Contents

Introduction

This thesis is in two parts. Part one concerns a smooth Riemannian manifold M^n with a codimension two submanifold Σ^{n-2} whose complement is spin. A removable singularities theorem is proven for square summable harmonic spinors associated to spin structures on $M - \Sigma$ which extend to M. It is shown that there may be a non-trivial finite dimensional vector space of L^2 harmonic spinors, with interesting new behaviour, associated to a non-extending spin structure on $M^n - \Sigma^{n-2}$. Bounds are obtained for the dimension of this null space for any spin structure on a punctured Riemann surface. These bounds are seen to be sharp in certain cases.

An index theorem for Dirac operators associated to non-extending spin structures on $M^n - \Sigma^{n-2}$ is proved when Σ has non-trivial normal bundle. This index is computable in terms of characteristic classes of M and Σ. It does not depend on Σ being orientable nor does it depend on the smooth metric on M. Reducing this index theorem modulo two gives a generalisation of the work of Esnault, Seade and Viehweg [ESV92] to the smooth category.

An index theorem is conjectured for non-extending spin structures on $M - \Sigma$ when Σ has non-trivial normal bundle. Evidence from a number of sources is presented for this.

In part two the author proves the uniqueness of the Sen form on the two monopole moduli space. This was conjectured by Sen in 1994 motivated by physical arguments from S-duality in non-abelian gauge theory.

Overview

The outline of this thesis is as follows. In chapters one to eight we study the Riemannian Dirac operator on certain incomplete manifolds. In chapter nine we study the index of another Dirac operator, the signature operator, on a specific complete manifold known as the reduced two monopole moduli space.

In **Chapter One** we introduce the concept of a non-extending spin structure on a smooth compact Riemannian manifold M^n with a codimension k submanifold removed. The primary object of study for chapters one to eight of this thesis is the codimension two case of this definition. We always assume that $M - \Sigma$ inherits a smooth metric from M so that $M - \Sigma$ is incomplete.

Having recalled some well known facts about spin structures, and presented our definition, we illustrate it with examples and remarks. The principal fact here is that non extending and extending spin structures on $M - \Sigma$ can be distinguished by the equivalence class of the spin structure induced upon the boundary circle of any two disc transverse to Σ^{n-2}.

We then present a detailed study of the unique non-extending spin structure on $\mathbb{C}P^1$ with two distinct points removed. We assume that this space inherits the Fubini-Study metric from $\mathbb{C}P^1$. This simple example is the key reference point for most of the results that follow. We show that the dimension of the L^2 kernel of the Dirac operator associated to the non-extending spin structure is one dimensional. This is the crux of our study of non-extending spin structures in the codimension two case: that there exists an interesting finite dimensional vector space of L^2 solutions for Dirac operators associated to them. We end chapter one by using the example to illustrate the cobordism classification at the start of the chapter.

Chapter Two contains the basic topological results for non-extending spin structures. The principal result is that $M^n - \Sigma^{n-2}$ has a non-extending spin structure if and only if there is a union of connected components of Σ which is characteristic. A submanifold is said to be characteristic if it represents the second Stiefel Whitney class of the manifold M. It is also shown that if $M - \Sigma$ is spin and Σ is of codimension two then M is orientable.

Examples are then presented to exhibit manifolds $M^n - \Sigma^{n-2}$ with non-extending spin structures where Σ is orientable and where Σ is unorientable. The fact that non-extending spin structures may be obtained by removing submanifolds of codimension less than two is briefly discussed and it is shown that in such cases the dimension of the L^2 kernel of the associated Dirac operator may be unbounded. From this point onwards only non-extending spin structures on manifolds $M - \Sigma$ where Σ is of codimension two are studied.

Finally in this chapter, we return to a point that was raised immediately after the definition of a non-extending spin structure. This point is that distinct spin structures may be abstractly equivalent as principal *Spin* bundles. This adds an extra subtlety to our classification of spin structures on $M^n - \Sigma^{n-k}$ since a non-extending spin structure may be topologically trivial and extending and non-extending spin structures may be isomorphic as principal *Spin* bundles. The discussion of this point is reserved until this section where we have enough examples to illustrate it fully. In particular we show that all spin structures on punctured Riemann surfaces are holomorphically trivial.

Chapter Three contains the basic analytical remarks for non-extending spin structures. It is here that we justify our classification of spin structures on $M^n - \Sigma^{n-2}$ as being extending or non extending. Precisely, we show that every L^2 solution to the Dirac equation on

$M^n - \Sigma^{n-2}$, which is a section associated to an extending spin structure, is actually the restriction of an L^2 harmonic spinor on the manifold M. This result is stated as a removable singularities theorem. We saw in the main example in chapter one that this fact does not hold true for non-extending spin structures .

Corollaries of the removable singularities theorem are then stated. In particular we have that the L^2 index of the Dirac operator associated to an extending spin structure on $M^n - \Sigma^{n-2}$ is just equal to the $\widehat{A}$ genus of M.

In Section 3.2 we consider the lack of self adjointness of the Dirac operator for non-extending spin structures on $M - \Sigma$. Because of this lack of self adjointness there is an ambiguity in the phrase 'harmonic spinor' since it may no longer be the case that $\dim\ker \not{D}^2 \cap L^2 \subset \dim\ker \not{D} \cap L^2$. Despite this we will continue to refer to any solution to the Dirac operator as a harmonic spinor. We give explicit examples of the lack of self adjointness of $\not{D}$. From these examples we are able to deduce that all L^2 harmonic spinors associated to a non-extending spin structure may not, in general, be included in any domain on which the Dirac operator is self adjoint. We conclude the chapter with the proof of an a priori growth lemma which is not used anywhere else in this thesis.

We begin **Chapter Four** by studying the simple case of non-extending spin structures on Riemann surfaces with points removed. We denote a Riemann surface of genus g with m distinct points removed by $S_g - m$. We show via an identification using the metric, that for any spin structure on $S_g - m$ the quantity

$$\dim\ker \not{D}^+ \cap L^2 - \dim\ker \not{D}^- \cap L^2 \tag{0.1}$$

is always zero. The main point is that for a non-extending spin structure this expression is not the index of any elliptic operator. This is a consequence of the lack of self adjointness comments in the previous chapter.

We then expand on an observation for the punctured sphere to show that the number of points around which the spinor connection has non trivial holonomy must be even for any spin structure on $S_g - m$. This enables us to obtain upper and lower bounds on the dimension of the vector space of L^2 harmonic spinors on $S_g - m$. These bounds are sharp for all spin structures on surfaces of genus zero or one. Thus we deduce $\dim\ker \not{D} \cap L^2$ for all of the spin structures on S_0 and S_1 with an arbitrary number of points removed. The bounds obtained are also seen to be sharp for all spin structures on punctured surfaces S_g with non trivial holonomy about g or more distinct points.

By considering a genus two surface with two points removed we show that, in general for genera $g \geqslant 2$, the dimension of the vector space $\ker \not{D} \cap L^2$ depends on the two distinct points removed. We remark on the striking similarity between the qualitative behaviour of $\dim\ker \not{D} \cap L^2$ for spin structures on $S_g - 2$ and S_{g+1}.

In Section 4.3 we show that a natural self adjoint index problem $\not{D} : L^2_1 \longrightarrow L^2$ for $S_g - m$ can be interpreted as the Atiyah-Patodi-Singer index on the manifold obtained by removing disjoint open discs about the deleted points in $\{m\}$. This index is natural in the sense that it automatically shows that there must be be an even number of points around which the spinor connection has non trivial holonomy. We close the chapter by recalling two distinct closures of the Dirac operator on incomplete manifolds. This remark is fundamental to all the chapters which follow and as always we illustrate it using the main example where things are clearest.

Chapter Five presents calculations of the explicit forms that a L^2 harmonic spinor on $M - \Sigma$ may take near Σ. Throughout this chapter it is assumed that Σ has trivial normal bundle and a suitably amenable metric near Σ. We draw a heavy distinction between which types of L^2 spinor may occur for extending and non-extending spin structures. The form of these spinors near Σ is key to the closure of the Dirac operator on $M^n - \Sigma^{n-2}$ whose index we calculate in chapter six. We describe this closure of the operator in detail in Section 5.3. It is stressed that this closure is not the same as the operator $\not{D} : L^2_1 \longrightarrow L^2$

for we allow the possibility of harmonic spinors which lie in $L^2 - L^2_1$ in both the domain and the range of the operator. Lastly, we recall that in two dimensions we could relate Atiyah-Patodi-Singer solutions on a manifold with boundary to L^2 solutions of spin structures on punctured surfaces. We end the chapter by showing that, in general, we cannot extend this relationship to higher dimensions.

Chapter Six contains the proof of an index theorem for the Dirac operator whose closure was described in chapter five. Explicitly, for M of even real dimension, the index of the Dirac operator associated to a non-extending spin structure on $M^{2n} - \Sigma^{2n-2}$ equals,

$$\begin{array}{lll} \widehat{A}(M) & \text{if} & \dim_{\mathbb{R}} M = 4k \quad \text{or,} \\ \frac{1}{2}\widehat{A}(\Sigma) & \text{if} & \dim_{\mathbb{R}} M = 4k+2. \end{array} \tag{0.2}$$

For this, and all the other results in this chapter, we assume that Σ has trivial normal bundle.

The theorem is proved by assuming, for simplicity, that Σ is spin and that $M^{2n} - \Sigma^{2n-2}$ has a specific form of metric near Σ. Since the proof uses a local heat kernel method on $M - \Sigma$, the result holds even when Σ is unorientable. By appealing to the work of [BS88] we argue that this theorem is independent of the smooth metric chosen on M. We emphasise that this index is fundamentally different from all other indices on incomplete manifolds because of the closure of the operator taken.

When Σ is spin the integrality of the invariants in equation 0.2 is proved independently by *Spin* cobordism arguments. Our theorem however gives an interpretation of these integers. When Σ is just orientable we are able to show that (0.2) equals a certain $Spin^{\mathbb{C}}$ index on M.

In Section 6.5.3 we illustrate the index theorem by a transparent example where we can clearly see precisely what the index counts in terms of the domain of the closure for the operator expounded in chapter five. We emphasise that this index is not an L^2 index: it does not equal

$$\dim\ker \not{D}^{+} \cap L^2 - \dim\ker \not{D}^{-} \cap L^2. \tag{0.3}$$

Another example is then presented which shows that this index breaks the quaternionic symmetry of the Clifford algebra in $8k+4$ dimensions. The key reason for this is again the particular closure of Dirac operator taken in chapter five. We explain in detail how symmetry is broken and conclude that when it is, the index theorem proves the existence of a harmonic spinor on $M^{8k+4} - \Sigma^{8k+2}$, associated to a non-extending spin structure, which is in $L^2 - L^2_1$.

The symmetry breaking in $8k+4$ dimensions is explored further by considering the index (0.2) modulo two when Σ is spin. It was observed in Remark 5.1.2 that in this case a non-extending spin structure on $M - \Sigma$ induces a spin structure upon Σ. By writing the index (0.2) as a limit of an Atiyah-Patodi-Singer index we are able to show that the $\widehat{A}$ genus of M is even if and only if the spin structure induced upon Σ is also even. This is because the index theorem modulo two equals the mod 2 index for the induced spin structure of Σ. This is a partial extension of the results of Esnault, Seade and Viehweg [ESV92] who showed this result for complex manifolds.

Chapter Seven presents preliminary calculations for the case where Σ has non trivial normal bundle in M. Here we express the Dirac operator and Spinor Laplacian in terms involving Clifford multiplication by the Euler class of the normal bundle of Σ in M. These preliminary calculations should allow us to prove an index theorem for non-extending spin structures on $M^{2n} - \Sigma^{2n-2}$ when Σ has non trivial normal bundle. This has not yet been carried out successfully by the author.

Chapter Eight contains the conjectured form of the index theorem in the case where Σ has non trivial normal bundle. In fact the conjecture is stated in the full generality that the author believes successful application of the methods in chapter six would yield. This

is an index for the Dirac operator associated to a non-extending spin structure on $M - \Sigma$ twisted by a complex line bundle over M. As before this is shown to equal a certain $Spin^{\mathbb{C}}$ index on M when Σ is orientable.

We present qualitative arguments as to why this index theorem holds. These arguments are based on the work in chapter six. The examples that we present to illustrate the conjectured index theorem are both in four dimensions. Because of this we present further arguments for the form of the index in these dimensions. These use an excision argument to deduce the form of the index. This excision result is conjectured as a lemma by analogy with a result due to Krönheimer and Mrowka [KM93].

Lastly, we present two simple applications of the conjectured theorem in four dimensions. In both examples we do not assume that Σ is orientable. Firstly we show that reducing the index modulo $\mathbb{Z}$ for an untwisted Dirac operator retrieves the generalisation of the Rohlin theorem, due to Guillou and Marin [GM77, GM86b], when M^4 has an unorientable characteristic submanifold. Lastly we calculate indices for the Dirac operator on $\mathbb{C}P^2 - \mathbb{R}P^2$ coupled to the $n/2^{th}$ power of the hyperplane bundle of $\mathbb{C}P^2 - \mathbb{R}P^2$. These indices, from our conjectured theorem, agree with S-duality test numbers of Sethi, Stern and Zaslow in [SSZ95] on the reduced two monopole moduli space. We make no attempt to explain this other than to remark that there is an SO_3 equivariant diffeomorphism from this space to $\mathbb{C}P^2 - \mathbb{R}P^2$.

In **Chapter Nine** we cease to study the solutions of the Dirac operator associated to non-extending spin structures and concentrate on another Dirac operator, the signature operator. Specifically, we study this operator on a certain complete manifold known as the reduced two monopole moduli space M^0_2. We use index techniques to show that there is a unique L^2 harmonic form on this space. This form is known for physical reasons as the Sen form and it was conjectured to be unique on this space in 1994.

Firstly we show that vanishing criteria apply so that the L^2 index of the signature operator

$$\dim\ker(d+d^*)_+ \cap L^2 - \dim\ker(d+d^*)_- \cap L^2 \tag{0.4}$$

will actually count the number of anti self dual harmonic forms on M_2^0. Furthermore, we use conformal invariance of the quantity (0.4), and the specific known form of the metric on M_2^0, to show that (0.4) is the index of a Fredholm operator. With this understood we then appeal to a known index theorem of Stern which reduces the problem to a calculation similar to those carried out in [SSZ95].

Chapter 1

Definitions and Basic Examples

1.1 Definition of a Non-extending Spin Structure.

This section presents the concept of a non-extending spin structure. A very simple example of a non-extending spin structure is then presented and two *local* characterisations of non-extending spin structures are explained.

Throughout this section we will be considering a compact smooth Riemannian manifold M^n and a closed codimension k submanifold $\Sigma^{n-k} \subset M^n$. Here n denotes the real dimension of M. The space we study is $M^n - \Sigma^{n-k}$ with metric obtained by restricting a smooth metric on M^n. We will **always** assume that $M^n - \Sigma^{n-k}$ is a spin manifold.

Recall that a spin structure on the manifold M^n is a double cover of the principal bundle of oriented frames which restricts fibrewise to the non trivial double cover $Spin_n \longrightarrow SO_n$. principal bundle of frames. Two distinct spin structures on M differ by a non trivial element of the cohomology group $H^1(M, \mathbb{Z}/2\mathbb{Z})$. With these facts in mind we make the main definition of this thesis.

Definition 1.1.1 If M^n is a spin manifold then a **non-extending spin structure** is a spin structure on $M^n - \Sigma^{n-k}$ which is **not** the restriction of a spin structure on M^n. If

M^n is not spin then every spin structure on $M^n - \Sigma^{n-k}$ is said to be non-extending. Spin structures on $M^n - \Sigma^{n-k}$ which **are** the restrictions of spin structures on M^n are said to be **extending.**

A word of caution is needed here. Recall the fact that $Spin$ bundles associated to distinct spin structures may be equivalent as principal bundles [LM89, p.85]. Thus our classification of spin bundles on $M - \Sigma$ as being non extending does not refer to a topological obstruction to the associated principal $Spin$ bundles being the restriction of a principal $Spin$ bundle on M. Having observed this subtlety we postpone further discussion of it until Section 2.2 when we shall have enough examples to illustrate it properly.

1.1.1 Examples and Remarks

Example 1.1.2 Whilst not an example of the definition, the spin structures on the unit circle are fundamental to our understanding. Since $H^1(S^1, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ there are precisely two such structures. The principal bundle of frames for the circle is simply $P_{SO}(S^1) = S^1$ and consequently the two spin structures on S^1 correspond to the two double covers of S^1. One of these spin structures is therefore the trivial double cover $S^1 \times \{-1, 1\}$ and as such it is disconnected. The other double cover is S^1 itself where the projection is the squaring map sending $e^{i\theta}$ to $e^{2i\theta}$ for $0 \leqslant \theta < 2\pi$. As a result this spin structure is connected.

Example 1.1.3 The simplest example of Definition 1.1.1 is the punctured unit disc $U - \{0\}$. The disc is contractible and $U - \{0\}$ is homotopic to the unit circle S^1. Thus the cohomology homomorphism,

$$i^* : H^1(U, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(U - \{0\}, \mathbb{Z}/2\mathbb{Z}), \tag{1.1}$$

induced by the inclusion $i : U - \{0\} \longrightarrow U$, maps $0 \longrightarrow \mathbb{Z}/2\mathbb{Z}$. Hence $U - \{0\}$ possesses one extending spin structure and one non-extending spin structure.

Take a polar coordinate system (r, θ) on $U - \{0\}$ and assume that $U - \{0\}$ inherits a metric $dr^2 + r^2 d\theta^2$ from U. The principal bundle of frames is trivial $P_{SO} = U - \{0\} \times S^1$ and the two spin structures are also both topologically trivial. This is simply because any fibre bundle over $U - \{0\}$, with connected fibre, is trivial. Writing both spin structures as $P_{Spin} = U - \{0\} \times S^1$, and denoting the fibre coordinate by e^{it}, we distinguish the distinct spin structures by their covering maps. These are

$$\pi_1(r, \theta, e^{it}) = (r, \theta, e^{i\theta} e^{2it}) \in P_{SO} \tag{1.2}$$

and

$$\pi_2(r, \theta, e^{it}) = (r, \theta, e^{2it}) \in P_{SO} \tag{1.3}$$

as in [Sha78, p.159].

With respect to the trivialisation of the principal bundle of frames by the global section $(\partial_r, r^{-1}\partial_\theta)$, the non-extending spin structure on $U - \{0\}$ corresponds to the projection π_1. This is because it is the restriction of the unique spin structure on U to $U - \{0\}$.

The two spin structures on $U - \{0\}$ may be distinguished by the fact that the pre-images $\pi_1^{-1}(1 \times S^1 \times 1) \subset P_{SO}$ and $\pi_2^{-1}(1 \times S^1 \times 1) \subset P_{SO}$ are connected and disconnected respectively (see Figures 1.1 and 1.2). Since we have trivialised the frame bundle with respect to the section $(\partial_r, r^{-1}\partial_\theta)$, these pre images are precisely the spin structures induced on the circle, as in Example 1.1.2, by considering it as the boundary of $U - \{0\}$.

Remark 1.1.4 In this Remark we show that non-extending spin structures have a *local* characterisation in terms of spin cobordism when the submanifold is of codimension two. We cannot get such a classification globally since if Σ^{n-2} is connected and N_ϵ is any open ϵ tubular neighbourhood of Σ^{n-2} then the boundary of this tube is trivially spin cobordant to zero.

We show here that when Σ is connected

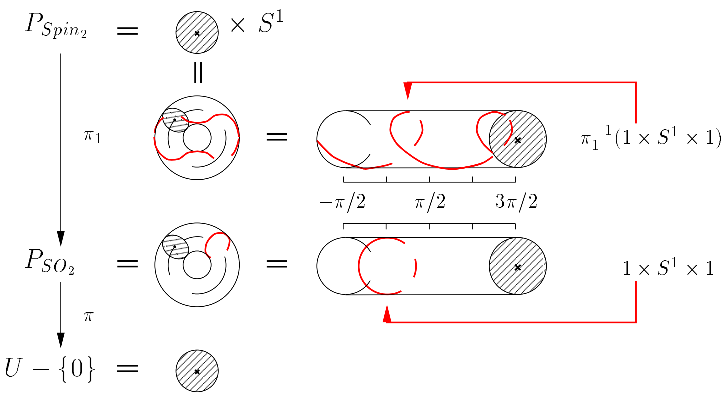


Figure 1.1: The Extending Spin Structure on the Punctured Plane

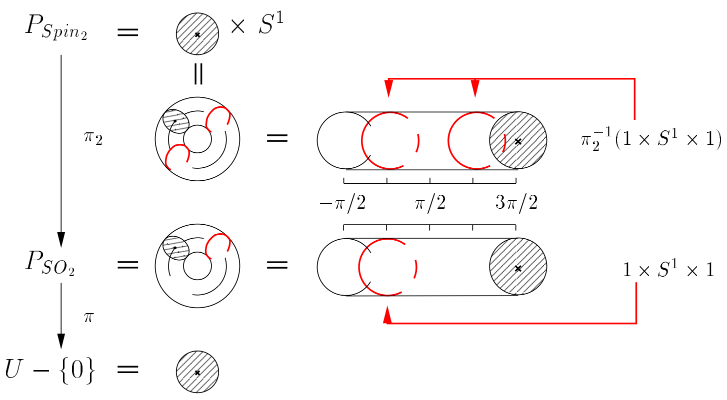


Figure 1.2: The Non-Extending Spin Structure on the Punctured Plane

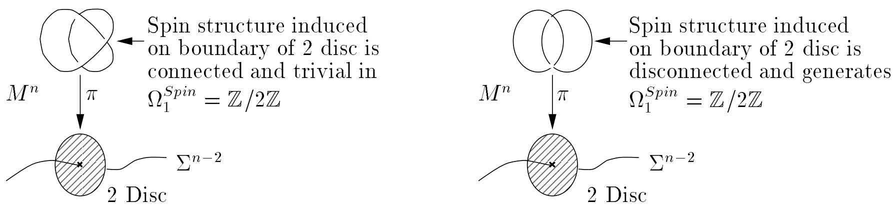
we can distinguish non-extending and extending spin structures on $M - \Sigma$ by the class of the spin structure induced on the boundary of any transverse two disc.

This local characterisation is easily generalised to the case where Σ is not connected. The statement is proven as follows.

We have a spin structure on the tangent bundle of $M - \Sigma$ which restricts to the boundary ∂N of the tubular neighbourhood N of Σ. Hence the obstruction to extending the spin structure to M is a relative second Stiefel Whitney class in $H^2(N, \partial N, \mathbb{Z}/2\mathbb{Z})$. This relative cohomology group is isomorphic to $H^2(U, \partial U, \mathbb{Z}/2\mathbb{Z})$, by the inclusion of any fibre, and to $H^0(\Sigma)$ by the Thom isomorphism theorem. If Σ is connected we have that $H^0(\Sigma, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Consequently, if Σ is connected and U is any two disc transverse to Σ^{n-2}, we may distinguish extending and non extending spin structures on $M - \Sigma$ by the spin structure induced upon the boundary of U.

We reinterpret this statement in terms of the *Spin* cobordism class of the spin structure on the boundary of U. Recall here how the two spin structures on the circle were obtained by restricting either the non extending or the extending spin structure on the punctured disc to the boundary (Example 1.1.3). Denote the one dimensional *Spin* cobordism group by Ω_1^{Spin}. As in Example 1.1.2 this group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by the connected spin structure on the circle. If Σ is connected, a non-extending spin structure on $M - \Sigma$ will therefore always induce the non trivial element of Ω_1^{Spin} upon the boundary of any two disc transverse to Σ. An extending spin structure on $M - \Sigma$ will induce the trivial element of Ω_1^{Spin}. Since spin structures on S^1 are just double covers we have the simple picture as in Figure 1.3.

Remark 1.1.5 Extending and non-extending spin structures can also be characterised locally by the natural S^1 action about the submanifold Σ^{n-2}. This fact will be relevant



Extending Spin Structure **Non-extending Spin Structure**

Figure 1.3: Induced Spin Structures

later in Remark 1.2.2. Suppose that Σ^{n-2} is connected. Take Fermi coordinates (r, θ, z) on a neighbourhood of a point $z_0 \in \Sigma$. Consider the natural S^1 action on this neighbourhood

$$e^{i\varphi} : (r, \theta, z) \to (r, \theta + \varphi, z). \tag{1.4}$$

Denote the disc through z_0 by U_0. Fix a trivialising section $(\partial_r, \partial_\theta)$ of the tangent bundle of U_0, where $\{\partial_r, \partial_\theta\}$ denotes an orthonormal basis of radial and angular tangent vectors. The principal S^1 bundle of frames is given by $P_{SO} = U_0 \times S^1$ and the local S^1 action (1.4) induces an action on P_{SO} given by

$$e^{i\varphi} : (r, \theta, w) \to (r, \theta + \varphi, w). \tag{1.5}$$

For the non-extending spin structure on $M^n - \Sigma^{n-2}$, the spin structure induced upon U_0, by restriction, is topologically $U_0 \times S^1$ with the covering map $\pi_2 : P_{Spin} \longrightarrow P_{SO}$ given by

$$\pi_2(r, \theta, z) = (r, \theta, z^2). \tag{1.6}$$

For this spin structure we can lift the local S^1 action from the principal bundle of frames to the spin structure itself by defining

$$e^{i\varphi} : (r, \theta, z) \to (r, \theta + \varphi, z). \tag{1.7}$$

The extending spin structure on $M^n - \Sigma^{n-2}$ induces the trivial element of $\Omega_1^{spin} = \mathbb{Z}/2\mathbb{Z}$ on the boundary of U_0. This has covering map $\pi_1 : U_0 \times S^1 \to U_0 \times S^1$ given by

$$\pi_1(r, \theta, z) = (r, \theta, e^{i\theta} z^2). \tag{1.8}$$

The natural action of S^1 does not lift to this spin structure. If it did there would be a continuous family of maps $g_\varphi : S^1 \to S^1$ such that,

$$g_\varphi^2(z) = e^{-i\varphi} z^2, \tag{1.9}$$

which is impossible (see [Sha78, p.161]).

In the terminology of [LM89, p.295] **global** S^1 actions on spin manifolds are classified as being **even** if they lift to actions on the spin structure and **odd** if they do not. Abusing this terminology we can state

For a non-extending spin structure on $M^n - \Sigma^{n-2}$ the natural local S^1 action about Σ^{n-2} is even. For an extending spin structure this S^1 action is odd.

Thus we can characterise extending and non-extending spin structures on $M^n - \Sigma^{n-2}$ in terms of the parity of the natural local S^1 action on transverse slices through points of Σ^{n-2}.

Remark 1.1.6 Since we made no assumption on the orientability of M^n in Definition 1.1.1 we cannot adapt the familiar definition of a spin structure [LM89, p.80] easily to our situation. This is because this definition involves consideration of the principal SO_n bundle of frames.

The primary focus of this thesis is on submanifolds of codimension two. As we shall see in Corollary 2.1.2, orientability of $M^n - \Sigma^{n-2}$ implies orientability of M^n. Restricting to codimension two submanifolds, and assuming Corollary 2.1.2, we can reformulate the usual definition of a spin structure as follows.

Definition 1.1.7 When $n \geqslant 3$ consider the fibration of the principal SO_n bundle of frames, $SO_n \longrightarrow P_{SO}(T_M) \xrightarrow{\pi} M$, where T_M denotes the tangent bundle of M. This has associated to it (via the Serre Spectral Sequence [LM89, p.81]) the exact sequence

$$0 \longrightarrow H^1(M) \xrightarrow{\pi^*} H^1(P_{SO}T_M) \longrightarrow H^1(SO_n) \longrightarrow H^2(M) \tag{1.10}$$

where $\mathbb{Z}/2\mathbb{Z}$ coefficients are understood. The non-trivial element of $H^1(SO_n) = \mathbb{Z}/2\mathbb{Z}$ is identified with the group $Spin_n$. Let $j : M^n - \Sigma^{n-2} \longrightarrow M^n$ denote inclusion. The following diagram commutes

$$\begin{array}{ccc} H^1(P_{SO}T_M) & \longrightarrow & H^1(SO_n) = \mathbb{Z}/2\mathbb{Z} \\ {\scriptstyle j^*}\big\downarrow & & \big\| \\ H^1(P_{SO}T_{M-\Sigma}) & \longrightarrow & H^1(SO_n) = \mathbb{Z}/2\mathbb{Z} \end{array}$$

Spin structures considered as double covers of the principal bundle of frames are identified with elements of $H^1(P_{SO})$ which map to $[Spin_n]$ under j^*. With respect to this identification **extending spin structures** are those elements of $H^1(P_{SO}T_{M-\Sigma})$ which lie in the image of j^*. The **non-extending spin structures** are those elements of $H^1(P_{SO}T_{M-\Sigma})$ which map to $[Spin_n]$ but **do not** lie in the image of j^*. This definition can easily be adapted to include the $n = 2$ case as in [LM89, p.81].

1.2 Main Example

In this section we present in detail the main example that will illustrate the comments in the basic sections which follow. Throughout this section, cohomology groups with $\mathbb{Z}/2\mathbb{Z}$ coefficients will be understood, unless explicitly stated otherwise.

1.2.1 Set Up

Consider the Riemannian sphere $\mathbb{C}P^1$ with the two points zero and infinity removed. This is an oriented complex one manifold and we equip it with the usual Fubini-Study metric

$$g = \frac{dz d\overline{z}}{(1 + z\overline{z})^2}. \tag{1.11}$$

Its frame bundle is a principal U_1 bundle P_{U_1}. A spin structure on the frame bundle P_{U_1} is a square root of this U_1 bundle [Hit74, Ati71]. Note that this is no longer a holomorphic classification because we are on a non-compact manifold, see Remark 2.2.2. Since $H^1(\mathbb{C}P^1 - \{0, \infty\}) = \mathbb{Z}/2\mathbb{Z}$ there are two choices of such a square root. Now $\mathbb{C}P^1$ is simply connected so $H^1(\mathbb{C}P^1) = 0$ and thus one of the spin structures is a non-extending spin structure. Observe here that $\mathbb{C}P^1 - \{0, \infty\}$ is just the punctured disc of Example 1.1.3 with a different metric.

1.2.2 Algebra

Identify the Clifford algebra of $\mathbb{R}^2$, denoted by Cl_2, with the quaternions as in [Mor96]

$$Cl_2 \cong \mathbb{H} \qquad e_1 \rightarrow j \qquad e_2 \rightarrow k \qquad e_1 e_2 \rightarrow i. \tag{1.12}$$

The subgroup Pin_2 of Cl_2 corresponds to the subgroup of $\mathbb{H}$ generated by all the elements of the form $\cos\theta . j + \sin\theta . k$ for $e^{i\theta} \in U_1$. Thus Pin_2 is the union of two circles in $\mathbb{H}$: the usual unit circle and j times it. The group $Spin_2 = SO_2 = U_1$ is just the group of all even elements of Pin_2. These are all the elements of the form $\cos\theta + i\sin\theta \in \mathbb{C} \subset \mathbb{H}$. Thus $Spin_2$ is the usual unit circle $U_1 \subset \mathbb{C} \subset \mathbb{H}$.

Identify $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ and define a map from $\mathbb{H}$ to $\mathbb{C}[2]$, the algebra of 2×2 complex matrices, by sending

$$\alpha + j.\beta \to \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}. \tag{1.13}$$

This extends naturally to an isomorphism of the complexified algebra, $Cl_2 \otimes \mathbb{C}$, with $\mathbb{C}[2]$. Considering $\mathbb{C}[2]$ as automorphisms of $\mathbb{C}^2$ we obtain the unique (up to isomorphism) irreducible finite dimensional complex representation of Cl_2. The spinor representation $\Delta : Spin_2 \to Aut\ \mathbb{C}^2$ is just this representation restricted to the subgroup $Spin_2$. Hence the action of $Spin_2$ on $\mathbb{C}^2$ is given by

$$z \to \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \quad \text{for all } z \in U^1. \tag{1.14}$$

1.2.3 Spinor Bundles

Form the spinor bundle $\not{S}$ as an associated bundle of the chosen spin structure $\not{S} = P_{Spin} \times_\Delta \mathbb{C}^2$. The representation (1.14) shows that $\not{S}$ is the direct sum of two complex line bundles. One line bundle is associated to P_{Spin} by the standard representation of U_1 whilst the other is associated via the conjugate representation. Since for U_1 valued transition functions $g_{\alpha\beta}^{-1} = \overline{g_{\alpha\beta}}$ the second line bundle is the inverse of the first. The line bundle associated to the conjugate of the standard representation is a square root of the canonical bundle $K^{1/2}$. The line bundle associated to the standard representation is therefore $K^{-1/2}$.

The complex volume element of the Clifford algebra is

$$\omega_{\mathbb{C}} = ie_1e_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.15}$$

so the grading of the spinor bundle into plus and minus eigenspaces of $\omega_{\mathbb{C}}$, namely $\not{S} = \not{S}^+ \oplus \not{S}^-$, is exactly the above decomposition into line bundles.

In summary we see that

$$\not{S}^+ = K^{1/2} \qquad \text{and} \qquad \not{S}^- = K^{-1/2}. \tag{1.16}$$

We will refer to sections of $\not{S}^+$ as **positive chiral spinors** and sections of $\not{S}^-$ as **negative chiral spinors**.

Having formed the spinor bundles it is important to note that the choice of spin structure (non-extending or extending) is now encoded in the choice of square root of the canonical bundle $K^{1/2}$.

Consider $\mathbb{C}P^1 - \{0, \infty\}$ with its usual two element set of coordinate charts: one a neighbourhood of zero and the other of infinity. Call these charts U_0 and U_∞ respectively. Note that since we work on the **punctured** sphere that we are taking $0 \notin U_0$ and $\infty \notin U_\infty$. The holomorphic transition function between these two charts is $z \to z^{-1} = w$.

The canonical bundle $K_{\mathbb{C}P^1}$ of $\mathbb{C}P^1$ has a meromorphic section s with two simple poles

$$s = \begin{cases} z^{-1}dz & \text{on} \quad U_0 \cup \{0\} \\ -w^{-1}dw & \text{on} \quad U_\infty \cup \{\infty\} \end{cases}. \tag{1.17}$$

This section restricts to give a global non-vanishing section of K.

By the fundamental work of Hitchin [Hit74] and Atiyah [Ati71], spin structures on a compact Kähler manifold are in one to one correspondence with holomorphic square roots of the canonical bundle. For $\mathbb{C}P^1$ the canonical bundle is H^{-2} so the unique spin structure on $\mathbb{C}P^1$ is the tautological Hopf bundle. The restriction of this holomorphic line bundle to $\mathbb{C}P^1 - \{0, \infty\}$ corresponds to one of the spin structures on $\mathbb{C}P^1 - \{0, \infty\}$: in this case it will be the extending spin structure (since by definition an extending spin structure is one which is the restriction of a spin structure on the ambient manifold).

Now dz is a meromorphic section of $K_{\mathbb{C}P^1}$ which restricts to a global holomorphic section of K on $\mathbb{C}P^1 - \{0, \infty\}$. The unique holomorphic square root of $K_{\mathbb{C}P^1}$ restricts to give a holomorphically trivial square root of K. **Denote a global holomorphic section of this root by** $(dz)^{1/2}$.

The other choice of square root of K corresponds to the twisting of the above square root with the non-trivial element of $H^1(\mathbb{C}P^1 - \{0, \infty\})$. This is a flat line bundle and the choice of $K^{1/2}$ corresponding to the non-extending spin structure can thus be globally holomorphically trivialised by $z^{1/2}(dz)^{1/2}$. Here, $z^{1/2}$ is the usual multi-form function on the punctured plane.

In conclusion, the positive spinor bundles associated to the extending and non-extending spin structures on $\mathbb{C}P^1 - \{0, \infty\}$ are holomorphically trivial. The spinor bundle associated to the non-extending spin structure is trivialised by $z^{1/2}(dz)^{1/2}$ and the one associated to the extending spin structure is trivialised by $(dz)^{1/2}$.

1.2.4 Kähler Structure

Since $\mathbb{C}P^1 - \{0, \infty\}$ is Kähler we have the standard identification of the positive Dirac operator $\not{D}^+ : \not{S}^+ \to \not{S}^-$ with $\overline{\partial} : K^{1/2} \to \overline{K}K^{1/2}$. Since the metric g is a global section of $K_{\mathbb{C}P^1} \otimes \overline{K_{\mathbb{C}P^1}}$ we have the identifications as shown in Figure 1.4.

