mathcal{M}(\Sigma)^{\mathrm{irr}}).$$

Because of compactness, we then see that

$$F(\partial T) = \overline{\operatorname{Im} F} - \operatorname{Im} F.$$

Hence there will be a $\tau_i \in \partial T$, such that

$$F(\tau_i) = P_{F_i,\tilde{g}_i}.$$

Since F has to be mapping class group equivariant, and the orbits of the mapping class group on the boundary are dense (see [19] p.117), we conclude that

$$F(\partial T) \subset \bigcup_{i=\rho(P_{F_2,\tilde{g}_2})}^{n} P_i(\mathcal{M}(\Sigma)^{\mathrm{irr}}).$$

This is a contradiction, since

$$\rho(F(\tau_1)) = \rho(P_{F_1,\tilde{g}_1}) < \rho(P_{F_2,\tilde{g}_2}).$$

◇

There is however a much more straightforward way to see that an extension of F cannot exist in the abelian case. There are Pseudo-Anosov diffeomorphisms f of surfaces of any genus greater than or equal to 2, which act trivially on cohomology. (See section 13 in [19].) Now such an f has exactly two fixed points τ^u and τ^s on $\overline{T}$, and the sequence $(f^m\tau) \subset T$ converges to τ^u for all $\tau \in \overline{T} - \{\tau^s\}$. (See section 12 [19].) Since f acts trivially on $H^1(\Sigma, \mathbb{C})$, we see that

$$F(f^m\tau) = H^{1,0}(\Sigma_\tau) \text{ for all } m \in N^+.$$

Hence we see that the limit set at τ^u will be $\overline{F(T)}$. This clearly shows why there can be no extension to the Thurston compactification of F in the abelian case.

Chapter 3

Geometric Quantization with respect to non-negative polarizations.

3.1 Leafwise elliptic complexes over fibrations

We shall start this chapter with a rather general discussion of leafwise elliptic complexes over fibrations. In the following sections we will then specialize to the case where one is studying the leafwise complex of a non-negative polarization with values in the prequantum line bundle over a compact symplectic manifold.

Let M be a compact manifold. Assume we have a locally trivial fibration $\pi : M \to B$ with compact fibre F. Let $E_i,\ i = 0, \ldots, n$ be vector bundles over M, such that they are locally trivial relative to π, i.e. there is around each point $b \in B$ a neighbourhood U such that

$$E_i|_{\pi^{-1}(U)} \cong E_i|_{\pi^{-1}(b)} \times U.$$

Suppose we are given a leafwise elliptic complex on M

$$0 \to C^\infty(M, E_0) \overset{d_0}{\to} C^\infty(M, E_1) \overset{d_1}{\to} \ldots \overset{d_{n-1}}{\to} C^\infty(M, E_n) \to 0. \tag{3.1}$$

We shall denote the cohomology groups of this complex by $H^i(M, d_i)$. For each $b \in B$ we can consider the elliptic complex

$$0 \to C^\infty(\pi^{-1}(b), E_0) \overset{d_0}{\to} C^\infty(\pi^{-1}(b), E_1) \overset{d_1}{\to} \ldots \overset{d_{n-1}}{\to} C^\infty(\pi^{-1}(b), E_n) \to 0. \tag{3.2}$$

Definition 3.1.1 *We say that the complex (3.1) is in the top filtration if the complex (3.2) is exact for all but a finite number of points on B. We will denote the set of points $b \in B$ for which (3.2) is not exact by S.*

It is of course necessary for the complex (3.1) to be in the top filtration in order for it to have finite cohomology groups. However, one can easily construct examples which show that this is not sufficient to guarantee that the cohomology groups are finite dimensional. We shall now describe a sufficient condition. We may assume that M has a metric and that E_i has a Hermitian structure. We then construct the adjoint

$$d_i^* : C^\infty(M, E_{i+1}) \to C^\infty(M, E_i).$$

by requiring that

$$\int_{\pi^{-1}(b)} < d_i\varphi, \psi > \;=\; \int_{\pi^{-1}(b)} < \varphi, d_i^*\psi > \;\; \forall b \in B$$

for all $\varphi \in C^\infty(M, E_i)$, and $\psi \in C^\infty(M, E_{i+1})$. Consider the Laplacian

$$\triangle_i = d_i^* d_i + d_i d_i^* : C^\infty(M, E_i) \to C^\infty(M, E_i)$$

Let (U, x) be an open coordinate neighbourhood around $b \in S$. If U is small enough, $\triangle_i(x)$ will be invertible for $x \in U - \{b\}$. We can then consider the function

$$f_v : U - \{b\} \to C^\infty(\pi^{-1}(b), E_i)$$

given by

$$f_v(x) = \triangle_i(x)^{-1}(v),$$

where v is some element of $C^\infty(\pi^{-1}(b), E_i)$. We are here using the local triviality of the vector bundle E_i relative to the fibration.

Definition 3.1.2 *We say that the complex (3.1) is transversal if the function g_v given by*

$$g_v(x) = |x|^2 f_v(x) \;\; \forall x \in U - \{b\}$$

is smooth on U for all $b \in S$ and $v \in \ker(\triangle_i|_{\pi^{-1}(b)})$.

We can now formulate

Theorem 3.1.1 *If we have a leafwise elliptic complex in the top filtration, which is transversal, then there is a natural inclusion*

$$r : H^i(M, d_i) \to H^i(\pi^{-1}(S), d_i)$$

just given by restriction.

Proof. Let $\varphi \in C^\infty(M, E_i)$, such that

$$d_i\varphi = 0.$$

Then $\varphi|_S$ defines a class in $H^i(\pi^{-1}(S), d_i)$ and hence gives a well-defined map

$$r : H^i(M, d_i) \to H^i(\pi^{-1}(S), d_i).$$

Now suppose

$$\varphi|_S \in \operatorname{Im} d_i|_S.$$

Locally, we do the following calculation. Suppose

$$\varphi(0) = d_i\psi$$

for some $\psi \in C^\infty(\pi^{-1}(b), E_i)$. Let

$$\tilde{\varphi} = \varphi - d_i(\rho\psi),$$

where ρ is a bump function on B with compact support in a neigbourhood of b. Since $\tilde{\varphi}(0) = 0$, there exists $\tilde{\varphi}'_j \in C^\infty(M, E_i)$, such that

$$\tilde{\varphi} = \sum_{j=1}^{n} x_j \tilde{\varphi}'_j.$$

Let $\rho_j = x_j^2 \tilde{\varphi}'_j$. Use the notation $W = \ker(\triangle_j(0))$ and $W^\perp = \operatorname{Im}(\triangle_j(0))$. Let p be the projection onto $W^\perp$ and set

$$\rho_j^2(x) = p(\rho_j(x)).$$

Consider

$$p\triangle_i(x)|_{W^\perp} : W^\perp \to W^\perp.$$

Now

$$p\triangle_i(0)|_{W^\perp} = \triangle_i(0) : W^\perp \to W^\perp,$$

is clearly invertible, so $p\triangle_i(x)|_{W^\perp}$ will be invertible for x in a neighbourhood of zero. That means we can find smooth map

$$\xi_j^2 : U \to W^\perp,$$

such that

$$p\triangle_i(x)|_{W^\perp}(\xi_j^2(x)) = \rho_j^2(x).$$

Consider

$$\rho_j^1(x) = \triangle_i(x)(\xi_j^2(x)) - \rho_j(x).$$

We calculate that

$$p(\rho_j^1(x)) = 0,$$

so $\rho_j^1(x) \in W$. Now

$$\begin{aligned} |x|^{-2}\rho_j^1(x) &= |x|^{-2}\triangle_i(x)(\xi_j^2(x)) - |x|^{-2}\rho_j(x) \\ &= x_j^2|x|^{-2}x_j^{-2}\triangle_i(x)(\xi_j^2(x))) - x_j^2|x|^{-2}\tilde{\varphi}_j'(x), \end{aligned}$$

which is seen to be smooth by the construction of ξ_j^2. By the assumption in theorem 3.1.1,

$$\xi_j^1(x) = \triangle_i^{-1}(x)(\rho_j^1(x))$$

will be smooth. Let

$$\tilde{\psi}_j'(x) = \xi_j^2(x) - \xi_j^1(x).$$

Then $\tilde{\psi}_j'$ is smooth and

$$\triangle_i\tilde{\psi}_j' = \rho_j.$$

Since $d_i\rho_j = 0$, it follows that $d_i\tilde{\psi}_j' = 0$ because $\ker \triangle_i(x) = 0$ for $x \in B - S$, so we get that

$$d_i(x_j^{-1}d_i^*\tilde{\psi}_j') = x_j\tilde{\varphi}_j'.$$

But

$$|x_j^{-1}d_i^*\tilde{\psi}_j'|^2 =< x_j^{-2}\triangle_i\tilde{\psi}_j', \tilde{\psi}_j' >=< \tilde{\varphi}_j', \tilde{\psi}_j' >$$

is smooth, so it follows that

$$\tilde{\psi}_j = x_j^{-1}d_i^*\tilde{\psi}_j'$$

is smooth. Thus we see that

$$\varphi = d_i(\sum_{j=1}^n \tilde{\psi}_j + \rho\psi).$$

This shows that r is an inclusion.

$\diamond$

In general, one cannot expect r to be an isomorphism, as we shall see in the next section. We shall now consider the example of a leafwise elliptic complex that one gets from the leafwise complex of a real reducible polarization on a symplectic manifold. We shall assume the reader is familiar with the definitions, the results and the notation of section 3.2 throughout the example.

Example: Let us first recall what the local analysis of section 3.2 gives for a reducible real polarization P on a compact symplectic manifold (M^{2n},ω). We have for a real polarization that

$$D = E = P.$$

B is the leaf space of P and $\pi_D : M \to B$ is a locally trivial T^n-fibration. The level k Bohr-Sommerfeld set of P is

$$S_k = \{b \in B | (\mathcal{L}^k \otimes \delta_P)|_{\pi_D^{-1}(b)} \text{ is trivial}\}.$$

For each point $b' \in B$, we can find a neighbourhood U of b' and symplectic action-angle (see [26]) coordinates (x,y) on $\pi_D^{-1}(U)$ and the P-connection in $\mathcal{L}^k \otimes \delta_P$ is given by

$$\nabla^P = d_D + i\sum_{i=1}^n kx_i dy_i$$

with respect to a unit length section of $\mathcal{L}^k \otimes \delta_P$. With this local expression for ∇^P, we shall now verify that the complex (3.4) for these real polarizations is in the top filtration and that it is transversal.

S_k is finite, so it is clear that (3.4) is in the top filtration, since the de Rham complex with values in a flat bundle on a torus is exact if and only if the flat bundle is non-trivial. We shall now choose a metric on M and show that (3.4) is transversal with respect to that metric. Fix around each $b \in S_k$ the special action-angle coordinates constructed above. Then construct a metric on M which makes $dx_1, \dots, dx_n, dy_1, \dots, dy_n$ an orthonormal basis for T^*M near $\pi_D^{-1}(b)$, $b \in S_k$. Using this metric and the the hermitian inner product in $\mathcal{L}$, we construct

$$(\nabla^P)^* : C^\infty(\Lambda^i P \otimes \mathcal{L}^k \otimes \delta_P) \to C^\infty(\Lambda^{i-1} P \otimes \mathcal{L}^k \otimes \delta_P).$$

and define

$$\triangle = \nabla^P(\nabla^P)^* + (\nabla^P)^*\nabla^P.$$

Let $\psi \in C^\infty(\Lambda^i P \times \mathcal{L}^k \otimes \delta_P)$ and suppose that locally around $b \in S_k$ we have

$$\psi = \sum_{|J|=i} \psi_J dy_J.$$

We then calculate

$$\begin{aligned}
\triangle(x)\psi &= \sum_i \nabla^P(-\frac{\partial \psi_J}{\partial y_i} + -ikx_i\psi_J)dy_i \wedge^* dy_J \\
&\quad + \sum_i (\nabla^P)^*(\frac{\partial \psi_J}{\partial y_i} + ikx_i\psi_J)dy_i \wedge dy_J \\
&= \sum_{i,j}(-\frac{\partial^2 \psi_J}{\partial y_j \partial y_i} - ikx_i\frac{\partial \psi_J}{\partial y_j} - ikx_j\frac{\partial \psi_J}{\partial y_i} + k^2 x_j x_i)(dy_j \wedge (dy_i \wedge^* dy_J) + dy_i \wedge^* (dy_j \wedge dy_J)).
\end{aligned}$$

Here $dy.\wedge^*$ denotes the adjoint of $dy.\wedge$. We have that

$$\ker \triangle(0) = \mathrm{Span}\{dy_J | \ |J| = i\}.$$

Moreover, we see that

$$\triangle(x)(dy_J) = k^2|x|^2 dy_J,$$

hence

$$\triangle(x)^{-1}(dy_J) = k^{-2}|x|^{-2} dy_J.$$

This clearly shows that (3.4) is transversal for these real polarizations. So by theorem 3.1.1, we get a new proof of the fact that the cohomology groups of the leafwise complex of a real polarization are finite dimensional. We note, however, that the conclusion of theorem 3.2.1 is stronger. From this we see that the injective map r in theorem 3.1.1 is in general not an isomorphism.

3.2 The leafwise complex of a non-negative reducible polarization.

In this section we shall study the differential geometry and the local structure of smooth polarizations on compact symplectic manifolds. This study will enable us to describe the cohomology groups of the leafwise complex with values in the prequantum bundle, associated to a polarization.

Let (M, ω) be a compact symplectic manifold. Suppose P is a non-negative smooth polarization on M. Throughout this chapter we shall only be concerned with smooth polarizations (see definition 1.3.1.). On a symplectic manifold one has a natural way to differentiate vector fields in a coisotropic foliation along the isotropic directions. Since a polarization is both isotropic and coisotropic, we can differentiate sections of $\overline{P}$ along directions in $\overline{P}$, by defining

$$\tilde{\nabla} : C^\infty(M, \overline{P}) \to C^\infty(M, \overline{P}^* \otimes \overline{P})$$

according to

$$i_{(\tilde{\nabla}_X Y)}\omega = i_X d(i_Y \omega) \ \forall X, Y \in C^\infty(M, \overline{P}). \tag{3.3}$$

We say that $\tilde{\nabla}$ is a $\overline{P}$-connection in $\overline{P}$. Now $\tilde{\nabla}$ induces a $\overline{P}$-connection in $\Lambda^n \overline{P}^*$.

