SPECIAL LAGRANGIAN GEOMETRY

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February 16, 2001

Thesis submitted in partial fulfilment of the requirements
for the degree *Doctor of Philosophy*
at the University of Oxford.

Hilary Term 2001

Abstract

This thesis is devoted to special Lagrangian submanifolds of Calabi-Yau manifolds. We study Donaldson's approach to the moduli space of special Lagrangian submanifolds, viewing it as a symplectic quotient of an infinite-dimensional space of maps. By combining geometric and analytic techniques, we produce from it a new proof of McLean's theorem for the existence of a smooth moduli space. We investigate the natural local geometry of this moduli space and give estimates for its curvature with respect to McLean's Hessian metric. By combining the *special* condition with the theory of singularities in smooth integrable systems we give a model for the singularities in smooth special Lagrangian T^n-fibrations with suitable regularity. We study the geometry of the base B near the discriminant locus Δ and show that $B \setminus \Delta$ is metrically non-complete and has locally finite volume near Δ. Motivated by recent advances in the study of topological mirror symmetry [Gro99b], we study a simple class of smooth T^3-fibrations $T^3 \to X \to S^3$ whose singular fibres occur over a knot (or link) $\mathfrak{k} \subset S^3$. These fibrations admit canonical duals which exhibit the topological features of mirror symmetry. The strong interplay between the topology of such fibrations on one side and the (co-) homology of their monodromy representation $\varrho\colon \pi_1(S^3 \setminus \mathfrak{k}) \to \mathrm{SL}(H_1(T^3, \mathbb{Z}))$ on the other can be exploited to compute the basic topological invariants (such as the fundamental group, characteristic classes, homology and cohomology) by purely algebraic methods from ϱ. To do this, we provide a formalism which enables us to find the relevant representations of knot groups and compute their (co-) homology efficiently. In particular, we classify all possible monodromy representations for torus knots. As an application, we obtain some concrete new T^3-fibrations on well-known six-manifolds such as $S^3 \times S^3 \to S^3$, degenerating over a trefoil knot.

Contents

Chapter 1

Introduction

1.1 Special Lagrangian fibrations and mirror symmetry

In 1982, Harvey and Lawson in their foundational paper [HL82] introduced the concept of *calibrated geometries*: Let a triple (X, g, Ω) be given where X is a smooth Riemannian manifold with metric g and $\Omega \in \mathcal{A}^p(X)$ a closed p-form such that

$$|\Omega_x(v_1, \ldots, v_p)| \leq |\operatorname{vol}_x(v_1, \ldots, v_p)| = |\det(g_x(v_i, v_j)_{i,j=1,\ldots,p})|^{1/2} \tag{1.1}$$

for all $x \in X$ and all $v_1, \ldots, v_p \in T_xX$. To exclude trivial cases we require that for every $x \in X$ there is a choice of tangent vectors $v_1, \ldots, v_p \in T_xX$ so that equality holds in (1.1). A p-dimensional oriented submanifold $\iota\colon L \to X$ is *calibrated* if $\iota^*\Omega = \operatorname{vol}_{\iota^*g}$, that is Ω restricts to the volume form of the induced metric on L. Stokes' theorem shows that calibrated submanifolds are minimal which originally made them interesting to Riemannian geometers.

Even though the definition of a calibration is rather general it turns out that in real life calibrations arise most naturally as covariantly constant differential forms on Riemannian manifolds with special holonomy. The simplest (and most easily obtainable) examples are complex submanifolds of Kähler manifolds which are calibrated by $\Omega = \omega^p/p!$ where ω is the Kähler form and p the complex dimension of the submanifold. This suggests that calibrated submanifolds of exceptional Riemannian manifolds should deserve the same attention as complex submanifolds in Kähler geometry. Over the past decade, Bryant, Joyce, Salamon and others have discovered many examples of Riemannian manifolds with exceptional holonomy (including compact ones) and thereby renewed the interest in calibrated submanifolds.

This thesis is devoted to *special Lagrangian submanifolds of Calabi-Yau manifolds*. A Calabi-Yau manifold is a Ricci-flat Kähler manifold X with a holomorphic n-form $\Omega^c \in \mathcal{A}^{n,0}(X)$. Such an Ω^c is parallel and thus the holonomy of the Ricci-flat metric g is reduced to a subgroup $\mathrm{Hol}_g(X) \subset \mathrm{SU}(n)$. With a suitable normalisation, the real part $\Omega_1 = \mathrm{Re}(\Omega^c)$ satisfies (1.1). A submanifold $L \to X$ is special Lagrangian if it is both Lagrangian with respect to the Kähler form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ and calibrated with respect to Ω_1.

Recently special Lagrangian submanifolds of Calabi-Yau manifolds have attracted a huge amount of interest from both physicists and mathematicians when, in 1996, Strominger, Yau and Zaslow [SYZ96] proposed a conjectural explanation of the *mirror symmetry phenomenon* which had been known to physicists for some time, but whose mathematical contents had until then been completely in the dark. If true, mirror symmetry can be understood and explained mathematically in terms of dual special Lagrangian torus fibrations on mirror pairs of Calabi-Yau three-folds.

Before explaining this let us sidestep for a moment and give a very crude description of the mirror principle in physics: We consider (super-) string theories living on a ten-dimensional universe, mathematically a manifold which is locally a product of four-dimensional Minkowski space and a compact complex Calabi-Yau three-fold X. As physicists say, the theory is "compactified on X". The Calabi-Yau condition on X is necessary for physical consistency. Mirror symmetry is the phenomenon that there are different string theories which are compactified on distinct Calabi-Yau spaces with no obvious *mathematical* connection between them, yet if each of them is interpreted appropriately they describe the *same physics*. In other words there are mirror pairs $(X, \check{X})$ of Calabi-Yau three-folds for which one type of string theory (say, "type II A") compactified on X is physically equivalent to another type ("type II B") compactified on $\check{X}$. Much effort is currently invested in understanding the mathematics of mirror symmetry. There is still no rigorous and universally accepted mathematical definition for this concept of "mirror-pairs". One basic property demanded by the underlying physics is that going to the mirror reflects the Hodge-diamond,

$$h^{p,q}(X) = h^{p,3-q}(\check{X}), \quad p, q = 0, \dots, 3. \tag{1.2}$$

This implies the well-known relation

$$\chi(X) + \chi(\check{X}) = 0 \tag{1.3}$$

for the Euler characteristics. Much less well-understood is the idea that there should also be a "mirror map" which identifies the complexified Kähler moduli of X with the holomorphic moduli of $\check{X}$, and vice versa. The latter statement is probably one of the most intriguing and challenging aspects of mirror symmetry for mathematicians; if the existence of such mirror pairs and mirror maps was established rigorously, the computation of holomorphic invariants of one Calabi-Yau manifold X could be translated into the computation of symplectic invariants of the mirror $\check{X}$. This would be a very powerful tool in the study of Calabi-Yau manifolds. Candelas, de la Ossa, Green and Parkes [C+91] have verified aspects of this relationship for the quintic and used the conjectural existence of a mirror map to predict the number of rational curves on a Calabi-Yau manifold from the Yukawa-coupling on the mirror. Some of these predictions have later been verified. This gives "experimental" evidence for mirror symmetry.

The conjecture of Strominger, Yau and Zaslow states roughly that given a mirror pair $(X, \check{X})$ of Calabi-Yau three-folds there should be special Lagrangian T^3-fibrations

$$\begin{array}{ccc} X & & \check{X} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \check{f}} & \\ & B & \end{array}$$

degenerating over a common discriminant locus $\Delta \subset B$ and such that for each $b \in B \setminus \Delta$ the fibres $F_b = f^{-1}(b)$ and $\check{F}_b = \check{f}^{-1}(b)$ are "dual" special Lagrangian three-tori.

Much progress has been made in the study of the topology of mirror pairs and the process of dualisation. M Gross has shown [Gro98] that for a class of fibrations $X \to B$ which he calls *simple* the cohomology of the total space X can be computed from the Leray spectral sequence and there is a notion of duality for these fibrations so that dual fibrations display the expected change in topology [Gro99b]. More recently [Gro99b] Gross has carried out a detailed study of the singularities of simple torus fibrations and defined the class of *well-behaved* fibrations in which every fibration has a unique dual. Moreover, he could show that the quintic

$$Q = \{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0\} \subset \mathbb{C}P^4$$

supports a well-behaved T^3-fibration and that dualising produces the right mirror $\check{Q} \cong Q/(\mathbb{Z}/5)$ at a topological level, without using any a priori knowledge. As a result there is a mathematically precise, canonical process of dualising for a certain category of T^3-fibred six-manifolds which exhibits the topological features of mirror symmetry and which contains at least one known non-trivial and physically relevant example of a mirror pair of Calabi-Yau three-folds.

Gross' work follows earlier work by Zharkov [Zha00] who proved the existence of T^3-fibrations on Calabi-Yau hypersurfaces in toric varieties. His construction depends on quite abstract analytical tools; this makes it difficult to address the question of dualisation. Also Ruan [Rua99a, Rua99b, Rua00] in a series of papers announced a construction of Lagrangian torus fibrations on a general quintic or hypersurface in a toric variety, thus taking the symplectic structure into account.

Constructing *special* Lagrangian fibrations on a compact Calabi-Yau manifold X still seems to be an intractable problem since in the compact case we do not know anything about Ricci-flat metrics on X except that there exists a unique one in each Kähler class in $H^{1,1}_{\mathbb{R}}(X)$, which is the statement of Yau's celebrated solution to the Calabi conjecture [Yau78]. However, there are non-compact Calabi-Yau manifolds with symmetries where one can obtain quite explicitly special Lagrangian submanifolds and fibrations, and at this time they are, however special they may seem, probably the best models to investigate metric questions. Joyce has studied special Lagrangian submanifolds and especially special Lagrangian cones in flat $\mathbb{C}^m$ and their singularities in a series of recent papers [Joy00a, Joy00b, Joy00c, Joy00d, Joy00e, Joy01].

1.2 A guide to the chapters

In **Chapter 2** we introduce Calabi-Yau manifolds and their special Lagrangian submanifolds. Following a suggestion of R Bryant we first study three-forms on a real six-dimensional vector space. We prove the following theorem.

Theorem 2.1.4 *Let $V \cong \mathbb{R}^6$ and $\Omega \in \Lambda^3 V^*$. Define a linear map $T_\Omega\colon V \to \Lambda^5 V^* \cong V \otimes \Lambda^6 V^*$ by $T_\Omega(X) = \iota(X)\Omega \wedge \Omega$. Then there is a non-linear function $\lambda\colon \Lambda^3 V^* \to (\Lambda^6 V^*)^2$ such that $T_\Omega^2\colon V \mapsto V \otimes (\Lambda^6 V^*)^2$ is given by*

$$T_\Omega^2 = \mathrm{Id} \otimes \lambda(\Omega).$$

If $\lambda(\Omega)$ is "negative" (see Definition 2.1.6), this can be used to define almost complex and complex structures with holomorphically trivial canonical bundle on real six-manifolds, and we show that the complex structure of a Calabi-Yau manifold is of this kind (see also [Hit00] for a proof different from ours). Carrying the linear algebra further we obtain the following classification of three-forms in dimension six.

Theorem 2.1.13 *Let $V \cong \mathbb{R}^6$ and $\Omega \in \Lambda^3 V^*$. Let $d = \dim(\mathrm{Ann}(\Omega))$ (see (2.2)). Then we can find a basis $e_1, \ldots, e_6$ for V^* so that Ω is given by one of the following normal forms, and this normal form is unique.*

d	Ω
0	$e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$
0	$e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_5 \wedge e_6 - e_2 \wedge e_6 \wedge e_4 - e_3 \wedge e_4 \wedge e_5$
1	$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_5$
3	$e_1 \wedge e_2 \wedge e_3$
6	0

We next move on to special Lagrangian submanifolds and review a construction of the moduli space of special Lagrangian submanifolds due to SK Donaldson

[Don99]. We obtain a natural complexification of the moduli space $\mathcal{S}$, together with a Kähler metric, by *formally* taking the symplectic (Kähler) reduction of a certain infinite dimensional space by the action of a group of diffeomorphisms. Our contribution is to provide the necessary analysis to back Donaldson's formal construction *analytically.*

Theorem 2.2.27 *Let (X, Ω^c, ω) be a Calabi-Yau n-fold and suppose (L, ν) is a smooth compact n-manifold with a fixed volume form $\nu \in \mathcal{A}^n(L)$. Let $\mathcal{M}^{k,\alpha}$ be the space of $C^{k,\alpha}$-embeddings $f\colon L \to X$ with $f^*\Omega^c = \nu$ and $f^*\omega$ exact, where $0 < \alpha < 1$ and $k \in \mathbb{N}$. Then $\mathcal{M}^{k,\alpha}$ is an infinite-dimensional Kähler manifold, invariant under the natural action of $\mathcal{D}_0^{k,\alpha}$ (the group of volume preserving $C^{k,\alpha}$-diffeomorphisms of L whose action on $H^*(L)$ is trivial). There is a moment map $\mu\colon \mathcal{M}_0^{k,\alpha} \to \mathrm{Lie}(\mathcal{D}_0^{k,\alpha})^*$ and the symplectic quotient*

$$\mathcal{S}_{\mathbb{C}} = \{\mathcal{M}_0^{k,\alpha} // \mathcal{D}_0^{k,\alpha}\}_{\mu=0}$$

is a smooth, finite-dimensional Kähler manifold which is a natural complexification of the moduli space of special Lagrangian submanifolds.

In the final section of the chapter we study the local geometry of $\mathcal{S}$ together with its natural Hessian Riemannian structure in the context of McLean's [McL98] and Hitchin's [Hit98] work, and we deduce some estimates for the metric and curvature of $\mathcal{S}$. We show that the geometry of $\mathcal{S}$ has properties similar to the special Kähler geometry on the base of an algebraically completely integrable system if the potential function of its Hessian metric satisfies a certain Monge-Ampère equation.

Theorem 2.3.8 *Suppose $h = \mathrm{Hess}_{\mathbf{u}}(\phi)$ is a metric on an m-manifold B which is of Hessian type with respect to a chart $\mathbf{u}\colon U \subset B \to \mathbb{R}^m$, and assume that at a point $b \in U$ we have*

$$\frac{\partial}{\partial u_k}\left[\det\left(\frac{\partial^2 \phi}{\partial u_i \partial u_j}\right)\right](b) = 0, \quad k = 1, \dots, m.$$

Then at b the metric h is Ricci non-negative, $\mathrm{Ric}_h(b) \geq 0$.

In **Chapter 3** we focus on symplectic aspects of *smooth* special Lagrangian fibrations, viewing them as integrable systems with singularities. We show that among Eliasson's classes of *non-degenerate* singularities [Eli90] there is just one type (called *focus-focus singularity*) which, in the case of a special Lagrangian fibration, is compatible with the *special* condition.

Theorem 3.1.3 *Let (X, Ω^c, ω) be a Calabi-Yau n-fold with a special Lagrangian fibration $f\colon X \to B$ of class C^3. Let $m \in X$ be a "weakly non-degenerate" (see Section 3.1) critical point of f. Then m is of focus-focus type.*

We show that a smooth "strongly non-degenerate" (see Section 3.1) special Lagrangian fibration degenerates over a codimension two discriminant locus. It has been a popular though not undisputed belief that special Lagrangian fibrations should display this behaviour. Moreover, we use Eliasson's classification of singularities to prove the following theorem.

Theorem 3.1.10 *Suppose B is compact and $f\colon X \to B$ is a special Lagrangian fibration with only strongly non-degenerate singularities. Then $\mathrm{Vol}_h(B) < \infty$, where h is the natural Riemannian metric on B.*

In particular, for an elliptic $K3$ surface this volume is a holomorphic invariant. Eliasson's normal form also allows us to show

Theorem 3.1.14 *The natural metric on the moduli space of a strongly non-degenerate special Lagrangian fibration is not complete.*

We then move on to study models for special Lagrangian fibrations due to Harvey-Lawson [HL82] and Ooguri-Vafa [OV96] where many of the features discussed here can be demonstrated explicitly.

In **Chapter 4** we aim at a more global study of T^3-fibrations, mainly from the topological point of view. Following [Gro99b] we give a concrete way of constructing T^3-fibred, compact, orientable, smooth six-manifolds which degenerate in a controlled manner over a knot or link $\mathfrak{k} \subset S^3$. More presicely, in a neighbourhood of a singular fibre the fibrations are equivalent to a product of fibrations

$$M^4 \times (S^1 \times \mathbb{R}) \xrightarrow{(p,\mathrm{pr}_2)} D^2 \times \mathbb{R} \tag{1.4}$$

where $p\colon M^4 \to D^2$ is a standard I_1-degeneration and $\mathrm{pr}_2\colon (S^1 \times \mathbb{R}) \to \mathbb{R}$ the projection. If such a fibration has a section we call it a *good* torus fibration.

The complement of a knot is an Eilenberg-MacLane space $K(\pi_1, 1)$. This implies a strong interplay between the topology of a good T^3-fibration and the (co-)homology of its knot group with coefficients in the monodromy representation, an aspect which is absent from Gross' work [Gro99b]. Therefore algebraic properties of knot groups translate into topological properties of good T^3-fibrations, and most of this chapter is aimed at using this relationship to construct torus fibrations out of group representations and study their properties. We give results about the fundamental group and characteristic classes of such fibrations, and we compute their homology and cohomology. These fibrations can be canonically dualised as described in [Gro99b], but due to the simplicity of the construction we can demonstrate in a very concrete and explicit way many features of mirror symmetry.

In **Chapter 5** we focus more specifically on torus knots. We give a complete classification of all possible *M-representations* of the group of a torus knot, by which we mean representations that can arise as the monodromy of a good T^3-fibration.

Theorem 5.1.8 - 5.1.10 *The trefoil knot $\mathfrak{t}(2,3)$ has precisely one trivial M-representation which factors through the Abelianisation $G(\mathfrak{t}(2,3))^{\mathrm{ab}}$ and one non-trivial M-representation whose image is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$. The $(4,3)$-torus knot has one trivial M-representation and an infinite family $(\varrho_k)_{k\in\mathbb{Z}}$ of non-trivial M-representations with $\varrho_{-k} = \varrho_k^*$. The group $G(p,q)$ of the (p,q)-torus knot $\mathfrak{t}(p,q)$ has non-trivial M-representations if and only if (after perhaps exchanging p and q) $p = 2^m p'$ and $q = 3^k q'$ with $\gcd(p',6) = \gcd(q',6) = \gcd(p',q') = 1$ and $m,k > 0$. In case $m = 1$ there is a bijection $M{-}\mathrm{Rep}(G(p,q)) \to M{-}\mathrm{Rep}(G(2,3))$ between the M-representations of $G(p,q)$ and $G(2,3)$. In case $m > 1$ there is a bijection $M{-}\mathrm{Rep}(G(p,q)) \to M{-}\mathrm{Rep}(G(4,3))$.*

We use this explicit knowledge of the M-representations of torus knot groups to construct concrete examples of T^3-fibred six-manifolds by the procedure described in Chapter 4. In particular, we prove the existence of good torus fibrations on certain spheres and products of spheres.

Theorem 5.3.10 *There are T^3-fibrations on $S^1 \times S^2 \times S^3$ and $S^3 \times S^3$ whose singularities are equivalent to (1.4).*

This might be interesting since these manifolds admit complex structures with vanishing first Chern class.

1.3 Acknowledgements

First and foremost, I would like to thank my supervisor Professor Nigel Hitchin for teaching me and guiding my research over the past three years. He would always offer help and encouragement, and his commitment and interest in my work were invaluable. I have been enjoying our collaboration throughout.

Among many other people who have helped along the way are Christian Bär, Martin Bridson, Mark Gross, Allen Hatcher, Dominic Joyce, Marc Lackenby, Graeme Segal, Ivan Smith, Ulrike Tillmann, and I would like to thank them as well. Especially I would like to thank Ivan Smith for his careful proof-reading of early versions of this thesis and helpful comments and discussions.

My studies in Oxford were financially supported by the Engineering and Physical Sciences Research Council (EPSRC) and by the Hochschulsonderprogramm III of the Deutscher Akademischer Austauschdienst (DAAD); I gratefully acknowledge their funding. Thanks to the Mathematical Institute for continued support and to New College for allowing me to use their piano.

1.4 List of Symbols

$\mathcal{A}^p(X)$	smooth p-forms on a manifold X	11
$\mathcal{A}^{p,q}(X)$	(p,q)-forms on a complex manifold X	21
$\mathcal{A}^{p,p}_{\mathbb{R}}(X)$	real (p,p)-forms on a complex manifold X	33
$\mathrm{Aff}(\mathbb{T}^n)$	the group of affine linear transformations of $\mathbb{T}^n$	73
$\mathrm{Ann}(\Omega)$	annihilator of a tensor, $\{X \mid \iota_X\Omega = 0\}$	13
$\widehat{B}$	universal cover of B	75
$\mathsf{B}G$	classifying space of the group G	74
$\mathcal{B}(F,G)$	space of bounded linear maps $F \to G$	29
b^p, $b^p(X)$	p-th Betti number (of X)	29
$[B,X]$	the set of homotopy classes of maps $B \to X$	74
$\mathcal{C}(A)$	conjugacy class of $A \in \mathrm{SL}(n,\mathbb{Z})$	81
$c_k(X)$	k-th Chern class of X	12
$\mathrm{Crit}(f)$	set of critical points of f	52
$\mathrm{Der}(G,M)$	derivations of a G-module M	108
$\mathrm{Der}_0(G,M)$	principal derivations of a G-module M	108
$\Delta, \Delta(f)$	discriminant locus of f	52
EXP	Riemannian exponential map $TX \to X$	34
G^{ab}	Abelianisation of a group G	102
$G(\mathfrak{k})$	$\pi_1(S^3 \setminus \mathfrak{k})$, the group of the knot $\mathfrak{k}$	80
$G(p,q)$, $G(\mathfrak{t}(p,q))$	group of the (p,q)-torus knot $\mathfrak{t}(p,q)$	104
$\mathrm{H}^{p,p}_{\mathbb{R}}(X)$	real p-forms of type (p,p)	6
$\mathrm{Hess}(\phi), \mathrm{Hess}_{\mathbf{u}}(\phi)$	Hessian of a function (with respect to a chart $\mathbf{u}$)	43
$h^{p,q}$, $h^{p,q}(X)$	(p,q)-th Hodge-number (of X)	4
$\mathrm{inj}(x), \mathrm{inj}_g(x)$	injectivity radius (of the metric g) at x	32
$\iota_X\Omega$, $\iota(X)\Omega$	contraction of a differential form Ω with a vector field X	12
$\mathcal{L}^1(B,b)$	loop-group of B with basepoint b	102
M^G (resp. M_G)	invariants (resp. coinvariants) of a G-module M	109
$p_k(X)$	k-th Pontrjagin class of X	86
$\mathrm{Ric}(X)$	Ricci *tensor* of X	7
$\mathrm{Scal}(X)$	scalar curvature of X	49
$\mathrm{Stab}(\Omega)$	stabiliser of a tensor under a group action	13
$\mathfrak{t}(p,q)$	(p,q)-torus knot	97
$\mathrm{Vect}(X)$	$\Gamma(TX)$, smooth vector fields on X	27
vol, vol_g	volume form (of a Riemannian metric g)	3
$\mathrm{Vol}(X)$, $\mathrm{Vol}_g(X)$	volume of X (with respect to a Riemannian metric g)	23
$V^{\perp_\omega}$	symplectic orthogonal complement	55
$\mathbb{Z}/p$	$\mathbb{Z}/p\mathbb{Z}$, the integers modulo p	5

Chapter 2

Calabi-Yau manifolds and special Lagrangians

In this chapter we introduce Calabi-Yau manifolds and their special Lagrangian submanifolds. We study complex structures on Calabi-Yau manifolds and the existence, smoothness and local geometry of the moduli space of special Lagrangian submanifolds.

2.1 Calabi-Yau manifolds

This thesis looks at Calabi-Yau manifolds very much from the viewpoint of differential, rather than algebraic geometry. This is a differential geometer's definition of Calabi-Yau manifolds.

Definition 2.1.1 *A* <u>*Calabi-Yau manifold*</u> *is a smooth manifold X satisfying one of the following three equivalent conditions:*

1. *X is a Ricci-flat Kähler manifold with holomorphically trivial canonical bundle.*
2. *X is a complex n-fold equipped with a real positive closed $(1,1)$-form ω and a nowhere vanishing holomorphic n-form Ω^c satisfying*

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega^c \wedge \overline{\Omega}^c. \tag{2.1}$$

3. (see [Hit99b]) *X is a $2n$-dimensional manifold equipped with a real two-form ω and a complex n-form $\Omega^c = \Omega_1 + i\Omega_2 \in \mathcal{A}^n(X) \otimes \mathbb{C} = \mathcal{A}^n_{\mathbb{C}}(X)$ so that*
   - *(a) ω and Ω^c are closed,* $\mathrm{d}\omega = \mathrm{d}\Omega_1 = \mathrm{d}\Omega_2 = 0$,
   - *(b) Ω^c is primitive,* $\omega \wedge \Omega_1 = \omega \wedge \Omega_2 = 0$,

(c) *Ω^c is locally decomposable, that is for small $U \subset X$ we can write*

$$\Omega^c|U = \theta_1 \wedge \cdots \wedge \theta_n$$

where $\theta_1, \ldots, \theta_n \in \mathcal{A}^1_{\mathbb{C}}(U)$ are local complex one-forms which give rise to a direct sum decomposition

$$\operatorname{span}\{\theta_j | 1 \le j \le n\} \oplus \operatorname{span}\{\bar{\theta}_j | 1 \le j \le n\} = T^*_{\mathbb{C}}M|_U,$$

(d) *the normalisation condition (2.1) holds,*

(e) *For all non-zero $\xi \in TX$ we have $i\left(\iota_\xi \Omega^c \wedge \iota_\xi \overline{\Omega}^c \wedge \omega\right) > 0$.*

The underlying complex manifold X of a Calabi-Yau manifold (X, Ω^c, ω) has trivial canonical class $c_1(X) = 0$. If $\dim_{\mathbb{C}}(X) = 2$, then X is always a hyper-Kähler manifold and has an S^2 of integrable complex structures behaving like the imaginary unit quaternions. However, hyper-Kähler manifolds always have even complex dimension $\dim_{\mathbb{C}}(X) = 2k$ and for $k > 1$ form a rather special subclass of all Calabi-Yau manifolds. In that sense $\dim_{\mathbb{C}}(X) = 3$ is the lowest dimension where we have "generic" Calabi-Yau manifolds, and for both mathematical and physical reasons this case deserves special attention. In complex dimension three the Hodge diamond for an irreducible compact Calabi-Yau manifold looks like

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

The only non-trivial Hodge numbers $h^{1,1} = h^{2,2}$ and $h^{1,2} = h^{2,1}$ have a geometrical meaning; a famous theorem, independently proved by Bogomolov [Bog78], Tian [Tia87] and Todorov [Tod89], states that the infinitesimal complex deformations of X are naturally identified with $H^{2,1}(X, \mathbb{C})$ while the Kähler moduli are $H^{1,1}_{\mathbb{R}}(X)$. The infinitesimal deformations of both the complex and the Kähler structures are unobstructed. Thus going back to the mirror symmetry conjecture mentioned in Chapter 1, the existence of isomorphisms, for a mirror pair $(X, \check{X})$, between the (complexified) Kähler moduli space of X and the complex moduli space of $\check{X}$ and vice versa, implies the exchange of Hodge numbers (1.2) and the relation (1.3) for the Euler characteristic. Finally we should mention Yau's famous affirmative answer to the Calabi conjecture that every Kählerian complex manifold X with $c_1(X) = 0$ admits a Ricci-flat metric, more precisely:

Theorem 2.1.2 (Yau [Yau78]) *Let X be a compact complex manifold with $c_1(X) = 0$, then every Kähler class in $H^{1,1}_{\mathbb{R}}(X)$ contains a unique Ricci-flat Kähler form.*

However, not every complex manifold with $c_1(X) = 0$ admits a Kähler metric. Yau's proof is based on existence theorems for solutions to non-linear partial differential equations. It is non-constructive and does not enable us to find solutions explicitly, not even to make any qualitative estimates. Explicit Calabi-Yau metrics have so far only been obtained on non-compact manifolds with symmetries, and we will study examples in Chapter 3.

2.1.1 Classification of three-forms in six dimensions

We give a classification of three-forms on a real six-dimensional vector space $V \cong \mathbb{R}^6$. In Paragraph 2.1.2 we will use the linear algebra developed here to define (almost) complex structures on real six-manifolds. The idea of using three-forms to define complex structures is due to R Bryant. Theorem 2.1.4 has also appeared in [Hit00] with a different proof, but the proof given here and the other results of this paragraph are entirely our original work.

Definition 2.1.3 *Let $V \cong \mathbb{R}^6$. We call a three-form $\Omega \in \Lambda^3 V^*$ non-degenerate if $\iota(X)\Omega \neq 0$ for all $X \in V \setminus \{0\}$. We call it d-fold degenerate if the annihilator of Ω*

$$\operatorname{Ann}(\Omega) = \{X \in V \mid \iota(X)\Omega = 0\} \tag{2.2}$$

has dimension $\dim(\operatorname{Ann}(\Omega)) = d$.

To be non-degenerate is a generic property for real three-forms in real dimension six. For $\Omega \in \Lambda^3 V^*$ let

$$\operatorname{Stab}^+(\Omega) = \{\varphi \in \mathrm{GL}^+(V) \mid \varphi^*\Omega = \Omega\}$$

be the stabiliser of Ω in $\mathrm{GL}^+(V)$, the orientation preserving automorphisms of V. Let $V \cong \mathbb{R}^6$ and $\Omega \in \Lambda^3 V^*$, and define a map $T_\Omega \colon V \to \Lambda^5 V^*$ by

$$T_\Omega(X) = \iota(X)\Omega \wedge \Omega \quad \forall\, X \in V. \tag{2.3}$$

Choose a volume form $\nu \in \Lambda^6 V^*$ and define from T_Ω an endomorphism $T^\nu_\Omega \colon V \to V$ by the formula

$$\iota(T^\nu_\Omega(X))\nu = T_\Omega(X). \tag{2.4}$$

Theorem 2.1.4 *There is a non-linear function $\lambda \colon \Lambda^3 V^* \to (\Lambda^6 V^*)^2$ such that $T^2_\Omega \colon V \mapsto V \otimes (\Lambda^6 V^*)^2$ is given by*

$$T^2_\Omega = \operatorname{Id} \otimes \lambda(\Omega). \tag{2.5}$$

Proof. Let $0 \neq X \in V$ be arbitrary. Let

$$W^* = \mathrm{Ann}(X) = \{\tau \in V^* \mid \tau(X) = 0\} \subset V^*$$

be its five-dimensional annihilator in V^*. Choose any $\theta \in V^*$ with $\theta(X) = 1$, hence $\theta \notin W^*$ and taking $U^* = \langle\theta\rangle$ the real line spanned by θ we get a decomposition $V^* = U^* \oplus W^*$. Also let $W = \ker(\theta) \subset V$ and let $\nu_0 \in \Lambda^5 W^*$ be the unique element with

$$\theta \wedge \nu_0 = \nu. \tag{2.6}$$

U^* and W depend on the (non-canonical) choice of θ, but the ambiguity arising from that does not cause any problems in what follows. Having fixed these choices we can find unique elements $\psi \in \Lambda^2 W^*$ and $\phi \in \Lambda^3 W^*$ such that

$$\Omega = \theta \wedge \psi + \phi. \tag{2.7}$$

So $\iota(X)\Omega = \psi$. Moreover, there are unique elements $r \in \mathbb{R}$ and $Y \in W$ such that

$$\begin{aligned} \iota(rX)\nu &= \psi \wedge \phi & (2.8)\\ \iota(Y)\nu_0 &= \psi \wedge \psi. & (2.9) \end{aligned}$$

$\theta(Y) = 0$ since $Y \in W$. If ψ is decomposable then $Y = 0$. *Claim:* We have

$$\iota(Y)\psi = 0. \tag{2.10}$$

Proof of claim: We prove this by contradiction and assume that $\iota(Y)\psi \neq 0$. First note that certainly

$$\iota(Y)\psi \wedge \psi = \frac{1}{2}\,\iota(Y)(\psi \wedge \psi) \overset{(2.9)}{=} \frac{1}{2}\,\iota(Y)\iota(Y)\nu_0 = 0.$$

Since ψ is a two-form, if $\iota(Y)\psi \neq 0$ and $\iota(Y)\psi \wedge \psi = 0$, then $\psi = \iota(Y)\psi \wedge \tau$ for some $\tau \in W^*$. Hence ψ is decomposable and $\psi \wedge \psi = 0$. But then (2.9) implies $Y = 0$ since ν_0 is non-degenerate and hence $\iota(Y)\psi = 0$, QEA. This proves (2.10). We can now compute

$$\begin{aligned} T_\Omega(X) &= \iota(X)\Omega \wedge \Omega \\ &= \psi \wedge (\theta \wedge \psi + \phi) && \text{by (2.7)} \\ &= \theta \wedge \psi \wedge \psi + \psi \wedge \phi \\ &= \theta \wedge \iota(Y)\nu_0 + \iota(rX)\nu && \text{by (2.8), (2.9)} \\ &= -\iota(Y) \wedge (\theta \wedge \nu_0) + \iota(rX)\nu && \text{because } Y \in W \\ &= \iota(-Y + rX)\nu && \text{by (2.6).} \end{aligned}$$

In the notation of (2.4), this implies

$$T_\Omega^\nu(X) = -Y + rX. \tag{2.11}$$

Next we compute $T_\Omega(Y)$. We find

$$\begin{aligned} T_\Omega(Y) &= \iota(Y)\Omega \wedge \Omega \\ &= \iota(Y)(\theta \wedge \psi + \phi) \wedge (\theta \wedge \psi + \phi) && \text{by (2.7)} \\ &= \iota(Y)\phi \wedge (\theta \wedge \psi + \phi) && \text{by (2.10)} \\ &= \theta \wedge \iota(Y)(\phi \wedge \psi) + \iota(Y)\phi \wedge \phi && \text{by (2.10).} \end{aligned}$$

Now observe that $\iota(Y)\phi \wedge \phi \in \Lambda^5 W^*$, so there is an $s \in \mathbb{R}$ such that

$$\iota(Y)\phi \wedge \phi = s\nu_0 = \iota(sX)\nu.$$

Using these definitions we can now compute

$$\begin{aligned} T_\Omega(Y) &= \theta \wedge \iota(Y)\iota(rX)\nu + \iota(sX)\nu && \text{by (2.8)} \\ &= \theta \wedge \iota(Y) r\nu_0 + \iota(sX)\nu && \text{by (2.6)} \\ &= -\iota(rY)(\theta \wedge \nu_0) + \iota(sX)\nu \\ &= \iota(-rY + sX)\nu && \text{by (2.6).} \end{aligned}$$

Thus we conclude from the definition in (2.4) that

$$T^\nu_\Omega(Y) = -rY + sX.$$

Putting everything together we obtain

$$(T^\nu_\Omega)^2(X) = T^\nu_\Omega(-Y + rX) = -(-rY + sX) + r(-Y + rX) = (-s + r^2)X.$$

The constants $s, r \in \mathbb{R}$ have no invariant meaning since they depend on our choices of volume forms and splittings, however, we can see that X is an eigenvector for $(T^\nu_\Omega)^2$. However, $X \in V$ was arbitrary, and a linear map for which *every* vector is an eigenvector is just a scalar. So we deduce

$$(T^\nu_\Omega)^2 = \lambda^\nu(\Omega)\,\mathrm{Id} \tag{2.12}$$

for some $\lambda^\nu(\Omega) \in \mathbb{R}$. Here $\lambda^\nu(\Omega)$ is homogeneous of degree -2 in ν, so let us define

$$\lambda(\Omega) = \lambda^\nu(\Omega)\nu^2 \in (\Lambda^6 V^*)^2. \tag{2.13}$$

Then $\lambda(\Omega)$ is independent of the choice of ν, and $T^2_\Omega = \mathrm{Id} \otimes \lambda(\Omega)$ which is (2.5). □

Corollary 2.1.5 *For all $t \in \mathbb{R}$ we have $\lambda(t\Omega) = t^4\lambda(\Omega)$ and $\lambda^{t\nu}(\Omega) = t^{-2}\lambda^\nu(\Omega)$. Hence the sign of $\lambda^\nu(\Omega)$ is independent of the choice of trivialisation $\nu\colon \Lambda^6 V^* \to \mathbb{R}$, and is therefore an invariant of $\Omega \in \Lambda^3 V^*$.*

Proof. Obvious from the definitions and the fact that $\lambda^\nu(\Omega)$ involves only the square $(\Lambda^6 V^*)^2$ of the top exterior power of V^*. □

On the basis of Corollary 2.1.5 we can make the following definition.

Definition 2.1.6 *We call a non-degenerate $\Omega \in \Lambda^3 V^*$ of positive type (respectively of negative type) if $\lambda^\nu(\Omega) > 0$ (respectively $\lambda^\nu(\Omega) < 0$) for some (and hence every) $\nu \in \Lambda^6 V^* \setminus \{0\}$.*

Proposition 2.1.7 *Let $\Omega \in \Lambda^3 V^*$ and $\varphi \in \mathrm{End}\,(V)$. Then for all $X \in V$*

$$\varphi^* T_\Omega(\varphi X) = T_{\varphi^*\Omega}(X). \tag{2.14}$$

If $\varphi \in \mathrm{GL}(V)$ and $\nu \in \Lambda^6 V^* \setminus \{0\}$ then

$$T^\nu_\Omega \circ \varphi = \varphi \circ T^{\varphi^*\nu}_{\varphi^*\Omega}. \tag{2.15}$$

Proof. Note that for all $\varphi \in \mathrm{End}\,(V)$, all $\Omega \in \Lambda^* V^*$ and $X \in V$ the relation

$$\varphi^*(\iota(\varphi X)\Omega) = \iota(X)\varphi^*\Omega \tag{2.16}$$

holds. Now let $\Omega \in \Lambda^3 V^*$ be given. Take $\varphi \in \mathrm{End}\,(V)$ as in the proposition and let $X \in V$ be arbitrary. Then

$$\begin{aligned}
\varphi^* T_\Omega(\varphi X) &= \varphi^*\left[\, \iota(\varphi X)\Omega \wedge \Omega \,\right] \\
&= \iota(X)\varphi^*\Omega \wedge \varphi^*\Omega && \text{by (2.16)} \\
&= T_{\varphi^*\Omega}(X).
\end{aligned}$$

This is (2.14). Now fix a volume form $\nu \in \Lambda^6 V^*$ as before and let $\varphi \in \mathrm{GL}(V)$. Since φ is invertible we can rewrite (2.14) as

$$\begin{aligned}
\iota(T^\nu_\Omega(\varphi X))\nu &= (\varphi^{-1})^* \left[\, \iota(T^{\varphi^*\nu}_{\varphi^*\Omega}(X))\varphi^*\nu \,\right] \\
&= \iota(\varphi \circ T^{\varphi^*\nu}_{\varphi^*\Omega}(X))\nu, && \text{by (2.16)}
\end{aligned}$$

and since ν is non-degenerate and $X \in V$ was arbitrary the relation (2.15) follows. □

Proposition 2.1.8 *For all $\Omega \in \Lambda^3 V^*$ and $\varphi \in \mathrm{End}\,(V)$ we have*

$$\det(\varphi)^2 \lambda(\Omega) = \lambda(\varphi^*\Omega). \tag{2.17}$$

Proof. The equation is trivial if $\det(\varphi) = 0$, so let us assume φ is invertible. Unwinding the definitions we obtain

$$\begin{aligned}
\det(\varphi)^2\, \mathrm{Id} \otimes \lambda(\Omega) &= \det(\varphi)^2\, T^2_\Omega \\
&= \det(\varphi)^2\, (T^\nu_\Omega)^2 \otimes \nu^2 \\
&= \left[\varphi \circ \left(T^{\varphi^*\nu}_{\varphi^*\Omega}\right)^2 \circ \varphi^{-1}\right] \otimes (\varphi^*\nu)^2 && \text{by (2.15)} \\
&= \mathrm{Id} \otimes \lambda^{\varphi^*\nu}(\varphi^*\Omega) \otimes (\varphi^*\nu)^2 \\
&= \mathrm{Id} \otimes \lambda(\varphi^*\Omega)
\end{aligned}$$

and hence $\det(\varphi)^2\lambda(\Omega) = \lambda(\varphi^*\Omega)$, as desired. $\square$

Corollary 2.1.9 *If $\Omega \in \Lambda^3V^*$ with $\lambda(\Omega) \neq 0$ then $\mathrm{Stab}^+(\Omega) \subset \mathrm{SL}(V)$.*

Proof. For all $\varphi \in \mathrm{Stab}^+(\Omega)$ we get

$$\lambda(\Omega) = \lambda(\varphi^*\Omega) = \det(\varphi)^2\lambda(\Omega)$$

and hence $\det(\varphi)^2 = 1$ since $\lambda(\Omega) \neq 0$. But since by definition $\mathrm{Stab}^+(\Omega) \subset \mathrm{GL}^+(V)$, this implies $\det(\varphi) = 1$ and hence $\mathrm{Stab}^+(\Omega) \subset \mathrm{SL}(V)$. $\square$

Proposition 2.1.10 *Let $\Omega \in \Lambda^3V^*$ such that $\lambda(\Omega) \neq 0$ and let $\varphi \in \mathrm{Stab}^+(\Omega)$. Then for all $\nu \in \Lambda^6V^*$ the commutator $[\varphi, T^\nu_\Omega] = 0$ vanishes.*

Proof. From Corollary 2.1.9 we get $\det(\varphi) = 1$, so $\varphi^*\nu = \nu$. Putting this together with $\varphi^*\Omega = \Omega$ into (2.15) the claim follows. $\square$

Notation 2.1.11 *Let $\Omega \in \Lambda^3V^*$ with $\lambda(\Omega) \neq 0$ and choose $\nu \in \Lambda^6V^* \setminus \{0\}$. We define an endomorphism $I^\nu_\Omega \colon V \to V$ by*

$$I^\nu_\Omega(X) = \frac{1}{\sqrt{|\lambda^\nu(\Omega)|}}\, T^\nu_\Omega(X). \tag{2.18}$$

We then have

$$I^{t\nu}_\Omega = \mathrm{sign}(t) I_\Omega, \qquad \forall t \in \mathbb{R} \setminus \{0\}. \tag{2.19}$$

So I^ν_Ω depends only on the orientation defined by ν. If we start with an oriented vector space V we will always assume that ν is chosen to be a positive volume form with respect to the given orientation, and we will simply write I_Ω instead of I^ν_Ω.

$(I^\nu_\Omega)^2$ is independent of $\nu \in \Lambda^6V^* \setminus \{0\}$ (and hence the choice of orientation) by (2.19). Thus we get

Corollary 2.1.12 *If Ω is of positive type then $(I^\nu_\Omega)^2 = \mathrm{Id}$ and I^ν_Ω is an involution, if Ω is of negative type then $(I^\nu_\Omega)^2 = -\mathrm{Id}$, and I^ν_Ω is a complex structure on V.*

Theorem 2.1.13 *Let $V \cong \mathbb{R}^6$ and $\Omega \in \Lambda^3V^*$. Let $d = \dim(\mathrm{Ann}(\Omega))$. Then we can find a basis $e_1, \ldots, e_6$ for V^* so that Ω adopts one of the following normal forms, and this normal form is unique.*

d	Ω
0	$e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$
0	$e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_5 \wedge e_6 - e_2 \wedge e_6 \wedge e_4 - e_3 \wedge e_4 \wedge e_5$
1	$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_5$
3	$e_1 \wedge e_2 \wedge e_3$
6	0

Proof. Throughout we assume that V is equipped with a fixed orientation and $\nu \in \Lambda^6 V^*$ is a positive volume form. Therefore we simply write I_Ω for I_Ω^ν.

Case 1: Assume $\lambda(\Omega) \neq 0$. If Ω is of *positive type*, then I_Ω in Notation 2.1.11 satisfies $I_\Omega^2 = \mathrm{Id}$ and hence $V = V_+ \oplus V_-$ splits into the (± 1)-eigenspaces of this involution. We can therefore find an eigenbasis $X_1, \ldots, X_l \in V_+$ and $X_{l+1}, \ldots, X_6 \in V_-$. For each $i = 1, \ldots, 6$ and any volume form $\nu \in \Lambda^6 V^*$, we then have

$$\begin{aligned} \iota(X_i)\Omega \wedge \iota(X_i)\Omega &= \iota(X_i)(\iota(X_i)\Omega \wedge \Omega) \\ &= \iota(X_i)(\iota(T_\Omega^\nu(X_i))\nu) \\ &= \pm\sqrt{\lambda^\nu(\Omega)}\, \iota(X_i)\iota(X_i)\nu \\ &= 0 \end{aligned} \tag{2.20}$$

since X_i is an eigenvector for I_Ω and hence for T_Ω^ν. Therefore $\iota(X_i)\Omega$ *is decomposable for all* $i = 1, \ldots, 6$, *and is never zero because* $\lambda(\Omega) \neq 0$. Let $e_1, \ldots, e_6 \in V^*$ be the dual basis. It is then easy to verify that decomposability (2.20) and $\lambda(\Omega) \neq 0$ imply

$$\Omega = \alpha\, e_1 \wedge e_2 \wedge e_3 + \beta\, e_4 \wedge e_5 \wedge e_6$$

for some $\alpha, \beta \neq 0$ (after possibly relabelling). Replacing e_1 by $\alpha\, e_1$ and e_4 by $\beta\, e_4$ establishes the normal form of the theorem. Note that $\dim(V_+) = \dim(V_-) = 3$ and

$$\mathrm{Stab}^+(\Omega) = \mathrm{SL}(V_+) \times \mathrm{SL}(V_-) \cong \mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R}). \tag{2.21}$$

If Ω is of *negative type* then the endomorphism I_Ω defined in Notation 2.1.11 is a complex structure on V which determines a subgroup $\mathrm{SL}(V, I_\Omega) \cong \mathrm{SL}(3, \mathbb{C})$ of $\mathrm{SL}(V)$ containing the I_Ω-linear automorphisms of V of unit determinant, and Corollary 2.1.9 and Proposition 2.1.10 imply $\mathrm{Stab}^+(\Omega) \subset \mathrm{SL}(V, I_\Omega)$. There is a complex linear isomorphism $\psi \colon (V, I_\Omega) \to \mathbb{C}^3$ which identifies the action of $\mathrm{SL}(V, I_\Omega)$ on V with the standard representation of $\mathrm{SL}(3, \mathbb{C})$ on $\mathbb{C}^3$, and $\mathrm{Stab}^+(\Omega)$ is isomorphic to a subgroup of $\mathrm{SL}(3, \mathbb{C})$. But $\dim(\mathrm{SL}(V, I_\Omega)) = \dim(\mathrm{SL}(3, \mathbb{C})) = 16$ and the orbit of Ω under the action of $\mathrm{GL}^+(V)$ is diffeomorphic to the homogeneous space

$$\frac{\mathrm{GL}^+(V)}{\mathrm{Stab}^+(\Omega)} \cong \mathrm{Orbit}(\Omega) \subset \Lambda^3 V^*.$$

Counting dimensions $\dim(\mathrm{GL}^+(V)) = 36$ and $\dim(\Lambda^3 V^*) = 20$ we conclude

$$\dim(\mathrm{Stab}^+(\Omega)) \geq 36 - 20 = 16 = \dim(\mathrm{SL}(V, I_\Omega)).$$

We conclude that $\mathrm{Stab}^+(\Omega)$ is open in $\mathrm{SL}(V, I_\Omega)$ and hence

$$\mathrm{Stab}^+(\Omega) = \mathrm{SL}(V, I_\Omega) \cong \mathrm{SL}(3, \mathbb{C}). \tag{2.22}$$

The only invariant subspace of the induced $\mathrm{SL}(3, \mathbb{C})$-action on $\Lambda^3\mathbb{C}^3$ is $\Lambda^{3,0}\mathbb{C}^3 \cong_{\mathbb{R}} \Lambda^{0,3}\mathbb{C}^3$. For a suitable $\mathbb{R}$-basis $\tilde{e}_1, \ldots, \tilde{e}_6$ of $(\mathbb{C}^3)^*$ and $e_i = \psi^*\tilde{e}_i$, the three-form Ω can then be written as

$$\begin{aligned} \Omega &= \psi^* \operatorname{Re}((\tilde{e}_1 + i\tilde{e}_4) \wedge (\tilde{e}_2 + i\tilde{e}_5) \wedge (\tilde{e}_3 + i\tilde{e}_6)) \\ &= e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_5 \wedge e_6 - e_2 \wedge e_6 \wedge e_4 - e_3 \wedge e_4 \wedge e_5 \end{aligned}$$

which is the form given in the table.

Case 2: Assume $\lambda(\Omega) = 0$ so that T_Ω and hence T_Ω^ν is not invertible. *Claim:*

$$\Omega \neq 0 \quad \Rightarrow \quad \mathrm{Ann}(\Omega) \subsetneqq \ker(T_\Omega).$$

Proof of claim: We certainly have $\mathrm{Ann}(\Omega) \subset \ker(T_\Omega)$, and $\dim(\ker(T_\Omega)) \geq 3$ since $T_\Omega^2 = 0$. Suppose $\mathrm{Ann}(\Omega) = \ker(T_\Omega)$, hence $\dim(\mathrm{Ann}(\Omega)) \geq 3$. Because $\Omega \neq 0$ we know $\dim(\mathrm{Ann}(\Omega)) \neq 6$ and it is then easy to see that $d = 3$ which means that Ω is decomposable. But this implies $T_\Omega \equiv 0$ and so $\ker(T_\Omega) = V \supsetneqq \mathrm{Ann}(\Omega)$. So we may assume for the rest that $\ker(T_\Omega)/\,\mathrm{Ann}(\Omega) \neq \{0\}$. The conclusion we want to draw from this is can find an $X \in V \setminus \{0\}$ such that

$$T_\Omega(X) = 0, \qquad \iota(X)\Omega \neq 0. \tag{2.23}$$

As before let $W^* = \mathrm{Ann}(X) \subset V^*$ be the 5-dimensional space of linear forms on V annihilating X, and choose $U^* = \langle\theta\rangle \subset V^*$ and $W \subset V$ as in Theorem 2.1.4. Then

$$\Lambda^3 V^* \cong \left(U^* \otimes \Lambda^2 W^*\right) \oplus \Lambda^3 W^*, \tag{2.24}$$
$$\Lambda^5 V^* \cong \left(U^* \otimes \Lambda^4 W^*\right) \oplus \Lambda^5 W^*. \tag{2.25}$$

Let us decompose $\Omega = \theta \wedge \psi + \phi$ in its components with respect to (2.24) with $\phi \in \Lambda^3 W^*$ and $\psi \in \Lambda^2 W^*$. The assumption $T_\Omega(X) = 0$ now translates into

$$0 = \iota(X)\Omega \wedge \Omega = \psi \wedge (\theta \wedge \psi + \phi) = \theta \wedge \psi \wedge \psi + \psi \wedge \phi. \tag{2.26}$$

Since $\theta \wedge \psi \wedge \psi \in U^* \otimes \Lambda^4 W^*$ and $\psi \wedge \phi \in \Lambda^5 W^*$ lie in different summands of the decomposition (2.25), this implies that both summands on the right hand side of (2.26) vanish individually:

$$\theta \wedge \psi \wedge \psi = 0, \tag{2.27}$$
$$\psi \wedge \phi = 0. \tag{2.28}$$

Moreover, since $\theta \in U^*$ and $\psi \wedge \psi \in \Lambda^4 W^*$, equation (2.27) implies $\psi \wedge \psi = 0$. Therefore $\psi \in \Lambda^2 W^*$ is decomposable. Thus let us choose linearly independent $e_1, e_2 \in W^*$ so that

$$\psi = e_1 \wedge e_2. \tag{2.29}$$

Here we use that $\iota(X)\Omega \neq 0$ and hence $\psi \neq 0$ by (2.23). Let $W_1^* = \mathrm{span}(e_1, e_2)$ and choose a complementary subspace W_2^* so that $W^* = W_1^* \oplus W_2^*$. If we decompose $\phi \in \Lambda^3 W^*$ into its components with respect to the direct sum decomposition

$$\Lambda^3 W^* = \bigoplus_{i=0}^{2} \Lambda^i W_1^* \otimes \Lambda^{3-i} W_2^*,$$

then the equations (2.28) and (2.29) tell us that the component of ϕ in $\Lambda^3 W_2^*$ vanishes because $\Lambda^2 W_1^* \otimes \Lambda^3 W_2^* \to \Lambda^5 W^*$ is an isomorphism and $\psi \neq 0$. Thus

$$\phi = (ae_1 + be_2) \wedge \rho + \psi \wedge \tau$$

with $\tau \in W_2^*$ and $\rho \in \Lambda^2 W_2^*$. We may assume without loss of generality that we have chosen e_1, e_2 so that $a = 1,\ b = 0$ since there is an $\mathrm{SL}(2, \mathbb{R})$-freedom of choice for the basis e_1, e_2 in (2.29), and we can scale ρ appropriately. $\rho \in \Lambda^2 W_2^*$ is decomposable as a two-form in three dimensions. Let $e_3, e_4 \in W_2^*$ be linearly independent with

$$\rho = \alpha\, e_3 \wedge e_4$$

and $\alpha = 1$ if $\rho \neq 0$ or $\alpha = 0$ otherwise. Now we can write

$$\Omega = \theta \wedge \psi + \phi = (\theta + \tau) \wedge (e_1 \wedge e_2) + e_1 \wedge (\alpha\, e_3 \wedge e_4).$$

$e_1, \ldots, e_4, \tau \in W^*$ and $\theta \notin W^*$, so $e_5 = \theta + \tau$ is linearly independent of $e_1, \ldots, e_4$, and we get

$$\Omega = (e_1 \wedge e_2 + \alpha\, e_3 \wedge e_4) \wedge e_5. \tag{2.30}$$

If $\alpha = 1$ we have $d = 1$ and the formula (2.30) for Ω coincides with corresponding normal form given in Theorem 2.1.13, if $\alpha = 0$ vanishes, we have $d = 3$ and again (2.30) reduces to the normal form given in the table, up to relabelling indices.

To complete the proof, the last step is to show that these normal forms are indeed inequivalent under the action of the full group $\mathrm{GL}(V)$. It is obvious that the degeneracy d is an invariant under the $\mathrm{GL}(V)$-action, and from Proposition 2.1.8 so is the sign of $\lambda^\nu(\Omega)$. Thus Theorem 2.1.13 classifies three-forms under the action of $\mathrm{GL}(V)$. □

Proposition 2.1.14 *The following are equivalent*

1. *Ω is degenerate.*

2. $\lambda(\Omega) = 0$,

3. $\ker(T_\Omega) \neq \{0\}$,

Proof. (1) $\Leftrightarrow$ (2): If Ω is degenerate then there is an $X \in V \setminus \{0\}$ with $\iota(X)\Omega = 0$. Hence $X \in \ker(T_\Omega)$ and certainly $\lambda(\Omega)X = T_\Omega^2(X) = 0$, so $\lambda(\Omega) = 0$. Conversely, by Proposition 2.1.8 it is enough to compute $\lambda(\Omega)$ for the normal forms in Theorem 2.1.13, and we find that $\lambda(\Omega) = 0 \Rightarrow d \geq 1$, hence Ω is degenerate. (2) $\Leftrightarrow$ (3): If $\lambda(\Omega) = 0$ then $T_\Omega^2 \equiv 0$ and so $\dim(\ker(T_\Omega)) \geq \frac{1}{2}\dim(V) = 3$. Conversely, if $\ker(T_\Omega) \neq \{0\}$ then $\ker(T_\Omega^2) \neq \{0\}$ and hence $T_\Omega^2 \equiv 0$ since $T_\Omega^2 = \mathrm{Id} \otimes \lambda(\Omega)$ is a scalar. Thus $\lambda(\Omega) = 0$. □

Corollary 2.1.15 *Let $V \cong \mathbb{R}^6$ and $\Omega \in \Lambda^3 V^*$ be non-degenerate. Then either Ω is of positive type and* $\mathrm{Stab}^+(\Omega) \cong \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ *or Ω is of negative type and* $\mathrm{Stab}^+(\Omega) \cong \mathrm{SL}(3,\mathbb{C})$.