In Figure 1.4 c denotes complex conjugation, and $c \circ g^{-1/2} = g^{-1/2} \circ c$. Hence, by using the Kähler coordinate system to identify $\not{D}^+$ with $\overline{\partial} : K^{1/2} \to \overline{K}K^{1/2}$, we see that we must identify $\not{D}^-$ with

$$c \circ g^{-1/2} \circ \overline{\partial} \circ c \circ g^{-1/2} : \overline{K}K^{1/2} \to K^{1/2}. \tag{1.18}$$

$$\begin{tikzcd}
\not{S}^- \cong \not{S}^+ \arrow[r, "\not{D}^+"] \arrow[r, bend left, "\not{D}^-"] \arrow[d, equal] & \not{S}^- \cong \not{S}^+ \arrow[d, equal, "g"] \\
K^{1/2}\overline{K} \cong K^{1/2} \arrow[r, "\overline{\partial}"] \arrow[loop below, "c \circ g^{-1/2}"] & K^{1/2}\overline{K} \cong K^{1/2} \arrow[loop below, "c \circ g^{-1/2}"]
\end{tikzcd}$$

Figure 1.4: Identifications

1.2.5 L^2 Solutions on $\mathbb{C}P^1 - \{0, \infty\}$

We now consider L^2 solutions to the Dirac operator on $\mathbb{C}P^1 - \{0, \infty\}$ **for the non-extending spin structure.** First we look for positive chiral solutions. The identification of $\not{D}^+$ with $\overline{\partial}$ shows that a solution of positive chirality must be of the form,

$$\sum_{n \in \mathbb{Z}} a_n z^{n+1/2} (dz)^{1/2}, \tag{1.19}$$

where the a_n are complex numbers. To be square summable it must also satisfy

$$\int_0^\infty \sum \frac{|a_n|^2 r^{2n+2}}{(1+r^2)} dr < \infty, \tag{1.20}$$

and we deduce from equation (1.20) that $a_n = 0$ unless $n = -1$. Thus we see that **there exists a positive chiral L^2 harmonic spinor on $\mathbb{C}P^1 - \{0, \infty\}$ associated to the non-extending spin structure. This spinor is unique in the sense that all other L^2 spinors will be linear multiples of it.**

This spinor is explicitly

$$\psi^+_{\text{ness}} := \begin{cases} z^{-1/2}(dz)^{1/2} & \text{on} \quad U_0 \\ w^{1/2}(-w^{-2}dw)^{1/2} & \text{on} \quad U_\infty \end{cases} \tag{1.21}$$

Under the identification $\not{D}^- = c \circ g^{-1/2} \circ \overline{\partial} \circ c \circ g^{-1/2}$ (see Figure 1.4) we note that any L^2 harmonic spinor of negative chirality must be of the form $g^{1/2}\overline{\varphi}$ where φ is an L^2 harmonic spinor of positive chirality (a section of $K^{1/2}$). This is simply because the isomorphisms

$$K^{1/2} \underset{g^{1/2}\circ c}{\cong} K^{1/2}\overline{K} \underset{g^{-1/2}\circ c}{\cong} K^{1/2} \tag{1.22}$$

both preserve the L^2 condition on $\mathbb{C}P^1 - \{0, \infty\}$ since g is smooth on $\mathbb{C}P^1$.
Therefore

there is a unique negative chiral L^2 harmonic spinor on $\mathbb{C}P^1 - \{0, \infty\}$ associated to the non-extending spin structure.

This takes the form

$$\psi^-_{\text{ness}} := g^{1/2}\overline{\psi^+_{\text{ness}}} = \frac{d\overline{z}(dz)^{1/2}}{\overline{z}^{1/2}(1+z\overline{z})} \quad \text{on} \quad U_0 \tag{1.23}$$

under the identification of $\not{S}^-$ with $\overline{K}K^{1/2}$.

The same methods show that **for the extending spin structure there are no L^2 solutions of the Dirac operator of either chirality.** This is easily seen by replacing n in equation (1.19) with $m - 1/2$ and checking that the bounds on m imposed by the L^2 condition at zero and infinity are mutually exclusive.

This absence of any harmonic spinors for the extending spin structure is symptomatic of the general Removable Singularities Theorem 3.1.1 and a vanishing theorem.

Remark 1.2.1 This calculation can be generalised to $\mathbb{C}P^1$ with an **arbitrary** pair of distinct points $\{z_0, z_\infty\}$ removed. This is because z_0 and z_∞ can be mapped biholomorphically to 0 and ∞ respectively by the Möbius transformation

$$z \longrightarrow \frac{1}{(z_0 - z_\infty)}\left(\frac{z - z_0}{z - z_\infty}\right). \tag{1.24}$$

Using this map, the unique positive chiral spinor associated to the non-extending spin structure on $\mathbb{C}P^1 - \{z_0, z_\infty\}$ is,

$$\frac{(dz)^{1/2}}{(z - z_\infty)^{1/2}(z - z_0)^{1/2}}. \tag{1.25}$$

Remark 1.2.2 In Remark 1.1.4 we presented a way of determining whether a spin structure on $M^n - \Sigma^{n-2}$ was extending or not. This involved taking a 2-disc transversely intersecting the submanifold Σ and seeing what the cobordism class of the spin structure induced upon the boundary of this disc was. The example here shows this clearly. Take equation (1.21) as a section of the non-extending spin structure. Denote the inclusion of the unit circle in the chart U_0 by $r : S^1 \to U_0$. The pullback of section (1.21) is

$$r^*(z^{-1/2}(dz)^{1/2}) = (e^{-i\theta}de^{i\theta})^{1/2}. \tag{1.26}$$

where $z = e^{i\theta}$ for $0 \leqslant \theta < 2\pi$. This is periodic, under $\theta \to \theta + 2\pi$, and is thus a section of the disconnected spin structure on the circle.

Similarly a section of the extending spin structure, say $(dz)^{1/2}$, pulls back to

$$r^*(dz)^{1/2} = e^{i\theta/2}(id\theta)^{1/2}. \tag{1.27}$$

This is aperiodic, under $\theta \to \theta + 2\pi$, and is thus a section of the connected spin structure. Another way to see this is to recall that the mod 2 index

$$\dim \ker \not{D}^+ \bmod 2 \in \mathbb{Z}/2\mathbb{Z} = \Omega_1^{Spin} = KO^{-1}(\text{pt}) \tag{1.28}$$

is a spin cobordism invariant. The Dirac operator on $r(S^1)$ is simply $i\partial_\theta$. Thus, sections in Ker $\not{D}^+$ are just a constant functions on the circle. For the disconnected spin structure on S^1 we therefore have a one dimensional kernel. For the connected spin structure, since spinors must be aperiodic functions, the kernel is trivial. Consequently, the pullback of the section $(dz)^{1/2}$ gives a section of the spin structure on S^1 which isn't spin cobordant to zero and the pullback of section (1.21) gives a section of the spin structure which generates Ω_1^{Spin}.

Chapter 2

Basic Topology

In this chapter we carry out some basic topological calculations concerning non-extending spin structures. The principal result is that

$M^n - \Sigma^{n-2}$ **has a non-extending spin structure if and only if there is a union of connected components of** Σ^{n-2} **which is characteristic (represents** $w_2(M)$**).**

Throughout this section, cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients will be understood, unless explicitly stated otherwise. We work primarily with codimension two submanifolds. Other codimensions are discussed in Section 2.1.2.

2.1 Cohomology Sequences

In this section we consider a compact n dimensional Riemannian manifold M^n and a closed codimension two submanifold $\Sigma^{n-2} \longrightarrow M^n$. The metric and embedding will be irrelevant in the topological remarks that we make. Since we only wish $M^n - \Sigma^{n-2}$ to be spin, we **do not** assume that either M^n or Σ^{n-2} are orientable. Ultimately M^n will be seen to be oriented. For completeness we start by recalling some well known facts in the form of a Lemma.

Lemma 2.1.1 There is the following long exact sequence in cohomology

$$\ldots \longrightarrow H^*(M-\Sigma) \longrightarrow H^{*+1-2}(\Sigma) \xrightarrow{\alpha^*} H^{*+1}(M) \xrightarrow{\beta^*} H^{*+1}(M-\Sigma) \to \ldots \quad (2.1)$$

Here cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients is understood. When $*+1-2=0$ and Σ is connected the non-trivial class in $H^0(\Sigma)=\mathbb{Z}/2\mathbb{Z}$ gets mapped to the cohomology class dual to Σ by α^*. This class is in turn mapped to the second Stiefel Whitney class of the normal bundle N of Σ by the restriction homomorphism $H^*(M) \to H^*(\Sigma)$.

$$\begin{array}{ccc} H^0(\Sigma) \xrightarrow{\alpha^*} H^2(M) \longrightarrow H^2(\Sigma) \\ 1 \longrightarrow \text{P.D.}[\Sigma] \longrightarrow w_2(N) \end{array} \quad (2.2)$$

The map β^* is the obvious restriction homomorphism.

Proof We split the proof into four steps. Steps one to three establish the sequence whilst the last step establishes the maps.

Step 1: [MS74, p.144]

The sequence follows from the long exact sequence of the pair $(M, M-\Sigma)$

$$\ldots \longrightarrow H^i(M, M-\Sigma) \xrightarrow{\gamma^*} H^i(M) \xrightarrow{\beta^*} H^i(M-\Sigma) \longrightarrow H^{i+1}(M, M-\Sigma) \xrightarrow{\gamma^*} \ldots \quad (2.3)$$

where γ^* and β^* are the cohomology maps induced from the natural restrictions.

Denote an ϵ tubular neighbourhood of Σ by T_ϵ. Define the normal bundle $\pi : N \longrightarrow \Sigma$ via the exact sequence

$$0 \longrightarrow T_\Sigma \longrightarrow T_M \mid_\Sigma \longrightarrow N \longrightarrow 0 \quad (2.4)$$

where T_M and T_Σ are the tangent bundles of M and Σ respectively. We identify Σ with the zero section of the normal bundle and denote the normal bundle with the zero set removed by N_0.

Step 2: $H^*(M, M-\Sigma) \cong H^*(N, N_0)$ **[MS74, p.117]**

Since the tubular neighbourhood T_ϵ and the complement $M-\Sigma$ are open subsets with union M and intersection $T_\epsilon - \Sigma$ there is an excision isomorphism

$$H^*(M, M-\Sigma) \cong H^*(T_\epsilon, T_\epsilon - \Sigma). \quad (2.5)$$

Let N_ϵ denote a neighbourhood of the zero section of the normal bundle. Define also $N_{\epsilon,0} = N_\epsilon \cap N_0$. Excision and the exponential map induce isomorphisms

$$H^*(N, N_0) \cong H^*(N_\epsilon, N_{\epsilon,0}) \cong H^*(T_\epsilon, T_\epsilon - \Sigma). \quad (2.6)$$

Combining (2.5) and (2.6) we obtain the isomorphism

$$H^*(M, M-\Sigma) \cong H^*(N, N_0), \quad (2.7)$$

which is independent of the ϵ neighbourhood chosen, and independent of the Riemannian metric on M.

Step 3: $H^*(\Sigma) \cong H^{*+2}(N, N_0)$

This is just the Thom isomorphism see [MS74, p.90]

$$H^*(\Sigma) \overset{\pi^*}{\cong} H^*(N) \overset{\cup\Phi_N}{\cong} H^{*+2}(N, N_0). \quad (2.8)$$

Here $\cup\Phi_N$ denotes the cup product with the Thom class of the normal bundle $\pi : N \longrightarrow \Sigma$. This establishes the sequence 2.1.

Step 4: The maps α^* and β^*

From the long exact sequence of the pair (2.3) the map $\beta^* : H^*(M) \longrightarrow H^*(M-\Sigma)$ is just the restriction map. The image of Φ_N in $H^2(M)$ under the above identification is by definition [MS74, p.120] the Poincaré dual of the homology class of the submanifold Σ^{n-2}. The Poincaré duality isomorphism

$$H_{n-2}(M) \xrightarrow{\text{P.D.}} H^2(M) \tag{2.9}$$

therefore maps

$$[\Sigma] \longrightarrow \gamma^* \Phi_N \tag{2.10}$$

where we consider the Thom class Φ_N as a class in $H^2(M, M - \Sigma)$ by Step Two and γ^* is the usual restriction homomorphism.

Under the further restriction $H^2(M) \longrightarrow H^2(\Sigma)$ the image of the Thom class gets mapped to the top Stiefel Whitney class $w_2(N)$ of the normal bundle [MS74, p.119]. In order to complete the proof of the lemma, we note that when Σ is connected the non-trivial class in $H^0(\Sigma) \cong \mathbb{Z}/2\mathbb{Z}$ is mapped to the Thom class Φ_N via the Thom isomorphism in Step Three. This is because Φ_N generates $H^n(N, N_0)$.

□

2.1.1 Remarks and Corollaries

Corollary 2.1.2 The manifold M is always orientable if Σ is of codimension two and $M - \Sigma$ is spin.

Proof This follows simply from part of the sequence above. Since $M - \Sigma$ is spin we have that $w_1(M - \Sigma) = 0$. The sequence (2.1) then shows that

$$\begin{aligned} 0 = H^1(N, N_0) \longrightarrow & H^1(M) \longrightarrow H^1(M - \Sigma) \\ & w_1 \longrightarrow w_1 \,|_{M-\Sigma} = 0 \end{aligned} \tag{2.11}$$

where the isomorphism comes from the fact that N is of rank two. Thus $w_1(M) = 0$ and M must be orientable. □

Remark 2.1.3 By the above corollary M is always orientable. Thus if Σ is orientable then its normal bundle must be too. Note that when Σ is orientable Lemma 2.1.1 holds with integer coefficients. The only difference here is that under the restriction $H^2(M, \mathbb{Z}) \longrightarrow H^2(\Sigma, \mathbb{Z})$ the Poincaré dual P.D.[Σ] of Σ maps to the Euler class of the normal bundle $e(N)$. The natural coefficient homomorphism $H^2(\Sigma, \mathbb{Z}) \longrightarrow H^2(\Sigma, \mathbb{Z}/2\mathbb{Z})$ then carries the Euler class $e(N)$ to $w_2(N)$.

Corollary 2.1.4 Suppose that $M^n - \Sigma^{n-2}$ has a non-extending spin structure where Σ is connected. Then the Poincaré dual in $H^2(M, \mathbb{Z}/2\mathbb{Z})$ of the homology class defined by Σ equals the second Stiefel Whitney class $w_2(M)$ of M

$$\text{P.D.}[\Sigma] = w_2(M). \tag{2.12}$$

If Σ is not connected the $w_2(M)$ must be supported on P.D.[Σ]. That is, there exists a non-empty subset of connected components of Σ, say $\{\Sigma_i\}$, such that $\text{P.D.}[\Sigma_0 \cup \ldots \cup \Sigma_n] = w_2(M)$.

Proof We split the proof into two cases: when M is spin and when it is not.

Step 1: When M is not spin

When Σ is connected then $H^0(\Sigma) \cong \mathbb{Z}/2\mathbb{Z}$ and the result follows from the exactness of the following portion of the long exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \text{P.D.}[\Sigma] & & \\
H^0(\Sigma) \arrow[u, phantom, "\Cup"] \arrow[r] & H^2(M) \arrow[r] & H^2(M - \Sigma) \arrow[r] & \ldots \\
\Phi_N \in H^2(N, N_\epsilon) \arrow[u, equal] \arrow[uu, bend left, to path={-| (\tikztotarget)}] & \omega_2 \neq 0 \arrow[r] & \omega_2|_{M-\Sigma} = 0 &
\end{tikzcd}$$

The proof for Σ disconnected is similar.

Step 2: When M is spin

By definition if $M - \Sigma$ has a non-extending spin structure then there exists an element $a \in H^1(M - \Sigma)$ which is not in the image of $H^1(M)$ under the restriction map β^*. From the exactness of the sequence (2.1) a maps via the coboundary map to a non-zero element of $H^0(\Sigma)$ and then to zero thereafter. From Lemma 2.1.1 we see that P.D.$[\Sigma] = 0 \in H^2(M)$ if Σ is connected as required. The disconnected case is similar □

Remark 2.1.5 From Corollary 2.1.4 we see that there is a close link between non-extending spin structures and characteristic submanifolds. Rohlin [GM86b] initially defined a characteristic submanifold to be a surface in a four manifold which represented w_2. There is an obvious higher dimensional analogue of this definition see for instance [ESV92]. Rewriting Corollary 2.1.4 in these terms gives

Corollary 2.1.6 The space $M^n - \Sigma^{n-2}$ has a non-extending spin structure if and only if there is a non-empty union of connected components of Σ which is characteristic.

Of course this non-extending spin structure need not be unique.

Remark 2.1.7 As $M^n - \Sigma^{n-2}$ is spin distinct spin structures upon it differ by a non trivial element of $H^1(M - \Sigma)$. Denote the number of distinct elements of $H^1(M)$ by b and the number of distinct elements of $H^1(M - \Sigma)$ by a. If M is spin then $M - \Sigma$ has $a - b$ non-extending spin structures because, by Lemma 2.1.1, $H^1(M) \longrightarrow H^1(M - \Sigma)$ is always injective. If M isn't spin then $M - \Sigma$ has a non-extending spin structures by definition.

Remark 2.1.8 Lemma 2.1.1 shows that every manifold with a non-extending spin structure is canonically a $Spin^{\mathbb{C}}$ manifold **when the submanifold Σ is oriented**. In this case the submanifold representing w_2 defines a complex line bundle L_Σ over M with first Chern class $c_1(L_\Sigma) = $ P.D.$[\Sigma]$. Then $w_2(M) = c_1(L_\Sigma)$ mod 2 and M has a canonical $Spin^{\mathbb{C}}$ structure with associated complex line bundle L_Σ. By Remark 2.1.3 pulling back L_Σ to Σ gives an oriented 2 plane bundle C^∞ isomorphic to N_Σ.

Corollary 2.1.9 If $M - \Sigma$ has a non-extending spin structure then the following are equivalent

- Σ is spin
- Σ is orientable
- Σ is co-oriented.

Proof It is an immediate from the Lemma 2.1.1 that if $M - \Sigma$ has a non-extending spin structure then $i^* w_2(M) = w_2(N)$ independent of whether M is spin or not. Using the Whitney product formula

$$\begin{aligned} i^* w_2(M) &= w_2(i^* T_M) = w_2(N \oplus T_\Sigma) \\ &= w_2(T_\Sigma) + w_2(N) w_2(T_\Sigma) + w_2(N), \end{aligned} \tag{2.13}$$

so that,

$$w_2(T_\Sigma) = w_1(N) w_1(T_\Sigma). \tag{2.14}$$

This completes the proof. □

2.1.2 Codimensions Other Than Two

Remark 2.1.10 We remark here that non-extending spin structures can not occur by removing submanifolds of codimension $\geqslant 3$. The whole of Lemma 2.1.1 holds verbatim for a codimension k submanifold Σ^{n-k} simply by replacing each occurrence of the number 2 in a cohomology group index by k. Using this generalisation of the lemma we see that if M

isn't spin then $M^n - \Sigma^{n-k}$ cannot be either. This follows since the modified lemma shows that $\beta^* : H^2(M) \longrightarrow H^2(M-\Sigma)$ is injective

$$\begin{aligned} 0 = & H^2(N,N_0) \longrightarrow H^2(M) \longrightarrow H^2(M-\Sigma) \\ & \qquad\qquad w_2 \longrightarrow w_2 \mid_{M-\Sigma} = 0. \end{aligned} \tag{2.15}$$

Furthermore, since

$$0 = H^1(N,N_0) \longrightarrow H^2(M) \longrightarrow H^1(M-\Sigma) \longrightarrow H^2(N,N_0) = 0 \tag{2.16}$$

is exact, $H^1(M)$ is isomorphic to $H^1(M-\Sigma)$. As a consequence even if M is spin we cannot obtain a non-extending spin structure by removing a submanifold of codimension $\geqslant 3$.

This can also be seen from an elementary deformation argument i.e. that

$$\pi_i(M^n - \Sigma^{n-k}) = \pi_i(M^n) \qquad \text{for } i \leqslant n-k-1. \tag{2.17}$$

Remark 2.1.11 We note here that by removing submanifolds of codimension one we can still obtain non-extending spin structures. This is however a completely different problem. For example we could consider spaces like $\mathbb{R}P^2 - \mathbb{R}P^1 = \mathbb{R}^2$. Considering $\mathbb{R}P^2$ as the quotient of $\mathbb{C}P^1$ with metric as in (1.11) exhibits $\mathbb{R}P^2 - \mathbb{R}P^1$ as the open unit disc in the complex plane with metric

$$\frac{dz d\overline{z}}{(1+z\overline{z})^2}. \tag{2.18}$$

Following Section 1.2.5 it is elementary to show that the dimension of the space of L^2 harmonic spinors on this space is unbounded. We therefore concentrate on the highest codimension in which non-extending spin structures occur and only consider (except in Remark 2.1.10 and here) the case where Σ is of codimension two.

2.1.3 Examples of Non-extending Spin Structures.

Example 2.1.12 Consider complex projective space $\mathbb{C}P^n$. The total Chern class of $\mathbb{C}P^n$ is well known to be

$$c(T_{\mathbb{C}P^n}) = (1 + c_1(H_n))^{n+1}. \tag{2.19}$$

Here $c_1(H_n)$ is the generator of $H^2(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}$: the first Chern class of the hyperplane bundle H_n of $\mathbb{C}P^n$. The coefficient homomorphism $H^*(M,\mathbb{Z}) \longrightarrow H^*(M, \mathbb{Z}/2\mathbb{Z})$ maps the total Chern class c to the total Stiefel Whitney class ([MS74, p.171]). Thus

$$w_2(\mathbb{C}P^n) = (n+1)c_1(H_n) \bmod 2. \tag{2.20}$$

Now consider the embedded hypersurface $\mathbb{C}P^{n-1}$ in $\mathbb{C}P^n$. It is elementary that

$$\text{P.D.}[\mathbb{C}P^{n-1}] = c_1(H_n) \tag{2.21}$$

in $H^2(\mathbb{C}P^n, \mathbb{Z})$. Therefore we see that when n is even $\mathbb{C}P^n$ is not spin but $\mathbb{C}P^n - \mathbb{C}P^{n-1}$ has a non-extending spin structure. As $\mathbb{C}P^n - \mathbb{C}P^{n-1} = \mathbb{C}^n$ this space is contractible **for all** n. Hence $\mathbb{C}P^n - \mathbb{C}P^{n-1}$ always has a unique spin structure. This spin structure is non-extending if and only if n is even.

Example 2.1.13 We exhibit here an example of a non-extending spin structure obtained by the removal of an unoriented codimension two submanifold. To do this we follow Lawson [Law82] who shows that $\mathbb{C}P^2$ is diffeomorphic to the twisted double $N_1 \cup_{\overline{\alpha}} T_4$ where N_1 is the unorientable 2 disc bundle over $\mathbb{R}P^2$ with Euler number 1 and T_4 is the 2 disc bundle over S^2 with Euler number 2. The glueing map $\overline{\alpha}$ is just $\overline{\alpha}[(x,y)] = [(y,x)]$. Explicitly

$$\begin{aligned} T_2 &= \{(x,y) \in S^2 \times D^3 : x.y = 0\} \\ T_4 &= T_2/(x,y) \sim (x,-y) \\ N_1 &= T_2/(x,y) \sim (-x,y) \end{aligned} \tag{2.22}$$

Under this identification the $\mathbb{R}P^2$ in N_1 is just the standard $\mathbb{R}P^2 \subset \mathbb{C}P^2$ and $S^2 \subset T_4$ represents $2[\mathbb{C}P^2]$ homologically. For all these facts see [Law82].

Hence removing the standard $\mathbb{R}P^2$ from $\mathbb{C}P^2$ leaves a space which retracts onto S^2. By the homotopy invariance of the Stiefel Whitney classes [LM89, p.86] we see that $w_2(\mathbb{C}P^2 - \mathbb{R}P^2) = w_2(S^2) = 0$.

Consequently $\mathbb{C}P^2 - \mathbb{R}P^2$ has a non-extending spin structure. It is the unique spin structure on this space since $H^1(\mathbb{C}P^2 - \mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) = 0$.

The Bockstein sequence shows that

$$H^2(\mathbb{C}P^2, \mathbb{Z}) \longrightarrow H^2(\mathbb{C}P^2, \mathbb{Z}/2\mathbb{Z}) \tag{2.23}$$

is surjective. Hence $H^2(\mathbb{C}P^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and so P.D.$[\mathbb{C}P^1] = w_2(\mathbb{C}P^1)$.

As $\mathbb{C}P^2 - \mathbb{R}P^2$ has a non-extending spin structure we see that the unoriented cohomology class P.D.$[\mathbb{R}P^2]$ equals $w_2(\mathbb{C}P^2)$. Thus the unoriented class P.D.$[\mathbb{R}P^2]$ generates $H^2(\mathbb{C}P^2, \mathbb{Z}/2\mathbb{Z})$ as well.

Note that we could use Corollary 2.1.9 here to see how $\mathbb{C}P^2 - \mathbb{R}P^2$ possessing a non-extending spin structure links the simple facts that $\mathbb{R}P^2$ is neither spin nor orientable. In this case however the remark is content free as any manifold of dimension less than or equal to three is trivially spin if and only if it is orientable [LM89, p.86].

Example 2.1.14 In the example above we showed that $H^2(\mathbb{C}P^2, \mathbb{Z}/2\mathbb{Z})$ was generated by either P.D.$[\mathbb{R}P^2]$ or P.D.$[\mathbb{C}P^1]$. Now $\mathbb{C}P^2$ can be considered as

$$S^2 \times S^2/(x,y) \sim (y,x) \tag{2.24}$$

see [Law82]. We can consider other quotients of $S^2 \times S^2$. In particular we consider the quotient of Habegger [Hab82] namely

$$M = S^2 \times S^2/(x,y) \sim (-x,-y). \tag{2.25}$$

This space is thoroughly investigated in [AL97].

The manifold M is an oriented four manifold. It has an embedded $\mathbb{R}P^2$ which is the image of the diagonal Δ in $S^2 \times S^2$. Since $[\Delta].[\Delta] = 2$ in $S^2 \times S^2$ we see that $[\mathbb{R}P^2].[\mathbb{R}P^2] = 1$ in M. We can deduce from this fact that M is not spin as follows. Denote P.D.$[\mathbb{R}P^2] \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ by γ. Then since $(\gamma \cup \gamma) = [\mathbb{R}P^2].[\mathbb{R}P^2] = 1$ we see that $w_2(M) \cup \gamma = \gamma \cup \gamma \neq 0$ so that $w_2 \neq 0$ and M is not spin.

There are two copies of S^2 in M. These are the images of $S^2 \times 1$ and $1 \times S^2$ under the quotient. Either of these can be taken together with $[\mathbb{R}P^2]$ to form a basis for $H_2(M, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (see [AL97]). Denoting this basis element by $[S^2]$ the intersection matrix for $H_2(M, \mathbb{Z}/2\mathbb{Z})$ is shown in [AL97] to be

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} [\mathbb{R}P^2] \,.\, [\mathbb{R}P^2] & [S^2] \,.\, [\mathbb{R}P^2] \\ [\mathbb{R}P^2] \,.\, [S^2] & [S^2] \,.\, [S^2] \end{pmatrix}. \tag{2.26}$$

Thus we see that $w_2 = $ P.D.$[S^2]$ whilst P.D.$[\mathbb{R}P^2] = 0$. In conclusion we see that $M - \mathbb{R}P^2$ does not have a non-extending spin structure whereas $M - S^2$ does. Contrast this with Example 2.1.13 where we could remove either a S^2 or $\mathbb{R}P^2$ to obtain a non-extending spin structure.

2.2 Equivalence of Spin Structures

One of the least salient aspects of spin structures is that distinct spin structures can be topologically equivalent as principal bundles. This fact was first pointed out by Milnor [LM89, p.84]. This result comes from the fact that spin structures on a spin manifold M are in one to one correspondence with elements of $H^1(M, \mathbb{Z}/2\mathbb{Z})$. To see how many topologically distinct spin structures there can be over M, one must consider the exact sequence [LM89, p.83]

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin_n \longrightarrow SO_n \longrightarrow 0. \tag{2.27}$$

This gives rise to an exact sequence

$$H^0(M, SO_n) \xrightarrow{\delta} H^1(M, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(M, Spin_n) \longrightarrow H^1(M, SO_n) \tag{2.28}$$

see [LM89, p.37]. Here $H^1(M, Spin_n)$ and $H^1(M, SO_n)$ are sets of topological equivalence classes of principal $Spin_n$ and SO_n bundles respectively. Since $\mathbb{Z}/2\mathbb{Z}$ is a field $H^1(M, \mathbb{Z}/2\mathbb{Z})$ can be thought of as either the topological equivalence classes of $\mathbb{Z}/2\mathbb{Z}$ bundles or as the first cohomology group of M with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

The sequence is exact in the sense that the sets of topological equivalence classes have distinguished elements: the equivalence class containing the trivial bundle. Choosing a spin structure on M gives an element of $H^1(M, Spin_n)$. The sequence shows that the topological equivalence classes of spin structures, when considered as principal $Spin_n$ bundles, are in one to one correspondence with elements of

$$H^1(M, \mathbb{Z}/2\mathbb{Z})/\delta H^0(M, SO_N). \tag{2.29}$$

The coboundary map δ can be surjective. For instance any class in $H^1(M, \mathbb{Z}/2\mathbb{Z})$ which is the mod 2 reduction of an integral class is in the image of δ [LM89, p.84].

Our classification of spin structures on $M - \Sigma$ into extending and non-extending is misleading in the sense that a non-extending spin structure may be topologically trivial. Furthermore, non-extending spin structures and extending spin structures can be isomorphic as principal $Spin$ bundles.

The terms non-extending and extending for spin structures on $M - \Sigma$ do not refer to a topological obstruction to the associated principal $Spin$ bundle on $M - \Sigma$ being the restriction of a principal $Spin$ bundle on M.

2.2.1 Examples

Example 2.2.1 The main example $\mathbb{C}P^1 - \{0, \infty\}$ illustrates clearly the main point of Section 2.2. On the complex manifold $\mathbb{C}P^1$ the unique spin structure corresponds to the unique holomorphic square root of the canonical bundle $K_{\mathbb{C}P^1}$ of $\mathbb{C}P^1$. For $\mathbb{C}P^1 - \{0, \infty\}$ the canonical bundle K is holomorphically trivial. The two spin structures on $\mathbb{C}P^1 - \{0, \infty\}$ correspond to two **holomorphically trivial** square roots of K. Holomorphic global sections of both roots are given on page 22. Hence, in this case, the non-extending and the extending spin structures are both holomorphically equivalent to the trivial bundle when considered as principal U_1 bundles.

Example 2.2.2 Any holomorphic line bundle on a punctured Riemann surface, $S_g - m$, is holomorphically trivial. This is because $H^1(S_g - m, \mathcal{O}) = H^2(S_g - m, \mathbb{Z}) = 0$ so that $H^1(S_g - m, \mathcal{O}^*) = 0$. Consequently, all spin structures on punctured Riemann surfaces will be holomorphically equivalent as principal U_1 bundles.

Example 2.2.3 For $n \geqslant 3$ consider the n sphere with an embedded $n-2$ sphere removed: $S^n - S^{n-2}$. We can regard $\mathbb{C}P^1 - \{0, \infty\}$ as the exceptional case $S^2 - S^0$. Now $S^n - S^{n-2}$ is homotopy equivalent to the unit circle so $H^1(S^n - S^{n-2}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus $S^n - S^{n-2}$ has two spin structures whereas S^n has only one.

The Bockstein sequence

$$\ldots \longrightarrow H^1(S^n - S^{n-2}, \mathbb{Z}) \longrightarrow H^1(S^n - S^{n-2}, \mathbb{Z}) \longrightarrow H^1(S^n - S^{n-2}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \tag{2.30}$$

shows that every class in $H^1(S^n - S^{n-2}, \mathbb{Z}/2\mathbb{Z})$ is the mod 2 reduction of an integral class. Such classes are in the image of the coboundary map δ in equation (2.28) see [LM89, p.84]. Thus the two spin structures, on $S^n - S^{n-2}$, when regarded as principal $Spin_n$ bundles are both topologically trivial. In the terminology used here, however, only one of them is extending.

Chapter 3

Basic Analysis

It was shown in the previous section that non-extending spin structures, when considered as principal *Spin* bundles, could be topologically trivial. In this section we present the basic analytical facts concerning non-extending spin structures. We concentrate on square summable (L^2) solutions of the Dirac operator on $M^n - \Sigma^{n-2}$.

The principal result in this section is that any L^2 solution for the Dirac operator on $M^n - \Sigma^{n-2}$, associated to the extending spin structure, is the restriction of a harmonic spinor on M.

This justifies the use of the term ‘extending spin structure’: its L^2 harmonic spinors extend to be spinors on M^n. For a non-extending spin structure we have already seen that an L^2 harmonic spinor on $M^n - \Sigma^{n-2}$ need not be the restriction of an L^2 spinor on M (see Section 1.2.5). This justifies the terminology of a non-extending spin structure.

We explain in detail the lack of self adjointness of the Dirac operator associated to non-extending spin structures on $M - \Sigma$ and prove an a priori lemma for the growth of solutions to such operators.

3.1 Removable Singularities

In this section we prove that every L^2 solution of the Dirac equation, associated to an extending spin structure on $M^n - \Sigma^{n-2}$, is just the restriction of a smooth spinor on M. The basic idea of the proof is as follows. Since the solution to the Dirac operator is L^2 it may be regarded as a distribution on M. Using suitably constructed bump functions, it is then possible to show that the spinor is a distributional solution to the Dirac equation on the whole manifold. Appealing to standard results of elliptic regularity the spinor is then seen to extend smoothly to the whole of M. This proof is, in essence, that of Harvey and Polking in [HP70] adapted to our situation.

Theorem 3.1.1 Let M^n be a compact Riemannian spin manifold and Σ^{n-2} a closed embedded submanifold of codimension two. Consider a metric on $M^n - \Sigma^{n-2}$ obtained by restricting a smooth metric on M. Let Ψ be an L^2 solution to the Dirac operator $\not{D}$ on $M - \Sigma$. If Ψ is associated to an extending spin structure on $M - \Sigma$ then it is the restriction of a smooth spinor on M which satisfies the Dirac equation on M.

Proof Since Ψ is an L^2 section of the spinor bundle $\not{S}$, associated to the *extending* spin structure, then it is *locally* a L^2 section of a spinor bundle $\not{S}_M$ on M. This is simply because the extending spin structure on $M^n - \Sigma^{n-2}$ is the restriction of a spin structure on M and so we have a natural inclusion $\not{S} \subset \not{S}_M$. In this way we may consider Ψ as a distribution on M. It is important to note that since Ψ is a section of a spinor bundle associated to an extending spin structure, **both** the spinor bundle **and**, the Dirac operator on $M - \Sigma$ have natural extensions to M. Specifically these are the spinor bundle $\not{S}_M$ and the Dirac operator $\not{D}_M$ on M.

Consider any compact subset $B \subset \Sigma$. Let $\mu : (0, \infty) \to \mathbb{R}$ be a smooth function with $0 \leqslant \mu \leqslant 1, \operatorname{supp} \mu \subset [0,2]$ and $\mu(t) = 1$ for $t \in [0,1]$. Let $d(x)$ be the infimum of the

geodesic distance between $x \in M$ and B. Then for every $\epsilon > 0$ set

$$\varphi_\epsilon(x) = \mu\left(\frac{d(x)}{\epsilon}\right). \tag{3.1}$$

Denote by B_ϵ the union of all the geodesic balls of radius ϵ around every point of B. Then for every $\epsilon > 0$, φ_ϵ is Lipschitz and hence differentiable almost everywhere on M. By construction B_ϵ has the following properties (see [Dod81]),

$$\begin{aligned} &0 \leqslant \varphi_\epsilon(x) \leqslant 1 && \text{for all} \quad x \in M \\ &\varphi_\epsilon(x) = 1 && \text{for all} \quad x \in B_\epsilon \\ &\mid d\varphi_\epsilon(x) \mid \leqslant C_0/\epsilon && \text{almost everywhere on } M \\ &\operatorname{supp} \varphi_\epsilon \subset B_{2\epsilon}, \end{aligned} \tag{3.2}$$

where C_0 is a constant independent of ϵ.