Assume a "normal half-form" bundle δ_P for the polarization P has been chosen. By definition (see e.g. [53]) a normal half-form bundle for P is a complex line bundle with a $\overline{P}$ connection, such that

$$\Lambda^n \overline{P}^* \cong \delta_P \otimes \delta_P$$

as complex line bundles with $\overline{P}$-connection.

Suppose $(\mathcal{L}, \nabla)$ is a prequantum line bundle over M. That means $\mathcal{L}$ is a Hermitian line bundle with a compatible connection ∇, such that the curvature two form of ∇ is ω. Recall (see e.g. [53] proposition (8.3.1)) that the prequantum line bundle exists if and only if $[\omega] \in H^2(M, \mathbb{R})$ is contained in the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$. Moreover, the inequivalent choices of the prequantum line bundle are parametrized by $H^1(M, U(1))$, when this condition is satisfied. By restricting the connection in $\mathcal{L}$, we get a $\overline{P}$-connection in the bundle $\mathcal{L}^k \otimes \delta_P$. We are interested in studying the leafwise complex:

$$0 \to C^\infty(\mathcal{L}^k \otimes \delta_P) \overset{\nabla^P}{\to} C^\infty(\overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P) \overset{\nabla^P}{\to} \dots \overset{\nabla^P}{\to} C^\infty(\Lambda^n \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P) \to 0. \tag{3.4}$$

We shall describe the cohomology groups of this complex for "special" polarizations. Let us now specify the kinds of polarizations on which we will concentrate.

Consider the two distributions

$$E = (P + \overline{P}) \cap TM$$

and

$$D = P \cap \overline{P} \cap TM.$$

By the definition of a polarization both D and E have constant rank, say m and $2n - m$ respectively.

Definition 3.2.1 *We say a polarization is reducible if E and D are integrable and reducible, meaning that the leafspaces $B = M/E$ and $M' = M/D$ are Hausdorff manifolds.*

In the rest of this chapter, we shall only consider reducible polarizations. For such polarizations we have the locally trivial projections

$$\pi_E : M \to B$$

and

$$\pi_D : M \to M'.$$

Furthermore we have a projection

$$\pi_B : M' \to B$$

which is a locally trivial fibration, such that

$$\pi_E = \pi_B \circ \pi_D.$$

The symplectic form gives us an isomorphism

$$\omega : T_pM \to T_p^*M,$$

and so we get an inclusion

$$T^*_{\pi_E(p)}B \overset{\pi_E^*}{\to} T_p^*M \overset{\omega^{-1}}{\to} T_pM,$$

the image being D_p. Hence we see that T_b^*B acts infinitesimally on $\pi_E^{-1}(b)$. But since $\pi_E^{-1}(b)$ is compact and T_b^*B is abelian, the infinitesimal action exponentiates to an action of T_b^*B on $\pi_E^{-1}(b)$. Let us denote $\pi_E^{-1}(b)/D = \pi_D(\pi_E^{-1}(b))$ by M_b'. Note that the polarization P induces a Kähler structure on M_b'. Let $p' \in M_b'$, then T_b^*B acts locally transitively, and therefore transitively on $\pi_D^{-1}(p')$. (See p.351 in [26]) This means that

$$\Lambda_{b,p'} = \{\alpha \in T_b^*B |\ \alpha|_{\pi_D^{-1}(p')} = \mathrm{Id}\}$$

is a discrete lattice of maximal rank in T_b^*B, and we have an identification

$$\pi_D^{-1}(p') \cong \frac{T_b^*B}{\Lambda_{b,p'}}.$$

Hence $\pi_D : \pi_E^{-1}(b) \to M_b'$ is a T^m fibre bundle.

We note that when we restrict the bundle $\mathcal{L}^k \otimes \delta_P$ to $\pi_D^{-1}(p')$, we get a flat line bundle on the torus $\pi_D^{-1}(p')$, i.e. an element of the dual torus of $\pi_D^{-1}(p')$. When we say that $\mathcal{L}^k \otimes \delta_P|_{\pi_D^{-1}(p')}$ is *trivial*, we mean trivial as a flat line bundle with connection.

Definition 3.2.2 *The level k Bohr-Sommerfeld set for P is defined to be*

$$S_k = \{b \in B|\ (\mathcal{L}^k \otimes \delta_P)|_{\pi_D^{-1}(p')} \text{ is trivial for all } p' \in M'_b\}.$$

In fact, if there is a $p' \in M'_b$ such that $(\mathcal{L}^k \otimes \delta)|_{\pi_D^{-1}(p')}$ is trivial, then $b \in S_k$. This follows from the fact that the symplectic annihilator of E is D. Moreover, we will see later in this section that S_k is a finite union of isolated points on B. This definition is a straightforward generalization of the classical definition of the Bohr-Sommerfeld set for a reducible real polarization (see [53] p. 201, p. 241 and pp. 251-53)

Suppose $b \in S_k$. By the definition of S_k, there will exist a holomorphic line bundle over M'_b, which we will denote $(\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}}, \overline{\partial}_b)$, such that $(\pi_D^*(\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}}), \pi_D^*\overline{\partial}_b)$ is isomorphic to $((\mathcal{L}^k \otimes \delta_P)|_{\pi_D^{-1}(b)}, \nabla^P)$. We have that

$$\begin{aligned} \pi_D^*((\mathcal{L}^k \otimes K_b^{\frac{1}{2}}) \otimes (\mathcal{L}^k \otimes K_b^{\frac{1}{2}})) &\cong \mathcal{L}^{2k}|_{\pi_D^{-1}(b)} \otimes \Lambda^n \overline{P}^* \\ &\cong \mathcal{L}^{2k}|_{\pi_D^{-1}(b)} \otimes \pi_D^* K_b. \end{aligned}$$

Here K_b is the canonical bundle of M'_b. This explains our notation $(\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}}, \overline{\partial}_b)$. (Note that $\mathcal{L}_b^k$ and $K_b^{\frac{1}{2}}$ may not exist, but their tensor product $\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}}$ will exist.)

Locally we have the following identifications. Let $b \in B$, then there is an open neighbourhood U of b, over which the fibrations are trivial, i.e.

$$\pi_E^{-1}(U) \cong \pi_E^{-1}(b) \times U$$

and

$$\pi_B^{-1}(U) \cong M'_b \times U.$$

Moreover, for each $p = (p', b') \in \pi_B^{-1}(U)$ there is a neighbourhood V of $p' \in M'_b$ such that

$$\pi_D^{-1}(V \times U) \cong T^m \times V \times U.$$

We will denote $\pi_D^{-1}(V \times U) = W$.

These trivializations show we can find a smooth section

$$\tilde{s} : V \times U \to W$$

of $\pi_D : W \to V \times U$. It is clear that $\tilde{s}_v^*\omega$, $v \in V$ is a smooth family of closed forms on U. Hence we can find a smooth family α_v, $v \in V$ of 1-forms, such that

$$-d_U \alpha_v = \tilde{s}_v^* \omega.$$

Now α_v is a section of T^*U, and therefore by the fiberwise action gives a diffeomorphism

$$\kappa_{\alpha_v} : T^m \times \{v\} \times U \hookleftarrow .$$

We define a new section $s_v : U \to W$, $v \in V$, by

$$s_v = \kappa_{\alpha_v} \tilde{s}_v.$$

We then have that

$$s_v^* \omega = 0, \ v \in V.$$

For each $v \in V$ we get a map

$$\chi_v : T^*U \to W,$$

given by

$$\chi_v(\xi) = -\xi s_v(x).$$

We have the following

Proposition 3.2.1 *$\chi_v : T^*U \to M$ is locally symplectic with respect to the standard symplectic structure on T^*U, i.e. $\chi_v^* \omega$ is the standard symplectic structure on T^*U.*

Proof. The proof is exactly the same as the proof of theorem 44.2 in [26] pp. 354-355.

$\diamond$

From this proposition it follows that we can therefore find smooth functions $x_i : V \times U \to \mathbb{R}$, such that $(x_1^v, \dots, x_m^v)$ form coordinates on U for all $v \in V$, and $dx_1^v, \dots, dx_m^v$ form a basis of $\Lambda_{U,v}$. There are uniquely defined smooth functions $y_1, \dots, y_m : W \to \mathbb{R}/\mathbb{Z}$, such that $(x_1^v, \dots, x_m^v, \chi_v^*(y_1), \dots, \chi_v^*(y_m))$ are canonical symplectic coordinates on $T^*U/\Lambda_{U,v}$. Define a 1-form α' on W by

$$\alpha' = \sum_{i=1}^{m} x_i dy_i.$$

It is then clear that for any $v \in V$, we have that

$$\chi_v^*(\omega - d\alpha') = 0.$$

The 1-form α' defines a connection in the trivial complex line bundle over W. Let us denote this bundle with connection $\mathcal{L}'$.

The restriction of $\mathcal{L}$ to W is topologically trivial, so there will be a non-zero section λ of $\mathcal{L}$ over W. Let α be the connection 1-form defined by λ :

$$\nabla \lambda = i\alpha \otimes \lambda.$$

Then $d\alpha = \omega$. For $v \in V$ let us normalize the coordinates x^v, such that

$$[\alpha|_{\pi_D^{-1}(v,b)}] = [\alpha'|_{\pi_D^{-1}(v,b)}]$$

as elements of $H^1(\pi_D^{-1}(v,b), \mathbb{R})$.

Proposition 3.2.2 *With the given normalization of the coordinates x, we have that $\mathcal{L}|_W \otimes (\mathcal{L}')^*$ is trivial as a bundle with connection along the fibres of D over $V \times U$.*

Proof. When we restrict to the image of χ_v, which is $\pi_D^{-1}(\{v\} \times U)$, we see that

$$d(\alpha|_{\pi_D^{-1}(\{v\}\times U)} - \alpha'|_{\pi_D^{-1}(\{v\}\times U)}) = \omega|_{\pi_D^{-1}(\{v\}\times U)} - d\alpha'|_{\pi_D^{-1}(\{v\}\times U)} = 0.$$

By the normalization of x we have that

$$[\alpha|_{\pi_D^{-1}(\{v\}\times U)} - \alpha'|_{\pi_D^{-1}(\{v\}\times U)}] \in H^1(\pi_D^{-1}(\{v\} \times U))$$

is zero. From this we conclude that $\mathcal{L} \otimes (\mathcal{L}')^*|_{\pi_D^{-1}(\{v\}\times U)}$ is trivial.

$\diamond$

For $b' \in U$ consider the bundle $\mathcal{L} \otimes (\mathcal{L}')^*|_{\pi_E^{-1}(b')\cap W}$. Since this bundle is trivial along the fibres of π_D, we see that there exists a prequantum bundle $\mathcal{L}_{b'}$ over $V \times \{b'\}$, such that

$$\mathcal{L}|_{\pi_E^{-1}(b')\cap W} \cong \mathcal{L}' \otimes \pi_D^*(\mathcal{L}_{b'}).$$

From this, it follows that we can find complex functions $z = (z_{m+1}, \ldots, z_n)$ on $V \times U$, such that

$$\overline{P} = \operatorname{Span}\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}, \frac{\partial}{\partial \overline{z}_{m+1}}, \ldots, \frac{\partial}{\partial \overline{z}_n}\},$$

and a smooth complex function

$$K : V \times U \to \mathbb{C},$$

such that

$$\tilde{\alpha} = \sum_{i=1}^{m} x_i dy_i + \sum_{i=m+1}^{n} \frac{\partial K}{\partial \overline{z}_i} d\overline{z}_i$$

is a connection form for $\mathcal{L}|_{\pi_E^{-1}(b')\cap W}$, $x = x(b')$. This and proposition 3.2.1 imply in particular that there exist smooth function $A_{ij}, B_{ij} : W \to \mathbb{C}$, such that

$$\begin{aligned} \omega \;\; = \;\; & \sum_{i=1}^{m} dx_i \wedge dy_i + \sum_{i=m+1}^{n} (A_{ij} + \frac{\partial^2 K}{\partial x_i \partial \overline{z}_j}) dx_i \wedge d\overline{z}_j \\ & + \sum_{i=m+1}^{n} B_{ij} dx_i \wedge dz_j + \sum_{i=m+1}^{n} \frac{\partial^2 K}{\partial z_i \partial \overline{z}_j} dz_i \wedge d\overline{z}_j. \end{aligned} \tag{3.5}$$

Since ω is closed, we must have that

$$\frac{\partial A_{ij}}{\partial y_k} = 0.$$

For any $(v, b') \in V \times U$ we can consider the flat bundle

$$(\Lambda^n \overline{P})^*|_{\pi_D^{-1}(v,b')}.$$

By using the explicit expression for ω, one easily verifies that

$$\tilde{\nabla}_{\frac{\partial}{\partial y_i}}\left(\frac{\partial}{\partial y_1}\wedge\ldots\wedge\frac{\partial}{\partial y_m}\wedge\frac{\partial}{\partial\overline{z}_{m+1}}\wedge\ldots\wedge\frac{\partial}{\partial\overline{z}_n}\right)=0.$$

Hence $(\Lambda^n\overline{P})^*|_{\pi_D^{-1}(v,b')}$ is trivial for all $(v,b')\in V\times U$. This means that the isomorphism class of $\delta_P|_{\pi_D^{-1}(v,b')}$, $(v,b')\in V\times U$ must be constant, since $(\Lambda^n\overline{P})^*|_{\pi_D^{-1}(v,b')}$ is trivial. That means that there exist constants x_i^0, $i=1,\ldots,m$, such that

$$\alpha''=\sum_{i=1}^m x_i^0 dy_i$$

defines a connection in the trivial bundle, say δ', over W, such that

$$\delta_P\otimes(\delta')^*|_{\pi_D^{-1}(v,b')},\ (v,b')\in V\times U$$

is trivial, hence there exists $\delta_{b'}$ a complex line bundle over $V\times\{b'\}$, such that

$$\delta_P|_{\pi_E^{-1}(b')\cap W}\cong\delta'\otimes\pi_D^*\delta_{b'}.$$

Note that $\delta'\otimes\delta'$ is by construction trivial. We then have that

$$\begin{aligned}
&\pi_D^*(\delta_{b'}\otimes\delta_{b'})\\
&\qquad\cong\ (\delta_P\otimes\delta_P)|_{\pi_E^{-1}(b')\cap W}\otimes(\delta'\otimes\delta')^*|_{\pi_E^{-1}(b')\cap W}\\
&\qquad\cong\ \Lambda^n\overline{P}^*\\
&\qquad\cong\ (\Lambda^m D)^*\otimes\pi_D^*(K_{b'}|_V)\\
&\qquad\cong\ \pi_D^*(K_{b'}|_V).
\end{aligned}$$

Here $K_{b'}$ is the canonical bundle of $M'_{b'}$. From this we conclude that $\delta_{b'}$ defines a square root of $K_{b'}$ over $V\times\{b'\}$. We shall use the notation $K_{b'}^{\frac{1}{2}}=\delta_{b'}$.