Proof. This is a byproduct of the proof of Theorem 2.1.13, see (2.21) and (2.22). Note that $\mathrm{Stab}^+(\Omega)$ is the connected component of the identity in the full stabiliser group $\mathrm{Stab}(\Omega) \subset \mathrm{GL}(V)$ which is not connected. □

2.1.2 Three-forms and complex structures

We will now use the results of the previous paragraph to describe a way of using three-forms $\Omega \in \mathcal{A}^3(X)$ on a real oriented six-manifold X which satisfy a pointwise non-degeneracy condition to define complex structures on X. Take a triple (X, Ω, ν) consisting of a real oriented six-manifold X with a fixed three-form Ω and positive volume form ν. To simplify our notation we write $\lambda^\nu(p)$ for $\lambda^\nu(\Omega_p)$. Then if $\lambda^\nu < 0$ all over X let us define

$$\Omega^c = \Omega + iI_\Omega^*\Omega. \tag{2.31}$$

Ω^c is of type $(3,0)$ with respect to the almost complex structure $I_\Omega = I_\Omega^\nu$.

Proposition 2.1.16 *Let X be an oriented six-manifold and $\Omega \in \mathcal{A}^3(X)$ a three-form of everywhere negative type. The almost complex structure I_Ω is integrable if and only if $d\Omega^c$ in (2.31) is of type* $(3,1)$.

Proof. (compare Hitchin [Hit98].) Denoting the space of differential forms of type (p,q) by $\mathcal{A}^{(p,q)}$ an almost complex structure is integrable if and only if

$$\mathrm{d}\mathcal{A}^{(p,q)} \subset \mathcal{A}^{(p+1,q)} \oplus \mathcal{A}^{(p,q+1)}. \tag{2.32}$$

So if I_Ω is integrable then $d\Omega^c \in \mathcal{A}^{(3,1)}$. Conversely since Ω^c is of type $(3,0)$ it is locally decomposable by non-zero $(1,0)$-forms $\Omega = \theta_1 \wedge \theta_2 \wedge \theta_3$. If $\mathrm{d}\Omega^c$ is a $(3,1)$-form then using $\mathcal{A}^{(p,q)} = \mathcal{A}^{(p,0)} \otimes \mathcal{A}^{(0,q)}$ we see

$$0 = (\mathrm{d}\Omega^c)^{(2,2)} = \sum_{k=1}^{3} (\mathrm{d}\theta_k)^{(0,2)} \otimes (\theta_{k+1} \wedge \theta_{k+2}).$$

This implies $\mathrm{d}\theta_k \in \mathcal{A}^{(2,0)} \oplus \mathcal{A}^{(1,1)}$ for $k = 1, 2, 3$ and this implies the integrability condition (2.32). □

We can apply these observations to a Calabi-Yau manifold (X, Ω^c, ω) with holomorphic three-form Ω^c and Kähler form ω. In the normalisation of Definition 2.1.1, the volume element of the Calabi-Yau metric is $\nu = \omega^3/6$.

Proposition 2.1.17 *Let (X, Ω^c, ω) be a Calabi-Yau 3-fold with holomorphic three-form Ω^c, complex structure J and volume element $\nu = \omega^3/6$. Then $\lambda^\nu \equiv -4$, and the (almost) complex structure I_Ω induced from $\Omega = \mathrm{Re}(\Omega^c)$ as in Notation 2.1.11 coincides with the original J.*

Proof. Fix $p \in X$ and let $(e_k, f_k)_{k=1,2,3}$ be an orthonormal basis of T_pX such that $Je_k = f_k$. On a Calabi-Yau manifold with Ricci-flat metric, we can choose this basis so that $\Omega_p = (e_1 + if_1) \wedge \cdots \wedge (e_3 + if_3)$ and thus

$$\Omega = \mathrm{Re}(\Omega^c) = e_1 \wedge e_2 \wedge e_3 - e_1 \wedge f_2 \wedge f_3 - e_2 \wedge f_3 \wedge f_1 - e_3 \wedge f_1 \wedge f_2.$$

It can then be seen directly that $T_\Omega^\nu(e_k) = 2f_k = 2Je_k$ and $T_\Omega^\nu(f_k) = -2e_k = -2Jf_k$, thus $\lambda^\nu \equiv -4$ and $I_\Omega = \frac{1}{2}T_\Omega^\nu = J$. □

The following proposition transfers the results of Paragraph 2.1.1 to manifolds and lists transformation properties of the objects we have defined under the action of the diffeomorphism group. The proofs are rather straightforward.

Proposition 2.1.18 *Let $\varphi\colon X \to X$ be a diffeomorphism of X, and set $\varepsilon = +1$ if φ is orientation preserving and $\varepsilon = -1$ otherwise. Let $\xi \in C^\infty(TX)$ be an arbitrary vector field on X and $\Omega \in \mathcal{A}^3(X)$. Let $\tau\colon \Lambda^3T^*X \to \Lambda^3T^*X$ be defined by $\tau(\Omega) = I_\Omega^*\Omega$. Then the following relations hold:*

$$\varphi_* I_{\varphi^*\Omega}(\xi) = \varepsilon\, I_\Omega(\varphi_*\xi) \tag{2.33}$$
$$\tau(\varphi^*\Omega) = \varepsilon\, \varphi^*\tau(\Omega) \tag{2.34}$$
$$D\tau|_\Omega(\mathcal{L}_\xi\Omega) = \mathcal{L}_\xi\tau(\Omega). \tag{2.35}$$

2.2 The moduli space of special Lagrangians

In this section we define special Lagrangian submanifolds, the main object of this thesis, and we focus on Donaldson's construction [Don99] of the moduli space of special Lagrangian submanifolds as a symplectic quotient of an infinite dimensional function space and compare it to McLean's work [McL98]. We use analytical methods for infinite dimensional manifolds to prove that Donaldson's construction (which is only formal in the sense that it deals with infinite-dimensional spaces as though they were finite-dimensional) does indeed produce a smooth manifold and that there is a naturally induced metric on the base which coincides with McLean's metric.

Definition 2.2.1 *Let X be a Calabi-Yau n-fold with Kähler form ω and holomorphic n-form $\Omega^c = \Omega_1 + i\Omega_2$. A real submanifold $\iota\colon L \to X$ with $\dim_{\mathbb{R}}(L) = n$ is called special Lagrangian if*

$$\iota^*\omega = 0 \quad \text{and} \quad \iota^*\Omega_2 = 0.$$

In Definition 2.1.1 there is an obvious freedom to rotate the holomorphic n-form by a constant phase factor $e^{i\vartheta}$, and one could define a Lagrangian submanifold L to be special if $(\sin(\vartheta)\Omega_1 + \cos(\vartheta)\Omega_2)$ vanishes on L for some $\vartheta \in [0, 2\pi]$. However, there is no substantial gain of generality and we work with Definition 2.2.1 to avoid cumbersome notation.

Corollary 2.2.2 (Harvey-Lawson [HL82]) *A special Lagrangian submanifold $L \to X$ is calibrated with respect to Ω_1.*

Corollary 2.2.3 (Harvey-Lawson [HL82]) *Special Lagrangian submanifolds are volume minimising within their homology class.*

Proof. This is a general property of calibrated submanifolds and follows easily from Stokes' theorem. Let g be the Ricci-flat metric on X, let $\iota\colon L \to X$ be special Lagrangian and let $\iota'\colon L' \to X$ be an n-cycle with $[L] = [L'] \in H_n(X)$. Then

$$\mathrm{Vol}(L) = \int_L \mathrm{vol}_{\iota^*g} = \int_L \Omega_1 = \int_{L'} \Omega_1 \leq \int_{L'} \mathrm{vol}_{\iota'^*g} = \mathrm{Vol}(L').$$

□

One of the few known results about special Lagrangians which is both fundamental to their study and also rather straightforward is McLean's theorem describing their deformations [McL98].

Theorem 2.2.4 (McLean [McL98]) *Let $L \to X$ be a closed special Lagrangian submanifold of a Calabi-Yau manifold X. The infinitesimal deformations of L can be canonically identified with $H^1(L, \mathbb{R})$. Moreover, infinitesimal deformations are unobstructed and the local moduli space is a smooth manifold $\mathcal{S}$ diffeomorphic to an open neighbourhood of the origin in $H^1(L, \mathbb{R})$ with a natural Riemannian metric induced from the L^2-inner product on harmonic one-forms on L.*

Special Lagrangian submanifolds are very hard to find; as noted in [Hit98] most known examples fall in one of the following categories:

- Complex Lagrangian submanifolds of hyper-Kähler manifolds [Hit99b],
- The fixed locus of an anti-holomorphic involution [Bry85],
- Some examples in non-compact Calabi-Yau manifolds, for instance [Bry99a, Bry99b, Fei99, OV96, Ste93, EH79], and many more.

Each of these examples depends on additional structures such as a hyper-Kähler metric or an isometric group action and should therefore be seen as non-generic. The reason why we have no "generic" examples, say on a compact Calabi-Yau manifold, is that we know very little about Ricci-flat metrics on them - Yau's Theorem 2.1.2 which grants their existence is non-constructive. On the other side, R Bryant has recently shown [Bry99a] that every real-analytic closed parallelisable Riemannian manifold can be isometrically embedded as a special Lagrangian submanifold in some (open) Calabi-Yau manifold (see also [Mat00] in this context).

2.2.1 Donaldson's construction

In [Don99] Donaldson constructs the moduli space of special Lagrangian submanifolds as a symplectic quotient of an infinite dimensional manifold by the action of a diffeomorphism group. His method is more abstract than McLean's, but has the advantage of giving us, in a natural way, a complexification of the (real) moduli space of special Lagrangian submanifolds together with a Kähler metric on it. Secondly, the conceptual framework of his approach is very general and widely applicable; indeed Atiyah and Bott's treatment of Yang-Mills connections [AB83] and Segal's description of Teichmüller space [Seg] are carried out in much the same spirit. This puts the moduli problem for special Lagrangians in a new and interesting wider context. On the other side, Donaldson applies methods from finite-dimensional geometry in a purely formal manner to infinite-dimensional function spaces. Therefore we cannot conclude that the moduli space we get is actually a smooth, finite-dimensional manifold: The usual symplectic quotient construction uses the implicit function theorem for finite-dimensional manifolds to prove the quotient has a manifold structure. There are various generalisations of this theorem to infinite dimensional manifolds, but extra work is needed to be able to apply them. Our goal in Paragraph 2.2.3 is to provide the necessary piece of analysis to fill this gap. But first we give a brief account of Donaldson's work. Part of this is also contained in [Hit99a].

Let X be a Calabi-Yau n-fold with holomorphic n-form $\Omega^c = \Omega_1 + i\Omega_2$, Kähler form ω and metric g. Let L be a real, orientable closed manifold with $\dim_{\mathbb{R}}(L) = n$, and we fix a volume form $\nu \in \mathcal{A}^n(L)$ once and for all. We define

$$\mathcal{M} = \{f\colon L \to X \mid f^*\Omega^c = \nu, \quad f^*\omega \text{ is exact }\}.$$

Let us assume $\mathcal{M} \neq \emptyset$. Working formally, we do not at this point specify what kind of maps we wish to consider, nor the sense in which the function space $\mathcal{M}$ is a manifold. This will be part of our analytic treatment in Paragraph 2.2.3. For each $f \in \mathcal{M}$ there is a natural isomorphism

$$T_f\mathcal{M} \cong \{V \in \Gamma(f^*TX) \mid \mathrm{d}f^*\iota_V\Omega^c = 0\}. \tag{2.36}$$

Remark 2.2.5 The notation $df^*\iota_V\Omega^c = 0$ is slightly inaccurate and really means this: Since $f^*\Omega^c = \nu$, we know that f is an immersion, thus a local embedding. So given $V \in \Gamma(f^*TX)$ and a small ball $B \subset L$, we can write $V|_B = v \circ f$ where $v \in \mathrm{Vect}(f(B))$ is a smooth vector field on $\tilde{B} = f(B) \subset X$. Then $d\iota_v\Omega^c \in \mathcal{A}^n_{\mathbb{C}}(\tilde{B})$ and we define $df^*\iota_V\Omega^c \in \mathcal{A}^n_{\mathbb{C}}(L)$ to be the n-form whose restriction to B is $f^*d\iota_v\Omega^c$.

So a tangent vector to $\mathrm{Map}(L,X)$ at f is a section of the pull-back bundle f^*TX, and the equation $f^*\Omega^c = \nu$ is preserved to first order if $0 = \mathcal{L}_V\Omega^c = d(f^*\iota_V\Omega^c)$ since $d\Omega^c = 0$. The symplectic structure and metric on X induce the following symplectic form $\tilde{\omega}$ and metric $\tilde{g}$ on $\mathcal{M}$

$$\tilde{\omega}(V,W) = \int_L \omega(V,W)\nu \tag{2.37}$$

$$\tilde{g}(V,W) = \int_L g(V,W)\nu. \tag{2.38}$$

Here the expression $\omega(V,W)$ means that we think of $V,W\colon L \to TX$ as sections of f^*TX as in (2.36), and contraction with ω yields a function $x \in L \mapsto \omega_{f(x)}(V(x),W(x))$ on L, compare Remark 2.2.5. It is obvious that $\tilde{\omega}$ and $\tilde{g}$ are non-degenerate. We also get an almost complex structure $\tilde{J}$ where $\tilde{J}V$ is $x \in L \mapsto JV(x) \in T_{f(x)}X$. Recall that $\Omega^c \in \mathcal{A}^{3,0}(X)$ is of type $(3,0)$ and so

$$\mathrm{d}f^*\iota_{J\xi}\Omega^c = i\;\mathrm{d}f^*\iota_\xi\Omega^c$$

hence one side vanishes if and only if the other does. Comparing with (2.36) this says that V is tangent to $\mathcal{M}$ if and only if $\tilde{J}V$ is. So $\mathcal{M}$ is an almost complex manifold. We would like to argue that, at least in a formal sense, $\tilde{\omega}$ is closed. Note that

$$\begin{array}{ccc} & \mathcal{M}\times L & \\ \swarrow \pi_1 & & \pi_2 \searrow \\ \mathcal{M} & & L \end{array}$$

is a fibre bundle in two ways, and

$$\Lambda^p T^*(\mathcal{M}\times L) = \bigoplus_{k=0}^{p} \Lambda^{p-k}(\pi_1^* T^*\mathcal{M}) \otimes \Lambda^k(\pi_2^* T^* L). \tag{2.39}$$

Denote by $(\pi_1)_*$ the push-down map along L which in terms of (2.39) is defined by

$$(\pi_1)_*(a\pi_1^*\theta \otimes \pi_2^*\psi)|_f = \left(\int_L a(f,\cdot)\ \psi\right)\theta \tag{2.40}$$

for $\theta \in \mathcal{A}^{p-n}(\mathcal{M})$, $\psi \in \mathcal{A}^n(L)$ and $a\colon \mathcal{M}\times L \to \mathbb{R}$ a function, and set

$$(\pi_1)_*(a\ \pi_1^*\theta \otimes \pi_2^*\psi) = 0$$

if $\deg(\psi) \neq n$. The evaluation map

$$ev\colon \mathcal{M}\times L \longrightarrow X$$

sends $(f,x)\mapsto f(x)$. Now $V_1, V_2 \in T\mathcal{M}$ have lifts $\bar{V}_1, \bar{V}_2 \in T(\mathcal{M}\times L)$ with $(\pi_1)_*\bar{V}_i = V_i$ and $(\pi_2)_*\bar{V}_i = 0$. Then

$$\begin{aligned} & \iota_{V_1}\iota_{V_2}(\pi_1)_*(ev^*\omega \wedge \pi_2^*\nu) \\ = \ & (\pi_1)_*\left([\iota_{\bar{V}_1}\iota_{\bar{V}_2} ev^*\omega] \wedge \pi_2^*\nu\right) \\ = \ & \int_L \left(\omega_f(ev_*\bar{V}_1, ev_*\bar{V}_2)\right)\ \nu \\ = \ & \int_L \omega(V_1,V_2)\nu \\ = \ & \tilde{\omega}(V_1,V_2) \end{aligned}$$

Hence $\tilde{\omega} = (\pi_1)_*(ev^*\omega \wedge \pi_2^*\nu)$. Obviously $(ev^*\omega)\wedge \pi_2^*\nu$ is closed, and the push-down of a closed form is again closed. So *formally* $(\mathcal{M}, \tilde{\omega}, \tilde{g})$ is an infinite-dimensional Kähler-manifold.

Let $\mathcal{D}$ be the group of volume-preserving diffeomorphisms of (L,ν). Then $\phi \in \mathcal{D}$ acts on $\mathcal{M}$ by sending $f \mapsto f\circ\phi^{-1}$ and on L by $x \mapsto \phi(x)$. This action on $\mathcal{M}$ preserves both the metric and the symplectic structure, and ev is $\mathcal{D}$-invariant under the combined action on $\mathcal{M} \times L$. For each $\xi \in \mathrm{Lie}(\mathcal{D})$ we get vector fields $V_\xi \in \mathrm{Vect}(\mathcal{M})$ and $W_\xi \in \mathrm{Vect}(L)$ giving the infinitesimal action. This identifies $\mathrm{Lie}(\mathcal{D})$ with the Lie-algebra $\mathcal{W}$ of divergence-free vector fields on L, that is $W_\xi \in \mathrm{Vect}(L)$ so that $\iota(W_\xi)\nu$ is closed. Since ν defines an isomorphism $\mathrm{Vect}(L) \to \mathcal{A}^{n-1}(L)$ we can equivalently identify $\mathrm{Lie}(\mathcal{D})$ with $\mathcal{Z}^{n-1}(L)$, the space of closed $(n-1)$-forms on L. We define

$$\mathcal{W}_0 = \{\xi \in \mathcal{W} \mid \iota_\xi \nu \text{ is exact } \} \cong \mathcal{B}^{n-1}(L),$$

the space of exact $(n-1)$-forms. Similarly the action of ξ on $\mathcal{M}$ corresponds to a vector field $V_\xi \in \mathrm{Vect}(\mathcal{M})$. There is an exact sequence of Lie algebras

$$\{1\} \longrightarrow \mathcal{W}_0 \longrightarrow \mathrm{Lie}(\mathcal{D}) \xrightarrow{\mathfrak{p}} \mathrm{H}^{n-1}(L, \mathbb{R}) \longrightarrow \{1\}$$

with $\mathfrak{p}$ sending every closed $(n-1)$-form to its deRham cohomology class. In [Hit99a], Hitchin shows (for $n = 3$, the case of general $n \in \mathbb{N}$ being similar) that there is a corresponding sequence of Lie-groups

$$\{1\} \longrightarrow \mathcal{D}_0 \longrightarrow \mathcal{D} \xrightarrow{\mathfrak{P}} \mathrm{H}^{n-1}(L, \mathbb{R}/\mathbb{Z}) \longrightarrow \{1\} \tag{2.41}$$

with $\mathrm{Lie}(\mathcal{D}_0) = \mathcal{W}_0$ and where the right geometric interpretation (in dimension three) of the action of $\mathcal{D}$ on L is that $\mathcal{D}$ acts naturally on a gerbe $\mathcal{G}$ with connection whose curvature is ν, thus $\mathcal{G} \otimes \phi^*\mathcal{G}^*$ is a flat gerbe on L whose connection is classified by an element of $H^{n-1}(L, \mathbb{R}/\mathbb{Z})$, and $\mathcal{D}_0$ is the subgroup which preserves the connection, that is, the diffeomorphisms for which $\mathcal{G} \otimes \phi^*\mathcal{G}^*$ is a flat gerbe with trivial flat connection. We do not need the language of gerbes for our present purpose but mention that in the physically relevant case $n = 3$ it provides an appropriate setting in which to formulate the SYZ approach to mirror symmetry. This is explained in detail in [Hit99a].

The only thing we want to prove about Donaldson's construction is that there exists a moment map for the $\mathcal{D}_0$-action on $\mathcal{M}$, that is a map

$$\mu\colon \mathcal{M} \longrightarrow \mathrm{Lie}(\mathcal{D}_0)^*$$

such that $\iota(V_\xi)\tilde{\omega} = -d(\mu\xi)$ for each $\xi \in \mathrm{Lie}(\mathcal{D}_0)$. In general, there is no moment map for the full action of $\mathcal{D}$. This theorem has appeared in [Hit99a] as well, however, the following proof is our original work and was carried out prior to the publication of [Hit99a]. Note that

$$\begin{array}{ccc} \mathcal{B}^{n-1}(L) \times \mathcal{A}^1(L)/\mathcal{Z}^1(L) & \longrightarrow & \mathbb{R} \\ (\theta, [\psi]) & \longmapsto & \int_L \theta \wedge \psi \end{array} \tag{2.42}$$

is a well-defined non-degenerate pairing, hence the dual of $\mathrm{Lie}(\mathcal{D}_0)$ is

$$\mathrm{Lie}(\mathcal{D}_0)^* \cong \frac{\mathcal{A}^1(L)}{\mathcal{Z}^1(L)} = \frac{\mathcal{A}^1(L)}{\ker \mathrm{d}\colon \mathcal{A}^1(L) \to \mathcal{A}^2(L)}.$$

By definition, $f^*\omega$ is exact for all $f \in \mathcal{M}$, hence there is a unique

$$\mu(f) \in \mathcal{A}^1(L)/\ker(\mathrm{d}) \cong \mathrm{Lie}(\mathcal{D}_0)^* \tag{2.43}$$

with $d\mu(f) = f^*\omega$.

Theorem 2.2.6 *The map $\mu\colon \mathcal{M} \to \mathrm{Lie}(\mathcal{D}_0)^*$ defined in (2.43) is the moment map for the action of $\mathcal{D}_0$ on $\mathcal{M}$.*

Proof. By definition of $\mathcal{M}$, $f^*\omega$ is exact for all $f \in \mathcal{M}$. So given f we can write $\omega = d\varphi$ in a tubular neighbourhood of $f(L)$; note that f is an immersion so that this notion is well-defined. The splitting

$$T^*(\mathcal{M} \times L) = \pi_1^* T^*\mathcal{M} \oplus \pi_2^* T^* L$$

decomposes $ev^*\varphi = \varphi_1 + \varphi_2$ and allows us to write the exterior derivative d on $\mathcal{M} \times L$ as the sum of two operators $d = d^{\mathcal{M}} + d^L$. Now let $\xi \in \mathrm{Lie}(\mathcal{D}_0)$. The vector field V_ξ has a unique lift $\bar{V}_\xi$ to $\mathrm{Vect}(\mathcal{M} \times L)$ such that $(\pi_1)_*\bar{V}_\xi = V_\xi$ and $(\pi_2)_*\bar{V}_\xi = 0$ and similarly we have a lift $\bar{W}_\xi$ of ξ with $(\pi_1)_*\bar{W}_\xi = 0$ and $(\pi_2)_*\bar{W}_\xi = W_\xi$. We note the following identities

$$\begin{aligned} ev_*(V_\xi) &= -ev_*(W_\xi) && (2.44)\\ (\pi_1)_*\mathrm{d}^{\mathcal{M}} &= \mathrm{d}(\pi_1)_* && (2.45)\\ (\varphi_2|\{f\} \times L) \bmod \ker(\mathrm{d}^L) &= \mu(f) && (2.46) \end{aligned}$$

where (2.44) is $\mathcal{D}$-invariance of ev, and (2.45) is obvious from (2.40) and (2.46) follows from the definition of φ_i. In particular, (2.44) implies $\varphi_1(V_\xi) + \varphi_2(W_\xi) = 0$ for all $\xi \in \mathrm{Lie}(\mathcal{D})$. Now we conclude

$$\begin{aligned} \iota(V_\xi)\tilde{\omega} &= \iota(V_\xi)(\pi_1)_*(ev^*\omega \wedge \pi_2^*\nu) \\ &= (\pi_1)_*\left\{\iota(\bar{V}_\xi)[ev^*\omega \wedge \pi_2^*\nu]\right\} \\ &= (\pi_1)_*\left\{\left(\iota(\bar{V}_\xi)ev^*\omega\right) \wedge \pi_2^*\nu\right\} \\ &= (\pi_1)_*\left\{-\left(\iota(\bar{W}_\xi)ev^*\omega\right) \wedge \pi_2^*\nu\right\} && \text{by (2.44)} \\ &= (\pi_1)_*\left\{-ev^*\omega \wedge \iota(\bar{W}_\xi)\pi_2^*\nu\right\} \\ &= (\pi_1)_*\left\{\left(-\mathrm{d}^{\mathcal{M}}\varphi_2 - \mathrm{d}^L\varphi_1\right) \wedge \iota(\bar{W}_\xi)\pi_2^*\nu\right\} \\ &\overset{(\dagger)}{=} -\mathrm{d}\left\{(\pi_1)_*\left[\varphi_2 \wedge \iota(\bar{W}_\xi)\pi_2^*\nu\right]\right\} && \text{by (2.45)} \\ &= -\mathrm{d}\left\{\int_L \mu(f) \wedge \iota(W_\xi)\nu\right\} && \text{by (2.46)} \\ &= -\mathrm{d}(\mu\xi). \end{aligned}$$

In $(\dagger)$ we use the fact that for $\xi \in \mathrm{Lie}(\mathcal{D}_0)$ the $(n-1)$-form $\iota(W_\xi)\nu \in \mathcal{A}^{n-1}(L)$ is exact and thus $(\pi_1)_*(d^L\varphi_1 \wedge \iota(\bar{W}_\xi)\pi_2^*\nu) = 0$. □

Now $\mu(f) = 0$ if and only if $f^*\omega = \mathrm{d}\varphi$ with $\varphi \in \ker(d)$, that is if and only if $f^*\omega = 0$, or f is Lagrangian. Thus $\mu^{-1}(\{0\}) \subset \mathcal{M}$ is the set of Lagrangian embeddings with

$f^*\Omega^c = \nu$, that is the special Lagrangian embeddings for which the induced metric has volume form ν. The latter restriction is not serious: Suppose $f\colon L \to X$ is any special Lagrangian submanifold and $[f^*\Omega^c] = [\nu] \in H^n(L, \mathbb{R})$. Then by Moser's theorem [Mos65] there is a $\phi \in \mathrm{Diff}(L)$ so that $\phi^*(f^*\Omega^c) = \nu$, so this description captures all special Lagrangian submanifolds whose total volume is $\nu[L]$. We can now formally take the symplectic quotient

$$\mathcal{S}_{\mathbb{C}} = \{\mathcal{M}//\mathcal{D}_0\}_{\mu=0}$$

which inherits a symplectic structure and Kähler metric (since $\mathcal{D}$ acts also isometrically) from $\mathcal{M}$. From [Hit99a], we know that $T = \mathcal{D}/\mathcal{D}_0 \cong H^{n-1}(L, \mathbb{R}/\mathbb{Z})$ and this torus acts on $\mathcal{S}_{\mathbb{C}}$. Then $\mathcal{S}_{\mathbb{R}} = \mathcal{S}_{\mathbb{C}}/T$ can be naturally thought of as the set of special Lagrangian submanifolds of X. McLean argues in [McL98] that this is a real manifold of dimension $b^1(L)$, and since $\dim(T) = b^{n-1}(L) = b^1(L)$ we conclude that $\mathcal{S}_{\mathbb{C}}$ is a Kähler manifold of complex dimension $b^1(L)$ with a Lagrangian T^n-fibration $\mathcal{S}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{R}}$.

2.2.2 Banach manifolds

This paragraph contains no new material. We collect the relevant facts about infinite-dimensional manifolds needed to give an analytic version of Donaldson's construction; a detailed account of this material can be found, for example, in Hamilton [Ham82] or Lang [Lan72]. Recall that a Banach space is a complete topological vector space whose topology is induced by a norm $\|\cdot\|$, hence a Banach space is Hausdorff and we shall only look at locally convex spaces. If F, G are Banach spaces, then so is $\mathcal{B}(F, G)$ the space of bounded linear maps $F \to G$ with the operator norm.

Definition 2.2.7 *A function $f\colon U \subset F \to G$ between open subsets of topological vector spaces F and G is differentiable if and only if the partial derivatives*

$$Df_x(\xi) = \lim_{t\to 0} \frac{f(x+t\xi) - f(x)}{t} \tag{2.47}$$

exist for all $(x, \xi) \in U \times F$. We define inductively the higher derivatives

$$D^k f = D(D^{k-1}f)\colon (U \times F^{k-1}) \times F = U \times F^k \longrightarrow G$$

in case $D^{k-1}f$ is a differentiable function on $U \times F^{k-1}$. Suppose F and G are Banach spaces, then f is called continuously differentiable or of class C^1 if $Df_x \in \mathcal{B}(F, G)$ for all $x \in U$ and the map

$$\begin{array}{rcl} U & \longrightarrow & \mathcal{B}(F, G) \\ x & \longmapsto & Df_x \end{array} \tag{2.48}$$

is continuous. By induction, we say f is k-times continuously differentiable or of class C^k if it is of class C^{k-1} and

$$D^{k-1}f\colon U \longrightarrow \mathcal{B}(F^{k-1}, G) \tag{2.49}$$

is of class C^1. Then $D^k f = D(D^{k-1}f)$ is a symmetric continuous multilinear map.

In (2.49) we have used the fact ([Lan72], Proposition I 4) that there is an isomorphism of Banach spaces

$$\mathcal{B}(F, \mathcal{B}(F, \mathcal{B}(F \cdots \mathcal{B}(F, G) \cdots))) \cong \mathcal{B}(F^k, G).$$

Note that the condition (2.48) implies that Df is continuous as a map $U \times F \to G$, but is much stronger than this if F is infinite-dimensional. It only makes sense in a Banach space; if for instance F, G are Fréchet and not Banach, then $\mathcal{B}(F, G)$ is never Fréchet and does not carry any useful topology. So while the definition of differentiability (2.47) makes sense in any topological vector space, we need the Banach space structure to be able to define *continuous* differentiability as above. A *smooth* function for us will always be a function on a Banach space which is C^∞.

Definition 2.2.8 *A Banach manifold is a set $\mathcal{M}$ equipped with an atlas (U_i, φ_i) whose charts take values in a Banach space F, whose transition functions are all smooth and such that for all $x, y \in \mathcal{M}$ the atlas contains sets U_x, U_y with $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$.*

As usual, there is a unique topology on such a manifold for which all charts are local homeomorphisms, and this topology is Hausdorff. Also the composition $g \circ f$ of smooth maps

$$U_1 \subset F_1 \xrightarrow{f} U_2 \subset F_2 \xrightarrow{g} U_3 \subset F_3$$

between open subsets of Banach spaces is again smooth ([Lan72], Proposition I 13) and so there is a well-defined notion of differentiability for maps $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ between Banach manifolds; f is of class C^∞ (of class C^k) if and only if $\varphi_i^{(2)} \circ f \circ (\varphi_j^{(1)})^{-1}$ is of the same class for all charts $\varphi_j^{(i)}$ of $\mathcal{M}_i$, whenever the composition is defined. There are implicit and inverse function theorems for Banach manifolds (see again Lang, [Lan72]).

Theorem 2.2.9 *Let F, G, H be Banach spaces and $U \subset F \times G$ open. Let $f\colon U \to H$ be a smooth map and $h \in H$. Assume that for every $(u_1, u_2) \in f^{-1}(\{h\})$ the differential $f_{*(u_1,u_2)}|\{0\} \times G$ induces an isomorphism $G \to H$ of Banach spaces (that is, it is bounded linear and has a bounded inverse). Then $f^{-1}(\{h\})$ is a Banach submanifold of class C^∞. If $f\colon U \subset F \to H$ is a smooth map and if $Df_u\colon F \to H$ is an isomorphism of Banach spaces for some $u \in U$, there is an open neighbourhood $V \subset H$ of $f(u)$ and a smooth map $g\colon V \to U$ such that $f \circ g = \mathrm{id}_V$. Since this is a local theorem it holds equally for C^∞-maps between Banach manifolds. There are analogous statements for C^k-maps.*

Note that there seems to be no unified terminology in the literature. Our notion of "continuous differentiability" sometimes appears as "differentiability in the sense of Fréchet". Due to its simplicity Theorem 2.2.9 is probably the most powerful tool available to prove the existence of manifold structures on infinite-dimensional spaces. However, it is strictly confined to the class of *Banach manifolds and C^∞-maps (C^k-maps)*, excluding Fréchet spaces like $C^\infty(V)$. In practice, if we are given a differentiable map $f\colon \mathcal{M} \to \mathcal{M}'$ between Banach manifolds, it is usually obvious what its derivative is, but the following questions which are trivial in the finite-dimensional case need to be checked carefully: 1. Does the map satisfy the strong notion (2.49) of continuous differentiability? 2. Does the closed subspace $F = \ker(Df_x) \subset T_x\mathcal{M}$ split, that is, is there a closed subspace $G \subset T_x\mathcal{M}$ such that $T_x\mathcal{M} = F \times G$? 3. Is $(Df_x|G)^{-1}$ bounded (continuous)? 2. is trivial if $T_x\mathcal{M}$ is a Hilbert-space, but not if it is Banach, and 3. is a kind of ellipticity condition on f.

The most important examples of infinite-dimensional Banach manifolds in differential geometry are function spaces or spaces of maps between smooth manifolds. Suppose (U, g) is a Riemannian manifold with Levi-Civita connection ∇, and $V \to U$ is a Riemannian vector bundle of finite rank with a compatible metric connection $\overline{\nabla}$. Then for each geodesic $\gamma\colon [a, b] \to U$ we can canonically identify $P_\gamma\colon V_{\gamma(b)} \xrightarrow{\cong} V_{\gamma(a)}$ by parallel transport with respect to $\overline{\nabla}$ along γ. A section $\sigma\colon U \to V$ is *Hölder continuous with coefficient α at $x \in U$* if

$$\sup_\gamma \frac{\|P_\gamma(\sigma(\gamma(b))) - \sigma(\gamma(a))\|_{V_x}}{|b-a|^\alpha} \quad < \quad c(x)$$

where the supremum is taken over all geodesics $\gamma\colon [a, b] \to U$ with $\gamma(a) = x$ and $\|\dot\gamma\|_g = 1$. It is simply called *Hölder continuous* if it is Hölder continuous at all $x \in U$. Moreover, in this case σ is *uniformly Hölder continuous* if there is a $c > 0$ so that we can choose $c(x) < c$ for all $x \in U$. We define $C^{k,\alpha}(V)$ to be the set of all $\sigma \in C^k(V)$ for which $\overline{\nabla}^k\sigma$ is a Hölder continuous section of the bundle $(\otimes^k T^*U) \otimes V$ with respect to the induced connection. Every Hölder continuous section over a compact Riemannian manifold is uniformly Hölder continuous, and if $V \to L$ is a Riemannian vector bundle over a *compact* Riemannian manifold L, then $C^k(V)$ and $C^{k,\alpha}(V)$ are naturally Banach spaces which are up to equivalence independent of the choice of metrics on L and V. (Over a non-compact base U, the *uniformly* Hölder continuous sections $\bar{C}^{k,\alpha}(V)$ still form a Banach space.) See for instance [Joy96].

Theorem 2.2.10 (Hamilton [Ham82]) *Let $p\colon Y \to Z$ be a fibre bundle whose total space Y and base Z are finite-dimensional smooth Riemannian manifolds and whose base Z is compact. Assume p is smooth. Then the space $C^k(Y)$ of C^k-sections and the space $C^{k,\alpha}(Y)$ of sections of class $C^{k,\alpha}$ are smooth Banach manifolds (if $\neq \emptyset$).*

Proof. *(Sketch)* Given $\sigma \in C^\infty(Y)$ the vertical bundle $\sigma^*\mathcal{V}$ at σ is a smooth Riemannian vector bundle over Z and hence $C^{k,\alpha}(\sigma^*\mathcal{V})$ is a (linear) Banach space. Let g be the Riemannian metric on Y (giving us the metric on $\sigma^*\mathcal{V}$) and choose ε so that the injectivity radius of g at $\sigma(x)$ satisfies $\mathrm{inj}(\sigma(x)) > \varepsilon$ for all $x \in Z$. Since Z is compact, it is possible to choose $\varepsilon > 0$. Using the smooth Riemannian exponential map associated to g and the Levi-Civita connection on the base one can construct a bijection φ_σ between the $C^{k,\alpha}$-sections σ' of Y with $\sup_x(\mathrm{dist}(\sigma'(x), \sigma(x))) < \varepsilon$ and an open neighbourhood of the origin in $C^{k,\alpha}(\sigma^*\mathcal{V})$. One can then check that for pairs σ, τ the transition functions $\varphi_\sigma \circ \varphi_\tau^{-1}$ are smooth maps. Since $C^\infty(Y)$ is C^0-dense in $C^k(Y)$, the domains of the charts φ_σ cover $C^{k,\alpha}(Y)$. Hence $C^{k,\alpha}(Y)$ is a smooth Banach manifold (the Hausdorff property is obvious). The C^k-case is similar. □

Corollary 2.2.11 *Let X and L be smooth finite-dimensional manifolds with L closed. Then $C^k(L, X)$ and $C^{k,\alpha}(L, X)$ are smooth Banach manifolds.*

Proof. Choose metrics on L and X. Then this becomes a special case of Theorem 2.2.10 where $Y \to Z$ is the trivial bundle $L \times X \to L$. □

We will mainly work in the $C^{k,\alpha}$ category because we only need standard existence and regularity results for elliptic differential operators, but no reflexiveness or the machinery of weak differentiability.

Suppose $V, W \to L$ are smooth Riemannian vector bundles of finite rank over a compact Riemannian manifold L. The Riemannian metric gives us L^2 inner products $\langle\cdot,\cdot\rangle^V, \langle\cdot,\cdot\rangle^W$ on $C^\infty(V)$ and $C^\infty(W)$. Suppose $P\colon C^\infty(V) \to C^\infty(W)$ is a linear elliptic differential operator of order m. We can then define the formal adjoint P^* by the usual rule

$$\langle Pv, w\rangle^W = \langle v, P^*w\rangle^V \qquad \text{for all } v \in C^\infty(V) \ \text{ and } \ w \in C^\infty(W).$$

P^* of course depends on the metrics of V, W and L, a different choice yields a naturally equivalent operator. However, our results later on will not depend on any particular choice. We can now quote two results about existence and regularity for elliptic operators.

Theorem 2.2.12 (Existence) (Besse [Bes87], Theorem 31, p.464) *Let V, W be smooth Riemannian vector bundles over a closed Riemannian manifold L and let $P\colon C^\infty(V) \to C^\infty(W)$ be a linear elliptic differential operator of order m. There is a natural extension of P to an operator $\bar{P}\colon C^{k+m,\alpha}(V) \to C^{k,\alpha}(W)$. Then* $\ker(\bar{P}), \ker(\bar{P}^*) \subset C^\infty(V)$*, and both kernels are finite-dimensional. Let $\theta \in C^{k,\alpha}(W)$ (respectively, $\theta \in C^\infty(W)$). The equation $\bar{P}\Theta = \theta$ has a solution in $C^{k+m,\alpha}(V)$ (respectively, in $C^\infty(V)$) if and only if θ is orthogonal to* $\ker(\bar{P}^*)$ *with respect to $\langle\cdot,\cdot\rangle^W$.*

There is a unique such solution which is orthogonal with respect to $\langle\cdot,\cdot\rangle^V$ *to* $\ker(\bar{P})$. *If* $\Theta \in L^2(V)$ *is any solution of the equation* $\bar{P}\Theta = \theta$, *then* $\Theta \in C^{k,\alpha}(V)$ *(respectively,* $\Theta \in C^\infty(V)$*).*

For this theorem it is essential that we use the Hölder spaces $C^{k,\alpha}(V)$ with $0 < \alpha < 1$ rather than the ordinary Banach spaces $C^k(V)$. In general, for a given $\theta \in C^k(W)$ orthogonal to $\ker(\bar{P}^*)$ there is not always a $\Theta \in C^{k+m}(V)$ with $\bar{P}\Theta = \theta$.

Theorem 2.2.13 (Regularity) (Besse [Bes87], Theorem 27, p.463) *We keep the notation of Theorem 2.2.12. Let* $0 < \alpha < 1$ *and* $k \geq 0$. *There are constants* $c_1, c_2, c_3 \geq 0$ *such that for every* $\Theta \in C^{k+m,\alpha}(V)$

$$\|\Theta\|_{C^{k+m,\alpha}} \leq c_1\|\bar{P}\Theta\|_{C^{k,\alpha}} + c_2\|\Theta\|_{C^0} \leq c_3\|\Theta\|_{C^{k+m,\alpha}}.$$

Moreover, if Θ *is* L^2*-orthogonal to* $\ker(\bar{P})$ *then there is such an estimate with* $c_2 = 0$.

The following Corollary can be easily deduced from Theorem 2.2.12 and Theorem 2.2.13. It tells us that we can do Hodge theory in $C^{k,\alpha}$.

Corollary 2.2.14 *In the situation of Theorem 2.2.12 there is a splitting*

$$\begin{aligned} C^{k,\alpha}(W) &= \bar{P}(C^{k+m,\alpha}(V)) \oplus \ker(\bar{P}^*) \\ C^\infty(W) &= \bar{P}(C^\infty(V)) \oplus \ker(\bar{P}^*) \end{aligned}$$

into closed subspaces, and $\bar{P}$ *has a continuous right inverse, that is there exists a bounded linear operator* $\bar{Q}\colon \operatorname{im}(\bar{P}) \cap C^{k,\alpha}(W) \to C^{k+m,\alpha}(V)$ *such that* $\bar{P}\bar{Q}$ *is the identity on* $\operatorname{im}(\bar{P}) \cap C^{k,\alpha}(W)$.

Proof. We define $\bar{Q}\theta = \Theta$ to be the unique solution of $\bar{P}\Theta = \theta$ which is L^2-orthogonal to $\ker(\bar{P})$ and whose existence is granted by Theorem 2.2.12. $\bar{Q}$ is obviously linear and by choosing it to be orthogonal to $\ker(\bar{P})$ we can apply Theorem 2.2.13 with $c_2 = 0$ to conclude that $\bar{Q}$ is continuous. □

2.2.3 Analytic construction of the moduli space

We would now like to back the formal construction of Paragraph 2.2.1 analytically. As before let (L, ν) be a closed smooth n-manifold with fixed smooth volume element $\nu \in \mathcal{A}^n(L)$, and X a Calabi-Yau n-fold with holomorphic n-form $\Omega^c \in \mathcal{A}^{n,0}(X)$ and Kähler form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$, as usual.

Theorem 2.2.15 *Let M_1, M_2 be smooth manifolds with M_1 compact, and let*

$$\mathcal{T}_i = \bigoplus_j (T^*M_i)^{\otimes j}$$

for $i = 1, 2$ be their tensor algebras. Take $0 \leq \alpha \leq 1$ and $k > 0$. For any smooth tensor-field $\Theta \in C^\infty(\mathcal{T}_2)$ (and any Riemannian structures on M_1, M_2) the map

$$\begin{aligned} \Phi\colon C^{k,\alpha}(M_1, M_2) &\longrightarrow C^{k-1,\alpha}(\mathcal{T}_1) \\ g &\longmapsto g^*\Theta \end{aligned}$$

is smooth. More generally this is true for $\Theta \in C^\infty(\mathcal{T}_2 \otimes \mathbb{C})$.

McLean [McL98] assumes this theorem for his proof that the moduli space of special Lagrangian submanifolds is smooth, however, he does not give a proof for it. The following argument can be used to fill this gap. We could not find a proof for Theorem 2.2.15 in the literature.

Proof. To be continuously differentiable is a local property and $C^\infty(M_1, M_2)$ is dense in $C^{k,\alpha}(M_1, M_2)$, so it is enough to take $f \in C^\infty(M_1, M_2)$ and prove the theorem for $\Phi|U$ where $U \subset C^{k,\alpha}(M_1, M_2)$ is an open neighbourhood of f. Choose a Riemannian metric h on M_2. Since $f(M_1)$ is compact, the injectivity radius of h satisfies

$$\varepsilon = \min\{\mathrm{inj}_h(f(x)) \mid x \in M_1\} > 0.$$

The Riemannian exponential map $\mathrm{EXP}\colon TM_2 \to M_2$ then identifies

$$\tilde{U} = \{\sigma \in C^{k,\alpha}(f^*TM_2) \mid \|\sigma\|_{C^0} < \varepsilon/2\} \tag{2.50}$$

which is an open neighbourhood of the zero section in $C^{k,\alpha}(f^*TM_2)$ with an open neighbourhood of f in $C^{k,\alpha}(M_1, M_2)$. Let $\tilde{\Theta} = \mathrm{EXP}^*\Theta$. We then have to show that

$$\begin{aligned} \tilde{\Phi}\colon \tilde{U} \subset C^{k,\alpha}(f^*TM_2) &\longrightarrow C^{k-1,\alpha}(\mathcal{T}_1) \\ \sigma &\longmapsto \sigma^*\tilde{\Theta} \end{aligned}$$

is smooth. It is enough to prove the theorem for $\Theta \in \mathcal{A}^1(M_2)$; the general case follows by composing with the map

$$\begin{aligned} C^{k-1,\alpha}(T^*M_1) \times \cdots \times C^{k-1,\alpha}(T^*M_1) &\longrightarrow C^{k-1,\alpha}(\mathcal{T}_1) \\ (\Theta_1, \ldots, \Theta_l) &\longmapsto \Theta_1 \otimes \cdots \otimes \Theta_l \end{aligned}$$

which is evidently continuous and multi-linear, hence smooth. The extension to $\mathbb{C}$-valued differential forms is trivial. Take a finite contractible cover $(V_i)_{i=1,\ldots,r}$ of M_1 (which we required to be compact) such that the closures of the images $\overline{f(V_i)}$ are all contained in coordinate domains $W_i \subset M_2$. Choose charts $\mathbf{x}^{(i)}: W_i \to \mathbb{R}^m$ (where $m = \dim(M_2)$) and let $\mathbf{y}^{(i)}: TM_2|W_i \to \mathbb{R}^m$ be linear coordinates on the fibres of the tangent bundle with respect to the trivialisation $TM_2|W_i \cong \mathbf{x}^{(i)}(W_i) \times \mathbb{R}^m$ induced by the sections $\partial/\partial x_1^{(i)}, \ldots, \partial/\partial x_m^{(i)}$. Without loss of generality we can assume that Θ is supported in just one of the covering sets, say $W = W_1$, because in general we can take a partition of unity on M_2 subordinate to the cover $W_1, \ldots, W_r, M_2 \setminus f(M_1)$ and apply the theorem to the forms $(\lambda_i \Theta)_{i=1,\ldots,r}$ separately. Let $V = V_1, \mathbf{x} = \mathbf{x}^{(1)}, \mathbf{y} = \mathbf{y}^{(1)}$. Thus $\tilde{\Theta}$ is of the form

$$\tilde{\Theta} = \sum_{j=1}^{m} \Big(a_j(x,y)\mathrm{d}x_j + b_j(x,y)\mathrm{d}y_j \Big).$$

Any section σ of $f^*TM_2|V$ can now be given by a function $\phi: V \to \mathbb{R}^m$ as $\sigma(p) = (f_1(p), \ldots, f_m(p), \phi_1(p), \ldots, \phi_m(p))$ with $f_j(p) = (x_j \circ f)(p)$. Fix such a σ and for $s = 1, \ldots, l$ let $\xi^{(s)} \in C^{k,\alpha}(f^*TM_2)$ be given for $p \in V$ by

$$\xi^{(s)}(p) = \sum_{j=1}^{m} \xi_j^{(s)}(p) \left. \frac{\partial}{\partial x_j} \right|_{f(p)}.$$

Due to local convexity there is a $\delta > 0$ such that $\sigma + t_1\xi^{(1)} + \cdots + t_l\xi^{(l)} \in \tilde{U}$ for all $t = (t_1, \ldots, t_l)$ with $\|t\| < \delta$. Write $t\xi = t_1\xi^{(1)} + \cdots + t_l\xi^{(l)}$. Then

$$\begin{aligned}
&\Big(D^l \tilde{\Phi}_\sigma \left(\xi^{(1)}, \ldots, \xi^{(l)}\right) \Big)(p) \\
= \;& \frac{d^l}{dt_1 \cdots dt_l} \left(\sigma(p) + t_1\xi^{(1)}(p) + \cdots + t_l\xi^{(l)}(p) \right)^* \tilde{\Theta} \Bigg|_{t=0} \\
= \;& \frac{d^l}{dt_1 \cdots dt_l} \Bigg[\sum_{j=1}^{m} \Big(a_j(f(p), \phi(p) + t\xi(p))\mathrm{d}f_j(p) + \\
&\qquad b_j(f(p), \phi(p) + t\xi(p)) \Big[\mathrm{d}\phi(p) + t_1 \mathrm{d}\xi_j^{(1)}(p) + \cdots + t_l \mathrm{d}\xi_j^{(l)}(p) \Big] \Big) \Bigg]_{t=0} \\
= \;& \sum_{j=1}^{m} \Bigg[\left(D^l a_j\right)_{\sigma(p)} \left(\xi^{(1)}, \ldots, \xi^{(l)}\right) \mathrm{d}f_j(p) + \left(D^l b_j\right)_{\sigma(p)} \left(\xi^{(1)}, \ldots, \xi^{(l)}\right) \mathrm{d}\phi_j(p) \\
&\qquad + \sum_{s=1}^{m} \left(D^{l-1} b_j\right)_{\sigma(p)} \left(\xi^{(1)}, \overset{s}{\ldots\wedge\ldots}, \xi^{(l)}\right) \mathrm{d}\xi_j^{(s)}(p) \Bigg].
\end{aligned}$$

From our choice of $\tilde{U}$ in (2.50) we know $\|\phi(p)\|_{C^0} < \varepsilon/2 < \varepsilon \leq \mathrm{inj}(f(p))$; hence the coefficient functions a_j, b_j and all their derivatives are defined on the ε-disk bundle of f^*TM_2 and therefore, by compactness of L, they are uniformly bounded in the C^l-norm on $\tilde{U}$ for every l, say

$$\max_{x\in\mathbf{x}(W),\,|y|<\varepsilon/2,\,0\leq s\leq l}\left\{\left|\frac{\partial^s a_j}{\partial y_{i_1}\cdots\partial y_{i_s}}(x,y)\right|,\ \left|\frac{\partial^s b_j}{\partial y_{i_1}\cdots\partial y_{i_s}}(x,y)\right|\right\} < c_l.$$

Also since f is smooth we have $\|df_j\|_{C^{k-1,\alpha}} < c$ in the $C^{k-1,\alpha}$-norm of T^*V for some $c > 0$. Finally, $\xi^{(s)} \in C^{k,\alpha}(V,\mathbb{R}^m)$ and $d\phi_j \in C^{k-1,\alpha}(T^*V)$. The product of Hölder continuous functions (sections) is again Hölder continuous with the same Hölder coefficient, and

$$\|d\xi^{(s)}\|_{C^{k-1,\alpha}} \leq c'\|\xi^{(s)}\|_{C^{k,\alpha}}$$

for some $c' > 0$. This yields an estimate

$$\left\|D\Phi_\sigma\left(\xi^{(1)},\ldots,\xi^{(l)}\right)\right\|_{C^{k-1,\alpha}} \leq \left(c_l c + c_l\|\mathrm{d}\phi\|_{C^{k-1,\alpha}} + c_{l-1}c'\right)\left\|\xi^{(1)}\right\|_{C^{k,\alpha}}\cdots\left\|\xi^{(l)}\right\|_{C^{k,\alpha}}.$$

Hence the derivative $D^l\Phi_\sigma$ is a bounded multi-linear map for all σ. We need to show it is continuous, that is

$$\|D^l\Phi_\sigma - D^l\Phi_{\sigma_n}\|_{\mathcal{B}(C^{k,\alpha}(f^*TM_2)^l,C^{k-1,\alpha}(\mathcal{T}_1))} \longrightarrow 0 \ \text{ as } \ \sigma_n \longrightarrow \sigma \ \text{ in } \ C^{k,\alpha}(f^*TM_2).$$

If the sections σ_n, σ are respectively given by $\phi_n, \phi \in C^{k,\alpha}(V,\mathbb{R}^m)$ as before, then

$$\|d\phi_n - d\phi\|_{C^{k-1,\alpha}} \leq c'\|\phi_n - \phi\|_{C^{k,\alpha}} \longrightarrow 0$$

and since the convergence $\phi_n \to \phi$ in the $C^{k,\alpha}$-norm implies uniform pointwise convergence of $\sigma_n(p) \to \sigma(p)$ by compactness of M_1 and the coefficient functions a, b are smooth, we get convergence

$$\left(D^l a\right)_{\sigma_n} \longrightarrow \left(D^l a\right)_\sigma \quad \text{and} \quad \left(D^l b\right)_{\sigma_n} \longrightarrow \left(D^l b\right)_\sigma$$

uniformly in M_1 and hence convergence in the operator norm. Hence $D^l\Phi$ is continuous in the sense of (2.48). $\square$

Let U and U' be (smooth) manifolds and $k \geq 1$. A map $\varphi \in C^k(U,U')$ is a C^k-*diffeomorphism* if φ is bijective and $D\varphi$ is nowhere degenerate. In this case the inverse function theorem shows that $\varphi^{-1} \in C^k(U',U)$ and that $D(\varphi^{-1})$ is nowhere degenerate, too, so φ^{-1} is a C^k-diffeomorphism as well. Let $\mathrm{Diff}^k(U,U')$ be the set of C^k-diffeomorphisms $\varphi\colon U \to U'$. (In particular, if $U = U'$ the set $\mathrm{Diff}^k(U,U)$ forms a group since it is obviously closed under composition). The next proposition gives an analogue for $C^{k,\alpha}$-*diffeomorphisms*

$$\varphi \in \mathrm{Diff}^{k,\alpha}(U,U') = \mathrm{Diff}^k(U,U') \cap C^{k,\alpha}(U,U'). \tag{2.51}$$

Proposition 2.2.16 *Let U, U' be smooth Riemannian manifolds. Let $k \geq 1$ and $\varphi \in \mathrm{Diff}^{k,\alpha}(U, U')$ (see (2.51)). Then $\varphi^{-1} \in \mathrm{Diff}^{k,\alpha}(U', U)$.*

Proof. We already know $\varphi^{-1} \in \mathrm{Diff}^{k}(U', U)$, so we only need to show that φ^{-1} satisfies a Hölder-condition with coefficient α. Note that

$$\begin{aligned} D\varphi\colon U \times \mathbb{R}^n &\longrightarrow U' \times \mathbb{R}^n = \varphi(U) \times \mathbb{R}^n \\ (x, \xi) &\longmapsto (\varphi(x), (D\varphi)_x(\xi)) \end{aligned}$$

is a diffeomorphism of class $C^{k-1,\alpha}$ (respectively, a Hölder continuous map if $k = 1$). Let $x \in U$ have a compact neighbourhood $K \subset U$ so that the image of the derivative $x \in K \mapsto (D\varphi)_x \in \mathrm{GL}(n, \mathbb{R})$ is compact. Matrix inversion is a smooth Hölder continuous map on each compact subset of $\mathrm{GL}(n, \mathbb{R})$. Since the composition of $C^{k,\alpha}$-maps is again $C^{k,\alpha}$, we get that

$$\begin{aligned} \varphi(K) \times \mathbb{R}^n &\longrightarrow K \times \mathbb{R}^n \\ (y, \eta) &\longmapsto (\varphi^{-1}(y), [(D\varphi)_{\varphi^{-1}(y)}]^{-1}(\eta)) \end{aligned}$$

is of class $C^{k-1,\alpha}$. But the right hand side is just $(\varphi^{-1}(y), D(\varphi^{-1})_y(\eta))$ and hence φ^{-1} is of class $C^{k,\alpha}$ as well. □

Theorem 2.2.17 *Minimal submanifolds of Calabi-Yau manifolds are real analytic manifolds. This is in particular true for special Lagrangian submanifolds.*

Proof. Every complex manifold is a real analytic manifold, and it follows from results in [dTK81] that for a Ricci-flat metric on a Calabi-Yau manifold the structure forms ω and Ω^c are analytic. Hence Calabi-Yau manifolds are analytic Riemannian manifolds since the metric is determined by ω and Ω^c. Minimal submanifolds of analytic Riemannian manifolds are always analytic, and so this is in particular true of special Lagrangian submanifolds of Calabi-Yau manifolds. □

In general, Kähler metrics on complex manifolds are of course not necessarily real analytic. A kind of converse statement is that every closed real analytic parallelisable Riemannian manifold can be isometrically embedded as a special Lagrangian submanifold into some (open) Calabi-Yau manifold [Bry99a].

Theorem 2.2.18 *Let $0 < \alpha < 1$. Then the sets*

$$\begin{aligned} \mathcal{M}^{k,\alpha} &= \{\, f \in C^{k,\alpha}(L, X) \mid f^*\Omega^c = \nu,\ f^*\omega \ \text{ is exact} \,\} \\ \mathcal{M}_0^{k,\alpha} &= \{\, f \in C^{k,\alpha}(L, X) \mid f^*\Omega^c = \nu,\ f^*\omega = 0 \,\} \end{aligned}$$

are smooth Banach manifolds.