Since M is compact the closure of B_ϵ, denoted $\overline{B}_\epsilon$, is compact for all $\epsilon > 0$. Now fix a smooth compactly supported section of $\not{S}_M$, $\phi_0 \in C_0^\infty(M, \not{S}_M)$. Fix B as being the set $\operatorname{supp} \phi_0 \cap \Sigma$. Then the distributional support of $\not{D}\Psi$ is contained in Σ. With the φ_ϵ constructed as above consider the distributional pairing of $\not{D}\Psi$ and ϕ_0. Since the Dirac operator is self adjoint on a compact manifold,

$$(\not{D}\Psi, \phi_0) = (\not{D}\Psi, \varphi_\epsilon\phi_0) = (\Psi, \not{D}(\varphi_\epsilon\phi_0)), \tag{3.3}$$

where the round brackets denote the distributional pairing. Using Hölders inequality one obtains that the right hand side of this expression is less than or equal to

$$\|\not{D}(\varphi_\epsilon\phi_0)\|_\infty . \|\chi_{B_\epsilon}\Psi\|_1. \tag{3.4}$$

Since $\overline{B}_\epsilon$ is compact in M the integral of its indicator function can be compared to n dimensional Lebesgue measure λ. Using Hölders inequality again shows that the above expression is in turn less than or equal to

$$\text{const.}\|\not{D}(\varphi_\epsilon\phi_0)\|_\infty . \lambda(B_\epsilon)^{\frac{1}{2}} . \|\chi_{B_\epsilon}\Psi\|_2. \tag{3.5}$$

The constant in this expression depends only upon B. By the construction of φ_ϵ this is less than a constant multiple of

$$\epsilon^{-1}\lambda\left(B_\epsilon\right)^{\frac{1}{2}}.\|\chi_{B_\epsilon}\Psi\|_2 = \left(\epsilon^{-2}\lambda\left(B_\epsilon\right)\right)^{\frac{1}{2}}.\|\chi_{B_\epsilon}\Psi\|_2. \tag{3.6}$$

Since Σ is a smooth submanifold, the limit as ϵ tends to zero of $\epsilon^{-2}\lambda\left(B_\epsilon\right)$, is precisely the $(n-2)$ dimensional Lebesgue measure of B. This is finite so for small $\epsilon > 0$,

$$(\not{D}\Psi, \phi_0) \leqslant \text{const.}\|\chi_{B_\epsilon}\Psi\|_2, \tag{3.7}$$

where again the constant depends only on B.

Lastly since Ψ is square summable on M, the right hand side can be made arbitrarily small. Thus $(\not{D}\Psi, \phi_0) = 0$ for all $\phi_0 \in C_0^\infty\left(M, \not{S}_M\right)$ and we conclude that Ψ is a distributional solution of $\not{D}$ on M. Using the standard results of elliptic theory, see [Mor66], Ψ is therefore a smooth solution to $\not{D}$ on M as required. □

3.1.1 Main Remark and Example

Remark 3.1.2 Note that in this proof we implicitly use the fact that the spin structure on $M - \Sigma$ is the restriction of a spin structure on M. This is because spinors are sections of vector bundles and for the above proof to work we must be able to interpret them as vector valued functions in a neighbourhood of Σ. Furthermore the Dirac operator on $M - \Sigma$ must have an interpretation on the whole of M as well.

For the extending spin structures we have these interpretations: we can consider the Dirac operator on $M - \Sigma$ acting on sections of $\not{S}$ as being the restriction of the Dirac operator on M acting on sections of $\not{S}_M$.

Remark 3.1.3 We cannot prove a removable singularities theorem for non-extending spin structures on $M - \Sigma$.

When M is not spin then this is clear since there is no spinor bundle which we can consider the section $\Psi \in L^2(M - \Sigma, \not{S}_{M-\Sigma})$ as being a distributional section of.

When M is spin then non-extending spin structures correspond to elements of $H^1(M - \Sigma)$ which are not in the restriction $H^1(M) \longrightarrow H^1(M - \Sigma)$. From the isomorphism

$$H^1(M - \Sigma) = \mathrm{Hom}(\pi_1(M - \Sigma), \mathbb{Z}/2\mathbb{Z}) \tag{3.8}$$

these elements are seen to be flat line bundles over $M - \Sigma$. Those not in the image of the restriction are flat line bundles with holonomy -1 about the submanifold Σ. From this we see that

when M is spin we can view a non-extending spin structure on $M - \Sigma$ as a twisting of an extending spin structure on M by a flat line bundle of order two on $M - \Sigma$.

Since a flat line bundle is C^∞ trivial there are bundles on M which we could possibly consider an L^2 solution, on the complement, as extending to be a distributional section of. The twisting, however, adds a non trivial holonomy term to the Dirac operator which is an obstruction to considering it as an operator on M. Consequently there is no regularity theorem to say that the solution extends smoothly. This is all immediate from the L^2 harmonic spinor associated to the non-extending spin structure exhibited in the main example.

Example 3.1.4 We illustrate here the point of view explained in Remark 3.1.3 using the basic Example 1.1.3. Recall that $U - \{0\}$ is the unit two disc, equipped with the usual flat metric, with the origin removed. On $U - \{0\}$ we have two natural coordinate systems (x, y) and (r, θ). The two corresponding orthonormal bases $(\widetilde{e_1}, \widetilde{e_2}) = (\partial_x, \partial_y)$ and $(e_1, e_2) = (\partial_r, r^{-1}\partial_\theta)$ are **both** trivialising sections of the tangent bundle $U - \{0\} \times \mathbb{R}^2$. Call these sections $\widetilde{s}$ and s respectively. Consider the principal SO_2 bundle of frames $P_{SO} = U - \{0\} \times SO_2$ as being trivialised by $\widetilde{s}$. With respect to this basis the trivialising section s is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.9}$$

The Levi-Civita connection on $U - \{0\}$ with respect to the basis $(\widetilde{e_1}, \widetilde{e_2})$ is the exterior derivative. With respect to (e_1, e_2) it is

$$d + \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix}. \tag{3.10}$$

The section $\widetilde{s}$ extends to a trivialising section of the principal bundle of frames for the un punctured disc. Both spin structures on $U - \{0\}$ are topologically trivial (see Example 1.1.3). The section $\widetilde{s}$ lifts to a global trivialising section of the extending spin structure and s lifts to a global trivialising section of the non-extending spin structure.

When we represent spinors associated to the non-extending spin structure as complex valued functions we are trivialising the spinor bundle with respect to the lift of s. The spinor derivative is

$$d + A_\theta d\theta = d + \frac{d\theta}{2} e_{12} \tag{3.11}$$

with respect to this basis (just (3.10) under the Lie algebra identification $\mathfrak{spin}_2 = \mathfrak{so}_2$). The A_θ term is the holonomy term that makes the flat connection non-trivial and prevents the Dirac operator extending.

For the non-extending spin structure we may change gauge and consider the spinor derivative locally as d. Doing this involves representing spinors with respect to the section which is the lift if s. The lifts of s are however

$$\pm(\cos\theta/2 + e_1 e_2 \sin\theta/2). \tag{3.12}$$

In this gauge sections of the spinor bundle will be aperiodic in theta so we may no longer consider spinors as complex valued functions.

3.1.2 Corollaries

Remark 3.1.5 Theorem 3.1.1 shows that for extending spin structures there is no interesting new behaviour to study when we restrict attention to L^2 harmonic spinors. The L^2 index of an extending spin structure on $M^n - \Sigma^{n-2}$ is the same as the L^2 index on the compact manifold M. Consequently, it equals the $\widehat{A}$ genus of the compact manifold M.

Remark 3.1.6 Theorem 3.1.1 has a trivial application to our main example. Using it we see that the index for the extending spin structure on $\mathbb{C}P^1$ with two points removed is zero. Further since the two sphere has strictly positive scalar curvature, the Bochner identity tells us that there are **no** harmonic spinors for the extending spin structure on $\mathbb{C}P^1$. This argument can therefore replace the explicit calculation in Section 1.2.5.

Remark 3.1.7 The Removable Singularities Theorem 3.1.1 shows that the L^2 index of the Dirac operator for any non-extending spin structure on $\mathbb{C}P^1$ with the three points $0, 1, \infty$ removed is always zero. The removal of the third point adds no new types of spin structure to those considered in the main example with two points removed. Specifically all the spin structures on $\mathbb{C}P^1 - \{0, 1, \infty\}$ have either spinor derivative with trivial holonomy about all the points or holonomy -1 about two of the points. Since there are three ways to chose these two points this accounts for all four spin structures on $\mathbb{C}P^1 - \{0, 1, \infty\}$. Therefore **any** L^2 harmonic spinor on $\mathbb{C}P^1 - \{0, 1, \infty\}$ must be smooth over at least one point.

For each of the three non-extending spin structures on $\mathbb{C}P^1 - \{0, 1, \infty\}$ we see therefore that the situation is unchanged from that in the main example: the vector space of L^2 solutions has dimension two and $\dim\ker \not{D}^+ \cap L^2 - \dim\ker \not{D}^+ \cap L^2 = 0$. Again this result holds for any three distinct points on $\mathbb{C}P^1$ since any three such points can be mapped biholomorphically to $0, 1$ and ∞. In Remark 4.2.1 we generalise these ideas to show that we must remove an even number of points from a Riemann surface to get new types of spin structures. In Example 4.2.5 we calculate the dimensions of $\dim\ker \not{D}^+ \cap L^2$ and

$\dim \ker \not{D}^- \cap L^2$ for all spin structures on $\mathbb{C}P^1$ with $2m$ distinct points removed.

Remark 3.1.8 The Removable Singularities Theorem for extending spin structures tells us that the usual vanishing theorems continue to hold. For example we have

Corollary 3.1.9 Let Ψ be an L^2 harmonic spinor associated to an extending spin structure on $M-\Sigma$. If M is compact and has non-negative scalar curvature then Ψ is parallel $\nabla\Psi = 0$. There are no harmonic spinors if the scalar curvature is strictly positive at some point.

Remark 3.1.10 We remark here how the criteria on M may be weakened. The removable singularities theorem holds verbatim if M is just assumed complete see [Dod81]. The corollary can then be rephrased thus.

Corollary 3.1.11 Let Σ be compact and M complete with non-negative bounded scalar curvature. There are no L^2 harmonic spinors on $M-\Sigma$ associated to an extending spin structure, if either the volume of $M-\Sigma$ is infinite or, if the scalar curvature is positive at some point.

To avoid any unnecessary confusion, we will **always** assume M to be compact in this thesis.

3.2 Lack of Self Adjointness for the Dirac Operator

Neighbourhoods, of points of Σ^{n-2}, are not simply connected in $M^n - \Sigma^{n-2}$. Since the metric on $M^n - \Sigma^{n-2}$ is obtained by restricting a smooth metric on M^n the space $M^n - \Sigma^{n-2}$ is not complete. This lack of completeness prevents the Dirac operator $\not{D}$ from being self adjoint in general (on a complete manifold the associated Dirac operator is **always** self adjoint [LM89, p.117]).

The main point of this section is to exhibit this lack of self adjointness and to show the sort of domain on which self adjointness may be asserted.
We start with the most fundamental ambiguity arising from the lack of self-adjointness.

3.2.1 Harmonic Spinors

When the Dirac operator is not self adjoint there is an ambiguity in the phrase 'harmonic spinor'. This phrase usually refers to a solution of the spinor Laplacian. On a complete manifold Ψ is a solution of $\not{D}_M$ if and only if Ψ is a solution of the spinor Laplacian $\not{D}^2_M$. Thus every harmonic spinor is a solution of the Dirac equation.
We will show that the Dirac operator on $M-\Sigma$ is not, in general, self adjoint. Thus it may no longer be the case that $\ker \not{D}^2 \subset \ker \not{D}$. For L^2 solutions associated to an extending spin structure on $M-\Sigma$ the Removable Singularities Theorem 3.1.1 tells us that $\ker_{L^2} \not{D} = \ker \not{D}_M$. The ambiguity thus only arises for non-extending spin structures where we may have spinors which are in $\ker \not{D}^2 \cap L^2$ but which are not in $\ker \not{D} \cap L^2$. **Despite this we will continue to call any solution to the Dirac operator a 'harmonic spinor'.**

3.2.2 A Vanishing Theorem

We investigate the lack of self adjointness of the Dirac operator by proving a vanishing theorem. To assert a vanishing theorem for non-extending spin structures we just need to justify integration by parts. This can be done if we restrict the domain of our spinors suitably. Denote by L^2_1 the first Sobolev space: the domain of spinors Ψ such that

$$\|\nabla\Psi\| + \|\Psi\| \tag{3.13}$$

is finite. Here, ∇ denotes the covariant spinor derivative, and $\| \, . \, \|$ is the usual L^2 norm. There is a natural inclusion $L^2_1 \subset L^2$.

Lemma 3.2.1 Let M have non-negative scalar curvature and let $M-\Sigma$ have a non-extending spin structure. Then every harmonic spinor on $M-\Sigma$ in the first Sobolev space L_1^2 is parallel. If the scalar curvature is strictly positive at some point then there are no harmonic spinors on $M-\Sigma$ in L_1^2.

Proof Choose an orthonormal basis $\{e_i\}|_{x_0}$ for the tangent space at some point $x_0 \in M-\Sigma$. Parallel translate this to obtain a geodesic basis $\{e_i\}$ in a neighbourhood of x_0. The Weitzenböck identity says that

$$\not{D}^2 = -\nabla_i\nabla_i + \frac{1}{4}\kappa, \tag{3.14}$$

at x_0, where κ is the scalar curvature of the manifold and ∇ is once again the spinor derivative. Thus,

$$\left\langle \Psi, \not{D}^2\Psi \right\rangle = -\left\langle \Psi, \nabla_i^2\Psi \right\rangle + \frac{1}{4}\left\langle \Psi, \kappa\Psi \right\rangle, \tag{3.15}$$

at x_0, where the angle brackets denote the Hermitian inner product on $\not{S}$. We have also that

$$d[\langle \Psi, \nabla_i\Psi\rangle\, e_i \llcorner \mu] = [\langle \nabla_i\Psi, \nabla_i\Psi\rangle + \langle \Psi, \nabla_i\nabla_i\Psi\rangle]\mu \tag{3.16}$$

where $e_i \llcorner \mu$ denotes the contraction of the i^{th} basis vector with the volume form of $M-\Sigma$. Combining these two pointwise expressions we see that for a harmonic spinor the following global identity holds

$$d[\langle \Psi, \nabla\Psi\rangle \llcorner \mu] = \langle \nabla\Psi, \nabla\Psi\rangle + \frac{1}{4}\langle \Psi, \kappa\Psi\rangle. \tag{3.17}$$

Denote an ϵ tubular neighbourhood of Σ by N_ϵ. Integrating equation (3.17) over N_ϵ we see that for all $\epsilon > 0$

$$\begin{aligned} \left|\int_{\partial N_\epsilon} \langle \Psi, \nabla\Psi\rangle \llcorner \mu\right| &\leqslant \int_{N_\epsilon} \left(|\nabla\Psi|^2 + \frac{1}{4}\langle \Psi, \kappa\Psi\rangle\right)\mu \\ &\leqslant \text{const.} \int_{N_\epsilon} \left(|\nabla\Psi|^2 + |\Psi|^2\right)\mu. \end{aligned} \tag{3.18}$$

Now since $\Psi \in L_1^2$ we see that the right hand side of equation (3.18) tends to zero as ϵ does. Lastly, integrating (3.17) over $M-N_\epsilon$, gives

$$\int_{\partial N_\epsilon} \langle \Psi, \nabla\Psi\rangle \llcorner \mu = \int_{M-N_\epsilon} \left(|\nabla\Psi|^2 + \frac{1}{4}\langle \Psi, \kappa\Psi\rangle\right)\mu. \tag{3.19}$$

By equation (3.18), above, as ϵ tends to zero we see that

$$\int_M \left(|\nabla\Psi|^2 + \frac{1}{4}\langle \Psi, \kappa\Psi\rangle\right)\mu = 0 \tag{3.20}$$

and the vanishing theorem is obtained in the usual way. □

3.2.3 Remarks

Remark 3.2.2 Using our main example we can see clearly what is going on. On $\mathbb{C}P^1 - \{0,\infty\}$ we have the positive chiral spinor represented by $\Psi = z^{-1/2}(dz)^{1/2}$ on U_0. This is not a parallel spinor since

$$\nabla(\Psi) = \nabla\left(\frac{dz^{1/2}}{z^{1/2}}\right) = (\partial + \overline{\partial})\left(\frac{dz^{1/2}}{z^{1/2}}\right) = -\frac{dz dz^{1/2}}{z^{3/2}} = \frac{-\Psi^3}{2} \tag{3.21}$$

This violates the vanishing Lemma 3.2.1. Alternatively a parallel spinor must have constant length which is clearly not the case here. Pointwise we have that

$$\langle \nabla_i\Psi, \Psi\rangle\,\mu = \langle \Psi, \nabla_i\Psi\rangle\,\mu - d[\langle \Psi, \Psi\rangle\, e_i \llcorner \mu]. \tag{3.22}$$

We see however that

$$\int_{B_\epsilon} d[\langle \Psi, \Psi\rangle\, e_i \llcorner \mu] = \int_{\partial B_\epsilon} \frac{d\theta}{(1+\epsilon^2)^2} = \frac{2\pi}{(1+\epsilon^2)^2} \tag{3.23}$$

which does not tend to zero as ϵ does. This shows that, in general, the spinor covariant derivative is not self adjoint.

We can see similarly that the Dirac operator is not self adjoint

$$\left(d\overline{z}dz^{1/2}, \not{D}(z^{-1/2}dz^{1/2})\right) = 0, \tag{3.24}$$

but

$$\left\langle \not{D}(d\overline{z}dz^{1/2}), z^{-1/2}dz^{1/2} \right\rangle = z\overline{z}^{-1/2}(1+z\overline{z})^2, \tag{3.25}$$

which is not even integrable near zero.

We conclude that all the L^2 harmonic spinors associated to a non-extending spin structure may not, in general, be included in any domain on which the Dirac operator is self adjoint.

3.3 A Growth Lemma

In this section we prove the following Lemma.

Lemma 3.3.1 If Ψ is a L^2 harmonic spinor, associated to a non-extending spin structure on $M^n - \Sigma^{n-2}$, then

$$|\Psi(y)| \leqslant \mathrm{const.} d(y, \Sigma)^{-1}. \tag{3.26}$$

In this expression $d(\ .\ ,\Sigma)$ is the geodesic distance of $y \in M - \Sigma$ from Σ and the constant is independent of y.

This Lemma is used nowhere else in this thesis.

Remark 3.3.2 This estimate is not optimal. As will be seen in Section 5.2.1 any L^2 harmonic spinor on $M - \Sigma$, with product metric, has growth of order $O(r^{-1/2})$. The method of proof only involves the presence of a flat connection with holonomy $e^{i\alpha}$, and not specifically the case $\alpha = \pi$. The estimate reflects this as follows.

Consider the Dirac operator, on the punctured disc, twisted by a flat line bundle of holonomy α

$$\not{D}^+_\alpha = -\left(\partial_r + \frac{1+\alpha}{2r} + \frac{i}{r}\partial_\theta\right). \tag{3.27}$$

This shows that, for such connections, L^2 solutions can vary as r^β for all $\beta > -1$.

Proof (Lemma 3.3.1) The growth estimate is proved first in the Euclidean case by approximating a Greens function on a thin annulus about Σ. The final step in the proof argues that for a general metric on $M - \Sigma$ the use of normal coordinates just adds error terms which do not effect the overall essence of the proof.

Step 1 Set up

For notational convenience we take $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant 2$ throughout this proof. Fix a point $a = (a_1 \dots a_n) \in M - \Sigma$. We work throughout on a neighbourhood of a point on Σ which contains a. Assume that the metric on this neighbourhood is Euclidean, and that we have coordinates $(x_1 \dots x_n) \equiv (x, y, z_1, \dots, z_{n-2})$, such that $\{x = y = 0\}$ is the intersection of Σ with the neighbourhood.

Choose a trivialising section for T_M given by $(e_1 \dots e_n) = (\partial_x, \partial_y, \partial_{z_1}, \dots, \partial_{z_{n-2}})$. Denote Clifford multiplication by e_j and then e_i by e_{ij}.

We always work in the gauge of the spinor bundle determined by the lift of this section.

There is a polar coordinate system about Σ, namely (r, θ, z_i), and a corresponding orthonormal basis for $T_{M-\Sigma} : (\partial_r, r^{-1}\partial_\theta, \partial_{z_i})$. Denote the elements of the Clifford algebra induced by this section by $\widetilde{e}_i$. Note that $e_{12} = \widetilde{e}_{12}$. Locally a section of the spinor bundle associated to a non-extending spin structure on $M - \Sigma$ is therefore given by αf where f is a function and α is the (locally trivialising) section, see equation (3.12)

$$\alpha = (\cos\theta/2 + e_{12}\sin\theta/2). \tag{3.28}$$

Let $r(y) = d(y, \Sigma)$ be the geodesic distance of the point y from Σ and define $p_0 = d(a, \Sigma)$. Let $p(y) = r(y - a)$. We may also assume that the unique point $b \in \Sigma$ such that $d(a, b) = p_0$ has coordinates $\{x = y = z_i = 0\}$.

Finally, choose ϵ and δ such that $0 < \epsilon < p_0/4$ and $0 < \delta < \epsilon$. We use these to define the sets shown in Figure 3.1.

Now $\not{D} = e_j \nabla_j = e_j \partial_j$ where ∂_j is the j^{th} coordinate derivative and thus,

$$\alpha^{-1}\not{D}(\alpha f) = \left(e_j\partial_j + \alpha^{-1}\not{D}(\alpha)\right) f = \left(e_j\partial_j + \frac{\widetilde{e}_1}{2r}\right) f. \tag{3.29}$$

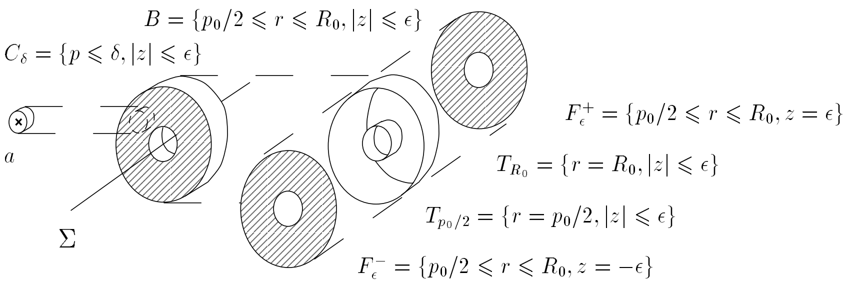


Figure 3.1: Definitions

Step 2

If αf is a harmonic spinor then

$$\left(\nabla_j\alpha^{-1}\right)\frac{a_i - y_i}{4\pi p^2} e_{ij}.\alpha f.\mu = d\left(\alpha^{-1}\frac{(a_i - y_i)}{4\pi p^2} e_{ij}.\alpha f.[e_j \llcorner \mu]\right). \tag{3.30}$$

We record that

$$\begin{aligned} |\alpha| &= O(1) \\ \alpha e_k &= e_k \alpha \qquad k \geqslant 3 \\ e_i \alpha &= \alpha^{-1} e_i \qquad 1 \leqslant i \leqslant 2 \end{aligned} \tag{3.31}$$

and

$$|\nabla_j \alpha^{-1}| = \begin{cases} O(1) & j \geqslant 3 \\ O(r^{-1}) & j = 1, 2. \end{cases} \tag{3.32}$$

Step 3

We show here that if expression (3.30) is integrated over $B - C_\delta$ then the limit exists as $\delta \longrightarrow 0$. Dividing (3.30) by ϵ and integrating over $B - C_\delta$ gives that

$$\int_{B - C_\delta} \left(\nabla_j\alpha^{-1}\right)\frac{y_i - a_i}{4\pi p^2 \epsilon} e_{ij}.\alpha f.\mu = \int_{\partial B - \partial C_\delta} \alpha^{-1}\frac{(y_i - a_i)}{4\pi p^2 \epsilon} e_{ij}.\alpha f.[e_j \llcorner \mu]. \tag{3.33}$$

We calculate first the explicit limit, as $\delta \longrightarrow 0$, of the integral over ∂C_δ. Recall that in general

$$e_j \llcorner \mu = \eta_j dS \tag{3.34}$$

where η_j is the j^{th} component of the unit normal to the surface and dS is the surface area element.

For ∂C_δ the unit normal is

$$\eta = \frac{(y_i - a_i)}{p} e_i \tag{3.35}$$

Thus $\alpha^{-1}.\eta = \eta.\alpha$ and we can rewrite the right hand side of (3.33) in terms of Clifford multiplication by the unit normal to ∂C_δ

$$\int_{\partial C_\delta} \alpha^{-1} \frac{(y_i - a_i)}{4\pi p^2 \epsilon} e_i.\eta.\alpha f.dS = \int_{\partial C_\delta} \frac{f}{4\pi p \epsilon} dS. \tag{3.36}$$

This expression tends uniformly to $-f(a + \delta_1)$, as $\delta \longrightarrow 0$, where $\delta_1 \longrightarrow 0$ uniformly as $\epsilon \longrightarrow 0$.

Consequently, not only may we take the limit of (3.33) as $\delta \longrightarrow 0$ but, we may divide (3.30) by ϵ and take the limit of the integral over B as $\epsilon \longrightarrow 0$. Writing this expression out gives

$$f(a) = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon} \int_B \left(\nabla_j \alpha^{-1}\right) \frac{(y_i - a_i)}{4\pi p^2} e_{ij}.\alpha f.\mu - \frac{1}{\epsilon} \int_{\partial B} \alpha^{-1} \frac{(y_i - a_i)}{4\pi p^2} e_{ij}.\alpha f.[e_j \llcorner \mu]. \tag{3.37}$$

It remains only to produce bounds on the two right hand terms. We consider these separately starting with the right hand term.

Step 4 Second Term

To show that the second term is bounded we split the boundary ∂B into faces F_ϵ^- and F_ϵ^+ and tubes T_{R_0} and $T_{\frac{p_0}{2}}$ (see Figure 3.1). We show that as ϵ tends to zero the face terms tend uniformly to zero and the tube terms are of the order $O(p_0^{-1/2})$.

The integral of the second term in this expression over the tubes is equal to

$$\frac{1}{\epsilon} \int_{T_{R_0} \cup T_{\frac{p_0}{2}}} \frac{(y^i - a^i)}{4\pi p^2} e_i.\eta.f dS \tag{3.38}$$

where

$$\eta = \sum_{i=1}^{2} \frac{(y_i)}{r} e_i \quad \text{is the unit normal.} \tag{3.39}$$

Since $r = p_0/2$ and $p \geqslant p_0/2$ on $T_{\frac{p_0}{2}}$ it is seen that ,

$$\begin{aligned} \left| \frac{1}{\epsilon} \int_{T_{\frac{p_0}{2}}} \frac{(y_i - a_i)}{4\pi p^2} e_i.\eta.f dS \right| &\leqslant \frac{1}{2\pi \epsilon p_0} \int_{T_{\frac{p_0}{2}}} |f| dS \\ &= \frac{1}{4\pi\epsilon} \int_{-\epsilon}^{\epsilon} \left(\int_{C(b)} |f| d\theta + \delta_2 \right) dz, \end{aligned} \tag{3.40}$$

where $C(b)$ is the circle $\{y \in M - \Sigma : d(y,b) = p_0/2\}$ and $\delta_2 \longrightarrow 0$ as $\epsilon \longrightarrow 0$. Letting $\epsilon \rightarrow 0$ therefore shows the right hand side to be equal to

$$\begin{aligned} \frac{1}{2\pi} \int_{C(b)} |f| d\theta &\leqslant \frac{1}{2\pi} \left(\int_{C(b)} |\, f \,|^2 \, d\theta \right)^{1/2} (\pi p_0)^{1/2} \quad \text{by Hölder} \\ &\leqslant \frac{1}{2\sqrt{\pi p_0}} \left(p_0^2 \int_{C(b)} |\, f \,|^2 \, d\theta \right)^{1/2}. \end{aligned} \tag{3.41}$$

Since f is L^2, Fubinis' theorem says that the term in parenthesis is bounded as $p_0 \longrightarrow 0$ (see [SS92]).

Following precisely the same method for the integral on T_{R_0}, and noticing that $r = R_0$ and $p \geqslant R_0 - p_0$ on T_{R_0}, it is seen that

$$\lim_{\epsilon \rightarrow 0} \left| \frac{1}{\epsilon} \int_{T_{R_0}} \frac{(x_o^i - y^i)}{4\pi p^2} e_i.\eta.f dS \right| \leqslant \frac{R_0^{3/2}}{\sqrt{2\pi}\,(R_0 - p_0)} \left(\int_{C(b)} |\, f \,|^2 \, d\theta \right)^{1/2}, \tag{3.42}$$

which is bounded independently of x_0.

Now consider the face terms in the boundary expression namely

$$\frac{1}{\epsilon} \int_{F_\epsilon^+ \cup F_\epsilon^-} \alpha^{-1} \frac{(y_i - a_i)}{4\pi p^2} e_{ij}.\alpha f\, [e_j \llcorner \mu] = \frac{1}{\epsilon} \int_{F_\epsilon^+ \cup F_\epsilon^-} \alpha^{-1} \frac{(y_i - a_i)}{4\pi p^2} e_i.\alpha.\eta.f dS. \tag{3.43}$$

The modulus of this expression is less than or equal to

$$\frac{1}{4\pi}\int_{p_0/2}^{R_0}\int_0^{2\pi}\left|\left(\frac{f\,(y-\epsilon)-f\,(y)}{\epsilon}\right)-\left(\frac{f\,(y+\epsilon)-f\,(y)}{\epsilon}\right)\right| rdrd\theta. \tag{3.44}$$

Consequently the face terms tend uniformly to zero as ϵ does.

Step 5 First Term

To show that the first term is bounded we split the integral over B into two pieces. These pieces are the sets $\{p > p_0/2\}$ and $\{p < p_0/2\}$. We show that each piece is of order p_0^{-1}. For the set $\{p > p_0/2\}$ we have that $r > p_0/2$. Using the expressions in (3.31) and (3.32) we see that the integral of the first term over this set is bounded by

$$\int_{p>p_0/2}\frac{|f|}{rp\epsilon}\mu \leqslant \frac{4}{p_0^2\epsilon}\int_{p>p_0/2}|f|\mu \leqslant \frac{4}{p_0^2}\left(\int_U |f|^2 rdrd\theta\right)^{1/2}\left(\frac{\pi p_0^2}{4}\right)^{1/2} = \text{const.}p_0^{-1}. \tag{3.45}$$

The last inequality is Hölders, U is the disc through b and the constant depends only on the L^2 norm.

For the set $\{p < p_0/2\}$ we can once again use equation (3.30) and to obtain

$$\int_{p<p_0/2}\left(\nabla_j\alpha^{-1}\right)\frac{y_i-a_i}{4\pi p^2\epsilon}e_{ij}.\alpha f.\mu = \int_{p=p_0/2}\left(\frac{\alpha^{-1}(a_i-y_i)e_{ij}.\alpha f}{4\pi p^2\epsilon}\right)\eta_j dS. \tag{3.46}$$

The face terms of the boundary vanish as $\epsilon \longrightarrow 0$ in a similar manner to those in (3.43). The outer unit normal to $\{p < p_0/2\}$ is simply $\eta = 2p_0^{-1}(y_k - a_k)e_k$ where $1 \leqslant k \leqslant 2$. Putting this into equation (3.46) shows that the integral of the first term over the set $p < p_0/2$ is bounded by

$$\int_{p=p_0/2}\frac{|f.dS|}{2\pi p_0\epsilon} \leqslant \frac{\text{const.}}{p_0}\left(p_0^2\int_{C(b)}|f|^2 d\theta\right)^{1/2} = \text{const.}p_0^{-1}. \tag{3.47}$$

Again the second inequality is Hölders and the constants are dependent only on the L^2 norm of the harmonic spinor.

Step 6

When the metric is no longer a product we simply use normal coordinates at a. Let $\gamma : [0,1] \longrightarrow M$ be the geodesic from b to a. Fix an orthonormal basis of $T_\Sigma|_b$ where b is the point of Σ where $d(a,b) = p_0$. Define $e_1 \in T_M|_b$ by $e_1 = \dot\gamma(0)$. Extend this basis to an orthonormal basis of $T_M|_b$ by adding $\{e_1, e_2\}$. Parallel translate this basis to a along the geodesic γ. Finally, use the resulting basis of $T_M|_a$ to form a normal coordinate system $\{x_1, \ldots, x_n\}$ centred on b. Approximate a fundamental solution to $\not{D}$ by using the Euclidean fundamental solution. Since

$$|\nabla e_i| = O(\epsilon + p) \tag{3.48}$$

in a suitable neighbourhood of b, Step 2 now yields that

$$d\left(\alpha_0^{-1}\frac{(a_i-y_i)}{4\pi p^2}e_{ij}.\alpha_0 f.[e_j \llcorner \mu]\right) = \left(\nabla_j(\alpha_0^{-1}) + \text{const.}(\epsilon+p)\right)\frac{(y_i-a^i)}{4\pi p^2}e_{ij}.\alpha_0 f.\mu. \tag{3.49}$$

when $\alpha_0 f$ is a harmonic spinor and α_0 is the trivialising section of the spinor bundle associated with the non-extending spin structure. Noting that

$$|\nabla(\alpha_0^{-1})| = \text{const.}\left(\frac{1}{r}+\epsilon+p\right) \tag{3.50}$$

we see that the above proof carries through with only minor changes: proving the result in general. $\square$

Chapter 4

Simplest Cases

In the spirit of the main example we investigate here the simplest examples of non-extending spin structures: those on Riemann surfaces with points removed.

Firstly, we make the simple observation that the difference between the dimensions of the vector spaces of positive and negative L^2 harmonic spinors is always zero. This is true for any spin structure on a punctured Riemann surface.

The main point of this chapter is that this difference is not, in general, the index of any operator. Remark 4.1.2 explains this.

Elementary means are then used to apply the Riemann-Roch theorem to the problem.

The main result of this chapter is that this gives an index which is a special case of that in Chapter 6. From this we are able to calculate bounds for the dimension of the L^2 null space in general and to ascertain the exact dimension in specific cases.

The constructions in this chapter are independent of those in subsequent chapters. **Only Remark 4.3.4 is fundamental to all the work which follows.**

4.1 Riemann Surfaces

For ease of notation we denote a Riemann surface with m **distinct** points removed by $S_g - m$. For any spin structure on $S_g - m$ we will show in Lemma 4.2.4 that the vector space $\ker \not{D} \cap L^2$ is finite dimensional. With this assumed the most basic result of this chapter is the following lemma.

Lemma 4.1.1 Denote the L^2 kernel of the positive Dirac operator by $\ker \not{D}^+ \cap L^2$. Use similar notation for the L^2 kernel of $\not{D}^-$. For **any** spin structure on $S_g - m$

$$\dim \ker \not{D}^+ \cap L^2 - \dim \ker \not{D}^- \cap L^2 = 0 \tag{4.1}$$

Proof For the extending spin structures the Kähler structure on the Riemann surface and the Removable Singularities Theorem 3.1.1 show that equation (4.1) equals

$$h^0(S_g, \mathcal{O}(K^{1/2})) - h^1(S_g, \mathcal{O}(K^{1/2})). \tag{4.2}$$

Here, $\mathcal{O}(L)$ denotes the sheaf of holomorphic sections of L, and $h^i = \dim H^i$. This expression is always zero by Kodaira-Serre duality.

For the non-extending spin structure we still have the same identifications as in the Example 1.2. The spinor bundles are

$$\not{S}^+ = K^{1/2} \qquad \not{S}^- = K^{-1/2} \overset{\otimes g}{\cong} \overline{K} K^{1/2}, \tag{4.3}$$

and the chiral restrictions of the Dirac operator are

$$\begin{aligned} \not{D}^+ &= g^{-1} \circ \overline{\partial} : K^{1/2} \longrightarrow K^{-1/2} \\ \not{D}^- &= g^{-1/2}\, c\, \overline{\partial}\, g^{1/2}\, c : K^{-1/2} \longrightarrow \end{aligned} \tag{4.4}$$

The g here is the metric on $S_g - m$. As in the main example, if Ψ is a L^2 holomorphic section of $K^{1/2}$ then $g^{-1/2}\overline{\Psi}$ is a L^2 section of $K^{-1/2}$ satisfying $\not{D}^-(g^{-1/2}\overline{\Psi}) = 0$ under the identification of $\not{D}^-$ in equation (4.4). Again this is simply because the C^∞ isomorphism

$g^{1/2} \circ c : K^{1/2} \longrightarrow \overline{K} K^{1/2}$ preserves the L^2 condition since g is a smooth global section of $K_{S_g}\overline{K_{S_g}}$. Hence (4.1) is zero once more. $\square$

Remark 4.1.2 The main point here is that for the non-extending spin structure (4.1) is not in general the index of any elliptic operator.