Consider now the bundle $\mathcal{L}\otimes\delta_P$ with its $\overline{P}$ connection ∇^P. We then have that

$$(\mathcal{L}\otimes\delta_P)|_{\pi_E^{-1}(b')\cap W}\cong(\mathcal{L}'\otimes\delta')|_{\pi_E^{-1}(b')\cap W}\otimes\pi_D^*(\mathcal{L}'_{b'}\otimes K_{b'}^{\frac{1}{2}})$$

and

$$\nabla^P=d_D+i\sum_{i=1}^m x_i dy_i+\pi_D^*(\overline{\partial}_{b'}).\tag{3.6}$$

We have here changed the coordinates (x_i) to $(x_i+x_i^0)$, and $\overline{\partial}_{b'}$ is the $\overline{\partial}$-operator in $\mathcal{L}'_{b'}\otimes K_{b'}^{\frac{1}{2}}$ over $V\times\{b'\}$. We will use the notation

$$\beta=\sum_{i=1}^m x_i dy_i.$$

It is clear from the expression for β that S_k is a discrete subset of B.

On the basis of this local analysis we can now prove the following factorization property.

Theorem 3.2.1 *We have that*

$$H^i(M, \nabla^P) = 0$$

for $0 \leq i < m$, *and there is a natural isomorphism*

$$(\pi_D)_* : H^i(M, \nabla^P) \to \bigoplus_{b \in S_k} H^{i-m}(M'_b, \overline{\partial}_b)$$

for $m \leq i \leq n$.

Since $\mathcal{L}^k \otimes \delta_P$ is trivial along the fibres of $\pi_D : \pi_E^{-1}(b) \to M'_b$ for $b \in S_k$, we get a well defined map

$$(\pi_D)_* : C^\infty(M, \Lambda^i \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P) \to \bigoplus_{b \in S_k} C^\infty(M'_b, \Lambda^{0,i-m}(M'_b) \otimes \mathcal{L}_b^k \otimes K_b^{\frac{1}{2}})$$

by restriction to

$$\bigcup_{b \in S_k} \pi_E^{-1}(b),$$

followed by integration along the fibres of

$$\pi_D : \bigcup_{b \in S_k} \pi_E^{-1}(b) \to \bigcup_{b \in S_k} M'_b.$$

We will see in the proof that it is this map which induces the isomorphism in the theorem.

Note that this theorem is a generalization of Sniatycki's result for reducible real polarization in [45].

Proof. The inclusion

$$i : D \to \overline{P}$$

induces a filtration on the complex (3.4), by letting

$$F^p(\Lambda^l \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P) = \mathrm{Ann}(\Lambda^{l-p+1} D \wedge \Lambda^{p-1} \overline{P} \otimes \mathcal{L}^k \otimes \delta_P)$$

and

$$F^p C^\infty(\Lambda^l \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P) = C^\infty(F^p(\Lambda^l \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P)).$$

Associated to this filtered complex we have a spectral sequence $\{E_r\}$, where

$$E_0^{p,q} = \frac{F^p C^\infty(\Lambda^{p+q} \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P)}{F^{p+1} C^\infty(\Lambda^{p+q} \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P)}$$

and the differential is induced from ∇^P.

Note that if $q > m$ then

$$E_0^{p,q} = 0,$$

and if $p > n - m$ then

$$E_0^{p,q} = 0.$$

By general theory we have that the spectral sequence converges to

$$E_\infty^{p,q} = \mathrm{Gr}^p(H^{p+q}(M, \nabla^P)).$$

We shall now compute this spectral sequence. Let $\varphi \in E_0^{p,q}$. We may then in the local coordinates $(x, y, z, \overline{z})$, constructed earlier in this section, represent

$$\varphi = [\sum_{|I|=p} \eta_I(x, z, \overline{z}) \wedge d\overline{z}_I],$$

where η_I is a q-form on T^m :

$$\eta_I = \sum_{|J|=q} \eta_I^J(x, y, z, \overline{z}) dy_J.$$

The differential

$$d_0 : E_0^{p,q} \to E_0^{p,q+1}$$

is then given by

$$d_0\varphi = [\sum_{|I|=p} (d_D + i\beta)\eta_I(x, z, \overline{z}) \wedge d\overline{z}_I].$$

Hence we see, that if $d_0\varphi = 0$ in $E_0^{p,q+1}$, then

$$(d_D + i\beta)\eta_I(x, z, \overline{z}) = 0. \tag{3.7}$$

By the expression for η_I we get thus that

$$\begin{aligned}&(d_D + i\beta)\eta_I(x, z, \overline{z})\\ &= \sum_{i=1}^{m} \sum_{|J|=q} (\frac{\partial \eta_I^J}{\partial y_i}(x, y, z, \overline{z}) + i x_i \eta_I^J(x, y, z, \overline{z})) dy_i \wedge dy_J.\end{aligned}$$

Following Śniatycki's local analysis of the leafwise complex of a real polarization in [45], we argue as follows using Fourier series:

Write

$$\eta_I^J(x, y, z, \overline{z}) = \sum_{n \in \mathbb{Z}^m} \eta_I^J(n)(x, z, \overline{z}) \exp(-i < n, y >). \tag{3.8}$$

Then (3.7) becomes

$$\sum_{i=1}^{m} (x_i - n_i) dy_i \wedge \left(\sum_{|J|=q} \eta_I^J(n)(x, z, \overline{z}) dy_J \right) = 0$$

for all $n \in \mathbb{Z}^m$. Suppose now that $q < m$. It then follows by lemma 3.7 in [45], that there exist $\psi_I^J(n)(x,z,\overline{z})$, for all $n \in \mathbb{Z}^m$, such that

$$\sum_{i=1}^{m}(x_i - n_i)dy_i \wedge \left(\sum_{|J|=q-1} \psi_I^J(n)(x,z,\overline{z})dy_J\right) = \eta_I^J(n)(x,z,\overline{z}),$$

and so we can construct $\psi_I^W \in C^\infty(W, \Lambda^{q-1}D^*)$ by defining

$$\psi_I^W = \sum_{|J|=q-1} \sum_{n\in\mathbb{Z}^m} \psi_I^J(n)(x,z,\overline{z}) \exp(-i < n, y >)dy_J.$$

By construction we have that

$$(d_D + i\beta)\psi_I^W = \eta_I.$$

Choose a partition of unity $\{V \times U, \rho_{V\times U}\}$ on M'. By pulling back to M, we get a partition of unity on M: $\{W, \rho_W\}$. Using this partition of unity, we can define $\psi \in E_0^{p,q}$ by

$$\psi = \sum_W \rho_W \sum_{|I|=p} \psi_I^W \wedge d\overline{z}_I.$$

Then

$$\begin{aligned} d_0\psi &= [\nabla^P \sum_W \rho_W \sum_{|I|=p} \psi_I^W \wedge d\overline{z}_I] \\ &= [\sum_W \rho_W \sum_{|I|=p} \eta_I \wedge d\overline{z}_I] \\ &= \varphi. \end{aligned}$$

Hence we see

$$E_1^{p,q} = 0,$$

if $q < m$.

It is clear that

$$E_0^{p,m} = \ker d_0,$$

so

$$E_1^{p,m} = \frac{E_0^{p,m}}{\operatorname{Im} d_0}.$$

We have a map

$$(\pi_D)_* : E_0^{p,m} \to \bigoplus_{b\in S_k} C^\infty(M_b', \Omega^{0,p}(\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}})),$$

given by

$$(\pi_D)_*(\varphi) = \sum_{b\in S_k} (\pi_D)_*\varphi(b)$$

where $(\pi_D)_*\varphi(b)$ is the integration along the fibres of the fibration $\pi_D : \pi_E^{-1}(b) \to M'_b$. Note that $(\pi_D)_*$ is in fact well-defined on $E_1^{p,m}$, so we get

$$(\pi_D)_* : E_1^{p,m} \to \bigoplus_{b \in S_k} C^\infty(M'_b, \Omega^{0,p}(\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}})). \tag{3.9}$$

In terms of the local Fourier series description of φ, we see that the condition $(\pi_D)_*\varphi = 0$ implies that

$$\eta_I^J(0)(0, z, \overline{z}) = 0.$$

That is precisely the condition one needs on η_I in order to be able, locally in the coordinates $(x, y, z, \overline{z})$, to solve the following equation for $\psi_I^J(n)$

$$i \sum_{i=1}^{m} (x_i - n_i) dy_i \wedge \left(\sum_{|J|=m-1} \psi_I^J(n)(x, z, \overline{z}) dy_J \right) = \eta_I^J(n)(x, z, \overline{z}) dy_J.$$

By exactly the same argument as in the case where $q < m$, we see that using a partition of unity and these local solutions, we can construct $\psi \in E_0^{p,m-1}$ such that $d_0\psi = \varphi$. Hence the map $(\pi_D)_*$ is injective.

Using the partition of unity $\{W, \rho_W\}$ one easily constructs an m-form ψ on M, such that

$$\int_{\pi_D^{-1}(p')} \psi = 1$$

for any $p' \in M'_b,\ b \in S_k$. For

$$\tilde{\varphi} \in \bigoplus_{b \in S_k} C^\infty(M_b, \Omega^p(\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}})),$$

we let

$$\varphi = (\pi_D)^*(\tilde{\varphi}) \wedge \psi.$$

Then $\varphi \in E_0^{p,m}$and $(\pi_D)_*\varphi = \tilde{\varphi}$. We thus see that (3.9) is an isomorphism.

Consider the map

$$d_1 : E_1^{p,m} \to E_1^{p+1,m}$$

given by

$$d_1\varphi = [\sum_{|I|=p} \overline{\partial}_x \eta_I(x, y, z, \overline{z}) d\overline{z}_I].$$

It is clear from the definition that

$$(\pi_D)_* \circ d_1 = \overline{\partial}_b \circ (\pi_D)_*.$$

Hence $(\pi_D)_*$ induces the following isomorphism on cohomology

$$E_2^{p,m} \cong \bigoplus_{b \in S_k} H^p(M_b, \overline{\partial}_b).$$

Hence we see that the spectral sequence degenerates at the E_2-term, and the general theory gives

$$E_2^{p,q} = \frac{F^p H^{p+q}(M, \nabla^P)}{F^{p+1} H^{p+q}(M, \nabla^P)}.$$

By a standard derivation this result implies the result stated in the theorem.

$\diamond$

3.3 A Lefschetz fixed point theorem for reducible non-negative polarizations.

We shall make the same assumption on the symplectic manifold (M, ω) and the polarization P as in section 3.2.

Suppose we have a symplectomorphism f of the symplectic manifold M. Assume we are given a lift of f to the line bundle $\mathcal{L}$. We shall require that f preserves P and that we are given a lift of f to the line bundle δ_P. f then induces an endomorphism of the cohomology groups of the complex (3.4).

Since f preserves P, it also preserves E and D, and hence induces a map on the leafspaces $M/E = B$ and $M/D = M'$. We will denote these maps $f_B : B \to B$ and $f' : M' \to M'$. It is clear that $f'_b : M'_b \to M'_{f_B(b)}$ will be holomorphic with respect to the Kähler structure induced on M'_b by P. Suppose $b \in B$ is a fixed point for $f_B : B \to B$. Let $p' \in \pi_B^{-1}(b)$. We shall now study f in the local coordinates around $\pi_D^{-1}(p')$ on M we constructed in section 3.2. In these coordinates we can write

$$f(x, y, z, \overline{z}) = (f_1(x, y, z, \overline{z}), f_2(x, z, \overline{z})(y), f_3(x, y, z, \overline{z}), \overline{f_3}(x, y, z, \overline{z})),$$

where

$$f_2(x, z, \overline{z}) : T^m \to T^m.$$

We calculate that

$$df(\frac{\partial}{\partial y_i}) = \sum_j \frac{\partial f_1^j}{\partial y_i}\frac{\partial}{\partial x_j} + \sum_j \frac{\partial f_2^j}{\partial y_i}\frac{\partial}{\partial y_j} + \sum_j \frac{\partial f_3^j}{\partial y_i}\frac{\partial}{\partial z_j} + \sum_j \frac{\partial \overline{f}_3^j}{\partial y_i}\frac{\partial}{\partial \overline{z}_j}$$

$$df(\frac{\partial}{\partial z_i}) = \sum_j \frac{\partial f_1^j}{\partial z_i}\frac{\partial}{\partial x_j} + \sum_j \frac{\partial f_2^j}{\partial z_i}\frac{\partial}{\partial y_j} + \sum_j \frac{\partial f_3^j}{\partial z_i}\frac{\partial}{\partial z_j} + \sum_j \frac{\partial \overline{f}_3^j}{\partial z_i}\frac{\partial}{\partial \overline{z}_j}$$

$$df(\frac{\partial}{\partial \overline{z}_i}) = \sum_j \frac{\partial f_1^j}{\partial \overline{z}_i}\frac{\partial}{\partial x_j} + \sum_j \frac{\partial f_2^j}{\partial \overline{z}_i}\frac{\partial}{\partial y_j} + \sum_j \frac{\partial f_3^j}{\partial \overline{z}_i}\frac{\partial}{\partial z_j} + \sum_j \frac{\partial \overline{f}_3^j}{\partial \overline{z}_i}\frac{\partial}{\partial \overline{z}_j}.$$

Since f preserves P, E and D, we can in particular conclude that

$$\frac{\partial f_1^j}{\partial y_i} = \frac{\partial f_1^j}{\partial z_i} = \frac{\partial f_1^j}{\partial \overline{z}_i} = 0$$

$$\frac{\partial f_3^j}{\partial y_i} = 0.$$

The symplectic form ω is also preserved by f, so we get from (3.5) that

$$\begin{aligned} f^*\omega &= \sum_{i=1}^m df_1^i \wedge df_2^i + \sum_{i=m+1}^n f^*(A_{ij} + \frac{\partial^2 K}{\partial x_i \partial \overline{z}_j}) df_1^i \wedge d\overline{f}_3^j \\ &+ \sum_{i=m+1}^n f^* B_{ij} df_1^i \wedge df_3^j + \sum_{i=m+1}^n f^* \frac{\partial^2 K}{\partial z_i \partial \overline{z}_j} df_3^i \wedge d\overline{f}_3^j \\ &= \omega. \end{aligned}$$

From this it follows that

$$\sum_{i=1}^n \frac{\partial f_1^i}{\partial x_j} \frac{\partial f_2^i}{\partial y_k} = \delta_{ik}. \tag{3.10}$$

We have the following identifications:

$$\begin{aligned} (d_B f)_{ij} &= \frac{\partial f_1^i}{\partial x_j} \\ (d_D f)_{ij} &= \frac{\partial f_2^i}{\partial y_j}, \end{aligned}$$

so (3.10) is just saying

$$(d_B f)^t d_D f = 1.$$

From this it follows that

$$\frac{\partial^2 f_2^i}{\partial y_k \partial y_j} = 0, \;\; k, j = 1, \ldots, m,$$

so we see that there exists $\Delta \in GL(m, \mathbb{Z})$ and a smooth map $c : V \times U \to \mathbb{R}^m$, such that

$$f_2^i(x, z, \overline{z})(y) = \sum_{j=1}^m \Delta_{ij} y_j + c_i(x, z, \overline{z}),$$

where $\Delta = d_D f$. Moreover, since

$$\operatorname{Det}(d_B f) \operatorname{Det}(d_D f) = 1,$$

we conclude that $\operatorname{Det}(d_D f)(p),\ p \in M$, only depends on $\pi_E(p) \in B$.