Proof. Take $\mathcal{M}^{k,\alpha}$ first. From Paragraph 2.2.2 we know that $C^{k,\alpha}(L,X)$ is a smooth Banach manifold. We will apply Theorem 2.2.9 to show that $\mathcal{M}^{k,\alpha}$ cuts out a smooth Banach submanifold. Note that the condition that $f^*\omega$ be exact determines an open subset of $C^{k,\alpha}(L,X)$, because if $f^*\omega$ is exact then the restriction of ω to a small open neighbourhood of $f(L)$ in X will be exact and hence $\tilde{f}^*\omega$ is exact for all $\tilde{f}$ which are C^0-close to f, and the C^0-norm is weaker than the $C^{k,\alpha}$-norm. Another way of saying this is that the cohomology class $[f^*\omega] \in H^2(L,\mathbb{R})$ is locally constant on $C^{k,\alpha}(L,X)$. Hence we only need to look at the equation $f^*\Omega^c = \nu$. Since Ω^c is smooth we know by Theorem 2.2.15 that

$$\begin{array}{rcc} \Phi\colon C^{k,\alpha}(L,X) & \longrightarrow & C^{k-1,\alpha}(\Lambda^n(T^*L)) \\ f & \longmapsto & (f^*\Omega^c) \end{array}$$

is smooth. Let $f \in \mathcal{M}^{k,\alpha}$. We claim that $\theta \in C^{k-1,\alpha}(\Lambda^n(T^*L))$ is in the image of $D\Phi_f$ if and only if θ is exact. For one direction, $D\Phi_f(\xi)$ is exact and of class $C^{k-1,\alpha}$ for all $\xi \in T_f(C^{k,\alpha}(L,X))$ because again the cohomology class $[f^*\Omega^c] \in H^n(L,\mathbb{C})$ is locally constant, so the infinitesimal change in $f^*\Omega^c$ as f varies is exact. Conversely, let an exact $\theta \in C^{k-1,\alpha}(\Lambda^n(T^*L))$ be given. Take any metric on L and let $P = d + \delta$. A priori, θ is the exterior derivative of a *differentiable* form. However, an exact form θ is L^2-orthogonal inside $C^{k-1,\alpha}(\Lambda^n(T^*L))$ to the space $\ker(P) = \ker(P^*)$ of harmonic forms and so from Corollary 2.2.14 we know that $\theta \in \operatorname{im}(P)$ and that there is a $\Theta \in C^{k,\alpha}(\Lambda^{n-1}(T^*L))$ with $d\Theta = P\Theta = \theta$.

We claim that there is a $V\colon L \to TX$ of class $C^{k,\alpha}$ with $\pi \circ V = f$ and $d(f^*(\iota_V\Omega^c)) = \theta$ (see Remark 2.2.5 for our notation). There is a slight subtlety about this: Since f is itself only of class $C^{k,\alpha}$ any differential form on L of the form $d(f^*\tilde{\Theta})$ is, a priori, only of class $C^{k-2,\alpha}$, and so it is not obvious that we can do that, given that $\theta \in C^{k-1,\alpha}$. We can remedy this in the following way: The image $f(L) = L' \subset X$ of an $f \in \mathcal{M}^{k,\alpha}$ is a calibrated, hence minimal submanifold, and by Theorem 2.2.17 such submanifolds are smooth. So there is a reparametrisation $\tilde{f}\colon L \to X$ which is smooth and such that $\tilde{f}(L) = f(L) = L'$. Since Ω^c is non-degenerate it is then obvious that we can find $\tilde{v} \in C^{k,\alpha}(TL')$ such that $d(\tilde{f}^*\iota_{\tilde{v}}\Omega^c) = \theta$. On the other side, since $f \in C^{k,\alpha}(L,X)$ and $\tilde{f} \in C^\infty(L,X)$ we have $\phi = f \circ \tilde{f}^{-1} \in \mathcal{D}^{k,\alpha}(L')$ and $f = \phi \circ \tilde{f}$. Let $v = \phi_*\tilde{v} \in C^{k-1,\alpha}(TL')$. Since ϕ is volume-preserving,

$$\begin{aligned} \theta &= \tilde{f}^*\mathrm{d}(\iota_{\tilde{v}}\Omega^c) \\ &= f^*(\phi^{-1})^*\mathrm{d}(\iota_{\tilde{v}}\,\mathrm{vol}_{L'}) \\ &= f^*\mathrm{d}(\iota_{\phi_*\tilde{v}}(\phi^{-1})^*\,\mathrm{vol}_{L'}) \\ &= f^*\mathrm{d}(\iota_v\,\mathrm{vol}_{L'}) \\ &= \mathrm{d}f^*(\iota_v\Omega^c). \end{aligned}$$

Since $d\iota_{\tilde{v}}\Omega^c$ is an exact form of class $C^{k-1,\alpha}$ on the smooth manifold L' and $\phi^{-1} \in \mathcal{D}^{k,\alpha}(L')$ by Proposition 2.2.16 we know that

$$\mathrm{d}(\iota_v\Omega^c)\big|_{L'} = (\phi^{-1})^*\mathrm{d}(\iota_{\tilde{v}}\Omega^c)\big|_{L'} \in C^{k-1,\alpha}(\Lambda^n T^*L')$$

and by elliptic regularity in Theorem 2.2.13 $\iota_v\Omega^c \in C^{k,\alpha}(\Lambda^n T^*L')$ and hence $v \in C^{k,\alpha}(TL')$. Thus $V = v\circ f \in T_f\mathcal{M}^{k,\alpha} \subset C^{k,\alpha}(L, TX)$ and $D\Phi_f(V) = d(f^*\iota_V\Omega) = \theta$ as desired.

By Corollary 2.2.14 the space $C^{k-1,\alpha}_{\text{exact}}(\Lambda^n(T^*L))$ of exact forms of class $C^{k-1,\alpha}$ is a Banach space. Also $T_f(C^{k,\alpha}(L,X)) = \ker(d) \oplus \operatorname{im}(\delta)$ splits into closed subspaces, and $D\Phi_f|\operatorname{im}(\delta) \to C^{k-1,\alpha}_{\text{exact}}(\Lambda^n(T^*L))$ is a continuous bijective linear map which has a continuous inverse (Green's operator Q); continuity of the inverse is equivalent to the validity of the elliptic estimates in Corollary 2.2.14. Hence by Theorem 2.2.9 $\mathcal{M}^{k,\alpha}$ is a smooth Banach manifold. The statement for $\mathcal{M}^{k,\alpha}_0$ follows by the same argument applied to ω. (The Ricci-flat Kähler form is indeed smooth.) □

From Corollary 2.2.11 and Proposition 2.2.16, if L is closed then $\mathcal{D}^{k,\alpha}(L)$, the volume preserving diffeomorphisms of L which are of class $C^{k,\alpha}$, form a topological group whose underlying manifold is a smooth Banach manifold (but not a Banach Lie-group since the group operations of multiplication and inversion are only continuous but not differentiable.) We need a Riemannian metric on L along the way to define $\mathcal{D}^{k,\alpha}(L)$, but the particular choice of metric is not relevant for what follows.

Proposition 2.2.19 *There is a short exact sequence*

$$\{\mathrm{id}\} \longrightarrow \mathcal{D}^{k,\alpha}_0 \longrightarrow \mathcal{D}^{k,\alpha} \xrightarrow{\beta} \mathrm{H}^{n-1}(L, \mathbb{R}/\mathbb{Z}) \longrightarrow \{0\} \tag{2.52}$$

analogous to (2.41) for which β is a smooth map of Banach manifolds.

Proof. This is very similar to the previous theorems; at the level of Lie-algebras we get an exact sequence because the deRham cohomology groups computed in $C^{k,\alpha}$ are the same as those computed in C^∞. There is a smooth homomorphism of Lie-groups corresponding to it, and the implicit function theorem implies that $\mathcal{D}^{k,\alpha}_0$ is a smooth Banach manifold. The crucial point is that β is defined by pulling back (locally defined) forms representing a cohomology class which is a smooth map by Theorem 2.2.15. □

Proposition 2.2.20 *The groups $\mathcal{D}^{k,\alpha}$ and $\mathcal{D}^{k,\alpha}_0$ operate continuously on the Banach manifolds $\mathcal{M}^{k,\alpha}$ and $\mathcal{M}^{k,\alpha}_0$.*

Proof. For $f \in \mathcal{M}^{k,\alpha}$ and $\phi \in \mathcal{D}^{k,\alpha}$ the composition $f\circ\phi^{-1}$ is again a map of class $C^{k,\alpha}$ by Proposition 2.2.16, and since ϕ is volume preserving $f\circ\phi^{-1} \in \mathcal{M}^{k,\alpha}$. Also $\mathcal{M}^{k,\alpha}_0$ is obviously preserved under the action of $\mathcal{D}^{k,\alpha}$. The actions of $\mathcal{D}^{k,\alpha}_0$ on $\mathcal{M}^{k,\alpha}$ and $\mathcal{M}^{k,\alpha}_0$ are just the induced ones. The operation is obviously continuous (but not of class C^1 since inversion is not of class C^1). □

Proposition 2.2.21 *Every $\mathcal{D}_0^{k,\alpha}$-orbit in $\mathcal{M}_0^{k,\alpha}$ contains a smooth map $f\colon L \to X$.*

Proof. We look at the orbit of a given $f_0 \in \mathcal{M}_0^{k,\alpha}$. Let us for simplicity assume that f_0 is an embedding, but the result remains true for immersions. Then $f_0(L)$ is a minimal submanifold, hence by Theorem 2.2.17 there is a smooth (in fact, analytic) map $f_1\colon L \to X$ with $f_0(L) = f_1(L)$. So f_0 and f_1 are diffeomorphisms onto their images, and $\phi = f_0^{-1} \circ f_1$ is a $C^{k,\alpha}$-diffeomorphism by Proposition 2.2.16. Hence $f_1 = f_0 \circ \phi = \phi^{-1} \cdot f_0$ is a smooth element in the $\mathcal{D}^{k,\alpha}$-orbit of f_0. Next we claim that there is a smooth $\psi \in \mathcal{D}^{k,\alpha}$ so that $f = f_1 \circ \psi$ is in the $\mathcal{D}_0^{k,\alpha}$-orbit of f_0. For that it is enough to show that in the short exact sequence (2.52) for every $t \in H^{n-1}(L, \mathbb{R}/\mathbb{Z})$ we have $\beta^{-1}(t) \cap C^\infty(L, L) \neq \emptyset$, because in this case we can take a smooth diffeomorphism ψ with $\beta(\psi) = \beta(\phi^{-1})$ and then

$$f = f_1 \circ \psi = f_0 \circ (\phi \circ \psi) = (\psi^{-1} \circ \phi^{-1}) \cdot f_0$$

is the desired smooth map. f is in the $\mathcal{D}_0^{k,\alpha}$-orbit of f_0 because $\beta(\phi \circ \psi) = 0$ which means $\phi \circ \psi \in \mathcal{D}_0^{k,\alpha}$. So choose a Riemannian metric on L. We know that

$$T_e\mathcal{D}^{k,\alpha} = \{\theta \in C^{k,\alpha}(\Lambda^{n-1}T^*L) \mid \mathrm{d}\theta = 0\} = \mathrm{d}C^{k+1,\alpha}(\Lambda^{n-2}T^*L) \oplus \mathcal{H}^{n-1} \qquad (2.53)$$

where the last equation follows from Theorem 2.2.13, and the splitting is orthogonal in the L^2 inner product. The contraction map

$$\xi \in C^{k,\alpha}(TL) \longmapsto \iota_\xi \nu \in C^{k,\alpha}(\Lambda^{n-1}T^*L),$$

with the volume form ν is an isomorphism of Banach spaces and yields a corresponding splitting $\mathcal{W}^{k,\alpha} = \mathcal{W}_0^{k,\alpha} \oplus \tilde{\mathcal{H}}^{n-1}$ of the space of divergence free vector fields on L of class $C^{k,\alpha}$, with $\mathcal{W}_0^{k,\alpha} = \mathrm{Lie}(\mathcal{D}_0^{k,\alpha})$. By elliptic regularity (Theorem 2.2.12), every harmonic $(n-1)$-form is smooth; hence so are the divergence free vector fields in $\tilde{\mathcal{H}}^{n-1}$. Let $\mathcal{U} \subset \mathcal{W}^{k,\alpha}$ be a small open neighbourhood of the origin, then the Riemannian exponential map EXP gives us a local diffeomorphism of $\mathcal{U}$ onto a neighbourhood U of $e \in \mathcal{D}^{k,\alpha}$ which is fibred by $b^{n-1}(L)$-dimensional submanifolds. Since EXP is smooth and $\tilde{\mathcal{H}}^{n-1} \subset \mathrm{Vect}(L) = C^\infty(TL)$, we have

$$S = \mathrm{EXP}(\{0\} \oplus \tilde{\mathcal{H}}^{n-1}) \subset C^\infty(L, L) \cap \mathcal{D}^{k,\alpha}.$$

Hence S is a $b^{n-1}(L)$-dimensional submanifold of $\mathcal{D}^{k,\alpha}$ with $S \subset C^\infty(L, L)$. Since the splitting (2.53) is orthogonal and the L^2 inner product gives us a C^0 metric on $\mathcal{D}^{k,\alpha}$ (which however does not induce the topology of $\mathcal{D}^{k,\alpha}$), S will be transversal to the map β in a small neighbourhood of e. Thus there is an open $S' \subset S$ such that $\beta\colon S' \to H^{n-1}(L, \mathbb{R}/\mathbb{Z})$ is a diffeomorphism onto a neighbourhood S'' of $0 \in H^{n-1}(L, \mathbb{R}/\mathbb{Z})$. Any such S'' generates all of $H^{n-1}(L, \mathbb{R}/\mathbb{Z})$, that is every $t \in H^{n-1}(L, \mathbb{R}/\mathbb{Z})$ is a word of finite length in elements of S'', say $t = t_1 \cdots t_r$. Then $\tau = \beta^{-1}(t_1) \cdots \beta^{-1}(t_r)$ is a smooth diffeomorphism of L with $\beta(\tau) = t$. $\square$

Proposition 2.2.22 *Let L and X be as before and $f \in C^\infty(L, X)$. Then the map $E_f \colon \mathcal{D}^{k,\alpha} \to C^{k,\alpha}(L, X)$ defined by sending $\psi \mapsto f \circ \psi$ is smooth.*

Proof. A neighbourhood of $\psi \in \mathcal{D}^{k,\alpha}$ can be identified with a neighbourhood V of the zero section in $C^{k,\alpha}(\psi^* TL)$. E_f is obviously continuous. Because f is smooth, $D^k E_f$ exists for all k and is easily seen to be given by

$$(D^k E_f)_\phi(\eta_1 \dots, \eta_k) = \left[p \mapsto (D^k f)_{\phi(p)}(\eta_1(p), \dots, \eta_k(p))\right].$$

Because L is compact and f is smooth, this depends continuously on ϕ and E_f is of class C^∞. □

Proposition 2.2.23 *For all smooth embeddings $f \in \mathcal{M}^{k,\alpha}$, the map*

$$\begin{aligned} E_f \colon \mathcal{D}_0^{k,\alpha} &\longrightarrow \mathcal{M}^{k,\alpha} \\ \phi &\longmapsto f \circ \phi \end{aligned}$$

is a smooth injective immersion. $E_f(\mathcal{D}_0^{k,\alpha})$ is a locally closed subset of $\mathcal{M}^{k,\alpha}$.

Proof. The map is obviously injective because f is, and it is smooth by Proposition 2.2.22. Since Df has everywhere maximal rank we conclude that

$$0 = (DE_f)_\phi(\eta) = [p \mapsto Df_{\phi(p)}(\eta(p))]$$

implies $\eta = 0$ and hence E_f is an immersion. Since $f \circ \phi$ is a smooth embedding there is a decomposition $TX|_L = T \oplus N$ into the tangent and normal bundles of $(f \circ \phi)(L)$ and hence we get a decomposition

$$T_{f\circ\phi}\mathcal{M}^{k,\alpha} = C^{k,\alpha}((f \circ \phi)^* T) \oplus C^{k,\alpha}((f \circ \phi)^* N)$$

into closed subspaces, and clearly $\mathrm{Im}(DE_f) = C^{k,\alpha}(T)$. This implies that the image of E_f is locally closed. □

Proposition 2.2.24 *The $\mathcal{D}_0^{k,\alpha}$-orbits in $\mathcal{M}_0^{k,\alpha}$ are smooth submanifolds.*

Proof. By Proposition 2.2.21 every $\mathcal{D}_0^{k,\alpha}$-orbit in $\mathcal{M}_0^{k,\alpha}$ contains a smooth map $f \colon L \to X$. Note that while the action of $\mathcal{D}_0^{k,\alpha}$ is only continuous, the orbit $\mathcal{D}_0^{k,\alpha} \cdot f \subset \mathcal{M}_0^{k,\alpha}$ is also the image of the map $E_f \colon \mathcal{D}_0^{k,\alpha} \to \mathcal{M}_0^{k,\alpha}$ which was shown to be an embedding with locally closed image in Proposition 2.2.23, hence the orbits are submanifolds. □

Theorem 2.2.25 *There is a slice for the $\mathcal{D}_0^{k,\alpha}$-action on $\mathcal{M}_0^{k,\alpha}$, that is we can find an $f \in \mathcal{M}_0^{k,\alpha}$ and a smooth local submanifold $S \subset \mathcal{M}_0^{k,\alpha}$ with $f \in S$ so that for each $\tilde{f}$ near f the orbit $\mathcal{D}_0^{k,\alpha} \cdot \tilde{f}$ intersects S in exactly one point.*

Proof. The proof is a repetition of the argument given in Proposition 2.2.21. Using similar arguments as in [McL98], if $f \in \mathcal{M}_0^{k,\alpha} \cap C^\infty(L,X)$ then

$$T_f\mathcal{M}_0^{k,\alpha} = \mathcal{W}_0^{k,\alpha} \oplus \tilde{\mathcal{H}}^{n-1} \oplus J\tilde{\mathcal{H}}^{n-1}$$

is a decomposition into Banach spaces, and the splitting is L^2-orthogonal. Composing elements of $\tilde{\mathcal{H}}^{n-1} \oplus J\tilde{\mathcal{H}}^{n-1} \cong \tilde{\mathcal{H}}_{\mathbb{C}}^{n-1}$ with EXP yields the slice S. Since J is $\mathcal{D}_0^{k,\alpha}$-equivariant, this is a complex manifold of complex dimension $b^{n-1} = b^1$. □

Theorem 2.2.26 *The metric $\tilde{g}$ and symplectic form $\tilde{\omega}$ defined on $\mathcal{M}_0^{k,\alpha}$ by the formulae (2.37) - (2.38) are of class $C^{k-1,\alpha}$.*

Proof. Evaluation $ev\colon \mathcal{M}_0^{k,\alpha} \times L \to X$ is of class $C^{k,\alpha}$, hence $ev^*\omega$ is a two-form of class $C^{k-1,\alpha}$ on $\mathcal{M}_0^{k,\alpha} \times L$. Tensoring with ν and push-down $(\pi_1)_*$ are continuous linear maps, hence smooth. Thus $\tilde{\omega} = (\pi_1)_*(ev^*\omega \wedge \nu)$ is of class $C^{k-1,\alpha}$. The case of $\tilde{g}$ is analogous. □

Theorem 2.2.27 *The quotient*

$$\mathcal{S}_{\mathbb{C}} = \mathcal{M}_0^{k,\alpha}/\mathcal{D}_0^{k,\alpha} = \{\mathcal{M}^{k,\alpha}//\mathcal{D}_0^{k,\alpha}\}_{\mu=0}$$

is a finite-dimensional smooth Kähler manifold with $\dim_{\mathbb{C}}(\mathcal{S}_{\mathbb{C}}) = b^{n-1}(L) = b^1(L)$. Hence $\mathcal{S}_{\mathbb{C}}$ is a smooth symplectic (Kähler-) quotient.

Proof. Theorem 2.2.25 identifies this quotient with a smooth, finite-dimensional, slice $S \subset \mathcal{M}_0^{k,\alpha}$. Theorem 2.2.26 induces a metric and symplectic structure on S which, a priori, is of class $C^{k-1,\alpha}$. However, $S \subset \mathcal{M}_0^\infty = \cap_k \mathcal{M}_0^{k,\alpha}$ and so

$$\mathcal{M}_0^\infty \subset \cdots \subset \mathcal{M}_0^{k+1,\alpha} \subset \mathcal{M}_0^{k,\alpha} \subset \mathcal{M}_0^{k-1,\alpha} \subset \cdots \subset \mathcal{M}_0^{1,\alpha}.$$

Thus $S \subset \mathcal{M}_0^{k,\alpha}$ is independent of $k \in \mathbb{N}$ and $\tilde{\omega}|S$ is also independent of $k \in \mathbb{N}$, so by letting $k \to \infty$ we get the result. □

The torus $T = H^{n-1}(L, \mathbb{R}/\mathbb{Z})$ has a fixed point free action on $\mathcal{S}_{\mathbb{C}}$, and the quotient

$$\pi\colon \mathcal{S}_{\mathbb{C}} \longrightarrow \mathcal{S}_{\mathbb{R}} = \mathcal{S}/T$$

can be naturally identified with the moduli space of special Lagrangian submanifolds.

Theorem 2.2.28 *π is a Riemannian submersion.*

Proof. This follows easily by unwinding the definitions. □

2.3 The local geometry of the moduli space

We will now discuss McLean's metric on the moduli space of special Lagrangians, following [Hit98]. We show that McLean's metric has positive Ricci-curvature if its potential satisfies a certain real Monge-Ampère equation and collect some of its elementary properties (see also Paragraph 3.1.2).

2.3.1 Hitchin's approach to the moduli space

In this paragraph we answer some questions emerging from Hitchin's approach to the local geometry of the moduli space of special Lagrangian submanifolds. In [Hit98] Hitchin showed that McLean's metric h on the moduli space of special Lagrangians has a very special form called a metric of *Hessian type*. We define:

Definition 2.3.1 *Let (B, h) be a Riemannian m-manifold. The Riemannian metric h is called a metric of Hessian type if around each $b \in B$ there is a local chart $\mathbf{u}\colon U \subset B \to \mathbb{R}^m$ and a (strictly convex) function $\phi\colon U \to \mathbb{R}$ such that*

$$h = \sum_{i,j=1}^{m} \frac{\partial^2 \phi}{\partial u_i \partial u_j}\, \mathrm{d}u_i \otimes \mathrm{d}u_j = \mathrm{Hess}_{\mathbf{u}}(\phi).$$

We point out that the operator $\mathrm{Hess}_{\mathbf{u}}$ is different from the Riemannian Hessian operator which involves the metric. Here and in the rest of this thesis we write $\mathrm{Hess}_{\mathbf{u}}(\phi)$ (or just $\mathrm{Hess}(\phi)$ when the choice of chart is obvious) simply to denote the second derivative with respect to $\mathbf{u}$. This is a natural operator on an affine manifold, that is a manifold with a distinguished set of charts whose coordinate transitions are given by affine transformations of $\mathbb{R}^m$. Here of course both the base and the Liouville tori of an integrable system are important examples of affine manifolds. Secondly, if $b \in B$ is a critical point of ϕ such that $d\phi(b) = 0$ then $\mathrm{Hess}_{\mathbf{u}}(\phi) \in \mathrm{Sym}^2(T_b^* B)$ is the same for all charts $\mathbf{u}$. In general the failure to be invariant makes it difficult to determine whether a given Riemannian manifold (B, h) is of Hessian type. See [Dui99, Ruu96, SY97] for a study of these metrics.

So let again (X, Ω^c, ω) be a Calabi-Yau manifold and let $\iota\colon L \to X$ be a special Lagrangian submanifold calibrated by $\Omega = \mathrm{Re}(\Omega^c)$ with $m = b^1(L)$. This is a sketch of the argument in [Hit98] how to realise McLean's metric h as the Hessian of a function. Every special Lagrangian submanifold has a canonical orientation which is induced from Ω. Let $A_1, \ldots, A_m$ be an integral basis for $H_1(L, \mathbb{Z})/\,\mathrm{Tors}$ and choose $\alpha_1, \ldots, \alpha_m \in H^1(L)$ in deRham cohomology with $\alpha_i[A_j] = \delta_{ij}$. Poincaré-duality yields dual bases $B_1, \ldots, B_m$ for $H_{m-1}(L, \mathbb{Z})/\,\mathrm{Tors}$ and $\beta_1, \ldots, \beta_m$ for $H^{m-1}(L)$.

Let B be a small open ball inside the local moduli space of special Lagrangian deformations of L and $\Phi\colon L \times B \to X$ the full family of deformations. We have

$$\Lambda^k(T^*(L \times B)) \cong \bigoplus_{p=0}^{k} \Lambda^p T^* L \otimes \Lambda^{k-p} T^* B$$

and so every $A \in H_p(L, \mathbb{Z})$ defines a push-down map

$$\pi_A\colon \Lambda^k(T^*(L \times B)) \longrightarrow \Lambda^{k-p} T^* B$$

mapping closed forms to closed forms. Therefore $\pi_{A_i}\Phi^*\omega$ is a closed 1-form and hence of the form du_i for some $u_i\colon B \to \mathbb{R}$, and similarly $\pi_{B_i}\Phi^*\Omega = dv_i$ for $i = 1, \ldots, m$. Both $(u_i)_i$ and $(v_i)_i$ are coordinates on B and unravelling the definitions one finds the expression $h = \Sigma_i du_i \otimes dv_i$ for McLean's metric. Now the map

$$\Psi\colon b \in B \longmapsto (\Sigma_i u_i \alpha_i, \Sigma_i v_i \beta_i) \in \mathrm{H}^1(L, \mathbb{R}) \times \mathrm{H}^{n-1}(L, \mathbb{R}) \tag{2.54}$$

is a Lagrangian embedding of B for the canonical symplectic structure on

$$\mathrm{H}^1(L, \mathbb{R}) \times \mathrm{H}^{n-1}(L, \mathbb{R}) = \mathrm{H}^1(L, \mathbb{R}) \times \mathrm{H}^1(L, \mathbb{R})^* = T^*(\mathrm{H}^1(L, \mathbb{R}))$$

as a cotangent bundle. So there is a $\phi\colon u(B) \subset H^1(L, \mathbb{R}) \to \mathbb{R}$ with $\Psi(B) = \mathrm{graph}(d\phi)$, hence $v_i(b) = \partial\phi/\partial u_i(u(b))$ and the metric is of Hessian form

$$h = \sum_{i,j=1}^{m} \frac{\partial^2 \phi}{\partial u_i \partial u_j} \, \mathrm{d}u_i \otimes \mathrm{d}u_j. \tag{2.55}$$

Let $\check{\Omega}_1 \in \Lambda^m H^1(L, \mathbb{R})$ and $\check{\Omega}_2 \in \Lambda^m H^{n-1}(L, \mathbb{R})$ be the canonical generators, then Hitchin defines an m-dimensional submanifold $L \subset H^1(L, \mathbb{R}) \times H^{n-1}(L, \mathbb{R})$ to be *special* if some linear combination of those vanishes identically on L.

Proposition 2.3.2 (Hitchin [Hit98]) $\Psi(B)$ *is a special Lagrangian submanifold in the sense that there exists a phase* $\vartheta \in [0, 2\pi]$ *so that*

$$\Psi^*[\cos(\vartheta)\check{\Omega}_1 + \sin(\vartheta)\check{\Omega}_2] \equiv 0$$

if and only if ϕ *satisfies an inhomogeneous Monge-Ampère equation*

$$\det\left(\frac{\partial^2 \phi}{\partial u_i \partial u_j}\right) = \mathrm{const}\,. \tag{2.56}$$

Hitchin asks the question whether this is true for every special Lagrangian submanifold. D Matessi [Mat00] has constructed a counterexample showing that this is generally not the case, however, we observe

Theorem 2.3.3 *Suppose $\iota\colon T^n \hookrightarrow X$ is a special Lagrangian n-torus so that $L = \iota(T^n)$ and all small special Lagrangian deformations of L are flat. Then L has a tubular neighbourhood $\mathcal{N}(L)$ with a special Lagrangian torus fibration $f\colon \mathcal{N}(L) \to U \subset H^1(L,\mathbb{R})$. Let $u\colon U \to \mathbb{R}^n$ be one of the coordinate charts on U as defined above and let the Hessian metric on U be given by ϕ. Then ϕ satisfies the Monge-Ampère equation (2.56).*

Proof. If $\iota(T^n) = L \subset X$ is flat then the harmonic 1-forms of $\iota^* g$ are pointwise linearly independent, so L has a saturated neighbourhood $\mathcal{N}(L) \subset X$ which is smoothly fibred by special Lagrangian tori and there is a submersion $f\colon \mathcal{N}(L) \to U \subset H^1(L,\mathbb{R})$. Let $\xi_1,\dots,\xi_n \in \mathrm{Vect}(\mathcal{N}(L))$ such that $\iota_{\xi_i}\omega = f^* du_i$. Then by construction the ξ_i are periodic with period one and commute because $d\omega = 0$. Choose a Lagrangian section $\sigma\colon B \to \mathcal{N}(T^n)$ and define multi-valued coordinates $\mathbf{x}\colon \mathcal{N}(T^n) \to \mathbb{R}^n/\mathbb{Z}^n$ by integrating the flow of $(\xi_1,\dots,\xi_n)$ with the boundary condition $\mathbf{x}\circ\sigma = 0$. Let $y_i = u_i \circ f$. Then it is well-known that $\omega = \Sigma_i dx_i \wedge dy_i$. Now let η_i be a lift of $\partial/\partial u_i$ so that $\iota_{\eta_i}\omega = dx_i$ and by McLean's theorem

$$h_b\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right) = \int_{f^{-1}(b)} \mathrm{d}x_i \wedge *\mathrm{d}x_j = \int_{f^{-1}(b)} g^{ij}\,\mathrm{vol}$$

But since the fibres are flat and the dx_i restrict to harmonic 1-forms on the fibres, $\mathbf{x}$ is a flat coordinate system and so the coefficients g^{ij} are constant along the fibres. Hence we get $h_{ij}(b) = g^{ij}(b)\,\mathrm{Vol}(f^{-1}(b))$. But special Lagrangian submanifolds are calibrated, so $\mathrm{Vol}(f^{-1}(b)) = V$ is independent of b and we get

$$\begin{aligned}
\det(h_{ij}(b)) &= \det(g^{-1}(b))V^n \\
&= \det(g^{-1}(b))V^{n-2}\left[\int_{f^{-1}(b)} \sqrt{\det(g)}\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\right]^2 \\
&= V^{n-2}\det(g^{-1}(b))\left(\sqrt{\det(g)}\right)^2\left[\int_{f^{-1}(b)} \mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\right]^2 \\
&= V^{n-2} \\
&= \mathrm{const}\,.
\end{aligned}$$

□

Fibrations with flat fibres are called *semi-flat*. However, we do not expect non-trivial special Lagrangian fibrations to be semi-flat because we have the following theorem:

Theorem 2.3.4 *Let $f\colon X \to B$ be a non-degenerate elliptic $K3$-surface with a hyper-Kähler metric g. Then g is nowhere semi-flat.*

Proof. An elliptic surface is non-degenerate if it has 24 singular fibres of Kodaira type I_1. Let F_0 be one of those. Then $F_0 \cong S^2/\{p_1, p_2\}$ is homeomorphic to a 2-sphere with two points identified. F_0 contains one critical point m_0 and $F_0 \setminus \{m_0\} \cong \mathbb{C}^\times$ is smoothly embedded. In a neighbourhood U of m_0 we can find a holomorphic coordinate (with respect to a suitable complex structure) $z\colon U \to \mathbb{C}^2$ so that $f(z) = z_1 z_2$ (for a suitable chart of B near $f(m_0)$). So

$$F_0 \cap U = \{z_1 z_2 = 0\}.$$

This is the union of two smoothly embedded disks which intersect transversally at m_0. Therefore F_0 is actually a smooth immersion $\iota\colon S^2 \to X$ which fails to be injective precisely at p_1, p_2. But then $\iota^* g$ is a smooth metric on S^2 and hence cannot be flat. In fact, since the hyper-Kähler structure is analytic the fibration must be nowhere semi-flat. □

Remark 2.3.5 We will later (in Chapter 4) study T^n-fibrations (particularly for $n = 3$) whose singular fibres are homeomorphic to $T^{n-2} \times F_0$ where F_0 is a standard I_1-degeneration as above. It is obvious that the argument carries over to that case and shows that there cannot be a metric on the total space of such a fibration for which all fibres are flat, unless there are no singular fibres at all. The reason is again that $\tilde{F}_0 = T^{n-2} \times S^2$ does not carry a flat metric, because $\tilde{F}_0$ is compact, hence complete, and for every flat complete Riemannian n-manifold the universal cover is isometric to Euclidean $\mathbb{R}^n$. But obviously the universal cover $\mathbb{R}^{n-2} \times S^2$ of $\tilde{F}$ is not diffeomorphic to $\mathbb{R}^n$.

Proposition 2.3.6 *Let $\mathbf{u}, \mathbf{v}\colon U \subset B \to \mathbb{R}^m$ be two sets of linear coordinates as in (2.54), and let $\nabla^{\mathbf{u}}$ and $\nabla^{\mathbf{v}}$ be the respective flat affine connections. Then the Levi-Civita connection ∇ of McLean's metric h is given by*

$$\nabla = \frac{1}{2}\left(\nabla^{\mathbf{u}} + \nabla^{\mathbf{v}}\right).$$

Proof. Let ${\Gamma_{ij}}^k$ be the Christoffel symbols of $\nabla^{\mathbf{v}}$ with respect to the linear chart $\mathbf{u}$. Then from $dv_j = \Sigma_k \phi_{jk} du_k$ we can compute

$$0 \;=\; \nabla^{\mathbf{v}}_{\partial/\partial u_i} \mathrm{d}v_j \;=\; \sum_k \Big(\phi_{ijk} - \sum_r \phi_{jr} {\Gamma_{ik}}^r\Big) \mathrm{d}u_k.$$

Thus ${\Gamma_{ij}}^k = \Sigma_r \phi^{kr} \phi_{rij}$. Since $\mathbf{u}$ is a flat affine chart for the connection $\nabla^{\mathbf{u}}$, we have $\nabla^{\mathbf{u}} du_j = 0$ for all $j = 1, \ldots, m$. Hence

$$\frac{1}{2}\left(\nabla^{\mathbf{u}}_{\partial/\partial u_i} + \nabla^{\mathbf{v}}_{\partial/\partial u_i}\right)(\mathrm{d}u_j) = -\frac{1}{2}\sum_k {\Gamma_{ik}}^j \mathrm{d}u_k = -\frac{1}{2}\sum_r \phi^{jr}\phi_{rik}\mathrm{d}u_k = \nabla_{\partial/\partial u_i}\mathrm{d}u_j$$

which is the Levi-Civita connection ∇ of h (see (2.61)). □

For a study of linear structures related by a Legendre transformation see [SY97] and [Ama85].

2.3.2 McLean's metric (I)

We use the *Hessian structure* of McLean's metric h to obtain some curvature estimates. In Paragraph 3.1.2 we will return to the geometry of this metric when studying the *linear structure* on the base. We begin by giving an alternative characterisation of Hessian metrics. On a Riemannian manifold (B,h) with Levi-Civita connection ∇ we define for each $X \in \mathrm{Vect}(B)$ an endomorphism $G(X)$ by

$$G(X)\colon Y \longmapsto \nabla_Y X \tag{2.57}$$

Then the following proposition holds.

Proposition 2.3.7 *Let h be a Riemannian metric on $U \subset \mathbb{R}^m$ and let $\partial_1, \ldots, \partial_m$ be the standard basis of TU. Then h is of Hessian type with respect to the natural linear structure on U if and only if $G(\partial_i)$ is self-adjoint with respect to h for all $i = 1, \ldots, m$.*

Proof. Assume that $G(\partial_i)$ is self-adjoint with respect to h for all $i = 1, \ldots, m$. Then

$$\begin{aligned} 0 &= h(G(\partial_i)\partial_j, \partial_k) - h(\partial_j, G(\partial_i)\partial_k) \\ &= \Gamma_{jik} - \Gamma_{kij} \\ &= \frac{1}{2}(\partial_i h_{jk} + \partial_j h_{ik} - \partial_k h_{ji}) - \frac{1}{2}(\partial_i h_{kj} + \partial_k h_{ij} - \partial_j h_{ki}) \\ &= \partial_j h_{ik} - \partial_k h_{ij} \end{aligned}$$

Thus there is a function ϕ_i with $h_{ij} = \partial\phi_i/\partial u_j$. But then the symmetry of h entails $\partial_i\phi_j = \partial_j\phi_i$ and so we can find a $\phi \in C^\infty(U)$ so that $\phi_i = \partial\phi/\partial u_i$. Thus $h = \mathrm{Hess}_{\mathbf{u}}(\phi)$. Conversely suppose $h = \mathrm{Hess}_{\mathbf{u}}(\phi)$ for some convex function ϕ, then

$$\begin{aligned} & h(G(\partial_i)\partial_j, \partial_k) - h(\partial_j, G(\partial_i)\partial_k) \\ =\ & h(\nabla_{\partial_j}\partial_i, \partial_k) - h(\partial_j, \nabla_{\partial_k}\partial_i) \\ =\ & \partial_j(h(\partial_i, \partial_k)) - \partial_k(h(\partial_j, \partial_i)) - h(\partial_i, [\partial_j, \partial_k]) \\ =\ & \phi_{ikj} - \phi_{jik} \\ =\ & 0. \end{aligned}$$

This means that $G(\partial_i)$ is self-adjoint with respect to h. $\square$

Theorem 2.3.8 *Suppose h is a metric on an m-manifold B which is of Hessian type with respect to a chart $\mathbf{u}\colon U \subset B \to \mathbb{R}^m$ and assume at a point $b \in U$ we have*

$$\frac{\partial}{\partial u_k}\left[\det\left(\frac{\partial^2\phi}{\partial u_i \partial u_j}\right)\right](b) = 0, \quad k = 1, \ldots, m, \tag{2.58}$$

then at b the metric h is Ricci non-negative, $\mathrm{Ric}_h(b) \geq 0$.

Proof. The proof is a straightforward check in local coordinates. For this proof we adopt physicists' convention of indexing forms with upper indices. We write ∂_i for $\partial/\partial u_i$ and $\phi_{i_1\ldots i_r}$ for $\partial_{i_1}\cdots\partial_{i_r}\phi$, so that the metric coefficients, somewhat ambiguously, satisfy $h_{ij}=\phi_{ij}$ and $h=\Sigma_{ij}\phi_{ij}du^i\otimes du^j$. By abuse of notation we also write h for the matrix (h_{ij}), and ϕ^{ij} for $((\phi_{..})^{-1})_{ij}$. The fact that $\phi_{i_1\ldots i_r}$ is totally symmetric in its indices causes some cancellation in the local coefficients of the Levi-Civita connection and its curvature, in particular if

$$
\begin{aligned}
\nabla_{\partial_i}\partial_j &= \sum_k \Gamma_{ij}{}^k\partial_k & (2.59)\\
R &= \sum_{i,j,k,l} R_{ijk}{}^l(\mathrm{d}u^i\wedge\mathrm{d}u^j)\otimes(\mathrm{d}u^k\otimes\partial_l) & (2.60)
\end{aligned}
$$

the coefficients are given by

$$
\Gamma_{ij}{}^k=\frac{1}{2}\sum_r\phi^{kr}\phi_{rij},\quad R_{ijk}{}^l=\frac{1}{4}\sum_{q,r,s}\phi^{qr}\phi^{ls}(\phi_{rsi}\phi_{qjk}-\phi_{rsj}\phi_{qik}). \tag{2.61}
$$

It is worth noting that in the formula for the curvature the fourth derivatives of ϕ cancel out. Now for any function $A\colon B\to\mathrm{GL}(m,\mathbb{R})$ the formula

$$
\partial_k\log(\det(A))=\mathrm{Trace}(A^{-1}\partial_kA) \tag{2.62}
$$

holds. In local coordinates, the tensor field $G(\partial_i)$ in (2.57) is given by

$$
G(\partial_i)=\sum_{i,j,k}\Gamma_{ji}{}^k\mathrm{d}u^j\otimes\partial_k,
$$

and by (2.59), (2.60), (2.61) and (2.62)

$$
\begin{aligned}
(m-1)\ \mathrm{Ric}_h(\partial_i,\partial_k) &= \sum_j R_{ijk}{}^j\\
&= \frac{1}{4}\sum_{p,q,r,j}\phi^{rp}\phi^{jq}(\phi_{pqi}\phi_{rjk}-\phi_{pqj}\phi_{rik})\\
&= \sum_{r,q}\Gamma_{qi}{}^r\Gamma_{rk}{}^q-\sum_p\Gamma_{ik}{}^p\partial_p\log\sqrt{\det(h)}\\
&= \mathrm{Trace}(G(\partial_i)G(\partial_k))-(G(\partial_i)\partial_k)\log\sqrt{\det(h)}.
\end{aligned}
$$

Now let $X\in T_bB$ be any unit vector and choose the affine chart $\mathbf{u}$ around $b\in B$ so that $\partial_1(b),\ldots,\partial_m(b)\in T_bB$ are orthonormal and $\partial_1(b)=X$. If (2.58) holds we have

$$
(m-1)\,\mathrm{Ric}_h(X,X)=(m-1)\,\mathrm{Ric}_h(\partial_1,\partial_1)=\mathrm{Trace}(G(\partial_1)^2).
$$

$G(\partial_i)$ is self-adjoint with respect to h by Proposition 2.3.7. Hence $G(\partial_i)^2(b)$ is a positive semi-definite symmetric endomorphism and therefore $\mathrm{Trace}(G(\partial_i)^2(b)) \geq 0$ and $\mathrm{Trace}(G(\partial_i)^2(b)) = 0$ if and only if $G(\partial_i(b)) \equiv 0$. □

This has been observed independently by Nannicini [Nan98].

Proposition 2.3.9 *Suppose a Hessian metric h on an affine m-manifold B satisfies (2.56) and is scalar-flat. Then h is flat and the Levi-Civita connection of h coincides with the natural flat affine connection on B.*

Proof. Assume that $\mathrm{Scal}(h) \equiv 0$. Since $\mathrm{Ric}_h \geq 0$ by Theorem 2.3.8 we conclude $\mathrm{Ric}_h \equiv 0$. Now let $\mathbf{u}\colon U \subset B \to \mathbb{R}^m$ be a local affine chart. Given $b \in U$ choose $A(b) \in \mathrm{GL}(m, \mathbb{R})$ so that the affine chart $\tilde{\mathbf{u}} = A(b)\mathbf{u}$ is orthonormal at b. In Theorem 2.3.8 we have shown that for such a chart $\tilde{\mathbf{u}}$ the associated basis $\tilde{\partial}_1, \ldots, \tilde{\partial}_m$ satisfies $G(\tilde{\partial}_i)_b = 0$ for all $i = 1, \ldots, m$. But this means that $\nabla_{\tilde{\partial}_i}\tilde{\partial}_j(b) = 0$ for all i, j. However, since $A(b)$ was a constant linear change of coordinates the original chart $\mathbf{u}$ satisfies $\nabla_{\partial_i}\partial_j(b) = 0$ as well and thus since $b \in U$ was arbitrary $\mathbf{u}$ is a flat coordinate system. So $K_h \equiv 0$ and the Levi-Civita connection reduces to the flat affine connection. □

An anti-holomorphic involution of a complex n-fold (X, J) is called *maximally real* if its fixed-point set $L \hookrightarrow X$ is a smooth real submanifold of maximal possible dimension $\dim_{\mathbb{R}}(L) = n$. If X is a Calabi-Yau manifold, then such an L is special Lagrangian [Bry85].

Proposition 2.3.10 *Let X be a compact Calabi-Yau n-fold and $L \hookrightarrow X$ the fixed point set of a maximally real anti-holomorphic involution τ with $\tau^*[\omega] = [-\omega]$. Let $U \subset H^1(L, \mathbb{R})$ be open in the local moduli space of special Lagrangian deformations of L and $\phi\colon U \to \mathbb{R}$ a generating function for the Hessian metric $h = \mathrm{Hess}(\phi)$ as in (2.55). Then*

$$\phi_{ijk} = 0, \qquad i, j, k = 1, \ldots, b^1(L).$$

In particular, the curvature tensor K_h of h vanishes at $0 \in U$ and (2.58) holds.

Proof. An anti-holomorphic involution τ on a compact Calabi-Yau manifold X with $\tau^*[\omega] = [-\omega] \in H^{1,1}_{\mathbb{R}}(X)$ is necessarily an isometry because for a fixed complex structure, say J, the Ricci-flat metric g and its pull-back τ^*g lie in the same Kähler-class. But for a compact Calabi-Yau manifold there is a unique such metric in each Kähler class, so $g = \tau^*g$. Therefore τ maps special Lagrangian submanifolds to special Lagrangian submanifolds and hence induces an isometry $\bar{\tau}$ on the (local) moduli space U which has a fixed point $\bar{\tau}(0) = 0$. Thus $h_{ij} = h_{ij} \circ \bar{\tau}$. Because $\tau|_L = \mathrm{id}_L$ and $TL^\perp \cong J(TL)$ the map τ acts as (-1) on the normal bundle of the fixed submanifold $L \subset X$. Therefore we have $\bar{\tau}_{*0} = -\,\mathrm{id}_{T_0U}$. But then

$$\mathrm{d}h_{ij}(0) = \mathrm{d}(h_{ij} \circ \bar{\tau})(0) = \bar{\tau}^*\mathrm{d}h_{ij}(0) = -\mathrm{d}h_{ij}(0)$$

and consequently $\Sigma_k \phi_{ijk} du_k(0) = dh_{ij}(0) = 0$. It then follows immediately from equation (2.61) for the coefficients of the curvature tensor that $K_h(0) = 0$. □

Given an arbitrary antiholomorphic involution τ on a Calabi-Yau n-fold X we can replace the Kähler form ω by $\frac{1}{2}(\tau^*\omega - \omega)$. Then τ is an isometry for this averaged Calabi-Yau metric. The most important situation to have in mind is that of a complete intersection $X \subset \mathbb{C}P^n$ with τ being complex conjugation.

Let $L \to X$ be an n-dimensional special Lagrangian submanifold of a Calabi-Yau n-fold, suppose $b^1(L) = m$ and identify the local moduli space of L with an open ball $U \subset H^1(L, \mathbb{R})$. In [McL98], McLean defines an n-form $\Theta \in \mathcal{A}^n(U)$ using the identification $T_pU \cong H^1(L, \mathbb{R})$ for all $p \in U$ by setting

$$\Theta(\xi_1, \ldots, \xi_n) = \int_L \xi_1 \wedge \cdots \wedge \xi_n. \tag{2.63}$$

McLean asks the question whether this form is closed or co-closed. We can give the following answer.

Proposition 2.3.11 *Θ is closed. If $m = n$ then Θ is co-closed with respect to h if and only if the Hessian metric $h = \mathrm{Hess}(\phi)$ satisfies the Monge-Ampère equation (2.56).*

Proof. Let $m \geq n$ since otherwise $\Theta = 0$ and there is nothing to prove. Let $u_1, \ldots, u_m \colon U \to \mathbb{R}$ be affine coordinates on U. Let $r_0, \ldots, r_n \in \{1, \ldots, m\}$ and $\xi_i = \partial/\partial u_{r_i}$. We simply use the formula

$$\mathrm{d}\Theta(\xi_0, \ldots, \xi_n) = \sum_{i=0}^{n} (-1)^i \xi_i \Theta(\xi_0, \overset{i}{\ldots\wedge\ldots}, \xi_n) + \sum_{0 \leq i < j \leq n} (-1)^{i+j} \Theta([\xi_i, \xi_j], \xi_0, \overset{i,j}{\ldots\wedge\ldots}, \xi_n)$$

and observe that, as a cohomology class, $\xi_i(u) \in T_uU \cong H^1(L, \mathbb{R})$ is independent of u, hence (2.63) shows that $\Theta(\xi_1(u), \ldots, \xi_n(u))$ is constant in u which annihilates the first sum. Furthermore $[\xi_i, \xi_j] = 0$ due to integrability and thus $d\Theta = 0$. For the second part note that, as we just indicated, Θ is not only closed but actually constant with respect to affine coordinates on U, that is the coefficients of Θ with respect to the canonical basis associated to an affine chart are constant. But then

$$*\Theta = *(c\, \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n) = \frac{c}{\det(h_{ij})} = \mathrm{const}\,.$$

for some $c \in \mathbb{R}$ and hence $d * \Theta = 0$. □

Chapter 3

Singularities of special Lagrangian fibrations

For topological reasons, every non-trivial special Lagrangian fibration $f\colon X \to B$ must have singular fibres, and this chapter is concerned with the study of these singularities. There are to date very few general results about the properties of the map f, and the lack of "generic" examples makes it difficult to make conjectures about the nature of f. Consequently even the most basic questions still remain unanswered, such as: Is the fibration $f\colon X \to B$ analytic, or at least smooth? What are its singular fibres?

Facing such difficulties we rather start from the other end by making a number of assumptions, and we ask what the implications are on a special Lagrangian fibration if these assumptions are satisfied. To justify this speculative approach, we have in mind that (assuming enough differentiability for special Lagrangian fibrations)

$$\left\{ \begin{array}{c} \text{elliptic } K3 \\ \text{surfaces} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{special} \\ \text{Lagrangian} \\ \text{fibrations} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{integrable} \\ \text{systems} \end{array} \right\}$$

(with apologies to set theorists). The singularities of the classes of fibrations on the left and right are well understood. It is then reasonable to conjecture that those properties which are enjoyed by every elliptic fibration and every generic integrable system should generally also be true for smooth special Lagrangian fibrations.

3.1 Singular points - a local normal form

We begin by giving a brief summary of a classification of non-degenerate singularities in integrable systems which was achieved by Eliasson [Eli90] and Tien Zung [TZ96, TZ97] and apply it to special Lagrangian fibrations. We show that within this classification there is just one type of singularity, called *focus-focus singularity*, which is compatible with the *special* condition. This implies that singularities arise in codimension two, and we can also derive some estimates of McLean's metric near points in the discriminant locus corresponding to focus-focus singular fibres.

The following notation will be used throughout Section 3.1. It agrees with the notation used in [Gro98, Gro99a, Gro99b]. Let $f\colon X \to B$ be a smooth (special Lagrangian) fibration on a Calabi-Yau n-fold X with Kähler form ω and holomorphic n-form $\Omega^c = \Omega_1 + i\Omega_2 \in \mathcal{A}^{n,0}(X)$. If $b \in B$ is a regular value then $F_b = f^{-1}(b)$ is a smooth special Lagrangian torus. Let $\mathrm{Crit}(f) = \{m \in X \mid \mathrm{rk}(f_{*m}) < n\}$, let $f(\mathrm{Crit}(f)) = \Delta \subset B$ be the discriminant locus of f, that is the set of singular values of f, and let $B_0 = B \setminus \Delta$ be the set of regular values. Finally set $X_0 = f^{-1}(B_0)$ and $X^{\#} = X \setminus \mathrm{Crit}(f)$, hence $X_0 \subset X^{\#} \subset X$.

3.1.1 Classification of singularities and special Lagrangians

Let (V, ω) be a real symplectic vector space with $\dim_{\mathbb{R}}(V) = 2n$. Then $\mathrm{Sym}^2(V^*, \omega)$ is a Lie-algebra which is isomorphic to $\mathfrak{sp}(n, \mathbb{R})$ in the following way: Let $\alpha, \beta \in \mathrm{Sym}^2(V^*, \omega)$ and let $x, y \in V$. Define $\alpha_x \in V^*$ and $x_\alpha \in V$ by

$$\alpha_x(y) = \alpha(x, y) = \omega(x_\alpha, y).$$

Then the Poisson-bracket $[\alpha, \beta] \in \mathrm{Sym}^2(V^*, \omega)$ is the quadratic form given by

$$[\alpha, \beta](x, y) = \alpha(x, y_\beta) - \beta(x, y_\alpha) \qquad \forall x, y \in V.$$

It is not hard to check that this bracket $[\cdot, \cdot]$ is skew and satisfies the Jacobi-identity. The Cartan subalgebras of $\mathrm{Sym}^2(V^*, \omega)$ are n-dimensional. Cartan subalgebras of $\mathfrak{sp}(n, \mathbb{R})$ enter into our problem in the following way: Suppose (X, ω) is a symplectic manifold and $f_1, f_2 \in C^\infty(X)$ are Poisson-commuting functions which are both critical at a point $m \in X$, that is $\{f_1, f_2\} \equiv 0$ and $df_1(m) = df_2(m) = 0$. Then the Hessians $\mathrm{Hess}(f_i)_m \in \mathrm{Sym}^2(T^*_m X, \omega_m)$ at $m \in X$ are well-defined symmetric bilinear forms on $T_m X$, and $\{f_1, f_2\} \equiv 0$ implies $[\mathrm{Hess}(f_1)_m, \mathrm{Hess}(f_2)_m] = 0$. In other words, for every family $f_1, \ldots, f_k$ of pairwise Poisson-commuting functions which are all critical at $m \in X$,

$$C = \mathrm{span}\{\mathrm{Hess}(f_1)_m, \ldots, \mathrm{Hess}(f_k)_m\} \subset \mathrm{Sym}^2(T^*_m X, \omega_m)$$

is an Abelian subalgebra of $\mathrm{Sym}^2(T^*_m X, \omega_m)$. If $\dim(C) = n$ then C is a Cartan subalgebra. These have a simple classification given in a classical theorem by Williamson:

Theorem 3.1.1 (Williamson [Wil36]) *For any Cartan subalgebra $C \subset \mathrm{Sym}^2(V^*, \omega)$ there is a symplectic system of coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on V such that C has a basis $q_1, \ldots, q_n$, given as $q_i = \mathrm{Hess}(v_i)$ (the Hessian of a function), and each v_i is one of the following:*

$$\begin{array}{lrcl} \textit{elliptic case:} & v_i & = & \frac{1}{2}(x_i^2 + y_i^2) \\ \textit{hyperbolic case:} & v_i & = & x_i y_i \\ \textit{focus-focus case:} & \left\{ \begin{array}{r} v_i \\ v_{i+1} \end{array} \right. & \begin{array}{c} = \\ = \end{array} & \begin{array}{l} x_i y_{i+1} - x_{i+1} y_i \\ x_i y_i + x_{i+1} y_{i+1} \end{array} \end{array}$$

Note that (x, y) are just linear coordinates on a vector space so that the Hessian of a function is well-defined. We can always label the coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ of V so that v_i is of elliptic type for $i = 1, \dots, k_e$, of hyperbolic type for $i = k_e + 1, \dots, k_e + k_h$ and of focus-focus type for $i = k_e + k_h + 1, \dots, k_e + k_h + 2k_f = n$. The numbers (k_e, k_h, k_f) are independent of any choices we made and classify the conjugacy classes of Cartan subalgebras of $\mathrm{Sym}^2(V^*, \omega)$. This is in distinction from the complex case where all Cartan subalgebras of $\mathfrak{sp}(n, \mathbb{C})$ are conjugate.

We now define two notions of non-degeneracy for critical points in integrable systems. The first is rather simple and weaker than the second version which is taken from Eliasson. Let $V \cong \mathbb{R}^n$ and $\alpha_1, \dots, \alpha_k \in \mathrm{Sym}^2(V^*)$. Let

$$S(\alpha_1, \dots, \alpha_k) = \{X \in V \mid \mathrm{rk}(\iota_X \alpha_1, \dots, \iota_X \alpha_k) < k\}$$

be the *singular set* of $(\alpha_i)_i$ which only depends on the subspace spanned by $(\alpha_i)_i$.

Definition 3.1.2 *Suppose $f_1, \dots, f_n$ are Poisson-commuting functions on a $2n$-dimensional symplectic manifold (X, ω) and $m \in X$ is a critical point with $\mathrm{rk}(f_{*m}) = k < n$, without loss of generality*

$$\mathrm{d}f_1(m) = \mathrm{d}f_2(m) = \cdots = \mathrm{d}f_{n-k}(m) = 0, \quad \mathrm{d}f_{n-k+1} \wedge \cdots \wedge \mathrm{d}f_n(m) \neq 0.$$

We then define

1. *m is weakly non-degenerate if there is a symplectic subspace $V \subset T_m X$ with $\dim(V) = 2n - 2k$ such that*

$$C = \mathrm{span}\{\mathrm{Hess}(f_1), \dots, \mathrm{Hess}(f_{n-k})\}$$

is a Cartan subalgebra of $\mathrm{Sym}^2(V, \omega_m|_V)$.

2. *m is strongly non-degenerate if*

$$C = \mathrm{Centr}(\mathrm{Hess}(f_1), \dots, \mathrm{Hess}(f_{n-k}))$$

is a Cartan subalgebra of $\mathrm{Sym}^2(T_m^* X, \omega_m)$ and

$$S(\mathrm{Hess}(f_1), \dots, \mathrm{Hess}(f_{n-k})) = S(C).$$

What we call strongly non-degenerate is what Eliasson simply calls non-degenerate. In case $k = 0$ the two definitions agree. Non-degeneracy is a generic condition for singularities in integrable systems [TZ96], however, we cannot prove whether it is also generic for singularities in *special* Lagrangian fibrations (see also Remark 3.1.8).