For the extending spin structure we can apply Theorem 3.1.1 to show that (4.1) is just the index of the Dirac operator on S_g. For a non-extending spin structure we saw in Section 3.2 that the vector space $\ker \not{D} \cap L^2$ could not, in general, be included in any domain on which the Dirac operator was self adjoint. Thus for the non-extending spin structure equation (4.1) is not in general equal to

$$\dim \ker A - \dim \operatorname{coker} A \tag{4.5}$$

for any operator A. As a consequence of this we cannot quote any form of duality to show that (4.1) is zero for the non-extending spin structure: we must identify the spaces using the ambient metric.

4.2 Bounds on the Dimension of the L^2 Null Space

Remark 4.2.1 The punctured Riemann surface $S_g - m$ is easily seen to be homotopic to the wedge product (denoted by Λ) of $2g + m - 1$ copies of S^1 (see Figure 4.1).

Thus $S_g - m$ has 2^{2g+m-1} spin structures $2^{2g}(2^{m-1} - 1)$ of which are non-extending.

The number of points in the set $\{m\}$ around which the spinor connection has non-trivial holonomy must be even. We see this as follows.

Removing one point from S_g induces no non-extending spin structures whilst removing two gives 2^{2g} non-extending spin structures. Removing three points generates essentially no new spin structures since the three ways of choosing the two points accounts for all the

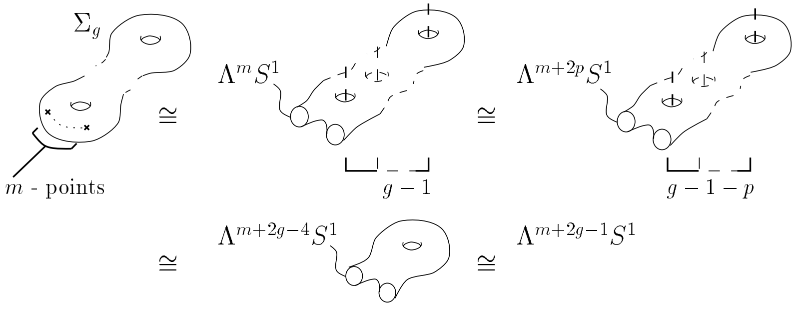


Figure 4.1: $S_g - m \sim \Lambda^{m+2g-1}S^1$

non-extending spin structures. In general the elementary identities

$$2^{2n} = \sum_{p=0}^{n} \binom{2n+1}{2p} \qquad \text{and} \qquad 2^{2n-1} = \sum_{p=0}^{n} \binom{2n}{2p} \tag{4.6}$$

show that removing $2n+1$ points from S_g gives non-trivial holonomy about at most 2n of them. Thus any spin structure on $S_g - (2n+1)$ can be considered as the restriction of a spin structure on $S_g - 2n$ and the Removable Singularities Theorem 3.1.1 shows that any L^2 harmonic spinor will extend smoothly over at least one of the points. Thus, in terms of the L^2 harmonic spinors, we get no new behaviour removing $2n+1$ points than we did removing $2n$ points.

We shall see later how our index theorem produces these results in a more general context. As a consequence of Remark 4.2.1 the hypothesis of the following Lemma loses no generality.

Lemma 4.2.2 There exist at least $2m$ linearly independent L^2 harmonic spinors on $S_g - 2m$ for any spin structure which has non-trivial holonomy around exactly $2m$ of its removed points.

Proof By Lemma 4.1.1 it is enough to consider solutions to the positive Dirac operator. We look for holomorphic sections of the spin line bundle $K^{1/2}$ which are of the form

$$f(z)\left(\frac{dz}{z}\right)^{1/2} \tag{4.7}$$

locally at the $2m$ distinct points $\{x_j\}$. These $2m$ points define a divisor D. Let s_0 be some section of $K^{1/2}$ with the behaviour above. If no such s_0 exists then the dimension of the null space is zero. If there is such a section denote the points where it vanishes by $p_1, \ldots, p_n$. These points may coincide and indeed might include some of the x_j.
Consider s_0^2. This is a meromorphic section of K with at most simple poles on D. Denote the line bundle of the divisor by D also and denote its tautological section by δ. The section δs_0^2 can then be interpreted as a holomorphic section of KD. Considering divisor classes

$$2\sum_i p_i = K + D \tag{4.8}$$

and so the line bundle L corresponding to the divisor $\sum p_i$ satisfies

$$L^2 = KD. \tag{4.9}$$

If s is any other L^2 holomorphic section of $K^{1/2}$ then ss_0 is also a meromorphic section of K with at most simple poles on D. Thus the positive L^2 null space can be embedded by the map $s \longrightarrow \delta s s_0$ as a subspace of $H^0(S_g, \mathcal{O}(KD))$. In particular it is seen to be finite dimensional. Now $\delta s s_0$ vanishes at $p_1, \ldots p_n$ and at further points $q_1, \ldots q_N$ which are the zeros of s. Also, $L \sim \sum p_i$ and $L^2 = KD$ so that $L \sim \sum q_i$ too. In other words **the zeros of all L^2 holomorphic sections of $K^{1/2}$ lie in a linear system equivalent to L.**
Conversely, if we denote by t a section of L with divisor $\sum p_i$ and let u by any other holomorphic section of L then us_0/t is an L^2 holomorphic section of $K^{1/2}$. It follows from this that our dimension, if non-zero, is equal to

$$\dim H^0(S_g, \mathcal{O}(L)). \tag{4.10}$$

The Riemann-Roch theorem says that

$$h^0(S_g, \mathcal{O}(L)) - h^1(S_g, \mathcal{O}(L)) = \deg\, L + (1-g) = m. \tag{4.11}$$

Consequently, if there is a positive L^2 solution, then

$$\dim\ker \not{D}^+ \cap L^2 = \dim H^0(S_g, \mathcal{O}(L)) \geqslant h^0(S_g, \mathcal{O}(L)) - h^1(S_g, \mathcal{O}(L)) = m. \tag{4.12}$$

In some cases, as we shall see below, the dimension of the kernel may vary, but this formula shows us that if there is at least one L^2 solution to the positive Dirac operator then there are at least m linearly independent ones. To complete the proof of the Lemma it remains only to show that there exists a solution when $m > 0$.

Recall that a non-extending spin structure can be regarded as an extending spin structure tensored with a flat line bundle of order two on $S_g - 2m$. Moreover, recall the assumption that the holonomy is -1 precisely around the $2m$ removed points. Choose a line bundle L on S_g such that $L^2 = KD$. There are 2^{2g} such choices exhausting all spin structures. Now

$$\deg\, L = \frac{1}{2}(\deg\, K + \deg\, D) = g - 1 + m \tag{4.13}$$

so, as in 4.12, for $m > 0$ the line bundle L has at least one holomorphic section s_L. Then $s_L/\sqrt{\delta}$ is the required section of $K^{1/2}$.

Having shown that equation (4.12) always holds, we obtain the lower bound on the dimension of the vector space of L^2 solutions on $S_g - 2m$ simply by recalling Lemma 4.1.1.

□

Remark 4.2.3 For the non-extending spin structure on $\mathbb{C}P^1$ with two distinct points removed we have already shown that this bound is sharp. Using the upper bound established below we will see that it is also sharp for any non-extending spin structure on any Riemann surface of genus 0 or 1 with any number of points removed.

Following Hitchin [Hit74] we may obtain an upper bound on the dimension of the L^2 null space.

Lemma 4.2.4 The dimension of the vector space $\ker \not{D}^+ \cap L^2$ on $S_g - m$ is at most $[g/2 + m]$, for $m \geqslant 1$, where square brackets denote the integer part. For the case $m = 0$ then the dimension is at most $[(g+1)/2]$.

Proof Let V be the vector space of all holomorphic sections of $K^{1/2}$ having at most singularities of the form $(dz/z)^{1/2}$ around m distinct points x_i. Multiplying any two sections together gives a map

$$\tilde{q} : V \times V \longrightarrow W \tag{4.14}$$

where W is the space of holomorphic sections having at most simple poles at the x_i. The map $\tilde{q}$ induces a differentiable map q on the product of the projective spaces

$$q : P(V) \times P(V) \longrightarrow P(W). \tag{4.15}$$

Points of $P(V)$ are zeros of holomorphic sections (effective divisors) of V. Now $q^{-1}\{\text{pt.}\}$ is finite since there is only a finite number of ways of dividing up the finite number of zeros of a holomorphic section of W. Thus $2\dim P(V) \leqslant \dim P(W)$. Now $W = H^0(S_g, \mathcal{O}(KL_{x_1} \ldots L_{x_m}))$ where L_{x_i} is the line bundle defined by x. By Riemann-Roch W has dimension

$$\deg K + 1 - g + m = g - 1 + m \tag{4.16}$$

assuming that $m \geqslant 1$. So $\dim V \leqslant (g + 2m)/2 = (g + m)/2$. If $m = 0$ then W has dimension g and so $\dim V \leqslant (g+1)/2$ completing the proof.

□

From Lemma 4.2.2 and Lemma 4.2.4 we see that $N = \dim\ker \not{D}^+ \cap L^2 + \dim\ker \not{D}^- \cap L^2$ on $S_g - m$ is such that

$$2 \times \left[\frac{m}{2}\right] \leqslant N \leqslant 2 \times \left[\frac{g+m}{2}\right]. \tag{4.17}$$

4.2.1 Examples and Remarks

Example 4.2.5 For surfaces of genus zero or one, the bounds above give exactly the dimensions of the L^2 null space, associated to the non-extending spin structures on $S_g - m$. This is most easily tabulated in terms of the number of points around which the spinor connection has holonomy -1. Call this number n. We showed in Remark 4.2.1 that n must be even.

Note that in the two cases, $g = 0$ or $g = 1$, n is equal to the dimension of the null space N.

Except for $n = 0$ all the spin structures in Figure 4.2 are non-extending. Thus for $\mathbb{C}P^1 - m$ there are spin structures with $0, 2, 4, \ \ldots$ and $2 \times [m/2]$ L^2 harmonic spinors.

There can of course be more than one spin structure with the same n. The table in Figure 4.3 is a simple consequence of Remark 4.2.1 for arbitrary g.

Remark 4.2.6 Recall here that equation (4.12) in the proof of Lemma 4.2.2 showed that

$$\dim\ker \not{D}^+ \cap L^2 = h^0(S_g, \mathcal{O}(L)). \tag{4.18}$$

The degree of the line bundle $K - L$ is negative precisely when

$$\deg L > 2(g-1). \tag{4.19}$$

Since the degree of L is $g - 1 + m$ we see that $K - L$ has negative degree when $m > g - 1$. Since

$$0 = h^0(S_g, \mathcal{O}(K-L)) = h^1(S_g, \mathcal{O}(L)) \tag{4.20}$$

n	$\dim\ker \not{D}^+ \cap L^2$	$\dim\ker \not{D}^- \cap L^2$
0	0	0
2	1	1
4	2	2
$\vdots$	$\vdots$	$\vdots$
$2 \times \left[\frac{m}{2}\right]$	$\left[\frac{m}{2}\right]$	$\left[\frac{m}{2}\right]$

Figure 4.2: Dimensions of the spaces of L^2 harmonic spinors on $S_g - m$ for $g = 0$ or 1

n	Number of spin structures on $S_g - m$ with this n
0	2^{2g}
2	$2^{2g}\begin{pmatrix} m \\ 2 \end{pmatrix}$
$\vdots$	$\vdots$
2p	$2^{2g}\begin{pmatrix} m \\ 2p \end{pmatrix}$
$\vdots$	$\vdots$
$2 \times \left[\frac{m}{2}\right]$	2^{2g} (m even) $2^{2g}m$ (m odd)

Figure 4.3: Number of spin structures on $S_g - m$ with non-trivial holonomy about n points

in these cases, Riemann Roch gives the dimension of $\ker \not{D}^+ \cap L^2$ on $S_g - 2m$ as precisely m. Thus, when $m = 1$ and $g = 0$ or 1 we obtain again the exact dimensions of the spaces of L^2 harmonic spinors. This observation also allows us to obtain the precise dimensions for higher genera g when $m \geqslant g$.

Remark 4.2.7 In view of Remark 4.2.6 we may probe the genus two case further. When $g = 2$ the surface S_g is hyperelliptic so we may describe S_2 as a two sheeted branched cover $p : S_2 \longrightarrow \mathbb{C}P^1$. Let $m = 1$ so that we consider S_2 with two points removed. As in 4.2.2 these points define a divisor D and this in turn defines another line bundle L, of degree 2, such that

$$L^2 = KD. \tag{4.21}$$

In this equation K denotes the canonical bundle of S_2. Since L is of degree 2

$$h^0(S_2, \mathcal{O}(K - L)) = \begin{cases} 0 & \text{when } L \neq K \\ 1 & \text{when } L = K, \end{cases} \tag{4.22}$$

see [GH94, p.247]. Thus, by Riemann Roch

$$\dim \ker \not{D}^+ \cap L^2 = \begin{cases} 1 & \text{when } L \neq K \\ 2 & \text{when } L = K. \end{cases} \tag{4.23}$$

By equation (4.21) if $L = K$ then $K = D$. However, $p^*\mathcal{O}(1) = K$ where $\mathcal{O}(1)$ is the line bundle defined by a single point on $\mathbb{C}P^1$. From this it is clear that if $\{p_1, p_2\}$ denotes the two distinct points removed from S_2 then

$$\dim \ker \not{D}^+ \cap L^2 = h^0(S_g, \mathcal{O}(L)) = \begin{cases} 1 & \text{if } \{p_1, p_2\} = p^{-1}(x) \text{ for } x \in \mathbb{C}P^1 \\ 2 & \text{otherwise.} \end{cases} \tag{4.24}$$

Hence we see that unlike the genus zero and one cases, the dimension of $\ker \not{D}^+ \cap L^2$, and consequently $\ker \not{D} \cap L^2$, is dependent on which two points we remove.

Recall here that, by Remark 4.2.6, spin structures on S_2 with non trivial holonomy around three or more removed points will have $\dim \ker \not{D}^+ \cap L^2$ fixed.

Remark 4.2.8 We observe here the curious stepping down of behaviour compared to the compact Riemann surface case described in [Hit74, p.17-19]. Table 4.4 compares $\dim \ker \not{D}^+ \cap L^2$, qualitatively, for spin structures on S_g and $S_g - 2$ for genera from zero to three.

g	S_g	$S_g - 2$
0	Zero	Fixed number
1	Fixed number	Fixed number
2	Fixed number	Varies depending on points
3	Varies depending on complex structure	-

Figure 4.4: Dimension of $\ker \not{D} \cap L^2$

4.3 Two Dimensional Riemannian Manifolds

In this section we set up an index problem for non-extending spin structures on punctured surfaces $M^2 - m$.

The main result is that the natural index for these spaces is the Atiyah-Patodi-Singer Index Theorem [APS75], for the manifold with boundary $M^2 - B_i$, where the B_i are certain pairwise disjoint neighbourhoods of the removed points. This index is not an L^2 index. With this index the fact that we must remove an even number of points to get new behaviour is transparent.

The main point of this section is to exhibit this index as a toy model of the index theorem we shall prove in the next chapter. We appeal to the Atiyah-Patodi-Singer index theorem solely to keep the analysis in Chapter 5 self contained. Only the basic Remark 4.3.4 is key to the work which follows.

4.3.1 Atiyah-Patodi-Singer Boundary Conditions

We consider the manifold $M^2 - m$, with metric of form $dr^2 + r^2 d\theta^2$, near to all of the removed points p_i around which the spinor derivative has non trivial holonomy.

Without loss of generality we may assume that the metric is of this form for all $r \in [0,1]$ and that $\{r = 0\} = \{p_i\}$.

Identify $\not{S}^+|_{r=\text{const.}}$ with $\not{S}^+|_{r=1}$ for all $r \in (0,1]$. Identify also $\not{S}^+$ with $\not{S}^-$ via Clifford multiplication by the unit normal tangent vector $e_i = \partial_r$. Under these identifications the positive Dirac operator is just

$$\not{D}^+ = e_1\left[\partial_r + \frac{1}{2r} + \frac{i}{r}\partial_\theta\right] =: e_1\left(\partial_r + \frac{1}{2r} + \frac{A}{r}\right). \tag{4.25}$$

Trivialise the tangent bundle near p_i by the section $(\partial_r, r^{-1}\partial_\theta)$. Locally we may decompose a spinor near p_i as $f^+ = a_n(r)e^{in\theta}$. Doing this we see that f^+ locally represents a section of

the spinor bundle associated to, the non-extending spin structure if $n \in \mathbb{Z}$ or, the extending spin structure if $n \in \mathbb{Z} + 1/2$. The a_n are smooth functions of r on $B_i - p_i$, where the B_i are pairwise disjoint neighbourhoods of the p_i, on which the metric is of the desired form. If $\not{D}^+ f^+ = 0$ we see that

$$\partial_r a_n = (n - 1/2)\frac{a_n}{r} \qquad \text{for all } n. \tag{4.26}$$

Integrating (4.26) gives that

$$a(r) = r^{(n-1/2)}a_n(1). \tag{4.27}$$

Now consider the manifold with boundary $\overline{M} := M - \cup B_i$. The Atiyah-Patodi-Singer index theorem allows solutions of the Dirac operator on $\overline{M}$ whose projection onto the positive, $\{\lambda > 0\}$, eigenspaces of $\not{D}_{\partial\overline{M}}$ are zero. Denote this projection by $1 - P_-$. In this simple case $\not{D}_{\partial\overline{M}} = A$ and if $(1 - P_-)f = 0$ then

$$f^+ = \sum_{n \geqslant 0} a_n(r)e^{in\theta}. \tag{4.28}$$

Integrating over the neighbourhood B_i we have that

$$\int_{B_i} |f^+|^2 r dr d\theta = 2\pi \sum_n \int_{(0,1]} r|a_n(r)|^2 dr = 2\pi \sum_n |a_n(1)|^2 \int_{(0,1]} r^{2n} dr. \tag{4.29}$$

Hence, for either spin structure, the section represented by f is L^2 if and only if $n > -1/2$. For the non-extending spin structure, where $n \in \mathbb{Z}$, we see that the Atiyah-Patodi-Singer boundary condition coincides exactly with the L^2 condition on the positive spinor bundle. Explicitly

if $M - m$ has a non-extending spin structure then harmonic spinors on $\overline{M}$ in $\not{S}^+$ which satisfy the Atiyah-Patodi-Singer boundary condition, $(1 - P_-)f = 0$, extend to L^2 harmonic spinors on $M - m$. Conversely every positive L^2 harmonic spinor on $M - m$, associated to a non-extending spin structure, satisfies the Atiyah-Patodi-Singer condition, $(1 - P_-)f = 0$, when restricted to $\overline{M}$.

This statement remains true for the extending spin structure.

4.3.2 Adjoint Atiyah-Patodi-Singer Boundary Conditions

Consider now the adjoint Atiyah-Patodi-Singer boundary conditions on $\not{S}^-$. These say that any solution on $\overline{M}$ must project onto the positive part of the spectrum of A i.e. $P_- f = 0$. With respect to our identification $\not{S}^- = e_1 \not{S}^+$, the adjoint operator is just

$$-e_1 \not{D}^+ e_1 = e_1 \left(\partial_r + \frac{1}{2r} - \frac{i}{r}\partial_\theta \right) = e_1 \left(\partial_r + \frac{1}{2r} - \frac{A}{r} \right). \tag{4.30}$$

Let $f^- = b_n(r)e^{in\theta}$ represent a harmonic spinor in $\not{S}^-$. If f^- satisfies the adjoint Atiyah-Patodi-Singer boundary condition then $P_- f^- = 0$ and

$$f^- = \sum_{n<0} b_n(r)e^{in\theta}. \tag{4.31}$$

Then by equation (4.30)

$$\partial_n b_n = -(n + 1/2)b_n, \tag{4.32}$$

for all n, so that

$$b_n(r) = r^{-n+1/2} b_n(1). \tag{4.33}$$

In this case the Atiyah-Patodi-Singer boundary condition translates as $n < 0$, however, f^- is L^2 if and only if $n < 1/2$.

When f^- is associated to the extending spin structure $n \in \mathbb{Z} + 1/2$ and the Atiyah-Patodi-Singer and L^2 conditions coincide once more. When f^- is associated to the non-extending spin structure then $n \in \mathbb{Z}$ so negative harmonic spinors which lie in the positive eigenspaces of A always extend to give L^2 spinors. The converse is not true.

Every negative L^2 spinor, associated to a non-extending spin structure, lies either in the positive eigenspaces of A or in the null space of A.

Thus, for the non-extending spin structure, not every L^2 harmonic spinor in $\not{S}^-$, on $M - m$, need satisfy the adjoint Atiyah-Patodi-Singer boundary condition when restricted to the boundary of $\overline{M}$

4.3.3 Remarks

Remark 4.3.1 As a consequence of Sections 4.3.1 and 4.3.2, the Atiyah-Patodi-Singer index theorem on $\overline{M}$, gives a valid index for any spin structure on $M - m$. It says that

$$\mathrm{index} \not{D} = \frac{h}{2} \in \mathbb{Z} \tag{4.34}$$

where h is the dimension of the null space of A. The dimension of the null space h is just the number of points p_i around which $\nabla^{\not{S}}$ has non-trivial holonomy. Thus, this index agrees with the elementary Riemann-Roch calculation (4.11), that we used to get a lower bound on the dimension of the L^2 null space.

As the index is always an integer the Atiyah-Patodi-Singer index shows immediately that we must have an even number number of points around which the connection has non-trivial holonomy. Thus approaching the index from this point of view is more natural as we do not need to go through the argument in Remark 4.2.1.

We stress that **this index is not an L^2 index** if h is non zero. From Sections 4.3.1 and 4.3.2 it is,

$$\dim\ker \not{D}^+ \cap L^2 - \dim\ker \not{D}^- \cap L^2 \cap H^\perp, \tag{4.35}$$

where $H^\perp$ is the orthogonal complement of the null space of A. This is equal in turn to

$$\dim\ker \not{D}^+ \cap L^2 - \dim\ker \not{D}^- \cap L^2_1. \tag{4.36}$$

Remark 4.3.2 For the Riemann-Roch result (4.11) it is not immediate how to interpret the second space $H^1(S_g, \mathcal{O}(L)) = (H^0(S_g, \mathcal{O}(LD^*)))^*$ as spinors of opposite chirality. From the Atiyah-Patodi-Singer interpretation things are clearer.

Recall that the space $H^0(S_g, \mathcal{O}(L))$ was identified with the space of L^2 solutions of $\not{D}^+$ in Lemma 4.2.2. The elements of $H^0(S_g, \mathcal{O}(LD^*))$ are just holomorphic sections of L which vanish on D. Thus, the elements of $H^1(S_g, \mathcal{O}(L))$ are interpreted as L^2_1 spinors of the opposite chirality. Here the duality is the usual pairing. This is simply because the Atiyah-Patodi-Singer index on $\overline{M}$ gives the index of the Dirac operator $\not{D} : L^2_1 \longrightarrow L^2$. In fact the calculation above calculates the index of the adjoint of this operator: the Atiyah-Patodi-Singer boundary conditions have been switched from the usual

$$\not{D} : C^\infty(\not{S}^+, P_+) \longrightarrow C^\infty(\not{S}^-, 1 - P_+) \tag{4.37}$$

to

$$\not{D} : C^\infty(\not{S}^+, 1 - P_-) \longrightarrow C^\infty(\not{S}^-, P_-) \tag{4.38}$$

to emphasise the link with the Riemann-Roch result. The notation here is as in [APS75] with P_+ denoting projection onto the positive eigenspaces of A, $\{\lambda \geqslant 0\}$, and P_- projection onto the negative eigenspaces $\{\lambda \leqslant 0\}$.

Remark 4.3.3 Since the Atiyah-Patodi-Singer index is the index of a boundary value problem we see that h, and thus the index (4.36), is invariant under change of orientation. We may use this fact and a simple counting argument to infer that
the dimension of the quotient vector space $\ker \not{D} \cap L^2 / \ker \not{D} \cap L^2_1$ **is always even for any spin structure on a punctured surface.**
This fact is also easily deduced from the identification between the chiral spinor bundles, via the metric, in Lemma 4.1.1.

For what follows it serves us to forget about the Atiyah-Patodi-Singer boundary value problem on $\overline{M}$ and reinterpret this index as the index of an operator on $M - m$. We have the following well known fact (see [LM89, p.116]) that is fundamental for all which follows.

Remark 4.3.4 Denote the space of L^2 sections of $\not{S}$ by $L^2(\not{S})$. This is the completion of the space of compactly supported sections of $\not{S}$, $C^\infty_0(\not{S})$ with respect to the Hilbert space norm $\| \ \|$. We can consider $\not{D}$ as a symmetric operator on $C^\infty_0(\not{S})$ (see 3.2.1) and take its closure in $L^2(\not{S})$. The domain of this closure consists of all $\phi \in C^\infty_0(\not{S})$ for which there is a sequence $\{\phi_n\} \subset C^\infty_0(\not{S})$ and an element "$\not{D}\phi$" $= \psi \in L^2$ such that $\|\phi_n - \phi\| \longrightarrow 0$ and $\|\not{D}\phi_n - \psi\| \longrightarrow 0$ in L^2. Denote this closure of $\not{D}$ in $L^2(\not{S})$ by D_0 and the domain of this closure by $\mathrm{dom}(D_0)$.
We can also consider **another** extension of this operator: we will denote this operator simply by D. The domain of D, $\mathrm{dom}(D)$, consists of all $\phi \in L^2(\not{S})$ such that the distributional image $\not{D}\phi$ is also in L^2. Note that $\mathrm{dom}(D_0) \subset \mathrm{dom}(D)$ and that $\mathrm{dom}(D_0)$ is dense. Now D_0 is simply the adjoint of D and vice versa. Recall that $\psi \in L^2(\not{S})$ is in the domain of the adjoint D_0^* if $\phi \longrightarrow (D_0\phi, \psi)$ is continuous on $\mathrm{dom}(D_0)$. As $\mathrm{dom}(D_0)$ is dense there exists a unique element $D_0^*\psi \in L^2(\not{S})$ such that $(D_0\phi, \psi) = (\phi, D_0^*\psi)$ for all $\phi \in \mathrm{dom}(D_0)$. Then $D_0^*\psi$ is just the distributional image of ψ under D_0 so that $D_0^* = D$.
On a complete manifold the Dirac operator is essentially self adjoint so $\mathrm{dom}(D^*) = \mathrm{dom}(D_0) = \mathrm{dom}(D)$ (see [LM89, p.117]) . The manifold we consider, namely $M - m$, is not complete and therefore the index of the operator $D : \not{S}^+ \longrightarrow \not{S}^-$ on $M - m$ is just

$$\dim \ker D^+ - \dim \ker D_0^-. \tag{4.39}$$

The lack of completeness of $M - m$ means that D_0 no longer has the same domain as D. The index (4.39) on $M - m$ is clearly that in Remark 4.3.1. This will be shown rigourously in Remark 6.5.3.

Example 4.3.5 The fact that $\mathrm{dom}(D) \not\subset \mathrm{dom}(D_0)$ is clear for the non-extending spin structure from the main example. The spinor $z^{-1/2}(dz)^{1/2}$ and $\not{D}(z^{-1/2}(dz)^{1/2}) = 0$ are in $L^2(\not{S})$ so that $z^{-1/2}(dz)^{1/2} \in \mathrm{dom}(D)$. The spinor $z^{-1/2}(dz)^{1/2}$ is, however, not in $\mathrm{dom}(D_0)$ by the lack of self adjointness shown in (3.24).

Chapter 5

Trivial Normal Bundle

In this chapter we investigate the form of harmonic spinors on $M^{2n} - \Sigma^{2n-2}$. We concentrate on the case where Σ has trivial normal bundle and an amenable metric near Σ.
The main results of this Chapter are the explicit forms that a L^2 harmonic spinor on $M - \Sigma$ may take near Σ.
The main point of this Chapter is to explain the closure of the Dirac operator we will study and how this exploits the growth of the harmonic spinors near Σ. The closure of the Dirac operator recalls Remark 4.3.4.

5.1 Set Up

In this section we investigate the form of L^2 harmonic spinors on $M^{2n} - \Sigma^{2n-2}$ where Σ has trivial normal bundle and $n \geqslant 2$. **We assume throughout this chapter that Σ is spin and that the metric is locally of the form**

$$dr^2 + r^2 d\theta^2 + ds_\Sigma^2 \tag{5.1}$$

near to Σ, where ds_Σ^2 is the metric on Σ . We may always patch such a metric onto M if Σ has trivial normal bundle.

Remark 5.1.1 From now on whenever we say that $M - \Sigma$ has a non-extending spin structure we will always assume that **every** two disc, transverse to Σ, has the generator of Ω_1^{spin} induced upon its boundary by the spin structure on $M - \Sigma$. When Σ is connected this is clear and we may always reduce the number of disconnected components of Σ such that this is true.

Remark 5.1.2 In assuming that Σ is spin we are assuming that the spin structure on Σ is induced from the non-extending spin structure on $M - \Sigma$. This spin structure on Σ is as follows. Denote an ϵ tubular neighbourhood of Σ by N and denote the tangent bundle of the boundary of this neighbourhood by $T_{\partial N}$. Since Σ has trivial normal bundle we can lift any section s of the tangent bundle T_Σ to $T_{\partial N}$ by mapping it to $\{\epsilon^{-1}\partial_\theta, s\}$. This induces an inclusion of the corresponding principal frame bundles

$$i : P_{SO}(T_\Sigma) \longrightarrow P_{SO}(T_{\partial N}). \tag{5.2}$$

There is a similar inclusion of $P_{SO}(T_{\partial N})$ in $P_{SO}(T_N)$.

These inclusions define reductions of the principal spin bundle on $M - \Sigma$ which are spin structures on ∂N_ϵ and Σ. **We refer to these spin structures simply as the spin structures induced on Σ and ∂N by the non-extending spin structure on $M - \Sigma$.**

The induced representations of the structure groups also allow us to consider the bundle of spinors of Σ, denoted by S_Σ, as being canonically embedded in the spinor bundle of $M - \Sigma$. This embedding will be implicitly assumed in all that follows.

Let V be any real vector space with basis $\{e_k\}$ where $3 \leqslant k \leqslant 2n$. Denote a basis of $\mathbb{R}^2$ by $\{e_1, e_2\}$. There is the following well known isomorphism of complexified Clifford algebras

$$\mathbb{C}l(V \oplus \mathbb{R}^2) = \mathbb{C}l(\mathbb{R}^2) \otimes_{\mathbb{C}} \mathbb{C}l(V) \tag{5.3}$$

defined, see [Mor96, p.16], by

$$\begin{aligned} e_j &\longmapsto \quad e_j \otimes 1 \qquad 1 \leqslant j \leqslant 2 \\ e_k &\longmapsto \quad ie_1.e_2 \otimes e_k \quad 3 \leqslant k \leqslant 2n. \end{aligned} \tag{5.4}$$

We saw in Section 1.2.2 that $\mathbb{C}l_2 = \mathbb{C}[2]$ so $\mathbb{C}l(V \oplus \mathbb{R}^2)$ can be represented as a 2×2 matrix algebra over $\mathbb{C}l(V)$. Denote the complex volume element of the vector space V by $\omega_{\mathbb{C}}^V$. Using the isomorphism (5.3) we see that

$$\omega_{\mathbb{C}}^{V \oplus \mathbb{R}^2} = ie_{12}\omega_{\mathbb{C}}^V = \begin{pmatrix} -\omega_{\mathbb{C}}^V & 0 \\ 0 & \omega_{\mathbb{C}}^V \end{pmatrix}. \tag{5.5}$$

Write an ϵ tubular neighbourhood, N, of Σ as $U \times \Sigma$ where U denotes a disc. Denote the spinor bundles on N, U and Σ by S, S_U and S_Σ respectively. Locally, near Σ we have the following splittings of the chiral spinor bundles

$$\begin{aligned} S^+ &= (S_U^+ \otimes S_\Sigma^+) \oplus (S_U^- \otimes S_\Sigma^-) \\ S^- &= (S_U^- \otimes S_\Sigma^+) \oplus (S_U^+ \otimes S_\Sigma^-) \end{aligned} \tag{5.6}$$

where $S_U^\pm$ and $S_\Sigma^\pm$ denote the eigenspaces of $\omega_{\mathbb{C}}^U$ and $\omega_{\mathbb{C}}^\Sigma$ respectively. This follows simply from equation (5.3).

Let $e_1 = \partial_r$ and $e_2 = r^{-1}\partial_\theta$ be orthonormal tangent vectors in T_U. Let $\{e_k\}$, for $3 \leqslant k \leqslant 2n$, form a local orthonormal basis for T_Σ. Then $s = (e_1, \ldots, e_{2n})$ is a local trivialising section of $T_{U \times \Sigma}$ which lifts locally to a section $\widetilde{s}$ of the spinor bundle.

For either spin structure we will always represent spinors and the operators which act upon them, with respect to this section $\widetilde{s}$.

With respect to the splitting (5.6), the representation (1.13) of $\mathbb{C}l_2$, and the section $\widetilde{s}$, the Dirac operator is locally given by

$$\not{D} = D_U + D_\Sigma = \begin{pmatrix} -D_\Sigma & -\partial_r - \frac{1}{2r} - \frac{i}{r}\partial_\theta \\ \partial_r + \frac{1}{2r} - \frac{i}{r}\partial_\theta & D_\Sigma \end{pmatrix} \tag{5.7}$$

mapping

$$\not{S}^+ = \begin{pmatrix} S_U^- \otimes S_\Sigma^- \\ S_U^+ \otimes S_\Sigma^+ \end{pmatrix} \longrightarrow \begin{pmatrix} S_U^- \otimes S_\Sigma^+ \\ S_U^+ \otimes S_\Sigma^- \end{pmatrix} = \not{S}^-. \tag{5.8}$$

The spinor Laplacian, with respect to the section $\widetilde{s}$, is

$$\not{D}_U^2 + \not{D}_\Sigma^2 = D_\Sigma^2 - \partial_r^2 - \frac{1}{r^2}\partial_\theta^2 - \frac{e_{12}}{r^2}\partial_\theta + \frac{1}{4r^2} - \frac{1}{r}\partial_r \tag{5.9}$$

Remark 5.1.3 We could also use the lift of the local trivialisation $(\partial_x, \partial_y, e_3, \ldots, e_{2n})$ to represent spinors with respect to. The change of basis (3.12), with respect to $\widetilde{s}$, is then simply

$$\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}. \tag{5.10}$$

5.2 The Form of L^2 Spinors

For future reference we make the following definition.

Definition 5.2.1 The local S^1 action on a tubular neighbourhood N of Σ induces a vector field V on $N - \Sigma$. Denote the Lie derivative of this vector field by L_V.