The following proposition computes the trace on the cohomology groups of any symplectomorphism that preserves P on the basis of the local description of f.

Proposition 3.3.1 *Let P be a reducible non-negative polarization on a compact symplectic manifold. Suppose f is a symplectomorphism of M that preserves P. Assume moreover we have a lift f^* of f to $\mathcal{L} \otimes \delta_P$. Then*

$$\sum_{i=0}^n (-1)^i \operatorname{Tr}(f^* : H^n(M, \nabla^P)) =$$

$$(-1)^m \sum_{\{b \in S_k | f_B(b) = b\}} \operatorname{Det}(d_D f)(b) \sum_{i=0}^{n-m} (-1)^i \operatorname{Tr}(f_b'^* : H^i(M_b', \overline{\partial}_b)),$$

where $f_b'^*$ *is the lift of* f_b' *to* $\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}}$, *constructed from* $f^*|_{\pi_E^{-1}(\{b\in S_k|f_B(b)=b\})}$ *the lift of* $f|_{\pi_E^{-1}(\{b\in S_k|f_B(b)=b\})}$ *to* $\mathcal{L}^k \otimes \delta_P|_{\pi_E^{-1}(\{b\in S_k|f_B(b)=b\})}$.

Note that there is no assumption on the nature of the fixed point set of f in this proposition.

Proof. Recall the spectral sequence from the proof of theorem 3.2.1. We have that the following diagram is commutative

$$\begin{array}{ccc} E_1^{p,m} & \xrightarrow{f^*} & E_1^{p,m} \\ {\scriptstyle (\pi_D)_*}\downarrow & & \downarrow{\scriptstyle (\pi_D)_*} \\ \bigoplus_{b\in S_k} C^\infty(M_b', \Omega^{0,p}(\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}})) & \xrightarrow{\bigoplus_{\{b\in S_k|f_B(b)=b\}} \mathrm{Det}(d_D f)(b) f_b'^*} & \bigoplus_{b\in S_k} C^\infty(M_b', \Omega^{0,p}(\mathcal{L}_b^k \otimes K_b^{\frac{1}{2}})). \end{array}$$

Hence we get

$$\mathrm{Tr}(f^* : E_2^{p,m} \to E_2^{p,m}) = \sum_{\{b\in S_k|f_B(b)=b\}} \mathrm{Det}(d_D f)(b)\,\mathrm{Tr}(f_b'^* : H^p(M_b', \overline{\partial}_b)).$$

The result follows from this equality.

◇

Using proposition 3.3.1, we can now give a fixed point formula for the trace of the action of f on the cohomology groups, in the case where f has simple fixed points.

Theorem 3.3.1 *Assume P is a reducible non-negative polarization on a compact symplectic manifold. Suppose f is a symplectomorphism of M that preserves P. We shall assume that f has simple fixed points. Assume moreover that we have a lift f^* of f to $\mathcal{L}\otimes\delta_P$. Then*

$$\sum_{i=0}^{n}(-1)^i\,\mathrm{Tr}(f^* : H^i(M, \nabla^P)) =$$
$$\sum_{\{p\in M|f(p)=p\}} \nu_p \sum_{i=0}^{n}(-1)^i \frac{\mathrm{Tr}(f_p^* : \Lambda^i \overline{P}_p^* \otimes \mathcal{L}_p^k \otimes \delta_{P_p})}{|\,\mathrm{Det}(1-df(p))|},$$

where

$$\nu_p = \mathrm{Sign}\,\mathrm{Det}(1-df_B(p)).$$

Before proving this theorem, we shall compute the number of fixed points for an affine diffeomorphism of an m-dimensional torus.

Let $\Delta \in GL(m,\mathbb{Z})$ and $y_0 \in \mathbb{R}^m$. Consider the map $F : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$F(y) = \Delta(y) + y_0.$$

This map induces a diffeomorphism of the torus. We will also call this diffeomorphism F.

Lemma 3.3.1 *Suppose* $\operatorname{Det}(\Delta - 1) \neq 0$. *Then* $F : T^m \to T^m$ *has finitely many fixed points and the number of fixed points is* $|\operatorname{Det}(\Delta - 1)|$.

Proof. We calculate

$$F(y) = y \quad \Leftrightarrow$$
$$\exists \lambda \in \mathbb{Z}^m : (\Delta - 1)y = y_0 + \lambda.$$

So we see that the number of fixed points is finite and it is given by the order of

$$\frac{(\Delta - 1)^{-1}\mathbb{Z}^m}{\mathbb{Z}^m},$$

which is the same as $|\mathbb{Z}^m/(\Delta - 1)\mathbb{Z}^m|$. It is a simple exercise to prove that

$$|\frac{\mathbb{Z}^m}{(\Delta - 1)\mathbb{Z}^m}| = |\operatorname{Det}(\Delta - 1)|.$$

$\diamond$

Using proposition 3.3.1 and lemma 3.3.1 it is easy to prove theorem 3.3.1

Proof of theorem 3.3.1. By proposition 3.3.1 and the holomorphic Lefschetz fixed point theorem we get

$$\sum_{i=0}^{n}(-1)^i \operatorname{Tr}(f^* : H^i(M, \nabla^P))$$
$$= (-1)^m \sum_{\{b \in S_k | f_B(b) = b\}} \operatorname{Det}(d_D f)(b) \sum_{\{p \in M'_b | f'_b(p) = p\}} \sum_{i=0}^{n-m} (-1)^i \frac{\operatorname{Tr}(f'^*_b : (\Lambda^{0,i} M'_b)_p \otimes (\mathcal{L}^k_b \otimes \delta_{P_b})_p)}{|\operatorname{Det}(1 - df'_b(p))|}.$$

Using lemma 3.3.1., we can write this as

$$\sum_{i=0}^{n}(-1)^i \operatorname{Tr}(f^* : H^i(M, \nabla^P))$$
$$= (-1)^m \sum_{\{b \in S_k | f_B(b) = b\}} \frac{\operatorname{Det}(d_D f)(b)}{|\operatorname{Det}(1 - d_D f)(b)|} \sum_{\{p \in \pi_E^{-1}(b) | f(p) = p\}}$$
$$\sum_{i=0}^{n-m} (-1)^i \frac{\operatorname{Tr}(f'^*_b : (\Lambda^{0,i} M'_b)_{\pi_D(p)} \otimes (\mathcal{L}^k_b \otimes \delta_{P_b})_{\pi_D(p)})}{|\operatorname{Det}(1 - df'_b(p))|}.$$

We note that if $b \in B$ is a fixed point for f_B, but b is not a Bohr-Sommerfeld leaf, then

$$\sum_{\{p \in \pi_E^{-1}(b) | f(p) = p\}} \sum_{i=0}^{n-m} (-1)^i \frac{\operatorname{Tr}(f'^*_b : (\Lambda^{0,i} M'_b)_{\pi_D(p)} \otimes (\mathcal{L}^k_b \otimes \delta_{P_b})_{\pi_D(p)})}{|\operatorname{Det}(1 - df'_b(p))|} = 0.$$

This follows by applying the usual Lefschetz fixed point theorem to the elliptic complex

$$0 \to C^\infty(\pi_E^{-1}(b), \mathcal{L}^k \otimes \delta_P) \xrightarrow{\nabla^P} \ldots \xrightarrow{\nabla^P} C^\infty(\pi_E^{-1}(b), \Lambda^n \overline{P}^* \otimes \mathcal{L}^k \otimes \delta_P) \to 0 \tag{3.11}$$

and the fact that the cohomology groups of this complex are all trivial. Combining this with the previous result, we get

$$\begin{aligned}&\sum_{i=0}^{n} (-1)^i \operatorname{Tr}(f^* : H^i(M, \nabla^P)) \\ &= (-1)^m \sum_{\{p \in M \mid f(p)=p\}} \frac{\operatorname{Det}(d_D f)(p)}{|\operatorname{Det}(1 - d_D f)(p)|} \sum_{i=0}^{n-m} (-1)^i \frac{\operatorname{Tr}(f_b'^* : (\Lambda^{0,i} M_b')_{\pi_D(p)} \otimes (\mathcal{L}_b^k \otimes \delta_{P_b})_{\pi_D(p)})}{|\operatorname{Det}(1 - df_b'(p))|}.\end{aligned}$$

From (3.10) we get that

$$\begin{aligned}\operatorname{Det}((1 - df_B)(d_D f)^t)& \\ &= \operatorname{Det}(d_D f - 1) \\ &= (-1)^m \operatorname{Det}(1 - d_D f).\end{aligned}$$

Since

$$\operatorname{Det}(1 - d_D f) = \sum_{i=0}^{m} (-1)^i \operatorname{Tr}(\Lambda^i(d_D f)),$$

we get that

$$\sum_{i=0}^{n} (-1)^i \operatorname{Tr}(f^* : H^i(M, \nabla^P)) = \sum_{\{p \in M \mid f(p)=p\}} \nu_p \sum_{i=0}^{n} (-1)^i \frac{\operatorname{Tr}(f^*(p) : \Lambda^i \overline{P}_p^* \otimes \mathcal{L}_p^k \otimes \delta_{P_p})}{|\operatorname{Det}(1 - df(p))|},$$

where

$$\nu_p = \operatorname{Sign} \operatorname{Det}(1 - df_B(p)).$$

Here we have used that

$$|\operatorname{Det}(1 - df(p))| = |\operatorname{Det}(1 - df_B(p))||\operatorname{Det}(1 - df_D(p))||\operatorname{Det}(1 - df_b'(p))|,$$

but this follows from the fact that f preserves D and E.

$\diamond$

3.4 An index theorem for reducible non-negative polarizations.

Suppose M is a compact symplectic manifold and assume that P' is a Kähler polarization on M. We can then identify $\delta_{P'} = K^{\frac{1}{2}}$, where K is the canonical bundle. The complex (3.4) for P' becomes the $\overline{\partial}$-complex with values in $\mathcal{L}^k \otimes K^{\frac{1}{2}}$, which is elliptic, and by the Hirzebruch-Riemann-Roch formula we get

$$\begin{aligned}\mathrm{Index}(\nabla^{P'}) &= \int_M \mathrm{ch}(\mathcal{L}^k \otimes K^{\frac{1}{2}}) \wedge \mathrm{ch}(K^{-\frac{1}{2}}) \wedge \widehat{A}(TM) \\ &= \int_M \mathrm{ch}(\mathcal{L}^k) \wedge \widehat{A}(TM).\end{aligned}$$

This formula shows that $\mathrm{Index}(\nabla^{P'})$ for Kähler polarizations P' on M only depends on the symplectic structure of M, since $\mathrm{ch}(\mathcal{L}^k) = \exp(k\omega)$.

We shall now prove the following analogue of the Hirzebruch-Riemann-Roch index theorem for reducible non-negative polarizations P, which satisfies the following some what technical assumption. We shall assume that the T^m bundle $\pi_D : M \to M'$ admits a flat connection. Note, that with this connection, we can define a closed 2-form ω_1 on M which in the local symplectic coordinates $(x, y, z, \overline{z})$ constructed in section 3.2 is given by

$$\omega_1 = \sum_{i=1}^{m} dx_i \wedge dy_i.$$

Notice that this condition is trivially satisfied for real polarizations and Kähler polarizations.