Theorem 3.1.3 *Let (X, Ω^c, ω) be a Calabi-Yau n-fold as before with a special Lagrangian fibration $f \colon X \to B$ of class C^3. Let $m \in X$ be a weakly non-degenerate critical point of f. Then m is of focus-focus type, $k_e(m) = k_h(m) = 0$.*

Proof. There are two steps in the argument: First we show that it is sufficient to prove the claim for the linearised fibration, in the second step we use the explicit form of the linearised fibration to show that if $k_e(m) + k_h(m) > 0$ then there is a *Lagrangian* plane $L_0 \subset T_mX$ with $\Omega^c|L_0 = 0$. If a complex form Ω^c of type $(n, 0)$ vanishes on a Lagrangian subspace of T_mX, then $\Omega^c(m) = 0$ - a contradiction for the holomorphic n-form in a Calabi-Yau manifold.

If m is a critical point of f, then $k = \mathrm{rk}(f_{*m}) < n$. We can choose a suitable local chart for B near $f(m)$ which makes f into an n-tuple of smooth Poisson commuting functions. We arrange our choices so that

$$f(m) = 0, \qquad \mathrm{d}f_1(m) = \cdots = \mathrm{d}f_{n-k}(m) = 0, \qquad \mathrm{d}f_{n-k+1} \wedge \cdots \wedge \mathrm{d}f_n \neq 0.$$

By weak non-degeneracy the Hessians of the critical Hamiltonians form a Cartan subalgebra of $\mathrm{Sym}^2(V, \omega|_V)$ for some symplectic subspace $V \subset T_mX$, say of type (k_e, k_h, k_f) with $k_e + k_h + 2k_f + k = n$. By Theorem 3.1.1 there is a set of symplectic coordinates $(x_1, \ldots, x_{n-k}, y_1, \ldots, y_{n-k})$ on $(V, \omega|_V)$ so that

$$\mathrm{span}\{\mathrm{Hess}(f_i)_m \mid i = 1, \ldots, n-k\} = \mathrm{span}\{\mathrm{Hess}(v_i)_0 \mid i = 1, \ldots, n-k\} \tag{3.1}$$

where the functions $v_i(x, y)$ are given by

$$v_i(x, y) = \begin{cases} \frac{1}{2}(x_i^2 + y_i^2) & \text{for} \quad i = 1, \ldots, k_e \\ x_i y_i & \text{for} \quad i = k_e + 1, \ldots, k_e + k_h \end{cases}, \tag{3.2}$$

and for $i = k_e + k_h + 2j - 1$ with $j = 1, \ldots, k_f$ they come in pairs of the form

$$v_i(x, y) = x_i y_{i+1} - x_{i+1} y_i \tag{3.3}$$

$$v_{i+1}(x, y) = x_i y_i + x_{i+1} y_{i+1}. \tag{3.4}$$

Without loss of generality, we may assume that $\mathrm{Hess}(f_i) = \mathrm{Hess}(v_i)$ (after replacing $f_V = (f_1, \ldots, f_{n-k})$ by $A \circ f_V$ for a suitable $A \in \mathrm{GL}(n-k, \mathbb{R})$, if necessary). In (3.1) as in all that follows we tacitly identify the cotangent space $T_0^*(T_mX)$ at the origin $0 \in T_mX$ with T_m^*X itself which should not lead to any confusion.

The symplectic structure gives us a splitting $T_mX = V \oplus V^{\perp_\omega}$ so that we can lift $(x_i, y_i)_{i=1,\dots,n-k}$ from the subspace V to T_mX, and we can complete them to (linear) symplectic coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on all of T_mX so that $dy_i = df_i(m)$ for $i > n-k$, because the f_i are Poisson commuting and thus $df_i|V = 0$ for $i > n-k$. Set $v_i = y_i$ for $i > n-k$. Choose a diffeomorphism $\Phi \colon U \subset T_mX \to X$ with $\Phi_{*0} = \mathrm{Id}$ to identify a neighbourhood of m in X with some small open neighbourhood U of $0 \in T_mX$. Thus we think of $f \colon U \to \mathbb{R}^n$ as a function on U and $\Omega^c \in \mathcal{A}^n_{\mathbb{C}}(U)$ as a (non-constant) complex n-form on U. We now have two fibrations, given by f and v respectively, on a small open neighbourhood of the origin in a symplectic vector space. We know that Ω_1 restricts to a non-vanishing form on the smooth part of the fibres of f, and we want to show that the same is true for v near the origin.

Since f was assumed to be in $C^3(X, B)$ we get $f_i(x,y) = v_i(x,y) + O(3)$ and

$$\mathrm{d}f_i(x,y) = \mathrm{d}v_i(x,y) + \rho_i(x,y) \tag{3.5}$$

where the error term $\rho_i \in \mathcal{A}^1(U)$ satisfies $\rho_i = O(2)$ for $i = 1, \dots, n-k$. Here by $\rho = O(k)$ we mean that for all $(x,y) \in U$ the expression $t^{-k}\rho(tx, ty)$ is bounded uniformly in (x,y) as $t \to 0$. We also need to define the following 1-forms on T_mX.

$$\theta_i(x,y) = \begin{cases} y_i \mathrm{d}x_i - x_i \mathrm{d}y_i & \text{for} \quad i = 1, \dots, k_e \\ x_i \mathrm{d}x_i - y_i \mathrm{d}y_i & \text{for} \quad i = k_e + 1, \dots, k_e + k_h \end{cases},$$

and for $i = k_e + k_h + 2j - 1$ where j runs through $1, \dots, k_f$ we set

$$\begin{aligned} \theta_i(x,y) &= x_i \mathrm{d}x_i + x_{i+1}\mathrm{d}x_{i+1} - y_i \mathrm{d}y_i - y_{i+1}\mathrm{d}y_{i+1} \\ \theta_{i+1}(x,y) &= x_i \mathrm{d}x_{i+1} - x_{i+1}\mathrm{d}x_i + y_i \mathrm{d}y_{i+1} - y_{i+1}\mathrm{d}y_i, \end{aligned}$$

finally we set $\theta_i = dx_i$ for $i = n-k+1, \dots, n$. Let $\tilde{U} = T_mX \setminus (\mathrm{Crit}(f) \cup \mathrm{Crit}(v))$. Since f is of class C^3, $\tilde{U}$ is dense in U (assuming U is small). Let $\mathcal{V}_0^* \subset T^*\tilde{U}$ be the subbundle generated by $\{df_1, \dots, df_n\}$ and $\mathcal{V}_1^* \subset T^*\tilde{U}$ the subbundle generated by $\{dv_1, \dots, dv_n\}$, and let $\mathcal{W}^* = \mathrm{span}\{\theta_1, \dots, \theta_n\}$. Define

$$\begin{aligned} r_i(x,y) &= x_i^2 + y_i^2 && \text{for} \quad i = 1, \dots, k_e + k_h \\ r_i(x,y) &= x_i^2 + y_i^2 + x_{i+1}^2 + y_{i+1}^2 && \text{for} \quad i = k_e + k_h + 1, \dots, k_e + k_h + 2k_f \\ r_i(x,y) &= 1 && \text{for} \quad i = n-k+1, \dots, n. \end{aligned}$$

One checks directly that $\mathcal{W}^*$ and $\mathcal{V}_1^*$ are transversal over $U \setminus \mathrm{Crit}(v)$. Let (x,y) be non-critical for v: If we take the standard volume form

$$\mathrm{vol} = \frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

then since $r_i(x,y) \neq 0$ for all $i = 1, \dots, n$ and $(x,y) \in U \setminus \mathrm{Crit}(v)$ we conclude

$$\theta_1 \wedge \cdots \wedge \theta_n \wedge \mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_n = \prod_{i=1}^n r_i(x,y)\,\mathrm{vol} \neq 0 \tag{3.6}$$

But then we can use (3.5) and (3.6) to obtain

$$\begin{aligned}
& \theta_1 \wedge \cdots \wedge \theta_n \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n(tx, ty) \\
= \; & \theta_1 \wedge \cdots \wedge \theta_n \wedge (\mathrm{d}v_1 + \rho_1) \wedge \cdots \wedge (\mathrm{d}v_n + \rho_n) \\
= \; & \prod_{i=1}^{n} r_i(tx, ty) \Big[\mathrm{vol} + \sum_{i=1}^{n} \underbrace{\frac{(\theta_i \wedge \rho_i)(tx, ty)}{r_i(tx, ty)}}_{\to 0 \text{ as } t \to 0} \wedge \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \overset{i}{\cdots} \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n + \cdots \Big].
\end{aligned}$$

By choosing $t > 0$ small enough we can make all terms in the bracket arbitrarily small, except the first one. Therefore the bracket will tend to vol $\neq 0$ for $t \to 0$ while the factor $\prod_i r_i(tx, ty) \neq 0$ for $t \neq 0$. So there is an $\varepsilon(x, y) > 0$ with

$$\theta_1 \wedge \cdots \wedge \theta_n \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n(tx, ty) \neq 0$$

for all $0 < t < \varepsilon(x, y)$. In particular, (tx, ty) is non-critical for f as well, and so $(tx, ty) \in \tilde{U}$ for $0 < t < \varepsilon(x, y)$. This means that $\tilde{U} \cup \{0\}$ is a star-shaped region with centre 0 where we have

$$T^*\tilde{U} = \mathcal{V}_0^* \oplus \mathcal{W}^* = \mathcal{V}_1^* \oplus \mathcal{W}^*.$$

We can therefore define an isomorphism of vector bundles $\phi\colon T^*\tilde{U} \to T^*\tilde{U}$ by sending $df_i \mapsto dv_i$ and keeping θ_i fixed for $i = 1 \dots n$.

Now assume $\Omega^c = \Omega_1 + i\Omega_2 \in \mathcal{A}^{n,0}(T_m X)$ is a smooth $(n, 0)$-form that restricts to the induced (real) volume form on the smooth part of each fibre of f. This implies

$$\Omega_1 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n(x, y) \neq 0$$

for all $(x, y) \in \tilde{U}$ and hence, since $\phi|\tilde{U}$ is an isomorphism,

$$\begin{aligned}
0 \; & \neq \; \phi(\Omega_1 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n)(x, y) \\
& = \; \phi(\Omega_1) \wedge dv_1 \wedge \cdots \wedge \mathrm{d}v_n(x, y) \\
& = \; \Omega_1 \wedge dv_1 \wedge \cdots \wedge \mathrm{d}v_n(x, y)
\end{aligned}$$

because ϕ leaves the component of Ω_1 lying in $\Lambda^n \mathcal{W}^*$ invariant.

Now suppose $k_e > 0$. For suitable $\varepsilon > 0$ let

$$\gamma_\varepsilon\colon t \in [-\varepsilon, \varepsilon] \longmapsto (\underbrace{0, \dots, 0}_{1,\dots,k_e}, \underbrace{\varepsilon, \dots, \varepsilon}_{k_e+1,\dots,n}, t, \underbrace{\varepsilon, \dots, \varepsilon}_{2,\dots,k_e}, \underbrace{0, \dots, 0}_{k_e+1,\dots,n}).$$

Then if $\varepsilon > 0$ is small enough we have $\gamma_\varepsilon(t) \in \tilde{U} \Leftrightarrow t \neq 0$ and

$$L_t = \left(\frac{\partial}{\partial x_1}\bigg|_{\gamma_\varepsilon(t)}, \dots, \frac{\partial}{\partial x_n}\bigg|_{\gamma_\varepsilon(t)} \right), \qquad |t| < \varepsilon$$

is a smooth family of everywhere linearly independent sections of $\gamma_\varepsilon^* TU$, and over $\tilde{U}$ the sections are tangent to the fibres of v. Then we can define the following smooth function $s\colon [-\varepsilon, \varepsilon] \to \mathbb{R}$:

$$s(t) = \Omega_1(\partial/\partial x_1, \ldots, \partial/\partial x_n)_{\gamma_\varepsilon(t)} \neq 0 \quad \text{for small } \ t \neq 0.$$

Now we have $s(t)s(-t) < 0$ for all $t \neq 0$ because if $(\partial/\partial x_1, \ldots, \partial/\partial x_n)_{\gamma_\varepsilon(t)}$ is a positively oriented basis of the vertical space of the fibre F_t at $\gamma_\varepsilon(t)$ then it is a negatively oriented basis of the same fibre F_t at $\gamma_\varepsilon(-t)$. (To see this let

$$\xi_1 = \frac{1}{\sqrt{x_1^2 + y_1^2}} \left(y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1} \right) \quad \text{for} \quad x_1^2 + y_1^2 > 0.$$

Then $\xi_1(\gamma_\varepsilon(t)) = \operatorname{sign}(t) \partial/\partial x_1$. However, $\xi_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$ restricted to F_t form a smooth basis field, and hence have the same orientation in $\gamma_\varepsilon(t)$ and $\gamma_\varepsilon(-t)$ which are points in the same fibre F_t.) But s is continuous (actually smooth), so $s(0) = 0$ and hence $\Omega_1 | L_0 = 0$. But also $\Omega_2 | L_0 = 0$ by continuity, and Ω^c vanishes on L_0. Now $\gamma_\varepsilon(0) = 0$ and because $D\Phi_0 = \mathrm{Id}$ we know that

$$L_0 = \operatorname{span}(\partial/\partial x_1, \ldots, \partial/\partial x_n)_m \subset T_m X$$

is a Lagrangian subspace. If an n-form of type $(n, 0)$ in a Kähler manifold vanishes on a Lagrangian plane $L_0 \subset T_m X$ then $\Omega^c_m = 0$. The case where we have $k_h > 0$ is similar. Since Ω^c is non-vanishing, there cannot be any elliptic or hyperbolic points, and all weakly non-degenerate critical points are focus-focus. □

If we have a strongly non-degenerate critical point we can say more. Eliasson proves the local statement and Tien-Zung the global part of the following theorem:

Theorem 3.1.4 (Eliasson [Eli90], Tien-Zung [TZ96]) *If $m \in X$ is a strongly non-degenerate critical point there are Darboux-coordinates (x, y) around m and a diffeomorphism $\psi\colon B \to \mathbb{R}^n$ such that $\psi_i \circ f = v_i(x, y)$ for $i = 1, \ldots, n-k$ and $\psi_i \circ f = y_i$ for $i = n-k+1, \ldots, n$, where the v_i are as in (3.2) - (3.4). The critical locus has a stratification by symplectic submanifolds with a locally free Hamiltonian T^{n-k}-action on each $2(n-k)$-dimensional stratum.*

This means that the integrable system is locally symplectomorphic (as a fibration) to the linearised system. This can be seen as a Morse lemma for Poisson commuting functions, and integrability, that is the existence of a maximal number of Poisson commuting functions, is crucial to be able to bring them in normal form simultaneously.

Proposition 3.1.5 *Suppose $f\colon X \to B$ is a special Lagrangian fibration where every critical point of f is strongly non-degenerate. Then $\operatorname{Crit}(f) \subset X$ is a union of symplectic codimension 4-submanifolds of X, and $\Delta \subset B$ is a union of codimension 2 submanifolds. In particular $\operatorname{Crit}(f)$ is orientable (see also Proposition 4.1.25).*

Proof. In the notation of Theorem 3.1.3 if we assume $f = v$ the critical locus of f is locally diffeomorphic to the symplectic subspace $\{0\} \times V^{\perp_\omega}$ via Φ. Since we can only have focus-focus singularities, $\dim(V) \geq 4$, and similarly we have chosen a chart on the base which maps Δ into $\{0\} \times \mathbb{R}^{n-2} \subset \mathbb{R}^n$. □

Note that there is however little evidence why special Lagrangian fibrations should *generically* be non-degenerate or smooth. In fact, $\operatorname{codim}(\Delta) \geq 2$ might only be a limiting case near large complex structure points of X, see [Joy00d].

Corollary 3.1.6 *Let X be compact. Then every stratum of the critical locus* $\operatorname{Crit}(f)$ *of a strongly non-degenerate special Lagrangian fibration $f\colon X \to B$ represents a non-trivial homology class in $H_{\mathrm{ev}}(X, \mathbb{R})$.*

Proof. By Proposition 3.1.5, $\operatorname{Crit}(f)$ is a (finite) union of symplectic submanifolds. Since X is compact and $\operatorname{Crit}(f)$ is closed, these have to be compact submanifolds. Because every stratum C of $\operatorname{Crit}(f)$ is even-dimensional and ∂C is again a union of even-dimensional strata, $\partial[C] = 0$ and C defines a homology class. For a stratum C of dimension $2k$ we have $\omega^k|C \neq 0$ and so $\omega^k[C]_{2k} \neq 0$ and hence $0 \neq [C] \in H_*(X, \mathbb{R})$. □

Proposition 3.1.7 *Let $f\colon X \to B$ be a special Lagrangian fibration on a Calabi-Yau three-fold with only strongly non-degenerate singularities and f proper. Then* $\operatorname{Crit}(f) \subset X$ *is a disjoint union of symplectic surfaces. If X is compact, every connected component of* $\operatorname{Crit}(f)$ *is diffeomorphic to T^2.*

Proof. In dimension 3 we can only have simple focus-focus singularities in a special Lagrangian fibration with only strongly non-degenerate singularities. Hence by Theorem 3.1.4 the stratification of (a connected component of) $\operatorname{Crit}(f)$ contains only one stratum in dimension 2 which is a symplectic surface, hence orientable. $\operatorname{Crit}(f)$ is clearly closed, so if X is compact then $\operatorname{Crit}(f)$ is a compact surface with a locally free action of S^1, hence $\chi(\operatorname{Crit}(f)) = 0$ and $\operatorname{Crit}(f) \cong T^2$. □

Remark 3.1.8 We expect that in general special Lagrangian fibrations $f\colon X \to B$ have some degenerate singularities. From [Gro98, Gro99b], we expect that a non-degenerate singular fibre F_{sing} with only focus-focus singular points is homeomorphic to $F_0 \times S^1$ where $F_0 \cong S^2/\{p_1, p_2\}$ is an I_1 singularity. We will show later (Theorem 4.1.30) that in this case the Calabi-Yau manifold X would need to have $\chi(X) = 0$.

3.1.2 McLean's metric (II)

Having studied McLean's metric h on the base B of a special Lagrangian fibration in Section 2.3 using its Hessian form, we now return to it once more to derive estimates for h from the symplectic normal forms in Theorem 3.1.4.

Theorem 3.1.9 *Let $b_0 \in \Delta(f)$ be a singular value of a special Lagrangian fibration $f\colon X \to B$, and suppose that $F_{b_0} \cap \mathrm{Crit}(f)$ contains only strongly non-degenerate critical points. Then b_0 has a neighbourhood $V \subset B$ such that*

$$\mathrm{Vol}_h(V \cap B_0) < \infty.$$

Proof. By Theorem 3.1.3 and Theorem 3.1.4, for each $m \in F_{b_0}$ exactly one of the following two cases is true:

1. m has a neighbourhood U_m in X such that all elements $m' \in U_m$ are regular (i.e. $\mathrm{rk}(f_{*m'}) = n$), and we can find Darboux coordinates of the form $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ with $y_i = f_i$.

2. m is a focus-focus singular point of Williamson type $(0, 0, k_f)$ with $k_f \leq n/2$ (and thus by Eliasson's theorem [Eli90] is contained in a $(2n-4k_f)$-dimensional symplectic submanifold of X which consists entirely of focus-focus points and which is a stratum of $\mathrm{Crit}(f)$). Such an m has a neighbourhood U_m on which we can find Darboux coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ around m and a chart $t\colon B \to \mathbb{R}^n$ so that the integrable system $f\colon X \to B$ takes the normal form

$$\begin{array}{lcll} f_i(x,y) & = & x_i y_{i+1} - x_{i+1} y_i, & \\ f_{i+1}(x,y) & = & x_i y_i + x_{i+1} y_{i+1}, & i = 1, 3, \ldots, 2k_f - 1 \\ f_i(x,y) & = & y_i, & i = 2k_f + 1, \ldots, n. \end{array} \tag{3.7}$$

By compactness of the fibres we can choose finitely many such sets $U_1, \ldots, U_r$ of type (1) and $U_{r+1}, \ldots, U_{r+s}$ of type (2) covering F_{b_0}. Again due to compactness they will cover all fibres F_b for $b \in V$, where V is a small enough neighbourhood of b_0. With respect to coordinates $t = (t_1, \ldots, t_n)\colon V \to \mathbb{R}^n$ the Riemannian metric h on B is given by a symmetric $n \times n$-Matrix (h_{ij}) defined on $V \cap B_0$, and

$$\mathrm{Vol}(V \cap B_0) = \int_{V \cap B_0} \sqrt{\det(h_{ij})}\, \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n.$$

Let us assume $t(V) \subset B_1(0)$. By McLean's result [McL98] the coefficient $h_{ij} = h(\partial/\partial t_i, \partial/\partial t_j)$ is given by

$$h_{ij}(b) = \int_{F_b} \iota(T_i)\omega \wedge \iota(T_j)\Omega_2.$$

where T_i and T_j are vector fields along F_b lifting $\partial/\partial t_i, \partial/\partial t_j$ (meaning $f_*(T_i) = \partial/\partial t_i$). If we choose Hamiltonian vector fields $(\xi_i)_{i=1,\dots,n}$ on $f^{-1}(V \cap B_0)$ so that $\iota_{\xi_i}\omega = df_i$ (where $f_i = t_i \circ f$) and if we set $g_{ij} = g(\xi_i, \xi_j)$ where g is the Ricci-flat metric on X then this can be expressed as

$$h_{ij}(b) = \int_{F_b} g^{ij} \,\mathrm{vol},$$

g^{ij} being the $n \times n$ matrix inverse to g_{ij}. We will now estimate this coefficient. Fix $b \in V \cap B_0$. We certainly have

$$|h_{ij}(b)| \le \int_{F_b} |g^{ij}| \,\mathrm{vol} \le \sum_{j=1}^{r+s} \int_{F_b \cap U_j} |g^{ij}| \,\mathrm{vol}\,.$$

To prove the theorem, we estimate the individual summands and the functions g^{ij} first.

Case 1: $1 \le k \le r$. We have smooth Darboux coordinates (x, y) on $U = U_k$ such that $t_i \circ f = y_i$ and $\xi_i = \partial/\partial x_i$ are our Hamiltonian vector fields. In this case g^{ij} which is the inverse matrix of $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ is a smooth symmetric matrix all of whose eigenvalues are positive and bounded away from zero (at least we can always achieve this by shrinking U a bit further if necessary), so g^{ij} is a smooth and *bounded* function on U. We can thus find a $K > 0$ so that

$$|g^{ij}(m')| < K$$

for all $i, j = 1, \dots, n$ and $m' \in U$. In the following let K be a generic constant which at every time is as large as we need it. The total volume of a fibre in our special Lagrangian fibration is finite and the same for all fibres. Thus we obtain the trivial estimate

$$\int_{F_{b_0} \cap U} |g^{ij}| \,\mathrm{vol} \le \int_{F_{b_0}} K \,\mathrm{vol} = K.$$

By choosing K large enough we may assume that this estimate holds for all $k = 1, \dots, r$.

Case 2: $r < k \le r + s$. If $m \in F_b$ is a point contained in some $U = U_{r+k}$ with $1 \le k \le s$ we do not have Darboux coordinates which are both adapted to the fibration and smooth over all of U, so we will use two sets of coordinates, one adapted, but singular along the singular fibres, the other smooth on all of U, but not adapted to the fibration. Let (x, y) be Darboux coordinates as in (3.7). Let

$$\begin{aligned}
p_i(x,y) &= \tfrac{1}{2}\left[\ln(x_i^2 + x_{i+1}^2) - \ln(y_i^2 + y_{i+1}^2)\right] & \\
p_{i+1}(x,y) &= \arctan\left(\tfrac{x_{i+1}}{x_i}\right) + \arctan\left(\tfrac{y_{i+1}}{y_i}\right) & \text{for } i = 1, 3, \dots, 2k_f - 1 \\
p_i(x,y) &= x_i & \text{for } i = 2k_f + 1, \dots, n.
\end{aligned}$$

One checks explicitly that outside $\mathrm{Crit}(f)$

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}f_i,$$

thus (p, f) is a Darboux coordinate system away from the critical locus. The corresponding Hamiltonian vector fields $\xi_i = \partial/\partial p_i$ are

$$\begin{aligned} \frac{\partial}{\partial p_i} &= x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i} + x_{i+1} \frac{\partial}{\partial x_{i+1}} - y_{i+1} \frac{\partial}{\partial y_{i+1}} \\ \frac{\partial}{\partial p_{i+1}} &= x_i \frac{\partial}{\partial x_{i+1}} - y_{i+1} \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_{i+1}} \end{aligned}$$

Define another metric g_0 on U by the requirement that the smooth basis field $\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n$ around m be orthonormal and choose $K > 0$ so that for all $m' \in U$ and $\eta \in T_{m'}X$ we have

$$g(\eta, \eta) \leq K\, g_0(\eta, \eta).$$

Since g_{ij} is symmetric and positive definite we have $(g_{ij})^2 \leq g_{ii} g_{jj}$ for all i, j and we only need to look at the diagonal terms g^{ii}. We have

$$|g^{ii}| = \left| \frac{\det(g_{jk})_{j,k \neq i}}{\det(g_{jk})} \right| \leq \frac{(n-1)!}{\det(g_{jk})} \prod_{j \neq i} g_{jj}$$

From our normal form we can deduce

$$\begin{aligned} g_{jj} &= g\left(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_j}\right) \\ &\leq K\, g_0\left(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_j}\right) \\ &= \begin{cases} K\,(x_i^2 + y_i^2 + x_{i+1}^2 + y_{i+1}^2) & j = 1, \ldots, 2k_f, \qquad i = 2[(j-1)/2] + 1 \\ K & j = 2k_f + 1, \ldots, n. \end{cases} \end{aligned}$$

Now choose a small $k > 0$ so that for each $m \in U$ and each oriented n-plane $W \subset T_m X$ the volume forms vol_W induced from the Ricci-flat metric g and vol_W^0 induced from g_0 satisfy

$$k\ \mathrm{vol}_W^0 \leq \mathrm{vol}_W$$

For $m' \notin \mathrm{Crit}(f)$ let $W_{m'}$ be the n-plane spanned by the $(\partial/\partial p_i)_{i=1,\ldots,n}$. We then get a lower bound

$$\begin{aligned} \sqrt{\det(g_{ij})(m)} &= \left| \left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\right)_m \llcorner \mathrm{vol}_{W_m} \right| \\ &\geq k \left| \left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\right)_m \llcorner \mathrm{vol}_{W_m}^0 \right| \\ &= k(x_1^2 + x_2^2 + y_1^2 + y_2^2) \cdots (x_{2k_f-1}^2 + x_{2k_f}^2 + y_{2k_f-1}^2 + y_{2k_f}^2). \end{aligned}$$

We can now estimate

$$\begin{aligned}
|g^{jj}| &\leq \frac{(n-1)!}{\det(g_{ik})} \prod_{k \neq j} g_{kk} \\
&\leq \begin{cases} \frac{K/k}{(x_i^2+y_i^2+x_{i+1}^2+y_{i+1}^2)} & j = 1, \ldots, 2k_f, \qquad i = 2[(j-1)/2]+1 \\ K & j = 2k_f+1, \ldots, n. \end{cases}
\end{aligned}$$

To obtain any useful estimate from that we proceed as follows: Note that

$$\begin{aligned}
& f_i(x,y)^2 + f_{i+1}(x,y)^2 \\
=\; & (x_i^2 + x_{i+1}^2)(y_i^2 + y_{i+1}^2) \\
\leq\; & \frac{1}{2}(x_i^2 + x_{i+1}^2)(y_i^2 + y_{i+1}^2) + \frac{1}{4}((x_i^2 + x_{i+1}^2)^2 + (y_i^2 + y_{i+1}^2)^2) \\
=\; & \frac{1}{4}(x_i^2 + x_{i+1}^2 + y_i^2 + y_{i+1}^2)^2
\end{aligned}$$

and therefore we can estimate

$$\frac{1}{x_i^2 + x_{i+1}^2 + y_i^2 + y_{i+1}^2} \leq \frac{1}{2\sqrt{f_i(x,y)^2 + f_{i+1}(x,y)^2}}.$$

Thinking of K as a generic constant, if $k < 1$ replace K by K/k. Together with the previous calculation this yields for all $m' \in U$

$$g^{jj}(m') \leq \begin{cases} \frac{K}{\sqrt{f_i(m')^2 + f_{i+1}(m')^2}} & j = 1, \ldots, 2k_f \qquad i = 2[(j-1)/2]+1 \\ K & j = 2k_f+1, \ldots, n. \end{cases}$$

Because the f_i are constant along the fibres, this depends only on the fibre F_b of m' where $f(m') = b$, but not on m' itself, and therefore we can continue

$$\begin{aligned}
& \int_{F_b \cap U} |g^{ii}| \operatorname{vol} \\
\leq\; & \int_{F_b \cap U} \frac{K}{\sqrt{f_i^2 + f_{i+1}^2}} \operatorname{vol} \\
\leq\; & \frac{K}{\sqrt{t_i^2(b) + t_{i+1}^2(b)}}
\end{aligned}$$

Again, we assume that these constants work for all U_{r+k}, $k = 1, \ldots, s$. Then we finally get $(j = i, i+1)$

$$h_{jj}(b) \leq \begin{cases} rK + \frac{sK}{\sqrt{(t_i^2(b) + t_{i+1}^2(b))}} & j = 1, \ldots, 2k_f, \qquad i = 2[(j-1)/2]+1 \\ (r+s)K & j = 2k_f+1, \ldots, n. \end{cases}$$

We can ignore the constant terms and get, for some suitably large $K > 0$,

$$
\begin{aligned}
\sqrt{\det(h_{ij}(b))} &\le \left[n! \prod_{i=1}^{n} h_{ii}(b) \right]^{1/2} \\
&\le \frac{K}{\sqrt{(t_1^2(b)+t_2^2(b))\cdots(t_{2k_f-1}^2(b)+t_{2k_f}^2(b))}}.
\end{aligned}
$$

We can integrate this explicitly and obtain for small $V \subset B_0$

$$
\begin{aligned}
\mathrm{Vol}(V \cap B_0) &\le \int_V \frac{K}{\sqrt{(t_1^2+t_2^2)\cdots(t_{2k_f-1}^2+t_{2k_f}^2)}}\,\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n \\
&\le \prod_{i=1}^{k_f} \int_{\{t_i^2+t_{i+1}^2<1\}} \frac{K}{\sqrt{t_1^2+t_2^2}}\, dt_i\, dt_{i+1} \\
&\le \left(\int_{r=0}^{r=1} \int_{\varphi=0}^{\varphi=2\pi} \frac{K}{r}\, d\varphi\, r dr \right)^{k_f} \\
&= (2\pi K)^{k_f}
\end{aligned}
$$

which is finite. This concludes the proof. $\square$

Theorem 3.1.10 *Suppose B is compact and $f\colon X \to B$ is a special Lagrangian fibration with only strongly non-degenerate singularities. Then* $\mathrm{Vol}_h(B) < \infty$.

We also note an application in *holomorphic* geometry.

Theorem 3.1.11 *Let $\Psi\colon S \to B$ be an elliptic $K3$-surface all of whose singular fibres are of Kodaira type I_1. Then* $\mathrm{Vol}_h(B) < \infty$.

Proof. The hyper-Kähler trick transforms $S \to B$ into a special Lagrangian fibration; this induces a Hessian metric h on B as before. For I_1-singularities we know that in a neighbourhood of each critical point m there are holomorphic coordinates (z_1, z_2) and there is a holomorphic coordinate t near $\Psi(m) \in B$ so that $t \circ \Psi = z_1 z_2$. Let

$$x_1 = \mathrm{Re}(z_1),\quad y_1 = \mathrm{Im}(z_2),\quad x_2 = \mathrm{Im}(z_1),\quad y_2 = \mathrm{Re}(z_2)$$

and $t = t_1 + it_2$. We find that

$$
\begin{aligned}
t_1 \circ \Psi &= x_1 y_2 - x_2 y_1 \\
t_2 \circ \Psi &= x_1 y_1 + x_2 y_2
\end{aligned}
$$

and hence we have got a smooth coordinate system $(x_i, y_i)_{i=1,2}$ near m in which the fibration Ψ is described by Eliasson's normal form (3.7), and the proof of Theorem 3.1.9 carries through because it only uses the smoothness of the metric on the ambient space and the local normal form of the fibration. This yields the result $\mathrm{Vol}_h(B) < \infty$. □

Theorem 3.1.12 *Suppose* $\Psi\colon S \to B$ *is an elliptic* $K3$*-surface all of whose singular fibres are of Kodaira type* I_1*. Then* $\mathrm{Vol}_h(B_0)$ *is a holomorphic invariant of* (S, I)*.*

Suppose $S \to B$ is an elliptic $K3$-surface $S \to B$ with respect to a complex structure I which determines a Kähler form $\omega = \omega_1$ and an I-holomorphic two-form $\Omega^c = \omega_2 + i\omega_3$. Then $\Omega^c \in \mathcal{A}^{2,0}(S)$ vanishes on all fibres and McLean's metric for the special Lagrangian fibration obtained by performing the hyper-Kähler trick is naturally determined by Ω^c alone and therefore depends only on the holomorphic structure, but not the metric. □

Proposition 3.1.13 *Suppose* $f\colon X \to B$ *is a special Lagrangian fibration with strongly non-degenerate singularities. Let* $b_0 \in B_0 \cap V$ *and* $b_1 \in \Delta_g \cap V$*. Then there is a path of finite length from* b_0 *to* b_1*.*

Proof. It is enough to consider the case $t(b) = (r, 0, \ldots, 0)$ and $t(b_0) = 0$. Then the path $\gamma\colon [0, r] \to V$ with $t \circ \gamma(s) = (r - s, 0, \ldots, 0)$ has length

$$\begin{aligned} L(\gamma) &\leq \int_0^r \sqrt{h_{11}(\gamma(s))}\, ds \\ &\leq \int_0^r \frac{K}{\sqrt{r-s}}\, ds \quad < \infty. \end{aligned}$$

This implies the Proposition. □

This has the following immediate consequence.

Theorem 3.1.14 *In the situation of Proposition 3.1.13,* (B_0, h) *is not complete.*

The results of Section 2.3 and Paragraph 3.1.2 are reminiscent of the natural geometry on the base space of an algebraically completely integrable Hamiltonian system, in other words of the geometry of *special Kähler manifolds.* These were first studied by DS Freed [Fre99]. A complete special Kähler manifold is necessarily flat; this was conjectured (more specifically for compact special Kähler manifolds) by Freed and proved by Lu [Lu99]. Moreover, special Kähler manifolds are Ricci-positive, and if they are scalar flat then they are flat.

3.2 A model for the singular fibres

Having looked at the local structure of non-degenerate singularities in a special Lagrangian fibration we now want to find a *semi-local* model; by this we mean a tubular neighbourhood of a singular fibre of a special Lagrangian fibration $f\colon X \to B$, local in the base, but global in the fibres. We first look at the classical model of a special Lagrangian fibration in $\mathbb{C}^3$ with (generically) non-degenerate focus-focus singularities which is due to Harvey and Lawson [HL82]. We then go on to describe a model of a special Lagrangian fibration with a singular fibre due to Ooguri and Vafa [OV96]. It is inspired by the Gibbons-Hawking Ansatz [GH78]. We compute McLean's metric on the base for this model. Besides that our main purpose is to make this model available for later use.

3.2.1 The Harvey-Lawson model

In [HL82], Harvey and Lawson define a special Lagrangian fibration $f\colon \mathbb{C}^3 \longrightarrow \mathbb{R}^3$ which sends

$$(z_1, z_2, z_3) \longmapsto (|z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2, \mathrm{Im}(z_1 z_2 z_3)).$$

It degenerates over a trivalent graph with three rays emanating from a vertex at the origin inside the hyperplane $\{x_3 = 0\}$ of $\mathbb{R}^3$.

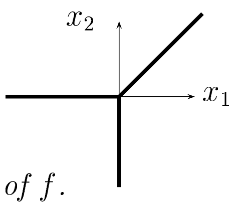

Figure 3.1. *The discriminant locus of f.*

The generic fibre is a submanifold diffeomorphic to $T^2 \times \mathbb{R}$. It is easy to check that a generic critical point of f is of focus-focus type. However, the singularity in the origin is degenerate because $\mathrm{Im}(z_1 z_2 z_3)$ vanishes to third order in $z = 0$. This is already clear from the topology of the graph Δ in Figure 3.1. It cannot be the discriminant locus of an integrable system with only strongly non-degenerate singularities because Δ is not a union of submanifolds of codimension ≤ 2.

3.2.2 The Ooguri-Vafa model

We define a function $V\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ by

$$V(x) = 1 + \frac{1}{2\|x\|}.$$

The only important piece of information about V is that it is a harmonic function, that is $\Delta V = 0$. Thus the form $F = *dV$ is itself closed. Let $\pi\colon M_0 \to \mathbb{R}^3 \setminus \{0\}$ be an S^1-bundle with Chern-class ± 1 over a pointed ball around $0 \in \mathbb{R}^3$, and let $\sigma \in \mathcal{A}^1(M_0)$ be a connection on M_0 with curvature $F = d\sigma = *dV$. Denote the vertical element by $d\tau$ (τ is itself not well-defined). We define a collection of two-forms

$$\begin{aligned}
\omega_1 &= V\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge (\sigma + \mathrm{d}\tau) && (3.8)\\
\omega_2 &= V\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge (\sigma + \mathrm{d}\tau) && (3.9)\\
\omega_3 &= V\mathrm{d}x_3 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge (\sigma + \mathrm{d}\tau). && (3.10)
\end{aligned}$$

Using $\Delta V = 0$ one can check that these forms are closed. They define a Ricci-flat metric on M_0 which extends smoothly to the one-point compactification

$$\bar{M}_0 = M_0 \cup \{\mathrm{pt}\}.$$

S^1 acts isometrically on $\bar{M}_0$ with *pt* an isolated fixed point. This is the Euclidean Taub-NUT metric.

The idea in [OV96] is to give this metric a translational symmetry by applying this construction to an S^1-bundle

$$\pi\colon M_0 \to U_0 \subset \mathbb{R}^3 \setminus \{(0,0,k) \mid k \in \mathbb{Z}\}.$$

In order to obtain a harmonic function with the appropriate singularities we cannot simply take $\Sigma_k V(x - ke_3)$ because this sum is divergent, however, for suitably chosen constants $c_k \in \mathbb{R}$ the sum $\Sigma_k(V(x-ke_3) - c_k)$ does converge to a harmonic function and the above construction produces a Ricci-flat metric on a neighbourhood U_0 of the line $\{x_1 = x_2 = 0\}$ in the compactification $\bar{M}_0$ of M_0. More precisely, one can take the potential

$$\begin{aligned}
V(x_1,x_2,x_3) &= \frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\frac{1}{\sqrt{x_1^2+x_2^2+(x_3-n)^2}} - \frac{1}{n}\right] + \frac{1}{2\sqrt{x_1^2+x_2^2+x_3^2}} - \frac{1}{2}\\
&= -\frac{1}{2}\ln(x_1^2+x_2^2) + \sum_{n\in\mathbb{Z}\setminus\{0\}} \exp(2\pi i n x_3) K_n(x_1,x_2) + c && (3.11)
\end{aligned}$$

for some $c \in \mathbb{R}$. The resulting metric has both an isometric S^1 action and an isometric action of $\mathbb{Z}$ by horizontal translations $x_3 \mapsto x_3 + 1$. Dividing out the $\mathbb{Z}$-action produces a special Lagrangian fibration $f\colon M \to U \subset \mathbb{R}^2$. The S^1-action descends to an action on the quotient which is free outside the critical point; its orbit represents the "vanishing cycle" of the fibration, and the singular fibre is topologically equivalent to an I_1-degeneration in an elliptic surface whose monodromy action on $H_1(F, \mathbb{Z})$ for a smooth fibre F is given, in suitable coordinates, by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Remark 3.2.1 The Ooguri-Vafa metric is not complete, contrary to the metrics of type Taub-NUT or ALE. This is essentially because the potential V fails to be positive everywhere. However, there is a similar construction of complete Ricci-flat Kähler metrics of infinite topological type using *sparsely distributed* mass centres [AKL89].

Remark 3.2.2 Let $M^{\#} = M \setminus \{m_0\}$ so that $f^{\#}\colon M^{\#} \to U$ is a submersion. Then there exists a lattice of maximal rank $\Lambda^* \subset T^*M^{\#}$ which is spanned by dx_3 and $\sigma + d\tau$. By duality we get a non-degenerate lattice of maximal rank $\Lambda \subset \mathcal{V}$ inside the vertical bundle of the fibration $f^{\#}$ (which is well-defined on $M^{\#}$).

Computing McLean's metric

We conclude this section by computing McLean's metric h explicitly for the special Lagrangian fibration of Ooguri and Vafa. This is a straightforward exercise in elementary differential geometry. The reason for giving it here is that, firstly, to our knowledge there are neither any general results nor any specific examples which tell us something about the asymptotic behaviour of h near the discriminant locus. Secondly, while entirely elementary the calculations are a bit tedious so that it might be worth making the results available here.

Proposition 3.2.3 *McLean's metric h on the base and its sectional curvature (Gauß curvature) K_h are given by*

$$\begin{aligned} h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) &= -\frac{1}{2}\log(x_1^2 + x_2^2)\delta_{ij} \\ K_h(x_1, x_2) &= \frac{-4}{(x_1^2 + x_2^2)(\ln(x_1^2 + x_2^2))^3}. \end{aligned}$$

Proof. Let $h_{ij} = h(\partial/\partial x_i, \partial/\partial x_j)$ and let $(X_i)_{i=1,2}$ be lifts of the basis vectors $\partial/\partial x_i$ to $M \setminus f^{-1}(0)$. We have

$$h_{ij}(x_1, x_2) = \int_{f^{-1}(x_1,x_2)} \iota(X_i)\omega_2 \wedge \iota(X_j)\omega_3$$

which is independent of the choice of lift. We let

$$V_0(x_1, x_2) = -\frac{1}{2}\log(x_1^2 + x_2^2)$$

be the zero-mode in the Fourier decomposition of the harmonic (on flat $\mathbb{R}^3 \setminus \{0\}$) potential function V in (3.11). Since V is independent of τ, we then get

$$h_{11}(x_1, x_2) = -\int_{f^{-1}(x_1,x_2)} V\,\mathrm{d}\tau \wedge \mathrm{d}x_3 = \int_{f^{-1}(x_1,x_2)} V\omega_1 = V_0(x_1, x_2),$$

and by similar calculations we find

$$h_{ij}(x_1, x_2) = V_0(x_1, x_2)\delta_{ij} = -\frac{1}{2}\log(x_1^2 + x_2^2)\delta_{ij}.$$

The Gauß curvature of a metric $h_{ij}(x_1, x_2) = V_0(x_1, x_2)\delta_{ij}$ is given by

$$K_h(x_1, x_2) = \frac{\left(\frac{\partial V_0}{\partial x_1}\right)^2 + \left(\frac{\partial V_0}{\partial x_2}\right)^2}{2V_0^3} - \frac{\frac{\partial^2 V_0}{\partial x_1^2} + \frac{\partial^2 V_0}{\partial x_2^2}}{2V_0^2}.$$

Note that V is harmonic if and only if each component in its Fourier decomposition is harmonic; so in this case V_0 is harmonic and since V_0 is independent of x_3 we find

$$\frac{\partial^2 V_0}{\partial x_1^2} + \frac{\partial^2 V_0}{\partial x_2^2} = 0$$

and hence $K_h \geq 0$, as we already know since the metric on the total space is hyper-Kähler, and $K_h(x_1, x_2) = 0$ implies $dV_0(x_1, x_2) = 0$. For our potential function,

$$K_h(x_1, x_2) = \frac{-4}{(x_1^2 + x_2^2)(\log(x_1^2 + x_2^2))^3}$$

as in the proposition. Note that $K_h(x_1, x_2) \to +\infty$ as $x \to 0$. □

Computation 3.2.4 *Let $D \subset \mathbb{R}^2$ be the unit disk. Then*

$$\mathrm{Vol}_h(D) = \frac{\pi}{2}.$$

Proof.

$$\begin{aligned}
\mathrm{Vol}_h(D) &= \int_D \sqrt{\det(h(x_1, x_2))}\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
&= -\frac{1}{2}\int_D \ln(x_1^2 + x_2^2)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
&= -\pi \int_{r=0}^{r=1} r\ln(r^2)\,dr \\
&= -\frac{\pi}{2}\left[r^2(\ln(r^2) - 1)\right]_{r=0}^{r=1} \\
&= \frac{\pi}{2}
\end{aligned}$$

□

Linear coordinates

Alternatively, we can work in linear coordinates on the base. Let

$$\alpha(x_1,x_2)=\frac{1}{2}-\frac{\operatorname{sign}(x_2)}{4}-\frac{1}{2\pi}\arctan\left(\frac{x_1}{x_2}\right)$$

be the smooth angle-function on $\mathbb{R}^2\setminus\{(t,0)|t\geq 0\}$. It is easy to check that

$$\begin{aligned}\xi_1 &= \frac{\alpha}{V_0}\frac{\partial}{\partial x_1}-\frac{1}{2\pi}\frac{\partial}{\partial x_2}\\ \xi_2 &= \frac{1}{V_0}\frac{\partial}{\partial x_1}\end{aligned}$$

are linear vector-fields on $\mathbb{R}^2\setminus\{0\}$ (with ξ_1 multi-valued). They are integrable because

$$[\xi_1,\xi_2]=-\left(\frac{\partial\alpha}{\partial x_1}+\frac{1}{2\pi}\frac{\partial V_0}{\partial x_2}\right)\frac{\partial}{\partial x_1}=0.$$

So there are (multi-valued) coordinates u_1,u_2 with $\xi_i=\partial/\partial u_i$. (We will not need these coordinates here, but the coordinate transformation can be given by

$$\begin{aligned}u_1(x_1,x_2) &= -2\pi x_2\\ u_2(x_1,x_2) &= -x_2\arctan\left(\frac{x_1}{x_2}\right)+\pi x_2-\frac{\pi|x_2|}{2}-\frac{x_1}{2}\ln(x_1^2+x_2^2)+x_1.\end{aligned}$$

To see this simply check $du_i(\xi_j)=\delta_{ij}$, hence $\xi_j=\partial/\partial u_j$. Also

$$\det(Du)_{(x_1,x_2)}=-\pi\ln(x_1^2+y_1^2)=2\pi\,V(x_1,x_2)>0$$

for small $x\neq 0$, so the coordinates are non-degenerate.

Let us now define a function

$$\phi(x_1,x_2)=-\frac{1}{4}\left[(x_1^2+x_2^2)\ln(x_1^2+x_2^2)-x_1^2-3x_2^2\right]. \tag{3.12}$$

We then find

$$\begin{aligned}\frac{\partial^2\phi}{\partial u_1^2} &= \left(\frac{\alpha}{V_0}\frac{\partial}{\partial x_1}-\frac{1}{2\pi}\frac{\partial}{\partial x_2}\right)\left(\frac{\alpha}{V_0}\frac{\partial\phi}{\partial x_1}-\frac{1}{2\pi}\frac{\partial\phi}{\partial x_2}\right)=\frac{\alpha^2}{V_0}+\frac{V_0}{4\pi^2}=h\left(\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_1}\right)\\ \frac{\partial^2\phi}{\partial u_1\partial u_2} &= \left(\frac{\alpha}{V_0}\frac{\partial}{\partial x_1}-\frac{1}{2\pi}\frac{\partial}{\partial x_2}\right)\left(\frac{1}{V_0}\frac{\partial\phi}{\partial x_1}\right)=\frac{\alpha}{V_0}=h\left(\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_2}\right)\\ \frac{\partial^2\phi}{\partial u_2^2} &= \frac{1}{V_0}\frac{\partial}{\partial x_1}\left(\frac{1}{V_0}\frac{\partial\phi}{\partial x_1}\right)=\frac{1}{V_0}=\ h\left(\frac{\partial}{\partial u_2},\frac{\partial}{\partial u_2}\right)\end{aligned}$$

Incidentally we can also verify that

$$\det(h(\xi_i, \xi_j)) = \frac{1}{V_0}\left(\frac{\alpha^2}{V_0} + \frac{V_0}{4\pi^2}\right) - \left(\frac{\alpha}{V_0}\right)^2 = \frac{1}{4\pi^2}$$

is constant, as it should be for the natural metric on the moduli space of special Lagrangian submanifolds in a hyper-Kähler manifold. Moreover, it is possible to compute the curvature of h in linear coordinates from the third derivatives. The identity (2.61) yields the formula

$$K_h\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = \frac{1}{4\det(\phi_{..})}\sum_{p,q}\phi^{pq}\left(\phi_{pij}\phi_{qij} - \phi_{pjj}\phi_{qii}\right).$$

Putting in the function ϕ of (3.12) confirms the result of Proposition 3.2.3. However, while feasible the computations in linear coordinates are quite tedious.

Finally, it is possible to change the potential by adding a harmonic function $w(x_1, x_2)$ so that

$$V_0 = -\frac{1}{2}\ln(x_1^2 + x_2^2) + w(x_1, x_2). \tag{3.13}$$

The calculations remain largely the same and

$$\begin{aligned}
h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) &= \left(\frac{1}{2}\ln(x_1^2 + x_2^2) + w(x_1, x_2)\right)\delta_{ij} \\
K_h(x_1, x_2) &= \frac{-\frac{1}{x_1^2+x_2^2} - \left(\frac{\partial w}{\partial x_1}\right)^2 - \left(\frac{\partial w}{\partial x_2}\right)^2}{2\left(\frac{1}{2}\ln(x_1^2 + x_2^2) + w(x_1, x_2)\right)^3} \\
\mathrm{Vol}_h(D) &= \frac{\pi}{2} + \int_D w(x_1, x_2)\, dx_1\, dx_2
\end{aligned}$$

provided (3.13) is positive on D.

Chapter 4

Good torus fibrations

In the spirit of [Gro99b] we describe a way of constructing six-manifolds by compactifying T^3-bundles over the complements of knots and links. If a link $\mathfrak{l}$ is *irreducible* (see Definition 4.1.31), its complement is an Eilenberg-MacLane space $K(\pi_1, 1)$, and the topology of these six-manifolds is closely related to the representation theory of the link group $G(\mathfrak{l})$. Therefore the topological invariants of the space can be identified with algebraic invariants of the representation which can be computed effectively.

4.1 Constructing smooth torus fibrations

We give a brief account of fibre bundles and fibrations. We introduce *good torus-fibrations* and study their topological properties.

4.1.1 Affine torus bundles

The notions of "fibre bundle" and "fibration", although fundamental, appear with slightly different meanings at different places in the literature. Therefore we want to spell out the definitions we are going to use here.

Definition 4.1.1 *Let X, B be topological spaces, $b_0 \in B$ and suppose $f\colon X \to B$ is a continuous map. Set $F = f^{-1}(b_0)$. Then f is called a topological fibre bundle or a topological F-bundle if for each $b \in B$ there exists a neighbourhood $U \subset B$ of b and a homeomorphism*

$$\varphi_U \colon f^{-1}(U) \longrightarrow U \times F$$

such that $f|_{f^{-1}(U)} = \mathrm{pr}_1 \circ \varphi_U$, where $\mathrm{pr}_1 \colon U \times F \to U$ is the projection map. If X and B are smooth manifolds and f is a smooth submersion then f is called a smooth fibre bundle (smooth F-bundle) or just a fibre bundle (F-bundle).

Thus by default a "fibre bundle" is always assumed to be a smooth submersion $f\colon X \to B$ of smooth manifolds unless we explicitly call it a "*topological* fibre bundle". We will be concerned with T^n-bundles, and we point out that a T^n-bundle in the sense of Definition 4.1.1 is simply a bundle of manifolds and not a principal $\mathbb{T}^n$-bundle. Here we use T^n to denote the n-torus as a manifold, while $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-torus as a group. A necessary (generally not sufficient) condition for a T^n-bundle to admit a principal $\mathbb{T}^n$-bundle structure is triviality of its monodromy. Hence we will always distinguish between the terms "T^n-bundle" and "principal $\mathbb{T}^n$-bundle".

Definition 4.1.2 *Let X, B be smooth manifolds. A surjective smooth map $f\colon X \to B$ is called a <u>fibration</u> if there is a smooth manifold F called the <u>typical fibre</u> and an open dense subset $B_0 \subset B$ such that $f_0\colon X_0 = f^{-1}(B_0) \to B_0$ is <u>an F-bundle</u>.*

In the literature, the definition of fibrations often includes additional nice properties, for instance that $B \setminus B_0$ be a finite union of smooth submanifolds, or that f should degenerate in a controllable way. In fact, as it stands Definition 4.1.2 is almost trivial. However, we do not need to worry about this here since we do not want to prove any results about arbitrary given fibrations, but instead we are going to *construct* fibrations which will have a number of good properties as a *result* of the way we construct them.

Definition 4.1.3 *Given fibre bundles (fibrations) $f\colon X \to B$ and $f'\colon X' \to B$, a diffeomorphism $\varphi\colon X \to X'$ is a <u>diffeomorphism of fibre bundles</u> <u>(fibrations)</u> if $f' \circ \varphi = f$. In case X and X' have sections $\sigma\colon B \to X$ and $\sigma'\colon B \to X'$, we call φ a <u>diffeomorphism of fibre bundles with section</u> <u>(fibrations with section)</u> if, in addition, $\sigma' = \varphi \circ \sigma$. For a Lie group G, an <u>isomorphism of principal G-bundles</u> is a G-equivariant diffeomorphism of the underlying fibre bundles.*

Thus we adopt the convention that a diffeomorphism of fibre bundles or fibrations (with section) is meant to cover the identity on the base space, unless stated otherwise.

Affine and linear structures on T^n-bundles

Since our ultimate goal is to dualise torus fibrations let us recall the notion of *dual tori.* If V is a vector space and $\Lambda \subset V$ a lattice of maximal rank, the dual of the torus $T = V/\Lambda$ is the torus $T^* = V^*/\Lambda^*$ where Λ^* is the lattice of linear forms on V which take integral values on Λ. So T and T^* are diffeomorphic as *manifolds*, but they inherit functorially different, *dual linear structures* from V and V^*. Thus we first need to equip the fibres of a T^n-bundle with a linear structure to have a meaningful notion of dual fibres.

Definition 4.1.4 *Let $f\colon X \to B$ be a T^n-bundle. An affine structure on X is a trivialisation $\{\varphi_i\colon f^{-1}(U_i) \to U_i \times \mathbb{T}^n\}$ with respect to a suitable cover $(U_i)_i$ of B so that the transition functions take values in $\mathrm{Aff}(\mathbb{T}^n)$, the group of affine linear transformations of $\mathbb{T}^n$. An affine structure is called a linear structure if there exists a trivialisation with transition functions in the subgroup $\mathrm{GL}(n,\mathbb{Z}) \subset \mathrm{Aff}(\mathbb{T}^n)$. A T^n-bundle together with an affine (linear) structure is called an affine (linear) T^n-bundle.*

A linear T^n-bundle is just an affine T^n-bundle with zero section. The construction in this chapter will produce smooth T^n-fibrations $f\colon X \to B$ with a natural affine structure away from its singularities. On an affine torus T, there is a preferred representative for every deRham cohomology class, given by a *linear differential*. Let us choose an origin $p_0 \in T$ and multi-valued linear functions

$$x = (x_1, \ldots, x_n)\colon T \to \mathbb{T}^n \text{ with } x(p_0) = 0.$$

Then for some integral basis $\alpha_1, \ldots, \alpha_n$ for $H_1(T,\mathbb{Z})$ we have $dx_i[\alpha_j] = \delta_{ij}$ and

$$\begin{array}{ccc} T & \longrightarrow & \dfrac{\mathrm{H}_1(T,\mathbb{R})}{\mathrm{H}_1(T,\mathbb{Z})} \cong \mathrm{H}_1(T,\mathbb{R}/\mathbb{Z}) \\ p & \longmapsto & \displaystyle\sum_i x_i(p)\alpha_i \end{array}$$

is an isomorphism of linear tori. This carries over to linear T^n-bundles.

Notation 4.1.5 *Let $f\colon X \to B$ be a fibre bundle. For $p \in \mathbb{N}_0$ let $\mathcal{H}_p(f,\mathbb{R}) \to B$ be the vector bundle whose fibre at $b \in B$ is $H_p(F_b,\mathbb{R})$, and let $\mathcal{H}^p(f,\mathbb{R}) \to B$ be the vector bundle whose fibre at $b \in B$ is $H^p(F_b,\mathbb{R})$, where $F_b = f^{-1}(b)$. We define similarly the fibre bundles $\mathcal{H}^p(f,R)$ and $\mathcal{H}_p(f,R)$ for $R = \mathbb{Z}$ and $R = \mathbb{R}/\mathbb{Z}$.*

We understand the sections of $\mathcal{H}^p(f,\mathbb{R})$ to be smooth sections of a vector bundle, while we write $R^p f_* \mathbb{R}$ for the local system whose sections are locally constant. Then for every linear T^n-bundle $f\colon X \to B$ with section, we can identify X with the associated bundle of affine tori $\mathcal{H}_1(f,\mathbb{R}/\mathbb{Z}) \to B$ by the above construction.