We now separate variables to calculate the possible forms of L^2 harmonic spinors locally near Σ. Firstly decompose spinors on $U \times \Sigma$ into eigenspaces of D_Σ, the Dirac operator associated to Σ. Denote a unit eigenspinor of D_Σ by ϕ_λ so that $D_\Sigma\phi_\lambda = \lambda\phi_\lambda$.
We may also decompose each spinor into eigenspaces of L_V thus $\phi_\lambda = \sum_n \phi_\lambda e^{in\theta}$. Hence we may write any spinor locally near Σ as

$$\Psi = \sum_{n,\lambda} \widetilde{s} e^{in\theta} \begin{pmatrix} f_n(r)\phi_\lambda^- \\ g_n(r)\phi_\lambda^+ \end{pmatrix}. \tag{5.11}$$

The functions $f_n(r)e^{in\theta}$ and $g_n(r)e^{in\theta}$ are considered as sections of either S_U^+ or S_U^- depending on the chirality of Ψ. **Because we have used the local trivialising section $\widetilde{s}$ with which to represent our spinor, $n \in \mathbb{Z}$ for the non-extending spin structure and $n \in \mathbb{Z} + 1/2$ for the extending spin structure.**

5.2.1 Harmonic Spinors Orthogonal to the Null Space of D_Σ

Let Ψ in equation (5.11) represent a harmonic spinor, associated to either spin structure, on $M^{2n} - \Sigma^{2n-2}$. By (5.9) the components of the spinor Ψ must locally satisfy

$$\begin{pmatrix} f''_n + \frac{1}{r}f'_n - \frac{(n+1/2)^2}{r^2}f_n & 0 \\ 0 & g''_n + \frac{1}{r}g'_n - \frac{(n-1/2)^2}{r^2}g_n \end{pmatrix} = \lambda^2 \begin{pmatrix} f_n \\ g_n \end{pmatrix} \tag{5.12}$$

near Σ. Here λ and n are the eigenspaces of D_Σ and L_V that we assume Ψ to lie in.
For simplicity we assume the $\lambda \neq 0$ so that in general

$$\begin{aligned} f_n(r) &= a_n J_{\pm|n+1/2|}(i\lambda r) \\ g_n(r) &= b_n J_{\pm|n-1/2|}(i\lambda r). \end{aligned} \tag{5.13}$$

In this expression J_n denotes the n^{th} Bessel function and a_n, b_n are suitably regularised complex constants. Recall that the n^{th} Bessel function $J_n(z)$ grows as $O(z^n)$ as $|z| \longrightarrow 0$.
Using the form of the Dirac operator (5.7) and the identities,

$$\partial_r J_\nu(i\lambda r) = \frac{\nu}{r}J_\nu(i\lambda r) - i\lambda J_{\nu+1}(i\lambda r) \tag{5.14}$$

$$\partial_r J_\nu(i\lambda r) + \frac{\nu}{r}J_\nu(i\lambda r) = i\lambda J_{\nu-1}(i\lambda r), \tag{5.15}$$

it is straightforward to show the following.

Positive L^2 harmonic spinors must, locally near Σ, be linear combinations of the following forms.

$$\widetilde{s}e^{in\theta} \begin{pmatrix} iJ_{n+1/2}(i\lambda r)\,\phi_\lambda^- \\ J_{n-1/2}(i\lambda r)\,\phi_\lambda^+ \end{pmatrix} \in \begin{pmatrix} S_U^- \otimes S_\Sigma^- \\ S_U^+ \otimes S_\Sigma^+ \end{pmatrix} = \not{S}^+ \quad \text{for } n \geqslant 1/2 \tag{5.16}$$

and

$$\widetilde{s}e^{in\theta}\begin{pmatrix} -iJ_{-(n+1/2)}\left(i\lambda r\right)\phi_\lambda^- \\ J_{-(n-1/2)}\left(i\lambda r\right)\phi_\lambda^+ \end{pmatrix} \in \begin{pmatrix} S_U^-\otimes S_\Sigma^- \\ S_U^+\otimes S_\Sigma^+ \end{pmatrix} = \not{S}^+ \quad \text{for } n\leqslant -1/2. \tag{5.17}$$

For the non-extending spin structure there are two extra possible forms that a positive L^2 spinor could take. These are

$$\widetilde{s}\begin{pmatrix} iJ_{1/2}\left(i\lambda r\right)\phi_\lambda^- \\ J_{-1/2}\left(i\lambda r\right)\phi_\lambda^+ \end{pmatrix} \quad \text{or} \quad \widetilde{s}\begin{pmatrix} -iJ_{-1/2}\left(i\lambda r\right)\phi_\lambda^- \\ J_{1/2}\left(i\lambda r\right)\phi_\lambda^+. \end{pmatrix} \tag{5.18}$$

Note that both these extra forms lie in the $n=0$ eigenspace of L_V. For the extending spin structure the possibilities in equation (5.18) do not occur.

To obtain the form of a negative L^2 harmonic spinor we may just interchange the roles of, ϕ_λ^- and ϕ_λ^+, and, S_Σ^- and S_Σ^+ in equations (5.16), (5.17) and (5.18).

We conclude that for the non-extending spin structure we may have L^2 harmonic spinors which grow like $O(r^{-1/2})$ as $r\longrightarrow 0$. For the extending spin structure we see that all L^2 spinors must be $O(r^{1/2})$.

This last observation is consistent with the Removable Singularities Theorem 3.1.1. For an extending spin structure the L^2 spinors extend smoothly across Σ.

5.2.2 Harmonic Spinors on the Null Space of D_Σ

If Ψ represents a harmonic spinor which lies in the null space of D_Σ then equation (5.12) holds with $\lambda=0$. From this it is elementary that such L^2 solutions of the Dirac operator must be of the form

$$\widetilde{s}e^{in\theta}\begin{pmatrix} a_n r^{|n+1/2|}\phi_0^- \\ 0 \end{pmatrix} \quad \text{if } n\leqslant -1/2, \text{ or} \quad \widetilde{s}e^{in\theta}\begin{pmatrix} 0 \\ b_n r^{|n-1/2|}\phi_0^+ \end{pmatrix} \quad \text{if } n\geqslant 1/2. \tag{5.19}$$

This is true for either spin structure. Recall that for the non-extending spin structure $n\in\mathbb{Z}$ and for the extending spin structure $n\in\mathbb{Z}+1/2$. Once again we have split the harmonic spinors into the eigenspaces induced by the S^1 action and a and b are complex numbers.

For the non-extending spin structure there is an extra possible form a L^2 solution can take near Σ . This is

$$\widetilde{s}\begin{pmatrix} a_0 r^{-1/2}\phi_0^- \\ b_0 r^{-1/2}\phi_0^+ \end{pmatrix} \tag{5.20}$$

and it lies in the zero eigenspace of L_V.

The possible forms, (5.18) and (5.20), that a L^2 harmonic spinor associated to a non-extending spin structure may take near Σ prevent the index we calculate in Theorem 6.4.1 from being an L^2 index. We explain this fact in Section 6.5.7.

5.3 Closure of the Operator

We have exhibited the explicit form of L^2 spinors associated to a non-extending spin structure on $M-\Sigma$ with product metric. In this section we construct a self adjoint operator whose index we will calculate in the next Chapter. We stress that this operator is not the same as $\not{D}: L_1^2 \longrightarrow L^2$ and thus gives a different index from all previous work. This is explained in depth in Remark 6.5.4. In particular it gives a computable index in terms of characteristic classes of M and Σ. More fundamentally, the index does not reflect the quaternionic structure of the Clifford algebra in $8k+4$ dimensions. This fact is explained fully in Section 6.6.

Consider a positive L^2 harmonic spinor, associated to a non-extending spin structure, which does not lie in the null space of D_Σ. This spinor must be of the form (5.16), (5.17) or (5.18). Its growth as $r\longrightarrow 0$ **can therefore be unbounded in only ever one component of the direct sum decomposition of $\not{S}^+$** (equation (5.6)). We set up an index to exploit this observation.

Recall the local form of the Dirac operator (5.7) and the two ways in which the Dirac operator on the punctured disc could be closed (Remark 4.3.4). Recall that one of these operators, D_0, was the closure of the Dirac operator on the space of compactly supported spinors with respect to the L^2 norm. The other operator D had as its domain the space of L^2 spinors whose distributional image was also in L^2. Recall how these two domains **were not equal** and how the operators were the adjoint of one another $D_0^* = D$.

Consider the closure of the Dirac operator on $M - \Sigma$ whose local form near Σ, with respect to the decomposition (5.6), is

$$\left(D_0^- \otimes 1\right) \oplus \left(D^+ \otimes 1\right) + \left(1 \otimes -D_\Sigma^-\right) \oplus \left(1 \otimes D_\Sigma^+\right) = \begin{pmatrix} -D_\Sigma^- & D^+ \\ D_0^- & D_\Sigma^+ \end{pmatrix} \tag{5.21}$$

when restricted to the positive spinor bundle, and

$$\left(D_0^- \otimes 1\right) \oplus \left(D^+ \otimes 1\right) + \left(1 \otimes -D_\Sigma^+\right) \oplus \left(1 \otimes D_\Sigma^-\right) = \begin{pmatrix} -D_\Sigma^+ & D^+ \\ D_0^- & D_\Sigma^- \end{pmatrix} \tag{5.22}$$

when restricted to the negative spinor bundle.

Since the Dirac operator on $M - \Sigma$ with this domain is the only operator which we now consider we will denote it simply by $\not{D}$. This operator is self-adjoint $\not{D} : \not{S}^+ \oplus \not{S}^- \longrightarrow \not{S}^- \oplus \not{S}^+$.

The index we calculate in Section 6 is the index of $\not{D}$. We do this using a local heat kernel argument.

For clarity, we may define the domain of $\not{D}$ in terms of growth conditions near Σ. Recall from Remark 4.3.4 how we could define the domains of D_0 and D in this way. The domain of D_0 contains all spinors which grow like $O(r^{1/2})$ near Σ and the domain of D contains all spinors which grow like $O(r^{-1/2})$. Consider the vector space of solutions which lie in the domain of the operator $\not{D}$. All positive spinors in this vector space must grow as

$$\begin{pmatrix} O(r^{1/2}) \\ O(r^{-1/2}) \end{pmatrix} \in \begin{pmatrix} S_U^- \otimes S_\Sigma^- \\ S_U^+ \otimes S_\Sigma^+ \end{pmatrix} = \not{S}^+, \tag{5.23}$$

near Σ, and all negative spinors must grow as

$$\begin{pmatrix} O(r^{1/2}) \\ O(r^{-1/2}) \end{pmatrix} \in \begin{pmatrix} S_U^- \otimes S_\Sigma^+ \\ S_U^+ \otimes S_\Sigma^- \end{pmatrix} = \not{S}^-. \tag{5.24}$$

For future reference we define two more self adjoint operators thus

$$\Delta_N := D_0 D \tag{5.25}$$

$$\Delta_D := D D_0. \tag{5.26}$$

With this notation the spinor Laplacian of the Dirac operator, see (5.9), takes the form

$$(1 \otimes \Delta_D^- + D_\Sigma^{2,-} \otimes 1) \oplus (1 \otimes \Delta_N^+ + D_\Sigma^{2,+} \otimes 1) = \begin{pmatrix} D_\Sigma^{2,-} + \Delta_D^- & 0 \\ 0 & D_\Sigma^{2,+} + \Delta_N^+ \end{pmatrix} \tag{5.27}$$

when restricted to the positive spinor bundle near the submanifold and

$$(1 \otimes \Delta_D^+ + D_\Sigma^{2,+} \otimes 1) \oplus (1 \otimes \Delta_N^- + D_\Sigma^{2,-} \otimes 1) = \begin{pmatrix} D_\Sigma^{2,+} + \Delta_D^+ & 0 \\ 0 & D_\Sigma^{2,-} + \Delta_N^- \end{pmatrix} \tag{5.28}$$

when restricted to the negative spinor bundle.

5.4 Atiyah-Patodi-Singer Solutions and L^2 Solutions

We noted in Section 4.3 that an index for non-extending spin structures on surfaces could be interpreted as the Atiyah-Patodi-Singer index on a manifold with boundary. We show here that this is not true, in general, for higher dimensions.

Firstly we consider an ϵ neighbourhood of Σ, namely N_ϵ, and calculate the eigenvalues of the Dirac operator associated to the boundary $\partial N_\epsilon = S_\epsilon^1 \times \Sigma$. Then we show that, for dimensions higher than two, harmonic spinors on the manifold with boundary $M - N_\epsilon$, which satisfy the Atiyah-Patodi-Singer boundary conditions, may not extend to L^2 solutions on $M - \Sigma$.

5.4.1 Eigenvalues of $D_{\partial N_\epsilon}$

The boundary Dirac operator on $M - N_\epsilon$ which the Atiyah-Patodi-Singer boundary conditions apply to is

$$A_\epsilon = \begin{pmatrix} -\frac{i}{\epsilon}\partial_\theta & D_\Sigma \\ D_\Sigma & \frac{i}{\epsilon}\partial_\theta \end{pmatrix}. \tag{5.29}$$

Off of the null space of D_Σ it is straightforward to show that, when $m \neq 0$, A_ϵ has unit eigenspinors of the form

$$\frac{e^{im\theta}}{\sqrt{2\beta_0(\beta_0 \pm m)}} \begin{pmatrix} (m \pm \beta_0)\phi_\lambda^- \\ \epsilon\lambda\phi_\lambda^+ \end{pmatrix} \tag{5.30}$$

with eigenvalues

$$\pm\frac{\beta_0}{\epsilon}. \tag{5.31}$$

Here $\beta_0 = \beta_0(\lambda, m, \epsilon) = \sqrt{m^2 + \lambda^2\epsilon^2}$ for all $m \neq 0$. Recall that m denotes the eigenspace of L_V which the spinor lies in and ϕ_λ represents a chiral unit eigenspinor of D_Σ with eigenvalue λ.

When $\lambda = 0$ and $m \neq 0$ there are unit eigenspinors of A_ϵ of form

$$e^{im\theta}\begin{pmatrix} 0 \\ \phi_0^+ \end{pmatrix} \qquad \text{and} \qquad e^{im\theta}\begin{pmatrix} \phi_0^- \\ 0 \end{pmatrix} \tag{5.32}$$

with eigenvalues $-m/\epsilon$ and m/ϵ respectively. Finally, when $m = 0$ there are unit eigenspinors

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \phi_\lambda^- \\ \phi_\lambda^+ \end{pmatrix} \tag{5.33}$$

with eigenvalue λ, for all λ in the spectrum of D_Σ.

5.4.2 Spectral Boundary Conditions

In two dimensions we saw, see page 72, that every Atiyah-Patodi-Singer solution, on the manifold with boundary $\overline{M}$, could be extended to an L^2 harmonic spinor on $M - m$. Here we show that propagating a harmonic spinor satisfying the Atiyah-Patodi-Singer boundary conditions on $M^{2n} - N_\epsilon$, where $n \geqslant 2$, to be solutions on $M^{2n} - \Sigma^{2n-2}$ need not give L^2 solutions for any $\epsilon > 0$.

Consider the positive spinor

$$\Phi = \frac{(1-\beta_0)e^{i\theta}}{\sqrt{2\beta_0(\beta_0 - 1)}}\begin{pmatrix} -iJ_{-3/2}(i\lambda r)\phi_\lambda^- \\ J_{-1/2}\phi_\lambda^+ \end{pmatrix} + \frac{\epsilon\lambda e^{-i\theta}}{\sqrt{2\beta_0(\beta_0+1)}}\begin{pmatrix} J_{-1/2}(i\lambda r)\phi_\lambda^- \\ -iJ_{-3/2}\phi_\lambda^+ \end{pmatrix} \tag{5.34}$$

where $\lambda \neq 0$. This spinor is not L^2 because of the $J_{-3/2}$ terms. The Atiyah-Patodi-Singer projection $(1 - P_-)$ onto the spectrum of the boundary operator corresponds to taking $(\beta_0 + m)$ in equation (5.30) and projecting onto these eigenspinors. An elementary calculation then shows that

$$(1 - P_-)\Phi = 0. \tag{5.35}$$

Using the form of the Dirac operator (5.7) and the identities (5.14) and (5.15) one can show that Ψ is also a solution of the Dirac operator. Thus Ψ satisfies the Atiyah-Patodi-Singer boundary condition on the positive spinor bundle, for all ϵ, but does not propagate to a L^2 spinor on $M^{2n} - \Sigma^{2n-2}$.

Chapter 6

An Index Theorem

In this Chapter we investigate the index of the Dirac operator on $M^{2n} - \Sigma^{2n-2}$ whose closure was described in Section 5.3.

The main result of this chapter is an index theorem for non-extending spin structures on $M^{2n} - \Sigma^{2n-2}$ for $n \geqslant 2$. This is Theorem 6.4.1 which expresses the index in terms of topological data on $M^{2n} - \Sigma^{2n-2}$. When Σ is spin we show that this index is equal to a $Spin^{\mathbb{C}}$ index in M.

The main point of this chapter is that this index is fundamentally different from all previous indices calculated for non-compact manifolds. The clearest manifestation of this is the breaking of the quaternionic symmetry of the Clifford algebra in $8k + 4$ dimensions. The key reason for this is the particular closure of the Dirac operator we took in Section 5.3.

Numerous corollaries are proved from Theorem 6.4.1. Chief among these is a partial extension of the results, on characteristic submanifolds, of Esnault, Seade and Viehweg [ESV92] from the holomorphic category to smooth manifolds. We relate the breaking of the quaternionic symmetry in $8k+4$ dimensions to the parity of the spin structure induced on Σ.

Throughout this chapter we assume as in Chapter 5 that Σ^{2n-2} has trivial normal bundle in M. We will also assume for simplicity that Σ^{2n-2} is spin and that the metric on $M - \Sigma$ is of the form (5.1) near Σ.

We will argue in Remark 6.5.1 that the results obtained in this chapter are independent of the metric on M. Furthermore in Remark 6.5.1 we show that we need not assume that Σ is either spin or orientable for the index theorem to hold.

6.1 Outline

The index of the Dirac operator is calculated using the heat equation method. This says, see for example [BGV92], that

$$\mathrm{Index}(\not{D}) = \mathrm{Tr}\ e^{-t\not{\partial}\not{\partial}^+} - \mathrm{Tr}\ e^{-t\not{\partial}\not{\partial}^-}. \tag{6.1}$$

where Tr denotes the global trace. To apply this method of computing the index we must show that the heat kernels of our operators are trace class and that the results of Fredholm theory can be applied.

Showing that the index problem is Fredholm is an application of the work of Bruning and Seeley [BS88]. The general framework of Bruning and Seeley also allows us to argue that our index is deformation invariant. These standard arguments are outlined in Section 6.2.

To actually compute the index we first show that we can approximate the trace (6.1), to an arbitrary order of t, as the trace of the heat kernel over the compact set $M - N_\epsilon$, plus the trace of *another* heat kernel over $N_\epsilon - \Sigma$. This is a standard argument from the work of [Che79, Cho85, BS88] and is outlined in Section 6.2.2.

The relevant heat kernels on $M - N_\epsilon$ and $N_\epsilon - \Sigma$ are calculated in Section 6.3. The index theorem is proved in Section 6.4.

6.2 Fredholm Framework

We show here that the operator $\not{D}$, closed as in Section 5.3, is Fredholm. From standard Fredholm theory we conclude from this that the eigenvalues of $\not{D}^2$ on $M - \Sigma$ are discrete and L^2 completeness of the eigenspinors of $\not{D}^2$ holds (see also [Cho85]). We also show that the heat kernel is trace class and that we may approximate the trace (6.1) by, the trace of the usual kernel on the compact set $M - N_\epsilon$, plus the trace of *another* heat kernel on N_ϵ.

All of these results are trivial modifications of the analysis of [Che79], [Cho85] and [BS88], which are the main references for this section. As such these results are completely standard and we only sketch the proofs by quoting these papers.

Remark 6.2.1 To obtain these results for the simple form of the metric (5.1) on M we need not quote [BS88] but could just rely on the analysis of Cheeger [Che79] and Chou [Cho85]. Ultimately however, we wish to show that the index theorem we prove is independent of the smooth metric on M. To do this requires the more general framework of 'regular singular operators' in [BS88] which generalises the original analysis of [Che79] and [Cho85] on conical singularities and allows us to consider perturbations of the metric. We do this in Section 6.5.1.

6.2.1 Parametrices

To show that $\not{D}$ is Fredholm we simply construct left and right parametrices for $\not{D}$ as in [BS88]. Recall the local form, near Σ, of the self adjoint extension we have taken of $\not{D}$

$$\not{D}^+ \oplus \not{D}^- = \begin{pmatrix} -D_\Sigma^- & D^+ \\ D_0^- & D_\Sigma^+ \end{pmatrix} \oplus \begin{pmatrix} -D_\Sigma^+ & D^+ \\ D_0^- & D_\Sigma^- \end{pmatrix}. \tag{6.2}$$

The key point is that both D_0 and D are simple examples of 'regular singular operators' in the framework of [BS88] i.e. they are both of the form

$$\partial_r + r^{-1} S_0(r) \tag{6.3}$$

where S_0 is a self adjoint first order elliptic differential operator (see [BS88, p.660] and compare equation (5.7)).

In fact in the notation of [BS88], $D_0 = D_{\min}$ and $D = D_{\max}$, the minimal and maximal extensions of the Dirac operator on the punctured disc. For consistency we will stick with our original notation for the operators.

By [BS88, p.666] there are right parametrices, *near* 0, for D_0 and D. By this we mean that there is an operator P_ϵ such that if ϕ is a spinor, supported in some ϵ neighbourhood of the punctured disc, then

$$D_0 P_\epsilon \phi = \phi \tag{6.4}$$

on the support of ϕ. Similarly there are left parametrices near zero for D_0 and D (see Lemmas 2.4 and 2.5 [BS88]). Since D_Σ is defined on a compact manifold we also have full left and right parametrices for this operator.

The decomposition of the operator (6.2) allows us to piece together these individual parametrices and construct parametrices, *near* Σ, for $\not{D}^+$ and $\not{D}^-$. For instance let ϕ^+ be a positive chiral spinor and let φ be a smooth bump function supported in N_ϵ and unity on $N_{\epsilon/2}$. Let $\widetilde{\varphi}$ be another bump function such that $\widetilde{\varphi}\varphi = \varphi$ and $\widetilde{\varphi} = 1$ in a neighbourhood of supp φ. Then there is a right parametrix P_ϵ^+ for $\not{D}^+$ near Σ such that

$$\not{D}^+ \widetilde{\varphi} P_\epsilon^+ \left(\varphi\phi^+\right) = (1 + R_\epsilon)\left(\varphi\phi^+\right). \tag{6.5}$$

By choosing ϵ suitably small it can be assumed (see [BS88, p.667]) that $\|R_\epsilon\| < 1/2$. We may construct in a similar manner a left parametrix for $\not{D}^+$ and also left and right parametrices for $\not{D}^-$ **near** Σ.

Away from Σ, on the compact subset $M - N_\epsilon$, we may pick any interior parametrix P_i. Here for definiteness P_i can be taken as the Greens operator for $\not{D}$ on the double of $M - N_\epsilon$ [APS75, p.55]. As is standard, we patch the two parametrices, P_ϵ^+ near Σ and P_i^+ on $M - N_\epsilon$, to begin constructing a parametrix to $\not{D}^+$ on $M^{2n} - \Sigma^{2n-2}$.

Take a bump function φ as above and set $\psi = 1 - \varphi$. Choose (as in [APS75, p.54]) another bump function $\widetilde{\psi}$ such that $\widetilde{\psi}\psi = \psi$ and so that the support of $\widetilde{\psi}$ is near the support of ψ. Define as in [BS88, p.667]

$$Q^+ = \widetilde{\varphi} P_\epsilon^+ \varphi + \widetilde{\psi} P_i^+ \psi. \tag{6.6}$$

Then

$$\not{D}^+ Q^+ = I + R_\epsilon + R_i, \tag{6.7}$$

where R_i is a compact operator that comes from the P_i^+ term of Q^+ and R_ϵ arises from the P_ϵ^+ term as in (6.5). We may choose ϵ small so that $\|R_\epsilon\| < 1/2$ and consequently assume that $I + R_\epsilon$ is invertible. Equation (6.7) then implies that

$$\not{D}^+ Q^+ (1 + R_\epsilon)^{-1} = I + R_i (I + R_\epsilon)^{-1} \tag{6.8}$$

so we have constructed a right parametrix for $\not{D}^+$.

In a totally analogous manner we can construct left and right parametrices for both $\not{D}^+$ and $\not{D}^-$. Thus the index for the operator $\not{D}^+ : \not{S}^+ \longrightarrow \not{S}^-$, whose closure was described in Section 5.3, is seen to be Fredholm **for either spin structure**.

6.2.2 Approximating the Trace

Having appealed to [BS88] to construct parametrices for $\not{D}$ we now appeal to the work of [Che79] and [Cho85] to approximate the trace of the heat kernels. Denote the, pointwise trace of the, heat kernel of $\not{D}$ on $M - \Sigma$ by tr $E_{M-\Sigma}(m, m, t)$. Denote the pointwise trace of the heat kernel on the space $\mathbb{R}^2 - \{0\} \times \Sigma$ simply by tr $E(m, m, t)$.

The work of Cheeger [Che79, p.578-581] and also Chou [Cho85, p.31] then gives the well known result that for all N there exists a constant K_N such that

$$\left| \int_{N_\epsilon} \text{tr } E_{M-\Sigma} - \int_{N_\epsilon} \text{tr } E \right| < K_N t^N. \tag{6.9}$$

We will show in equation (6.50) that for $t > 0$

$$\left| \int_{N_\epsilon} \text{tr } E \right| < \infty. \tag{6.10}$$

Hence

$$\int_{M-\Sigma} \text{tr } E_{M-\Sigma} = \int_{M-N_\epsilon} \text{tr } E_{M-\Sigma} + \int_{N_\epsilon} \text{tr } E + O(t^N). \tag{6.11}$$

This is the key fact used to calculate the indices in Cheeger [Che79] and Chou [Cho85]. Thus we may approximate the trace (6.1), to an arbitrary order of t, by evaluating the contribution from a compact piece, $M - N_\epsilon$, and from a piece 'near' to Σ.

The contribution of the compact piece $M - N_\epsilon$ is well understood so the study of the trace of the heat kernel $E_{M-\Sigma}$ is reduced to the study of the trace of the heat kernel on $\mathbb{R}^2 - \{0\} \times \Sigma$. We exhibit an explicit pointwise expression for this kernel in Section 6.3. This approach also shows that

$$\int_{M-\Sigma} \text{tr } E_{M-\Sigma} < \infty. \tag{6.12}$$

and hence that the heat kernel of $\not{D}$ is trace class. In turn this implies that our index problem is Fredholm, but only for the product metric (5.1) on $M - \Sigma$ (see Remark 6.2.1).

6.3 Heat Kernels

In Section 6.2.2 we saw that we could reduce the study of the index (6.1) to the study of the heat kernel on $\mathbb{R}^2 - \{0\} \times \Sigma$.

In this section we calculate explicit pointwise expressions for the heat kernels of the Dirac operator on $\mathbb{R}^2 - \{0\} \times \Sigma$**.** This is done for both the extending and non-extending spin structures.

Recall the form (5.27) of the spinor Laplacian near Σ. There is a well known expression for the pointwise asymptotic expansion of the heat kernel $e^{-t\not{\partial}^2_\Sigma}$. By [BGV92, p.144] this expansion, on the diagonal of $\Sigma \times \Sigma$, varies as $t \longrightarrow 0$ like

$$\frac{1}{(4\pi t)^{n-1}} \sum_{k \geqslant 0} \alpha_k(x) t^k. \tag{6.13}$$

The coefficients $\alpha_k(x)$ have explicit expressions as local sections of the Clifford bundle of Σ. In particular, the pointwise supertrace

$$\text{tr }_\Sigma \left(e^{-t\not{\partial}^{2,+}_\Sigma} - e^{-t\not{\partial}^{2,-}_\Sigma} \right) \tag{6.14}$$

over Σ is well known to be

$$\frac{1}{(2\pi i)^{n-1}} \widehat{A}(\Sigma)_{[2n-2]}, \tag{6.15}$$

where $\widehat{A}(\Sigma)_{[2n-2]}$ denotes the coefficient of the top degree form in the Hirzebruch $\widehat{A}$ polynomial applied to the Pontrjagin forms of the Riemannian metric on Σ^{2n-2}.

Because of this expansion for $e^{-t\not{\partial}^2_\Sigma}$ it is enough for us to calculate $e^{-t\Delta_N}$ and $e^{-t\Delta_D}$ to obtain a pointwise expression for $e^{-t\not{\partial}^2}$ on $\mathbb{R}^2 - \{0\} \times \Sigma$.

We surmise the results of this section as four lemmas.

Lemma 6.3.1 (Non-extending Spin Structure) The heat kernel for $e^{-t\Delta_D}$ on the punctured plane for the non-extending spin structure is

$$K_{\Delta_D}(r_1, \theta_1, r_2, \theta_2) = \sum_{n \in \mathbb{Z}} \frac{1}{4\pi t} e^{-(r_1^2 + r_2^2)/4t} \left(I_{|n-1/2|}\left(\frac{r_1 r_2}{2t}\right) \circ P_+ + I_{|n+1/2|}\left(\frac{r_1 r_2}{2t}\right) \circ P_- \right) e^{in(\theta_2 - \theta_1)} \tag{6.16}$$

Here I_n refers to the n^{th} modified Bessel function and $P_\pm$ is the projection onto the plus or minus spinor bundles of the punctured plane. Note that in expressing this heat kernel we are using the same trivialising section of the spinor bundle as for (5.9) and (5.7). Recall the notation L_V for the Lie derivative of the vector field induced from the local circle action about Σ. Denote projection onto the n eigenspace of L_V by P_n.

On the diagonal ($r_1 = r_2, \theta_1 = \theta_2$) expression (6.16) reduces to

$$K_{\Delta_D} \circ P_n = \frac{1}{4\pi t} e^{-r^2/2t} \left(I_{|n-1/2|}\left(\frac{r^2}{2t}\right) \circ P_+ + I_{|n+1/2|}\left(\frac{r^2}{2t}\right) \circ P_- \right). \tag{6.17}$$

Lemma 6.3.2 (Non-extending Spin Structure) In the notation of Lemma 6.3.1 the heat kernel for $e^{-t\Delta_N}$ on the punctured plane for the non-extending spin structure is

$$\begin{aligned} K_{\Delta_N}(r_1,\theta_1,r_2,\theta_2) = \\ \sum_{n\in\mathbb{Z}} \frac{1}{4\pi t} e^{-(r_1^2+r_2^2)/4t} \left(I_{|n-1/2|}\left(\frac{r_1 r_2}{2t}\right) \circ P_+ + I_{|n+1/2|}\left(\frac{r_1 r_2}{2t}\right) \circ P_- \right) e^{in(\theta_2-\theta_1)} \\ + \frac{1}{4\pi t} e^{-(r_1^2+r_2^2)/4t} I_{-1/2}\left(\frac{r_1 r_2}{2t}\right) \left[1 + e^{i(\theta_2-\theta_1)} \circ P_+ + e^{i(\theta_1-\theta_2)} \circ P_-\right]. \end{aligned} \tag{6.18}$$

On the diagonal this expression reduces to

$$K_{\Delta_N} \circ P_n = \frac{1}{4\pi t} e^{-r^2/2t} \left(I_{|n-1/2|}\left(\frac{r^2}{2t}\right) \circ P_+ + I_{|n+1/2|}\left(\frac{r^2}{2t}\right) \circ P_- \right) + \delta_n \tag{6.19}$$

where

$$\delta_n = \begin{cases} \frac{1}{4\pi t} e^{-r^2/2t} I_{-1/2}\left(\frac{r^2}{2t}\right) \circ P_+ & \text{if } n = 1 \\ \frac{1}{4\pi t} e^{-r^2/2t} I_{-1/2}\left(\frac{r^2}{2t}\right) & \text{if } n = 0 \\ \frac{1}{4\pi t} e^{-r^2/2t} I_{-1/2}\left(\frac{r^2}{2t}\right) \circ P_- & \text{if } n = -1. \end{cases} \tag{6.20}$$

Lemma 6.3.3 (Extending Spin Structure) In the notation of Lemma 6.3.1 the heat kernel for $e^{-t\Delta_D}$ on the punctured plane for the extending spin structure is

$$\begin{aligned} K_{\Delta_D}(r_1,\theta_1,r_2,\theta_2) = \\ \sum_{n\in\mathbb{Z}+1/2} \frac{1}{4\pi t} e^{-(r_1^2+r_2^2)/4t} \left(I_{|n-1/2|}\left(\frac{r_1 r_2}{2t}\right) \circ P_+ + I_{|n+1/2|}\left(\frac{r_1 r_2}{2t}\right) \circ P_- \right) e^{in(\theta_2-\theta_1)} \end{aligned} \tag{6.21}$$

On the diagonal this expression reduces to

$$K_{\Delta_D} \circ P_n = \frac{1}{4\pi t} e^{-r^2/2t} \left(I_{|n-1/2|}\left(\frac{r^2}{2t}\right) \circ P_+ + I_{|n+1/2|}\left(\frac{r^2}{2t}\right) \circ P_- \right). \tag{6.22}$$

where n is now an element of $\mathbb{Z} + 1/2$.

Lemma 6.3.4 (Extending spin structure) In the notation of Lemma 6.3.1 the heat kernel for $e^{-t\Delta_N}$ on the punctured plane for the extending spin structure is

$$\begin{aligned} K_{\Delta_N}(r_1,\theta_1,r_2,\theta_2) = \\ \sum_{n\in\mathbb{Z}+1/2} \frac{1}{4\pi t} e^{-(r_1^2+r_2^2)/4t} \left(I_{|n-1/2|}\left(\frac{r_1 r_2}{2t}\right) \circ P_+ + I_{|n+1/2|}\left(\frac{r_1 r_2}{2t}\right) \circ P_- \right) e^{in(\theta_2-\theta_1)}. \end{aligned} \tag{6.23}$$

The expression on the diagonal therefore agrees with that in Lemma 6.3.3.

Proof (Lemmas 6.3.1, 6.3.2, 6.3.3 and 6.3.4)

The proof uses a transform to calculate the kernels. First we find all the eigenspinors of the relevant Laplacian. Then we use the Hankel transform to obtain the kernels.

Step 1: Eigenspinors of the Laplacian

We already know the expression for the spinor Laplacian on the punctured disc

$$\Delta = -\partial_r^2 - \frac{1}{r^2}\partial_\theta^2 - \frac{1}{r}\partial_r - \frac{e_{12}}{r^2}\partial_\theta + \frac{1}{4r^2}. \tag{6.24}$$

Note that $-e_{12}$ is either $+i$ or $-i$ depending on whether we consider the Laplacian acting on the positive or negative chiral bundle respectively. Denote the composition $\Delta \circ P_n$ by Δ_n.

Decompose a chiral spinor as $\Psi^{\pm} = \sum_n f_n(r)e^{in\theta}$ where the parity reflects which bundle we consider the spinor a section of. If $\Psi^{\pm}$ is an eigenspinor of Δ_n, with eigenvalue μ, then f_n must satisfy

$$f_n'' + \frac{1}{r}f_n' + \left[\mu^2 - \frac{(n \mp 1/2)^2}{r^2}\right] f_n = 0 \tag{6.25}$$

so that

$$f_n = J_{+|n\mp 1/2|}(\mu r) \qquad \text{or} \qquad f_n = J_{-|n\mp 1/2|}(\mu r). \tag{6.26}$$

Here J_n denotes the n^{th} Bessel function and the choice of signs is once again consistent with the parity of the spinor.

Step 2: Domains of Δ_N and Δ_D

We ascertain here which of the eigenspinors are in the domains of Δ_N and which are in the domain of Δ_D. Recalling that the n^{th} Bessel function grows like $O(r^n)$ as $r \longrightarrow 0$ we conclude that the maximal set of eigenspinors in each domain, for the non-extending spin structure, are

$$\mathrm{dom}\Delta_D \supset \left\{J_{+|n\mp 1/2|}(\mu r)e^{in\theta} \in \mathcal{S}^{\pm} \text{ for all } n \in \mathbb{Z}\right\} \tag{6.27}$$
$$\mathrm{dom}\Delta_N \supset \mathrm{dom}\Delta_D \cup \left\{J_{-1/2}(\mu r) \in \mathcal{S}^{\pm}, J_{-1/2}(\mu r)e^{i\theta} \in \mathcal{S}^{+}, J_{-1/2}(\mu r)e^{-i\theta} \in \mathcal{S}^{-}\right\}.$$

That these sets span follows from the Fredholm nature of our index problem (see Section 6.2 and also [Cho85, p.31]). For the extending spin structure we have that the domains of the two Laplacians are equal

$$\mathrm{dom}\Delta_D = \mathrm{dom}\Delta_N \supset \left\{J_{+|n\mp 1/2|}(\mu r)e^{in\theta} \in \mathcal{S}^{\pm} \text{ for all } n \in \mathbb{Z}\right\}. \tag{6.28}$$

Step 3: Hankel Transform

See also [Che79], [Cho85] and [Sne74]. Recall that the Hankel Transform is an isometry from $L^2((0,\infty), rdr)$ to $L^2((0,\infty), \lambda d\lambda)$ defined for $n > -1$ by

$$H_n(g)(\lambda) := \int_0^\infty J_n(\lambda r)g(r).rdr. \tag{6.29}$$

The Hankel Inversion formula, see [Che79, Cho85, Sne74], says that

$$H_n\left[H_n(g)\right](r) = g(r) \tag{6.30}$$

for $n > -1$. It is clear from (6.29) that

$$H_{|n\mp 1/2|}(\Delta_n^{\pm} g)(\lambda) = \lambda^2 H_{|n\mp 1/2|}(g)(\lambda) \tag{6.31}$$

for all n and that also,

$$H_{-1/2}(\Delta_n^{\pm} g)(\lambda) = \lambda^2 H_{-1/2}(g)(\lambda). \tag{6.32}$$

Denote the spinor bundle on the punctured plane by $S_0^+ \oplus S_0^-$. We decompose any L^2 spinor as

$$\Psi = \sum_n \left(f_n^+ \oplus f_n^-\right) e^{in\theta}, \tag{6.33}$$

where $f_n^+ \in S_0^+$ and $f_n^- \in S_0^-$, and then define two maps as follows. Let

$$F_D(\Psi)(\mu) := \sum_n \left(H_{|n-1/2|}(f_n^+)(\mu) \oplus H_{|n+1/2|}(f_n^-)(\mu)\right) e^{in\theta} \tag{6.34}$$

where $n \in \mathbb{Z}$ for the non-extending spin structure and $n \in \mathbb{Z} + 1/2$ for the extending spin structure. Let $F_N(\Psi)(\mu) := F_D(\Psi)(\mu)$ for the extending spin structure. Lastly, denote projection onto $S_0^{\pm}$ by $P_{\pm}$ and define

$$\begin{aligned} F_N(\Psi)(\mu) := F_D(\Psi)(\mu) &+ \left(H_{-1/2}(f_0^+)(\mu) \oplus H_{-1/2}(f_0^-)(\mu)\right) \\ &+ P_+\left(H_{-1/2}(f_1^+)(\mu)e^{i\theta} \oplus 1\right) + P_-\left(1 \oplus H_{-1/2}(f_{-1}^-)(\mu)e^{-i\theta}\right) \end{aligned} \tag{6.35}$$

for the non-extending spin structure.