Theorem 3.4.1 *Let M be a compact symplectic manifold. Suppose we have a reducible non-negative polarization P on M such that the T^m bundle $\pi_D : M \to M'$ admits a flat connection. The index of the complex (3.4) is given by*

$$\mathrm{Index}(\nabla^P) = (-1)^m \int_M \mathrm{ch}(\mathcal{L}^k) \wedge \widehat{A}(TM). \tag{3.12}$$

From the proof it will follow that if P is a real polarization, then the right hand side of (3.12) just reduces to the volume of $(M, k\omega)$. So in short the theorem is in that case saying that the cardinality of the level k Bohr-Sommerfeld set is given by the volume of $(M, k\omega)$:

$$\# S_k = k^n \,\mathrm{Vol}(M).$$

Proof. First of all we notice that using the symplectic form, we get an isomorphism

$$\begin{aligned}TM &\cong D \oplus \pi_D^* TM' \\ &\cong \pi_E^* T^* B \oplus \pi_D^* TM'.\end{aligned}$$

Recall from section 3.2 that we constructed a lattice inside $\pi_B^*(T^*B)$, hence the bundle $\pi_B^*(T^*B)$ is flat. This means that

$$\hat{A}(TM) = \pi_D^*(\hat{A}(TM')).$$

The left hand side of (3.12) becomes

$$\begin{aligned}
&\int_M \exp(k\omega) \wedge \hat{A}(TM) \\
&\qquad = \int_B (\pi_E)_*(\exp(k\omega) \wedge \hat{A}(TM)) \\
&\qquad = \int_B (\pi_B)_*((\pi_D)_*(\exp(k\omega)) \wedge \hat{A}(TM'))
\end{aligned}$$

Using the local expression (3.5) for the symplectic form we have that

$$\omega = \omega_1 + \omega_2 + \omega_3,$$

where

$$\omega_1 = \sum_{i=1}^{m} dx_i \wedge dy_i$$

$$\omega_2 = \sum_{i=m+1}^{n} (A_{ij} + \frac{\partial^2 K}{\partial x_i \partial \overline{z}_j}) dx_i \wedge d\overline{z}_j + \sum_{i=m+1}^{n} B_{ij} dx_i \wedge dz_j$$

$$\omega_3 = \sum_{i=m+1}^{n} \frac{\partial^2 K}{\partial z_i \partial \overline{z}_j} dz_i \wedge d\overline{z}_j.$$

Here ω_3 is a local expression for ω_b, the reduction of ω to M'_b.

We then see that

$$\begin{aligned}
&(\pi_D)_*(\exp(k\omega)) \\
&\qquad = (\pi_D)_*(\sum_{i=0}^{n} \frac{1}{i!} \sum_{j=0}^{i} \begin{pmatrix} i \\ j \end{pmatrix} (k\omega_1)^j (k(\omega_2 + \omega_3))^{i-j}) \\
&\qquad = k^m dx_1 \wedge \ldots \wedge dx_m \sum_{i=m}^{n} \frac{1}{(i-m)!} (k(\omega - \omega_1))^{i-m}.
\end{aligned}$$

Note that $\omega - \omega_1$ is closed and basic with respect to π_D, hence it is the pull back of a form on M'. We shall also denote that form by $\omega - \omega_1$. Let Λ be the submanifold of $\pi_B^*(T^*B)$, which consists of points in $\pi_B^*(T^*B)$, that acts trivially on the corresponding fibres. We will use the notation p for the projection $p : \pi_B^*(T^*B)/\Lambda \to M'$.

Let us now define a section $s_k : M' \to \pi_B^*(TB)/\Lambda^*$ by the following requirement. Note that any $\alpha_{p'} \in \Lambda_{p'}$ represents a homology class in $H_1(\pi_D^{-1}(p'), \mathbb{Z})$, for $p' \in \pi_B^{-1}(b)$. We then define $s_k(p') \in \pi_B^*(T_bB)/\Lambda_{p'}^*$ by

$$\exp(2\pi i \alpha_{p'}(s_k(p'))) = \mathrm{Hol}_{[\alpha_{p'}]}(\mathcal{L}^k \otimes \delta_P|_{\pi_D^{-1}(p')}),$$

for $p' \in \pi_B^{-1}(b)$. Here $\mathrm{Hol}_{[\alpha_{p'}]}(\mathcal{L}^k \otimes \delta_P|_{\pi_D^{-1}(p')})$ is the holonomy of the flat bundle $\mathcal{L}^k \otimes \delta_P|_{\pi_D^{-1}(p')}$ around $[\alpha_{p'}]$. Let $s_0 : M' \to \pi_B^*(TB)/\Lambda$ be the zero-section.

We now note that

$$\pi_B(p(s_0(B) \cap s_k(B))) = S_k, \;\; k = 1, 2, \ldots.$$

Let us assume that we have chosen an orientation on B. If B is not orientable, we will just lift the whole situation to the orientation cover of B. Since M'_b is oriented, we get an orientation on M'. From the local study of the polarization we know there exist coordinates around each point $p' \in M'$, such that s_k is given by

$$s_k(x, z, \overline{z}) = \sum_{i=1}^{n} (kx_i + x_i^0) \frac{\partial}{\partial x_i}.$$

We can assume that $(x, z, \overline{z})$ are positive coordinates on M'.The orientation of M' gives a canonical orientation of $\pi_B^* TB$. From the coordinate expression of s_k we see that is possible to orient $s_k(M')$, such that $s_k(M')$ and $s_0(M')$ always intersect positively in $\pi_B^*(TB)/\Lambda^*$. Now define a smooth n-form α on $\pi_B^*(TB)/\Lambda^*$ by

$$\alpha_b = e_1 \wedge \ldots \wedge e_n,$$

where $(e_1, \ldots, e_n)$ is a positive basis of $\Lambda_{p'} \subset \pi_B^*(T^*_{\pi_B(p')}B)$. It is clear that α is the Poincaré dual of the zero section of $\pi_B^*(TB)/\Lambda^*$. Suppose we have a closed $2(n-m)$-form β on M'. We then get by Poincaré duality, that

$$\begin{aligned}
\int_{p(s_k(M') \cap s_0(M'))} \beta \\
&= \int_{s_k(M') \cap s_0(M')} p^*(\beta) \\
&= \int_{s_k(M')} \alpha \wedge p^*(\beta) \\
&= \int_{M'} s_k^*(\alpha \wedge p^*(\beta)) \\
&= \int_{M'} s_k^*(\alpha) \wedge \beta.
\end{aligned}$$

The local expression for s_k shows that

$$s_k^*(\alpha) = k^m dx_1 \wedge \ldots \wedge dx_m.$$

Let β be the closed form, which in local coordinates is given by

$$\beta = \sum_{i=m}^{n} \frac{1}{(i-m)!} (k(\omega - \omega_1))^{i-m} \wedge \hat{A}(TM').$$

By the previous calculations we get that

$$\begin{aligned}
&\int_M \exp(k\omega) \wedge \hat{A}(TM) \\
&\qquad = \int_{p(s_k(M')\cap s_0(M'))} \sum_{i=m}^{n} \frac{1}{(i-m)!}(k(\omega-\omega_1))^{i-m} \wedge \hat{A}(TM') \\
&\qquad = \sum_{p\in S_k} \int_{M_b'} \sum_{i=0}^{n-m} \frac{1}{i!}(k\omega_b)^i \wedge \hat{A}(TM_b').
\end{aligned}$$

Here we have used that

$$\hat{A}(i^*(TM')) = \hat{A}(TM_b'),$$

for the inclusion $i : M_b' \to M'$.

By theorem 3.2.1 and the Hirzebruch-Riemann-Roch theorem applied to $(M_b', \overline{\partial}_b)$, we thus get

$$\mathrm{Index}(\nabla^P) = (-1)^m \int_M \mathrm{ch}(\mathcal{L}^k) \wedge \widehat{A}(TM).$$

$\diamond$

In connection with this theorem we should mention that Guillemin and Sternberg in [27] have worked out explicitly the quantization of coadjoint orbits of $U(n)$ with respect to some non-trivial (singular) real polarizations. They find that the quantization agrees with the usual Kähler quantization.

3.5 Geometric quantization of tori with respect to linear polarizations.

Let us now consider the case where M, the symplectic manifold, is a torus, i.e.

$$M = V/\Lambda,$$

where V is a vector space equipped with a symplectic structure, and Λ is a discrete lattice in V of maximal rank. Let P be a real linear polarization on M. Here Linear means that P is induced from a Lagrangian subspace P of V, i.e. in the notation of chapter 1 we have that $P \in L_n(V \otimes \mathbb{C})$. Assume that ω is an integral form on M. This ensures that there exists a prequantum line bundle $\mathcal{L}$. In fact, we can make an explicit construction of $\mathcal{L}$. Choose an integral basis $\lambda = (\lambda_1, \dots, \lambda_{2n})$ for Λ. Let $x = (x_1, \dots, x_{2n})$ be the dual coordinates on V, corresponding to $\lambda_1, \dots, \lambda_{2n}$. We assume λ is chosen such that

$$\omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}.$$

Define

$$\alpha = \frac{1}{2}\sum_{i=1}^{n}(x_i dx_{n+i} - x_{n+i} dx_i).$$

Then

$$\omega = d\alpha.$$

α defines a connection ∇ in the trivial line bundle $\tilde{\mathcal{L}} = V \times \mathbb{C}$. Suppose we are given the theta characteristics $\epsilon : \Lambda \to \{1, -1\}$ such that

$$\epsilon_{\lambda_1+\lambda_2} = \epsilon_{\lambda_1}\epsilon_{\lambda_2}\exp(\pi i \omega(\lambda_1, \lambda_2)).$$

This enables us to define an action of Λ on $\tilde{\mathcal{L}}$ by

$$\lambda(v, z) = (v + \lambda, e_\lambda(v) z),$$

where

$$e_\lambda(v) = \epsilon_\lambda \exp(\pi i \omega(\lambda, v)).$$

Then $\mathcal{L} = \tilde{\mathcal{L}}/\Lambda$ is a unitary complex line bundle with compatible connections on M, whose curvature is ω. By the discussion on pp. 232-233 in [53] we can in this case choose δ_P to be trivial, i.e. such that $\mathcal{L}^k \otimes \delta_P \cong \mathcal{L}^k$ as complex line bundles with $\overline{P}$-connections.

For any polarization on M we can consider the space of distributional sections of $\mathcal{L}^k$, which are covariant constant along P:

$$\mathcal{H}_P = \{\varphi \in D(M, \mathcal{L}^k) | \varphi(\nabla_X s) = 0 \ \forall X \in C^\infty(P),\ s \in C^\infty(\mathcal{L}^k)\}.$$

Lemma 3.5.1 *Let P be a real Lagrangian subspace of V. P then induces a real polarization on M, and we get that $\mathcal{H}_P$ is finite dimensional and*

$$\dim \mathcal{H}_P = k^n.$$

Proof. Suppose first that P is a reducible polarization. One then easily verifies that elements of $\mathcal{H}_P$ have to be supported on the Bohr-Sommerfeld leaves. It is then easy to see that $\mathcal{H}_P$ is isomorphic to the top cohomology group of the complex (3.4)

$$\mathcal{H}_P \cong H^n(M, \nabla^P).$$

Hence by theorem 3.4.1 we get for these polarizations that

$$\dim \mathcal{H}_P = k^n.$$

For a general Lagrangian subspace P we can find a Lagrangian subspace P' of V, which is transversal to P and which induces a reducible polarization on M. We shall now see how one can construct an isomorphism

$$U : \mathcal{H}_P \to \mathcal{H}_{P'},$$

and therefore obtain the desired result.

So, suppose P_1 and P_2 are two transverse Lagrangian subspaces of V. We can then find a Lagrangian subspace Y which is tranversal to both P_1 and P_2. Let

$$\rho_i : V \to Y$$

be the projection $V = P_i \oplus Y \to Y$. Clearly the sections of $\tilde{\mathcal{L}}^k$ covariant constant along P_i can be identified with sections of $\tilde{\mathcal{L}}^k|_Y$. Let s be a covariant constant section of $\tilde{\mathcal{L}}^k|_Y$. Using s we identify $C_c^\infty(\tilde{\mathcal{L}}|_Y)$ with $C_c^\infty(Y)$. Extend s to a section s_i of $\tilde{\mathcal{L}}^k$ by extending covariant constantly along P_i. We assume that s is of unit length, hence so is s_i. Let (q,p) be symplectic coordinates on V such that the map ρ_1 is given by $(q,p) \mapsto q$. Suppose that (q',p) are symplectic coordinates on V such that

$$\rho_2(q,p) = q'.$$

Then

$$q'_i = q_i + \sum_{j=1}^{n} S_{ij} p_j,$$

where S is a symmetric matrix and non-singular, since P_1 and P_2 are transversal.

Let $\mathcal{S}(Y)$ be the Schwartz class functions on Y. Consider the operator

$$U : \mathcal{S}(Y) \to \mathcal{S}(Y)$$

given by

$$U(f)(q) = \int_Y \exp(i\pi k < Sp, p >) f(q + Sp) dp$$

for $f \in \mathcal{S}(Y)$.

Since S is invertible, we get that

$$U(f)(q) = \exp(i\pi k < S^{-1}q, q >) \int_Y \exp(i\pi k < p, Sp - 2q >) f(Sp) dp.$$

Now define operators $V, W : \mathcal{S}(Y) \to \mathcal{S}(Y)$ by

$$V(f)(p) = \exp(i\pi k < S^{-1}p, p >) f(p),$$

$$W(f)(p) = f(Sp),$$

and the Fourier transform $F : \mathcal{S}(Y) \to \mathcal{S}(Y)$ by

$$F(f)(q) = \int_Y \exp(-2\pi i k < p, q >) f(p) dp.$$

Since all these maps are isomorphisms on $\mathcal{S}(Y)$, we conlude that

$$U = V \circ F \circ W \circ V$$

is an isomorphism on $\mathcal{S}(Y)$.