General T^n-bundles

We digress for a moment and ask whether given an arbitrary T^n-bundle $f\colon X \to B$ we can always equip it with an affine or linear structure. The answer is yes for $n \leq 3$, due to the following theorem.

Theorem 4.1.6 *Let $n \leq 3$ and $\iota\colon \mathbb{T}^n \hookrightarrow \mathrm{Diff}_0(\mathbb{T}^n)$ be the inclusion of the n-torus into its own group of diffeomorphisms with trivial action on homology. Then ι is a homotopy equivalence.*

For $n = 1$, this is trivial, for $n = 2$ it has been well-known for some time and the proof can be found, for instance, in Earle-Eells [EE69]. For $n = 3$ the result has been proved by Hatcher several years ago, and a new version of his proof can be found in [Hat99]. Nothing at all is known whether or not this is true for $n = 4$. It is however known that the statement is not true for all $n \in \mathbb{N}$.

Corollary 4.1.7 *For $n \leq 3$, every T^n-bundle carries an affine structure.*

Proof. The short exact sequence

$$\{1\} \longrightarrow \mathrm{Diff}_0(\mathbb{T}^n) \longrightarrow \mathrm{Diff}(\mathbb{T}^n) \longrightarrow \mathrm{GL}(n, \mathbb{Z}) \longrightarrow \{1\}$$

is split and presents the diffeomorphism group of $\mathbb{T}^n$ as a semi-direct product

$$\mathrm{Diff}(T^n) \cong \mathrm{Diff}(\mathbb{T}^n) \cong \mathrm{Diff}_0(\mathbb{T}^n) \rtimes \mathrm{GL}(n, \mathbb{Z}).$$

For any base manifold B, the T^n-bundles over B are classified up to isomorphism by $[B, \mathsf{B}\,\mathrm{Diff}(\mathbb{T}^n)]$, the set of homotopy classes of maps $B \to \mathsf{B}\,\mathrm{Diff}(\mathbb{T}^n)$ to the classifying space. By Theorem 4.1.6, $\iota\colon \mathbb{T}^n \to \mathrm{Diff}_0(\mathbb{T}^n)$ is a homotopy equivalence and thus

$$[B, \mathsf{B}\,\mathrm{Diff}(\mathbb{T}^n)] \cong [B, \mathsf{B}(\mathbb{T}^n \rtimes \mathrm{GL}(n, \mathbb{Z}))].$$

The latter is naturally the space of affine T^n-bundles because the action of $\mathbb{T}^n \rtimes \mathrm{GL}(n, \mathbb{Z})$ on $\mathbb{T}^n$ preserves the affine structure of $\mathbb{T}^n$. So for $n \leq 3$, every T^n-bundle is diffeomorphic to an affine T^n-bundle. $\square$

Monodromy, orientation, sections

We want to argue that affine T^n-bundles with section are classified by their monodromy representation and describe the obstruction for an affine T^n-bundle to admit a section. Some of this is well-known and only included for the sake of completeness.

Definition 4.1.8 *Let M and M' be modules over a ring R and G a group. Two homomorphisms $\varrho\colon G \to \mathrm{Aut}_R(M)$ and $\varrho'\colon G \to \mathrm{Aut}_R(M')$ are equivalent if there is an isomorphism of R-modules $A\colon M' \to M$ such that for all $g \in G$ and $\xi \in M'$ we have $\varrho(g)A\xi = A\varrho'(g)\xi$. A representation (of G on M over R) is an equivalence class of homomorphisms $G \to \mathrm{Aut}_R(M)$.*

The most important situation will be the monodromy representation associated to a fibre bundle.

Definition 4.1.9 *Let $f\colon X \to B$ be a fibre bundle and $F_b = f^{-1}(b)$ a fibre for some fixed $b \in B$. Let $p \in \mathbb{N}$. Then $h\colon \mathcal{H}_p(f,\mathbb{Z}) \to B$ is a covering space of (B,b) with structure group $H_p(F_b,\mathbb{Z})$. Consequently, for every loop $\gamma\colon [0,1] \to B$ with $\gamma(0) = \gamma(1) = b$ and every $\xi \in H_p(F_b,\mathbb{Z}) = h^{-1}(b)$ there is a unique lift $\hat{\gamma}\colon [0,1] \to \mathcal{H}_*(f,\mathbb{Z})$ with $\hat{\gamma}(0) = \xi$. We define $\varrho(\gamma) = \hat{\gamma}(1) \in H_p(F_b,\mathbb{Z})$. Then $\varrho(\gamma)$ depends only on the homotopy class of γ and therefore induces a linear action of $\pi_1(B,b)$ on $H_p(F_b,\mathbb{Z})$ called the monodromy action. Equivalently this is the holonomy of the Gauß-Manin connection on $\mathcal{H}_p(f,\mathbb{R})$.*

An affine T^n-bundle with section is determined up to diffeomorphism by its monodromy. This is well-known and we formulate it in the following theorem.

Theorem 4.1.10 *Let B be a manifold with fundamental group $G = \pi_1(B)$. There is, via monodromy, the following bijective correspondence:*

$$\begin{gathered}\{\text{ representations } \ \pi_1(B) \to \mathrm{GL}(n,\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^n)\ \}\\ \updownarrow \\ \{\text{ diffeomorphism classes of affine } T^n\text{-bundles with section over } B\ \}\end{gathered}$$

Proof. Let $\varrho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ be a homomorphism. Let $p\colon \widehat{B} \to B$ be the universal cover. $\widehat{B}$ is a principal G-bundle, and we can associate to it the fibre bundle $X = \widehat{B} \times_\varrho \mathbb{T}^n$ where G acts on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ via ϱ. This will have a section since G acts linearly (preserving the origin), and the monodromy is given by ϱ. Suppose $\varrho'\colon G \to \mathrm{GL}(n,\mathbb{Z})$ is an equivalent homomorphism, that is, there is an $A \in \mathrm{GL}(n,\mathbb{Z})$ with $A\varrho(g) = \varrho'(g)A$ for all $g \in G$, and $X' = \widehat{B} \times_{\varrho'} \mathbb{T}^n$. We define $\hat{\Phi}\colon \widehat{B}\times\mathbb{T}^n \to \widehat{B}\times\mathbb{T}^n$ by $\mathrm{id} \times A$, sending $(\hat{b},\hat{\xi}) \mapsto (\hat{b}, A\hat{\xi})$. This covers a diffeomorphism $\Phi\colon X \to X'$ which maps $(b,\xi) \mapsto (p' \circ \hat{\Phi})(\hat{b},\hat{\xi})$ where $(\hat{b},\hat{\xi}) \in p^{-1}((b,\xi))$ is any lift. It is well-defined because for any other lift $(\hat{b}_0,\hat{\xi}_0)$ there is a $g \in G$ with $(\hat{b}_0,\hat{\xi}_0) = (g\hat{b}, \varrho(g)\hat{\xi})$ and so $(g\hat{b}, A\varrho(g)\hat{\xi}) = (g\hat{b}, \varrho'(g)A\hat{\xi})$, and the p'-projections agree.

Conversely, given an affine T^n-bundle $f\colon X \to B$ with section σ we can identify it with $\mathcal{H}_1(f,\mathbb{R}/\mathbb{Z})$. For a fixed $b \in B$, trivialise $H_1(F_b,\mathbb{Z}) \cong \mathbb{Z}^n$ which represents the monodromy by a homomorphism $\varrho\colon G \to \mathrm{GL}(n,\mathbb{Z}) \cong \mathrm{Aut}_\mathbb{Z}(H_1(F_b,\mathbb{Z}))$, and the original bundle is diffeomorphic to $\widehat{B} \times_\varrho \mathbb{T}^n$. If $\varphi\colon X \to X'$ is a diffeomorphism of affine torus bundles with section then the isomorphism $A = (\varphi|_{F_b})_*\colon H_1(F_b,\mathbb{Z}) \to H_1(F'_b,\mathbb{Z})$ satisfies $A \circ \varrho' = \varrho \circ A$ and hence the monodromies are equivalent in $\mathrm{GL}(n,\mathbb{Z})$.

The two constructions relating representations of $\pi_1(B)$ to diffeomorphism classes of affine T^n-bundles with section and vice versa are clearly inverse to each other. □

Without a section the monodromy in a T^n-bundle does not uniquely determine the bundle, there is still a freedom of regluing local trivialisations with a 1-cocycle of affine transformations.

Theorem 4.1.11 *To each affine T^n-bundle $f\colon X^{n+m} \to B^m$ we can associate a class $c(f) \in H^2(B, \mathcal{H}_1(f, \mathbb{Z}))$ (see Notation 4.1.5) which we call the Chern-class of $f\colon X \to B$ such that:*

1. *For each map $\varphi\colon B' \to B$ we have $c(\varphi^* f) = \varphi^* c(f)$.*
2. *The following three statements are equivalent*
   (a) *There is an isomorphism of affine T^n-bundles $X \cong \mathcal{H}_1(f, \mathbb{R}/\mathbb{Z})$,*
   (b) *$X \to B$ has a section,*
   (c) *$c(f) = 0$.*
3. *There is an affine T^n-bundle $f\colon X \to B$ with $c(f) \neq 0$.*
4. *The bundle $f\colon X \to B$ can be reconstructed up to isomorphism from the bundle $\mathcal{H}_1(f, \mathbb{R}/\mathbb{Z})$ and the Chern-class $c(f) \in H^2(B, \mathcal{H}_1(f, \mathbb{Z}))$.*

Proof. Let $f\colon X^{n+m} \to B^m$ be an affine T^n-bundle with monodromy ϱ. Choose a contractible cover $\mathcal{U} = (U_i)_i$ of B and local sections $\sigma_i\colon U_i \to X|_{U_i}$. These sections induce trivialisations

$$\phi_i\colon X|_{U_i} \to \mathcal{H}_1(f, \mathbb{R}/\mathbb{Z})|_{U_i} \tag{4.1}$$

in the following way: For each $b \in B$ there is a natural isomorphism between $\mathcal{H}_1(f, \mathbb{R})_b = H_1(F_b, \mathbb{R})$ and the space of linear vector fields on F_b. A linear vectorfield $v \in \Gamma(TF_b)$ operates on the fibre F_b by a diffeomorphism $\Phi(v, 1)$, given by integration of the flow of v up to time $t = 1$. Given any $x \in F_b$ we have $\Phi(v, 1)(x) = x$ if and only if $v \in H_1(F_b, \mathbb{Z})$ is integral in which case $\Phi(v, 1) = \mathrm{id}$, and so we get a fixed-point free and transitive action of

$$\frac{\mathrm{H}_1(F_b, \mathbb{R})}{\mathrm{H}_1(F_b, \mathbb{Z})} \cong \mathrm{H}_1(F_b, \mathbb{R}/\mathbb{Z}) \cong \mathcal{H}_1(f, \mathbb{R}/\mathbb{Z})_b$$

on F_b for each $b \in B$. So for each $b \in U_i$ and $x \in F_b$ there is a unique $\phi_i(x) \in \mathcal{H}_1(f, \mathbb{R}/\mathbb{Z})_b$ so that $x = \phi_i(x) \cdot \sigma_i(b)$. This is the trivialisation (4.1). We derive from it the following definition of the Chern-class: There is a 1-chain $\tau_{ij} \in C^1(\mathcal{U}, \mathcal{H}_1(f, \mathbb{R}/\mathbb{Z}))$ defined as $\tau_{ij} = \sigma_i/\sigma_j$ (which is meant to say that $\sigma_i = \tau_{ij} \cdot \sigma_j$) for which the boundary condition $\tau_{ij} + \tau_{jk} + \tau_{ki} = 1$ is obviously satisfied. Any other choice of local sections of X over $\mathcal{U}$ is of the form $\sigma_i' = \rho_i \cdot \sigma_i$ and thus $\tau_{ij}' = \tau_{ij} + \rho_i - \rho_j$, or equivalently $\tau = \tau' + \delta\rho$. Hence f defines a unique cohomology class $\tilde{c}(f) = [\tau] \in H^1(\mathcal{U}, \mathcal{H}_1(f, \mathbb{R}/\mathbb{Z}))$. Now in the long exact sequence

$$\mathrm{H}^1(\mathcal{U}, \mathcal{H}_1(f, \mathbb{R})) \longrightarrow \mathrm{H}^1(\mathcal{U}, \mathcal{H}_1(f, \mathbb{R}/\mathbb{Z})) \xrightarrow{\delta^*} \mathrm{H}^2(\mathcal{U}, \mathcal{H}_1(f, \mathbb{Z})) \longrightarrow \mathrm{H}^2(\mathcal{U}, \mathcal{H}_1(f, \mathbb{R}))$$

the coboundary map δ^* is an isomorphism since $\mathcal{H}_1(f,\mathbb{R})$ is a sheaf of (continuous) sections of a vector bundle, hence a soft sheaf, and $H^p(\mathcal{U},\mathcal{H}_1(f,\mathbb{R})) = \{0\}$ for all $p > 0$. Therefore we can define the Chern-class of f to be $c(f) = \delta^*(\tilde{c}(f))$.

(1) The naturality of this definition is implicit in the construction, if $\varphi\colon B' \to B$ is any map and $f'\colon X' = \varphi^*X \to B'$ the pull-back of X we get a cover $\mathcal{U}' = \phi^{-1}\mathcal{U}$ and local sections $\sigma_i' = \sigma_i \circ \varphi$. We get $\tau_{ij}' = \tau_{ij} \circ \varphi$ and thus $[\tau'] = \varphi^*[\tau]$ which naturally represents a class in $H^1(\mathcal{U}',\mathcal{H}_1(f',\mathbb{R}/\mathbb{Z}))$, and consequently $c(\varphi^* f) = c(f') = \varphi^* c(f)$.
(2) The implication $(a) \Rightarrow (b)$ is trivial since $\mathcal{H}_1(f,\mathbb{R}/\mathbb{Z})$ has a zero-section. For $(b) \Rightarrow (c)$ observe that if $X \to B$ has a section σ we can take $\sigma_i = \sigma|U_i$, hence $\tau = 0$ and consequently $c(f) = \delta([\tau]) = 0$. For $(c) \Rightarrow (a)$ choose local sections σ_i of $X|_{U_i}$ and construct the 1-cocycle τ as before. Since $c(f) = \delta^*([\tau]) = 0$ and δ^* is an isomorphism we know $[\tau] = 0$ or equivalently $\tau = \delta\rho$ for some 0-chain ρ. Thus $\sigma_i' = (-\rho_i)\cdot\sigma_i$ is the restriction to U_i of a global section σ', and the map sending $x \in F_b$ to the unique element $\phi'(x) \in H_1(F_b,\mathbb{R}/\mathbb{Z}) \cong \mathcal{H}_1(f,\mathbb{R}/\mathbb{Z})_b$ with $x = \phi'(x)\cdot\sigma'(b)$ is a global diffeomorphism of affine T^n-bundles

$$\phi'\colon X \to \mathcal{H}_1(f,\mathbb{R}/\mathbb{Z}).$$

(3) This follows from (2) since there are certainly affine T^n-bundles which do not admit sections (such as every non-trivial principal S^1-bundle).

(4) Let τ be a 1-cocycle with $\delta^*([\tau]) = c(f)$. Then

$$X \cong \frac{\bigcup_i \mathcal{H}_1(f,\mathbb{R}/\mathbb{Z})|_{U_i} \times \{i\}}{\sim}$$

where $(\xi,i) \sim (\eta,j)$ if and only if ξ and η lie in the same fibre of $\mathcal{H}_1(f,\mathbb{R}/\mathbb{Z})$ and $\xi = \eta + \tau_{ij}$. □

This theorem is an adaptation of Theorem 2.1 in Duistermaat's paper on global action-angle coordinates [Dui80] in a completely integrable system $X \to B$ without singular fibres. In this case the symplectic structure is used to induce an affine T^n-bundle structure on the total space X together with a linear action on the fibres. Once we have this affine structure, the definition of the Chern-class is essentially the same as above. For a non-singular integrable system, the $\mathbb{Z}^n$-bundle $\mathcal{H}_1(f,\mathbb{Z})$ can be identified with a lattice $\Lambda \subset T^*B$ so that in this case the Chern class can also be thought of as an element in $H^2(B,\Lambda)$. If ϱ is trivial, we have

$$\mathcal{H}_1(f,\mathbb{Z}) \cong B \times \mathbb{Z}^n \tag{4.2}$$

and $c(f) \in H^2(B,\mathbb{Z}^n) = H^2(B,\mathbb{Z})^n$. However, the identification of the Chern class with a sequence of n elements in $H^2(B,\mathbb{Z})$ is not canonical since it depends on a choice of trivialisation (4.2), except if $n = 1$ in which case a choice of orientation on B and X determines a canonical trivialisation in (4.2), and $c(f)$ is of course the ordinary Chern-class of the (principal) circle bundle $S^1 \to X \to B$.

Proposition 4.1.12 *Let $f\colon X^{m+n} \to B^m$ be an affine T^n-bundle whose base B^m is orientable. Then X^{m+n} is orientable if and only if the monodromy ϱ takes values in* $\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{GL}(n,\mathbb{Z})$.

Proof. Let $\mathcal{V} \to B$ be the vertical bundle of $f\colon X \to B$. We have

$$\Lambda^{m+n}T^*X \cong \Lambda^n\mathcal{V}^* \otimes f^*\Lambda^m T^*B.$$

Since B is orientable, $f^*(\Lambda^m T^*B)$ is trivial, and T^*X is orientable if and only if $\Lambda^n\mathcal{V}^*$ is trivial. But $\wedge^n\varrho^* = \det(\varrho^*)$ and

$$\Lambda^n\mathcal{V}^* \cong f^*(\widehat{B} \times_{\det\varrho^*} \mathbb{R}).$$

So orientability is equivalent to the triviality of $\det(\varrho^*) = \det(\varrho)$, that is the monodromy representation ϱ taking values in $\mathrm{SL}(n,\mathbb{Z})$. □

Now let $\varrho\colon \pi_1(B) \to \mathrm{GL}(n,\mathbb{Z})$ be any representation and define $X_\varrho = \widehat{B} \times_\varrho \mathbb{T}^n$. We let $H^*(B,\varrho)$ be the cohomology of B with coefficients in ϱ. Let

$$\mathcal{Z}(\varrho) = \{\, Z \in \mathrm{GL}(n,\mathbb{Z}) \mid [\varrho(\gamma), Z] = 0 \ \text{ for all } \ \gamma \in \pi_1(B) \,\}.$$

If $\varrho(\pi_1(B)) \subset \mathrm{SL}(n,\mathbb{Z})$ define

$$\mathcal{Z}^+(\varrho) = \mathcal{Z}(\varrho) \cap \mathrm{SL}(n,\mathbb{Z}).$$

Then $\mathcal{Z}(\varrho)$ can be naturally thought of as the automorphisms of the linear T^n-bundle X_ϱ. If $\varrho(\pi_1(B)) \subset \mathrm{SL}(n,\mathbb{Z})$, then ϱ is orientable by Proposition 4.1.12, and $\mathcal{Z}^+(\varrho)$ are the orientation preserving automorphisms of X_ϱ.

Theorem 4.1.13 *The isomorphism classes of affine T^n-bundles with monodromy ϱ correspond bijectively to the elements of $H^2(B,\varrho)/\mathcal{Z}(\varrho)$. If X_ϱ is orientable, then $H^2(B,\varrho)/\mathcal{Z}^+(\varrho)$ parametrises the isomorphism classes of oriented affine T^n-bundles with monodromy ϱ.*

Proof. Let $f\colon X \to B$ be any affine T^n-bundle with monodromy ϱ and let $c(f) \in H^2(B,\mathcal{H}_1(f,\mathbb{Z}))$ be its Chern-class. $\mathcal{H}_1(f,\mathbb{Z})$ is a $\mathbb{Z}^n$-bundle with section, and so there is an isomorphism

$$\Psi\colon \widehat{B} \times_\varrho \mathbb{Z}^n \longrightarrow \mathcal{H}_1(f,\mathbb{Z}). \tag{4.3}$$

This induces an isomorphism $\Psi^*\colon H^2(B, \mathcal{H}_1(f, \mathbb{Z})) \to H^2(B, \varrho)$ sending $c(f)$ to some element $\tilde{\mathbf{c}}(f) \in H^2(B, \varrho)$. If Ψ' is any other choice of isomorphism in (4.3) then $\Psi^{-1} \circ \Psi'$ is an automorphism of $\widehat{B} \times_\varrho \mathbb{Z}^n$ and therefore the image

$$\mathbf{c}(f) = \tilde{\mathbf{c}}(f) \bmod \mathcal{Z}(\varrho) \in \mathrm{H}^2(B, \varrho)/\mathcal{Z}(\varrho)$$

is well-defined. Conversely given an element of $H^2(B, \varrho)/\mathcal{Z}(\varrho)$ we can choose representatives and construct an affine T^n-bundle which is well-defined up to isomorphism. The argument for $\mathcal{Z}^+(\varrho)$ is analogous. □

For example, if $B = S^2$ and $n = 1$ then ϱ is trivial and an affine T^1-bundle is just a principal S^1-bundle, hence orientable. Now $H^2(S^2, \varrho) = \mathbb{Z}$ and $\mathcal{Z}(\varrho) = \{1, -1\}$, $\mathcal{Z}^+(\varrho) = \{1\}$. So the Chern-class of an *oriented* affine T^1-bundle over S^2 becomes an element of $H^2(S^2, \varrho)/\mathcal{Z}^+(\varrho) = H^2(S^2, \mathbb{Z})$ which is of course the ordinary Chern-class of a principal S^1-bundle over S^2, while its Chern-class as an affine bundle (ignoring the orientation) is an element of $H^2(S^2, \mathbb{Z})/\{\pm 1\}$, reflecting the fact that principal S^1-bundles with opposite Chern-class are diffeomorphic manifolds.

For two oriented T^n-bundles $X \to B$ and $X' \to B$ together with $b \in B$, choose trivialisations

$$\psi\colon \mathrm{H}_1(F_b, \mathbb{Z}) \to \mathbb{Z}^n \quad \text{and} \quad \psi'\colon \mathrm{H}_1(F'_b, \mathbb{Z}) \to \mathbb{Z}^n$$

such that $\psi' \circ \psi^{-1}$ is orientation preserving. Via ψ and ψ' we can identify the monodromies with homomorphisms

$$\varrho,\ \varrho'\colon \pi_1(B, b) \longrightarrow \mathrm{SL}(n, \mathbb{Z}).$$

Proposition 4.1.14 *There is a diffeomorphism of fibre bundles $\varphi\colon X \to X'$ which preserves the orientations if and only if ϱ and ϱ' are conjugate in* $\mathrm{SL}(n, \mathbb{Z})$.

Proof. A diffeomorphism $\varphi\colon X \to X'$ of affine T^n-bundles (which by definition acts as the identity on the base) is orientation preserving if and only if $h_\varphi = (\varphi|F_b)_*\colon H_1(F_b, \mathbb{Z}) \to H_1(F'_b, \mathbb{Z})$ preserves the orientations induced by ψ and ψ'. But $\varrho = h_\varphi^{-1} \circ \varrho' \circ h_\varphi$ and hence this is equivalent to ϱ and ϱ' being conjugate in $\mathrm{SL}(n, \mathbb{Z})$. □

4.1.2 Knots and torus fibrations

We now introduce the class of *good* T^3-fibrations which display particularly simple singularities. They form a subclass of Gross' *well-behaved* fibrations [Gro99b].

Knots and links

We recall the definitions of knots and links. We will work in the smooth category even though, from a knot-theorist's viewpoint, it may be more natural to give definitions in the category of piecewise linear topological spaces and maps.

Definition 4.1.15 *A knot is an equivalence class of embeddings $S^1 \to S^3$ where two such embeddings $\mathfrak{k}, \mathfrak{k}'$ are identified if and only if there are (not necessarily orientation preserving) diffeomorphisms φ of S^3 and φ' of S^1 with $\varphi \circ \mathfrak{k} \circ \varphi' = \mathfrak{k}'$. A link with μ components is an equivalence class of embeddings $S^1 \sqcup \cdots \sqcup S^1 \to S^3$ of μ copies of S^1 where the equivalence relation is defined accordingly. To each knot or link $\mathfrak{k}$ we associate its knot group (link group) $G(\mathfrak{k}) = \pi_1(S^3 \setminus \mathfrak{k})$.*

We will sometimes be imprecise and not distinguish between a knot as an equivalence class of embeddings and a particular embedding $\mathfrak{k}\colon S^1 \to S^3$ representing it, or even just the image $\mathfrak{k}(S^1) \subset S^3$. Also, the equivalence relation as given in Definition 4.1.15 identifies every knot $\mathfrak{k}$ with its *inverse* $\bar{\mathfrak{k}} = \mathfrak{k} \circ \varphi'$ and its *mirror* $\mathfrak{k}^* = \varphi \circ \mathfrak{k}$, obtained by composing $\mathfrak{k}$ with an orientation reversing diffeomorphism φ' of S^1, respectively an orientation reversing diffeomorphism φ of S^3. This is a coarser notion of equivalence than the one commonly found in the literature where it is usually required that the diffeomorphisms φ, φ' in Definition 4.1.15 be always orientation preserving. In this latter case, a knot may or may not be equivalent to its inverse (respectively, mirror); if it is, it is called *invertible* (respectively, *amphichiral*). For instance, torus knots which we will consider later on are invertible but not amphichiral. However, in our applications, such a distinction is unnecessary and we consider $\mathfrak{k}, \bar{\mathfrak{k}}, \mathfrak{k}^*$ and $\bar{\mathfrak{k}}^*$ *a priori* to be the same knots.

Singularities

We intend to construct T^3-fibrations $f\colon X \to S^3$ which are only allowed to develop one particularly simple type of singularity and whose discriminant locus is a smoothly embedded knot or link $\mathfrak{l} \subset S^3$. More precisely, for each critical value $x \in \mathfrak{l}$ we want there to be an open ball $U \subset S^3$ around x and a commutative diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\Phi} & M^4 \times (S^1 \times \mathbb{R}) \\ \downarrow & & \downarrow{\scriptstyle (p,\mathrm{pr}_2)} \\ U & \xrightarrow{\phi} & D^2 \times \mathbb{R} \end{array} \tag{4.4}$$

where $p\colon M^4 \to D^2$ is topologically equivalent to an I_1-singularity with a focus-focus singular fibre F_0 over $0 \in D$, like the Ooguri-Vafa fibration, $\mathrm{pr}_2\colon S^1 \times \mathbb{R} \to \mathbb{R}$ is the projection onto the second factor, and $\phi(U \cap \mathfrak{l}) = \{0\} \times \mathbb{R}$.

Symplectic structure

We can define a symplectic structure on our model singularity (4.4) by setting

$$\omega = \omega_3 + \mathrm{d}\vartheta \wedge \mathrm{d}r \tag{4.5}$$

where ω_3 is the (lift to the product $M^4 \times (S^1 \times \mathbb{R})$ of the) canonical symplectic structure on M^4 given in (3.10), and (ϑ, r) is the projection $M^4 \times (S^1 \times \mathbb{R}) \to S^1 \times \mathbb{R}$. With respect to this symplectic structure ω, the T^3-fibration f is Lagrangian.

Monodromy

In the model singularity (4.4), the local monodromy action of a generator γ of $\pi_1(U \setminus \mathfrak{l}) \cong \mathbb{Z}$ on the homology $H_1(F, \mathbb{Z})$ of a smooth fibre $F \cong T^3$ is given, with respect to a suitable choice of basis for $H_1(F, \mathbb{Z}) \cong \mathbb{Z}^3$, by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{Z}). \tag{4.6}$$

This leads to a restriction on the possible monodromy $\varrho\colon \pi_1(S^3 \setminus \mathfrak{l}) \to \mathrm{SL}(H_1(F, \mathbb{Z}))$ in a T^3-fibration $f\colon X \to S^3$ degenerating over $\mathfrak{l} \subset S^3$ with all singularities locally diffeomorphic to (4.4): A necessary condition for an affine T^3-bundle $f_0\colon X_0 \to S^3 \setminus \mathfrak{l}$ to allow a compactification of the local form (4.4) is that every *meridian* in $G(\mathfrak{l}) = \pi_1(S^3 \setminus \mathfrak{l})$ must be mapped into the conjugacy class $\mathcal{C}(A)$ of A. (See page 101 for a picture of the trefoil knot and its meridians.) We thus make the following definition.

Definition 4.1.16 *Let $\mathfrak{l} \subset S^3$ be a knot or link. By an M-homomorphism we mean a homomorphism $\varrho\colon G \to \mathrm{GL}(3, \mathbb{Z})$ of the group $G = \pi_1(S^3 \setminus \mathfrak{l})$ of $\mathfrak{l}$ which has the property that each meridian is mapped to $\mathcal{C}(A)$. More generally, if Λ is a free $\mathbb{Z}$-module of rank $\mathrm{rk}(\Lambda) = 3$, we call $\varrho\colon G \to \mathrm{GL}(\Lambda)$ an M-homomorphism if $\psi\varrho\psi^{-1}$ is an M-homomorphism for some (hence any) trivialisation $\psi\colon \Lambda \to \mathbb{Z}^3$. Two homomorphisms $\varrho, \varrho'\colon G \to \mathrm{GL}(\Lambda)$ are conjugate if there is a fixed $A \in \mathrm{GL}(\Lambda)$ with $\varrho' = A^{-1}\varrho A$. An M-representation is a conjugacy class of M-homomorphisms.*

So the monodromy representations which arise in T^3-fibrations whose singularities are of the local form (4.4) are M-representations.

Local sections

We are interested in constructing fibrations with section, so we check that our local model is a fibration with smooth section:

Proposition 4.1.17 *The Ooguri-Vafa fibration $p\colon M^4 \to D^2$ has a smooth section. Any two smooth sections are smoothly homotopic.*

Proof. It is enough to show that for small enough $\varepsilon > 0$ there is a smooth local section of M^4 on the ε-disk D^2_ε around 0, and any two smooth local sections σ_1, σ_2 on D^2_ε are smoothly homotopic. Observe that $\mathrm{rk}(p_{*m}) = 2$ for all $m \in M^4 \setminus \{m_0\}$, where $m_0 \in F_0$ is the focus-focus singular point. In particular, the rank of f_* is maximal along the singular fibre F_0, except in $m_0 \in F_0$. So through every $m \in F_0 \setminus \{m_0\}$ we can find a smoothly embedded 2-disk which is transversal to p and which will be the graph of a smooth section. Finally let σ_1, σ_2 be two local sections. Smoothness implies $\sigma_i(0) \neq m_0$ for $i = 1, 2$, so there is a $\delta > 0$ such that the graphs of σ_1, σ_2 do not meet a δ-ball $B_\delta(m_0)$ around the focus-focus singular point. For $\varepsilon > 0$ small enough, $M|_{D_\varepsilon} \setminus B_\delta(m_0) \cong D^2_\varepsilon \times (\mathbb{R} \times S^1)$, and p becomes projection onto D^2_ε. So our local sections can be identified with smooth functions $D^2_\varepsilon \to \mathbb{R} \times S^1$, and any two such functions are smoothly homotopic since D^2_ε is contractible and $\mathbb{R} \times S^1$ is connected. ☐

The compactification

We can now explain how to construct T^3-fibrations by compactifying T^3-bundles over link-complements.

Notation 4.1.18 *For the rest of this paragraph we adopt the notation of [Gro99b]. If $f\colon X \to B$ is a T^n-fibration we let $\Delta \subset B$ be the discriminant locus of f, that is the set of critical values of f. We denote the smooth part of f by $f_0\colon X_0 \to B_0$. This is a smooth T^n-bundle over $B_0 = B \setminus \Delta$. Moreover, we write $C = f^{-1}(\Delta) = X \setminus X_0$ for the union of the singular fibres and* $\mathrm{Crit}(f) \subset C$ *for the set of critical points of f. We let $X^\# = X \setminus \mathrm{Crit}(f)$ and $f^\#\colon X^\# \to B$ is the restriction $f|_{X^\#}$. Finally $\mathcal{V} \to X^\#$ is the vertical bundle, $\mathcal{V} = \ker(f^\#_*)$.*

Theorem 4.1.19 *Let $f\colon X \to B$ be the model singularity (4.4) with $X \cong M^4 \times (S^1 \times \mathbb{R})$ and $B \cong D^2 \times \mathbb{R}$. Then every automorphism $\varphi_0\colon X_0 \to X_0$ of affine T^3-bundles with section extends to a diffeomorphism $\varphi^\#\colon X^\# \to X^\#$ of fibrations with section.*

Proof. Let $\omega \in \mathcal{A}^2(X)$ be the symplectic form given in (4.5) which makes $f^\#$ into a submersion with Lagrangian fibres. We therefore get from ω the usual isomorphisms $\iota_x\colon \mathcal{V}_x \cong T^*_{f(x)}B$ for all $x \in X^\#$, sending a vertical vector $\xi_x \in \mathcal{V}_x = \ker(f^\#_{*x})$ to the unique $\theta \in T^*_{f(x)}B$ so that $(f^*\theta)(x) = \iota(\xi_x)\omega$. If $\sigma\colon B \to X$ is a smooth section of the fibration, then $\sigma(B) \subset X^\#$ and we can define $\Xi = \sigma^*\mathcal{V}$. Then

$$\Xi_b = \mathcal{V}_{\sigma(b)} \overset{\iota_{\sigma(b)}}{\cong} T^*_{f(\sigma(b))}B = T^*_b B. \tag{4.7}$$

As in the case of a non-singular integrable system, we call a vertical vector field $v \in C^\infty(\mathcal{V})$ a *linear vector field* if there is a one-form $\theta \in \mathcal{A}^1(B)$ so that $\iota(v)\omega = f^*\theta$, or equivalently $v(x) = \iota_x^{-1}\theta(f(x))$ for all $x \in X^\#$. Combining with (4.7) we see that for all $b \in B$ we can then think of Ξ_b as the space of linear vector fields on $F_b = f^{-1}(b)$. Since we have restricted to $X^\#$ where the projection map $f^\#$ has maximal rank this includes the singular fibres.

Linear vector fields commute and are nowhere vanishing because ω is non-degenerate and $d\omega = 0$. Thus there is a a fibre-preserving action of $\mathcal{A}^1(B)$ on $X^\#$ which is locally free and transitive on each fibre. Thus for each $b \in B$ the isotropy subgroup $\Lambda_b \subset \Xi_b$ of the action of Ξ_b on F_b is discrete and $F_b \cong \Xi_b/\Lambda_b$. Then $\Xi_b \cong \widehat{F}_b$ is the universal cover of F_b and $\Lambda_b = \pi_1(F_b, \sigma(b)) = H_1(F_b, \mathbb{Z})$. Thus $\Lambda|B_0$ coincides with the integral lattice $\mathcal{H}_1(f_0, \mathbb{Z})$ of the smooth part of the fibration, and the monodromy in the $\mathbb{Z}^3$-bundle $\Lambda|B_0$ where $B_0 = B \setminus \Delta$ is the standard monodromy ϱ of the T^3-bundle $X_0 \to B_0$.

An automorphism φ of affine T^3-bundles with section is determined by its (linear) action on a fibre. Fix a basepoint $b_0 \in B_0$ and a trivialisation $\psi\colon \Xi \to B \times \mathbb{R}^3$ so that $\psi(\Lambda_{b_0}) = \mathbb{Z}^3$. Then φ is determined by an element $Z \in \mathrm{GL}(3,\mathbb{Z})$ acting on F_{b_0}. Conversely, such a Z defines an automorphism of affine T^3-bundles if and only if $[Z, \varrho(\gamma)] = 0$ where $\varrho(\gamma) \in \mathrm{GL}(3,\mathbb{Z})$ defines the monodromy action of a generator $\gamma \in \pi_1(B_0, b_0) \cong \mathbb{Z}$ on Λ_{b_0}. For $b \in \Delta$ we have $\Lambda_b \cong \Lambda_b/\Lambda_b^\varrho$, hence Λ_b is Z-invariant and the automorphism $\hat{\varphi}\colon \Xi \to \Xi$ defined by $(b,\xi) \mapsto (b, Z\xi)$ descends to a diffeomorphism $\varphi^\#$ of the quotient $\Xi/\Lambda \cong X^\#$. □

Corollary 4.1.20 *The vertical bundle $\mathcal{V} \subset TX^\#$ of a T^3-fibration $f\colon X \to S^3$ whose singularities are locally of the form (4.4) and $TX^\#$ itself are trivial bundles.*

Proof. The proof of Theorem 4.1.19 shows that $X^\# = \Xi/\Lambda$, so there is a trivial flat connection along $\mathcal{V}|F$ for each fibre F, hence for each $b \in B$ and $x \in F_b = f^{-1}(b)$ we can identify $\mathcal{V}_x \cong \mathcal{V}_{\sigma(b)}$ by parallel transport and thus $\mathcal{V} \cong f^*(\sigma^*\mathcal{V}) = f^*\Xi$ which is trivial as the pull-back of a bundle from S^3 (since $\pi_2(\mathrm{GL}(3,\mathbb{R})) = \{1\}$). Then

$$TX^\# \cong \mathcal{V} \oplus f^*TS^3|_{X^\#} \cong f^*(\Xi \oplus TS^3)|_{X^\#} \tag{4.8}$$

is a trivial vector bundle. □

Theorem 4.1.21 *Let $\mathfrak{l} \subset S^3$ be a link with μ components in S^3 and fix an M-representation $\varrho\colon G \to \mathrm{GL}(3,\mathbb{Z})$ of its group $G = \pi_1(S^3 \setminus \mathfrak{l})$. Let $f_0\colon X_0 \to S^3 \setminus \mathfrak{l}$ be the unique affine T^3-bundle with section associated to ϱ as in Theorem 4.1.10. Then there is a compactification of X_0 to a T^3-fibration $f\colon X \to S^3$ with section which is locally of the form (4.4).*

Proof. Let $\hat{f}_1\colon \hat{X}_1 \to \hat{B}_1 \cong D^2 \times \mathbb{R}$ be the model singularity (4.4) with discriminant locus $\Delta \cong \{0\} \times \mathbb{R}$. Choose $b \in \hat{B}_1 \setminus \Delta$ and a trivialisation $H_1(F_b, \mathbb{Z}) \cong \mathbb{Z}^3$ (where

$F_b = f^{-1}(b)$) so that the monodromy takes values in $\mathrm{SL}(3,\mathbb{Z})$. In particular, the actions of a meridian and a longitude are respectively given by matrices $A = \varrho([m])$ and $Z = \varrho([\ell]) \in \mathrm{SL}(3,\mathbb{Z})$. Since $m\ell m^{-1}\ell^{-1} = 1$ and therefore $[A, Z] = 0$, there is a unique diffeomorphism $\hat{\varphi}$ of fibre bundles with section whose action on F_b is given by Z. By Theorem 4.1.19, $\hat{\varphi}$ extends to a diffeomorphism $\hat{\varphi}^{\#}$ of the fibration $\hat{X}_1^{\#} \to B_1$. Now let $\varphi = \hat{\varphi}^{\#} \circ \tau$ where $\tau\colon \hat{X}_1^{\#} \to \hat{X}_1^{\#}$ is translation in horizontal direction,

$$\tau(x', \vartheta, r) = (x', \vartheta, r+1) \quad \text{for all} \quad (x', \vartheta, r) \in (M^4)^{\#} \times (S^1 \times \mathbb{R}) \cong \hat{X}_1^{\#}.$$

Then φ is fixed point free and hence the quotient $X_1^{\#} = \hat{X}_1^{\#}/\mathbb{Z}$ (where $a \cdot x = \varphi^a(x)$) is a smooth manifold which has the structure of a smooth T^3-fibration over a solid torus $B_1 \cong D^2 \times \mathbb{R}$. There is a unique compactification to a smooth T^3-fibred six-manifold X_1 with singularities locally equivalent to (4.4). The monodromy in the smooth part is given by $\varrho|\langle \ell, m\rangle$. The smooth part of $X_1 \to B_1$ is diffeomorphic, as an affine T^3-fibration, to $X_0|_{\mathcal{N}(\mathfrak{l})\cap(S^3\setminus\mathfrak{l})}$. Hence we can glue together X_0 and X_1 over the intersection $\mathcal{N}(\mathfrak{l}) \cap (S^3 \setminus \mathfrak{l})$ which produces a smooth T^3-fibration with section and the right type of singularities. □

Topology of T^3-fibrations

The question about orientability (or $w_1(X) \in H^1(X, \mathbb{Z}/2)$, the first Stiefel-Whitney class) of a T^3-fibration as above turns out to be very easy:

Proposition 4.1.22 *The total space of a T^3-fibration $f\colon X \to B$ with B orientable and singularities of type (4.4) is orientable if and only if $\varrho(G(\mathfrak{l})) \subset \mathrm{SL}(H_1(F,\mathbb{Z}))$.*

Proof. It follows directly from Proposition 4.1.12 that the smooth T^3-bundle $f|_{B\setminus\Delta}$ is orientable if and only if $\varrho(G) \subset \mathrm{SL}(H_1(F,\mathbb{Z}))$. But $f^{-1}(\Delta) \subset X$ is contained in a finite union of submanifolds of codimension ≥ 2, and so X is orientable if and only if $X_0 = f^{-1}(B \setminus \Delta)$ is. □

Corollary 4.1.23 *A T^3-fibration $f\colon X \to S^3$ whose discriminant locus is a knot or link $\mathfrak{l} \subset S^3$ and whose singularities are of the form (4.4) is orientable.*

Proof. The group $G = \pi_1(S^3 \setminus \mathfrak{l})$ is generated by meridians which must be mapped into $\mathcal{C}(A) \subset \mathrm{SL}(3,\mathbb{Z})$ under the monodromy representation ϱ, hence $\varrho(G) \subset \mathrm{SL}(3,\mathbb{Z})$ and Proposition 4.1.22 applies. □

Definition 4.1.24 *Let $f\colon X \to S^3$ be an affine T^3-fibration with section and with singularities as in (4.4) arising over a knot or link $\mathfrak{l} \subset S^3$. Then we call f a good T^3-fibration.*

Proposition 4.1.25 *The critical locus* $\mathrm{Crit}(f)$ *of a good* T^3*-fibration* $f\colon X \to S^3$ *over a link* $\mathfrak{l} \subset S^3$ *with* μ *components is diffeomorphic to* μ *copies of* T^2*, in particular,* $\mathrm{Crit}(f)$ *is orientable.*

Proof. Without loss of generality assume $\mu = 1$. Then $\mathrm{Crit}(f)$ is a smooth S^1-bundle over $\mathfrak{l} \cong S^1$, so it is enough to show that the monodromy action of $[\mathfrak{l}]$ on the class of a fibre $\gamma \cong S^1$ is trivial. Let $\iota\colon \gamma \to X$ be inclusion and take a smooth T^3-fibre $F \subset X$ of f together with a suitable trivialisation $\psi\colon H_1(F, \mathbb{Z}) \cong \mathbb{Z}^3$ so that the monodromy action of a fixed meridian m on $H_1(F, \mathbb{Z})$ is given by (4.6). Then $\iota_*([\gamma]) = \psi^{-1}(e_3)$. The monodromy action of a longitude ℓ must commute with (4.6). This implies that the monodromy action of ℓ preserves $\psi^{-1}(e_3)$ modulo the vanishing cycle $\psi^{-1}(e_1)$ (see (A.32) for the centraliser of (4.6)), and hence monodromy in $\mathrm{Crit}(f)$ around $\mathfrak{l}$ preserves $[\gamma]$. □

Proposition 4.1.26 *In a* T^3*-fibration* $f\colon X \to B$ *with compact orientable base* B *and with section* σ*, a typical fibre* $\iota\colon F \hookrightarrow X$ *carries a non-trivial, non-torsion homology class* $\iota_*([F]) \in H_3(X, \mathbb{Z})$*.*

Proof. We just observe that $\sigma_*([B]) \in H_3(X, \mathbb{Z})$ is non-zero because $0 \neq [B] = f_*\sigma_*([B]) \in H_3(B, \mathbb{Z})$. On the other side, since σ is a section, there is a non-trivial intersection number

$$\sigma_*([B]) \cdot \iota_*([F]) = \pm 1,$$

and hence $\iota_*([F])$ is nonzero and not a torsion class. It is easy to find examples of T^3-fibrations without section where $\iota_*([F]) = 0$ (see also Section 4.2). □

Theorem 4.1.27 *The underlying six-manifold* X *of a good* T^3*-fibration* $f\colon X \to S^3$ *has vanishing second Stiefel-Whitney class,* $w_2(X) = 0$*.*

Proof. (cf. [Gro99b]) By Proposition 4.1.12, a good T^3-fibration is orientable. Since $\mathrm{Crit}(f) \subset X$ is of codimension ≥ 4, we have $w_2(X) = w_2(X^\#)$. Now

$$TX^\# = \mathcal{V} \oplus (f^*TS^3)|_{X^\#}, \tag{4.9}$$

and of course f^*TS^3 is trivial, while $\mathcal{V}$ is trivial from Corollary 4.1.20. Thus $w_2(X^\#) = 0$. □

Corollary 4.1.28 *If* $X \to S^3$ *is a good* T^3*-fibration then* X *has an almost complex structure and a compatible almost symplectic structure.*

Proof. In [Wal66], Theorem 9, Wall shows that a smooth oriented six-manifold X has an almost complex structure if and only if $W_3(X) = 0$. By Theorem 4.1.27, $w_2(X) = 0$ and so $W_3(X) = \delta(w_2(X)) = 0$ where δ is the Bockstein homomorphism in the long exact cohomology sequence associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Given an almost complex structure J, take a hermitian metric $\langle \cdot, \cdot \rangle$ and define a compatible almost symplectic structure $\omega(X, Y) = \langle JX, Y \rangle$. □

Theorem 4.1.29 *In a six-manifold X carrying a good T^3-fibration, the critical locus represents the first Pontryagin class:*

$$\pm 2[\mathrm{Crit}(f)] = p_1(X) \in H^4(X, \mathbb{Q})$$

under Poincaré-duality.

Proof. This theorem is also contained in [Gro99b], but since we are in the simpler situation of good T^3-fibrations, we can avoid the use of K-theory.

Let $f\colon X \to S^3$ be a good T^3-fibration. Then by Corollary 4.1.20, the bundle $TX^{\#}$ is trivial; moreover, the isomorphism (4.8) gives us a preferred trivialisation for every trivialisation of TS^3. Every such trivialisation defines a relative Pontryagin class $p_{\mathrm{rel}}(X, X^{\#}) \in H^4(X, X^{\#}, \mathbb{Z})$. There is a short exact sequence

$$\mathrm{H}^3(X, \mathbb{Z}) \xrightarrow{\phi} \mathrm{H}^3(X^{\#}, \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^4(X, X^{\#}, \mathbb{Z}) \xrightarrow{\psi} \mathrm{H}^4(X, \mathbb{Z}).$$

The image $p_1(X) = \psi(p_{\mathrm{rel}}(X, X^{\#}))$ is the first Pontryagin class of X. (It is independent of the chosen trivialisation of TS^3 because any other choice is given by a map $t\colon S^3 \to \mathrm{SO}(3)$ and changes $p_{\mathrm{rel}}(X, X^{\#})$ by $\delta((f^{\#})^* t^* \omega)$ where $\omega \in H^3(\mathrm{SO}(3))$ is the canonical degree-3 class. But $(f^{\#})^*(t^*\omega) = \phi(f^*(t^*\omega))$ and hence $\delta((f^{\#})^* t^* \omega) = 0$.) But now $H^4(X, X^{\#}, \mathbb{Z}) \cong H_2(X \setminus X^{\#}, \mathbb{Z}) \cong \mathbb{Z}$ and $p_1(X)$ is represented by some multiple of the generator $\mathrm{Crit}(f) \in H_2(X \setminus X^{\#}, \mathbb{Z})$. This is a universal constant and can be determined from any example. So let $S \to \mathbb{C}P^1$ be an elliptic $K3$ surface and let $h\colon X = S \times T^2 \to \mathbb{C}P^1 \times S^1$. Note that again $T(\mathbb{C}P^1 \times S^1)$ is trivial. Since a generic elliptic $K3$ surface has 24 singular fibres, we obtain

$$48 = |p_1(S)| = |p_1(S \times T^2)| = 24 |p_1(f^{-1}(\mathcal{N}(\Delta)))|$$

and thus $p_1(f^{-1}(\mathcal{N}(\Delta))) = \pm 2$. □

Theorem 4.1.30 *For any T^3-fibration $f\colon X \to B$ whose discriminant locus is a closed one-dimensional submanifold $\Delta \subset B$ and with singularities of type (4.4), the Euler-characteristic vanishes, $\chi(X) = 0$.*

Proof. For a topological fibre bundle $f\colon X \to B$ with fibre F we have the formula $\chi(X) = \chi(F)\chi(B)$. Write the total space of our T^3-fibration as $X = X_0 \cup X_1$ where $X_0 \to B \setminus \Delta$ is the smooth part of the fibration and $X_1 \to \mathcal{N}(\Delta)$ is the singular fibration over an open solid torus. Then

$$\chi(X) = \chi(X_0) + \chi(X_1) - \chi(X_0 \cap X_1).$$

Now $\chi(X_0) = \chi(T^3)\chi(B \setminus \Delta) = 0$ because it is T^3-fibred. X_1 is homotopy equivalent to a topological fibre bundle $X_1' \to S^1$ with fibre $F' \cong I_1 \times S^1$, hence $\chi(X_1) = \chi(F')\chi(S^1) = 0$, and $\chi(X_0 \cap X_1) = 0$ because the intersection retracts onto a compact smooth five-manifold (a T^3-bundle over T^2). Thus $\chi(X) = 0$. $\square$

Definition 4.1.31 *We call a link $\mathfrak{l} \subset S^3$ irreducible if for each open ball $V \subset S^3$ with $\partial V \cap \mathfrak{l} = \emptyset$ we have either $\mathfrak{l} \subset V$ or $\mathfrak{l} \subset S^3 \setminus \overline{V}$.*

A link is irreducible precisely if its complement is an irreducible three-manifold. The next theorem describes the homotopy type of the complement of an irreducible link.

Theorem 4.1.32 (Papakyriakopoulos [Pap57]) *If $\mathfrak{l} \subset S^3$ is an irreducible link, then $U = S^3 \setminus \mathfrak{l}$ is an Eilenberg-MacLane space $K(\pi_1(U), 1)$.*

This will allow us to reduce the topological problem of computing the (co-) homology of link complements and bundles over them to a purely algebraic problem.

Theorem 4.1.33 *Suppose $\mathfrak{l}$ is an irreducible link and $b \in S^3 \setminus \mathfrak{l}$. Let $G = \pi_1(S^3 \setminus \mathfrak{l}, b)$ and $\varrho\colon G \to \mathrm{SL}(3, \mathbb{Z})$ an M-representation. If*

$$\frac{\mathrm{H}_1(F_b, \mathbb{Z})}{\{\varrho(g)\xi - \xi \mid \xi \in \mathrm{H}_1(F_b, \mathbb{Z}),\ g \in G\}} = \{0\}, \tag{4.10}$$

the total space X of the good T^3-fibration (with section) $f\colon X \to S^3$ associated to ϱ is simply connected.

The module on the left hand side of (4.10) is the 0-th homology of G with coefficients in the G-module $H_1(F_b, \mathbb{Z})$, thus equivalently (4.10) can be expressed as $H_0(G, \varrho) = \{0\}$ (see Paragraph 5.2.1 for definitions).

Proof. Let $b \in S^3 \setminus \mathfrak{l}$ and $\iota\colon F_b \to X$ the inclusion of the smooth fibre over b in X. We claim that (4.10) implies that for every loop γ in F_b with base point $\sigma(b)$, the push-forward $\iota_*\gamma$ is null-homotopic in X. A geometric way of saying this is that "$H_1(F_b, \mathbb{Z})$ is generated by vanishing cycles". Note that G is generated by finitely many meridians $g_1, \dots, g_l$ and

$$(g_i g_j)\xi - \xi = [g_i(g_j\xi) - (g_j\xi)] + [g_j\xi - \xi],$$

thus by induction on the length of words in $g_1, \dots, g_l$ we conclude from (4.10) that every $\xi \in H_1(F_b, \mathbb{Z})$ is of the form

$$\xi = \sum_{i=1}^{r} [\varrho(g_{k_i})\xi_i - \xi_i], \quad k_i \in \{1, \dots, l\}, \ \xi_i \in \mathrm{H}_1(F_b, \mathbb{Z}). \tag{4.11}$$

The image of the map

$$(\varrho(g_i) - \mathrm{Id})\colon \mathrm{H}_1(F_b, \mathbb{Z}) \to \mathrm{H}_1(F_b, \mathbb{Z})$$

is a $\mathbb{Z}$-module $E_i \subset H_1(F_b, \mathbb{Z})$ with $\mathrm{rk}(E_i) = 1$, and by (4.10) and (4.11),

$$E_1 + \dots + E_l = \mathrm{H}_1(F_b, \mathbb{Z}).$$

In our local model in Section 3.2, we could find a fibre-preserving S^1-action in a neighbourhood of every focus-focus singularity which is free outside the singular points, and whose fixed points are precisely the focus-focus points. If $x \in F_b$ the orbit of x under the S^1-action represents a non-trivial homology class $c_i \in H_1(F_b, \mathbb{Z})$ which generates E_i (the "vanishing cycle" for g_i). Now let $[\gamma] \in \pi_1(F_b, \sigma(b))$, then since $H_1(F_b, \mathbb{Z}) \cong \pi_1(F_b, \sigma(b))$ we can write the representing loop

$$\gamma = \sum_{i=1}^{l} r_i \gamma_i,$$

thus realising $[\gamma]$ as a linear combination of vanishing cycles. But now the collapsing S^1-actions yields a homotopy from γ to a constant loop, and $\iota_*(\pi_1(F_b, \sigma(b))) = \{0\} \subset \pi_1(X, \sigma(b))$. Now let $\gamma\colon S^1 \to X$ be an arbitrary (say, piecewise linear or smooth) loop. Since $f^{-1}(\mathfrak{l})$ is contained in a finite union of codimension 2 submanifolds of X, we can move γ slightly inside its homotopy class to make it disjoint from $C = f^{-1}(\mathfrak{l})$. It then represents an element of $\pi_1(X \setminus C)$. Now we have the following homotopy exact sequence:

$$\begin{tikzcd}
\pi_2(S^3 \setminus \mathfrak{l}) \arrow[r] \arrow[d, "\cong"'] & \pi_1(F_b) \arrow[r] \arrow[d, "\cong"'] & \pi_1(X \setminus C) \arrow[r] \arrow[d, "\cong"'] & \pi_1(S^3 \setminus \mathfrak{l}) \arrow[r] \arrow[d, "\cong"'] & \pi_0(F_b) \arrow[d, "\cong"'] \\
\{0\} \arrow[r] & \mathrm{H}_1(F_b, \mathbb{Z}) \arrow[r] & \mathrm{H}_1(F_b, \mathbb{Z}) \rtimes G \arrow[r] & G \arrow[r] & \{*\}
\end{tikzcd}$$

Because $X \to S^3$ has a section σ this sequence splits and we can write

$$\gamma = (\iota_* \gamma_1)(\sigma_* \gamma_2)$$

with $[\gamma_1] \in \pi_1(F_b, \sigma(b))$ and $[\gamma_2] \in \pi_1(S^3 \setminus \mathfrak{l}, b)$. However, we just said that under the inclusion $\pi_1(X \setminus C, \sigma(b)) \to \pi_1(X, \sigma(b))$ the element $[\iota_* \gamma_1] \mapsto 0 \in \pi_1(X)$ and $[\sigma_* \gamma_2] \mapsto 0 \in \pi_1(X)$ because S^3 is simply connected. Thus $\pi_1(X) = \{1\}$. ☐

We note that the total space of a T^n-fibration without section may be simply connected even if (4.10) is not satisfied, see Section 4.2.

4.2 Torus fibrations without section

There is a more general construction than the one given in the previous section where we do not require the existence of a section. We briefly comment on this case.

A trivial example in dimension three

Let us construct an S^1-bundle over S^2 by gluing together two trivial S^1-bundles $D_i \times S^1$, where $D_1 \cong D_2 \cong D^2$ are two-dimensional disks, by a diffeomorphism $\varphi\colon \partial D_1 \times S^1 \to \partial D_2 \times S^1$ of S^1-bundles, not necessarily preserving the sections $\sigma_i\colon t \in \partial D_i \mapsto (t,1) \in \partial D_i \times S^1$. As a bundle diffeomorphism, $\varphi\colon \partial D_1 \times S^1 \to \partial D_2 \times S^1$ is of the form $\varphi(t,\theta) = (t, a_\varphi(t,\theta))$. We derive from that a map $A_\varphi\colon \partial D^2 \cong S^1 \to S^1$ given by $A_\varphi(t) = a_\varphi(t,1)$. A_φ is the ratio of the sections σ_2 and $\varphi \circ \sigma_1$, thus φ preserves the sections if and only if $A_\varphi \equiv 1$.