With these maps defined equation (6.31) shows that for either spin structure

$$F_N(\Delta_N\Psi)(\mu) = \mu^2 F_N(\Psi) \qquad \text{for all } \Psi \in \text{dom } \Delta_N \text{ and,} \tag{6.36}$$

$$F_D(\Delta_D\Psi)(\mu) = \mu^2 F_D(\Psi) \qquad \text{for all } \Psi \in \text{dom } \Delta_D. \tag{6.37}$$

By Hankel inversion we have, at least formally, that

$$e^{-t\Delta_N}\Psi = F_N^{-1}\left(e^{-\mu^2 t}F_N(\Psi)(\mu)\right) \qquad \text{and} \tag{6.38}$$

$$e^{-t\Delta_D}\Psi = F_D^{-1}\left(e^{-\mu^2 t}F_D(\Psi)(\mu)\right). \tag{6.39}$$

Writing this out in full we see from (6.34) that

$$\begin{aligned} K_{\Delta_D}(r_1,\theta_1,r_2,\theta_2) = \sum_n \int_0^\infty \frac{1}{2\pi} e^{-t\mu^2} J_{|n-1/2|}(\mu r_2) J_{|n-1/2|}(\mu r_1) e^{in(\theta_2-\theta_1)} P_+ \mu d\mu \\ + \sum_n \int_0^\infty \frac{1}{2\pi} e^{-t\mu^2} J_{|n+1/2|}(\mu r_2) J_{|n+1/2|}(\mu r_1) e^{in(\theta_2-\theta_1)} P_- \mu d\mu, \end{aligned} \tag{6.40}$$

for either spin structure. The identity $K_{\Delta_N} = K_{\Delta_D}$ holds for the extending spin structure and from (6.35)

$$\begin{aligned} K_{\Delta_N}(r_1,\theta_1,r_2,\theta_2) &= K_{\Delta_D}(r_1,\theta_1,r_2,\theta_2) \\ &+ \int_0^\infty \frac{1}{2\pi} e^{-t\mu^2} J_{-1/2}(\mu r_2) J_{-1/2}(\mu r_1)\left(1 + e^{i(\theta_2-\theta_1)}P_+ + e^{-i(\theta_2-\theta_1)}P_-\right)\mu d\mu \end{aligned} \tag{6.41}$$

for the non-extending spin structure. Webers' second exponential integral [Wat95, p.395] says that

$$\int_0^\infty \exp(-p^2\mu^2) J_\nu(a\mu) J_\nu(b\mu) \mu d\mu = \frac{1}{2p^2}\exp\left(-\frac{a^2+b^2}{4p^2}\right) I_\nu\left(\frac{ab}{2p^2}\right) \tag{6.42}$$

for $\mathrm{Re}(\nu) > -1$ and $|\arg p| < \pi/4$.

Thus we may simplify the kernels (6.40) and (6.41) to the expressions that appear in Lemmas 6.3.1, 6.3.2, 6.3.3 and 6.3.4. This completes the proofs of these Lemmas.

□

Remark 6.3.5 Since the Hankel Transform is the Fourier Transform on $\mathbb{R}^2 - \{0\}$ in polar coordinates we may also obtain these results using the Fourier Transform. To emphasise the similarity to the work of Cheeger [Che79] and Chou [Cho85] on conical singularities, we use the Hankel transform and Bessel functions.

6.4 Proof of the Index Theorem

We have shown that our closure of the Dirac operator on $M^{2n} - \Sigma^{2n-2}$ is Fredholm and argued that that its index can be calculated by evaluating, separately, heat kernel contributions from a tubular neighbourhood of Σ and its complement. Having explicit expressions for these kernels it remains only to bring all this information together to prove the main result of this chapter.

Theorem 6.4.1 For $n \geqslant 2$ let M^{2n} be a compact manifold and Σ^{2n-2} a codimension two spin submanifold of M^{2n} such that $M^{2n} - \Sigma^{2n-2}$ has a non-extending spin structure. Assume that Σ has trivial normal bundle and the metric near the submanifold is *locally* of the form $dr^2+r^2d\theta^2+ds^2_{\Sigma}$. The index of the Dirac operator, associated to the non-extending spin-structure, and closed as in Section 5.3, is equal to

$$\begin{aligned} &\widehat{A}(M) && \text{when} \quad \dim_{\mathbb{R}} M = 4k \\ &\frac{1}{2}\widehat{A}(\Sigma) && \text{when} \quad \dim_{\mathbb{R}} M = 4k+2. \end{aligned} \tag{6.43}$$

Proof We saw in equation (6.11) that the trace of the heat kernel could be approximated, to an arbitrary order of t, by the trace over a compact piece and the trace of the heat kernel on $\mathbb{R}^2 - \{0\} \times \Sigma$ over a tubular neighbourhood of Σ.

The trace of the heat kernel on the compact piece gives a contribution $\int_{M-N} \widehat{A}(M)$. The piece near to Σ is explicitly the integral over $N - \Sigma$ of the pointwise expression

$$\mathrm{tr}e^{-t\left(1\otimes\Delta_D^- + D_\Sigma^{2,-}\otimes 1\right)\oplus\left(1\otimes\Delta_N^+ + D_\Sigma^{2,+}\otimes 1\right)} - \mathrm{tr}e^{-t\left(\Delta_D^-\otimes 1 + 1\otimes D_\Sigma^{2,+}\right)\oplus\left(\Delta_N^+\otimes 1 + 1\otimes D_\Sigma^{2,-}\right)}. \tag{6.44}$$

Since

$$\mathrm{tr}e^{-tA\oplus B} = \mathrm{tr}e^{-tA} + \mathrm{tr}e^{-tB} \tag{6.45}$$

and

$$\mathrm{tr}e^{-t(1\otimes A + B\otimes 1)} = \mathrm{tr}e^{-tA}.\mathrm{tr}e^{-tB} \tag{6.46}$$

the expression (6.44) is equal to

$$\mathrm{tr}_\Sigma\left(e^{-tD_\Sigma^{2,+}} - e^{-tD_\Sigma^{2,-}}\right)\mathrm{tr}_0\left(e^{t\Delta_N^+} - e^{-t\Delta_D^-}\right) \tag{6.47}$$

where tr_Σ and tr_0 denote the pointwise traces on Σ and the punctured plane $\mathbb{R}^2 - \{0\}$ respectively. Now use the explicit pointwise expressions of Lemmas 6.3.1 and 6.3.2 for the relevant heat kernels associated to the non-extending spin structure. These show that, for small t, the pointwise trace (6.47) is asymptotically equal to

$$\frac{1}{2\pi t}e^{-r^2/2t}I_{-1/2}\left(r^2/2t\right)\left[\frac{1}{(2\pi i)^{n-1}}\widehat{A}(\Sigma)_{[2n-2]}\right]. \tag{6.48}$$

From the definition of the modified Bessel function $I_{-1/2}$

$$\frac{1}{2\pi t}e^{-r^2/2t}I_{-1/2}\left(r^2/2t\right) = \frac{1}{r\pi\sqrt{4\pi t}}\left[1 + e^{-r^2/t}\right]. \tag{6.49}$$

From this it is clear that as $t \to 0$

$$\int_U \frac{1}{2\pi t}e^{-r^2/2t}I_{-1/2}\left(r^2/2t\right) r dr d\theta \sim \frac{1}{2} + \frac{1}{\sqrt{\pi t}} + \mathrm{const}.t^{1/2} \tag{6.50}$$

for any radius disc U. This is because as $t \longrightarrow 0$ expression (6.49) tends to

$$\frac{1}{\sqrt{\pi t}} + \frac{\delta_0}{2} \tag{6.51}$$

where δ_0 is the Dirac delta distribution supported at $\{r = 0\}$.

This completes the proof of the theorem.

□

6.5 Remarks and Examples

Remark 6.5.1 Since we have constructed a pointwise approximation to the heat kernel we see that our assumptions on Σ in Theorem 6.4.1 may be weakened. This fact is noted also in [Che79, p.589]. The heat kernel proof holds, with only minor alterations, when Σ is neither spin nor orientable. In these cases we must interpret the integral of the form (6.48) over Σ in a suitable way. To do this the differential form must either, canonically take values in the orientation bundle of Σ or we must consider the appropriate form, invariant under the natural involution, on a cover of Σ. **In this way we see that the index Theorem 6.4.1 holds for any characteristic submanifold with trivial normal bundle in M.** In Section 6.5.1 we argue that Theorem 6.4.1 is independent of the smooth metric on M.

Remark 6.5.2 The heat kernels for the extending spin structure can be substituted into the proof of Theorem 6.4.1. Doing so we recover that the L^2 index for the extending spin structure on $M^{2n} - \Sigma^{2n-2}$ is $\widehat{A}(M)[M]$ which is consistent with the Removable Singularities Theorem 3.1.1.

Remark 6.5.3 When $n = 2$ we can use the heat kernels in Section 6.3 to establish the index in equation (4.34) for punctured surfaces. This was derived in Section 4.3 via the Atiyah-Patodi-Singer index theorem. We commented in Remark 4.3.4 that this index was

$$\dim \ker D^+ - \dim \ker D_0^- \tag{6.52}$$

and equation (6.50) proves this statement rigourously. In this way we see the index of Section 4.3 as a special case of the index Theorem 6.4.1.

Remark 6.5.4 Indices of non-complete manifolds have been studied before. We explain here why this index differs from all previous work.
The most closely related work to our situation is that of Krönheimer and Mrowka in four dimensions [KM93]. They study the index of operators, on manifolds like $M^4 - \Sigma^2$, with non-trivial holonomy terms about the submanifold. As in the cases considered here the metric on $M - \Sigma$ is inherited from a smooth metric on M. The index they study is, however, the index of an operator which maps from the first Sobolev space L_1^2 to L^2.

The index calculated here differs from this work because, in dimensions greater than two, we allow the possibility of solutions which are L^2 but not in L_1^2, in both the domain and the range of the operator.
This is most simply expressed by considering the growth of possible solutions near to Σ. Krönheimer and Mrowka show that for a covariant derivative, with non-trivial holonomy about Σ, the first Sobolev space L_1^2 norm is equivalent to

$$\|\frac{1}{r}f\| + \|df\|. \tag{6.53}$$

Here f is a section of the relevant bundle and the norm is the usual L^2 norm. In equations (5.18) and (5.20) we saw that we allowed the possibility of harmonic spinors in our domain and range which were $O(r^{-1/2})$ near Σ. Spinors which grow like $O(r^{-1/2})$ will not have (6.53) finite. All other index theorems on non complete manifolds [BS88, Cho85, Che79] differ in a similar manner.

6.5.1 Deformation Invariance

The principle reason for appealing to the general theory of [BS88], to show that the operator was Fredholm, lies in the fact that this work allows us to argue that the index theorem holds for any metric on M when Σ has trivial normal bundle.
Consider $\Sigma \subset M$, with trivial normal bundle, and any smooth metric g_1 on M. Since Σ has trivial normal bundle we may always patch a metric, of form (5.1), onto M where g_Σ is taken to be the metric induced on Σ by g_1. Denote this metric on M by g_0.
Consider the continuous one parameter family of metrics

$$(1-t)g_0 + tg_1. \tag{6.54}$$

Lemma 2.7 of [BS88] says that as t varies from 0 to 1 we obtain a continuous family of operators, closed locally near Σ as in Section 5.3, which always lies in the space of Fredholm operators. Thus as we deform the metric from g_0 to g_1 on $M - \Sigma$ the index is unchanged. Hence the Theorem 6.4.1 holds for any smooth matric on M when Σ has trivial normal bundle. Actually, this requires an extra argument since the bundle of spinors varies as the metric does. This technicality is easily resolved as in [Ati71, p.51].

Remark 6.5.5 The $\widehat{A}$ genus is not in general an integer and, for a spin manifold of dimension $8k$, it is not necessarily an even integer. In this remark we explain the integrality of (6.43) in these dimensions independently of the index theorem. Firstly we deal with the case when $\dim \Sigma = 8k$ and Σ is spin. With these assumptions we may deduce that the $\widehat{A}$ genus of Σ is even by a simple cobordism argument.

The spin structure on $M - \Sigma$ induces a spin structure on $\Sigma \times S^1$ making $\Sigma \times S^1$ a spin boundary. There is a graded ring homomorphism [LM89, p.151]

$$\mathrm{ind}_* : \Omega_*^{\mathrm{Spin}} \longrightarrow KO^{-*}(\mathrm{pt}) \tag{6.55}$$

such that

$$\mathrm{ind}_{8k+1}(\Sigma \times S^1) = \mathrm{ind}_{8k}(\Sigma)\mathrm{ind}_1(S^1) \tag{6.56}$$

is a spin cobordism invariant. Since $\Sigma \times S^1$ is spin cobordant to zero the expression (6.56) vanishes.

We saw in Remark 1.1.4 that a non-extending spin structure induced the disconnected spin structure on S^1. Since this spin structure doesn't bound $\mathrm{ind}_1(S^1) = 1 \in \mathbb{Z}/2\mathbb{Z} = KO^{-1}(\mathrm{pt})$. The vanishing of (6.56) then gives that $\mathrm{ind}_{8k}(\Sigma) = \widehat{A}(\Sigma) \in \mathbb{Z} = KO^{-8}(\mathrm{pt})$ is zero modulo two and thus the $\widehat{A}$ genus must be even.

When Σ is just orientable we have the following Lemma which shows the integrality of (6.43) in general.

Lemma 6.5.6 Suppose that $j : \Sigma^{n-2} \to M^n$ is an embedding of a compact oriented submanifold of M. Suppose that Σ^{n-2} has trivial normal bundle and that $M^n - \Sigma^{n-2}$ has

a non-extending spin structure. Then the index (6.43) of the Dirac operator $\not{D}$, see (5.7), agrees with the $Spin^{\mathbb{C}}$ index

$$\widehat{A}(M^n)e^{v/2}[M^n] \tag{6.57}$$

Proof Let $v = \mathrm{P.D.}[\Sigma^{n-2}] \in H^2(M^n, \mathbb{Z})$ so that $c_1(N) = j^*(v)$ where N is the normal bundle of Σ in M. The total Pontrjagin class p satisfies (see [Hir95, p.87])

$$j^*p(M^n) = p(\Sigma^{n-2})p(N). \tag{6.58}$$

However

$$p(N) = j^*(1 + v^2) \tag{6.59}$$

so that

$$p(\Sigma^{n-2}) = j^*\left((1+v^2)^{-1}p(M^n)\right). \tag{6.60}$$

The $\widehat{A}$ polynomials $\widehat{A}_j(p_1, \ldots, p_j)$ are generated by the multiplicative sequence corresponding to the power series

$$\frac{\sqrt{z}/2}{\sinh \sqrt{z}/2}. \tag{6.61}$$

see [Hir95, p.197]. Thus

$$\sum_{i=0}^{\infty} \widehat{A}_i\left(p_1(\Sigma^{n-2}), \ldots, p_i(\Sigma^{n-2})\right) = j^*\left[\frac{2\sinh v/2}{v}\sum_{i=0}^{\infty}\widehat{A}_i\left(p_1(M^n), \ldots, p_i(M^n)\right)\right] \tag{6.62}$$

so that

$$\frac{1}{2}\widehat{A}(\Sigma^{n-2})[\Sigma^{n-2}] = \left(\sinh v/2 \sum_{i=0}^{\infty}\widehat{A}_i(M^n)\right)[M^n]. \tag{6.63}$$

In the trivial normal bundle case the index of the operator $\not{D}$ is, by Theorem 6.4.1, equal to

$$\frac{1}{2}\widehat{A}(\Sigma^{n-2})[\Sigma^{n-2}] + \widehat{A}(M^n)[M^n] = \widehat{A}(M^n)\left(1 + \sinh v/2\right)[M^n], \tag{6.64}$$

where the equality comes from equation (6.63). This expression is seen to equal the $Spin^{\mathbb{C}}$ index as follows.

Case One $\dim M = 4n$

In this case the powers of v arising from the term $\sinh v/2$ are all odd. Hence the right hand side of equation (6.64) is just $\widehat{A}(M)[M]$ as is equation (6.57).

Case One $\dim M = 4n+2$

In this case only terms with odd powers of v will contribute to the index. Thus, both equation (6.64) and equation (6.57) are seen to equal

$$\sum_{p+q=m} \frac{1}{(2p+1)!} \left(\frac{v}{2}\right)^{2p+1} \widehat{A}_q[M]. \tag{6.65}$$

This completes the proof of Lemma 6.5.6. □

6.5.2 Hermitian Bundle Twisting

We have concentrated thus far on the untwisted Dirac operator associated to a non-extending spin structure on $M - \Sigma$. We may however, also consider this Dirac operator twisted by a Hermitian connection on a complex vector bundle F over M. In this remark we show how this twisting affects the form of the index.

Note that it is important to our discussion that the complex vector bundle is well defined over M. Otherwise, twisting the Dirac operator may remove the obstruction to the operator and the bundle extending over M. For example, on $\mathbb{C}P^2 - \mathbb{C}P^1$ the Dirac operator $\not{D}^+$ is identified [LM89, p.400] with

$$\overline{\partial} + \overline{\partial}^* : \Lambda^{0,\mathrm{even}} \otimes K^{1/2} \longrightarrow \Lambda^{0,\mathrm{odd}} \otimes K^{1/2}, \tag{6.66}$$

where K is the canonical bundle of $\mathbb{C}P^2 - \mathbb{C}P^1$. Since $\mathbb{C}P^2$ is not spin this unique spin structure is non extending (see also Example 2.1.12). Denote the hyperplane bundle of $\mathbb{C}P^2 - \mathbb{C}P^1$ by H and twist the Dirac operator by the connection on $H^{3/2} = K^{-1/2}$. This coupled Dirac operator on $\mathbb{C}P^2 - \mathbb{C}P^1$ is just $\overline{\partial} + \overline{\partial}^* : \Lambda^{0,\mathrm{even}} \longrightarrow \Lambda^{0,\mathrm{odd}}$ which is the

restriction of the $\overline{\partial}$ operator on $\mathbb{C}P^2$ acting on the restriction of a well defined bundle on $\mathbb{C}P^2$. In such cases a simple adaptation of Removable Singularities Theorem 3.1.1 shows that the L^2 index of this operator on $\mathbb{C}P^2 - \mathbb{C}P^1$ can be calculated by applying the $Spin^{\mathbb{C}}$ index theorem on $\mathbb{C}P^2$.

Consider now the basic case where F is a complex line bundle over M. Assume that Σ is as in Theorem 6.4.1 and that on some neighbourhood, $N = U \times \Sigma$, of Σ the line bundle F is separable as $L_1 \otimes L_2$, where $L_1 \longrightarrow U$ is a line bundle over the disc and $L_2 \longrightarrow \Sigma$ is a line bundle over Σ. On this tubular neighbourhood the spinor Laplacian for the coupled Dirac operator $\not{D}^2_{\otimes F}$ is of the form,

$$\Delta^2 + D^2_\Sigma + i\left(F^1_{12}e_{12} + \sum_{3\leqslant j<k} F^2_{jk}e_{jk}\right), \tag{6.67}$$

where F^1 and F^2 are the curvatures of L_1 and L_2 and $F_{jk} = F(e_j, e_k)$.

We may close this operator in a manner analogous to that in Section 5.3. This gives a spinor Laplacian whose chiral restrictions are of the form,

$$\not{D}^{2,\pm}_{\otimes F} = \begin{pmatrix} \Delta^-_D + D^{2,\mp}_\Sigma - F^1_{12} + iF^2_{jk}e_{jk}\omega^\mp_\Sigma & 0 \\ 0 & \Delta^+_N + D^{2,\pm}_\Sigma + F^1_{12} + iF^2_{jk}e_{jk}\omega^\pm_\Sigma \end{pmatrix}, \tag{6.68}$$

near Σ. Recall the $\omega^\pm_\Sigma$ denotes restriction of Clifford multiplication by the volume form on Σ to the spinor bundles $S^\pm_\Sigma$. Compare this expression to that in equation (5.27).

The analytical framework of Section 6.2 carries over with only minor changes for this operator. The pointwise trace

$$\mathrm{tr}_{U\times\Sigma}\left(e^{-t\not{D}^{2,+}_{\otimes F}} - e^{-t\not{D}^{2,-}_{\otimes F}}\right) \tag{6.69}$$

is equal to

$$\mathrm{tr}_U\left(e^{-t\Delta^+_N}e^{-tF^1_{12}} - e^{-t\Delta^-_D}e^{tF^1_{12}}\right)\mathrm{tr}_\Sigma\left(e^{-tD^{2,+}_{\otimes L_2}} - e^{-tD^{2,-}_{\otimes L_2}}\right) \tag{6.70}$$

on the tubular neighbourhood of Σ. Here we have denoted the heat kernel of the Dirac operator on Σ, twisted by the line bundle $L_2 \longrightarrow \Sigma$, by $e^{-tD^2_{\otimes L_2}}$. As in the proof of Theorem 6.4.1,

$$\int_U \mathrm{tr}_U \left(e^{-t\Delta_N^+} e^{-tF^1_{12}} - e^{-t\Delta_D^-} e^{tF^1_{12}} \right) r dr d\theta \sim \frac{1}{\sqrt{\pi t}} + \frac{1}{2} + O(t^{1/2}), \tag{6.71}$$

as $t \longrightarrow 0$. Consequently, by following the proof of Theorem 6.4.1, the index is seen to equal

$$\int_M \widehat{A}(M)\mathrm{Ch}(F) + \frac{1}{2}\int_\Sigma \widehat{A}(\Sigma)\mathrm{Ch}(i^*F) \tag{6.72}$$

where $i : \Sigma \longrightarrow M$ is the embedding and Ch denotes the Chern character. As before this index theorem will be independent of the smooth metric on M and the orientability of Σ. Furthermore, it is straightforward to extend the proof sketched above to the case of an arbitrary line bundle F over M.

6.5.3 Examples

Example 6.5.7 The most transparent example of the index (6.43) is on the space $\mathbb{C}P^1 - \{0, \infty\} \times K3$. Here $K3$ denotes a $K3$ surface with Kähler metric. By Remark 2 of [Hit74, p.20] such a $K3$ surface has precisely two positive spinors. Denote these by ϕ_1^+ and ϕ_2^+. There are no negative harmonic spinors. These facts are independent of the Kähler metric chosen on the $K3$.

Recall that the punctured $\mathbb{C}P^1$ has two L^2 harmonic spinors associated to its non-extending spin structure: one of each chirality Ψ^+_{ness} and Ψ^-_{ness} (see equations (1.21) and (1.23)). Thus the L^2 harmonic spinors on the product $\mathbb{C}P^1 - \{0, \infty\} \times K3$ are simply

$$\begin{aligned} &\Psi^+_{\mathrm{ness}} \otimes \phi_1^+, \quad \Psi^+_{\mathrm{ness}} \otimes \phi_2^+ \quad \in \mathcal{S}^+ \\ &\Psi^-_{\mathrm{ness}} \otimes \phi_1^+, \quad \Psi^-_{\mathrm{ness}} \otimes \phi_2^+ \quad \in \mathcal{S}^-. \end{aligned} \tag{6.73}$$

We saw in the main example that both Ψ^+_{ness} and Ψ^-_{ness} grew like $O(r^{-1/2})$ near to the removed points. Considering the operator whose index we calculated we see that the index theorem will count both $\Psi^+_{\mathrm{ness}} \otimes \phi_1^+$ and $\Psi^+_{\mathrm{ness}} \otimes \phi_2^+$, but not $\Psi^-_{\mathrm{ness}} \otimes \phi_1^+$ or $\Psi^-_{\mathrm{ness}} \otimes \phi_2^+$. Recall here the growth conditions, equations (5.23) and (5.24), of the spinors in the domain of the operator.

As a consequence, the difference

$$\dim \ker \not{D}^+ \cap L^2 - \dim \ker \not{D}^- \cap L^2, \tag{6.74}$$

is zero for the unique non-extending spin structure on $\mathbb{C}P^1 - \{0, \infty\} \times K3$, but the index of the operator (5.21) will be two. This easily verifies Theorem 6.4.1 in this case since

$$\widehat{A}(K3)[K3] = 2. \tag{6.75}$$

This example shows clearly how the index differs from the formal L^2 difference. It also shows clearly how some harmonic spinors in $L^2 - L^2_1$ are in the domain of $\not{D}$ but not all.

Remark 6.5.8 The example which follows is highly pertinent. In $8k+4$ dimensions the Clifford Algebra has a quaternionic structure, see [LM89, p.37], which is maintained in the spin representation. Consequently the complex dimension of the vector space of L^2 harmonic spinors associated to a non-extending spin structure, on $M^{8k+4} - \Sigma^{8k+2}$, must be even. This is well known. Example 6.5.9 shows that the index we have calculated breaks the symmetry [AS68] of the representation. How it breaks this symmetry is explained in detail in Section 6.6.

Example 6.5.9 Consider a cubic curve C in $\mathbb{C}P^2$. Since $K_{\mathbb{C}P^2} = \mathcal{O}(-3)$ the holomorphic line bundle defined by C, which we denote by $[C]$, is therefore $-K_{\mathbb{C}P^2}$. If $p : M \longrightarrow \mathbb{C}P^2$ denotes the surface obtained by blowing up a point $x \in C$, then the exceptional divisor $E = \mathbb{C}P^1$ (the inverse image of x under p) has self-intersection -1 (see [GH94, p.475]). If

C is non singular at x then its inverse image under p is $C' + E$ for a curve C' isomorphic to C. Pulling back the holomorphic line bundle $[C]$ we have that

$$p^*[C] = [C' + E]. \tag{6.76}$$

By considering local two forms it can be shown [GH94, p.187] that

$$K_M = p^* K_{\mathbb{C}P^2} + [E] \tag{6.77}$$

where K_M denotes the canonical bundle of M.
When C is the cubic in $\mathbb{C}P^2$ the above gives that

$$[C'] = p^*[C] - [E] = -\left(p^* K_{\mathbb{C}P^2} + [E]\right) = -K_M. \tag{6.78}$$

Hence, from (6.77),

$$c_1[C'] = -c_1\left(p^* K_{\mathbb{C}P^2}\right) - c_1\left([E]\right) \tag{6.79}$$

so that

$$\begin{aligned} c_1^2\left([C']\right)[M] &= c_1^2\left(K_{\mathbb{C}P^2}\right)[\mathbb{C}P^2] + c_1^2[E][M] \\ &= 9 - 1 \end{aligned} \tag{6.80}$$

Thus, blowing up a point on C adds another class to H^2. If we blow up nine distinct points on C, the resulting surface Σ, is such that $c_1^2[\Sigma] = 0$. Hence Σ has trivial normal bundle in M. On a four manifold the second Chern class is just the Euler characteristic class. The extra class in H^2 obtained by blowing up a point will increase the Euler characteristic by one. Therefore $c_2(M) = c_2(\mathbb{C}P^2) + 9 = 12$.
From equation (6.78) $[\Sigma] = -K_M$ and since $K_M = w_2(M) \mod 2$, for any complex manifold M, we see that removing Σ from M gives a non-extending spin structure on $M - \Sigma$.
The index Theorem 6.4.1 says that the index of the operator (5.21) is equal to

$$\widehat{A}(M)_{[4]} = -\frac{p_1}{24} = \frac{2c_2 - c_1^2}{24} = \frac{c_2}{12} = 1 \tag{6.81}$$

for any smooth metric on M.

6.6 Symmetry Breaking

We saw in Example 6.5.9 that the index (6.43) could be odd in $8k+4$ dimensions.
The reason why the index does not reflect the quaternionic nature of the spin representations in $8k+4$ dimensions is because the quaternionic structure does not, in general, preserve the domain on which the operator is closed.
The subtlety is due to the null space of the Lie derivative of the vector field induced by the local S^1 action. We explain this in detail below. Recall from (5.20) that L^2 harmonic spinors, in the null space of D_Σ, may take the form near Σ

$$\cdots\ \overbrace{\begin{pmatrix} e^{-i\theta} r^{1/2}\phi_0^- \\ 0 \end{pmatrix}}^{n=-1},\ \overbrace{\underset{\substack{\| \\ \mathbf{A}^+}}{\begin{pmatrix} r^{-1/2}\phi_0^- \\ 0 \end{pmatrix}},\ \underset{\substack{\| \\ \mathbf{B}^+}}{\begin{pmatrix} 0 \\ r^{-1/2}\phi_0^+ \end{pmatrix}}}^{n=0},\ \overbrace{\begin{pmatrix} 0 \\ e^{i\theta} r^{1/2}\phi_0^+ \end{pmatrix}}^{n=1},\ \cdots \tag{6.82}$$

in $\not{S}^+$ or

$$\cdots\ \overbrace{\begin{pmatrix} e^{-i\theta} r^{1/2}\phi_0^+ \\ 0 \end{pmatrix}}^{n=-1},\ \overbrace{\underset{\substack{\| \\ \mathbf{A}^-}}{\begin{pmatrix} r^{-1/2}\phi_0^+ \\ 0 \end{pmatrix}},\ \underset{\substack{\| \\ \mathbf{B}^-}}{\begin{pmatrix} 0 \\ r^{-1/2}\phi_0^- \end{pmatrix}}}^{n=0},\ \overbrace{\begin{pmatrix} 0 \\ e^{i\theta} r^{1/2}\phi_0^- \end{pmatrix}}^{n=1},\ \cdots \tag{6.83}$$

in $\not{S}^-$. Off of the null space of D_Σ harmonic spinors may also take the form (5.18).
Recall that the value of n denotes the decomposition into eigenspaces associated with the local circle action and $\phi_0^\pm$ denotes a harmonic spinor in $S_\Sigma^\pm$.
The main point here is that $\mathbf{A}^+$ and $\mathbf{A}^-$ are **not** in the domain of the Dirac operator whose closure was described in 5.3. All of the other forms of spinors, in equations (6.82) and (6.83) are in dom $\not{D}$. With respect to our representation the quaternionic structure

may without loss of generality be of the form

$$\mathbf{j} = \begin{pmatrix} 0 & -e_4 \\ -e_4 & 0 \end{pmatrix}. \tag{6.84}$$

This structure is parallel so $[\not{D}, \mathbf{j}\,] = 0$.

Denote by W the vector space of all solutions to the Dirac operator which lie in the domain of our closure 5.3. Then

$$W = (\mathbf{j}(W) \cap W) \oplus (\mathbf{j}(W) \cap W)^{\perp}, \tag{6.85}$$

where the orthogonal complement is taken in W. The point is that the subspace $(\mathbf{j}(W) \cap W)^{\perp}$ may be non trivial so that $\mathbf{j}(W) \not\subset W$.

This is clear since if $M - \Sigma$ has a non-extending spin structure there is the possibility that there may be L^2 spinors like $\mathbf{B}^+$. Now $\mathbf{j}.\mathbf{B}^+$ is also harmonic and L^2 however

$$\mathbf{j}.\mathbf{B}^+ = \text{const. } \mathbf{A}^+ \notin W \tag{6.86}$$

so

$$B^+ \in (\mathbf{j}(W) \cap W)^{\perp}. \tag{6.87}$$

Thus the quaternionic structure does not preserve the domain on which our Dirac operator is defined.

Remark 6.6.1 The L^2 harmonic spinors which lie in $(\mathbf{j}(W) \cap W)^{\perp}$ on an $8k+4$ manifold all lie in the zero eigenspace of L_V. For the product metric the L^2 harmonic spinors with this form near to Σ are the only harmonic spinors which are in $L^2 - L^2_1$. Thus we conclude that the following corollary holds.

Corollary 6.6.2 Let M be a manifold of dimension $8k+4$ and let $\Sigma^{8k+2} \subset M^{8k+4}$ be a codimension two submanifold such that $M - \Sigma$ has a non-extending spin structure. Assume

that Σ has trivial normal bundle and equip $M - \Sigma$ with a metric which is locally of form (5.1) near Σ. If the index (6.43) is odd then there exists an L^2 harmonic spinor Ψ on $M - \Sigma$ which is not in L^2_1. Furthermore such spinors satisfy $L_V \Psi = 0$ near Σ where L_V is the Lie derivative of the vector field induced on a tubular neighbourhood by the local S^1 action.

6.6.1 The mod 2 Index as a Limit

We consider here the reduction of the index (6.43) modulo 2 in $8k+4$ dimensions. By writing this index as a limit of the Atiyah-Patodi-Singer index we are able to show that the parity, modulo 2, of $\widehat{A}(M)$ equals the parity of the spin structure induced upon Σ.

Consider a spin submanifold $\Sigma^{8k+2} \subset M^{8k+4}$, with trivial normal bundle, such that $M - \Sigma$ has a non extending spin structure. Express the index 6.4.1 as follows

$$\widehat{A}(M) = \lim_{\epsilon \to 0} \left[\text{APS index } \partial\!\!\!/|_{M_\epsilon} - \int_{\partial M_\epsilon} (*) + \frac{h_\epsilon + \eta_\epsilon}{2} \right] \in \mathbb{Z} \tag{6.88}$$

Here APS Index $\partial\!\!\!/|_{M_\epsilon}$ is the Atiyah-Patodi-Singer index on the manifold with boundary $M_\epsilon = M - N_\epsilon$ where N_ϵ is an ϵ neighbourhood of Σ. The terms h_ϵ and η_ϵ are the usual eta invariant and null space dimension associated to this global boundary value problem. The term $\int_{\partial M_\epsilon} (*)$ represents the usual boundary correction term. We must include this term because the product metric (5.1) on $U \times \Sigma$ is not the product metric on $\mathbb{R} \times S^1 \times \Sigma$, which is the context of the original Atiyah-Patodi-Singer problem [APS75].

Since Σ is of dimension $8k+2$ we have [AS69, p.148] that

$$\omega^{\Sigma}_{\mathbb{C}} \partial\!\!\!/_{\Sigma} = -\partial\!\!\!/_{\Sigma} \omega^{\Sigma}_{\mathbb{C}} \tag{6.89}$$

where $\omega^{\Sigma}_{\mathbb{C}}$ is Clifford multiplication by the $\mathbb{C}$-volume element of Σ. Hence the non zero eigenvalues of D_Σ come in pairs $\pm\lambda$.

Since we assume that the normal bundle of Σ is trivial we may equip $M - \Sigma$ with a metric of form (5.1) near Σ. The explicit form of the eigenspinors (5.30) for the non-extending spin structure shows then that

$$h_\epsilon = h_\Sigma \qquad \text{and} \qquad \eta_\epsilon = 0, \tag{6.90}$$

where h_Σ is the dimension of the null space of the Dirac operator on Σ.

The dimension of this null space is with respect to the spin structure induced upon Σ by the non-extending spin structure on $M - \Sigma$. See Remark 5.1.2.