We have an action of Λ on $\tilde{\mathcal{L}}^k$. Using s_i to identify sections of $\tilde{\mathcal{L}}^k$, which are covariant constant along P_i with functions on Y, we get two actions ^{*_1} and ^{*_2} of Λ on $\mathcal{S}(Y)$. Let us describe these two actions. For $\lambda \in \Lambda$ we write $\lambda = \lambda_1 + \lambda_2$, where $\lambda_2 \in P_1$ and $\lambda_1 \in Y$ (note that we cannot assume that λ_1 or λ_2 are in Λ.). For $f \in \mathcal{S}(Y)$ we have that

$$\lambda^*(fs_1)(q,p) = \epsilon_\lambda \exp(i\pi k(< \lambda_1, p > - < \lambda_2, q >)) f(q + \lambda_1) s_1(q + \lambda_1, p + \lambda_2).$$

We then calculate that

$$\begin{aligned} & s_1(q + \lambda_1, p + \lambda_2) \\ &= \exp(-i\pi k < p + \lambda_2, q + \lambda_1 >) s_1(q + \lambda, 0) \\ &= \exp(-i\pi k < p + \lambda_2, q + \lambda_1 >) s_1(q, 0) \\ &= \exp(-i\pi k < p + \lambda_2, q + \lambda_1 >) \exp(i\pi k < p, q >) s_1(q, p), \end{aligned}$$

hence

$$\lambda^*(fs_1)(q,p) = \epsilon_\lambda \exp(-i\pi k < \lambda_2, \lambda_1 >) \exp(-i2\pi k < \lambda_2, q >) f(q + \lambda_1) s_1(q,p).$$

So $\lambda^{*_1} : \mathcal{S}(Y) \to \mathcal{S}(Y)$ is given by

$$\lambda^{*_1}(f)(q) = \epsilon_\lambda \exp(-i\pi k < \lambda_2, \lambda_1 >) \exp(-i2\pi k < \lambda_2, q >) f(q + \lambda_1).$$

We also have that

$$\lambda^*(fs_2)(q,p) = \epsilon_\lambda \exp(i\pi k(<\lambda_1, p> - <\lambda_2, q>))f(q+\lambda_1+S(p+\lambda_2))s_2(q+\lambda_1, p+\lambda_2).$$

We compute that

$$\begin{aligned}
&s_2(q+\lambda_1, p+\lambda_2)\\
&\quad = \exp(-i\pi k < p+\lambda_2, q+\lambda_1+Sp+S\lambda_2 >)s_2(q+\lambda_1+Sp+S\lambda_2, 0)\\
&\quad = \exp(-i\pi k < p+\lambda_2, q+\lambda_1+Sp+S\lambda_2 >)s_2(q+Sp, 0)\\
&\quad = \exp(-i\pi k < p+\lambda_2, q+\lambda_1+Sp+S\lambda_2 >)\exp(i\pi k < p, q+Sp >)s_2(q,p),
\end{aligned}$$

hence

$$\begin{aligned}
&\lambda^*(fs_2)(q,p)\\
&\quad = \epsilon_\lambda \exp(-i\pi k < \lambda_2, \lambda_1+S\lambda_2 >)\exp(-i2\pi k < \lambda_2, q+Sp >)f(q+Sp+\lambda_1+S\lambda_2)s_2(q,p),
\end{aligned}$$

which gives us the action $\lambda^{*_2} : \mathcal{S}(Y) \to \mathcal{S}(Y)$ given by

$$\lambda^{*_2}(f)(q) = \epsilon_\lambda \exp(-i\pi k < \lambda_2, \lambda_1+S\lambda_2 >)\exp(-i2\pi k < \lambda_2, q >)f(q+\lambda_1+S\lambda_2).$$

We now compute

$$\begin{aligned}
&\lambda^{*_2}U(f)(q)\\
&\quad = \epsilon_\lambda \exp(-i\pi k < \lambda_2, \lambda_1+S\lambda_2 >)\\
&\quad\ \exp(-i2\pi k < \lambda_2, q >)\int_Y \exp(i\pi k < Sp, p >)f(q+\lambda_1+S(p+\lambda_2))dp
\end{aligned}$$

and

$$\begin{aligned}
&U\lambda^{*_1}(f)(q)\\
&\quad = \epsilon_\lambda \exp(-i\pi k < \lambda_2, \lambda_1 >)\\
&\quad\ \int_Y \exp(i\pi k < Sp, p >)\exp(-i2\pi k < \lambda_2, q+Sp >)f(q+Sp+\lambda_1)dp\\
&\quad = \epsilon_\lambda \exp(-i\pi k < \lambda_2, \lambda_1 >)\\
&\quad\ \int_Y \exp(i\pi k < S(p+\lambda_2), p+\lambda_2 > -i2\pi k < \lambda_2, q+S(p+\lambda_2) >)f(q+\lambda_1+S(p+\lambda_2))dp.
\end{aligned}$$

Since

$$\begin{aligned}
&\epsilon_\lambda \exp(-i\pi < \lambda_2, \lambda_1 >)\exp(i\pi k < S(p+\lambda_2), p+\lambda_2 > -i2\pi k < \lambda_2, q+S(p+\lambda_2) >)\\
&\quad = \epsilon_\lambda \exp(-i\pi k < \lambda_2, \lambda_1+S\lambda_2 >)\exp(i\pi k < Sp, p >)\exp(-i2\pi k < \lambda_2, q >),
\end{aligned}$$

we see that

$$\lambda^{*_2}U(f)(q) = U\lambda^{*_1}(f)(q),$$

i.e. U intertwines the two actions.

Let $\mathcal{S}'(Y)$ be the dual of $\mathcal{S}(Y)$, i.e. the space of temperate distributions. Using the projection π, we can identify $\mathcal{H}_{P_i}$ with the subspace of $\mathcal{S}'(Y)$ which is invariant under the action $*_i$ of Λ on $\mathcal{S}'(Y)$. Note that $\pi^*(\mathcal{H}_{P_i}) \subset \mathcal{S}'(Y)$, since the multipliers, which define $\mathcal{L}^k$, have unit norm. Since $U : \mathcal{S}(Y) \to \mathcal{S}(Y)$ is an isomorphism, it induces an isomorphism

$$U' : \mathcal{S}'(Y) \to \mathcal{S}'(Y).$$

Since U' intertwines the two actions of Λ on $\mathcal{S}'(Y)$, we see that U' induces an isomorphism

$$U : \mathcal{H}_{P_1} \to \mathcal{H}_{P_2}.$$

$\diamond$

As mentioned in the proof of this theorem, we have for reducible polarizations a natural isomorphism

$$\mathcal{H}_P = H^n(M, \nabla^P).$$

Now if $f : M \to M$ is a symplectomorphism, which preserves P and has a lift f^* to $\mathcal{L}^k$, we get that

$$\mathrm{Tr}(f^* : \mathcal{H}_P \to \mathcal{H}_P) = \mathrm{Tr}(f^* : H^n(M, \nabla^P) \to H^n(M, \nabla^P)).$$

By theorem 3.3.1 we then get, if f also has simple fixed points, that

$$\mathrm{Tr}(f^* : \mathcal{H}_P \to \mathcal{H}_P) = \sum_{\{p \in M | f(p) = p\}} \nu_p \sum_{i=0}^{n} (-1)^i \frac{\mathrm{Tr}(f_p^* : \Lambda^i P_p^* \otimes \mathcal{L}_p^k)}{|\,\mathrm{Det}(1 - df(p))|},$$

where

$$\nu_p = \mathrm{Sign}\,\mathrm{Det}(1 - df_B(p)).$$

For a general Lagrangian subspace we make the following conjecture.

Conjecture 3.5.1 *Suppose that f preserves a linear real polarization P on M. We then have that*

$$\mathrm{Tr}(f^* : \mathcal{H}_P \to \mathcal{H}_P) = \sum_{\{p \in M | f(p) = p\}} \nu_p \sum_{i=0}^{n} (-1)^i \frac{\mathrm{Tr}(f_p^* : \Lambda^i P_p^* \otimes \mathcal{L}_p^k)}{|\,\mathrm{Det}(1 - df(p))|},$$

where

$$\nu_p = \mathrm{Sign}\,\mathrm{Det}(1 - df_B(p)),$$

and $df_B(p)$ is the transverse derivative at p.

Let us now consider the space $L(V\otimes\mathbb{C})$ of linear non-negative Lagrangian subspaces of $V\otimes\mathbb{C}$. This space is most conveniently parametrized by the following set of complex $n\times n$ matrices

$$L(V\otimes\mathbb{C})\cong\{\Omega\in M_{n,n}(\mathbb{C})|\Omega^t=\Omega,\ \operatorname{Im}\Omega\geq 0\}. \tag{3.13}$$

In the symplectic coordinates chosen in the beginning of this section the correspondence is given by

$$P=\operatorname{Span}_{\mathbb{C}}\{X_1,\dots,X_n\}\in L(V\otimes\mathbb{C}),$$

where

$$X_i=-\sum_k\Omega_{ik}\frac{\partial}{\partial x_k}+\frac{\partial}{\partial x_{n+i}},\ i=1,\dots,n.$$

We are interested in giving an explicit basis for the spaces $\mathcal{H}_P,\ P\in L_0(V\otimes\mathbb{C})$. Note that the set $L_0(V\otimes\mathbb{C})$ corresponds, in the matrix description, to the open dense subset where $\operatorname{Im}\Omega>0$. In order to describe a basis for $\mathcal{H}_P$, let us consider the following gauge transformation in $\tilde{\mathcal{L}}$:

$$g=\exp(-i\pi k\sum_i(x_i+\sum_j\Omega_{ij}x_{n+j})x_{n+i}). \tag{3.14}$$

One finds that $g^*\nabla^P$ is trivial along the leaves of P, i.e.

$$g\nabla_{X_i}g^{-1}=X_i,$$

and the multipliers transform to

$$g^*e_{\lambda_i}=\epsilon_{\lambda_i},\ i=1,\dots,n$$

and

$$g^*e_{\lambda_{n+i}}=\epsilon_{\lambda_{n+i}}\exp(-i2\pi kz_i-i\pi k\Omega_{ii}),\ i=1,\dots,n,$$

where

$$z_i=x_i+\sum_j\Omega_{ij}x_{n+j}. \tag{3.15}$$

Hence we see because of elliptic regularity, that elements of $\mathcal{H}_P$ are functions $s:V\to\mathbb{C}$, which are holomorphic in the coordinates $(z_1,\dots,z_n)$ and which satisfy

$$\begin{aligned}s(v+\lambda_i)&=\epsilon_{\lambda_i}s(v),\ i=1,\dots,n\\ s(v+\lambda_{n+i})&=\epsilon_{\lambda_{n+i}}\exp(-i2\pi kz_i-i\pi k\Omega_{ii})s(v),\ i=1,\dots,n.\end{aligned} \tag{3.16}$$

These are precisely the classical level k theta functions (see e.g. [24]). Let $a, b \in (\frac{1}{2}\mathbb{Z})^n$ such that

$$\epsilon_{\lambda_i} = \exp(i2\pi a_i),\ \epsilon_{\lambda_{n+i}} = \exp(-i2\pi b_i),\ i = 1, \dots, n.$$

We have the following explicit basis $\theta^{a,b}_\delta$, $\delta \in (\mathbb{Z}/k\mathbb{Z})^n$ for $\mathcal{H}_P$ given by

$$\begin{aligned} &\theta^{a,b}_\delta((x_1, \dots, x_{2n}), \Omega) \\ &\quad = \sum_{l \in \mathbb{Z}^n} \exp(i\pi k < l + \frac{\delta}{k} + a, \Omega(l + \frac{\delta}{k} + a) > +2\pi i k < l + \frac{\delta}{k} + a, z + b >) \end{aligned} \tag{3.17}$$

where

$$z = (z_1, \dots, z_n)$$

and z_i is given by (3.15).

We notice that (3.17) still defines $\theta^{a,b}_\delta$ as a temperate distribution if we relax the condition on Ω to $\operatorname{Im}\Omega \geq 0$. We denote the space of temperate distributions on V by $S'(V)$. The convergence of the sum in (3.17) is then in the weak topology on $S'(V)$. We also have that

$$X_i[\theta^{a,b}_\delta(\cdot, \Omega)] = 0,$$

and (3.16) still holds. Clearly (3.14) makes sense for any matrix Ω, so we can interpret $\theta^{a,b}_\delta(\cdot, \Omega)$ as distributional sections of $\mathcal{L}^k$ over M, which are covariant constant along the directions of the non-negative polarization P corresponding to Ω. It is easy to see that $\theta^{a,b}_\delta(\cdot, \Omega)$, $\delta \in (\mathbb{Z}/k\mathbb{Z})^n$, are linearly independent. Now by lemma 3.5.1 we know that $\dim \mathcal{H}_P = k^n$ for $P \in L_n(V \otimes \mathbb{C})$, hence we see that $\theta^{a,b}_\delta(\cdot, \Omega)$, $\delta \in (\mathbb{Z}/k\mathbb{Z})^n$, form a basis for $\mathcal{H}_P$ for the real polarizations $P \in L_n(V \otimes \mathbb{C})$.

From this we thus see that, when a smooth family of complex polarizations P_τ degenerates into a real polarization P_∞, the basis b_τ for $\mathcal{H}_{P_\tau}$, consisting of theta functions, varies smoothly. Moreover this basis b_τ converges to a basis b_∞ of $\mathcal{H}_{P_\infty}$ in the weak topology on the space of distributional sections of $\mathcal{L}^k$. We shall use these results in the next section, when we study the genus one case of Jones-Witten theory.

We note that if P_∞ is a reducible polarization, then $\mathcal{H}_{P_\infty}$ consists of distributional sections, which are covariant constant along the directions in P_∞, and supported on the Bohr-Sommerfeld leaves of P_∞. Hence we see in this case that the holomorphic sections in $\mathcal{H}_{P_\tau}$ develop into delta-functions with support on the Bohr-Sommerfeld leaves, as P_τ converges to P_∞.