Proposition 4.2.1 *Let φ be a diffeomorphism of trivial S^1-bundles as above, and let $f_\varphi\colon X_\varphi \to S^2$ be the S^1-bundle obtained by gluing together two copies of $D^2 \times S^1$ along φ. Then X_φ is diffeomorphic to a circle-bundle of degree k over S^2, where $k = \deg(A_\varphi)$.*

Proof. Standard arguments in topology show that up to isomorphism the S^1-bundle X_φ depends only on the isotopy class of $A_\varphi\colon S^1 \to S^1$ which is determined by the degree $k = \deg(A_\varphi)$. It is then easy to see that the associated complex line bundle has a section $\sigma\colon S^2 \to X_\varphi$ with $\sigma_1\colon z \in D_1 \mapsto (z,1) \in D_1 \times \mathbb{C}$ and $\sigma_2\colon z \in D_2 \mapsto (z, z^k) \in D_1 \times \mathbb{C}$, where $\sigma_i = \sigma|D_i$. Thus σ has a zero of order k at $0 \in D_2$ and X_φ is the unit circle bundle of a unitary line bundle of degree k over S^1. □

The Chern-class $c_1(X_\varphi)$ of the circle-bundle X_φ coincides with the Chern-class $c(X_\varphi)$ defined in Theorem 4.1.11, see page 79.

An example in dimension four

Let $D^2 \subset \mathbb{C}$ be the unit disk and let $M_1 \to D^2$ and $M_2 \to \bar{D}^2$ be two copies of an I_1-degeneration such as the Ooguri-Vafa model, where $\bar{D}^2$ carries the reversed orientation and thus the orientation on $M = M_1 = M_2$ induces opposite orientations on the vertical bundles $\mathcal{V}_1$, $\mathcal{V}_2$.

Let $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ and $S^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ be the unit spheres. Then the Hopf map $H\colon S^3 \to S^2$ can be given with respect to the standard coordinates (z_1, z_2) of $\mathbb{C}^2$ in the form

$$H(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2).$$

The suspension ΣH of the Hopf map sends $\Sigma S^3 = S^4 \subset \mathbb{C}^2 \times \mathbb{R}$ to $\Sigma S^2 = S^3 \subset \mathbb{C}^2$.

$$\Sigma H(z_1, z_2, t) = \left(\cos\left(\frac{\pi t}{2}\right) \frac{2 z_1 \bar{z}_2}{|z_1|^2 + |z_2|^2},\ \cos\left(\frac{\pi t}{2}\right) \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2} + i \sin\left(\frac{\pi t}{2}\right)\right).$$

Composing this again with the Hopf-map we get the following map $f = H \circ \Sigma H \colon S^4 \to S^2$.

$$\begin{aligned} f(z_1, z_2, t) &= \left(\cos^2\left(\frac{\pi t}{2}\right) \frac{2 z_1 \bar{z}_2}{|z_1|^2 + |z_2|^2} \left[\frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2} - i \sin\left(\frac{\pi t}{2}\right)\right],\right. \\ &\qquad \left. \cos^2\left(\frac{\pi t}{2}\right) \frac{4|z_1 z_2|^2 - (|z_1|^2 - |z_2|^2)^2}{(|z_1|^2 + |z_2|^2)^2} - \sin^2\left(\frac{\pi t}{2}\right)\right) \end{aligned}$$

Computation 4.2.2 *f is smooth, and $f|_{S^4 \setminus \{\mathsf{N}, \mathsf{S}\}}$ is a submersion. The poles N, S are focus-focus singularities.*

Proof. This can be checked easily in local coordinates. f is real analytic with a removable singularity at the poles. Indeed, if we write

$$f(z_1, z_2, t) = (f_1(z_1, z_2, t) + i f_2(z_1, z_2, t), f_3(z_1, z_2, t)) \in S^2 \subset \mathbb{C} \times \mathbb{R}$$

and $z_1 = \sqrt{\pi}(x_1 + i x_2)$ and $z_2 = -\sqrt{\pi}(y_1 + i y_2)$, then we get the Taylor expansions

$$\begin{aligned} f_1(x) &= x_1 y_2 - x_2 y_1 + O(4) \\ f_2(x) &= x_1 y_1 + x_2 y_2 + O(4) \\ f_3(x) &= -1 + O(4). \end{aligned}$$

So $\mathrm{Hess}(f_1)$ and $\mathrm{Hess}(f_2)$ have the normal form for a focus-focus singularity. □

Computation 4.2.3 *The T^2-fibration $f \colon S^4 \to S^2$ constructed above has one singular fibre $F_0 = f^{-1}((0, 0, -1))$. F_0 is the union of two smoothly embedded copies of S^2 which intersect transversally at $\mathsf{N}, \mathsf{S} \in S^4$.*

Proof. We have $f(\mathsf{N}) = f(\mathsf{S}) = (0, 0, -1)$, so there is just one singular fibre over the south pole $\mathsf{S}' = (0, 0, -1)$ of S^2. It is determined by the single equation

$$\cos^2\left(\frac{\pi t}{2}\right) \frac{4|z_1 z_2|^2 - (|z_1|^2 - |z_2|^2)^2}{(|z_1|^2 + |z_2|^2)^2} - \sin^2\left(\frac{\pi t}{2}\right) = -1.$$

This is equivalent to

$$\cos^2\left(\frac{\pi t}{2}\right) \frac{8|z_1 z_2|^2}{(|z_1|^2 + |z_2|^2)^2} = 0.$$

Hence $f(z_1, z_2, t) = \mathsf{S}'$ if and only if $t = \pm 1$ or $z_1 z_2 = 0$. Since $t = \pm 1$ implies $z_1 z_2 = 0$ we can omit the first equation and get the singular fibre

$$F_{\mathsf{S}'} = S^4 \cap \{(z_1, z_2, t) \in \mathbb{C}^2 \times \mathbb{R} \mid z_1 z_2 = 0\}.$$

$F_{\mathsf{S}'}$ is the union of two smoothly embedded 2-spheres $S^4 \cap \{z_1 = 0\}$ and $S^4 \cap \{z_2 = 0\}$ intersecting transversely in the poles of S^4.

Remark 4.2.4 Since the discriminant locus $\Delta = \{\mathsf{S}'\} \subset S^2$ is just a point and $\pi_1(S^2 \setminus \{\mathsf{S}'\}) = \{1\}$, the monodromy in this fibration is trivial, and $S^4 \setminus F_{\mathsf{S}'}$ is the trivial T^2-bundle $f_0 \colon D^2 \times T^2 \to D^2$. This is an example where the topology of a singular fibre in a torus fibration is not determined by the monodromy around it. So a T^n-bundle may have non-diffeomorphic smooth compactifications (f_0 can be compactified to $\tilde{f} \colon T^2 \times S^2 \to S^2$). However, we can deform f smoothly to obtain a T^2-fibration on S^4 with two singular fibres of the simplest possible type, namely simple focus-focus singularities.

Remark 4.2.5 The T^2-fibration $f \colon S^4 \to S^2$ has obviously no section. Let $f \colon X \to S^2$ be the four-manifold obtained by gluing together two I_1-singularities by a diffeomorphism φ which preserves the sections, hence X is the unique good T^2-fibration with section over S^2 whose discriminant locus $\Delta = \{P, Q\}$ contains precisely two points. Using Kirby-calculus [GS99] one can show that

$$X \cong (S^2 \times S^2)\#(S^1 \times S^3).$$

Up to homeomorphism, rather than diffeomorphism, this would also follow from results in 4-manifold theory, see for instance the book by Freedman and Quinn [FQ90], Theorem 10.7A, p.173.

Using arguments similar to those in Theorem 4.1.11, one can associate to each diffeomorphism φ of affine T^n-bundles $X \to B$ and $X' \to B$ with sections σ, σ' a *relative Chern-class* $c_{\text{rel}}(\varphi)$ which measures the failure of φ to be isotopic to a diffeomorphism preserving the section. There is a natural section $\tilde{c}(\varphi) \in H^0(B, \mathcal{H}_1(f, \mathbb{R}/\mathbb{Z}))$ defined by the property $\varphi^{-1} \circ \sigma' = \tilde{c}(\varphi) \cdot \sigma$. Changing φ inside its isotopy class changes $\tilde{c}(\varphi)$ by the $\bmod \mathbb{Z}$ reduction of a section of $\mathcal{H}_1(f, \mathbb{R})$. So we define

$$c_{\text{rel}}(\varphi) = \delta^* \tilde{c}(\varphi) \in \mathrm{H}^1(B, \mathcal{H}_1(f, \mathbb{Z})).$$

Theorem 4.2.6 *An diffeomorphism $\varphi \colon X \to X'$ of affine T^n-bundles as above is isotopic to a section-preserving diffeomorphism if and only if $c_{\text{rel}}(\varphi) = 0$. Given $B = B_0 \cup B_1$ and T^n-fibrations $f_i \colon X_i \to B_i$ with section together with a diffeomorphism of T^3-bundles $\varphi \colon X_0|_{B_0 \cap B_1} \to X_1|_{B_0 \cap B_1}$, the diffeomorphism type of the T^n-fibration $X_\varphi = X_0 \cup_\varphi X_1$ only depends on $c_{\text{rel}}(\varphi) \in H^1(B_0 \cap B_1, \mathcal{H}_1(f, \mathbb{Z}))$.*

4.3 Mirror symmetry

In this section we define the mirror of a T^3-fibration and show that mirror pairs exhibit the change in topology expected from physics. This has been carried out in much greater generality in [Gro99b]. There Gross defines a category of *well-behaved* T^3-fibrations, characterised by a number of sheaf-theoretical properties and regularity assumptions, and he shows that these fibrations can be canonically dualised and that dualising exchanges the even and odd cohomologies. We now aim to give a more elementary description of the mirror phenomenon for the concrete examples of fibrations constructed above. In particular, we do not impose the sheaf theoretical simplicity assumption which in a sense incorporates *a priori* the appearance of the mirror phenomenon, but rather derive it explicitly from our construction.

Theorem 4.3.1 *Let $f\colon X \to B$ be a T^3-fibration with compact orientable base, section σ and singularities as in (4.4). Let $f_0\colon X_0 \to B_0 = B\setminus\Delta$ be its smooth part. Assume that*

$$\mathrm{H}^0(B_0, R^1 f_{0*}\mathbb{Z}) \cong \{0\} \cong \mathrm{H}^0(B_0, R^2 f_{0*}\mathbb{Z}). \tag{4.12}$$

Then the Leray spectral sequence of $f_0\colon X_0 \to B_0$ for cohomology degenerates at E_2. Similarly, if

$$\mathrm{H}_0(B_0, R_1 f_{0*}\mathbb{Z}) \cong \{0\} \cong \mathrm{H}_0(B_0, R_2 f_{0*}\mathbb{Z}), \tag{4.13}$$

the Leray spectral sequence of f_0 for homology degenerates at E^2.

Proof. The condition (4.12) says that the monodromy representation $\varrho\colon \pi_1(B_0, b) \to \mathrm{SL}(H_1(F_b,\mathbb{Z}))$ and its dual ϱ^* have no *invariants*, while (4.13) entails the vanishing of the *coinvariant modules* for ϱ and ϱ^*. For good T^3-fibrations this is equivalent to saying that ϱ, ϱ^* have no cohomology, respectively homology, in degree zero (see Paragraph 5.2.1).

We have $X_0 \cong \widehat{B}_0 \times_\varrho \mathbb{T}^3$. Under the assumptions of the theorem the only possible non-zero differential at E_2-level is $d_2\colon H^0(B_0, R^3 f_{0*}\mathbb{Z}) \to H^2(B_0, R^2 f_{0*}\mathbb{Z})$. However, $\sigma_*[B] \in H_3(X,\mathbb{Z})$ is a non-torsion homology class, so its Poincaré dual θ restricts to a non-trivial class on the fibres. Then $\theta|X_0$ is non-trivial as well and so induces a generator $0 \neq \tilde{\theta} \in H^0(B_0, R^3 f_{0*}\mathbb{Z}) \cong \mathbb{Z}$ which must therefore survive to E_∞. So $d_2 = 0$ and the sequence degenerates.

In homology the only possible non-trivial boundary operator is $\partial_2\colon E^2_{2,2} \to E^2_{0,3}$. But we know from Proposition 4.1.26 that the homology class $[F]$ of a fibre is non-trivial in X and hence in X_0. On the other side it induces a generator of $E^2_{0,3} \cong \mathbb{Z}$ which must therefore survive to E^∞. This implies that $\partial_2 = 0$. So the sequence degenerates. □

The following theorem shows that dualising T^3-bundles reflects the corresponding Leray spectral sequence.

Theorem 4.3.2 *Let $f\colon X \to B$ and $\check{f}\colon \check{X} \to B$ be the T^3-bundles $X = \widehat{B} \times_\varrho \mathbb{T}^3$ and $\check{X} = \widehat{B} \times_{\varrho^*} \mathbb{T}^3$ where $\varrho\colon \pi_1(B) \to \mathrm{SL}(3,\mathbb{Z})$ is a representation which satisfies the conditions (4.12) and (4.13). Suppose B is compact and orientable. Then there are non-canonical isomorphisms*

$$\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q}) \cong \mathrm{H}^{\mathrm{odd}}(\check{X},\mathbb{Q}), \quad \mathrm{H}^{\mathrm{odd}}(X,\mathbb{Q}) \cong \mathrm{H}^{\mathrm{ev}}(\check{X},\mathbb{Q}).$$

Proof. (cf. [Gro99b]) Since ϱ, ϱ^* take values in $\mathrm{SL}(3,\mathbb{Z})$ the representation $\wedge^3\varrho = \wedge^3\varrho^*$ is trivial and $R^3 f_*\mathbb{Q} \cong \mathbb{Q} \cong R^3\check{f}_*\mathbb{Q}$. Thus there is a non-degenerate pairing

$$R^p f_*\mathbb{Q} \otimes R^{3-p} f_*\mathbb{Q} \longrightarrow \mathbb{Q}$$

so that we get isomorphisms

$$R^p f_*\mathbb{Q} \cong (R^{3-p} f_*\mathbb{Q})^* \cong R^{3-p}\check{f}_*\mathbb{Q}.$$

Because the Leray spectral sequences for f, $\check{f}$ degenerate at E_2 level, we have

$$\begin{aligned}
\mathrm{H}^{\mathrm{ev}}(X,\mathbb{Q}) &\cong \bigoplus_{p+q \text{ ev}} \mathrm{H}^p(B, R^q f_*\mathbb{Q}) \\
&\cong \bigoplus_{p+q \text{ ev}} \mathrm{H}^p(B, R^{3-q}\check{f}_*\mathbb{Q}) \\
&\cong \bigoplus_{p+q \text{ odd}} \mathrm{H}^p(B, R^q\check{f}_*\mathbb{Q}) \\
&\cong \mathrm{H}^{\mathrm{odd}}(\check{X},\mathbb{Q})
\end{aligned}$$

and similarly $H^{\mathrm{odd}}(X,\mathbb{Q}) \cong H^{\mathrm{ev}}(\check{X},\mathbb{Q})$. □

Next let us look at the topological fibre bundle $f_1\colon X_1 \to U_1 = \mathcal{N}(\Delta)$ and the intersection $f_{01}\colon X_0 \cap X_1 \to U_0 \cap U_1$.

Proposition 4.3.3 *Let F be a topological space. Let $F \longrightarrow X \xrightarrow{f} T^k$ be a topological F-bundle over the k-torus and ϱ^* the action of $\pi_1(T^k)$ on $H^*(F,\mathbb{Q})$. Then*

$$\mathrm{H}^*(X,\mathbb{Q}) \cong \mathrm{H}^*(F,\mathbb{Q})^{\varrho^*} \otimes \mathrm{H}^*(T^k,\mathbb{Q})$$

Proof. Let $k=1$ and cover the circle $T^1 = S_0 \cup S_1$ by two connected open intervals with $S_0 \cap S_1 = S_{01} \sqcup S_{10}$. Then $X_i = X|_{S_i} \simeq F$ and $X|_{S_0\cap S_1} \simeq F \sqcup F$ are homotopy equivalences, and the Mayer-Vietoris sequence takes the form

$$\cdots \mathrm{H}^p(X,\mathbb{Q}) \longrightarrow \mathrm{H}^p(F,\mathbb{Q}) \oplus \mathrm{H}^p(F,\mathbb{Q}) \xrightarrow{\psi_p} \mathrm{H}^p(F,\mathbb{Q}) \oplus \mathrm{H}^{p-1}(F,\mathbb{Q}) \cdots$$

with $\psi_p(\theta,\sigma) = (\theta-\sigma, \theta-\varrho_p^*(\sigma))$ where ϱ_p^* is the monodromy action of the fundamental class of T^1 on $H^p(F,\mathbb{Q})$. Thus

$$\begin{aligned}\ker(\psi_p) &= \{(\theta,\theta)\in \operatorname{diag}(\mathrm{H}^p(F,\mathbb{Q})\oplus \mathrm{H}^p(F,\mathbb{Q})) \mid \theta\in\ker(\varrho_p^*-\mathrm{Id})\}\\ \operatorname{im}(\psi_p) &= \{(\theta,\sigma)\in \mathrm{H}^p(F,\mathbb{Q})\oplus \mathrm{H}^p(F,\mathbb{Q}) \mid \sigma-\theta\in\operatorname{im}(\varrho_p^*-\mathrm{Id})\}\end{aligned}$$

and therefore

$$\ker(\psi_p)\cong\ker(\varrho_p^*-\mathrm{Id}),\quad \operatorname{im}(\psi_p)\cong \mathrm{H}^p(F,\mathbb{Q})\oplus\operatorname{im}(\varrho_p^*-\mathrm{Id})$$

Since we work with rational coefficients,

$$\mathrm{H}^p(X,\mathbb{Q})\cong\ker(\psi_p)\oplus\operatorname{coker}(\psi_{p-1}).$$

But $\dim(\operatorname{coker}(\psi_{p-1}))=\dim(\ker(\psi_{p-1}))$ and so by counting dimensions the map

$$(\theta,\sigma)\in \mathrm{H}^p(F,\mathbb{Q})^{\varrho_p^*}\oplus \mathrm{H}^{p-1}(F,\mathbb{Q})^{\varrho_p^*}\longmapsto \theta+\sigma\wedge f^*\omega\in \mathrm{H}^p(X,\mathbb{Q})$$

for a generator $\omega\in H^1(S^1,\mathbb{Q})$ is an isomorphism. The case $k>1$ follows by induction. This is generally false for $\mathbb{Z}$-coefficients due to torsion. □

Proposition 4.3.4 *Let $f\colon X\to B$ be the model I_1-singularity (4.4), let $F\hookrightarrow X$ be a smooth fibre and $F_0\hookrightarrow X$ a singular fibre over $b_0\in\Delta\subset B$, and let ϱ be the monodromy action of $\pi_1(B\setminus\Delta)\cong\mathbb{Z}$. Then*

$$\mathrm{H}^*(X,\mathbb{Q})\cong \mathrm{H}^*(F_0,\mathbb{Q})\cong \mathrm{H}^*(F,\mathbb{Q})^{\varrho}.$$

Proof. X retracts onto F_0. The second isomorphism can be checked explicitly in the local model in Section 3.2. □

Proposition 4.3.5 *There is a canonical embedding*

$$\mathrm{H}^*(X_1,\mathbb{Q})\hookrightarrow \mathrm{H}^*(X_0\cap X_1,\mathbb{Q}).$$

Proof. Let $[\ell],[m]\in\pi_1(U_0\cap U_1)$ denote the classes of longitude and meridian. Then

$$\begin{aligned}\mathrm{H}^*(X_1,\mathbb{Q}) &\cong \mathrm{H}^*(F_{\mathrm{sing}},\mathbb{Q})^{[\ell]}\otimes \mathrm{H}^*(U_1,\mathbb{Q})\\ &\cong (\mathrm{H}^*(F,\mathbb{Q})^{[m]})^{[\ell]}\otimes \mathrm{H}^*(U_1,\mathbb{Q})\\ &\hookrightarrow \mathrm{H}^*(F,\mathbb{Q})^{\langle[m],[\ell]\rangle}\otimes \mathrm{H}^*(U_0\cap U_1,\mathbb{Q})\\ &\cong \mathrm{H}^*(X_0\cap X_1,\mathbb{Q}).\end{aligned}$$

by Proposition 4.3.4. More precisely,

$$\mathrm{H}^p(X_1,\mathbb{Q})\cong \mathrm{H}^p(F_{\mathrm{sing}},\mathbb{Q})^{[\ell]}\oplus\Big(\mathrm{H}^{p-1}(F_{\mathrm{sing}},\mathbb{Q})^{[\ell]}\wedge P([m])\Big)$$

where $P([m])$ is the Poincaré-dual of $[m]\in H_1(X_1,\mathbb{Z})$. □

Proposition 4.3.6 *There are isomorphisms*

$$\begin{aligned}\mathrm{H}^{\mathrm{ev}}(X_1,\mathbb{Q}) &\cong \mathrm{H}^{\mathrm{odd}}(\check{X}_1,\mathbb{Q})\\ \mathrm{H}^{\mathrm{odd}}(X_1,\mathbb{Q}) &\cong \mathrm{H}^{\mathrm{ev}}(\check{X}_1,\mathbb{Q}).\end{aligned}$$

Proof. Let F_{sing} and $\check{F}_{\mathrm{sing}}$ be singular fibres of X_1 and $\check{X}_1$, respectively, and $F \cong T^3$ a smooth fibre. Let $\varrho_p = \wedge^p \varrho$ be the monodromy action on $H^p(F,\mathbb{Q})$. Then

$$\begin{aligned}\mathrm{H}^p(F_{\mathrm{sing}},\mathbb{Q}) &\cong \mathrm{H}^p(F,\mathbb{Q})^{\varrho_p}\\ &\cong (\Lambda^p\mathrm{H}^1(F,\mathbb{Q}))^{\wedge^p\varrho_1}\\ &\cong (\Lambda^{3-p}\mathrm{H}^1(\check{F},\mathbb{Q})^{\wedge^{3-p}\varrho_1^*}\\ &\cong \mathrm{H}^{3-p}(\check{F},\mathbb{Q})^{\varrho_p^*}\\ &\cong \mathrm{H}^{3-p}(\check{F}_{\mathrm{sing}},\mathbb{Q}).\end{aligned}$$

By Proposition 4.3.4 we obtain

$$\begin{aligned}&\bigoplus_{p\text{ even}} \mathrm{H}^p(X_1,\mathbb{Q})\\ \cong\ &\bigoplus_{p\text{ even}} \left[\mathrm{H}^p(F_{\mathrm{sing}},\mathbb{Q})\otimes\mathrm{H}^0(S^1,\mathbb{Q})\oplus\mathrm{H}^{p-1}(F_{\mathrm{sing}},\mathbb{Q})\otimes\mathrm{H}^1(S^1,\mathbb{Q})\right]\\ \cong\ &\bigoplus_{p\text{ even}} \left[\mathrm{H}^{3-p}(\check{F}_{\mathrm{sing}},\mathbb{Q})\otimes\mathrm{H}^0(S^1,\mathbb{Q})\oplus\mathrm{H}^{2-p}(\check{F}_{\mathrm{sing}},\mathbb{Q})\otimes\mathrm{H}^1(S^1,\mathbb{Q})\right]\\ \cong\ &\bigoplus_{p\text{ odd}} \left[\mathrm{H}^{p}(\check{F}_{\mathrm{sing}},\mathbb{Q})\otimes\mathrm{H}^0(S^1,\mathbb{Q})\oplus\mathrm{H}^{p-1}(\check{F}_{\mathrm{sing}},\mathbb{Q})\otimes\mathrm{H}^1(S^1,\mathbb{Q})\right]\\ \cong\ &\bigoplus_{p\text{ odd}} \mathrm{H}^p(\check{X}_1,\mathbb{Q})\end{aligned}$$

which we wanted to show. □

Now let us use the abbreviation

$$\tilde{\mathrm{H}}_0^{\mathrm{ev}} = \bigoplus_{p+q\text{ ev}} \mathrm{H}^p(\pi_1(B_0),\varrho_q^*),\quad \tilde{\mathrm{H}}_0^{\mathrm{odd}} = \bigoplus_{p+q\text{ odd}} \mathrm{H}^p(\pi_1(B_0),\varrho_q^*). \tag{4.14}$$

Here $\tilde{H}_0^{\mathrm{ev}}$ is meant to be the vector space H_0^{ev} together with a fixed decomposition as a direct sum of subspaces which is given by the Leray spectral sequence. Define $\tilde{H}_{01}^{\mathrm{ev}}$, $\tilde{H}_{01}^{\mathrm{odd}}$ similarly. Then the Mayer-Vietoris principle for the sheaf-cohomology modules in (4.14) yields a commutative diagram whose rows are long exact and whose vertical arrows are isomorphisms:

$$\begin{tikzcd}
\tilde{\mathrm{H}}_0^{\mathrm{ev}}\oplus\tilde{\mathrm{H}}_1^{\mathrm{ev}} \arrow[r, "\tau_1"] \arrow[d, "\psi_1"'] & \tilde{\mathrm{H}}_{01}^{\mathrm{ev}} \arrow[r, "\tau_2"] \arrow[d, "\psi_2"'] & \tilde{\mathrm{H}}^{\mathrm{odd}} \arrow[r, "\tau_3"] & \tilde{\mathrm{H}}_0^{\mathrm{odd}}\oplus\tilde{\mathrm{H}}_1^{\mathrm{odd}} \arrow[r, "\tau_4"] \arrow[d, "\psi_4"'] & \tilde{\mathrm{H}}_{01}^{\mathrm{odd}} \arrow[r, "\tau_5"] \arrow[d, "\psi_5"'] & \tilde{\mathrm{H}}^{\mathrm{ev}} \\
\tilde{\mathrm{H}}_0^{\mathrm{odd}}\oplus\tilde{\mathrm{H}}_1^{\mathrm{odd}} \arrow[r, "\tau_4"] & \tilde{\mathrm{H}}_{01}^{\mathrm{odd}} \arrow[r, "\tau_5"] & \tilde{\mathrm{H}}^{\mathrm{ev}} \arrow[r, "\tau_6"] & \tilde{\mathrm{H}}_0^{\mathrm{ev}}\oplus\tilde{\mathrm{H}}_1^{\mathrm{ev}} \arrow[r, "\tau_1"] & \tilde{\mathrm{H}}_{01}^{\mathrm{ev}} \arrow[r, "\tau_2"] & \tilde{\mathrm{H}}^{\mathrm{odd}}
\end{tikzcd} \tag{4.15}$$

Here we use that while the isomorphisms given in Theorem 4.3.2 are not canonical at the level of deRham cohomology they are natural for the modules arising in the Leray spectral sequence and they commute with the homomorphisms in the Mayer-Vietoris sequence. We now claim

Proposition 4.3.7 *There are maps $\psi^{\mathrm{ev}}\colon \tilde{H}^{\mathrm{ev}} \to \tilde{H}^{\mathrm{odd}}$ and ψ^{odd} in the opposite direction which make the diagram (4.15) commute.*

Proof. Choose splittings

$$\tilde{\mathrm{H}}^{\mathrm{odd}} = \ker(\tau_3) \oplus C, \quad \tilde{\mathrm{H}}^{\mathrm{ev}} = \ker(\tau_6) \oplus C'.$$

Let $x \in \ker(\tau_3)$. Then there is a $y \in \tilde{H}^{\mathrm{ev}}_{01}$ with $\tau_2(y) = x$. Set $\psi^{\mathrm{odd}}(x) = (\tau_5 \circ \psi_2)(y)$. This is well-defined since any other choice y' satisfies $y - y' \in \mathrm{im}(\tau_1)$ and hence $\psi_2(y-y') \in \mathrm{im}(\tau_4)$ which implies $(\tau_5 \circ \psi_2)(y-y') = 0$ by exactness. Now let $x \in C$ and $y = (\psi_4 \circ \tau_3)(x)$. Claim: $y \in \mathrm{im}\,\tau_6$. Otherwise $\tau_1(y) \neq 0$, however, $\tau_1(y) = (\psi_5 \circ \tau_4 \circ \tau_3)(x) = 0$, a contradiction. Since $\tau_6|C'$ is injective there is a unique $z \in C'$ with $\tau_6(z) = y$. Set $\psi^{\mathrm{odd}}(x) = z$. This defines a map $\psi^{\mathrm{odd}}\colon \tilde{H}^{\mathrm{odd}} \to \tilde{H}^{\mathrm{ev}}$ which makes (4.15) commute. □

We can now put all this together and conclude

Theorem 4.3.8 *In a good T^3-fibration with (4.12) and (4.13) there are non-canonical isomorphisms*

$$\begin{aligned} \mathrm{H}^{\mathrm{odd}}(X) &\cong \mathrm{H}^{\mathrm{ev}}(\check{X}) \\ \mathrm{H}^{\mathrm{ev}}(X) &\cong \mathrm{H}^{\mathrm{odd}}(\check{X}). \end{aligned}$$

Proof. Apply the five-lemma to (4.15) to conclude that

$$\psi^{\mathrm{ev}}\colon \tilde{\mathrm{H}}^{\mathrm{ev}} \longrightarrow \tilde{\mathrm{H}}^{\mathrm{odd}} \quad \text{and} \quad \psi^{\mathrm{odd}}\colon \tilde{\mathrm{H}}^{\mathrm{odd}} \longrightarrow \tilde{\mathrm{H}}^{\mathrm{ev}}$$

are isomorphisms. There are non-canonical (because they involve a choice of splitting) isomorphisms $H^{\mathrm{odd}} \cong \tilde{H}^{\mathrm{odd}}$ and $H^{\mathrm{ev}} \cong \tilde{H}^{\mathrm{ev}}$. □

Chapter 5

Knot groups and T^3-fibrations

We provide some tools to determine the M-representations of knot groups in concrete cases. In particular, we focus on the family $(\mathfrak{t}(p,q))_{p,q}$ of torus knots which form an important infinite class of knots. Torus knots enjoy a number of strong properties, and we use these to describe explicitly the M-representations and associated manifolds given by the construction of Chapter 4.

5.1 M-representations

The purpose of this section is to determine the M-representations of torus knots. We begin by providing some simple facts from linear algebra in the first paragraph; they are completely elementary and rather unsurprising, however, since their statements are of somewhat special nature we could not find them in the literature and give the proofs here. In the second paragraph we use this to classify the M-representations of torus knots.

5.1.1 Tools from linear algebra

If $\mathcal{N}(F_0)$ is a normal neighbourhood of a focus-focus singular fibre F_0 in a T^n-fibration and $F \subset \mathcal{N}(F_0)$ is a smooth fibre, then with respect to a suitable trivialisation $H_1(F,\mathbb{Z}) \cong \mathbb{Z}^n$ the local monodromy around F_0 is given by

$$A_n = \begin{pmatrix} 1 & 1 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{Id}_{n-2} \end{pmatrix} \in \mathrm{SL}(n,\mathbb{Z}), \tag{5.1}$$

as mentioned in (4.6). We would like to find a simple description of the conjugacy class of A in $\mathrm{SL}(n,\mathbb{Z})$ for $n = 2, 3$. This is elementary for $A_2 \in \mathrm{SL}(2,\mathbb{Z})$.

Proposition 5.1.1 *A matrix $\tilde{A} \in \mathrm{SL}(2,\mathbb{Z})$ is conjugate to A_2 in $\mathrm{SL}(2,\mathbb{Z})$ if and only if it is of the form*

$$\tilde{A} = \begin{pmatrix} 1-ab & a^2 \\ -b^2 & 1+ab \end{pmatrix} \tag{5.2}$$

where $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$.

We omit the proof which is similar to, but much simpler than that of Theorem 5.1.4. This proposition has a simple well-known corollary. Let

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Corollary 5.1.2 *Every $A \in \mathrm{SL}(2,\mathbb{Z})$ satisfies $M^{-1}AM = (A^{-1})^T$. Hence every representation in $\mathrm{SL}(2,\mathbb{Z})$ is conjugate to its transpose inverse, in other words, self-dual.*

We now turn to dimension $n = 3$. We remark that $\mathcal{C}_{\mathrm{SL}(3,\mathbb{Z})}(A_3) = \mathcal{C}_{\mathrm{GL}(3,\mathbb{Z})}(A_3)$, that is the conjugacy classes of A_3 in $\mathrm{SL}(3,\mathbb{Z})$ and in $\mathrm{GL}(3,\mathbb{Z})$ coincide, while $\mathcal{C}_{\mathrm{SL}(2,\mathbb{Z})}(A_2) \subsetneqq \mathcal{C}_{\mathrm{GL}(2,\mathbb{Z})}(A_2)$. Let us from now on drop subscripts and write $A = A_3$. We let $\langle\cdot,\cdot\rangle$ be the standard bilinear form on $\mathbb{Z}^3$, given by

$$\langle \xi, \eta \rangle = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3.$$

Definition 5.1.3 *We call an element $\xi = (\xi_1, \ldots, \xi_n)^T \in \mathbb{Z}^n$ <u>primitive</u> if*

$$\gcd(\xi_1, \ldots, \xi_n) = 1,$$

that is, if it is not a non-trivial multiple of any other element in the lattice.

Let the conjugacy class $\mathcal{C}(A) = \mathcal{C}_{\mathrm{SL}(3,\mathbb{Z})}(A)$ of A in $\mathrm{SL}(3,\mathbb{Z})$ and a certain set $\mathcal{S} \subset \mathbb{Z}^3 \times \mathbb{Z}^3$ be given by

$$\begin{aligned} \mathcal{C}(A) &= \left\{ M^{-1}AM \mid M \in \mathrm{SL}(3,\mathbb{Z}) \right\} && (5.3) \\ \mathcal{S} &= \left\{ (\eta,\xi) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \mid \eta, \xi \text{ are primitive and } \langle \eta, \xi \rangle = 0 \right\} && (5.4) \end{aligned}$$

and let $P\mathcal{S} = \mathcal{S}/\pm$, identifying $(\eta,\xi) \equiv (-\eta,-\xi)$. We can then prove

Theorem 5.1.4 *There is a bijective map $\Phi\colon P\mathcal{S} \to \mathcal{C}(A)$, defined by*

$$\Phi\colon [\eta,\xi] \longmapsto \mathrm{Id} + \eta\xi^T.$$

Proof. We know that any matrix $B \in \mathcal{C}(A)$ satisfies

$$\operatorname{rk}(B - \mathrm{Id}) = 1 \tag{5.5}$$
$$\det(B) = 1 \tag{5.6}$$

By (5.5) and because $B - \mathrm{Id}$ is an integral matrix we can write

$$B - \mathrm{Id} = \eta\xi^T. \tag{5.7}$$

with $\eta, \xi \in \mathbb{Z}^3$. In our case, such η, ξ are in fact primitive, because suppose that for instance $\eta = m\tilde{\eta}$ with $m \in \mathbb{Z}, \tilde{\eta} \in \mathbb{Z}^3$, then $B - \mathrm{Id} = m(\tilde{\eta}\xi^T)$. Now take $M \in \mathrm{SL}(3, \mathbb{Z})$ such that $M^{-1}BM = A$, thus $A - \mathrm{Id} = M^{-1}(B - \mathrm{Id})M = m(M^{-1}(\tilde{\eta}\xi^T)M)$. Therefore m divides all coefficients of $A - \mathrm{Id}$ and hence $m = \pm 1$. One finds that

$$\det(B) = \det\left(\mathrm{Id} + \eta\xi^T\right) = 1 + \langle \eta, \xi \rangle,$$

and hence (5.6) implies $\langle \eta, \xi \rangle = 0$. Obviously η and ξ are determined up to sign by B. We conclude that $[\eta, \xi] \in P\mathcal{S}$, and is the unique element such that $\Phi([\eta, \xi]) = B$.

Now conversely let $[\eta, \xi] \in P\mathcal{S}$ be arbitrary. We have to show that $B = \Phi([\eta, \xi])$ is conjugate in $\mathrm{SL}(3, \mathbb{Z})$ to A. This means we have to find $M \in \mathrm{SL}(3, \mathbb{Z})$ such that

$$M^{-1}\eta\xi^T M = A - \mathrm{Id} = e_1 e_2^T.$$

We therefore want to find an $M \in \mathrm{SL}(3, \mathbb{Z})$ solving the equations

$$M^{-1}\eta = e_1, \quad M^T\xi = e_2.$$

The first of these equations simply says that η has to be the first column in M. Since ξ is primitive, we can find $\rho = (\rho_1, \rho_2, \rho_3)^T \in \mathbb{Z}^3$ such that

$$\langle \rho, \xi \rangle = \rho_1\xi_1 + \rho_2\xi_2 + \rho_3\xi_3 = 1.$$

Let $\kappa = \eta \times \rho$ where the cross product is as in $\mathbb{R}^3$. Then κ is primitive since if κ was divisible, then so would be $\kappa \times \xi$, however,

$$\kappa \times \xi = (\eta \times \rho) \times \xi = \langle \eta, \xi \rangle \rho - \langle \rho, \xi \rangle \eta = -\eta$$

is primitive where we have used that $\langle \eta, \xi \rangle = 0$ for elements of $P\mathcal{S}$. This says that the three 2×2 principal minors $(\kappa_1, \kappa_2, \kappa_3)$ of the matrix

$$\begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \rho_1 & \rho_2 & \rho_3 \end{pmatrix}$$

satisfy $\gcd(\kappa_1, \kappa_2, \kappa_3) = 1$ and hence we can solve the equation $\langle \tilde{\sigma}, \kappa \rangle = 1$ by some $\tilde{\sigma} \in \mathbb{Z}^3$. But the latter sum is just $\det(\eta, \rho, \tilde{\sigma}) = 1$. Now set $m = \langle \tilde{\sigma}, \xi \rangle$. Then if we replace $\sigma = \tilde{\sigma} - m\rho$ and set

$$M = (\eta, \rho, \sigma) = \begin{pmatrix} \eta_1 & \rho_1 & \sigma_1 \\ \eta_2 & \rho_2 & \sigma_2 \\ \eta_3 & \rho_3 & \sigma_3 \end{pmatrix}$$

we will have

$$\det(M) = \det(\eta, \rho, \sigma) = \det(\eta, \rho, \tilde{\sigma} - m\rho) = \det(\eta, \rho, \tilde{\sigma}) = 1$$

and the desired equations $Me_1 = \eta$ and $M^T\xi = e_2$ follow which is all we had to show. □

Note that for every $B \in \mathcal{C}(A)$ the inverse matrix B^{-1} is contained in $\mathcal{C}(A)$ as well. For A itself conjugation with $D = \operatorname{diag}(-1, 1, -1)$ realises $D^{-1}AD = A^{-1}$, and in general it follows because conjugation and inversion commute.

Proposition 5.1.5 *Let $\iota\colon B \mapsto B^{-1}$ be matrix inversion on $\mathcal{C}(A)$ and let $\mathsf{I}\colon P\mathcal{S} \to P\mathcal{S}$ be the involution defined by $\mathsf{I}([\eta, \xi]) = [-\eta, \xi]$. Then*

$$\mathsf{I} = \Phi^{-1} \circ \iota \circ \Phi.$$

Proof. We only have to check that

$$\begin{aligned} \Phi([-\eta, \xi])\Phi([\eta, \xi]) &= (\operatorname{Id} - \eta\xi^T)(\operatorname{Id} + \eta\xi^T) \\ &= \operatorname{Id} + \eta\xi^T - \eta\xi^T - (\eta\xi^T)(\eta\xi^T) \\ &= \operatorname{Id} - \eta(\xi^T\eta)\xi^T \\ &= \operatorname{Id} - \langle \eta, \xi \rangle \eta\xi^T \\ &= \operatorname{Id} \end{aligned}$$

where we use $\langle \eta, \xi \rangle = 0$. □

$\mathcal{C}(A)$ is obviously invariant under conjugation $\kappa\colon \mathrm{SL}(3, \mathbb{Z}) \times \mathcal{C}(A) \to \mathcal{C}(A)$ sending $(M, B) \mapsto M^{-1}BM$, so more specifically if we restrict the conjugating matrix M to lie in $\mathcal{C}(A)$ as well we obtain, via Φ, a map $\mathsf{K}\colon P\mathcal{S} \times P\mathcal{S} \to P\mathcal{S}$ such that

$$\Phi \circ \mathsf{K} = \kappa \circ (\Phi \times \Phi).$$

We now describe this map.

Proposition 5.1.6 *Suppose $[\eta, \xi], [\tilde{\eta}, \tilde{\xi}] \in P\mathcal{S}$ are given and $M = \Phi([\eta, \xi]), B = \Phi([\tilde{\eta}, \tilde{\xi}])$. Then*

$$M^{-1}BM = \Phi([\tilde{\eta} - \langle \xi, \tilde{\eta} \rangle \eta, \tilde{\xi} + \langle \tilde{\xi}, \eta \rangle \xi]).$$

Proof. Again the proof is straightforward:

$$
\begin{aligned}
& M^{-1}BM \\
= \; & \Phi([-\eta,\xi])\Phi([\tilde\eta,\tilde\xi]\Phi([\eta,\xi]) \\
= \; & (\mathrm{Id} - \eta\xi^T)(\mathrm{Id} + \tilde\eta\tilde\xi^T)(\mathrm{Id} + \eta\xi^T) \\
= \; & \mathrm{Id} + \tilde\eta\tilde\xi^T - (\eta\xi^T)(\tilde\eta\tilde\xi^T) + (\tilde\eta\tilde\xi^T)(\eta\xi^T) - (\eta\xi^T)(\eta\xi^T) - (\eta\xi^T)(\tilde\eta\tilde\xi^T)(\eta\xi^T) \\
= \; & \mathrm{Id} + \Big[\tilde\eta\tilde\xi^T - \langle\xi,\tilde\eta\rangle\eta\tilde\xi^T + \langle\tilde\xi,\eta\rangle\tilde\eta\xi^T - \langle\xi,\tilde\eta\rangle\langle\tilde\xi,\eta\rangle\eta\xi^T\Big] \\
= \; & \mathrm{Id} + \Big(\tilde\eta - \langle\xi,\tilde\eta\rangle\eta\Big)\Big(\tilde\xi + \langle\tilde\xi,\eta\rangle\xi\Big)^T .
\end{aligned}
$$

Thus we obtain

$$\mathsf{K}([\eta,\xi],[\tilde\eta,\tilde\xi]) = [\tilde\eta - \langle\xi,\tilde\eta\rangle\eta, \tilde\xi + \langle\tilde\xi,\eta\rangle\xi].$$

The expression on the right is invariant under the substitutions $(\xi,\eta) \to (-\xi,-\eta)$ and $(\tilde\xi,\tilde\eta) \to (-\tilde\xi,-\tilde\eta)$ and hence well-defined. $\square$

This can be used to translate the Wirtinger presentation of a knot group into simple equations in $P\mathcal{S}$.

Remark 5.1.7 This description also shows that the centraliser of $B = \Phi([\eta,\xi]) \in \mathcal{C}(A)$ inside $\mathcal{C}(A)$ is the image under Φ of the set

$$\Big\{ [\tilde\eta,\tilde\xi] \in P\mathcal{S} \;\Big|\; \langle\tilde\xi,\eta\rangle = \langle\xi,\tilde\eta\rangle = 0 \Big\}.$$

5.1.2 Torus knots

We use the tools provided in the previous section to classify all M-representations for the infinite class of torus knots. The geometric meaning of this classification is that we give a complete list of all possible monodromy representations which can arise in T^3-fibrations with simple focus-focus singularities over a torus knot.

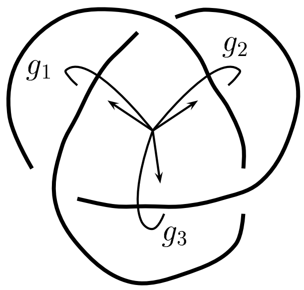

Figure 5.1: *The trefoil knot* $\mathsf{t}(2,3)$ *(or* 3_1*) and its meridians* g_1, g_2, g_3*.*

Theorem 5.1.8 *Let $\mathfrak{k} = \mathfrak{t}(2,3) \subset S^3$ be the trefoil knot and $G = \pi_1(S^3 \setminus \mathfrak{k})$. Then the M-representations of G can be classified in the following way: There is one trivial representation which factorises through the Abelianisation $G^{\mathrm{ab}} = G/[G,G] \cong \mathbb{Z}$, and one non-trivial representation whose image is isomorphic to* $\mathrm{SL}(2,\mathbb{Z})$.

Proof. The knot group of the trefoil has a Wirtinger presentation

$$G = \left\langle\, g_1, g_2, g_3 \;\middle|\; g_2 g_1 = g_1 g_3 = g_3 g_2 \,\right\rangle.$$

We rewrite these relations as

$$g_1^{-1} g_2 g_1 = g_3 \tag{5.8}$$
$$g_3^{-1} g_1 g_3 = g_2 \tag{5.9}$$
$$g_2^{-1} g_3 g_2 = g_1 \tag{5.10}$$

and note that G is in fact generated by g_1 and g_2 alone due to (5.8). Also the relation (5.10) is a consequence of (5.8) and (5.9) so that we only need to satisfy the first two. We use the convention that for a manifold with basepoint (B,b) and elements $\gamma_1, \gamma_2 \in \mathcal{L}^1(B,b)$ of its loop-group the composition $\gamma_1\gamma_2$ is the loop going *first* through γ_2 and *then* through γ_1. This way our M-representations $\pi_1(S^3 \setminus \mathfrak{k}) \to \mathrm{SL}(3,\mathbb{Z})$ will be homomorphisms rather than anti-homomorphisms if we let $\mathrm{SL}(3,\mathbb{Z})$ act on $\mathbb{Z}^3$ from the *left*, as usual.

We first show that for $\Lambda \cong \mathbb{Z}^3$, every non-trivial M-homomorphism on Λ belongs to a family $\varrho_{abc}\colon G \to \mathrm{SL}(\Lambda)$ which is parametrised bijectively by all $(a,b,c) \in \mathbb{Z}^3$. Given any M-homomorphism $\varrho_\Lambda\colon G \to \mathrm{SL}(\Lambda)$ we can find a trivialisation $\Lambda \to \mathbb{Z}^3$ so that ϱ_Λ becomes an M-homomorphism $\varrho\colon G \to \mathrm{SL}(3,\mathbb{Z})$ of the form

$$\varrho(g_1) = \Phi([e_1, e_2]) = A \tag{5.11}$$
$$\varrho(g_2) = \Phi([\eta, \xi]) \tag{5.12}$$

for some $[\eta,\xi] \in P\mathcal{S}$. Then $\varrho(g_3)$ is determined by (5.8). Namely, using Proposition 5.1.6 and (5.11), (5.12) we can express $\varrho(g_3)$ as

$$\begin{aligned} \varrho(g_3) &= \varrho(g_1)^{-1}\varrho(g_2)\varrho(g_1) \\ &= \Phi([\eta - \langle e_2, \eta\rangle e_1, \xi + \langle \xi, e_1\rangle e_2]) \\ &= \Phi([\eta - \eta_2 e_1, \xi + \xi_1 e_2]). \end{aligned} \tag{5.13}$$

If we apply ϱ to (5.9) and use (5.11) and (5.13), we obtain the equation

$$\begin{aligned} \varrho(g_2) &= \varrho(g_3)^{-1}\varrho(g_1)\varrho(g_3) \\ &= \Phi\left([e_1 - \langle \xi + \xi_1 e_2, e_1\rangle\left(\eta - \eta_2 e_1\right), e_2 + \langle e_2, \eta - \eta_2 e_1\rangle\left(\xi + \xi_1 e_2\right)]\right) \\ &= \Phi\left([e_1 - \xi_1(\eta - \eta_2 e_1), e_2 + \eta_2(\xi + \xi_1 e_2)]\right) \end{aligned}$$

Substituting this into (5.12) yields the equation

$$[\eta, \xi] = [e_1 - \xi_1(\eta - \eta_2 e_1), e_2 + \eta_2(\xi + \xi_1 e_2)] \tag{5.14}$$

which is a necessary and sufficient condition for (5.11) - (5.12) to define an M-homomorphism. Now observe that the right hand side of (5.14) remains unchanged if we replace (ξ, η) by $(-\xi, -\eta)$, hence if we choose the sign appropriately (5.14) is equivalent to the following two equations

$$\begin{aligned}(1 + \xi_1)\eta &= (1 + \xi_1\eta_2)e_1 && (5.15)\\ (1 - \eta_2)\xi &= (1 + \xi_1\eta_2)e_2. && (5.16)\end{aligned}$$

We first want to find all solutions to these equations in $\mathcal{S}$. Let us look at (5.15) to begin with. We distinguish the two cases whether the sides of the equation are or are not zero. If they are not we can argue as follows: η is a multiple of e_1, hence being primitive it must equal $\eta = \pm e_1$. But then $\eta_2 = 0$ and (5.16) says $\xi = e_2$. Therefore, $\xi_1 = 0$ and going back to (5.15) this determines the sign of $\eta = e_1$. So $\varrho(g_2) = \Phi([e_1, e_2]) = A$ which gives us the trivial representation ϱ_{trivial} (trivial in the sense that it factors through G^{ab}). So now we assume that both sides in (5.15) are zero. This means $\xi_1 = -1$ and $\eta_2 = 1$. Then because both η and ξ have a coefficient ± 1 the primitivity condition places no further restriction on what the other $\eta_j, \xi_j \in \mathbb{Z}$ can be, and the only other equation to be satisfied is

$$0 = \langle \eta, \xi \rangle = -\eta_1 + \xi_2 + \eta_3\xi_3.$$

So let $a, b, c \in \mathbb{Z}$ be any integers and let us make an Ansatz $\eta_1 = a$, $\eta_3 = b$, $\xi_3 = c$. Having fixed those coefficients, $\xi_2 = a - bc$ is uniquely determined. This yields

$$\varrho(g_2) = \mathrm{Id} + \begin{pmatrix} a \\ 1 \\ b \end{pmatrix} \Big(-1, a - bc, c \Big) = \begin{pmatrix} 1-a & a^2 - abc & ac \\ -1 & a - bc + 1 & c \\ -b & ab - b^2 c & bc + 1 \end{pmatrix}. \tag{5.17}$$

Finally it is easy to determine the image of g_3 from (5.8) or (5.13); it is given by

$$\varrho(g_3) = \begin{pmatrix} 2-a & 1 - 2a + a^2 - abc + bc & ac - c \\ -1 & a - bc & c \\ -b & ab - b - b^2 c & bc + 1 \end{pmatrix}. \tag{5.18}$$

Now conversely, using the Wirtinger relations (5.8), (5.9) and (5.10) one can easily check that for all $a, b, c \in \mathbb{Z}$ (5.17), (5.18) extends to a homomorphism $\varrho_{abc} \colon G \to \mathrm{SL}(3, \mathbb{Z})$ that sends all meridians into $\mathcal{C}(A)$.

Are any of these M-homomorphisms equivalent? We check that the matrix

$$C = \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix}$$

commutes with A, and $C^{-1}\varrho_{abc}C = \varrho_{000}$, and all the M-homomorphisms ϱ_{abc} are equivalent via conjugation to the M-homomorphism ϱ_{000} sending

$$g_1 \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 \longmapsto \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other side, ϱ_{000} is certainly not equivalent to the trivial (cyclic) M-homomorphism ϱ_{trivial} with $\varrho_{\text{trivial}}(g_j) = A$ for $j = 1, 2, 3$ since $\varrho_{\text{trivial}}(G) \cong \mathbb{Z}$ while the matrices $A = \varrho_{000}(g_1)$ and $A' = \varrho_{000}(g_2)$ are well-known to generate a subgroup of $\mathrm{SL}(3, \mathbb{Z})$ isomorphic to $\mathrm{SL}(2, \mathbb{Z})$. $\square$

The second example that we need later on is the $(4, 3)$-torus knot. It appears as 8_{19} in standard tables of knot-diagrams for knots with low crossing numbers.

Theorem 5.1.9 *Let $\mathfrak{k} \subset S^3$ be the $(4, 3)$-torus knot and $G = \pi_1(S^3 \setminus \mathfrak{k})$ its group. Then the M-representations of G are given by the trivial M-representation ϱ_{trivial} and an infinite family ϱ_k of non-Abelian M-representations parametrised by $k \in \mathbb{Z}$, with $\varrho_{-k} = (\varrho_k^{-1})^T = \varrho_k^*$.*

The proof is similar to that of Theorem 5.1.8, but slightly more technical. We postpone the details to Appendix A.1. The representations ϱ_k are given explicitly in terms of the images of a set of generators for $G(4, 3)$ on page 132.

We can now use the classification of the M-representations for the $(2, 3)$ and $(4, 3)$-torus knots to classify *all* M-representations for the (p, q)-torus knot.

Theorem 5.1.10 *The group $G(p, q)$ of the (p, q)-torus knot $\mathfrak{t}(p, q)$ has non-trivial M-representations if and only if (after perhaps exchanging p and q) $p = 2^m p'$ and $q = 3^k q'$ with $\gcd(p', 6) = \gcd(q', 6) = \gcd(p', q') = 1$ and $m, k > 0$. In case $m = 1$ there is a bijection $M-\mathrm{Rep}(G(p, q)) \to M-\mathrm{Rep}(G(2, 3))$ between the M-representations of $G(p, q)$ and $G(2, 3)$. In case $m > 1$ there is a bijection $M-\mathrm{Rep}(G(p, q)) \to M-\mathrm{Rep}(G(4, 3))$.*

Sketch of proof. We only have an overkill proof for this theorem, we would therefore like to give the key idea here which is quite simple, and postpone the details to Appendix A.3. First note that the knot group $G(p, q) = \langle\, x, y \mid x^p = y^q \,\rangle$ has a non-trivial centre $\mathcal{Z}(G(p, q)) = \langle x^p \rangle = \langle y^q \rangle$ which is an infinite cyclic group generated by

$x^p = y^q$. Hence the image $Z = \varrho(x^p) = \varrho(y^q)$ of the generator must commute with $\varrho(G_{p,q}) \subset \mathrm{SL}(3,\mathbb{Z})$ and in particular commutes with $\varrho(g_1) = A$. This restricts the possible values of Z to elements of the centraliser $\mathcal{Z}(A)$ which can be written down easily (see Lemma A.3.1). It turns out that all elements of $\mathcal{Z}(A)$ are special linear matrices with eigenvalues ± 1. Let $X = \varrho(x)$ and $Y = \varrho(y)$ so that $Z = X^p = Y^q$. Because $\mathrm{spec}(Z) \subset \{1, -1\}$, the eigenvalues of X (respectively Y) have to be p-th (respectively q-th) roots of ± 1. But also they need to be roots of the characteristic polynomial χ_X (respectively χ_Y) which has integer coefficients, hence the elements of $\mathrm{spec}(X)$ (respectively $\mathrm{spec}(Y)$) are algebraic of degree two over $\mathbb{Q}$ (since we can always split off a linear factor corresponding to the eigenvalue $\lambda = \pm 1$). This reduces the possible eigenvalues of X and Y to a finite set (essentially the primitive roots of unity of orders $2, 3, 4, 6$), and the remainder of the proof consists in applying elementary linear algebra (Jordan normal forms) to establish the classification given in the theorem. $\square$

As an example that a knot generally need not have any non-trivial M-representations we mention the four-knot 4_1, or figure-eight knot, whose knot group has a Wirtinger-presentation

$$G(4_1) = \left\langle\, g_1, g_2, g_3, g_4 \mid g_4^{-1} g_2 g_4 = g_3,\; g_1^{-1} g_4 g_1 = g_3,\; g_2^{-1} g_4 g_2 = g_1,\; g_3^{-1} g_2 g_3 = g_1 \,\right\rangle$$

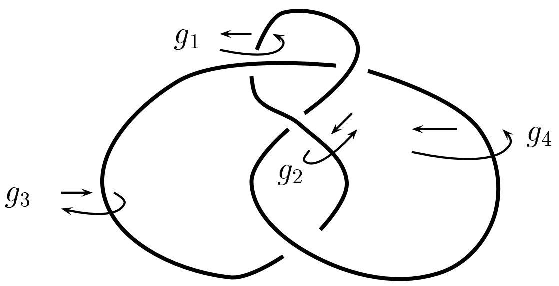

Figure 5.2 *The four knot 4_1 (or figure-eight knot).*

Proposition 5.1.11 *The four-knot has no non-trivial M-representations.*

Proof. Working as before we make the Ansatz $\varrho(g_1) = \Phi([e_1, e_2])$ and $\varrho(g_4) = \Phi([\eta, \xi])$. Then the equation $g_1^{-1} g_4 g_1 = g_3$ implies $\varrho(g_3) = \Phi([\eta - \eta_2 e_1, \xi + \xi_1 e_2])$ and $g_3^{-1} g_2 g_3 = g_1$ implies $\varrho(g_2) = \Phi([e_1 + \xi_1 \eta, e_2 - \eta_2 \xi])$. If we substitute this into $g_2^{-1} g_4 g_2 = g_1$ we obtain

$$(\eta_2 \pm 1) e_1 + (\eta_2 \xi_1 - 1)\eta = 0, \tag{5.19}$$
$$(-\xi_1 \pm 1) e_2 + (\eta_2 \xi_1 - 1)\xi = 0. \tag{5.20}$$

If e_1 and η are linearly dependent, then $\eta_2 = 0$ and (5.19) and (5.20) become $\eta = \pm e_1$, $\xi = \pm e_2$, and so $\varrho(g_4) = \Phi([\pm e_1, \pm e_2]) = \varrho(g_1)$. In this case ϱ is trivial and factors through G^{ab}. If e_1 and η are linearly independent, then from (5.19) we get $\eta_2 \pm 1 = \eta_2\xi_1 - 1 = 0$, hence $\eta_2 = \mp 1$ and $\xi_1 = \mp 1$. Substituting into (5.20) we get $2e_2 = 0$, a contradiction. So there are no non-trivial M-representations. □

Remark 5.1.12 We are not aware of any knots apart from torus knots which admit non-trivial M-representations. Based on the small number of cases that we have checked it would go too far to conjecture that such cannot exist, however, they will clearly be much harder to find. Even though the Wirtinger presentation of a torus knot looks just as complicated as that of any other knot with similar crossing number (thus presenting us with a fairly complex system of equations), torus knot groups have nevertheless a much simpler structure which is reflected in the alternative presentation $G(p,q) = \langle\, x, y \,|\, x^p = y^q \,\rangle$ by just two generators and one relation. This simplicity, though not apparent in the Wirtinger presentation, is responsible for the existence of many M-representations. It will be much more difficult to find M-representations for general knots.