The boundary term $\int(*)$ vanishes in the limit as $\epsilon \longrightarrow 0$. The easiest way to see this is by noting that if $M - \Sigma$ has an extending spin structure then the explicit form of the eigenspinors (5.30) gives that $h_\epsilon = \eta_\epsilon = 0$ for all $\epsilon > 0$. The quaternionic structure associated to the $8k+4$ spinor representation preserves the Atiyah-Patodi-Singer boundary value problem. Similarly when M^{8k+4} is spin its $\widehat{A}$ genus is always even. Thus reducing (6.88) modulo 2 for the extending spin structure gives that

$$\lim_{\epsilon \longrightarrow 0} \int_{\partial M_\epsilon} (*) = 0 \bmod 2. \tag{6.91}$$

A calculation shows that in general this boundary term vanishes as $\epsilon \longrightarrow 0$ irrespective as to whether M is spin.

For a spin structure on $M - N_\epsilon$ induced from a non-extending spin structure on $M - \Sigma$ the the quaternionic structure associated to the $8k+4$ spinor representation again preserves the Atiyah-Patodi-Singer boundary value problem however **it need not** preserve the domain on which the operator $\not{D}$ is defined. Thus the Atiyah-Patodi-Singer index is always even but the one in Theorem 6.4.1 may not be. Putting these facts together we obtain that in $8k+4$ dimensions

$$\widehat{A}(M)[M] = \frac{h_\Sigma}{2} \bmod 2. \tag{6.92}$$

With Σ^{8k+2} spin the index of the Dirac operator is zero modulo 2 so that $h_\Sigma/2 = \dim\ker \not{D}^+_\Sigma$. Thus $h_\Sigma/2$ is the usual mod 2 index of $\not{D}_\Sigma$.

We argued that the index (6.43) for the non-extending spin structure was deformation invariant in Section 6.5.1. We conclude therefore that the reduction modulo 2 of the index for the spin structure induced upon a characteristic submanifold of dimension $8k+2$ is independent of the Riemannian metric on it. This is of course a special case of the result of Atiyah [Ati71, p.57]

Proposition (3.3) On a spin manifold of dimension $8k+2$ the harmonic spinors H form a complex vector space and $\dim_{\mathbb{C}} H \bmod 2$ is independent of the Riemannian metric.

We may also however use the observation (6.92) to relate the parity of the $\widehat{A}$ genus of M to the type of spin structure induced on Σ. This gives us two basic corollaries.

Corollary 6.6.3 Assume that $M^{8k+4} - \Sigma^{8k+2}$ has a non-extending spin structure where Σ is spin with trivial normal bundle. Suppose that the non-extending spin structure induces a spin structure on Σ via the inclusion $T_\Sigma \subset N$, where N is the unit normal bundle. Then the following are equivalent

- the $\widehat{A}$ genus of M is even
- the index (6.43) is even
- the spin structure induced on Σ is even
- $\dim(\mathbf{j}(W) \cap W)^\perp$ is even.

Here $(\mathbf{j}(W) \cap W)^\perp$ is as in (6.85).

Corollary 6.6.4 If $M^{8k+4} - \Sigma^{8k+2}$ is as in Corollary 6.6.3, then the non-extending spin structure on $M - \Sigma$ induces an odd spin structure on Σ if and only if the $\widehat{A}$ genus of M is odd.

When the index is odd recall from Corollary 6.6.2 that this implies the existence of a L^2 harmonic spinor associated to the non-extending spin structure on $M-\Sigma$ which is not in L^2_1.

Note that the even and odd in the above Corollaries do not refer to the 'local' even and oddness with respect to the local S^1 action in Remark 1.1.5, but to the spin structure induced upon Σ.

6.6.2 The Work of Esnault, Seade and Viehweg

We show here that the interpretation of the index modulo 2, in Section 6.6.1, provides a partial extension of the results of Esnault, Seade and Viehweg [ESV92]. They studied characteristic (smooth) divisors on complex manifolds of (complex) dimension $4k+2$. These they defined as divisors of complex codimension one submanifolds $\Sigma^{8k+2}\subset M^{8k+4}$ such that

$$\mathrm{P.D.}[\Sigma] = K_M^{-1}\otimes L^2 \tag{6.93}$$

where L is some line bundle on M. Of course when equation (6.93) holds then P.D.$[\Sigma]$ mod $2 = w_2(M)$ so that $M-\Sigma$ has a non-extending spin structure. Recall here equation (4.9). The authors of [ESV92] observed that when Σ was a characteristic divisor, Σ induced a canonical spin structure upon itself. By the adjunction formula

$$K_\Sigma = K_M|_\Sigma \otimes \mathrm{P.D.}[\Sigma]|_\Sigma = L^2|_\Sigma. \tag{6.94}$$

As in Lemma 6.5.6 we will now denote P.D.$[\Sigma]$ by ν. One of the principal results of [ESV92] has as a corollary that the $Spin^{\mathbb{C}}$ index of the Dirac operator $\not{D}_{\nu^{1/2}}$, reduced modulo 2, equals the mod 2 index of the Dirac operator on Σ for the spin structure it induces upon itself. When $\dim_{\mathbb{C}} M = 4k+2$ we may write this as

$$\chi(M,L) = \sum_{i=0}^{2k} h^i(\Sigma, L) \mod 2 \tag{6.95}$$

where $\chi(M,L)$ is the Euler characteristic of the sheaf of germs of holomorphic sections of L and $h^i(\Sigma,L) = \dim H^i(\Sigma,\mathcal{O}(L))$.

In Remark (b) of [ESV92, p.22] the authors ask whether this theorem holds true in general for $Spin^{\mathbb{C}}$ manifolds of real dimension $8k+4$. The index theorem for non-extending spin structures goes some way to answering this question when Σ has trivial normal bundle. We saw in Remark 2.1.8 that if Σ was orientable and $M-\Sigma$ had a non-extending spin structure then M was $Spin^{\mathbb{C}}$. Since P.D.$[\Sigma] = K_M \bmod 2$ we may define a line bundle L such that $K_M\otimes \mathrm{P.D.}[\Sigma] = L^2$. Then by Lemma 6.5.6 the index theorem for the non-extending spin structure equals the $Spin^{\mathbb{C}}$ index of $\not{D}_{\nu^{1/2}}$. By the argument in Section 6.6.1 the index for the non-extending spin structure is also equal to the mod 2 index of the Dirac operator on Σ for the spin structure induced on Σ as in Remark 5.1.2. This proves the conjecture of Esnault, Seade and Viehweg when Σ has trivial normal bundle.

Remark 6.6.5 We remark here that in Conjecture 8.2.1 we suggest that it may be possible to extend this result, not only to characteristic manifolds with non trivial normal bundle but, to **singular** divisors using the same heat kernel methods of Chapter 6.

Chapter 7

Non Trivial Normal Bundle

In this chapter we present preliminary calculations for a general index theorem for non-extending spin structures on $M^{2n} - \Sigma^{2n-2}$, when $n \geqslant 2$. Here we allow Σ to have non trivial normal bundle. Once again we assume an amenable form of metric near Σ, or equivalently, on the normal bundle of Σ. We will also assume that Σ is spin.

The main results in this chapter are expressions for the Dirac operator and spinor Laplacian for such a metric. These are given in equations (7.29) and (7.31), which differ from (5.7) and (5.9) in that they include terms involving Clifford multiplication by the Euler class of the normal bundle of Σ in M.

7.1 Set Up

Let $\pi : N \to \Sigma$ denote the normal bundle of Σ. Let $\pi : N_1 \to \Sigma$ denote the unit disc bundle of tangent vectors normal to the submanifold Σ. Let $\pi : P_{U_1} \to \Sigma$ denote the principal U_1 bundle of unit normal vectors. Lastly denote the tangent bundle of the boundary of a tubular neighbourhood of Σ by M_0. Consider the simplest form of a metric on N_1

$$dr^2 + 4\pi^2 r^2 \varphi \otimes \varphi + ds^2_\Sigma. \tag{7.1}$$

Here φ denotes the global angular form on P_{U_1}. Thus $\alpha := 2\pi i\varphi$ is a U_1 connection on P_{U_1} and $d\varphi = -\pi^* e$ where e is the Euler class of the normal bundle of Σ. Choose an orthonormal basis for T_Σ and denote this by $e_3, \ldots, e_{2n}$. Denote the lift of tangent vectors from T_Σ to M_0 by a tilde.

Extend $\widetilde{e_3} \ldots \widetilde{e_{2n}}$ to an orthonormal basis of M_0 by adding $\widetilde{V} := \varphi^*/2\pi$. Parallel translate these tangent vectors radially and extend to an orthonormal basis of N_1 by adding ∂_r. Denote parallel translation of elements of M_0 along radial lines by a bar. In what follows we do not distinguish between $e_i, \widetilde{e}_i$ or $\overline{\widetilde{e}_i}$. We also denote $\overline{\widetilde{V}}$ by V for simplicity. Define

$$A := \sum_{3 \leqslant i < j \leqslant 2n} F_{ij} e_i . e_j. \qquad \text{where} \qquad F_{ij} := F(e_i, e_j) = d\alpha(e_i, e_j). \tag{7.2}$$

7.2 Operator Expressions

Using the set up above we may calculate the form of the Dirac operator and the spinor Laplacian with respect to the lift of the local section $(\partial_r, V, e_3, \ldots, e_{2n})$. We start by working out the form of the covariant spinor derivative from which it is straightforward to find expressions for the operators.

7.2.1 Dirac Operator

We record the following Lemma for the Levi Civita connection.

Lemma 7.2.1

$$\nabla_{\partial_r}\partial_r = \nabla_{\partial_r} e_i = \nabla_{\partial_r} V = \nabla_{e_i}\partial_r = 0 \qquad \nabla_V \partial_r = \frac{V}{r} \tag{7.3}$$

$$\nabla_V V = -\frac{\partial_r}{r} \qquad \nabla_{e_i} e_j = \overline{\left(\widetilde{\nabla^{\Sigma}_{e_i} e_j}\right)} - rF_{ij}.V. \tag{7.4}$$

$$\nabla_{e_i} V = \sum_j rF_{ij}.e_j. \qquad \nabla_V e_i = \sum_j rF_{ij}.e_j. \tag{7.5}$$

Proof These expressions follow from the fact that

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle \tag{7.6}$$

for pairwise orthonormal vector fields X, Y and Z. The vector fields in our orthonormal basis satisfy

$$[\partial_r, \partial_r] = [\partial_r, e_i] = [V, e_i] = 0 \quad \text{and} \quad [\partial_r, V] = -\frac{V}{r}. \tag{7.7}$$

Equations (7.3) follow from this.

Also

$$2(\nabla_V V, e_i) = ([e_i, V], V) = g_{U_1}\left([e_i, \widetilde{V}], \widetilde{V}\right) = 0 \tag{7.8}$$

since $[e_i, \widetilde{V}]$ is still horizontal. This gives that $\nabla_V V = -r^{-1}\partial_r$.

The rest of the the Lemma uses the fact that $\pi : P_{U_1} \longrightarrow \Sigma$, with the metric (7.1), is a Riemannian submersion where $\{e_3 \ldots e_{2n}\}$ are horizontal and $\widetilde{V}$ is the vertical vector field. From this we can conclude [dC92, p.186] the relationship between the two connections $\widetilde{\nabla}$ and ∇.

$$\widetilde{\nabla}_{\widetilde{e}_i}\widetilde{e}_j = \widetilde{\nabla_{e_i} e_j} + \frac{1}{2} g_{U_1}\left([\widetilde{X}, \widetilde{Y}], V\right). \tag{7.9}$$

We have also that

$$\begin{aligned} 2g_{U_1}\left([e_i, \widetilde{V}], \widetilde{V}\right) &= -g_{U_1}\left([\widetilde{V}, e_i], \widetilde{V}\right) - g_{U_1}\left([\widetilde{V}, e_i], \widetilde{V}\right) \\ &= g_{U_1}\left(\widetilde{\nabla}_{\widetilde{V}}\widetilde{V}, e_i\right) = g_\Sigma\left(e_i, \pi_*\left(\widetilde{\nabla}_{\widetilde{V}}\widetilde{V}\right)\right) \end{aligned} \tag{7.10}$$

From (7.6) we see that

$$\begin{aligned} 2r(\nabla_{e_i} V, e_j) &= g_{U_1}\left([e_i, \widetilde{V}], e_j\right) - g_{U_1}\left([\widetilde{V}, e_j], e_i\right) - r^2 g_{U_1}\left([e_i, e_j], \widetilde{V}\right) \\ &= 2g_{U_1}\left(\widetilde{\nabla_{e_i}}\widetilde{V}, e_j\right) + (1 - r^2)\, g_{U_1}\left([e_i, e_j], \widetilde{V}\right). \end{aligned} \tag{7.11}$$

Now

$$\begin{aligned}
F_{ij} &= F(e_i,e_j) = d\alpha(e_i,e_j)\\
&= -\frac{1}{2}\alpha[e_i,e_j]\alpha(\widetilde{V}) = -\frac{1}{2}g_{U_1}\left([e_i,e_j],\widetilde{V}\right)\\
&= -g_{U_1}\left(\widetilde{\nabla_{e_i}e_j},\widetilde{V}\right) = g_{U_1}\left(e_j,\widetilde{\nabla_{e_i}\widetilde{V}}\right)\\
&= g_{\Sigma}\left(e_j,\pi_*(\widetilde{\nabla_{e_i}\widetilde{V}})\right). \qquad (7.12)
\end{aligned}$$

Putting the previous two equations together we obtain that

$$2\left\langle \nabla_{e_i}V,e_j\right\rangle = 2rF_{ij} \qquad (7.13)$$

and hence the expression for $\nabla_{e_i}V$.

The expression for $\nabla_V e_i$, and the coefficient of V in $\nabla_{e_i}e_j$, are obtained in a similar manner. The coefficients for the e_k in $\nabla_{e_i}e_j$ follow simply because

$$\langle [e_i,e_j],e_k\rangle = \overline{g_{\Sigma}(\widetilde{[e_1,e_j]},e_k)}. \qquad (7.14)$$

This completes the proof.

□

From now on we denote the Clifford algebra elements by $e_1 := \partial_r$ and $e_2 := V$ to avoid any confusion.

The above expressions imply the form of the spinor derivative

Lemma 7.2.2

$$\nabla_{\partial_r}\overline{\phi} = 0 \qquad (7.15)$$

$$\nabla_V\overline{\phi} = \left(\frac{rA}{2}+\frac{e_{12}.}{2r}\right)\overline{\phi} \qquad (7.16)$$

$$\nabla_{e_i}\overline{\phi} = \overline{\left(\widetilde{\nabla^{\Sigma}_{e_i}}\phi\right)} + \frac{r}{2}\sum_k F_{ik}\ V.e_k.\overline{\phi} \qquad (7.17)$$

Proof Equation (7.15) holds since $\overline{\phi}$ is obtained from ϕ by radial parallel translation. By the definition of the spinor derivative

$$\begin{aligned}
\nabla_V\left(\overline{\phi}\right) = \frac{1}{2}\sum_{1\leqslant j<k\leqslant 2n-2} e_{jk}\left\langle \nabla_V e_j,e_k\right\rangle.\overline{\phi} + \frac{1}{2}\sum_{1\leqslant k\leqslant 2n-2}\partial_r.e_k\left\langle \nabla_V\partial_r,e_k\right\rangle.\overline{\phi}\\
+\frac{1}{2}\sum_{1\leqslant k\leqslant 2n-2}V.e_k\left\langle \nabla_V V,e_k\right\rangle.\overline{\phi} + \frac{1}{2}\partial_r.V\left\langle \nabla_V\partial_r,V\right\rangle.\overline{\phi}. \qquad (7.18)
\end{aligned}$$

By the previous lemma this equals

$$\frac{r}{2}\sum_{j<k}e_{jk}F_{jk}\overline{\phi} + \frac{1}{2r}\partial_r.V.\overline{\phi} = \left(\frac{rA}{2}+\frac{e_{12}}{2r}\right).\overline{\phi}. \qquad (7.19)$$

Similarly

$$\begin{aligned}
\nabla_{e_i}\overline{\phi} &= \frac{1}{2}\sum_{3\leqslant j<k\leqslant 2n} e_{jk}\left\langle \nabla_{e_i}e_j,e_k\right\rangle.\overline{\phi}\\
&+\frac{1}{2}\sum_{3\leqslant k\leqslant 2n} e_{2k}\left(\nabla_{e_i}V,e_k\right).\overline{\phi} + \frac{1}{2}e_{12}.\left\langle \nabla_{e_i}\partial_r,V\right\rangle.\overline{\phi}\\
&= \frac{1}{2}\sum_{3\leqslant j<k\leqslant 2n} e_{jk}\left\langle \overline{\widetilde{\nabla_{e_i}e_j}},e_k\right\rangle + \frac{1}{2}\sum_{3\leqslant k\leqslant 2n} V.e_k.F_{ik}\overline{\phi}. \qquad (7.20)
\end{aligned}$$

The first term is just $\overline{\nabla^{\Sigma}_{e_i}\phi}$ which completes the proof. □

We record for completeness that the following identities also hold.

Lemma 7.2.3

$$\overline{[e_i,e_k]} = [e_i,e_k] + 2\left(r-1\right)F_{ik}.e_1 \qquad (7.21)$$

$$\overline{[e_i,\widetilde{V}]} = r\left[e_i,V\right] \qquad (7.22)$$

$$\nabla_V A = [\nabla_V,A] = [e_1.\nabla_V,A] = 0 \qquad (7.23)$$

$$\nabla_{\partial_r}A = [\nabla_{\partial_r},A] = [e_0.\nabla_{\partial_r},A] = 0 \qquad (7.24)$$

$$A^2 = 2\sum_{a<b<c<d}e_{abcd}F_{ab}F_{cd} - \sum_{a<b}F^2_{ab} \qquad (7.25)$$

Proof All these identities are elementary, for example,

$$\begin{aligned}\overline{[e_i.e_k]} &= \overline{\tilde{\nabla}_{e_i} e_k} - \overline{\tilde{\nabla}_{e_k} e_i} \\ &= \nabla_{e_i} e_k - \nabla_{e_k} e_i + 2(r-1) F_{ik}.V \\ &= [e_i, e_k] + 2(r-1) F_{ik}.V \end{aligned} \tag{7.26}$$

proves equation (7.21) and (7.22) follows similarly. Equation (7.23) holds since

$$\begin{aligned}(\nabla_V A) &= \nabla_V \left(\sum_{j<k} e_{jk} F_{jk} \right) = \sum_{j<k} (\nabla_V e_j)\, e_k F_{jk} + e_j\, (\nabla_V e_k)\, F_{jk} \\ &= \frac{1}{2} \sum_{j,k,p} e_{pk} F_{jk} \left(F_{jp} + F_{pj} \right) = 0. \end{aligned} \tag{7.27}$$

Expression (7.24) is straightforward. Lastly, equation (7.25) follows from the symmetries of the Clifford algebra and the curvature tensor.

□

We can write any spinor on $U \times \Sigma$ locally as $f(r,\theta)\,\overline{\phi}$ where ϕ is a spinor on Σ lifted to the tubular neighbourhood and extended by parallel translation. Thus the lemmas above give the form of the Dirac operator

Lemma 7.2.4

$$\nabla_{\partial_r} = \partial_r \quad \nabla_V = \left(\frac{\partial_\theta}{r} + \frac{rA}{2} + \frac{e_0.e_1.}{2r} \right) \quad \nabla_{e_i} = \left(\overline{\widetilde{\nabla^{\Sigma}_{e_i}}} + \frac{r}{2} e_1. \sum_k F_{ik}\; e_k. \right) \tag{7.28}$$

$$\not{D} = e_1. \left(\partial_r + \frac{1}{2r} \right) + e_2. \left(\frac{\partial_\theta}{r} - \frac{rA.}{2} \right) + \overline{\widetilde{D_\Sigma}} \tag{7.29}$$

7.2.2 Spinor Laplacian

Using the results of Section 7.2.1 we may deduce the form of the spinor Laplacian.

Lemma 7.2.5

$$\not{D}^2 = \overline{\widetilde{D_\Sigma} \widetilde{D_\Sigma}} - \left(\partial_r + \frac{1}{2r} \right)^2 - \left(\frac{\partial_\theta}{r} - \frac{rA}{2} \right)^2 \tag{7.30}$$

$$- \frac{e_{12}.}{r} \left(\frac{\partial_\theta}{r} + \frac{rA}{2} \right) + \frac{re_2}{2} [\not{D}, A]. \tag{7.31}$$

Proof It is a straightforward application of the previous Lemmas that

$$\begin{aligned} e_1 \nabla_{e_1} \left(\not{D} f \overline{\phi} \right) = &- \left(f'' + \frac{f'}{2r} - \frac{f}{2r^2} \right) \overline{\phi} + \\ & e_{12} \left(\frac{\partial_\theta f'}{r} - \frac{\partial_\theta f}{r^2} - \frac{(f'r+f)}{2} A \right) \overline{\phi} + e_1.f' \left(\overline{\widetilde{\not{D}_\Sigma} \phi} \right) \end{aligned} \tag{7.32}$$

$$\begin{aligned} e_2 \nabla_{e_2} \left(\not{D} f \overline{\phi} \right) = &- \left(\frac{f'}{2r} + \frac{f}{4r^2} + \frac{\partial_\theta^2 f}{r^2} - \frac{r^2 A^2 f}{4} \right) \overline{\phi} + e_1 \frac{f}{2r} \overline{\widetilde{\not{D}_\Sigma} \phi} \\ &- e_{12} \left(\frac{\partial_\theta f'}{r} + \frac{rf'A}{2} + \frac{A.f}{2} \right) \overline{\phi} + e_2 \left(\frac{\partial_\theta f}{r} + \frac{rA.f}{2} \right) \overline{\widetilde{\not{D}_\Sigma} \phi} \end{aligned} \tag{7.33}$$

$$\begin{aligned} e_i \nabla_{e_i} \left(\not{D} f \overline{\phi} \right) = &\left(\partial_\theta f.A - \frac{r^2.A^2.f}{2} + f \overline{\widetilde{\not{D}_\Sigma} \widetilde{\not{D}_\Sigma}} \right) \overline{\phi} - e_1 \left(f' + \frac{f}{2r} \right) \overline{\widetilde{\not{D}_\Sigma} \phi} \\ &+ e_2 \frac{rf}{2} [\not{D}, A] \overline{\phi} - e_2 \left(\frac{\partial_\theta f}{r} + \frac{rf.A}{2} \right) \overline{\widetilde{\not{D}_\Sigma} \phi} + e_{12} \left(rf' + \frac{f}{2} \right) A.\overline{\phi}. \end{aligned} \tag{7.34}$$

Putting these three expressions together completes the proof of the Lemma. □

Let P_m denote projection onto the m eigenspace of the Lie derivative of the vector field induced from the U_1 action. Substituting $r = e^{-s}$ we may rewrite these last two expressions as

Lemma 7.2.6

$$\not{D} \circ P_m = \overline{\widetilde{D_\Sigma}} - e^s e_1. \left(\partial_s - \frac{1}{2} \right) + ie^s e_2 \left(m + \frac{ie^{-2s} A}{2} \right) \tag{7.35}$$

$$\begin{aligned} \not{D}^2 \circ P_m = &\overline{\widetilde{D_\Sigma} \widetilde{D_\Sigma}} - e^{2s} \partial_s^2 + e^{2s} \left(m - \frac{ie_1.e_2}{2} \right)^2 \\ &+ i \left(m + \frac{ie_1.e_2}{2} \right) A - \frac{e^{-2s} A^2}{4} + \frac{e_2 e^{-s}}{2} [\not{D}, A] \end{aligned} \tag{7.36}$$

When the normal bundle is trivial $A \equiv 0$ and we return to the product case of equation (5.9).

Chapter 8

Conjectures

The preliminary calculations in Chapter 7 should allow us to prove an index theorem for non-extending spin structures on $M^{2n} - \Sigma^{2n-2}$ when Σ has non trivial normal bundle.

This has not yet been carried out successfully by the author.

Using the index theorem for the trivial normal bundle case as a model we may, however, deduce enough qualitative behaviour to conjecture the form of such a theorem. This is Conjecture 8.1.1. Having argued why this theorem should hold we then present two simple consequences of it. Firstly we show that in four dimensions the conjectured theorem gives a simple, analytic, proof of the unorientable Rohlin theorem of Guillou and Marin [GM80]. Secondly we show how to obtain indices for the Dirac operator on $\mathbb{C}P^2 - \mathbb{R}P^2$ twisted with half powers of the hyperplane bundle. This is a 'model' case for the S-duality test numbers of [SSZ95] on the reduced two monopole moduli space and was one of the primary motivations for this work. We make no attempt to explain why these numbers agree with those in [SSZ95], other than to remark that there is an SO_3 equivariant diffeomorphism from $\mathbb{C}P^2 - \mathbb{R}P^2$ to the two monopole moduli space [Hit95].

8.1 Main Conjecture

The main conjecture of this chapter is the following theorem.

Conjectured Theorem 8.1.1 Let M^{2n} be a smooth compact manifold with smooth metric and let $i : \Sigma \longrightarrow M$ be a closed embedded codimension two submanifold. Let F be any complex line bundle over M and let $M - \Sigma$ have a non-extending spin structure. Denote the Dirac operator associated to the non-extending spin structure, coupled to F by $\not{D}_{\otimes F}$. The index of this operator, closed as in Section 5.3, is then **conjectured** to be

$$\text{index } \not{D}_{\otimes F} = \int_M \widehat{A}(M)\mathrm{Ch}(F) + \frac{1}{2}\int_\Sigma \widehat{A}(\Sigma)\mathrm{Ch}(i^*F)\left(\tanh(\nu/4) + 1\right). \tag{8.1}$$

Here $\widehat{A}$ represents the $\widehat{A}$ characteristic class, Ch denotes the Chern character and ν denotes the Euler class of the normal bundle of Σ in M. When Σ is unorientable the integral over Σ is interpreted as the integral of the appropriate form, invariant under the natural involution, on a cover of Σ.

Remark 8.1.1 When Σ is orientable then the index above is just the $Spin^{\mathbb{C}}$ index for $\not{D}_{\otimes\nu/2\otimes F}$. This follows in a manner similar to Lemma 6.5.6.

Proof Recalling equation (6.62) shows that

$$\frac{1}{2}\left(\tanh\nu/4 + 1\right)\widehat{A}(\Sigma^{n-2})[\Sigma^{n-2}] = j^*\left(\frac{(e^{\nu/2} - 1)}{\nu}\widehat{A}(M^n)\right)[\Sigma^{n-2}] \tag{8.2}$$

which proves this remark. □

8.2 Arguments for the Conjecture

In this section we present the principal arguments that we use to justify Conjecture 8.1.1. These stem from the preliminary calculations in Chapter 7 and the proof of the index theorem for the trivial normal bundle case in Section 6.

8.2.1 Qualitative Behaviour

We start by considering the untwisted Dirac operator and a non-extending spin structure on $M - \Sigma$ where Σ is spin. The expression for the Dirac operator (7.29) differs from the trivial normal bundle case only in the addition of a single term involving Clifford multiplication by the Euler form of the normal bundle. This term is of order $O(r)$ and thus should give growth conditions for L^2 harmonic spinors near Σ similar to those exhibited in Section 5.2.1. More importantly, we still have the local decomposition of the spinor bundle (5.6) and may thus still consider the closure of the Dirac operator described in Section 5.3. This operator will again by self adjoint.

The fact that our operator is now an $O(r)$ perturbation of the trivial bundle case shows that the index of this closure is Fredholm by appealing to the work of Bruning and Seeley, as in Section 6.2. Moreover, we should again be able to calculate the index by approximating the heat kernels over a compact set in $M - \Sigma$ and a tubular neighbourhood of Σ. The standard pointwise expression for the heat kernel on the compact piece would contribute the first term in (8.1). The expression for the spinor Laplacian (7.31) could then be used to approximate the heat kernel close to Σ. The details of this contribution are argued below. These are the qualitative arguments for proving an index theorem in the non trivial normal bundle case.

8.2.2 Small Eigenvalues

The form of the spinor Laplacian (7.31) is of an $O(r)$ perturbation of the trivial normal bundle Laplacian (5.9) plus the operator

$$i\left(m + \frac{ie_{12}}{2}\right)A. \tag{8.3}$$

Recall that A involves Clifford multiplication by the Euler form of the normal bundle.

It is natural to suggest that we can approximate a heat kernel for this Laplacian, to an arbitrary order of t, by ignoring the terms of order r. **Recall that in the trivial normal bundle case the contribution from the heat kernel near the submanifold came solely from the small eigenvalues of L_V (see Section 6.3).**

Precisely these were the $n = -1, 0$ and 1 eigenspaces of L_V (see equation (6.20)). The contributions from these eigenvalues were the only ones which were supported on Σ as $t \longrightarrow 0$.

A cursory assessment of the heat kernel for (7.36) suggests similar behaviour. These small eigenvalues are the **only** difference between the two parts of the direct sum in the domain of the Dirac operator. The Removable Singularities Theorem 3.1.1 suggests that the terms which lie outside the small eigenspaces will contribute nothing to the index as they did in the trivial normal bundle case.

Schematically therefore the contribution of the heat kernel near to Σ should be

$$\frac{1}{2}\int_{\Sigma} \widehat{A}(\Sigma)P(\nu), \tag{8.4}$$

where $P(\nu)$ is a polynomial in the Euler class of the normal bundle of Σ.

The Form of the Polynomial

To determine the coefficients of the polynomial $P(\nu)$ the author considered the cases $M - \Sigma = \mathbb{C}P^{2n} - \mathbb{C}P^{2n-1}$ and concluded that the first four coefficients were most likely to be

$$2^2(2^2-1)\frac{B_1}{2!}\left(\frac{\nu}{4}\right) - 2^4(2^4-1)\frac{B_2}{4!}\left(\frac{\nu}{4}\right)^3 + 2^6(2^6-1)\frac{B_3}{6!}\left(\frac{\nu}{4}\right)^5 - 2^8(2^8-1)\frac{B_4}{8!}\left(\frac{\nu}{4}\right)^7 + \ldots \tag{8.5}$$

These are recognisable as the coefficients generated by the power series $\tanh \nu/4$, see [MS74, p.282], where the B_i are the Bernoulli numbers as described in [MS74, p.281]. In light of Remark 8.1.1 it is obvious that the resulting expression always gives an integer for (8.1) in the case where Σ is orientable.

As in Remark 6.5.1 a rigourous proof of the above argument using heat kernel methods would extend, with suitable modifications, to the case when Σ is unorientable. By comparison with Section 6.5.2 the extension of this argument to twistings of the Dirac operator, associated to the non-extending spin structure, by a complex line bundle over M would complete a proof of Conjecture 8.1.1.

Conjecture 8.2.1 The observation that the contribution from the kernel near the submanifold should come only from the small eigenspaces of L_V (those terms that have support on Σ as $t \longrightarrow 0$) may extend the scope of the conjectured theorem further. All of the topological comments in Chapter 2 remain true when Σ is no longer smooth but is a singular divisor [CG75, p.86]. The heat kernel analysis should then have parallels in this case. Further preliminary calculations would however need to be made to see if the presence of singularities contributed significantly to the form of any index of a non-extending spin structure.

8.3 Four Dimensions

In four dimensions we may argue the form of the untwisted index in Theorem 8.1.1 by following the work of Krönheimer and Mrowka.

8.3.1 Excision and Surgery

In the work of Krönheimer and Mrowka [KM93] the index theorem they obtain is constructed by appealing to an excision argument and then performing surgery on the manifold $M^4 - \Sigma^2$ to show that certain characteristic classes enter linearly in the index formula. By explicit calculation of the index in several examples the coefficients of these linear terms are then established.

In the case of four dimensions we could pursue a similar argument for the index of a non-extending spin structure when Σ has non-trivial normal bundle. Krönheimer and Mrowka [KM93, p.809] say that the proof of excision in [DK90] carries over, from manifolds to orbifolds. Hence they can apply it to the index of their holonomic singular operator on $M^4 - \Sigma^2$. Because of this result we **should** be able to prove that an excision result holds for the closure of our operator. Explicitly (compare Krönheimer and Mrowka [KM93, p.809]) we assume the following Lemma.

Conjectured Lemma 8.3.1 Denote the index for a non-extending spin structure on $M^4 - \Sigma^2$ by $i(M, \Sigma)$ denote the $\widehat{A}$ genus of M^4 by $i(M)$. Consider the manifolds $M_1 - \Sigma_1$ and $M_2 - \Sigma_2$ and assume that both have non-extending spin structures. The difference

$$i(M_i, \Sigma_i) - i(M_i) \tag{8.6}$$

is the same provided that the surfaces Σ_1 and Σ_2 have diffeomorphic neighbourhoods or equivalently the same genus and self-intersection.

Appealing to this excision Lemma we may show that in four dimensions the non-extending spin structure index must depend linearly on the self intersection and genus of the submanifold.

Firstly we can form the connected sum $M^4 \# \mathbb{C}P^2$ and take the characteristic submanifold to be $\Sigma \# \mathbb{C}P^1$, see Figure 8.1. This increases the self intersection number of the submanifold by one whilst fixing its genus and decreasing the $\widehat{A}$ genus by $1/8$. Excision shows therefore that the self intersection number enters the expression $i(M, \Sigma)$ linearly with constant coefficient.

Similarly we may form the connected sum $M \# S^4$ and take characteristic submanifold $\Sigma \# T^2$ where T^2 is an embedded torus in S^4, see Figure 8.1. This increases the genus of the characteristic submanifold by one leaving its self intersection, and the $\widehat{A}$ genus of the

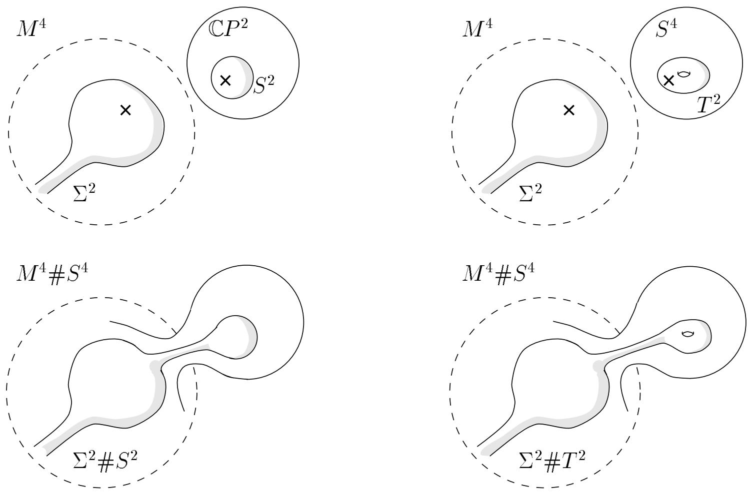


Figure 8.1: Surgery

four manifold, fixed. Excision shows that the genus enters the expression $i(M, \Sigma)$ linearly with constant coefficients. In conclusion if the excision result 8.3.1 holds then the shape of the index in four dimensions must be

$$i(M, \Sigma) = \widehat{A}(M) + c_0 \Sigma.\Sigma + c_1 g + c_2, \tag{8.7}$$

where the c_i are constants.

By considering the non-extending spin structure on $S^4 - S^2$ and the trivial normal bundle index Theorem 6.4.1 we see that $c_2 = 0$. Similarly, considering $S^2 - \{0, \infty\} \times T^2$ shows that $c_1 = 0$. Hence it remains only to determine c_0. By considering $\mathbb{C}P^2 - \mathbb{C}P^1$ we see that this constant must be of the form $1/8 + n$ where $n \in \mathbb{Z}$.

8.3.2 Obstructions to Surgery

Note that there is a topological obstruction here to performing surgery. We can form the connected sum $M^4\#\mathbb{C}P^2$ with characteristic submanifold $\Sigma\#\mathbb{C}P^1$ **only** because we consider both $M^4-\Sigma^2$ and $\mathbb{C}P^2-\mathbb{C}P^1$ as having non-extending spin structures. As these both restrict to the boundaries of transverse two discs to give the generator of Ω_1^{spin}, the patching of the spin structures can be made. This patching is however not always possible. Consider for example, as in [KM93], a neighbourhood U of a point $y\in\Sigma$. Take a small ball B inside U disjoint from Σ. Inside B, let T^2 be a torus in some standard embedding. We cannot modify Σ by joining it via a pipe to T^2 because the spin structure on $B-T^2$ will be extending. Thus it will not patch to a non-extending spin structure around Σ.

8.4 Examples

In this section we present two applications of the conjectured index theorem.