3.6 The Witten invariant for mapping cylinders of tori.

In this section we apply our Lefschetz fixed point theorem to calculate the Witten invariant of Σ_f, where $f \in SL(2,\mathbb{Z})$, and Σ is a genus one surface. Let G be a compact simple and simply connected Lie group. T will denote the maximal torus and W the Weyl group. Let $\mathrm{Det}(w)$, $w \in W$, denote the alternating character on W. Recall that we normalize the bi-invariant inner product $B(\cdot,\cdot)$ on t, by requiring that the highest root has length $\sqrt{2}$. Λ_I denotes the integer lattice. Since $\pi_1(\Sigma)$ is an abelian group on two generators, we can explicitly describe the moduli space $\mathcal{M}(\Sigma)$ of flat G-connections in some principal G-bundle over Σ, namely

$$\mathcal{M}(\Sigma) = (T^n \times T^n)/W.$$

Let

$$V = \mathsf{t} \oplus \mathsf{t}.$$

Then

$$T \times T = V/\Lambda,$$

where

$$\Lambda = \Lambda_I \oplus \Lambda_I.$$

A trivialization of the principal G-bundle over Σ identifies V with a subspace of $\mathcal{A}_\Sigma$, which induces a symplectic form on V:

$$\omega((\xi_1,\eta_1);(\xi_2,\eta_2)) =< \xi_1,\eta_2 > - < \xi_2,\eta_1 > .$$

The 1-form

$$\Theta_A(a) = -i\pi\omega(A,a),\ A \in V,\ a \in V,$$

induces a connection in the trivial bundle $V \times \mathbb{C} = \tilde{\mathcal{L}}$. We can lift the action of Λ action on V to $\tilde{\mathcal{L}}$ by

$$\lambda((A,z)) = (A+\lambda, e_\lambda(A)z),$$

where

$$e_\lambda(A) = \epsilon_\lambda \exp(-\pi i\omega(A,\lambda)),$$

and $\epsilon_\lambda \in \{1,-1\}$ are the theta characteristics satisfying

$$\epsilon_{\lambda_1+\lambda_2} = \epsilon_{\lambda_1}\epsilon_{\lambda_2}\exp(i\pi\omega(\lambda_1,\lambda_2)).$$

Λ preserves the connection, and we get a hermitian bundle with a compatible connection, whose curvature is ω. By taking the trivial lift of the action of W on V, we get an action of W on $\tilde{\mathcal{L}}$, which respects the action of Λ and preserves the connection. This means that $\mathcal{L}$ is actually the prequantum bundle over $\mathcal{M}(\Sigma)$. The natural model for Teichmüller space of Σ is the upper half plane S. (We have changed notation, to avoid using T for both the maximal torus and the upper half plane.) A point τ in S induces a complex structure on Σ. The complex structure τ induces a complex polarization P_τ on V. If we let $\lambda_1,\dots,\lambda_n,\lambda_{n+1},\dots\lambda_{2n}$ be a basis for $\Lambda=\Lambda_I\oplus\Lambda_I$, we then get the dual coordinates $x_1,\dots x_n,x_{n+1},\dots,x_{2n}$ on V. In these coordinates, using the identification (3.13), we have that

$$P_\tau=\operatorname{diag}(\tau).$$

We define

$$\mathcal{H}_{P_\tau}=H^0(T\times T,\mathcal{L}^{k+h}).$$

Here $H^0(T\times T,\mathcal{L}^{k+h})$ denotes holomorphic sections with respect to the polarization P_τ on $T\times T$. The bundle $\mathcal{H}_{P_\tau}$, $\tau\in S$, is equipped with a projectively flat connection, and as is explained in [8] the sections parallel under this connection are the theta functions $\theta_{\gamma,k+h}$ defined by

$$\theta_{\gamma,k'}(\tau,u)=\sum_{\alpha\in\Lambda_I}\exp(i\pi k'\tau<\alpha+\frac{\gamma}{k'},\alpha+\frac{\gamma}{k'}>+2\pi ik'<u,\alpha+\frac{\gamma}{k'}>),\ \forall u\in\mathfrak{t}\otimes\mathbb{C},$$

where γ is a positive weight of G. It is explained in [8] that the vector space one associates to $\mathcal{M}(\Sigma)$ at level k is the Weyl invariant part of $H^0(T\times T,\mathcal{L}^k)$, denoted $H^0(T\times T,\mathcal{L}^k)^W$. However the connection is not the one induced from $H^0(T\times T,\mathcal{L}^k)$. As is argued in [8], one finds that the Weyl anti-invariant part of $H^0(T\times T,\mathcal{L}^h)$ is one dimensional. We denote the Weyl anti-invariant part of $H^0(T\times T,\mathcal{L}^{k'})$ by $H^0(T\times T,\mathcal{L}^{k'})^{-W}$. Let $s\in H^0(T\times T,\mathcal{L}^h)^{-W}-\{0\}$. Then multiplication by s defines an isomorphism

$$s:H^0(T\times T,\mathcal{L}^k)^W\to H^0(T\times T,\mathcal{L}^{k+h})^{-W}.$$

Using this isomorphism and the anti-invariant projectively flat connection in $H^0(T\times T,\mathcal{L}^{k+h})$, we get a projectively flat connection in $H^0(T\times T,\mathcal{L}^k)^W$, and the parallel sections are the Weyl invariant level k theta functions

$$\psi_{\gamma,k}=\frac{\theta^-_{\gamma+\rho,k+h}}{\theta^-_{\rho,h}}.$$

Here $\theta^-_{\gamma,k}$ is the Weyl anti-invariant part

$$\theta^-_{\gamma,k}=\frac{1}{|W|}\sum_{w\in W}\operatorname{Det}(w)\theta_{w(\gamma),k},$$

ρ is half the sum of the positive roots, and h is the dual Coxeter number.

Let $\overline{S}$ denote the natural compactification of S. So $\overline{S}$ is homeomorphic to a closed disc, and it parametrizes non-negative linear polarizations on Σ.

Now recall that by the discussion concluding section 3.5 we have that the theta functions extend over $\overline{S}$ to give a continuous basis for all the spaces $\mathcal{H}_{P_\tau}$, $\tau \in \overline{S}$. From this we see that if $f \in SL(2,\mathbb{Z})$, and $w \in W$, then the action of wf on the covariant constant sections of $\mathcal{H}_{P_\tau}$, $\tau \in S$, is just given by wf's action on $\mathcal{H}_{P_{\tau_w}}$, where τ_w is a fixed point for the action of wf on $\overline{S}$.

We thus get the following expression for the Witten invariant of Σ_f:

$$Z(\Sigma_f) = \frac{1}{|W|} \sum_{w \in W} \mathrm{Det}(w) \, \mathrm{Tr}(wf^* : \mathcal{H}_{P_{\tau_w}}).$$

Independently of whether τ_w lies in S or on the boundary of $\overline{S}$, we can apply our Lefschetz fixed point theorem to get (if P_{τ_w} is real and not reducible, we appeal to our conjecture 3.5.1.)

$$\begin{aligned} Z(\Sigma_f) &= \frac{1}{|W|} \sum_{wfA=A,\ A \in T\times T} \sum_{i=0}^{n} (-1)^i \frac{\mathrm{Tr}(wf_A^* : \Lambda^i P_A \otimes \mathcal{L}^{k+h})}{|\,\mathrm{Det}(1-wf)|} \\ &= \frac{1}{|W|} \sum_{wfA=A,\ A \in T\times T} \frac{\mathrm{Tr}(f_A^* : \mathcal{L}^{k+h})}{|\,\mathrm{Det}(1-wf)|^{\frac{1}{2}}}. \end{aligned}$$

This is the same formula Jeffrey arrives at, using a formal argument with path integrals. (The formula actually differ by a factor $\exp(2\pi i \frac{c}{24})^n$, due to the fact that we have not made sure $Z(\Sigma_f)$ was calculated with respect to Atiyah's standard 2-framing.) In [32] Jeffrey did the number theoretical complicated verification that this formula actually agrees with the combinatorial expression for $Z(\Sigma_f)$, given by the expression of f expanded in the S and T matrices. We here take this as a verification of our conjectural Lefschetz fixed point theorem 3.5.1 for real linear polarizations.

3.7 The Bohr-Sommerfeld set and conformal field theory.

In this section we shall consider some of the singular non-negative polarizations, that were constructed in chapter 2 on the moduli spaces of flat connections in a principal bundle over a closed surface Σ. Let P be a principal G bundle over a closed surface Σ. G is a compact, simply connected and simple Lie group. Chose a maximal torus T of G. Let γ be a simple closed curve on Σ. Assume for simplicity that γ is non-seperating. Suppose we have a metric $\tilde{g}$ with infinite ends on the complement $\tilde{\Sigma} = \Sigma - \gamma$ (see section 2.5). This induces a singular non-negative polarization $P_{[\gamma],\tilde{g}}$ on $\mathcal{M}(\Sigma)$, the moduli space of flat connections in P. By theorem 2.5.1 and the results of section 2.4 we have that the restriction of $P_{[\gamma],\tilde{g}}$ to the open dense subset $\mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}$ is a reducible non-negative smooth polarization on the smooth symplectic manifold $\mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}$.

Let A be a flat connection in P representing a point $[A] \in \mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}$. Since $\mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}$ is a subset of $\mathcal{M}(\Sigma)^{\mathrm{irr}}$, we have that

$$T_{[A]}\mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}} \cong H^1(\Sigma, d_A),$$

and

$$D([A]) = P_{[\gamma],\tilde{g}}([A]) \cap \overline{P_{[\gamma],\tilde{g}}}([A]) \cong H^1_F(\Sigma, d_A)$$

is the isotropic foliation associated to $P_{[\gamma],\tilde{g}}$. By the discussion at the beginning of section 2.4 and lemma 2.4.1 we have an explicit description of the leaves of D. (For notation see section 2.4.) We have that $[\tilde{A}] \in \mathcal{M}(\tilde{\Sigma})^c_c(\mathrm{irr})$ and

$$\pi_c^{-1}([\tilde{A}]) \cong T,$$

since $c \in C_G^{\mathrm{reg}}$, so $Z(\tilde{A}) \cong T$, and $\Gamma_{\tilde{A}} = Z(G)$ acts by conjugation hence trivially on T.

In analogy with definition 3.2.2, we define

Definition 3.7.1 *The level k Bohr-Sommerfeld set for $P_{[\gamma],\tilde{g}}$ on $\mathcal{M}(\Sigma)$ is defined to be*

$$S_k = \{c \in C_G^{\mathrm{reg}} |\ \mathcal{L}^{k+h}|_{\pi_c^{-1}([\tilde{A}])} \text{ is trivial for all } [\tilde{A}] \in \mathcal{M}(\Sigma)^c_c(\mathrm{irr})\}.$$

Here h is the dual Coexeter number of G. We refer to [8] for the explanation of the level shift from k to $k+h$.

We can in fact give a very explicit expression for the level k Bohr-Sommerfeld set, $S_k \subset C_G^{\mathrm{reg}}$. Let $\alpha_1, \ldots, \alpha_m$ be the simple roots of G. Let Λ_I be the integral lattice of $\mathfrak{t}$.

Lemma 3.7.1 *The level k Bohr-Sommerfeld set of $P_{[\gamma],\tilde{g}}$ on $\mathcal{M}(\Sigma)$ is given by*

$$\{[\exp \xi] \in C_G^{\mathrm{reg}} | \xi \in \frac{1}{k+h}\Lambda_I,\ 0 < \frac{1}{2\pi}\frac{2\alpha_i(\xi)}{B(\alpha_i, \alpha_i)} < 1\}. \tag{3.18}$$

If G is simply laced we get the simpler expression

$$S_k = \{c \in C_G^{\mathrm{reg}} | c^{k+h} \in Z(G)\}.$$

Proof. Let U_γ be a tubular neighbourhood of γ on Σ. Assume we have coordinates $(\theta, r) : U_\gamma \to S^1 \times [0,1]$, such that $\gamma = r^{-1}(\frac{1}{2})$. Let A be a flat connection in P representing a point $[A] \in \mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}$. By changing A by a gauge transformation, we can assume that

$$A|_{U_\gamma} = A_\gamma d\theta, A_\gamma \in \mathfrak{t}.$$

Now we have that $\tilde{A} = A|_{\tilde{\Sigma}}$, and $c = h(A)$. We shall now use "broken" gauge tranformations to give an explicit integral basis of $H_1(\pi_c^{-1}([\tilde{A}]), \mathbb{Z})$. Since $c \in C_G^{\mathrm{reg}}$, we have that $Z(\exp(A_\gamma)) = T$. Let $\xi_1, \dots, \xi_m$ be an intergral basis for Λ_I. Consider a smooth function ρ on $[\frac{1}{2}, 1]$ which is constant equal to one in a neighbourhood of $\frac{1}{2}$ in $[\frac{1}{2}, 1]$ and zero in a neighbourhood of 1 in $[\frac{1}{2}, 1]$. Then define the following "broken" gauge transformation of P

$$g_t^i(\theta, r) = \begin{cases} e & r \in [0, \frac{1}{2}) \\ \exp(t\rho(r)\xi_i) & r \in [\frac{1}{2}, 1]. \end{cases}$$

Since $g_t^i(\frac{1}{2})(A|_\gamma) = A|_\gamma$, we see that $(g_t^i)^*(A)$ gives a smooth curve of flat connections in P on Σ. Note that g_1^i is a gauge transformation, so $\{(g_t^i)^*(A)\}$ are closed curves in $\pi_c^{-1}([\tilde{A}])$. Under the isomorphism

$$\pi_c^{-1}([\tilde{A}]) \cong T$$

these curves are mapped to a set of generators of $H_1(T, \mathbb{Z})$, hence the curves

$$\{(g_t^i)^*(A)\}, \ i = 1, \dots, m$$

form an integral basis for $H_1(\pi_c^{-1}([\tilde{A}]), \mathbb{Z})$. In order to determine if $c \in C_G^{\mathrm{reg}}$ is in the level k Bohr-Sommerfeld set, we just need to calculate the holonomy of $\mathcal{L}^{k+h}$ around the curves $\{(g_t^i)^*(A)\}$ for all $[A] \in \mathcal{M}(\Sigma)$ such that $h([A]) = c$.

Let $Y = \Sigma \times I$. We get a principal G-bundle P_Y over Y by trivially lifting P to Y. In P_Y we get a connection in P, by letting

$$A_Y^i(\sigma, t) = (g_t^i)^*(A)(\sigma).$$

Using the gauge transformation g_1^i, we can construct a principal G-bundle $P_{Y_{\mathrm{Id}}}$ over $Y_{\mathrm{Id}} = \Sigma \times S^1$, and A_Y^i gives us a connection $A_{Y_{\mathrm{Id}}}^i$ in $P_{Y_{\mathrm{Id}}}$. By the definition of the line bundle $\mathcal{L}$ (see section 1.1) we have that the holonomy of $\mathcal{L}^{k+h}$ around the curve $\{(g_t^i)^*(A)\}$ is

$$\exp(i2\pi(k+h)CS(A_{Y_{\mathrm{Id}}}^i)) \in \mathbb{C}.$$

So we get $c \in S_k$ is equivalent to

$$\exp(i2\pi(k+h)CS(A_{Y_{\mathrm{Id}}}^i)) = 1, \ i = 1, \dots, m,$$

for all $[A] \in \mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}$, such that $h([A]) = c$. Since

$$(g_t^i)^*(A)(\theta, r) = \begin{cases} A_\gamma d\theta & r \in [0, \frac{1}{2}) \\ A_\gamma d\theta + t\rho'(r)\xi_i dr & r \in [\frac{1}{2}, 1] \end{cases},$$

we see that

$$dA^i_{Y_{\mathrm{Id}}}((\theta, r), t) = \rho'(r)\xi_i dt \wedge dr.$$

So we get

$$\begin{aligned} CS(A^i_{Y_{\mathrm{Id}}}) &= \frac{1}{4\pi^2}\int_{Y_{\mathrm{Id}}} (B(A^i_{Y_{\mathrm{Id}}} \wedge dA^i_{Y_{\mathrm{Id}}}) + \frac{2}{3}B(A^i_{Y_{\mathrm{Id}}} \wedge [A^i_{Y_{\mathrm{Id}}} \wedge A^i_{Y_{\mathrm{Id}}}])) \\ &= \frac{1}{4\pi^2}\int_0^1\int_0^1\int_0^1 \rho'(r)B(A_\gamma, \xi_i)d\theta \wedge dr \wedge dt \\ &= \frac{1}{4\pi^2}B(A_\gamma, \xi_i). \end{aligned}$$

Hence we see $c \in S_k$ is equivalent to

$$\frac{1}{4\pi^2}(k+h)B(A_\gamma, \xi_i) \in \mathbb{Z},\ i = 1, \ldots, m. \tag{3.19}$$

B induces an isomorphism

$$i : \mathsf{t} \to \mathsf{t}^*$$

given by

$$i(\xi)(\xi') = B(\xi, \xi').$$

Since the set of weights are by definition the subset $\hat{T} \subset \mathsf{t}^*$, which take on integer multiples of 2π on Λ_I, we see that (3.19) is equivalent to

$$\frac{1}{2\pi}(k+h)i(A_\gamma) \in \hat{T} \tag{3.20}$$

By corollary 5.61 in [1] we get that (3.20) is equivalent to

$$(k+h)\frac{1}{2\pi}\frac{2\alpha_i(A_\gamma)}{B(\alpha_i, \alpha_i)} \in \mathbb{Z},\ i = 1, \ldots, m. \tag{3.21}$$

Since we are interested in $[\exp A_\gamma] \in C_G^{\mathrm{reg}}$, we see that (3.21) precisely gives the expression (3.18) for the Bohr-Sommerfeld set.