Remark 5.1.13 Despite the difficulties in finding M-representations described in Remark 5.1.12, there is a saving grace: Let $\mathfrak{k} = \mathfrak{k}_1 \# \cdots \# \mathfrak{k}_l$ be a composite knot. Then its group

$$G(\mathfrak{k}) = G(\mathfrak{k}_1) *_{\langle g_1 \rangle} \cdots *_{\langle g_{l-1} \rangle} G(\mathfrak{k}_l)$$

is an *amalgamated sum* (see Definition 5.2.3) of the knot groups of its prime factors. Any collection ϱ_i of M-representations for $\mathfrak{k}_i$ induces an M-representation of $\mathfrak{k}$ as follows: Define $\varrho | G(\mathfrak{k}_1) = \varrho_1$. Suppose we have defined ϱ on the subgroup

$$G_r(\mathfrak{k}) = G(\mathfrak{k}_1) *_{\langle g_1 \rangle} \cdots *_{\langle g_{r-1} \rangle} G(\mathfrak{k}_r)$$

for some $1 \leq r < l$. Then since $\varrho | G_r(\mathfrak{k})$ and ϱ_{r+1} are M-representations, $\varrho(g_r)$ and $\varrho_{r+1}(g_r)$ are conjugate, hence for a suitable $C \in \mathrm{SL}(3, \mathbb{Z})$ and $\varrho'_{r+1} = C^{-1}\varrho_{r+1}C$ we have $\varrho(g_r) = \varrho'_{r+1}(g_r)$ and we can extend ϱ from $G_r(\mathfrak{k})$ to $G_{r+1}(\mathfrak{k})$ by ϱ'_{r+1}. This allows us to construct a huge number of M-representations if we consider composite knots where (some of) the factors are torus knots with non-trivial M-representations.

5.2 The topology of T^3-fibrations

We give a brief account of the basic facts about group (co-) homology and how to compute it in concrete cases. We apply this to torus bundles over knot complements.

5.2.1 Group homology and cohomology

Recall from elementary topology that for any manifold X and module M over a commutative ring R with identity we can define its singular homology and cohomology with coefficients in M using the standard chain complex $C_*(X, M) = C_*(X) \otimes_R M$ where $C_*(X)$ is the free module over R generated by simplicial maps. The resulting theory has the following isomorphisms which can be found in Spanier [Spa66].

Proposition 5.2.1 *Let X be a compact n-manifold with boundary and M a module over some ring R. Then for all $q \geq 0$*

$$\begin{array}{rcccl} \mathrm{H}_q(X, M) & \cong & \mathrm{H}_q(X \setminus \partial X, M) & \cong & \mathrm{H}^{n-q}(X, \partial X, M) \\ \mathrm{H}_q(X, \partial X, M) & \cong & \mathrm{H}^{n-q}(X \setminus \partial X, M) & \cong & \mathrm{H}^{n-q}(X, M) \end{array}$$

Much of the following will use the homology and cohomology of a group G with coefficients in a module M over the group ring $\mathbb{Z}G$. It is defined as the (co-) homology of the (co-) chain complex obtained as the image of any free resolution of M over $\mathbb{Z}G$ under the functor $- \otimes_G M$ (the functor $\mathrm{Hom}_G(-, M)$).

Notation 5.2.2 *For a discrete group G and a module M over its group ring $\mathbb{Z}G$ we denote the <u>(co-) homology of G with coefficients in M</u> by $H_*(G, M)$ (by $H^*(G, M)$). If M is a $\mathbb{Z}$-module with a representation $\varrho\colon G \to \mathrm{Aut}_{\mathbb{Z}}(M)$ making it into a $\mathbb{Z}G$-module, we also write $H_*(G, \varrho)$ (and $H^*(G, \varrho)$) for the corresponding group (co-) homology to emphasise the choice of representation.*

The following theorem establishes the existence of Mayer-Vietoris sequences for *amalgams*. We first give the following definition.

Definition 5.2.3 *Suppose we are given groups A, G_1, G_2 and embeddings $\alpha_i\colon A \to G_i$ for $i = 1, 2$. There is a group G called the <u>amalgamated sum</u> (of G_1 and G_2 along A) which fits into a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & G_1 \\ {\scriptstyle\alpha_2}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ G_2 & \xrightarrow{\beta_2} & G \end{array} \tag{5.21}$$

and which is uniquely characterised by the following universal property: Given a group K and homomorphisms $\gamma_i\colon G_i \to K$ with $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$ there is a unique homomorphism $\varphi\colon G \to K$ so that $\varphi \circ \beta_i = \gamma_i$. We write $G = G_1 *_A G_2$.

We remark that Definition 5.2.3 is still a sensible definition without the requirement that α_1 and α_2 be injective, however, if they are injective there is a nice link between algebra and topology which is given by *Whitehead's theorem.* This theorem says that every amalgamation diagram (5.21) can be realised by a diagram of $K(\pi_1, 1)$-complexes for the corresponding groups with the arrows replaced by inclusions. This implies the existence of a Mayer-Vietoris sequence for (co-) homology. We will need a version of this principle which works for (co-) homology with coefficients. It was first noted in [Lyn50] and a proof can be found in [Swa69].

Theorem 5.2.4 (Lyndon [Lyn50], Swan [Swa69]) *Let $G = G_1 *_A G_2$ be a free product with amalgamated subgroup A, and let $\varrho\colon G \to \mathrm{Aut}_{\mathbb{Z}}(M)$ be a representation of G on a $\mathbb{Z}$-module M, making M into a $\mathbb{Z}G$-module. Then the following naturally defined Mayer-Vietoris sequences are exact:*

$$\cdots \to \mathrm{H}_{n+1}(G, \varrho) \to \mathrm{H}_n(A, \varrho) \to \mathrm{H}_n(G_1, \varrho) \oplus \mathrm{H}_n(G_2, \varrho) \to \mathrm{H}_n(G, \varrho) \to \cdots$$
$$\cdots \to \mathrm{H}^{n-1}(A, \varrho) \to \mathrm{H}^n(G, \varrho) \to \mathrm{H}^n(G_1, \varrho) \oplus \mathrm{H}^n(G_2, \varrho) \to \mathrm{H}^n(A, \varrho) \to \cdots$$

The next theorem is again well-known. It establishes the fundamental link between the (co-) homology (with coefficients) of a $K(\pi_1(B), 1)$-space B and the group (co-) homology (with coefficients) of $\pi_1(B)$.

Theorem 5.2.5 *Let B be a $K(\pi_1(B), 1)$ space and $\varrho\colon \pi_1(B) \to \mathrm{Aut}(M)$ be a representation of its fundamental group on a (left-) module M over an associative ring R with identity. Then for all p*

$$\mathrm{H}^p(B, M) \cong \mathrm{H}^p(\pi_1(B), \varrho), \quad \mathrm{H}_p(B, M) \cong \mathrm{H}_p(\pi_1(B), \varrho),$$

where the left-hand sides denote the singular (co-) homology with coefficients in the local system determined by the representation ϱ on M, and the right-hand sides are the group (co-) homology of $\pi_1(B)$ with respect to ϱ.

In all our applications, $R = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. There is a standard procedure using the *bar resolution* of M over the group ring $\mathbb{Z}G$ which, in principle, allows to compute all homology and cohomology groups with coefficients in a module M. In practice, this can still be complicated, but at least for degrees 0 and 1 we can give an easy explicit description after introducing another piece of notation.

Definition 5.2.6 *Let G be a group and M a G-module. A derivation is a function $d\colon G \to M$ such that for all $g, h \in G$ we have $d(gh) = dg + g \cdot dh$. A derivation is called principal if it is of the form $g \mapsto g \cdot \xi - \xi$ for some fixed $\xi \in M$. We let $\mathrm{Der}(G, M)$ be the derivations of G in M and $\mathrm{Der}_0(G, M)$ the principal derivations.*

Note that if G operates trivially on M then derivations are just group homomorphisms, $\mathrm{Der}(G, M) \cong \mathrm{Hom}(G, M)$.

Proposition 5.2.7 *Let G be a group and M a G-module. Then*

$$\begin{aligned}
\mathrm{H}_0(G,M) &= \frac{M}{\langle g\xi - \xi \mid g \in G, \xi \in M\rangle} = M_G \\
\mathrm{H}^0(G,M) &= \{\xi \in M \mid \forall g \in G\colon g\cdot\xi = \xi\} = M^G \\
\mathrm{H}^1(G,M) &= \frac{\mathrm{Der}(G,M)}{\mathrm{Der}_0(G,M)}
\end{aligned}$$

$H_0(G,M)$ is the largest quotient of M on which G acts trivially (the space of *coinvariants*), while $H^0(G,M)$ is the largest submodule of M on which G acts trivially (the space of *invariants*). Also the following will be useful.

Definition 5.2.8 *For a group G we define the* <u>*cohomological dimension*</u>

$$\mathrm{cd}(G) = \max\{n \in \mathbb{N} \mid \mathrm{H}^n(G,\varrho) \neq \{0\} \text{ for some }-\text{representation } \varrho \text{ of } G\}.$$

This may or may not be finite for a given group G, for instance, $\mathrm{cd}(\mathbb{Z}/n) = \infty$ for all $n \in \mathbb{N}$. In general the computation of group cohomology can be very difficult, even for the most familiar groups. One simple case however is that of a free cyclic group.

Proposition 5.2.9 *Suppose G is a free cyclic group and M is a G-module. Then*

$$\begin{aligned}
\mathrm{H}^0(G,M) &\cong \mathrm{H}_1(G,M) \cong M^G \\
\mathrm{H}^1(G,M) &\cong \mathrm{H}_0(G,M) \cong M_G \\
\mathrm{H}^i(G,M) &\cong \mathrm{H}_i(G,M) \cong \{0\} \quad \text{for } i > 1.
\end{aligned}$$

Proof. See any book on group homology, for instance [Bro82], p. 58. □

Using Proposition 5.2.7, these modules can be easily computed.

Proposition 5.2.10 *Let G be a group and suppose there exists an n-dimensional smooth manifold B which is a $K(G,1)$ space. Then $\mathrm{cd}(G) \leq n$ with equality if and only if B is closed.*

Proof. See [Bro82], Proposition 8.1, p.210. □

By Theorem 4.1.32, if G is the group of a knot or irreducible link $\mathfrak{l} \subset S^3$ then $S^3 \setminus \mathfrak{l}$ is a $K(G,1)$. So Proposition 5.2.10 implies the following corollary.

Corollary 5.2.11 *The group of a knot or irreducible link has no cohomology with coefficients in degree > 2 for any representation.*

A connected Riemann surface Σ of genus $g > 0$ is covered by the disc, hence a $K(\pi_1(\Sigma),1)$ space. Going back to Theorem 4.1.11, we conclude

Corollary 5.2.12 *Let Σ be an open Riemann surface of genus $g > 0$ and $n \in \mathbb{N}$. Then every affine T^n-bundle over Σ has a section.*

Proof. Let $f\colon X \to \Sigma$ be an affine T^n-bundle with fibre $F \cong T^n$, monodromy $\varrho\colon \pi_1(\Sigma) \to \mathrm{Aut}_{\mathbb{Z}}(H_1(F, \mathbb{Z}))$ and Chern-class $c(f) \in H^2(\Sigma, \varrho)$. Since $g > 0$, the Riemann surface Σ is a $K(\pi_1(\Sigma), 1)$ space. Since Σ is open, Proposition 5.2.10 implies $H^2(\pi_1(\Sigma), \varrho) = \{0\}$ and hence $c(f) = 0$. Then Theorem 4.1.11 says that f has a section. □

This applies in particular to the case $\Sigma = S^2 \setminus \{p_1, \dots, p_k\}$ with $k \geq 2$; the two-sphere with k points removed is a Riemann surface of genus $k - 1 > 0$. Hence if $f\colon X \to S^2$ is any T^n-fibration with discrete discriminant locus $\Delta = \{p_1, \dots, p_k\} \subset S^2$ and $n \leq 3$, its smooth part $f_0\colon X_0 \to S^2 \setminus \{p_1, \dots, p_k\}$ is completely determined by the monodromy representation.

5.2.2 (Co-) homology of link complements and T^3-fibrations

Let $U_1 \subset S^3$ be a tubular neighbourhood of the knot (or link) $\mathfrak{l}$ and let $U_0 = S^3 \setminus \mathcal{K}(\mathfrak{l})$ where $\mathcal{K}(\mathfrak{l}) \subset U_1$ is a compact "thickening" of $\mathfrak{l}$. Let $X_i = f^{-1}(U_i)$ for $i = 0, 1$. So $X = X_0 \cup X_1$ is the total space of our fibration. Then if we use the Mayer Vietoris sequence

$$\cdots \to \mathrm{H}^p(X, \mathbb{Z}) \to \mathrm{H}^p(X_0, \mathbb{Z}) \oplus \mathrm{H}^p(X_1, \mathbb{Z}) \xrightarrow{\psi_p} \mathrm{H}^p(X_0 \cap X_1, \mathbb{Z}) \to \mathrm{H}^{p+1}(X, \mathbb{Z}) \to \cdots \tag{5.22}$$

we can compute the cohomology of X in terms of the cohomology of X_0 and X_1. In particular we obtain the following formula for the Betti numbers:

$$b^p(X) = b^p(X_0) + b^p(X_1) + b^{p-1}(X_0 \cap X_1) - \mathrm{rk}(\psi_p) - \mathrm{rk}(\psi_{p-1}).$$

The following Lemma describes the integral cohomology of a link complement.

Proposition 5.2.13 *Let $\mathfrak{l} \subset S^3$ be a link with μ components and $U_0 = S^3 \setminus \mathcal{K}(\mathfrak{l})$ the complement of a small compact normal neighbourhood of $\mathfrak{l}$. Then the integral (co-) homology of U_0 is given by*

p	0	1	2	3
$\mathrm{H}^p(U_0, \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^\mu$	$\mathbb{Z}^{\mu-1}$	0
$\mathrm{H}_p(U_0, \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^\mu$	$\mathbb{Z}^{\mu-1}$	0

This is well-known. $H_1(U_0, \mathbb{Z})$ is generated by the images in homology of the meridians in $\pi_1(U_0)$, while $H_2(U_0, \mathbb{Z})$ is generated by the boundary components $T_1, \dots, T_\mu$ of U_0, subject to the relation $[T_1] + \cdots + [T_\mu] = 0$.

5.3 Examples

In this section we show how to obtain T^3-fibrations of the kind discussed earlier on manifolds such as $S^3 \times S^3$ and $S^1 \times S^2 \times S^3$ degenerating over torus knots. We give a detailed account of the trefoil in the first paragraph and review other torus knots in the second.

5.3.1 The trefoil knot

We investigate the T^3-fibrations obtained from the non-trivial M-representation $\varrho\colon G(\mathfrak{t}(2,3)) \to \mathrm{SL}(2,\mathbb{Z})$ of the trefoil knot $\mathfrak{t}(2,3) \subset S^3$ (see Theorem 5.1.8). The following rather curious fact can be found in [Mil71].

Theorem 5.3.1 *The complement of the trefoil knot $\mathfrak{t}(2,3) \subset S^3$ is diffeomorphic to the homogeneous space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$.*

Proof. We outline a proof due to Quillen. First note that $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ is naturally the space of lattices $\Lambda \subset \mathbb{C}$ with unit volume. To each such Λ we can associate its Weierstraß $\wp$-function which satisfies the differential equation

$$\left(\frac{\partial \wp(z)}{\partial z}\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ are defined by the formulas

$$g_2(\Lambda) = 60 \sum_{\omega \in \Lambda\setminus\{0\}} \frac{1}{\omega^4}, \quad g_3(\Lambda) = 140 \sum_{\omega \in \Lambda\setminus\{0\}} \frac{1}{\omega^6}. \tag{5.23}$$

Given a lattice Λ and $t > 0$ the invariants of Λ and of the rescaled lattice $t\Lambda$ are related by

$$g_2(t\Lambda) = t^{-4} g_2(\Lambda), \quad g_3(t\Lambda) = t^{-6} g_3(\Lambda).$$

This is obvious from (5.23). All that matters for us now is that given a lattice Λ of unit volume, g_2, g_3 cannot both vanish simultaneously, and we can rescale the lattice by a unique $t > 0$ so that $(g_2(t\Lambda), g_3(t\Lambda)) \in S^3 \subset \mathbb{C}^2$. This defines a smooth map

$$\psi\colon \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SL}(2,\mathbb{Z})} \longrightarrow S^3$$

mapping $\Lambda \mapsto (g_2(t\Lambda), g_3(t\Lambda))$ for an appropriate $t > 0$. A pair $(g_2, g_3) \in S^3$ can arise in this manner if and only if the discriminant Δ satisfies

$$\Delta = 27g_3^2 - g_2^3 \neq 0.$$

ψ is injective because Λ is characterised uniquely by g_2 and g_3. Thus the image of ψ is the intersection of S^3 with the complement of the complex curve $\{z_1^3 = 27z_2^2\} \subset \mathbb{C}^2$. This curve is well-known to intersect S^3 in a $(2,3)$-torus knot, that is a trefoil knot (see [Mil68]). $\square$

The previous result suggests a nice way of obtaining a T^2-fibred five-manifold. Let g_2, g_3 be the usual coordinates on $\mathbb{C}^2$ and $S^3 \subset \mathbb{C}^2$ be the standard sphere of elements with $|g_2|^2 + |g_3|^2 = 1$. Let $[x:y:z]$ be homogeneous coordinates on $\mathbb{C}P^2$. Then we define $X \subset \mathbb{C}P^2 \times S^3$ by the equation

$$4x^3 - g_2 x z^2 - g_3 z^3 - y^2 z = 0. \tag{5.24}$$

Proposition 5.3.2 *X is a smooth compact T^2-fibred five-manifold.*

Proof. Use inhomogeneous coordinates to write X as the vanishing locus of a smooth function with 0 as a regular value. For instance, on $U = \{[x:y:z] \in \mathbb{C}P^2 \mid z \neq 0\}$ we can choose inhomogeneous coordinates $u = x/z$ and $v = y/z$ and write $X \cap (U \times S^3)$ as $f_U^{-1}(0)$ for the smooth function

$$f_U \colon (u, v, g_2, g_3) \in \mathbb{C}^2 \times S^3 \longmapsto 4u^3 - g_2 u - g_3 - v^2 \in \mathbb{C}.$$

The unitary condition $g_2\bar{g}_2 + g_3\bar{g}_3 = 1$ ensures that there are no critical points in $f^{-1}(0)$. The implicit function theorem then shows that X is smooth, and it is obviously compact. $\square$

Proposition 5.3.3 *The T^2-fibration $X \to S^3$ has a section and its monodromy is given by the unique non-trivial M-representation $\varrho\colon G(\mathfrak{t}(2,3)) \to \mathrm{SL}(2,\mathbb{Z})$.*

Proof. See Theorem 5.1.8 for the representation ϱ. The T^2-fibration X has a section $\sigma\colon S^3 \to X$ mapping

$$(g_2, g_3) \in S^3 \longmapsto ([0:1:0], (g_2, g_3)) \in X.$$

$\sigma(g_2, g_3)$ is the point at infinity of $f^{-1}(g_2, g_3)$. Now recall that $G = \langle\, x, y \mid x^2 = y^3 \,\rangle$. There is a short exact sequence

$$\{1\} \longrightarrow 2\mathcal{Z}(G) \longrightarrow G \longrightarrow \mathrm{SL}(2,\mathbb{Z}) \longrightarrow \{1\}.$$

The restriction of ϱ to $2\mathcal{Z}(G) = \langle x^4 \rangle = \langle y^6 \rangle$ is trivial, that is $\varrho(x^4) = \varrho(y^6) = \mathrm{Id}$. Thus ϱ descends to a representation of $G/2\mathcal{Z}(G) = \mathrm{SL}(2,\mathbb{Z})$ which is easily seen to be the standard representation of $\mathrm{SL}(2,\mathbb{Z})$. Thus we can write the T^2-bundle with section X' associated to ϱ as

$$X' = \mathrm{SL}(2,\mathbb{R}) \times_{\mathrm{SL}(2,\mathbb{Z})} \mathbb{T}^2.$$

By construction, we have $X' \cong X_0 = X \setminus f^{-1}(\mathfrak{t}(2,3))$. $\square$

Theorem 5.3.4 *X is a spin manifold, that is the second Stiefel-Whitney class $w_2(X) = 0$ vanishes.*

Proof. Let $\mathcal{N} \subset T(\mathbb{C}P^2 \times S^3)|_X$ be the normal bundle of the submanifold, then

$$w_2(X) + w_2(\mathcal{N}) = w_2(\mathbb{C}P^2 \times S^3)|_X = (\mathrm{pr}_1^* w_2(\mathbb{C}P^2))|_X = (\mathrm{pr}_1^* c_1(-3\mathsf{H}))|_X \bmod 2$$

where the hyperplane H generates $H_2(\mathbb{C}P^2, \mathbb{Z}) \cong H_2(\mathbb{C}P^2 \times S^3, \mathbb{Z})$ and $\mathrm{pr}_1 \colon \mathbb{C}P^2 \times S^3 \to S^3$ is projection. But since the equation (5.24) is a section of $\mathcal{O}(-3)$ we get

$$w_2(\mathcal{N}) = (-3c_1(\mathsf{H}) \bmod 2) = (c_1(\mathsf{H}) \bmod 2),$$

and thus $w_2(X) = 0$ which means X is spin (see also Theorem 4.1.27). $\square$

We would now like to identify this manifold. We first compute its integral homology. We define the following open subsets:

$$\begin{array}{rclr} U_0 & = & S^3 \setminus \mathfrak{t}(2,3), & X_0 = f^{-1}(U_0), \\ U_1 & = & \mathcal{N}(\mathfrak{t}(2,3)), & X_1 = f^{-1}(U_1). \end{array} \qquad (5.25)$$

As usual, $\mathcal{N}(\mathfrak{t}(2,3))$ denotes a tubular neighbourhood of the knot.

Theorem 5.3.5 *The integral homology of the five-manifolds X_0, X_1 and $X_0 \cap X_1$ is as follows:*

p	$\mathrm{H}_p(X_0)$	$\mathrm{H}_p(X_1)$	$\mathrm{H}_p(X_0 \cap X_1)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2$
2	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$
4	0	0	$\mathbb{Z}$
5	0	0	0

Proof. Let $G = G(\mathfrak{t}(2,3))$ be the trefoil group and let us think of $M \cong (\mathbb{Z}^2, \varrho)$ as a $\mathbb{Z}G$-module with the natural action induced from ϱ. Then

$$\mathrm{H}_n(G, \varrho) \cong \mathrm{H}_n(U_0, M) \cong \mathrm{H}_n(\overline{U}_0, M) \qquad (5.26)$$

is an equality between the group homology of G with coefficients in the representation ϱ and the ordinary homology of $U_0 \simeq \overline{U}_0$ with coefficients in the local system defined by M.

Homology of X_0. We can compute $H_*(X_0,\mathbb{Z})$ from the Leray spectral sequence of the fibre bundle whose E^2-term is the following:

$$\begin{array}{cccc} \mathbb{Z} & \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z}/2 & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 & 0 \end{array} \tag{5.27}$$

Here $E^2_{p,q} \cong H_p(G, \wedge^q \varrho)$ by (5.26) for $0 \leq p \leq 3$ and $0 \leq q \leq 2$. Since $\wedge^0 \varrho = \wedge^2 \varrho$ is trivial we have $E^2_{p,0} \cong E^2_{p,2} \cong H_p(U_0, \mathbb{Z})$ and so the top and bottom rows are given by Proposition 5.2.13. For the middle row we have from Proposition 5.2.7

$$E^2_{0,1} = \mathrm{H}_0(G,\varrho) = \frac{M}{\{g\xi - \xi \mid \xi \in M, \ \ g \in G\}} \cong \frac{\mathbb{Z}^2}{\{\varrho(g)\xi - \xi \mid \xi \in \mathbb{Z}^2, \ \ g \in G\}}.$$

However, with respect to the standard basis $M \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and meridians g_1, g_2 as in Chapter 4 we have $\varrho(g_2)e_1 - e_1 = -e_2$ and $\varrho(g_1)e_2 - e_2 = e_1$, thus the right hand side is trivial and $H_0(G,\varrho) \cong E^2_{0,1} \cong \{0\}$. Also $H_3(G,\varrho) \cong E^2_{3,1} \cong \{0\}$ follows from Corollary 5.2.11. To compute $E^2_{2,1}$ and $E^2_{1,1}$ we use the Mayer-Vietoris sequence in Theorem 5.2.4. We can write $G = \langle x \rangle *_{\langle x^2 = y^3 \rangle} \langle y \rangle$ as the amalgamated sum of two cyclic groups $G_1 = \langle x \rangle$ and $G_2 = \langle y \rangle$ along the cyclic subgroup $A \cong \langle x^2 \rangle \cong \langle y^3 \rangle$. By Proposition 5.2.9 cyclic groups have no homology in degree > 1 and the first term which has a chance of being non-zero is $H_2(G,\varrho) \cong E^2_{2,1}$, so the tail of this sequence looks like

$$0 \longrightarrow E^2_{2,1} \longrightarrow \mathrm{H}_1(A,\varrho) \xrightarrow{\psi_1} \mathrm{H}_1(G_1,\varrho) \oplus \mathrm{H}_1(G_2,\varrho) \longrightarrow \cdots . \tag{5.28}$$

By Proposition 5.2.9 and Proposition 5.2.7

$$\begin{array}{rcccl} \mathrm{H}_1(A,\varrho) & \cong & \mathrm{H}^0(A,\varrho) & \cong & \{\xi \in \mathbb{Z}^2 \mid \varrho(x^2)\xi = \xi\} \\ \mathrm{H}_1(G_1,\varrho) & \cong & \mathrm{H}^0(G_1,\varrho) & \cong & \{\xi \in \mathbb{Z}^2 \mid \varrho(x)\xi = \xi\} \\ \mathrm{H}_1(G_2,\varrho) & \cong & \mathrm{H}^0(G_2,\varrho) & \cong & \{\xi \in \mathbb{Z}^2 \mid \varrho(y)\xi = \xi\} \end{array}$$

We check from the matrices given in Theorem 5.1.8 that

$$\begin{array}{rcccl} \varrho(x) & = & \varrho(g_3 g_2 g_1) & = & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \varrho(y) & = & \varrho(g_3 g_2) & = & \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \\ \varrho(x^2) & = & \varrho(y^3) & = & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \end{array}$$

We conclude that none of the representations induced from ϱ on A, G_1 and G_2 has invariants, and thus

$$\mathrm{H}_1(G_1,\varrho) \cong \mathrm{H}_1(G_2,\varrho) \cong \mathrm{H}_1(A,\varrho) \cong \{0\}.$$

Hence $E^2_{2,1} \cong \mathrm{H}_2(G,\varrho) \cong \{0\}$. But also $\operatorname{coker}(\psi_1) = \{0\}$ in (5.28) and so $E^2_{1,1} \cong \mathrm{H}_1(G,\varrho)$ is the kernel of the map ψ_0 in the short exact sequence

$$0 \to E^2_{1,1} \to \mathrm{H}_0(A,\varrho) \xrightarrow{\psi_0} \mathrm{H}_0(G_1,\varrho) \oplus \mathrm{H}_0(G_2,\varrho) \to \mathrm{H}_0(G,\varrho) \cong \{0\}. \tag{5.29}$$

We compute these 0-th homology groups from Proposition 5.2.7. Note that

$$\begin{aligned} B = \varrho(x) - \mathrm{Id} &= \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \\ B' = \varrho(y) - \mathrm{Id} &= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \\ B'' = \varrho(x^2) - \mathrm{Id} = \varrho(y^3) - \mathrm{Id} &= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}. \end{aligned}$$

Since $\det(B) = 2$ and $\det(B') = 1$ we deduce that

$$\begin{aligned} \mathrm{H}_0(G_1,\varrho) &\cong \frac{\mathbb{Z}^2}{B\mathbb{Z}^2} \cong \mathbb{Z}/2, \\ \mathrm{H}_0(G_2,\varrho) &\cong \frac{\mathbb{Z}^2}{B'\mathbb{Z}^2} \cong \{0\}, \\ \mathrm{H}_0(A,\varrho) &\cong \frac{\mathbb{Z}^2}{B''\mathbb{Z}^2} \cong \frac{\mathbb{Z}^2}{2\mathbb{Z}^2} \cong \mathbb{Z}/2 \times \mathbb{Z}/2. \end{aligned}$$

Thus (5.29) becomes a short exact sequence of groups

$$0 \longrightarrow E^2_{1,1} \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

The only group fitting into this sequence is $E^2_{1,1} \cong \mathbb{Z}/2$. It is obvious that the boundary operator $\partial_2 \colon E^2_{p,q} \to E^2_{p-2,q+1}$ of the E^2-term must be zero since all non-trivial homology is concentrated in degree 0 and 1. Hence the homology X_0 is the total homology of $E^2_{p,q}$ which is the homology given in the proposition.

Homology of X_1. The singular fibration $X_1 \to U_1$ is homotopy equivalent to a topological fibre bundle of singular elliptic curves of type I_1 over the circle. Note that a longitude ℓ acts on $M \cong \mathbb{Z}^2$ via

$$\varrho(\ell) = \begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix}. \tag{5.30}$$

Let F_0 be a singular fibre. Then $H_p(F_0,\mathbb{Z}) \cong \mathbb{Z}$ for $p = 0,1,2$ and $H_p(F_0,\mathbb{Z}) = \{0\}$ for $i > 2$, and (5.30) implies that the monodromy around ℓ acts as (-1) on $H_1(F_0,\mathbb{Z}) \cong \mathbb{Z}$. Cover S^1 with two open intervals U', V' and let $U = X_1|_{U'}$ and $V = X_1|_{V'}$ so that

$$\mathrm{H}_p(U,\mathbb{Z}) \oplus \mathrm{H}_p(V,\mathbb{Z}) \cong \mathrm{H}_p(U \cap V,\mathbb{Z}) \cong \mathrm{H}_p(F_0,\mathbb{Z}) \oplus \mathrm{H}_p(F_0,\mathbb{Z}) \cong \mathbb{Z}^2$$

for $p = 0, 1, 2$. Choosing appropriate trivialisations the map ψ_p in the Mayer-Vietoris sequence

$$\cdots \to \mathrm{H}_{p+1}(X_1, \mathbb{Z}) \to \mathrm{H}_p(U \cap V, \mathbb{Z}) \xrightarrow{\psi_p} \mathrm{H}_p(U, \mathbb{Z}) \oplus \mathrm{H}_p(V, \mathbb{Z}) \to \mathrm{H}_p(X_1, \mathbb{Z}) \to \cdots \tag{5.31}$$

takes the form $(a, b) \in \mathbb{Z}^2 \mapsto (a - b, a - \mu_p b) \in \mathbb{Z}^2$ where μ_p is the monodromy action on $H_p(F_0, \mathbb{Z})$. We have $\mu_0 = \mu_2 = 1$ and $\mu_1 = -1$. Because $H_*(F_0, \mathbb{Z})$ is torsion free and all modules are Abelian, the Mayer-Vietoris sequence splits at all of its boundary operators, and

$$\mathrm{H}_p(X_1, \mathbb{Z}) = \ker(\psi_{p-1}) \oplus \operatorname{coker}(\psi_p) \tag{5.32}$$

for all p. Let $\psi^\pm \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ be the map $(a, b) \mapsto (a - b, a \mp b)$, then

$$\begin{aligned} \ker(\psi^+) &= \operatorname{diag}(\mathbb{Z}^2) \cong \mathbb{Z}, \\ \operatorname{coker}(\psi^+) &= \frac{\mathbb{Z}^2}{\operatorname{diag}(\mathbb{Z}^2)} \cong \mathbb{Z}, \\ \ker(\psi^-) &= \{(a, b) \in \mathbb{Z}^2 \mid a + b = a - b = 0\} \cong \{0\}, \qquad \text{(no 2 torsion)} \\ \operatorname{coker}(\psi^-) &= \frac{\mathbb{Z}^2}{\{(a - b, a + b) \mid a, b \in \mathbb{Z}\}} \cong \mathbb{Z}/2. \end{aligned}$$

Now $\psi_0 = \psi_2 = \psi^+$, $\psi_1 = \psi^-$ and $\psi_p = 0$ for $p > 2$. Together with (5.32) this implies the claim.

Homology of $X_0 \cap X_1$. We compute again the Leray spectral sequence for the T^2-bundle over $U_0 \cap U_1 \simeq T^2$. We claim that the E^2-term has the following form:

$$\begin{array}{ccc} \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} \\ \mathbb{Z}/2 & \mathbb{Z}/2 & 0 \\ \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} \end{array} \tag{5.33}$$

Again, the top and bottom rows are just the integral homology of $U_0 \cap U_1 \simeq T^2$. For the middle row note that the inclusion $U_0 \cap U_1 \hookrightarrow U_0$ is an inclusion of $K(\pi_1, 1)$-complexes corresponding to the embedding $G' = \langle m, \ell \rangle \to G$ of the subgroup of G generated by a longitude ℓ and a meridian m into G. Hence let $\varrho' = \varrho | G'$. Then $E^2_{p,1} \cong H_p(G', \varrho')$ for $p = 0, 1, 2$. As before we compute $H_0(G', \varrho')$ from Proposition 5.2.7 and equations (5.1) and (5.30) which give explicitly the actions of m and ℓ on M with respect to the standard trivialisation of $M \cong \mathbb{Z}^2$. We find

$$E^2_{0,1} \cong \mathrm{H}_0(G', \varrho') \cong \mathbb{Z}/2.$$

Note that $\varrho \sim \varrho^*$ by Corollary 5.1.2 and

$$E^2_{2,1} \cong \mathrm{H}_2(G', \varrho') \cong \mathrm{H}^0(G', (\varrho')^*) \cong \mathrm{H}^0(G', \varrho') \cong \{0\}$$

by Poincaré-duality in Proposition 5.2.1 ((5.30) shows that ϱ' has no invariants). Finally in order to compute $E^2_{1,1}$ we again use Poincaré-duality and find

$$E^2_{1,1} \cong \mathrm{H}_1(G', \varrho') \cong \mathrm{H}^1(G', \varrho').$$

So we need to determine the derivations of ϱ'. We make an Ansatz

$$d(m) = \mu = \begin{pmatrix} r \\ s \end{pmatrix}, \quad d(\ell) = \lambda = \begin{pmatrix} t \\ u \end{pmatrix}.$$

This defines a derivation if and only if $0 = d(1) = d(m\ell m^{-1}\ell^{-1})$ which by virtue of the relations $d(gh) = dg + g \cdot dh$ and $d(g^{-1}) = -g^{-1} \cdot dg$ transforms by expansion into the equation

$$\begin{aligned} (\varrho'(m) - \mathrm{Id})\lambda &= (\varrho'(\ell) - \mathrm{Id})\mu \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} &= \begin{pmatrix} -2 & 6 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix}. \end{aligned}$$

The most general solution is given by choosing $r, t \in \mathbb{Z}$ arbitrary and setting $s = 0$, $u = -2t$. Therefore $\mathrm{Der}(\varrho') \cong \mathbb{Z}^2$. The corresponding derivation $d_{r,t}$ given by $\mu = (r, 0)$ and $\lambda = (t, -2r)$ is principal if and only if there is $\kappa \in \mathbb{Z}^2$ so that

$$\mu = (\varrho'(m) - \mathrm{Id})\kappa \quad \text{and} \quad \lambda = (\varrho'(\ell) - \mathrm{Id})\kappa.$$

The unique solution to these equations is

$$\kappa = \begin{pmatrix} -\frac{t}{2} - 3r \\ r \end{pmatrix}$$

which is integral only if $t \equiv 0 \bmod 2$, hence the principal derivations correspond to the submodule $\mathbb{Z} \oplus 2\mathbb{Z} \subset \mathbb{Z}^2$ and so

$$\mathrm{H}_1(G', \varrho') \cong \mathrm{H}^1(G', \varrho') \cong \left(\frac{\mathbb{Z}^2}{2\mathbb{Z} \oplus \mathbb{Z}}\right) \cong \mathbb{Z}/2.$$

Thus we find the term

$$E^2_{1,1} \cong \mathbb{Z}/2$$

which completes the computation of the last missing term in the spectral sequence. (It is easy to check that for the other obvious cover of T^2 we get

$$\mathrm{H}^1(V, M) \oplus \mathrm{H}^1(V', M) \cong \mathrm{H}^1(V \cap V', M) \cong (\mathbb{Z}/2)^4$$

and $\mathrm{im}(\psi_1) \cong (\mathbb{Z}/2)^3$, hence $\mathrm{coker}(\psi_1) \cong \mathbb{Z}/2$ as before.) Again the sequence degenerates because $E^2_{2,1} = \{0\}$, and the other arrow $\partial_2 \colon E^2_{2,0} \to E^2_{0,1}$ vanishes because

$$\mathrm{H}_2(U_0 \cap U_1, \mathbb{Z}) \xrightarrow{\sigma_*} E^2_{2,0} \xrightarrow{\partial_2} E^2_{0,1}$$

is an exact sequence. Here σ is a section with $\sigma_* = (f_*|E^2_{2,0})^{-1}$, hence σ_* is surjective and $\partial_2 = 0$. The homology is thus given by the total homology of the E^2-term which is as described in the proposition. □

Theorem 5.3.6 *The integral homology of X is as follows:*

p	0	1	2	3	4	5
$\mathrm{H}_p(X,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$

Proof. We apply the Mayer-Vietoris sequence

$$\cdots \to \mathrm{H}_{p+1}(X,\mathbb{Z}) \to \mathrm{H}_p(X_0 \cap X_1,\mathbb{Z}) \xrightarrow{\psi_p} \mathrm{H}_p(X_0,\mathbb{Z}) \oplus \mathrm{H}_p(X_1,\mathbb{Z}) \to \mathrm{H}_p(X,\mathbb{Z}) \to \cdots \tag{5.34}$$

to $X = X_0 \cup X_1$. Let $[m], [\ell] \in H_1(U_0 \cap U_1, \mathbb{Z})$ be the classes of a meridian and a longitude, respectively, let F be a fibre and σ a section of $X_0 \to U_0$. Now obviously $H_0(X,\mathbb{Z}) = \mathbb{Z}$. We also know already that $H_1(X,\mathbb{Z}) = \{0\}$ because Theorem 4.1.33 states that X is simply connected. But observe that $H_1(X_0 \cap X_1, \mathbb{Z})_{\mathrm{free}}$ is generated by $\sigma_*[m]$ and $\sigma_*[\ell]$, while $H_1(X_0,\mathbb{Z}) \cong \mathbb{Z}(\sigma_*[m])$ and $H_1(X_1,\mathbb{Z})_{\mathrm{free}} \cong \mathbb{Z}(\sigma_*[\ell])$. Then $\psi_1(\sigma_*[m]) = (\sigma_*[m], 0)$ and $\psi_1(\sigma_*[\ell]) = (0, -\sigma_*[\ell])$ in $H_1(X_0,\mathbb{Z}) \oplus H_1(X_1,\mathbb{Z})$. This implies that ψ_1 induces an isomorphism of the free parts

$$\mathrm{H}_1(X_0 \cap X_1,\mathbb{Z})_{\mathrm{free}} \xrightarrow{\cong} \mathrm{H}_1(X_0,\mathbb{Z}) \oplus \mathrm{H}_1(X_1,\mathbb{Z})_{\mathrm{free}}. \tag{5.35}$$

Let $[\tau_1] \in H_1(X_1,\mathbb{Z})$ and $[\tau_2] \in H_1(X_0 \cap X_1,\mathbb{Z})$ be the unique torsion elements. Since ψ_1 has to be onto in order for $H_1(X,\mathbb{Z})$ to vanish, we conclude $[\tau_1] \in \mathrm{im}(\psi_1)$. Given the isomorphism (5.35) this is only possible if $\psi_1([\tau_2]) = [\tau_1]$ for the torsion class, and thus ψ_1 is an isomorphism of $\mathbb{Z}$-modules. So $\mathrm{coker}(\psi_1) = \ker(\psi_1) = \{0\}$. Therefore $H_2(X,\mathbb{Z}) \cong \mathrm{coker}(\psi_2)$. Note that

$$\mathrm{H}_2(X_0,\mathbb{Z})_{\mathrm{free}} \cong \mathrm{H}_2(X_1,\mathbb{Z}) \cong \mathbb{Z}[F]$$

is generated by the class of a fibre, while for a generator $T \in H_2(U_0 \cap U_1, \mathbb{Z})$

$$\mathrm{H}_2(X_0 \cap X_1,\mathbb{Z})_{\mathrm{free}} \cong \mathbb{Z}[F] \oplus \mathbb{Z}(\sigma_*[T]).$$

Thus $\psi_2([F]) = ([F], -[F])$ while $\psi_2(\sigma_*[T]) = 0$. Now denote the degree-2 torsion classes by $[\tau_3] \in H_2(X_0,\mathbb{Z})$ and $[\tau_4] \in H_2(X_0 \cap X_1,\mathbb{Z})$. Suppose $\psi_2([\tau_4]) = [\tau_3]$. In this case we get

$$\begin{aligned} \ker(\psi_2) &= \mathbb{Z}([\sigma_* T]) \cong \mathbb{Z}, && (5.36)\\ \mathrm{coker}(\psi_2) &= \frac{\mathbb{Z}[F] \oplus \mathbb{Z}[F] \oplus \mathbb{Z}/2[\tau_3]}{\{(a[F], -a[F]) \mid a \in \mathbb{Z}\} \oplus \mathbb{Z}/2[\tau_3]} \cong \frac{\mathbb{Z}^2}{\mathrm{diag}(\mathbb{Z}^2)} \cong \mathbb{Z} && (5.37) \end{aligned}$$

and therefore

$$\mathrm{H}_2(X,\mathbb{Z}) \cong \mathrm{coker}(\psi_2) \cong \mathbb{Z}[T] \cong \mathbb{Z}.$$

Otherwise if $\psi_2([\tau_4]) \neq \tau_3$ then $\psi_2([\tau_4]) = 0$ since it is a torsion class. But then we obtain $H_2(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. This is a contradiction since the torsion subgroup $H_2(X,\mathbb{Z})_{\mathrm{tors}} \cong \frac{1}{2}T \oplus \frac{1}{2}T$ of a simply connected spin five-manifold is always decomposable as a direct sum of two isomorphic torsion modules (see [Sma62]). So we can rule out the case $[\tau_4] \in \ker(\psi_2)$ and $H_2(X,\mathbb{Z}) \cong \mathbb{Z}$. For the remaining homology groups $H_p(X,\mathbb{Z})$ with $p > 2$ we only give a sketch since Smale's Theorem 5.3.8 implies that they are already uniquely determined by $H_2(X,\mathbb{Z})$. For $Y = X_0, X_1, X_0 \cap X_1$ there is an isomorphism

$$\tau_{[F]} \colon \mathrm{H}_1(Y,\mathbb{Z})_{\mathrm{free}} \longrightarrow \mathrm{H}_3(Y,\mathbb{Z}).$$

This isomorphism sends the classes of m and ℓ to $[f^{-1}(m)]$ and $[f^{-1}(\ell)]$. Then

$$\psi_3 \circ \tau_{[F]} = (\tau_{[F]}, \tau_{[F]}) \circ \psi_1,$$

hence ψ_3 is an isomorphism as is ψ_1. Thus

$$\mathrm{H}_3(X,\mathbb{Z}) = \ker(\psi_2) \cong \mathbb{Z}$$

by (5.37). For $i > 3$, we have $\psi_i = 0$ trivially, and

$$\mathrm{H}_4(X,\mathbb{Z}) = \mathrm{coker}(\psi_4) = \frac{\{0\}}{\{0\}} = \{0\}$$

since $H_4(X_i,\mathbb{Z}) = \{0\}$ for $i = 0, 1$. Finally

$$\mathrm{H}_5(X,\mathbb{Z}) = \ker(\psi_4) = \mathrm{H}_4(X_0 \cap X_1,\mathbb{Z}) \cong \mathbb{Z},$$

which is also clear otherwise since X is compact, orientable and $\dim(X) = 5$. □

Corollary 5.3.7 *X is an (integral) homology $S^2 \times S^3$.*

We are now in a position to show that X is diffeomorphic to $S^2 \times S^3$. This uses the following theorem of Smale:

Theorem 5.3.8 (Smale [Sma62]) *There is a bijective correspondence between the category of smooth, closed, orientable, simply connected, 5-dimensional spin manifolds and the category of finitely generated Abelian groups. The correspondence is realised by sending a manifold X in this category to the group $F \oplus \frac{1}{2}T$, where $F \oplus T = \mathrm{H}_2(X,\mathbb{Z})$ is a decomposition of the second integral homology of X into its free and its torsion subgroups, and $T = \frac{1}{2}T \oplus \frac{1}{2}T$ is a direct sum decomposition of the torsion subgroup.*

This is an analogue for five-manifolds of Wall's classification theorem, however, it is easier to apply since we do not need to know the ring-structure on homology, and it is also more general in that it includes manifolds with torsion in homology.

Theorem 5.3.9 *X is diffeomorphic to $S^2 \times S^3$.*

Proof. The five-manifold X is smooth and compact by Proposition 5.3.2 and simply connected by Theorem 4.1.33, it is oriented by Proposition 4.1.12. X is torsion free and has the right Betti-numbers by Theorem 5.3.5. X is spin from Theorem 5.3.4. Hence the claim follows by applying Theorem 5.3.8. □

This gives us torus fibrations on well-known manifolds. We formulate this in the following theorem.

Theorem 5.3.10 *There are T^3-fibrations on $S^1 \times S^2 \times S^3$ and $S^3 \times S^3$ whose singularities are locally equivalent to (4.4).*

Proof. For the first we simply take the product $S^1 \times X$ and extend the fibration map $f\colon X \to S^3$ trivially, for the second we take the Hopf map $H\colon S^3 \to S^2$ and let $\tilde{f}\colon S^3 \times S^3 \to S^3$ be given by $\tilde{f} = f \circ (H, \mathrm{id})$. Note that this fibration has no section, while the first one obviously has one. □

5.3.2 General torus knots

In this section we give a brief account how to deal with other torus knots. We focus on $\mathfrak{t}(4,3)$ first and then show reduce the general case to either $\mathfrak{t}(2,3)$ or $\mathfrak{t}(4,3)$.First observe that we can easily modify the argument given in Paragraph 5.3.1 to obtain a smooth compactification of a T^2-bundle over $S^3 \setminus \mathfrak{t}(p,q)$ for any pair of coprime integers $(p,q) \in \mathbb{N}^2$ such that $2|p$ and $3|q$. As in (5.24), we define $X_{p,q} \subset \mathbb{C}P^2 \times S^3$ by the equation

$$4x^3 - g_2^{q/3}xz^2 - g_3^{p/2}z^3 - y^2z = 0.$$

and let again the fibration $f\colon X_{p,q} \to S^3$ be given by $f = \mathrm{pr}_2|X_{p,q}$ where $\mathrm{pr}_2\colon \mathbb{C}P^2 \times S^3 \to S^3$ is projection onto the second factor. We can realise $X_{p,q}$ as the vanishing locus of a smooth section $\sigma \in C^\infty(\mathrm{pr}_1^*\mathcal{O}(-3))$ which is transversal to the zero-section, hence $X_{p,q}$ is a smooth compact submanifold of $\mathbb{C}P^2 \times S^3$. For generic $(g_2, g_3) \in S^3$ the fibre $f^{-1}(g_2, g_3)$ is an elliptic curve in $\mathbb{C}P^2$ determined by the Weierstraß equation with coefficients $g_2^{q/3}$ and $g_3^{p/2}$. The fibre is singular if and only if the discriminant vanishes, that is

$$\Delta = 27\left(g_3^{p/2}\right)^2 - \left(g_2^{q/3}\right)^3 = 27g_3^p - g_2^q = 0.$$

From [Mil68] the discriminant locus is a (p,q)-torus knot. The argument for the vanishing of the second Stiefel-Whitney class in Paragraph 5.3.1 carries over directly to $X_{p,q}$. As before one can check that $X_{p,q}$ is the canonical compactification of the T^2-bundle associated to the unique non-trivial self-dual M-representation $\varrho_{p,q}\colon G(p,q) \to \mathrm{SL}(2,\mathbb{Z})$ which takes values in $\mathrm{SL}(2,\mathbb{Z})$. From Paragraph 5.1.2 and Theorem 4.1.33 we deduce that $X_{p,q}$ is simply connected. To summarise:

Theorem 5.3.11 *$X_{p,q} \subset \mathbb{C}P^2 \times S^3$ is a smooth, compact, orientable submanifold with $\pi_1(X_{p,q}) = \{1\}$ and $w_2(X_{p,q}) = 0$. There is a T^2-fibration $f\colon X_{p,q} \to S^3$ with singularities of type (4.4) which degenerates over a (p,q)-torus knot $\mathfrak{t}(p,q) \subset S^3$.*

The $(4,3)$-torus knot

Recall from Theorem 5.1.9 and its proof in Section A.1 that the group $G = G(\mathfrak{t}(4,3))$ of the $(4,3)$-torus knot $\mathfrak{t}(4,3)$ has a unique non-trivial M-representation $\varrho = \varrho_0\colon G \to \mathrm{SL}(2,\mathbb{Z})$ which can be given, with respect to a presentation $G = \langle x, y \mid x^4 = y^3\rangle$, by

$$\varrho(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varrho(y) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \tag{5.38}$$

see Appendix A.1. Note that $x^4 = y^3 = \mathrm{Id}$ and the meridian $m = x^{-1}y$ is mapped to the standard monodromy matrix A_2 given in (5.1). We now study the T^2-bundle $f\colon X \to S^3$ associated to this representation. As before, let

$$\begin{array}{rcll} U_0 & = & S^3 \setminus \mathfrak{t}(4,3), & X_0 = f^{-1}(U_0), \\ U_1 & = & \mathcal{N}(\mathfrak{t}(4,3)), & X_1 = f^{-1}(U_1). \end{array}$$

and denote by M the $\mathbb{Z}G$-module given by the monodromy representation ϱ.

Theorem 5.3.12 *The integral homology of X_0, X_1 and $X_0 \cap X_1$ is given by the following table:*

p	$\mathrm{H}_p(X_0)$	$\mathrm{H}_p(X_1)$	$\mathrm{H}_p(X_0 \cap X_1)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$
2	$\mathbb{Z}^3$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
3	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}^3$
4	0	0	$\mathbb{Z}$
5	0	0	0

Proof. Homology of X_0. The T^2-bundle $f_0 \colon X_0 \to U_0$ has a Leray spectral sequence with the following E^2-term:

$$\begin{array}{cccc} \mathbb{Z} & \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 & 0 \end{array} .$$

Again the top and bottom rows are the homology of U_0 as given in Proposition 5.2.13. Also $E^2_{0,1} = \{0\}$ because $\operatorname{im}(\varrho)$ contains the monodromy matrices A_2 and A_2^T and obviously

$$\operatorname{im}(A_2 - \mathrm{Id}) + \operatorname{im}(A_2^T - \mathrm{Id}) = M.$$

As usual, $E^2_{3,1} = \{0\}$ by Corollary 5.2.11. Let again $G_1 = \langle x \rangle$ and $G_2 = \langle y \rangle$ and write G as the amalgamated sum $G = G_1 *_A G_2$ along the cyclic subgroup $A \cong \langle x^4 \rangle \subset G_1$ and $A \cong \langle y^3 \rangle \subset G_2$. Applying Theorem 5.2.4 to this presentation, we get the following exact sequence:

$$0 \longrightarrow \mathrm{H}_2(G, \varrho) \longrightarrow \mathrm{H}_1(A, \varrho) \xrightarrow{\psi_1} \mathrm{H}_1(G_1, \varrho) \oplus \mathrm{H}_1(G_2, \varrho) \longrightarrow \cdots$$

Now by Proposition 5.2.9

$$\mathrm{H}_1(A, \varrho) \cong \mathrm{H}^0(A, \varrho) \cong M^A \cong M$$

because $\varrho(x^4) = \varrho(y^3) = \mathrm{Id}$ and the restriction of ϱ to A acts trivially on M. On the other side, it is obvious from (5.38) that there are no invariants in M for the actions of G_1 or G_2, hence

$$\mathrm{H}_1(G_i, \varrho) \cong \mathrm{H}^0(G_i, \varrho) = \{0\}, \quad i = 1, 2. \tag{5.39}$$

This implies $H_2(G, \varrho) \cong M \cong \mathbb{Z}^2$. But also the vanishing of the modules (5.39) implies that the end of the Mayer-Vietoris sequence is

$$0 \to \mathrm{H}_1(G, \varrho) \to \mathrm{H}_0(A, \varrho) \to \mathrm{H}_0(G_1, \varrho) \oplus \mathrm{H}_0(G_2, \varrho) \to \mathrm{H}_0(G, \varrho). \tag{5.40}$$

$H_0(A, \varrho) \cong M \cong \mathbb{Z}^2$ because $\varrho|A$ is trivial. As to G_1 and G_2 we observe that

$$\begin{array}{rcccl} B & = & \varrho(x) - \mathrm{Id} & = & \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \\ B' & = & \varrho(y) - \mathrm{Id} & = & \begin{pmatrix} -1 & 1 \\ -1 & -2 \end{pmatrix} . \end{array}$$

Hence $\det(B) = 2$ and $\det(B') = 3$ and so

$$\mathrm{H}_0(G_1, \varrho) \oplus \mathrm{H}_0(G_2, \varrho) \cong \frac{\mathbb{Z}^2}{B\mathbb{Z}^2} \oplus \frac{\mathbb{Z}^2}{B'\mathbb{Z}^2} \cong \mathbb{Z}/2 \times \mathbb{Z}/3 \cong \mathbb{Z}/6.$$

Therefore the exact sequence (5.40) is

$$\{0\} \longrightarrow \mathrm{H}_1(G,\varrho) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}/6 \longrightarrow \{0\}. \tag{5.41}$$

The only $\mathbb{Z}$-module which fits into (5.41) is $H_1(G,\varrho) \cong E^2_{1,1} \cong M \cong \mathbb{Z}^2$, hence the result for X_0.

Homology of X_1. X_1 is homotopy equivalent to a bundle $F_0 \to X_1 \to S^1$ of I_1-fibres. The longitude ℓ of $\mathfrak{t}(4,3)$ operates on M via

$$\varrho(\ell) = \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix}. \tag{5.42}$$

and hence the monodromy action of $\pi_1(S^1) \cong \mathbb{Z}[\ell]$ on $H_*(F_0,\mathbb{Z})$ is trivial. $H_i(F_0,\mathbb{Z}) \cong \mathbb{Z}$ for $i = 0,1,2$ and $\cong \{0\}$ otherwise, so if we write $X_1 = U \cup V$ in a standard manner, we get

$$\mathrm{H}_p(U \cap V,\mathbb{Z}) \cong \mathrm{H}_p(F_0,\mathbb{Z}) \oplus \mathrm{H}_p(F_0,\mathbb{Z}) \cong \mathbb{Z}^2$$

and the map ψ_p in the Mayer-Vietoris sequence (5.31) is $(a,b) \in \mathbb{Z}^2 \mapsto (a-b, a-b) \in \mathbb{Z}^2$ and thus $\ker(\psi_p) \cong \mathrm{coker}(\psi_p) \cong \mathbb{Z}$ for $p = 0,1,2$. Because $H_p(U \cap V,\mathbb{Z})$ is free Abelian, we get $H_p(X_1,\mathbb{Z}) = \ker(\psi_{p-1}) \oplus \mathrm{coker}(\psi_p)$. This implies the result.