Recall that if G is simply laced, we have chosen the normalization such that $B(\alpha_i, \alpha_i) = 2,\ i = 1, \ldots, m$. Now by proposition 5.3 in [1] we see that (3.21) and therefore (3.19) are equivalent to

$$(\exp A_\gamma)^{k+h} \in Z(G).$$

Since $[A] \in \mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}$, we clearly have that $[\exp A_\gamma] \in C_G^{\mathrm{reg}}$.

◇

This lemma gives a bijection between the Bohr-Sommerfeld set S_k and the set

$$L' = \left\{ \xi \in \frac{1}{2\pi}\Lambda_I \mid 0 < 2\frac{\alpha_i(\xi)}{B(\alpha_i,\alpha_i)} < k+h,\ i = 1,\dots,m \right\}.$$

Now let $\rho = \frac{1}{2}\sum_{i=1}^{l} \alpha_i$, where $\alpha_1,\dots,\alpha_l$ are the positive roots of G. Now the map $\rho : \mathfrak{t} \to \mathfrak{t}$, which maps $\xi \mapsto \xi - \rho$, gives in fact a bijection of the set L' with

$$L = \left\{ \xi \in \frac{1}{2\pi}\Lambda_I \mid 0 \leq 2\frac{\alpha_i(\xi)}{B(\alpha_i,\alpha_i)} \leq k,\ i = 1,\dots,m \right\}. \tag{3.22}$$

This follows from Lemma 5.58 in [1] and a fact, which follows directly from the definition of h, namely that

$$B(\alpha_m, \rho) = h - 1,$$

where α_m is the highest root of G.

Now the set L is precisely the "labelling set" of the conformal field theory one gets from the Lie group G (see [44] for the definition of conformal field theory.) For that theory the labelling set is the set of level k positive energy irreducible representations of the loop group LG, which by formula (9.3.6) in [37] is precisely the set L.

We shall end this section with some conjectural remarks on the potential consequences of lemma 3.7.1 and the degeneration results of section 2.6. By the theorem 2.6.2, we have that the non-negative polarization $P_{[\gamma],\tilde{g}}$ on $\mathcal{M}(\Sigma)$ is the limit of a sequence of Kähler polarizations on $\mathcal{M}(\Sigma)$, induced from Teichmüller space. Hence, if the holomorphic sections of the prequantum line bundle $\mathcal{L}^{k+h}$, over the moduli space $\mathcal{M}(\Sigma)$, behave as in the genus one case in section 3.5 and 3.6, we would get a natural isomorphism (see definition 1.3.2)

$$Z(\Sigma) \cong \bigoplus_{c \in S_k} H^0(\mathcal{M}(\tilde{\Sigma})^c_c, \mathcal{L}_c^{k+h}), \tag{3.23}$$

where $\mathcal{L}_c$ is the line bundle $(\pi_c)_*(\mathcal{L}|_{h^{-1}(c)})$, and the holomorphic structure on $\mathcal{M}(\tilde{\Sigma})^c_c$ is induced from $P_{[\gamma],\tilde{g}}$, i.e. from the conformal structure induced on $\tilde{\Sigma}$ by $\tilde{g}$. We recognize this as the factorization axiom from the conformal field theory, associated to G (see [44]).

The understanding we got in section 3.5 and 3.6 of the asymptotics of the covariant constant sections of $\mathcal{H}$ over Teichmüller space S in the genus one case relies heavily on the fact, that we know explicit expresions for the covariant constant sections. At present no such expressions are known for the higher genus cases.

Let us here just indicate how one can construct a map

$$i_1 : H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^{k+h}) \to \bigoplus_{c \in S_k} H^0(\mathcal{M}(\tilde{\Sigma})^c_c, \mathcal{L}_c^{k+h}). \tag{3.24}$$

and when $\mathcal{M}(\Sigma)$ is smooth, a map in the opposite direction

$$i_2 : \bigoplus_{c\in S_k} H^0(\mathcal{M}(\tilde{\Sigma})^c_c, \mathcal{L}^{k+h}_c) \to H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^{k+h}) \tag{3.25}$$

Here $\mathcal{M}(\Sigma)_\sigma$ is $\mathcal{M}(\Sigma)$ with the complex structure induced from a point σ in Teichmüller space.

Note that there is a unique section

$$\alpha \in C^\infty(\Lambda^m D)|_{\cup_{c\in S_k} h^{-1}(c)\cap\mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}},$$

which is covariant constant with respect to the D-connection $\tilde{\nabla}$ defined by (3.3) and is normalized such that

$$\int_{\pi_c^{-1}([\tilde{A}])} \alpha = 1$$

for all $[\tilde{A}] \in \mathcal{M}(\tilde{\Sigma})^c_c(\mathrm{irr}),\ c \in S_k$. If $s \in H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^{k+h})$, then

$$\bigoplus_{c\in S_k} (\pi_c)_*(s|_{\pi_c^{-1}(\mathcal{M}(\tilde{\Sigma})^c_c(\mathrm{irr}))}\alpha) \in \bigoplus_{c\in S_k} L_2(\mathcal{M}(\tilde{\Sigma})^c_c(\mathrm{irr}), \mathcal{L}^{k+h}_c).$$

We have an orthogonal projection

$$P_c : L_2(\mathcal{M}(\tilde{\Sigma})^c_c(\mathrm{irr}), \mathcal{L}^{k+h}_c) \to H^0(\mathcal{M}(\tilde{\Sigma})^c_c, \mathcal{L}^{k+h}_c),$$

and we can define the map (3.24) by

$$i_1(s) = \bigoplus_{c\in S_k} P_c((\pi_c)_*(s|_{\pi_c^{-1}(\mathcal{M}(\tilde{\Sigma})^c_c(\mathrm{irr}))}\alpha)).$$

Assume now that $\mathcal{M}(\Sigma)$ is smooth. Let us denote the s Sobolev space completion of $C^\infty(\mathcal{M}(\Sigma), \mathcal{L}^{k+h})$ by $H_s(\mathcal{M}(\Sigma), \mathcal{L}^{k+h})$. We can construct a map

$$i : \bigoplus_{c\in S_k} L_2(\mathcal{M}(\tilde{\Sigma})^c_c(\mathrm{irr}), \mathcal{L}^{k+h}_c) \to H_{-s}(\mathcal{M}(\Sigma), \mathcal{L}^{k+h}),$$

provided s is sufficiently large by letting

$$i(\bigoplus_{c\in S_k} s_c)(s) = \bigoplus_{c\in S_k} \int_{\mathcal{M}(\tilde{\Sigma})^c_c} < (\pi_c)_*(s|_{\pi_c^{-1}(\mathcal{M}(\Sigma)^c_c(\mathrm{irr}))}\alpha), s_c > \omega_c^l.$$

Here ω_c is the symplectic form on $\mathcal{M}(\tilde{\Sigma})^c_c$, and $2l$ is the dimension of $\mathcal{M}(\tilde{\Sigma})^c_c$. By standard Fredholm theory applied to the $\overline{\partial}$-complex with values in $\mathcal{L}^{k+h}$ on $\mathcal{M}(\Sigma)_\sigma$, we have that

$$H_{-s}(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^{k+h}) = \ker \overline{\partial}_\sigma \oplus \mathrm{Im}\, \overline{\partial}^*_\sigma \overline{\partial}_\sigma.$$

Thus, we get a well defined projection

$$R_\sigma : H_{-s}(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^{k+h}) \to H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^{k+h}).$$

We can now define (3.25) by

$$i_2 = R_c \circ i : \bigoplus_{c \in S_k} H^0(\mathcal{M}(\tilde{\Sigma})^c_c, \mathcal{L}^{k+h}_c) \to H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}^{k+h}),$$

If these maps were indeed isomorphisms, they would seem to indicate that the factorization axiom (3.23) in some sense is local with respect to diffeomorphisms. Let us illustrate this with an example.

Let γ be a simple closed curve on Σ. Let f_n be the diffeomorphism of Σ which preforms n Dehn twist around γ. Note that the restriction of f_n to $\tilde{\Sigma} = \Sigma - \gamma$ is homotopic to the identity on $\tilde{\Sigma}$. We assume that a trivialization of P has been chosen. In section 1.1 we constructed a lift $f_n^* : \mathcal{L} \to \mathcal{L}$ of $f_n : \mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma)$. This lift induces a lift $(\mathrm{Id})^*$ of the the identity map on $\mathcal{M}(\Sigma)^c_c$, $c \in S_k$, to the line bundle $\mathcal{L}^{k+h}_c = (\pi_c)_*(\mathcal{L}^{k+h}|_{h^{-1}(c)})$ over $\mathcal{M}(\Sigma)^c_c$, $c \in S_k$.

Let U_γ be a tubular neighbourhood of γ on Σ. Assume we have coordinates $(\theta, r) : U_\gamma \to S^1 \times [0,1]$, such that $\gamma = r^{-1}(\frac{1}{2})$. Let ρ be a function on $[0,1]$ which is zero in a neighbourhood of 0 in $[0,1]$, and equal to one in a neighbourhood of 1 in $[0,1]$. We can then explicitly write down an expression for f_n

$$f_n(\theta, r) = (\theta + n\rho(r), r).$$

It is clear that f_n preserves the non-negative polarization $P_{[\gamma],\tilde{g}}$ on $\mathcal{M}(\Sigma)$. Here $\tilde{g}$ is any metric with infinite ends on $\tilde{\Sigma}$. Let A be a flat connection in the bundle P which represents a point $[A] \in \mathcal{M}(\Sigma)_{[\gamma],\mathrm{reg}}$. We can assume that

$$A|_{U_\gamma} = A_\gamma d\theta,\ A_\gamma \in \mathsf{t}.$$

Suppose that $[A]$ is in a level k Bohr-Sommerfeld leaf and assume

$$h_\gamma([A]) = c \in C_G.$$

The line bundle $\mathcal{L}^{k+h}$ is then trivial over the fibre $\pi_c^{-1}(\pi_c([A]))$. Now we have that

$$f_n^*(A) = A_\gamma d\theta + n\rho'(r) A_\gamma dr. \tag{3.26}$$

So on the 3-manifold Σ_{f_n}, we can now define the following connection in the bundle $P_{f_n} = P \times S^1$:

$$A_{f_n}(t) = A_\gamma d\theta + tn\rho'(r) A_\gamma dr,\ t \in [0,1].$$

By the definition of the lift of f_n to $\mathcal{L}^{k+h}$ (see section 1.1), it follows that $(\mathrm{Id})^*_{\pi_c([A])}$ is multiplication by $\exp(2\pi i(k+h)CS(A_{f_n})) \in \mathbb{C}$. We calculate that

$$CS(A_{f_n}) = \frac{1}{4\pi^2} nB(A_\gamma, A_\gamma).$$

Hence we see that $(\mathrm{Id})^*_{\pi_c([A])}$ is multiplication by

$$\exp(2\pi i(k+h)n\frac{1}{4\pi^2}B(A_\gamma, A_\gamma)) \in \mathbb{C}.$$

Now, if we assume that (3.23) is natural, we get using lemma 3.7.1 the following conjectural expression for the Witten invariant of Σ_{f_n}.

$$Z(\Sigma_{f_n}) = \sum_{\xi \in L} \exp(2\pi i n \frac{1}{k+h} B(\xi+\rho, \xi+\rho)) \dim H^0(\mathcal{M}(\tilde{\Sigma})^{c(\xi)}_{c(\xi)}, \mathcal{L}^{k+h}_{c(\xi)}),$$

where

$$c(\xi) = \left[\exp(2\pi \frac{\xi+\rho}{k+h})\right] \in C_G,$$

and L is defined by (3.22). In this expression we have ignored the framing issue.

Let us conclude this section by remarking that one should in principle be able to construct the maps (3.24) and (3.25) for any non-negative polarization $P_{F,\tilde{g}}$, F being a Strebel measured foliation F and $\tilde{g}$ a metric with infinite ends on the complement of a cylindrical leaf choice for F. Note that by Thurston's classification theorem of diffeomorhisms theorem 2.1.4 quite a large class of diffeomorphisms, namely the reducible periodic diffeomorphisms, preserves such a pair $(F, \tilde{g})$ and so preserves the non-negative polarization $P_{F,\tilde{g}}$. It therefore seems interesting to study further the maps (3.24) and (3.25), for the non-negative polarizations $P_{F,\tilde{g}}$ on the moduli space $\mathcal{M}(\Sigma)$, especially when one has in mind theorem 13.1 in [11] and the degeneration result of theorem 2.6.2, since the moduli spaces $\mathcal{M}(\Sigma)_\sigma$, $\sigma \in T$ in fact are algebraic varieties.

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