Homology of $X_0 \cap X_1$. Here we use again the Leray spectral sequence which we claim has the following E^2-term:

$$\begin{matrix} \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} \end{matrix}\,.$$

Of course $E^2_{0,p} \cong E^2_{2,p} \cong H_p(T^2,\mathbb{Z})$ and we only need to check $E^2_{1,p} \cong H_p(G',\varrho)$ for $G' = \langle m,\ell\rangle \subset G$ and $p = 0,1,2$, where we again write ϱ for the action of G' induced from G by restriction. Choose the standard basis $M \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Then $I = \mathrm{im}(\varrho(m) - \mathrm{Id}) = \mathbb{Z}e_1$ and $\mathrm{im}(\varrho(\ell) - \mathrm{Id}) = 12I$, hence

$$\mathrm{H}_0(G',\varrho) \cong \frac{\mathbb{Z}^2}{\mathbb{Z}e_1} \cong \mathbb{Z}e_2 \cong \mathbb{Z}.$$

Similarly the space of invariants is $H^0(G',\varrho) = \mathbb{Z}e_1$ and so

$$\mathrm{H}_2(G',\varrho) \cong \mathrm{H}^0(G',\varrho^*) \cong \mathrm{H}^0(G',\varrho) \cong M^{G'} \cong \mathbb{Z}.$$

Finally let us compute $E^2_{1,1}$. We first look at $H^1(G',\varrho)$. To find the derivations of ϱ we make an Ansatz $d(m) = \mu$, $d(\ell) = \lambda$ with $\mu,\lambda \in \mathbb{Z}^2$ and find that this extends to a derivation if and only if

$$(\varrho(m) - \mathrm{Id})\lambda = (\varrho(\ell) - \mathrm{Id})\mu$$

which is equivalent to the equation $\lambda_2 = 12\mu_2$. Hence $\mathrm{Der}(\varrho) \cong \mathbb{Z}^3$. For $r, s, t \in \mathbb{Z}$ the derivation d_{rst} given by

$$d_{rst}(m) = \mu = \begin{pmatrix} r \\ s \end{pmatrix}, \quad d_{rst}(\ell) = \lambda = \begin{pmatrix} t \\ 12s \end{pmatrix}$$

is principal if and only if $\mu = (\varrho(m) - \mathrm{Id})\kappa$ and $\lambda = (\varrho(\ell) - \mathrm{Id})\kappa$ for some $\kappa \in \mathbb{Z}^2$ which is solvable precisely if $s = 0$ and $t = 12r$. Hence

$$\mathrm{H}^1(G', \varrho) \cong \frac{\mathrm{Der}(G', \varrho)}{\mathrm{Der}_0(G', \varrho)} \cong \mathbb{Z}^2.$$

But now by Poincaré-duality

$$\mathrm{H}_1(G', \varrho) \cong \mathrm{H}^1(G', \varrho^*) \cong \mathrm{H}^1(G', \varrho) \cong \mathbb{Z}^2.$$

This completes the proof. □

Theorem 5.3.13 *The integral homology of $X = X_0 \cup X_1$ is as follows:*

p	0	1	2	3	4	5
$\mathrm{H}_p(X, \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}$

Proof. This is very similar to Theorem 5.3.6, and we use again the notation (5.25) and the Mayer-Vietoris sequence (5.34). Obviously $H_0(X, \mathbb{Z}) = \mathbb{Z}$ and $H_1(X, \mathbb{Z}) = \{0\}$ because X is simply connected by Theorem 4.1.33. Let $\sigma\colon S^3 \to X$ be a section. Let $T \subset U_0 \cap U_1$ be a homologically non-trivial two-torus, $F \hookrightarrow X_0$ a fibre and let $\gamma \subset F$ represent a primitive monodromy-invariant class in $H_1(F, \mathbb{Z})^{\langle m, \ell \rangle}$, where $m, \ell \subset U_0 \cap U_1$ are again a meridian and longitude, respectively. Then $H_2(X_0 \cap X_1, \mathbb{Z}) \cong \mathbb{Z}^4$ has a set of generators

$$\mathrm{H}_2(X_0 \cap X_1, \mathbb{Z}) \cong \mathbb{Z}\sigma_*[T] \oplus \mathbb{Z}(\gamma \times \sigma_*[m]) \oplus \mathbb{Z}(\gamma \times \sigma_*[\ell]) \oplus \mathbb{Z}[F] \cong \mathbb{Z}^4$$

where $\gamma \times \sigma_*[m]$ means a two-cycle supported on an S^1-bundle over m with fibre homologous to γ, and similarly $\gamma \times \sigma_*[\ell]$. The fibre-class $[F] \in H_2(X_0, \mathbb{Z})$ is a non-trivial and primitive (due to the section) homology class. Now

$$\begin{aligned} \mathrm{H}_2(X_0, \mathbb{Z}) &\cong \mathbb{Z}(\gamma \times \sigma_*[m]) \oplus \mathbb{Z}(\gamma \times \sigma_*[\ell]) \oplus \mathbb{Z}[F] \cong \mathbb{Z}^3 \\ \mathrm{H}_2(X_1, \mathbb{Z}) &\cong \mathbb{Z}(\gamma \times \sigma_*[\ell]) \oplus \mathbb{Z}[F] \cong \mathbb{Z}^2 \end{aligned}$$

and with respect to this trivialisation, the map $\psi_2\colon H_2(X_0 \cap X_1, \mathbb{Z}) \cong \mathbb{Z}^4 \to H_2(X_0, \mathbb{Z}) \oplus H_2(X_1, \mathbb{Z}) \cong \mathbb{Z}^5$ is given by $\psi_2(r, s, t, u) = ((s, t, u), (-t, -u))$ and thus

$\operatorname{coker}(\psi_2) \cong \mathbb{Z}^2$. Furthermore, if we write the general element of $H_1(X_0 \cap X_1, \mathbb{Z})$ as $a\sigma_*[m] + b\sigma_*[\ell] + c[\gamma]$ and if we use

$$\begin{aligned} \mathrm{H}_1(X_0, \mathbb{Z}) &\cong \mathbb{Z}(\sigma_*[m]) \cong \mathbb{Z} \\ \mathrm{H}_1(X_1, \mathbb{Z}) &\cong \mathbb{Z}(\sigma_*[\ell]) \oplus \mathbb{Z}[\gamma] \cong \mathbb{Z}^2, \end{aligned}$$

then $\psi_1 \colon \mathbb{Z}^3 \to \mathbb{Z} \oplus \mathbb{Z}^2$ becomes the map $(a, b, c) \mapsto (a, (b, c))$ and hence $\ker(\psi_1) = \{0\}$. Thus

$$\mathrm{H}_2(X, \mathbb{Z}) \cong \ker(\psi_1) \oplus \operatorname{coker}(\psi_2) \cong \mathbb{Z}^2.$$

Smale's Theorem 5.3.8 then yields the remaining homology groups. □

The following corollary follows from Smale's Theorem 5.3.8 in the same way as the corresponding statements in Paragraph 5.3.1.

Corollary 5.3.14 *X is diffeomorphic to $(S^2 \times S^3) \# (S^2 \times S^3)$.*

Proof. X is closed, spin, and simply connected by Theorem 5.3.11. The description of the integral homology in Theorem 5.3.13 together with Smale's Theorem 5.3.8 yields the result. □

Note that $H_2((S^2 \times S^3) \# (S^2 \times S^3), \mathbb{Z}) \cong \mathbb{Z}^2$. For each $r, s \in \mathbb{Z}$ let

$$h^{r,s} \colon X^{r,s} \longrightarrow (S^2 \times S^3) \# (S^2 \times S^3) \tag{5.43}$$

be an S^1-bundle with $c_1(X^{r,s}) = (r, s)$ (having chosen suitable trivialisations and orientations). Then $X^{r,s} \cong X^{r',s'}$ are diffeomorphic as affine torus fibrations if and only if $\{|r|, |s|\} = \{|r'|, |s'|\}$ (see Theorem 4.1.11).

Theorem 5.3.15 *Let $r \geq s \geq 0$ be positive integers. There are T^3-fibrations with singularities of type (4.4)*

$$f^{r,s} \colon X^{r,s} \longrightarrow S^3 \tag{5.44}$$

degenerating over a $(4, 3)$-torus knot $\mathfrak{t}(4, 3) \subset S^3$.

Proof. Set $f^{r,s} = f \circ h^{r,s}$ where $h^{r,s}$ is defined in (5.43). In particular, for $r = s = 0$ we have $X^{r,s} = S^1 \times ((S^2 \times S^3) \# (S^2 \times S^3))$. □

The (p, q)-torus knot

We only give a brief sketch and argue that we do not get any new manifolds by using other torus knots.

Theorem 5.3.16 *Let $p, q \in \mathbb{N}$ be coprime positive integers with $2|p$ and $3|q$, let $\mathfrak{k} = \mathfrak{t}(p,q) \subset S^3$ be the (p,q)-torus knot and let $f\colon X \to S^3$ be the T^2-bundle with section associated to the unique M-representation $\varrho = \varrho_0\colon G(\mathfrak{k}) \to \mathrm{SL}(2,\mathbb{Z})$. Then either $4 \not| p$ and $X \cong S^2 \times S^3$ or $4|p$ and $X \cong (S^2 \times S^3)\#(S^2 \times S^3)$.*

Sketch of proof. Let $G(\mathfrak{k}) = \langle x, y \,|\, x^p = y^q \rangle$ and $G(2,3) = \langle \bar{x}, \bar{y} \,|\, \bar{x}^2 = \bar{y}^3 \rangle$. Let $\bar{\varrho}\colon G(2,3) \to \mathrm{SL}(2,\mathbb{Z})$ the unique non-trivial M-representation of $G(2,3)$. Then (see Appendix A.3) $\varrho(x) = \bar{\varrho}(\bar{x})^{\pm 1}$ and $\varrho(y) = \bar{\varrho}(\bar{y})^{\pm 1}$. Also note that the image of a longitude ℓ under our representation is

$$\varrho(\ell) = \begin{pmatrix} -1 & pq \\ 0 & -1 \end{pmatrix} = \bar{\varrho}(\bar{\ell})^{pq/6}.$$

One can then check that all calculations in Section 5.3 remain valid for the representation $\varrho\colon G(\mathfrak{k}) \to \mathrm{SL}(2,\mathbb{Z})$. The case of (p,q)-torus knots with $4|p$ is similar. □

Corollary 5.3.17 *There are infinitely many pairwise inequivalent T^3-fibrations with singularities of type (4.4)*

$$\begin{array}{rcl} S^3 \times S^3 & \longrightarrow & S^3 \\ S^1 \times S^2 \times S^3 & \longrightarrow & S^3 \\ S^1 \times ((S^2 \times S^3)\#(S^2 \times S^3)) & \longrightarrow & S^3, \end{array}$$

and more generally such fibrations exist on every S^1-bundle over $S^2 \times S^3$ or $(S^2 \times S^3)\#(S^2 \times S^3)$.

5.4 Open problems

We conclude this chapter with a few open questions that emerge from our work on topological T^3-fibrations.

1. Since our original motivation was to learn something about mirror symmetry, an obvious question is whether our good T^3-fibrations can ever be special Lagrangian. As a first step, one should therefore ask whether their total spaces can ever be symplectic, and if so, whether the torus fibrations can be made Lagrangian. Note that none of our explicit examples involving torus knots was even cohomologically symplectic. We have found a number of algebraic conditions on the cohomology of an M-representation which are necessary for the existence of an $[\omega] \in H^2(X,\mathbb{R})$ with $[\omega]^3 \neq 0$, however, they did not seem to be very useful and we do not give them here. If X is symplectic and $f\colon X \to B$

is a *Lagrangian* good T^3-fibration then the critical set $\mathrm{Crit}(f) \subset X$ should be a collection of *symplectic* T^2's, each of which would hence need to represent a non-trivial homology class in $H_2(X, \mathbb{Z})$. This imposes some restrictions on the M-representations.

2. In a smooth Lagrangian fibration the set of regular values $B \setminus \Delta$ in the base has a natural flat affine connection whose monodromy is determined by the symplectic structure. It would be interesting to know more about the existence of such flat linear structures on knot complements. Most knot complements have a *hyperbolic* structure, however, torus knots considered here do not. One should probably expect that a generic special Lagrangian fibration is not generally good.

3. While we have not found any good T^3-fibrations on symplectic manifolds, many of our examples (like $S^3 \times S^3$ and $S^1 \times S^2 \times S^3$) are complex. An obvious question to ask is whether we can make these T^3-fibrations totally real with respect to some complex structure.

4. Since most of our examples of good T^3-fibrations were connected sums of spheres and products of spheres, one could ask whether good T^3-fibrations exist on $\#^n(S^3 \times S^3)$ for $n \geq 2$. These manifolds admit complex structures with holomorphically trivial canonical bundle [LT96].

5. Recall that Gross' well-behaved T^3-fibration on the quintic Q has more complicated singular fibres than our good fibrations, namely "positive" and "negative" singular fibres with Euler characteristic ± 1 which arise over isolated points (the "dissident" points) in the base. This is necessary for topological reasons because $\chi(Q) = -200$ does not vanish. However, there are Calabi-Yau manifolds X with $\chi(X) = 0$, and there is no obvious obstruction for such an X to support a good T^3-fibration. It would be interesting to find actual Calabi-Yau manifolds with good T^3-fibrations. Such examples are likely to exist.

6. In Theorem 3.1.12 we showed that the (finite) total volume of McLean's metric on the moduli-space of an elliptic $K3$ surface is a holomorphic invariant. It would be interesting to identify this number or understand better its geometric significance. We believe it may be related to the holomorphic prepotential function of the special Kähler structure.

Appendix A

Computations with knot groups

In this appendix we give some computations with knot groups which serve to verify the assertions made earlier. All of this is purely mechanical and can be easily skipped if the reader is willing to believe the claims made in Chapter 5.

A.1 The M-representations of $G(\mathfrak{t}(4,3))$

We give a proof of Theorem 5.1.9 which describes the M-representations in $\mathrm{SL}(3,\mathbb{Z})$ of the torus knot $\mathfrak{t}(4,3)$.

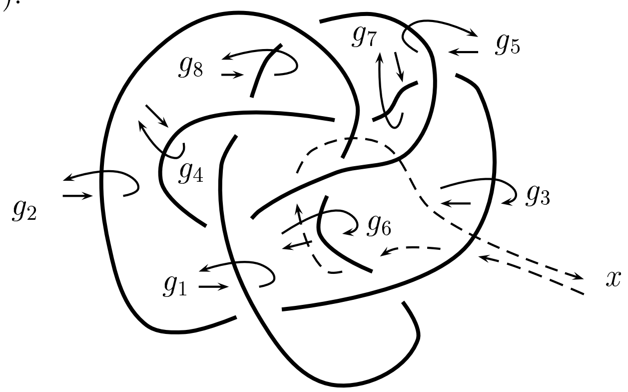

Figure A.1: *The* $(4,3)$*-torus knot* $\mathfrak{t}(4,3)$ *(or* 8_{19}*).*

Proof. From this knot-diagram we obtain the following Wirtinger-presentation of G:

$$g_1^{-1}g_3g_1 = g_2 \qquad \text{(A.1)} \qquad\qquad g_2^{-1}g_8g_2 = g_5 \qquad \text{(A.3)}$$
$$g_3^{-1}g_6g_3 = g_1 \qquad \text{(A.2)} \qquad\qquad g_4^{-1}g_1g_4 = g_8 \qquad \text{(A.4)}$$

$$g_5^{-1}g_2g_5 = g_6 \quad (A.5) \qquad g_1^{-1}g_5g_1 = g_4 \quad (A.7)$$
$$g_2^{-1}g_4g_2 = g_7 \quad (A.6) \qquad g_5^{-1}g_7g_5 = g_3 \quad (A.8)$$

(If the diagram does not convince the reader that this is the $(4,3)$-torus knot, see Appendix A.2 where we prove formally that (A.1) - (A.8) is a Wirtinger presentation of $\mathfrak{t}(4,3)$. Note that $\mathfrak{t}(4,3)$ has no alternating diagram.) The thin arrows stand for meridians - loops g_i which originate in the reader's eye, go straight to the beginning of the correspondingly labelled arrow, encircle the knot in the direction of the arrow and return directly to the base-point.

Suppose we are given an M-representation ϱ with $\varrho(g_i) = G_i \in \mathrm{SL}(3,\mathbb{Z})$. Let us write G_1, G_3, G_5 as

$$G_1 = \Phi([\eta_1,\xi_1]), \quad G_3 = \Phi([\eta_3,\xi_3]), \quad G_5 = \Phi([\eta_5,\xi_5])$$

for certain $(\xi_i,\eta_i)_{i=1,3,5} \in \mathcal{S}$ (see (5.4) for our notation). Given those matrices, all other G_j's are uniquely determined, and we use five of the Wirtinger relations to find the images of the other meridians (in the following order)

$$\begin{array}{llllll} G_2 \overset{(A.1)}{=} G_1^{-1}G_3G_1 & G_6 \overset{(A.2)}{=} G_3G_1G_3^{-1} & G_8 \overset{(A.3)}{=} G_2G_5G_2^{-1} \\ G_4 \overset{(A.7)}{=} G_1^{-1}G_5G_1 & G_7 \overset{(A.6)}{=} G_5G_3G_5^{-1} & . \end{array} \quad (A.9)$$

Every M-representation of G is of that form, and for given $(\eta_j,\xi_j)_{i=1,3,5} \in \mathcal{S}$ this is an M-representation if and only if the remaining three Wirtinger-relations are satisfied, that is

$$G_8 = G_4^{-1}G_1G_4, \quad G_6 = G_5^{-1}G_2G_5, \quad G_7 = G_2^{-1}G_4G_2. \quad (A.10)$$

Set $r_{ij} = \langle \xi_i, \eta_j \rangle$. Let us define

$$\begin{aligned}
(\eta_2,\xi_2) &= (\eta_3 - r_{13}\eta_1, \xi_3 + r_{31}\xi_1)\\
(\eta_4,\xi_4) &= (\eta_5 - r_{15}\eta_1, \xi_5 + r_{51}\xi_1)\\
(\eta_6,\xi_6) &= (\eta_1 + r_{31}\eta_3, \xi_1 - r_{13}\xi_3)\\
(\eta_7,\xi_7) &= (\eta_3 + r_{53}\eta_5, \xi_3 - r_{35}\xi_5)\\
(\eta_8,\xi_8) &= (\eta_5 + r_{25}\eta_2, \xi_5 - r_{52}\xi_2)
\end{aligned}$$

Proposition 5.1.6 then tells us that $G_j = \Phi([\eta_j,\xi_j])$ are the unique matrices satisfying (A.9). Moreover, $r_{25} = r_{35} + r_{31}r_{15}$ and $r_{52} = r_{53} - r_{13}r_{51}$, so in the last equation the pair (η_8,ξ_8) can be expressed in terms of $(\eta_j,\xi_j)_{j=1,3,5}$ as

$$\begin{aligned}
\eta_8 &= -r_{13}(r_{35} + r_{31}r_{15})\eta_1 + (r_{35} + r_{31}r_{15})\eta_3 + \eta_5\\
\xi_8 &= -r_{31}(r_{53} - r_{13}r_{51})\xi_1 - (r_{53} - r_{13}r_{51})\xi_3 + \xi_5.
\end{aligned}$$

The necessary and sufficient conditions (A.10) now become

$$
\begin{aligned}
(r_{13}(r_{35}+r_{31}r_{15})+1+r_{15}r_{51})\eta_1-(r_{35}+r_{31}r_{15})\eta_3-(1+r_{51})\eta_5 &= 0 \quad \text{(A.11)}\\
(r_{31}(r_{53}-r_{13}r_{51})+1+r_{51}r_{15})\xi_1+(r_{53}-r_{13}r_{51})\xi_3-(1-r_{15})\xi_5 &= 0 \quad \text{(A.12)}\\
(1+r_{13})\eta_1-(1-r_{31})\eta_3+(r_{53}-r_{13}r_{51})\eta_5 &= 0 \quad \text{(A.13)}\\
(1-r_{31})\xi_1-(1+r_{13})\xi_3-(r_{35}+r_{31}r_{15})\xi_5 &= 0 \quad \text{(A.14)}\\
(r_{15}-r_{35}r_{13})\eta_1+(1+r_{35})\eta_3-(1-r_{53})\eta_5 &= 0 \quad \text{(A.15)}\\
-(r_{51}+r_{53}r_{31})\xi_1+(1-r_{53})\xi_3-(1+r_{35})\xi_5 &= 0. \quad \text{(A.16)}
\end{aligned}
$$

Case 1: Suppose η_1, η_2, η_3 are linearly independent. Then the equations involving the η's can only be satisfied if all coefficients vanish. This is equivalent to requiring

$$r_{15}=r_{31}=r_{53}=1, \quad r_{51}=r_{13}=r_{35}=-1.$$

Since we are eventually only interested in M-representations, we may assume without loss of generality that $\eta_1=e_1, \xi_1=e_2$. We can then write

$$\eta_3=\begin{pmatrix} a\\ -1\\ b\end{pmatrix}, \quad \xi_3=\begin{pmatrix} 1\\ c\\ d\end{pmatrix}, \quad \eta_5=\begin{pmatrix} e\\ 1\\ f\end{pmatrix}, \quad \xi_5=\begin{pmatrix} -1\\ g\\ h\end{pmatrix}$$

for some $a,\dots,h\in\mathbb{Z}$. Primitivity poses no further restriction since ± 1 is always a coefficient, however, $r_{jj}=\langle\xi_j,\eta_j\rangle=0$ for $j=3,5$ and $r_{53}=-r_{35}=1$ imposes the conditions

$$
\begin{aligned}
a-c+bd &= 0, && \text{(A.17)}\\
g-e+fh &= 0, && \text{(A.18)}\\
c+e+fd &= -1, && \text{(A.19)}\\
-a-g+bh &= 1. && \text{(A.20)}
\end{aligned}
$$

So choose $b,d,e,f,h\in\mathbb{Z}$ subject to the relation $(b+f)(d+h)=0$. This means that (A.20) follows from (A.17) - (A.19), and we can use the first three equations to set $g=e-fh$, $a=-e+(f+b)h-1$ and $c=-e-fd-1$. This yields

$$
\begin{aligned}
G_1 &= \begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\\
G_3 &= \begin{pmatrix} -e+(b+f)h & (1+e+fd)(1+e+(f+b)d) & -(1+e+(f+b)d)d\\ -1 & 2+e+fd & -d\\ b & -b(1+e+fd) & 1+bd\end{pmatrix},\\
G_5 &= \begin{pmatrix} 1-e & e^2-efh & eh\\ -1 & 1+e-fh & h\\ -f & ef-f^2h & 1+fh\end{pmatrix}.
\end{aligned}
$$

To confirm these computations, it is a straightforward check (say, for a computer-algebra programme) that these matrices do indeed define an M-homomorphism, so these are precisely the M-homomorphisms sending $g_1 \mapsto A$. Furthermore, we define the matrices

$$C = \begin{pmatrix} 1 & e & h \\ 0 & 1 & 0 \\ 0 & f & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which are easily seen to satisfy $[C, G_1] = [D, G_1] = 0$. Let $\varepsilon = 0$ if $b + d + f + h \geq 0$ and $\varepsilon = 1$ otherwise. We find that by conjugation $\tilde{G}_j = D^{-\varepsilon} C^{-1} G_j C D^{\varepsilon}$ we can reduce this to the form

$$\tilde{G}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{G}_3 = \begin{pmatrix} 0 & 1 & -|h+d| \\ -1 & 2 & -|h+d| \\ |b+f| & -|b+f| & 1 \end{pmatrix}, \quad \tilde{G}_5 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To summarise, every M-homomorphism $\varrho\colon G \to \mathrm{SL}(3, \mathbb{Z})$ is conjugate in $\mathrm{SL}(3, \mathbb{Z})$ to an M-homomorphism of the family $(\varrho_k)_{k \in \mathbb{Z}}$, given by sending

$$g_1 \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and sending the remaining generator

$$g_3 \longmapsto \begin{pmatrix} 0 & 1 & k \\ -1 & 2 & k \\ 0 & 0 & 1 \end{pmatrix} \text{ if } k < 0, \quad \text{and} \quad g_3 \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ k & -k & 1 \end{pmatrix} \text{ if } k \geq 0.$$

This corresponds to choosing $(\eta_1, \xi_1) = (e_1, e_2)$, $(\eta_3, \xi_3) = (-e_2, e_1)$ and $(\eta_5, \xi_5) = ([e_1 + e_2 + ((k - |k|)/2)e_3], [(e_1 - e_2 + ((k + |k|)/2)e_3)])$. One can verify explicitly that for every ϱ_k the *dual homomorphism* $\varrho_k^* = (\varrho_k^{-1})^T$ is conjugate to ϱ_{-k}. It is then easy to check that two such M-homomorphisms ϱ_k and $\varrho_{k'}$ are conjugate if and only if $k = k'$: Firstly, ϱ_0 is the unique M-homomorphism with $\varrho(G) \cong \mathrm{SL}(2, \mathbb{Z})$ and therefore not equivalent to any other ϱ_k with $k \neq 0$. For all $k > 0$ we have

$$\mathrm{H}_0(G, \varrho_k) \cong \mathbb{Z}/k\mathbb{Z}, \quad \mathrm{H}^0(G, \varrho_k) \cong \mathbb{Z}, \quad \mathrm{H}^0(G, \varrho_{-k}) \cong \{0\}.$$

The group (co-) homology modules (for definitions see Paragraph 5.2.1) depend only on the representation but not a particular presentation of the group. From the last two isomorphisms we see that if there are $k, k' \in \mathbb{Z}$ with $\varrho_k \simeq \varrho_{k'}$ then necessarily k and k' have the same sign. If $k, k' > 0$ then the first isomorphism shows that $k = k'$. If $k, k' < 0$ then $\varrho_{-k} \simeq \varrho_k^* \simeq \varrho_{k'}^* \simeq \varrho_{-k'}$ and hence again $k = k'$.

Case 2: Suppose the η's are linearly dependent. Then either ϱ is trivial or $\mathrm{rk}(\eta_1, \eta_3, \eta_5) = 2$, because if $\eta_1 = \pm\eta_3 = \pm\eta_5$ then the image of ϱ lies in an Abelian group and hence ϱ factors through $G^{\mathrm{ab}} \cong \mathbb{Z}$ and hence is trivial. The argument goes as follows: Due to linear dependence,

$$0 = \det(r_{ij}) = r_{13}r_{35}r_{51} + r_{15}r_{31}r_{53}. \tag{A.21}$$

Take the inner product of (A.13) and (A.14) with η_3, ξ_3 respectively and add them up. By (A.21) we obtain $r_{13} + r_{31} = 0$. Similarly the inner product of (A.15) and (A.16) with η_5, ξ_5 respectively yields $r_{35} + r_{53} = 0$. Finally, use these two relations to eliminate r_{31} and r_{53} from (A.11) and (A.12), then taking inner products with η_1, ξ_1 and adding together we get $(r_{15} + r_{51})(-1 - r_{13}^2) = 0$ and hence $r_{15} + r_{51} = 0$, thus (r_{ij}) is skew symmetric. Because $\mathrm{rk}(\eta_1, \eta_3, \eta_5) = 2$, the coefficient vectors in (A.11), (A.13) and (A.15) which can now be written

$$\begin{aligned}
R_1 &= (r_{13}(r_{35} + r_{13}r_{51}) + 1 - r_{51}^2, -(r_{35} + r_{13}r_{51}), -(1 + r_{51})), && \text{(A.22)}\\
R_2 &= ((1 + r_{13}), -(1 + r_{13}), -(r_{35} + r_{13}r_{51})), && \text{(A.23)}\\
R_3 &= (-(r_{51} + r_{13}r_{35}), (1 + r_{35}), -(1 + r_{35})) && \text{(A.24)}
\end{aligned}$$

are pairwise linearly dependent. We see in (A.23) and (A.24) that $R_{2,1} + R_{2,2} = R_{3,2} + R_{3,3} = 0$. Hence because R_1, R_2, R_3 are pairwise linearly dependent, $R_{i,1} = -R_{i,2} = R_{i,3}$ for all $i = 1, 2, 3$. Thus from (A.22) - (A.24) we get

$$\begin{aligned}
r_{35} + r_{13}r_{51} + 1 + r_{51} &= 0, && \text{(A.25)}\\
-1 - r_{13} - r_{35} - r_{13}r_{51} &= 0, && \text{(A.26)}\\
-r_{51} - r_{13}r_{35} + 1 + r_{35} &= 0. && \text{(A.27)}
\end{aligned}$$

Adding these three equations $(A.25) + (A.26) + (A.27)$ yields

$$0 = 1 + r_{35} - r_{13} - r_{13}r_{35} = (1 - r_{13})(1 + r_{35})$$

and hence $r_{35} = -1$ or $r_{13} = 1$. In the first case, $R_1 = R_2 = R_3 = 0$ and

$$r_{15} = r_{31} = r_{53} = 1, \quad r_{51} = r_{13} = r_{35} = -1$$

as before. We can then proceed as in case 1 and get the representation ϱ_0. In the second case we obtain from (A.23) and (A.24) the equations

$$r_{15} = r_{31} = r_{53} = -1, \quad r_{51} = r_{13} = r_{35} = 1$$

which have the opposite signs. We make the same Ansatz as in case 1, and it turns out that we obtain the same solutions as before. □

A.2 The Wirtinger presentation of $\mathfrak{t}(4,3)$

We show that the knot $\mathfrak{k}$ in Figure A.1 whose group $G(\mathfrak{k})$ has the Wirtinger-presentation (A.1) - (A.8) which we derived from a knot-diagram is indeed the (4, 3)-torus knot $\mathfrak{t}(4,3)$. First set

$$x = g_5g_6g_3 \quad \text{and} \quad y = g_5g_6g_3g_1 \tag{A.28}$$

and let $G' = \langle x, y\rangle \subset G$. Then it follows that

$$\begin{array}{rcll}
g_5g_3 &=& g_7g_5 & \text{by (A.8)}\\
\Rightarrow g_2g_5g_3 &=& g_2g_7g_5 & \\
g_5g_6g_3 &=& g_4g_2g_5 & \text{by (A.5), (A.6)}\\
\Rightarrow g_1g_5g_6g_3 &=& g_1g_4g_2g_5 & \\
&=& g_4g_8g_2g_5 & \text{by (A.4)}\\
&=& g_4g_2g_5g_5 & \text{by (A.3)}\\
&=& g_2g_7g_5g_5 & \text{by (A.6)}\\
&=& g_2g_5g_3g_5 & \text{by (A.8)}\\
&=& g_5g_6g_3g_5 & \text{by (A.5)}\\
\Rightarrow g_1g_5g_6g_3g_3g_1 &=& g_5g_6g_3g_5g_3g_1 & \\
g_1g_5g_6g_3g_1g_2 &=& g_5g_6g_3g_5g_6g_3 & \text{by (A.1)}\\
\Rightarrow g_1g_5g_6g_3g_1g_2g_5 &=& g_5g_6g_3g_5g_6g_3g_5 & \\
g_1g_5g_6g_3g_1g_5g_6 &=& g_5g_6g_3g_5g_6g_3g_5 & \text{by (A.5)}\\
g_1g_5g_6g_3g_1g_5g_6g_3g_1 &=& g_5g_6g_3g_5g_6g_3g_5g_6g_3 & \text{by (A.2)}\\
g_1g_5g_6g_3g_1g_5g_6 &=& g_5g_6g_3g_5g_6g_3g_5 & \text{by (A.5)}\\
g_1g_5g_6g_3g_1g_5g_6g_3g_1 &=& g_5g_6g_3g_5g_6g_3g_5g_6g_3 & \text{by (A.2)}\\
\Rightarrow g_5g_6g_3g_1g_5g_6g_3g_1g_5g_6g_3g_1 &=& g_5g_6g_3g_5g_6g_3g_5g_6g_3g_5g_6g_3 & \\
\Rightarrow (g_5g_6g_3g_1)(g_5g_6g_3g_1)(g_5g_6g_3g_1) &=& (g_5g_6g_3)(g_5g_6g_3)(g_5g_6g_3)(g_5g_6g_3) & \\
y^3 &=& x^4 &
\end{array}$$

Note that we have only used seven out of eight relations ((A.7) was not used); but that is no problem since in a Wirtinger presentation (at least) one relation is always redundant. Hence $x^4 = y^3$, and $G' = \langle x, y\rangle$ is a subgroup of G which is isomorphic to the group of the (4, 3)-torus knot. But we also find that

$$\begin{array}{rcll}
g_1 &=& x^{-1}y \in G' & \text{(A.29)}\\
g_3g_1 &=& g_6g_3 = g_5^{-1}x & \\
\Rightarrow g_5g_3 &=& xg_1^{-1} \in G' & \\
\Rightarrow g_2 &=& g_5g_6g_5^{-1} & \\
&=& (g_5g_3)g_1(g_5g_3)^{-1} \in G' & \\
\Rightarrow g_3 &=& g_1g_2g_1^{-1} \in G' & \text{(A.30)}\\
\Rightarrow g_5 &=& (g_5g_3)g_3^{-1} \in G' & \text{(A.31)}
\end{array}$$

From (A.29), (A.30) and (A.31) we know that $g_1, g_3, g_5 \in G'$, but these elements generate G, so $G' = G$ and the group $G(4,3)$ of the $(4,3)$-torus knot $\mathfrak{t}(4,3)$ is isomorphic to G.

Theorem A.2.1 (Burde-Zieschang [BZ66]) *A non-trivial knot whose group G has a non-trivial centre is a torus knot.*

The group G has obviously non-trivial centre $\mathcal{Z}(G) = \langle x^4 \rangle = \langle y^3 \rangle$, hence $\mathfrak{k}$ is a torus knot. But then it is obvious that $\mathfrak{k} = \mathfrak{t}(4,3)$, because two (unoriented) torus knots $\mathfrak{t}(p,q)$ and $\mathfrak{t}(p',q')$ are equivalent (up to change of orientation of the complement) if and only if $(|p'|,|q'|) = (|p|,|q|)$ or $(|q|,|p|)$. It is more generally true that any prime knot (and all torus knots are prime ([BZ85], Corollary 7.11, p. 92)) is determined by its group, as the following two theorems show.

Theorem A.2.2 (Whitten, Gonzales-Acuña [Whi87]) *If $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are prime knots in S^3 and $\pi_1(S^3 \setminus \mathfrak{k}_1) \cong \pi_1(S^3 \setminus \mathfrak{k}_2)$ are isomorphic groups, then $S^3 \setminus \mathfrak{k}_1$ and $S^3 \setminus \mathfrak{k}_2$ are homeomorphic spaces.*

Theorem A.2.3 (Gordon, Luecke [GL89]) *If $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are unoriented knots in in S^3 and there is an orientation preserving homeomorphism between their complements, then $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are equivalent (as unoriented knots).*

A.3 The M-representations of $G(\mathfrak{t}(p,q))$

In this section we give the details of the proof of Theorem 5.1.10. We first collect some auxiliary results in the first paragraph and then carry out the proof in the second.

A.3.1 Lemmas for the proof

The following lemma describes the centraliser $\mathcal{Z}(A)$ in $\mathrm{SL}(3,\mathbb{Z})$ of our monodromy matrix A given in (5.1).

Lemma A.3.1 *An element of* $\mathrm{SL}(3,\mathbb{Z})$ *commutes with A if and only if it is a matrix of the form*

$$C^{\pm}(a,b,c) = \begin{pmatrix} \pm 1 & a & b \\ 0 & \pm 1 & 0 \\ 0 & c & 1 \end{pmatrix}, \quad a,b,c \in \mathbb{Z} \tag{A.32}$$

for some $a,b,c \in \mathbb{Z}$.

These matrices will play an important role when we determine the representations of torus knot groups. We will need to know about the roots of these matrices in $\mathrm{SL}(3,\mathbb{Z})$.

Lemma A.3.2 *Let $a, b, c \in \mathbb{Z}$ and an integer $p > 0$ be given.*

1. *There exists an $X \in \mathrm{SL}(3,\mathbb{Z})$ such that $X^p = C^+(a,b,c)$ and $\mathrm{spec}(X) = \{1\}$ if and only if $p|\gcd(a,b,c)$. In this case, X is unique with these properties and given by $X = C^+(\tilde{a},\tilde{b},\tilde{c})$, where*
$$\tilde{a} = \frac{2pa - (p+1)bc}{2p^2}, \quad \tilde{b} = \frac{b}{p}, \quad \tilde{c} = \frac{c}{p}. \tag{A.33}$$

2. *There exists an $X \in \mathrm{SL}(3,\mathbb{Z})$ such that $X^p = C^-(a,b,c)$ and all eigenvalues of X are real (hence $\mathrm{spec}(X) \subset \{1,-1\}$) if and only if p is odd and $p|(2a + (p-1)bc)$. In this case X is unique with these properties and given by $X = C^-(\tilde{a},\tilde{b},\tilde{c})$ with*
$$\tilde{a} = \frac{2a + (p-1)bc}{2p}, \quad \tilde{b} = b, \quad \tilde{c} = c. \tag{A.34}$$

Proof. The proof is a straightforward calculation where we think of $\mathrm{SL}(3,\mathbb{Z}) \subset \mathrm{GL}(3,\mathbb{C})$ and use Jordan normal forms. Note that $\mathrm{spec}(C^+(a,b,c)) = \{1\}$ and $\mathrm{spec}(C^-(a,b,c)) = \{1,-1\}$, for any $a,b,c \in \mathbb{Z}$. □

Let $p \in \mathbb{N}$ and suppose $R \in \mathrm{SL}(3,\mathbb{Z})$ satisfies $R^p = C^\pm(a,b,c)$ with $\mathrm{spec}(R) \not\subset \mathbb{R}$. Then R has two complex conjugate eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}$ and one real eigenvalue $\tilde{\lambda} \in \mathbb{R}$. Then $\tilde{\lambda}^p = \pm 1$ implies $\tilde{\lambda} = \pm 1$ and the characteristic polynomial $\chi_R(x) = (x \pm 1)q(x)$ splits over $\mathbb{Z}$ into a linear factor and a quadratic term $q(x)$.

Lemma A.3.3 *The only primitive p-th roots of ± 1 satisfying a quadratic equation $X^2 + aX + b = 0$ with $a, b \in \mathbb{Z}$ arise for $p = 2,3,4,6$ and are given by*

p	$+1$	-1
2	-1	$i, -i$
3	$-\frac{1}{2}\left(1 \pm i\sqrt{3}\right)$	$\frac{1}{2}\left(1 \pm i\sqrt{3}\right)$
4	$i, -i$	n/a
6	$\frac{1}{2}\left(1 \pm i\sqrt{3}\right)$	n/a

Proof. Let $\lambda \in \mathbb{C}^*$ be a (non-real) root of ± 1 and suppose $f \in \mathbb{Z}[X]$ is a quadratic polynomial with leading coefficient 1 such that $f(\lambda) = 0$. Then $f(X) = X^2 + 2\,\mathrm{Re}(\lambda)X + \lambda\bar{\lambda} = X^2 + 2\,\mathrm{Re}(\lambda)X + 1$. This implies $\mathrm{Re}(\lambda) \in \{0, \pm 1/2, \pm 1\}$, and it is just a short check to verify the above list. □

Corollary A.3.4 *Let $p \in \mathbb{N}$ and $X \in \mathrm{SL}(3,\mathbb{Z})$ with $X^p = \mathrm{Id}$. If $\gcd(p,6) = 1$ then $X = \mathrm{Id}$.*

Proof. Since $\mathrm{spec}(\mathrm{Id}) = \{1\}$ every eigenvalue of X must be a p-th root of unity. If $\gcd(p,6) = 1$, then from Lemma A.3.3 the only p-th root of unity which can arise as an eigenvalue of an element of $\mathrm{SL}(3,\mathbb{Z})$ is $\lambda_X = 1$. Hence $X = \mathrm{Id}$. □

Fact A.3.5 *If $A \in \mathrm{SL}(3,\mathbb{Z})$ has a complex eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then $\mathrm{spec}(A) = \{1/\lambda\bar{\lambda}, \lambda, \lambda\}$, hence A is separable and in particular diagonalisable.*

Fact A.3.6 *Let $A \in \mathrm{SL}(3,\mathbb{Z})$ act on $\mathbb{C}^3 = \mathbb{Z}^3 \otimes_{\mathbb{Z}} \mathbb{C}$ and $p \in \mathbb{N}$, then $\xi \in \mathbb{C}^3$ is an eigenvector for A if and only if it is an eigenvector for A^p, as the Jordan normal form for A shows.*

A.3.2 Proof of Theorem 5.1.10

In order to classify the M-representations of a general torus knot group we first assume that we are given a non-trivial M-homomorphism $\varrho\colon G(p,q) \to \mathrm{SL}(3,\mathbb{Z})$ and derive a number of necessary conditions on the possible values of (p,q) and on the homomorphism ϱ itself. These conditions will be restrictive enough so that we can then find all M-representations by checking a finite number of possible candidates for M-homomorphisms ϱ as above and deciding which ones are equivalent and therefore define the same M-representation.

Definitions and notation

Let us begin by defining the objects needed for the proof and by setting up some notation. As usual, let (p,q) be a fixed pair of coprime positive integers and let $G(p,q)$ be the group of the (p,q)-torus knot $\mathfrak{t}(p,q)$. We fix elements $x, y \in G(p,q)$ which generate the group in the usual manner

$$G(p,q) = \left\langle\, x, y \;\middle|\; x^p = y^q \,\right\rangle. \tag{A.35}$$

Since $\gcd(p,q) = 1$ we can find $r, s \in \mathbb{Z}$ with

$$ps + qr = 1. \tag{A.36}$$

We pick any such pair (r,s) and define $g_1 = x^r y^s$. Then g_1 is a meridian; it depends on our previous choice of $x, y \in G(p,q)$, but not on the particular choice of solution (r,s) of equation (A.36).

Suppose we are given an M-representation of $G(p,q)$. Such a representation is the equivalence class of an M-homomorphisms $\varrho\colon G(p,q) \to \mathrm{SL}(3,\mathbb{Z})$ with $\varrho(g_1) = A$ as defined in (4.6). Moreover, let us define elements $X, Y, Z \in \mathrm{SL}(3,\mathbb{Z})$ by

$$\begin{aligned} X &= \varrho(x) \\ Y &= \varrho(y) \\ Z &= \varrho(x^p) = \varrho(y^q) \end{aligned}$$

The proof

Unfortunately, this is not the kind of theorem that is true because there is some deep reason why it has to be true, rather it is the kind of theorem that happens to be true due to an accidental lack of counterexamples. We regret that we have therefore only found a rather tedious overkill proof.

The key observation for the proof is that the knot group $G(p,q)$ has a non-trivial centre which is the infinite cyclic group

$$\mathcal{Z}(G(p,q)) = \langle x^p \rangle = \langle y^q \rangle.$$

This is an algebraic characterisation of torus knots - they are precisely the knots with non-trivial centre [BZ66]. Because $x^p = y^q \neq 1$ is central in $G(p,q)$, the image $Z = \varrho(x^p) = \varrho(y^q)$ must commute with every element of $\mathrm{Im}(\varrho) \subset \mathrm{SL}(3,\mathbb{Z})$, in particular it has to commute with $A = \varrho(g_1)$, that is $Z \in \mathcal{Z}(A)$. The centraliser $\mathcal{Z}(A)$ of A in $\mathrm{SL}(3,\mathbb{Z})$ is described in Lemma A.3.1. From there we conclude that there are $a, b, c \in \mathbb{Z}$ so that either $Z = C^+(a,b,c)$ or $Z = C^-(a,b,c)$. On the other side, $Z = X^p = Y^q$, hence in order to obtain necessary conditions on our M-representation ϱ we determine the roots of Z in $\mathrm{SL}(3,\mathbb{Z})$.

1. Suppose $Z = C^-(a,b,c)$ for some $a, b, c \in \mathbb{Z}$.

   The characteristic polynomial of Z is $\chi_Z(\lambda) = (1+\lambda)^2(1-\lambda)$, hence $\mathrm{spec}(Z) = \{1, -1\}$ and the minimal polynomial is $\mu_Z(\lambda) = (1+\lambda)^m(1-\lambda)$, with $m \in \{1, 2\}$. We look at the cases $m = 1, 2$ separately.

   (a) $m = 2$. Then μ_Z is not separable and so Z is not diagonalisable. By Fact A.3.5 we then have $\mathrm{spec}(X), \mathrm{spec}(Y) \subset \mathbb{R}$ and hence Lemma A.3.2 tells us that p, q are odd and $p|(2a+(p-1)bc)$, $q|(2a+(q-1)bc)$. Moreover, Z has *unique* roots X, Y, R of orders p, q, pq, respectively, and hence $X = R^q$, $Y = R^p$ and $\varrho\colon G(p,q) \to \langle R \rangle$ is cyclic.

   (b) $m = 1$. Since $\gcd(p,q) = 1$ we can write

   $$p = 2^j p' \quad \text{and} \quad q = 3^k q'$$

with $\gcd(p',q') = \gcd(p',6) = \gcd(q',6) = 1$. We then know from Lemma A.3.3 that $\operatorname{spec}(X^{2^j}) \subset \mathbb{R}$. So X^{2^j} is the unique p'-th root of Z in $\mathrm{SL}(3,\mathbb{Z})$ which is given by Lemma A.3.2. Now $m = 1$ implies $2a = bc$, so (A.34) shows $X^{2^j} = Z$. Thus, since X is a 2^j-th root of Z its spectrum contains 2^j-th roots of (-1), hence *primitive* 2^{j+1}-th roots of unity. This implies $j \leq 1$, by Lemma A.3.3. If $j = 0$ then ϱ is cyclic with image $\langle Y \rangle$, hence we can assume $j = 1$. Also the only $3^k q'$-th roots of (-1) which can arise as eigenvalues of $Y \in \mathrm{SL}(3,\mathbb{Z})$ are $(1 \mp i\sqrt{3})/2$ which implies $Y^3 = Y^{3^k q'} = Y^q = Z$. Also $Z = Z^{-1}$ because Z is diagonalisable with $\operatorname{spec}(Z) = \{1,-1\}$, thus $X^2 = X^{-2} = Y^3 = Y^{-3} = Z$. Now let

$$G(2,3) = \langle \bar{x}, \bar{y} \,|\, \bar{x}^2 = \bar{y}^3 \rangle$$

be the trefoil group. Then, given ϱ, we obtain a homomorphism

$$\psi_\varrho \colon G(2,3) \longrightarrow \mathrm{SL}(3,\mathbb{Z})$$

by sending $\bar{x} \mapsto X^{\pm 1}$ and $\bar{y} \mapsto Y^{\pm 1}$. By choosing the exponents carefully we can ensure that ψ_ϱ is an M-homomorphism. More precisely, we know that the element $x^r y^s \in G(p,q)$ is represented by a meridian if

$$ps + qr = 2p's + 3^k q' r = 1. \tag{A.37}$$

We can always solve this with $r \equiv \pm 1 \bmod 4$ and $s \equiv \pm 1 \bmod 6$, indeed any solution will satisfy $r \equiv \pm 1 \bmod 4$ and $s \equiv \pm 1$ or $\pm 2 \bmod 6$. But if $s \equiv \pm 2 \bmod 6$ we can take the solution $(r',s') = (r+p, s-q)$ which will be of the form we want. Now let $\bar{r}, \bar{s} \in \{1,-1\}$ so that $\bar{r} \equiv r' \bmod 4$ and $\bar{s} \equiv s' \bmod 6$, then it is easy to check that

$$\psi_\varrho(\bar{x}) = X^{\bar{r}}, \quad \psi_\varrho(\bar{y}) = Y^{-\bar{s}}$$

is an M-homomorphism, and obviously non-trivial. Hence ψ_ϱ is the unique non-trivial M-representation ϱ_{trefoil} of $G(2,3)$. Conversely, if $p = 2p'$ and $q = 3^k q'$ with $p', q', 6$ pairwise coprime we define a unique M-homomorphism of $G(p,q)$ by sending $x \mapsto \varrho_{\text{trefoil}}(\bar{x}^{\bar{r}})$ and $y \mapsto \varrho_{\text{trefoil}}(\bar{y}^{-\bar{s}})$. Uniqueness of the extension follows because any homomorphism from $G(p,q)$ to another group is determined by the values it takes on generators x, y. This establishes an explicit bijection between the M-representations of $G(p,q)$ and $G(2,3)$. In particular we have just one Abelian and one non-Abelian M-representation.

2. Suppose $Z = C^+(a,b,c)$ for some $a,b,c \in \mathbb{Z}$.

   Now the characteristic polynomial of Z is $\chi_Z(\lambda) = (1-\lambda)^3$, hence $\operatorname{spec}(Z) =$

$\{1\}$, and the minimal polynomial of Z is $\mu_Z(\lambda) = (1-\lambda)^m$ with $m \in \{1,2,3\}$. In the following, we think of Z as a linear transformation of $\mathbb{C}^3 = \mathbb{Z}^3 \otimes_{\mathbb{Z}} \mathbb{C}$. Thus Z has a unique eigenvalue $\lambda_Z = 1$ and corresponding eigenspace $E_1(Z) \subset \mathbb{C}^3$. Note that $\dim_{\mathbb{C}}(E_1(Z)) = 4 - m$. We make a further case distinction for $m = 1,2,3$.

(a) $m = 3$. This is the case $\dim(E_1(Z)) = 1$. Then by Fact A.3.6 $X = \varrho(x)$ (respectively $Y = \varrho(y)$) also has a one-dimensional space of eigenvectors over $\mathbb{C}$, and thus has a unique eigenvalue λ_X (respectively λ_Y). Since X and Y cannot be diagonalisable Fact A.3.5 implies $\operatorname{spec}(X), \operatorname{spec}(Y) \subset \mathbb{R}$ and from $\lambda_X^3 = \det(X) = 1$ we conclude $\lambda_X = 1$, and similarly $\lambda_Y = 1$. Hence Lemma A.3.2 implies that $p, q \mid \gcd(a,b,c)$ and X, Y are the *unique* p-th and q-th roots of Z in $\mathrm{SL}(3,\mathbb{Z})$. But since $\gcd(p,q) = 1$ we also have $(pq) \mid \gcd(a,b,c)$ and Lemma A.3.2 gives us a (pq)-th root R of Z which, by uniqueness of the roots X and Y, satisfies $X = R^q$, $Y = R^p$. Hence the representation is cyclic: $\varrho\colon G(p,q) \to \langle R \rangle$.

(b) $m = 2$ In this case, $\dim(E_1(Z)) = 2$. As before $\operatorname{spec}(X), \operatorname{spec}(Y) \subset \mathbb{R}$ by Fact A.3.5 and Fact A.3.6. Recall that $\gcd(p,q) = 1$, so p and q cannot both be even. Let us assume, without loss of generality, that q is odd. We distinguish according to the parity of p.

i. p is odd. The only real root of $\lambda_Z = 1$ of *odd* order is $\lambda_X = \lambda_Y = 1$, hence if p, q are both odd we get $\operatorname{spec}(X) = \operatorname{spec}(Y) = \{1\}$, and Lemma A.3.2 tells us that the required roots X, Y exist if and only if $p, q \mid \gcd(a,b,c)$, and in this case they are uniquely determined. But if $p, q \mid \gcd(a,b,c)$ then $(pq) \mid \gcd(a,b,c)$ and Lemma A.3.2 also gives us a (pq)-th root $R \in \mathrm{SL}(3,\mathbb{Z})$ of Z. By uniqueness of the roots X, Y we find $X = R^q$ and $Y = R^p$, hence $\varrho\colon G(p,q) \to \langle R \rangle$ is cyclic.

ii. p is even. Since q is odd, the only real q-th root of $\lambda_Z = 1$ is $\lambda_Y = 1$, hence $\operatorname{spec}(Y) = \{1\}$ and by Lemma A.3.2 there is a unique such $Y \in \mathrm{SL}(3,\mathbb{Z})$ with $Y^q = Z$, as before. Now recall that $Z = C^+(a,b,c)$. The assumption $\dim(E_1(Z)) = 2$ implies $bc = 0$. Without loss of generality assume $b = 0$ (the case $c = 0$ being similar). Also recall the equation $X^r Y^s = A$ and thus $X^r = AY^{-s}$. Inserting A as in (5.1) and the value of Y given by Lemma A.3.2 yields

$$X^r = AY^{-s} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-sa}{q} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-sc}{q} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 - \frac{sa}{q} & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{sc}{q} & 1 \end{pmatrix}.$$

But then $\operatorname{spec}(X^r) = \{1\}$. If p is even, (A.36) shows that r is odd and hence because $\operatorname{spec}(X) \subset \mathbb{R}$ we have $\operatorname{spec}(X) = \{1\}$. But then we can apply Lemma A.3.2 as before and X has to be the unique p-th root of Z with $\operatorname{spec}(X) = \{1\}$. Moreover, there is an $R \in \mathrm{SL}(3,\mathbb{Z})$ with $X = R^q$, $Y = R^p$ and $\varrho\colon G(p,q) \to \langle R \rangle$ is cyclic, as in (i).

(c) $m = 1$. In this case $Z = \mathrm{Id}$. Therefore both X and Y need to be diagonalisable over $\mathbb{C}$ (Fact A.3.6), and they satisfy $X^p = Y^q = \mathrm{Id}$. From Corollary A.3.4, if $\gcd(p,6) = 1$ (respectively $\gcd(q,6) = 1$) then $X = \mathrm{Id}$ (respectively $Y = \mathrm{Id}$), and $\varrho\colon G(p,q) \to \langle Y \rangle$ (respectively $\varrho\colon G(p,q) \to \langle X \rangle$) is cyclic. So we can assume $\gcd(p,6) \neq 1 \neq \gcd(q,6)$. Since $\gcd(p,q) = 1$ we have, after exchanging p, q if necessary,

$$p = 2^j p' \quad \text{and} \quad q = 3^k q' \quad \text{with} \quad j,k \geq 1 \quad \text{and} \quad \gcd(p',q') = 1. \tag{A.38}$$

Then X^{2^j} is a p'-th root of Id and by Lemma A.3.2 we find $X^{2^m} = Z = \mathrm{Id}$, thus the eigenvalues of X are 2^j-th roots of unity. By Lemma A.3.3 they have to be primitive roots of order 2^d with $d = 0, 1$ or 2. In the same way the eigenvalues of Y are third roots of unity and hence either $Y = \mathrm{Id}$ in which case ϱ is cyclic or $\operatorname{spec}(Y) = \{1, \lambda_Y, \bar{\lambda}_Y\}$ where λ_Y is a primitive third root of unity. Thus $X^{2^d} = Y^3 = \mathrm{Id}$.

i. $d = 0$. In this case we have a cyclic representation with image $\langle Y \rangle$.

ii. $d = 1$. Then there is a $C \in \mathrm{SL}(3,\mathbb{Z})$ with $C^{-1}XC = \operatorname{diag}(-1,-1,1)$ and thus, since r is odd,

$$C^{-1}X^rY^sC = C^{-1}XY^sC = C^{-1}AC = \mathrm{Id} + \eta\xi^T$$

for some $(\xi,\eta) \in \mathcal{S}$, where we use Theorem 5.1.4 again. Thus we get

$$C^{-1}Y^sC = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -\eta_1 \\ -\eta_2 \\ \eta_3 \end{pmatrix} (\xi_1, \xi_2, \xi_3).$$

But then if $(\xi_1,\xi_2) \neq (0,0)$ the vector $\nu = (-\xi_2, \xi_1, 0)^T$ is a (-1) eigenvector of Y^s, and if $\xi_1 = \xi_2 = 0$ then $(1,1,0)$ is a (-1) eigenvector. But we know

$$(-1) \notin \operatorname{spec}(Y) = \operatorname{spec}(Y^s) = \operatorname{spec}(C^{-1}YC),$$

so the case $d = 1$ cannot occur.

iii. $d = 2$. In this case $X^4 = Y^3 = \mathrm{Id}$ and $X^2 = -\mathrm{Id}$. Suppose (r,s) is a solution of the equation $2^m p's + 3^k q'r = 1$. Then $s \equiv \pm 1 \bmod 3$ and $r \equiv \pm 1 \bmod 4$ and hence given any M-homomorphism ϱ of $G(p,q)$ we get, by making a unique choice of signs, an M-representation of $G(4,3)$ by mapping $\bar{x} \mapsto X^{\pm 1}$, $\bar{y} \mapsto Y^{\pm 1}$ as before. So the M-representations of $G(p,q)$ correspond bijectively to the M-representations (ϱ_k) of $G(4,3)$

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