8.4.1 The Unorientable Rohlin Theorem

Conjecture 8.4.1 The Unorientable Rohlin Theorem of Guillou and Marin is the index theorem for non-extending spin structures in four dimensions reduced modulo the integers.

We work in four dimensions. Let $M^4-\Sigma^2$ have a non-extending spin structure. The conjectured form of the index theorem for non-extending spin structures says that,

$$\widehat{A}(M)[M]+\frac{1}{8}\int_{\Sigma}\epsilon_N\in\mathbb{Z}, \tag{8.8}$$

where ϵ_N is the Euler form of the normal bundle N_Σ of Σ. The Euler form ϵ_N naturally lies in $\Omega^2(\Sigma,\Lambda^2N)$. Since we have made no assumptions on the orientability of Σ we must interpret this integral appropriately.

Case One: Σ orientable

When Σ is orientable then so is N (since $M-\Sigma$ having a non-extending spin structure forces M to be orientable). In this case the integral has its usual meaning as Λ^2N is trivial. Hence we obtain that

$$\widehat{A}(M^4)+\frac{\Sigma.\Sigma}{8}\in\mathbb{Z} \tag{8.9}$$

where $\Sigma.\Sigma$ is the self intersection number of Σ. Since $8\widehat{A}(M^4)=-p_1(M)$ we re-derive the Rohlin Theorem in the same way one can from the $Spin^{\mathbb{C}}$ index theorem.

Case Two: Σ unorientable

When Σ is unorientable then we must consider the line bundle which the Euler form takes values in. Since Σ is unorientable we can only integrate forms which take values in the orientation bundle of Σ. This line bundle is $(\Lambda^2T_\Sigma)^*\otimes|\Lambda^2T_\Sigma|$.

The Riemannian metric on Σ obtained by restricting the smooth metric on M trivialises $|\Lambda^2T_\Sigma|$. This metric also identifies $\left(\Lambda^kT_\Sigma\right)^*$ with Λ^kT_Σ. Now $T_M|_\Sigma=T_\Sigma\oplus N_\Sigma$ so we see that

$$\left(\Lambda^2T_\Sigma\right)\otimes\left(\Lambda^2N_\Sigma\right)=\left(\Lambda^4T_M\right)|_\Sigma. \tag{8.10}$$

The orientation of M trivialises Λ^4T_M so

$$\Lambda^2N_\Sigma=\left(\Lambda^2T_\Sigma\right)^*=\Lambda^2T_\Sigma. \tag{8.11}$$

Thus Λ^2N_Σ can be identified with the orientation bundle of Σ and $\int_\Sigma\epsilon_N$ is well defined.

It is important to note that this identification is made using the orientation of M.

The integral $\int_\Sigma\epsilon_N$ can be evaluated by counting the number of zeros of a generic section of N counting multiplicity. This is because the multiplicity of a zero is defined without using an orientation (locally it is a map $N_y\longrightarrow N_y$).

Hence $\int_N \epsilon_N$ is equal to the number of zeros of a normal vector field to Σ. This is in turn equal to the self intersection of Σ in M. Thus the conjectured index theorem says, once again, that

$$\widehat{A}(M^4) + \frac{\Sigma.\Sigma}{8} \in \mathbb{Z}. \tag{8.12}$$

Here $\Sigma.\Sigma$ is the self intersection number (sometimes called the normal Euler number) of the unoriented submanifold Σ. Rearranging and reducing modulo eight gives that

$$\sigma(M) - \Sigma.\Sigma = 0 \mod 8, \tag{8.13}$$

where $\sigma(M)$ is the signature of M. This is the **unoriented** Rohlin Theorem of Guillou and Marin.

8.4.2 S-Duality Test Numbers

Example 8.4.2 We showed in Example 2.1.14 that $\mathbb{C}P^2 - \mathbb{R}P^2$ had a non-extending spin structure. We consider here indices for the Dirac operator twisted by the line bundle $L^n = H^{n/2}$ where H^n is the n^{th} power of the hyperplane bundle of $\mathbb{C}P^2$.

Case One: n **odd**

Because $K_{\mathbb{C}P^2} = H^{-3}$ the twisted Dirac operator is identified with

$$\begin{aligned} \not{D}_{L^n} &= \overline{\partial} + \overline{\partial}^* : \Lambda^{0,\text{even}} \otimes K^{1/2} \otimes L^n \longrightarrow \Lambda^{0,\text{odd}} \otimes K^{1/2} \otimes L^n \\ &= \overline{\partial} + \overline{\partial}^* : \Lambda^{0,\text{even}} \otimes H^{(n-3)/2} \longrightarrow \Lambda^{0,\text{odd}} \otimes H^{(n-3)/2}. \end{aligned} \tag{8.14}$$

When n is odd this is just the coupled Dirac operator associated to an extending $Spin^{\mathbb{C}}$ structure. The Removable Singularities Theorem 3.1.1 implies that for odd n

$$\text{index}\left(\not{D}^+_{L^n}|_{\mathbb{C}P^2 - \mathbb{R}P^2}\right) = \text{index}\left(\not{D}^+_{L^n}|_{\mathbb{C}P^2}\right) \tag{8.15}$$

where the index on the right is a $Spin^{\mathbb{C}}$ index on $\mathbb{C}P^2$. The Atiyah-Singer Index Theorem for a $Spin^{\mathbb{C}}$ manifold M^4 with associated line bundle L says that

$$\text{Index}\left(\not{D}^+_L\right) = \int_M \left(\frac{1}{2}c_1(L)^2 - \frac{1}{24}p_1(T_M)\right). \tag{8.16}$$

Hence

$$\begin{aligned} \text{Index}\left(\not{D}^+_{L^n}\right) &= \int_{\mathbb{C}P^2} \frac{n^2}{8}.c_1(H)^2 + \frac{1}{24}\left(6c_1(H)^2 - 9c_1(H)^2\right) \\ &= \left(\frac{n^2-1}{8}\right)\int_{\mathbb{C}P^2} c_2(H)^2 = \left(\frac{n^2-1}{8}\right) \end{aligned} \tag{8.17}$$

for all odd n.

Case Two: $n = 2 \mod 4$

When n is even, the index we consider is the index of the Dirac operator associated to the non-extending spin structure on $\mathbb{C}P^2 - \mathbb{R}P^2$ coupled to H^m where $m = 2n$.

We consider first the term $c_1(i^*H^m) = i^*(mc_1(H))$ where i^* is the homomorphism

$$\begin{aligned} i^* : H^2(\mathbb{C}P^2, \mathbb{Z}) &\longrightarrow H^2(\mathbb{R}P^2, \mathbb{Z}) \\ \mathbb{Z} &\longrightarrow \mathbb{Z}/2\mathbb{Z} \end{aligned} \tag{8.18}$$

induced by the inclusion $i : \mathbb{R}P^2 \longrightarrow \mathbb{C}P^2$. It is straightforward to show that

$$i^*c_1(H) = 1. \tag{8.19}$$

We do this as follows. Firstly it is enough to consider the tautological bundle H^* since $i^*c_1(H^*) = -i^*c_1(H) = i^*c_1(H)$ in $\mathbb{Z}/2\mathbb{Z}$. The complex line bundle H^* is the line bundle whose fibre over the point $[a,b,c] \in \mathbb{R}P^2$ is (za, zb, zc) for all $z \in \mathbb{C}$. This is simply the complexification $T_{\mathbb{C}}$ of the real tautological bundle T of $\mathbb{R}P^2$. Now the first Chern class of $T_{\mathbb{C}}$ is the Euler class of the underlying real bundle. The realisation of $T_{\mathbb{C}}$ is simply $T + T$ and thus it remains only to calculate the Euler number $e(T+T)$ to establish equation (8.19).

We calculate the Euler number of $T+T$ by counting the number of zeros of a generic section. A point of $\mathbb{R}P^2$ is a straight line l in $\mathbb{R}^3$ through zero. Thus a point in the the fibre of $T+T$, above l, is a pair of points of l.

Construct a section of $T+T$ by fixing two points, x and y in $\mathbb{R}^3$ such that x, y and 0 are not collinear, and projecting them onto l to get the two required elements of l. This section has zeros precisely where the projections of x and y are zero.

There is a unique such line l through zero which gives a zero of this section. Denote the line segments from zero to x and from zero to y by A and B. The unique line which gives a zero of the section is the intersection of the two planes, through zero, for which A and B are defining perpendiculars. As $H^2(\mathbb{R}P^2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ the sign of this zero is irrelevant. Hence equation (8.19) is proved and we see from this that

$$i^* c_1(H^m) = \begin{cases} 0 & m \text{ even} \\ 1 & m \text{ odd.} \end{cases} \tag{8.20}$$

The case where m is even corresponds to $n = 0$ mod 4. Appealing to the conjectured form of our index theorem we obtain that the index of $\not{D}_{L^n}$ for this value of n is simply

$$\frac{n^2}{8} - \frac{1}{8} + \frac{1}{8}\int_{\mathbb{R}P^2} \epsilon_N \tag{8.21}$$

where ϵ_N is the Euler form of the normal bundle of $\mathbb{R}P^2$ in $\mathbb{C}P^2$. As in 8.4.1, since N is unorientable ϵ_N takes values in the orientation bundle of $\mathbb{R}P^2$ namely $\Lambda^2 T_{\mathbb{R}P^2}$. Therefore, the integral over $\mathbb{R}P^2$ in equation (8.21) is equal to the number of zeros of a normal vector field to $\mathbb{R}P^2$ (the 'normal Euler number' or 'self intersection' of $\mathbb{R}P^2$ in $\mathbb{C}P^2$).

For the standard embedding $\mathbb{R}P^2 \longrightarrow \mathbb{C}P^2$ multiplication by i defines $T_{\mathbb{R}P^2} = N_{\mathbb{R}P^2}$ so the number of zeros of a normal vector field to $\mathbb{R}P^2$ equals the number of zeros of a tangent vector field. This is just the Euler number of $\mathbb{R}P^2$ and $\chi_{\mathbb{R}P^2} = 1$.

Hence, by the conjectured form of the index theorem, when $n = 0 \mod 4$ the index of $\not{D}_{L^n}$ for the non-extending spin structure equals $n^2/8$.

Chapter 9

The Sen Form

In this chapter we study another Dirac operator, the signature operator, on a certain complete manifold known as the reduced two monopole moduli space.

The main result of this chapter is that this manifold possesses a L^2 harmonic two form which is unique up to a constant multiple. This form is known for physical reasons as the Sen form of this space.

This is shown by using a vanishing theorem to prove that the L^2 index of the signature operator counts the number of L^2 harmonic forms. We then show that this index is the index of a Fredholm operator and lastly appeal to a known index theorem to calculate it. This final calculation is in the same spirit as those in [SSZ95].

9.1 Background

In [Sen94] Sen proposed, among other things, the existence of a unique square summable harmonic form on the reduced moduli space of SU_2 monopoles with magnetic charge two. We denote this space by M_2^0. The Atiyah-Hitchin manifold, $\widetilde{M}_2^0$, is a complete hyperkähler manifold whose metric has been explicitly calculated [AH88]. The metric can be expressed in the following form

$$ds^2 = f(r)^2 \, dr^2 + a_1(r)^2 \, \sigma_1^2 + a_2(r)^2 \, \sigma_2^2 + a_3(r)^2 \, \sigma_3^2, \tag{9.1}$$

Here the σ_i are a dual frame to the usual basis $\{X_1, X_2, X_3\}$ for the Lie algebra of SO_3. These left invariant vector fields satisfy $[X_i, X_j] = -\epsilon_{ijk} X_k$ and their duals satisfy $d\sigma_i = \frac{1}{2}\sigma_j \wedge \sigma_k$. Explicitly the σ_i can be expressed by the SO_3 Euler angles, $0 \leqslant \theta < \pi$, $0 \leqslant \varphi < 2\pi$, $0 \leqslant \psi < 2\pi$ as

$$\begin{aligned} \sigma_1 &= -\sin\psi \; d\theta + \cos\psi \; \sin\theta \; d\varphi, \\ \sigma_2 &= \cos\psi \; d\theta + \sin\psi \; \sin\theta \; d\varphi, \\ \sigma_3 &= d\psi + \cos\theta \; d\varphi. \end{aligned} \tag{9.2}$$

The space M_2^0 is the quotient of $\widetilde{M}_2^0$ under the identification

$$(r, \theta, \varphi, \psi) = (r, \pi - \theta, \varphi + \pi, -\psi) \, . \tag{9.3}$$

Since the manifold $\widetilde{M}_2^0$ is hyperkähler its curvature is anti self dual. Consequently, the coefficients of the metric satisfy the equation [AH88]

$$\frac{2a_2a_3}{f}\frac{da_1}{dr} = (a_2 - a_3)^2 - a_1^2, \tag{9.4}$$

as well as the equations obtained by cyclically permuting a_1, a_2 and a_3.
We may [GM86a] take $f = -a_2/r$ with the range of the radial coordinate r being from π to ∞. With this choice of radial coordinate the coefficients of the metric have the following large r asymptotics [GM86a]

$$\begin{aligned} a_1 = a_2 &= r\sqrt{1 - \frac{2}{r}} + O\left(e^{-r}\right), \\ a_3 &= -\frac{2}{\sqrt{1 - 2/r}} + O\left(e^{-r}\right). \end{aligned} \tag{9.5}$$

Near the bolt coordinate singularity at $r = \pi$ the coefficients are, up to terms of order e^{-r}, asymptotically [GM86a]

$$a_1 = 2(r - \pi), \qquad a_2 = \frac{1}{2}(r - \pi) + \pi, \qquad a_3 = \frac{1}{2}(r - \pi) - \pi. \tag{9.6}$$

9.2 Set Up

In this section we use a vanishing theorem and a rescaling argument to show that the number of L^2 harmonic forms on M_2^0 can be calculated as the index of a Fredholm operator. This is the content of Sections 9.2.1 and 9.2.2.

9.2.1 Vanishing Theorems

To show the uniqueness of the Sen form on M_2^0 we consider the index of the signature operator $d + d^*$ which maps between the bundle of self dual forms Ω_+ and the bundle of anti self dual forms Ω_-. Recall that the bundles $\Omega_\pm$ are the eigenbundles of the complex volume form acting by Clifford multiplication on the space of forms Ω^*. This operator acts as $i^{p(p-1)}*$ on p forms. We study the L^2 index of $d + d^*$, namely

$$L^2 \text{ index } (d + d^*)_+ = \dim\ker(d + d^*)_+ \cap L^2 - \dim\ker(d + d^*)_- \cap L^2, \tag{9.7}$$

where $(d + d^*)_\pm$ is the restriction of $d + d^*$ to $\Omega_\pm$. We have the following Lemma.

Lemma 9.2.1 The manifold $\widetilde{M}_2^0$ possesses no L^2 harmonic forms which are not two forms. The L^2 index (9.7) of the signature operator on $\widetilde{M}_2^0$ is invariant under conformal rescaling to other complete metrics. Furthermore, this L^2 index counts the number of middle dimensional L^2 anti self dual harmonic forms on $\widetilde{M}_2^0$.
Proof Since the manifold $\widetilde{M}_2^0$ is hyperkähler its Ricci tensor vanishes. Thus, the general Bochner identity for the Laplace-Beltrami operator on all but the two forms reduces to

$$(d + \delta)^2 = \nabla^*\nabla. \tag{9.8}$$

Consequently, any L^2 harmonic form which is not a two form must be covariantly constant. Katos' inequality and the fact that the manifold has infinite volume then tell us that any L^2 harmonic form on $\widetilde{M}_2^0$ must be a two form.

Denote the space of L^2 harmonic two forms in $\Omega_\pm$ by $H^2_\pm$. For two forms the L^2 condition is independent of the conformal class of the metric. Furthermore, since the manifold is complete a form is harmonic if and only if it is both closed and coclosed [dR84, p.158]. As a result, the L^2 index of the signature operator (9.7) equals

$$\dim H^2_+ - \dim H^2_-, \tag{9.9}$$

and is invariant for all **complete** metrics conformally equivalent to the Atiyah-Hitchin metric (9.1). Denote $\dim H^2_\pm$ by $h^2_\pm$.

The Atiyah-Hitchin metric (9.1) is anti self dual so the Weyl tensor vanishes, as an endomorphism acting on self dual two forms. Using the same Bochner argument, as above, we see from this that there can be no self dual L^2 harmonic forms on $\widetilde{M_2^0}$. Thus $h^2_+ = 0$, and the L^2 index of the signature operator on $\widetilde{M_2^0}$ will count the number of middle dimension square summable harmonic forms. These harmonic forms will necessarily be anti self dual by the choice of orientation. $\square$

Remark 9.2.2 It is clear that Lemma 9.2.1 holds for any complete hyperkähler manifold with unbounded volume.

9.2.2 Fredholm Framework.

For equation (9.9) to have meaning we must check that the vector spaces $H^2_\pm$ are finite dimensional. We show this by exploiting the invariance of these spaces under conformal rescaling to other complete metrics.

Firstly we rescale the metric on $\widetilde{M_2^0}$ for large r. This rescaled metric has the advantage that a particularly neat differential inequality (9.2.5) holds. From this inequality it is then standard to show that the kernel and cokernel of the signature operator on $\widetilde{M_2^0}$ are finite dimensional. This is shown in Lemma 9.2.6. Lastly, we show that equation (9.9) is actually the index of a Fredholm operator. The results of this section are summarised in Corollary 9.2.8.

The rescaling we choose on $\widetilde{M_2^0}$ is to divide the coefficients of the Atiyah-Hitchin metric (9.1) by a_2^2 (as in [SSZ95, p.494]). This gives a metric of form,

$$\begin{aligned} ds^2 &= \frac{f(r)^2}{a_2^2} dr^2 + \sum_1^3 \frac{a_i^2}{a_2^2}\sigma_i^2 \\ &= \frac{dr^2}{r^2} + (1+\alpha_1)\,\sigma_1^2 + \sigma_2^2 + \left(\frac{4}{(r-2)^2} + \alpha_2\right)\sigma_3^2 \\ &= dt^2 + (1+\alpha_1)\,\sigma_1^2 + \sigma_2^2 + \left(\frac{4}{(e^t-2)^2} + \alpha_2\right)\sigma_3^2, \end{aligned} \tag{9.10}$$

by choosing a new radial coordinate $\exp(t) = r$. Here the α_i are doubly exponentially decreasing functions of t: they are of order $O(e^{-e^t})$.

We denote the manifold with this metric as M_r.

Remark 9.2.3 We need not worry whether this rescaling is well defined over the bolt of $\widetilde{M_2^0}$ for we may use a bump function, supported away from the bolt, to give a well defined rescaling which descends to M_2^0.

The manifold M_r will also be anti self dual, since this is a conformally invariant property. It is also straightforward to check that this manifold is complete.

Lemma 9.2.4 The manifold with the rescaled metric M_r is complete.

Proof The Hopf-Rinow theorem states that the manifold M_r is complete if and only if every bounded closed set is compact. There is a natural diffeomorphism between M_r and $\widetilde{M_2^0}$ which has the form $t \longrightarrow \ln t$ for large t. Since there is a non-zero lower bound on the coordinate t, and since it is known that $\widetilde{M_2^0}$ is complete [Tau85], this map will send closed and bounded sets to compact sets in $\widetilde{M_2^0}$. Hence every closed and bounded set in M_r is compact. $\square$

For the rescaled manifold M_r we have the following elementary inequality.

Lemma 9.2.5 Denote the L^2 norm on $M_{\rm r}$ by $\| \,.\, \|$ and the first Sobolev space by $L^2_1(M_{\rm r})$. Fix R_0 large and denote the region of $M_{\rm r}$ where $R_0 \leqslant r \leqslant R_0+1$ by A. Denote the L^2 norm over this region by $\| \,.\, \|_A$. Then there exists a positive constant, dependent on R_0, such that

$$\|\phi\|_A^2 + \|\nabla\phi\|^2 \geqslant \text{const.}\|\phi\|^2 \tag{9.11}$$

for all forms $\phi \in L^2_1(M_{\rm r})$.

Proof This estimate is similar to those in [PT82, p.234] and [Tau85, p.248]. The volume form of the rescaled metric $d\mu_{\rm r}$ has large r asymptotics like,

$$d\mu_{\rm r} = \frac{2dr\sigma_{123}}{r^2} + O(r^{-1}), \tag{9.12}$$

where the notational simplification $\sigma_i\sigma_j =: \sigma_{ij}$ has been invoked and the order refers to the point wise norm $|\,.\,|_r$ on the rescaled manifold. Consider a compactly supported form $\phi \in \Omega_0^*$ on $M_{\rm r}$ whose support lies in $M_{\rm r,\,End} := \{r \geqslant R_0\}$ for some suitably large R_0. Integration by parts gives that

$$\int \frac{|\phi|_r^2}{r^2} dr\sigma_{123} \leqslant 2\int \frac{|\phi|_r |d|\phi|_r|_r}{r^2} dr\sigma_{123}. \tag{9.13}$$

Using the asymptotic form of the metric, Hölders inequality and Katos' inequality it can be concluded that there exists a constant C_0 such that

$$\|\phi\|^2 \leqslant C_0\|\nabla\phi\|^2 \tag{9.14}$$

for all ϕ in $L^2_1(M_{\rm r})$ with support in $M_{\rm r,\,End}$. Here the constant depends only on R_0. Fix R_0 and choose a bump function β such that supp $\beta \subset M_{\rm r,\,End}$ and supp $d\beta \subset A$, where A is the region $\{R_0 \leqslant r \leqslant R_0+1\} \subset M_{\rm r,\,End}$. Assume also that $\beta = 1$ on $M_{\rm r,\,End} - A$.

Split any form $\phi \in L^2_1$ into 'internal' and 'external' parts thus

$$\phi = (1-\beta)\phi + \beta\phi =: \phi_{\rm in} + \phi_{\rm ex}\,. \tag{9.15}$$

It is immediate from this definition that

$$\|\phi\|^2 \leqslant \|\phi_{\rm in}\|^2 + \|\phi_{\rm ex}\|^2 + \|\phi\|_A^2. \tag{9.16}$$

Using equation (9.14) there exists a constant C_1 such that

$$\|\nabla\phi_{\rm ex}\|^2 \leqslant 2\left(\|d\beta\wedge\phi\|^2 + \|\beta.\nabla\phi\|^2\right) \leqslant C_1\left(\|\nabla\phi\|^2 + \|\phi\|_A\right). \tag{9.17}$$

Since $\phi_{\rm in}$ has compact support on $M_{\rm r}$, and the bilinear form $(\psi,\phi)\mapsto(\nabla\psi,\nabla\phi)$ is positive definite on Ω_0^*, standard elliptic theory (see [Mor66, § 5.2]) gives that there exists a constant $C_2 = C_2(R_0) > 0$ such that

$$\|\phi_{\rm in}\|^2 \leqslant C_2\|\nabla\phi_{\rm in}\|^2. \tag{9.18}$$

Estimates (9.18) and (9.14) can now be put together to give a global estimate. Firstly

$$\|\nabla\phi\|^2 = \|\nabla\phi_{\rm in}\|^2 + \|\nabla\phi_{\rm ex}\|^2 + 2(\nabla\phi_{\rm in}, \nabla\phi_{\rm ex}). \tag{9.19}$$

Equation (9.17) implies that

$$-2(\nabla\phi_{\rm in}, \nabla\phi_{\rm ex}) \leqslant \frac{1}{2}\|\nabla\phi_{\rm in}\|^2 + 2\|\nabla\phi_{\rm ex}\|^2 \leqslant \frac{1}{2}\|\nabla\phi_{\rm in}\|^2 + 2C_1\|\nabla\phi\|^2 + 2C_1\|\phi\|_A^2, \tag{9.20}$$

from which (9.14), (9.18) and (9.19) show that

$$\begin{aligned} 2C_1\|\phi\|_A^2 + (2C_1+1)\|\nabla\phi\|^2 &\geqslant \frac{1}{2}\|\nabla\phi_{\rm in}\|^2 + \|\nabla\phi_{\rm ex}\|^2 \\ &\geqslant \frac{1}{2C_2}\|\phi_{\rm in}\|^2 + \frac{1}{C_0}\|\phi_{\rm ex}\|^2. \end{aligned} \tag{9.21}$$

Using equation (9.16) completes the proof of the lemma. □

Lemma 9.2.5 is enough to enable us to show that there are only a finite number of harmonic forms on $\widetilde{M}_2^0$.

From now on we will denote the signature operator $d + d^*$ simply by D.

Lemma 9.2.6 The dimensions of the L^2 kernel and cokernel of the signature operator D on $\widetilde{M}_2^0$ are finite.

Proof By the vanishing criteria in Lemma 9.2.1 we need only consider the operator restricted to the middle dimensional forms $\Omega^2(\widetilde{M}_2^0)$. By the conformal invariance of this space it is enough to show that there are only a finite number of L^2 harmonic forms in $\Omega^2(M_{\mathrm{r}})$. On $\Omega^2(M_{\mathrm{r}})$ the Bochner identity reduces to [FU91, p.174]

$$D^2 = \nabla^*\nabla - 2W_- + R/3 \tag{9.22}$$

where W_- denotes the Weyl tensor and R the scalar curvature for the rescaled manifold. Thus for every harmonic form $\phi \in \ker D \cap L^2$ we see that

$$\nabla^*\nabla\phi = 2W_-\phi - R\phi/3. \tag{9.23}$$

A calculation shows that for large r on M_{r} that

$$|W_-| = O(r^{-1}) \quad \text{and} \quad |R| = O(r^{-1}), \tag{9.24}$$

where here the norms referred to are the operator norms of W_- and R as endomorphisms of Ω^*. It is immediate from this and (9.23) that,

$$\ker D \cap L^2 = \ker D \cap L_1^2, \tag{9.25}$$

so generality is not lost by assuming that $\|\nabla\phi\|^2 + \|\phi\|^2 = 1$ for our solution.

Using the decay of the scalar curvature and the Weyl tensor we see that there exist constants C_0 and C_1 such that,

$$\|\nabla\phi\|^2 \leqslant C_0\|\phi\|_{B_t}^2 + C_1 e^{-t}\left(1 - \|\phi\|_{B_t}^2\right), \tag{9.26}$$

where $\|\ .\ \|_{B_t}$ is the L^2 norm over the bounded set $\{r \leqslant e^t\}$ for some t. The constants may be chosen to be independent of t for $e^t \geqslant R_0$. From Lemma 9.2.5 we see that

$$1 - \|\phi\|_{B_t}^2 \leqslant 1 = \|\phi\|^2 + \|\nabla\phi\|^2 \leqslant \text{const.}\ \left(\|\nabla\phi\|^2 + \|\phi\|_A^2\right). \tag{9.27}$$

From equation (9.26) we see that there exists a constant C_2 such that

$$\|\nabla\phi\|^2 \leqslant C_0\|\phi\|_{B_t}^2 + \frac{C_2}{e^t}\left(\|\nabla\phi\|^2 + \|\phi\|_A^2\right). \tag{9.28}$$

Thus, by fixing $e^t \geqslant R_0$ such that $C_2 e^{-t} \leqslant 1/2$ we see that there exists a constant, dependent only on t, such that for all $\phi \in \ker D \cap L^2$

$$\|\nabla\phi\|^2 \leqslant C_3\|\phi\|_{B_{t+1}}^2. \tag{9.29}$$

We can now follow the strategy of Gromov and Lawson (see [GL83, p.327] and [LM89, p.314]) to complete the proof. Consider now the uniform C^r norm $\|\phi\|_{C^r,B_t}$ over B_t. Using equation (9.29), the Friedrichs Lemma [GL83, Theorem 3.5] and Lemma 9.2.5 there are constants, depending only on t, such that

$$\|\phi\|_{C^r,B_t} \leqslant C_4\|\phi\|_{B_{t+1}} \leqslant C_5\left(\|\nabla\phi\| + \|\phi\|_A\right) \leqslant C_6\|\phi\|_{B_t} \leqslant C_6. \tag{9.30}$$

Now for any $\epsilon > 0$ consider an ϵ net $\{x_i\}_{i=1}^d$ of the ball B_{t+1}. If $\dim\ker D \cap \Omega^2 \cap L^2(M_{\mathrm{r}}) > d$ then there exists an element $\psi \in \ker D \cap L^2$ such that $\psi(x_i) = 0$ for all $1 \leqslant i \leqslant d$ and $\|\nabla\psi\|^2 + \|\psi\|^2 = 1$. By the choice of ψ we have that $|\psi|_r \leqslant \epsilon.C_6$ on B_{t+1}. Thus $\|\psi\|_{B_{t+1}} \leqslant \epsilon.C_6|B_{t+1}| = \epsilon.C_7(t)$. Consequently, if $\ker D \cap \Omega^2 \cap L^2(M_{\mathrm{r}})$ is not finite then for all $\epsilon > 0$ there exists an $\psi \in \ker D \cap \Omega^2 \cap L^2(M_{\mathrm{r}})$ such that $\|\nabla\psi\|^2 + \|\psi\|^2 = 1$ and, by equation (9.29),

$$\|\nabla\psi\|^2 \leqslant \epsilon.\text{const}. \tag{9.31}$$

By equation (9.11) this contradicts that $\|\nabla\phi\|^2 + \|\phi\|^2 = 1$ showing that $\ker D \cap \mathrm{L}^2(M_{\mathrm{r}})$ is finite. □

Since the bounded operator $D : L_1^2(\widetilde{M}_2^0) \mapsto L^2(\widetilde{M}_2^0)$ has finite dimensional kernel and cokernel it only remains to show that it has closed range to show that D is Fredholm. This is an immediate consequence of the following lemma (see for example [Kat95, § IV.5.2]).

Lemma 9.2.7 There exists a positive constant such that $\|D\phi\| \geqslant$ const. $(\|\nabla\phi\| + \|\phi\|)$ for all $\phi \in L^2_1(M_{\mathrm{r}})$ which are L^2_1 orthogonal to $\ker D$.

Proof Suppose that such a constant does not exist. Then there exists a sequence $\{\phi_i\}_{i\in\mathbb{N}}$ such that

$$\|D\phi_i\| \to 0 \tag{9.32}$$

$$\langle \nabla\phi_i, \nabla\eta\rangle_2 + \langle \phi_i, \eta\rangle_2 = 0, \qquad \text{for all } \eta \in \ker D \cap L^2, \tag{9.33}$$

$$\|\nabla\phi_i\| + \|\phi_i\| = 1 \tag{9.34}$$

By equation (9.34) there exists an L^2_1 weakly convergent subsequence with weak limit ϕ. The Rellich lemma says that $\phi_i \to \phi$ strongly on any bounded set. Equation (9.32) gives that $D\phi = 0$ and thus equation (9.33) implies that $\phi = 0$. Recalling equation (9.29) and Lemma 9.2.5 we see that $\lim \|\nabla\phi_i\| + \|\phi_i\| = 0$ which contradicts equation (9.34). $\square$

We draw the results of this section together by stating

Corollary 9.2.8 The signature operator $D : L^2_1(\widetilde{M^0_2}) \longrightarrow L^2(\widetilde{M^0_2})$ on the complete manifold $\widetilde{M^0_2}$ is Fredholm. Its index is equal to $-h_-$ where h_- is the dimension of the space of L^2 anti self dual harmonic two forms on $\widetilde{M^0_2}$. Since $\widetilde{M^0_2}$ is the double cover of the reduced two monopole moduli space M^0_2 the index of $D : L^2_1(\widetilde{M^0_2}) \longrightarrow L^2(\widetilde{M^0_2})$ will equal twice the number of L^2 harmonic forms on M^0_2.

9.3 Index Formula

To establish the uniqueness of the Sen form on the manifold M^0_2 we are left to perform the calculation of the index of the signature operator on $\widetilde{M^0_2}$. To do this we recall the modifications of [Ste89] required to calculate an L^2 index on a non compact, complete manifold. **Henceforth any index referred to will be an L^2 index.**

To emphasise the similarity between this calculation and those in [SSZ95] we will actually perform the calculation on the rescaled manifold M_{r}. We may do this because of the conformal invariance of the index (9.9). This is explained in Remark 9.3.2. As M_{r} is complete we have the following theorem of Stern [Ste89, p.242]

Theorem 9.3.1 If the continuous spectrum of the signature operator D on M_{r} is bounded away from zero then

$$\text{index } D = \lim_{L\to\infty}\left(\lim_{s\to 0}\int_{t<L} \mathrm{tr}\tau e^{-sD^2} d\mu_{\mathrm{r}} - \int_0^\infty \int_{t=L} \mathrm{tr}c^0\tau D e^{-sD^2} d\sigma ds\right). \tag{9.35}$$

Here $d\mu_{\mathrm{r}}$ denotes the volume form of the rescaled metric, e^{-sD^2} the heat operator associated to D^2, τ Clifford multiplication by the volume element on the bundle of forms, c^0 Clifford multiplication by the one form dt and $d\sigma$ the induced volume form on the submanifold defined by $t = L$.

Remark 9.3.2 To show that the signature operator on the rescaled manifold M_{r} has continuous spectrum bounded away from zero we could argue in a manner akin to Lemma 9.2.7. Note, however, that since we have shown $D : L^2_1(\widetilde{M^0_2}) \longrightarrow L^2(\widetilde{M^0_2})$ to be Fredholm we can invoke the theorem for this operator. We saw in Lemma 9.2.1 that the left hand side of equation (9.35) was invariant under conformal rescaling to other complete metrics. We will see below that the first term of the right hand side of equation (9.35) is also conformally invariant. Consequently, we may conclude that the expression for the L^2 index of D is valid on M_{r}.

Calculating the value of the first term in (9.35) is straightforward. The local Atiyah-Singer index theorem says that

$$\lim_{L\to\infty}\left(\lim_{s\to 0}\int_{t<L} \mathrm{tr}\tau e^{-sD^2} d\mu_{\mathrm{r}}\right) = \frac{-1}{24\pi^2}\int \mathrm{Tr}\,(R\wedge R) = \frac{1}{24\pi^2}\int \mathrm{Tr}\,(R\wedge *R). \tag{9.36}$$

Here R denotes the curvature tensor of M_{r}. Since the integrand is conformally invariant we can calculate it simply by integrating the corresponding terms over the original Atiyah-Hitchin metric. We find (see also [SSZ95, GH96]) that

$$\mathrm{Tr}\,(R \wedge R) = d\left[\left(16ABC - 4\,(A+B+C-1)^2\right)\sigma_{123}\right], \tag{9.37}$$

where

$$\begin{aligned} A &= \left(a_2^2 + a_3^2 - a_1^2\right)/2a_2a_3 \\ B &= \left(a_3^2 + a_1^2 - a_2^2\right)/2a_1a_3 \\ C &= \left(a_1^2 + a_2^2 - a_3^2\right)/2a_1a_2. \end{aligned} \tag{9.38}$$

Using Stokes' theorem and the asymptotic formulas for the a_i, in equations (9.5) and (9.6), gives that

$$\frac{1}{24\pi^2}\int \mathrm{Tr}\,(R \wedge R) = -\frac{4}{3}. \tag{9.39}$$

This accords with the known fact [Dan94, GM86a] that the Atiyah-Hitchin manifold retracts onto the two sphere. Note that since we wish to evaluate the L^2 index of the signature operator on M_2^0 we have divided by a factor of two to account for the $\mathbb{Z}/2\mathbb{Z}$ identification. As in [SSZ95] to complete the index calculation we must calculate the defect term

$$\delta_D := \lim_{L\to\infty}\int_0^\infty \int_{t=L} \mathrm{trc}^0 \tau D e^{-sD^2} d\sigma ds. \tag{9.40}$$

9.3.1 Asymptotics

We now proceed, as in [SSZ95], to calculate the defect term. We start by calculating the asymptotic form of the signature operator on M_r. In Section 9.4 we use this asymptotic form of the operator to approximate a parametrix for the heat kernel e^{-tD^2} suitable for establishing (9.40). Finally, in Section 9.5.1 we use this approximation to explicitly evaluate (9.40) and prove the uniqueness of the Sen form.

Fix an orthonormal frame for the tangent bundle of M_r, specifically $Y_0 = \partial_t$, $Y_1 = (1+\alpha_3)X_1$, $Y_2 = X_2$ and $Y_3 = (e^t/2 - 1 + \alpha_4)X_3$. This frame does not descend under the identification (9.3). Here once again the α_i are doubly exponentially decreasing. Modulo these terms of order $O(e^{-e^t})$, we have the commutation relations (see also [SSZ95, p.495])

$$[Y_1, Y_2] = -\frac{2}{(e^t-2)}Y_3 \qquad [Y_3, Y_1] = -\frac{(e^t-2)}{2}Y_2 \tag{9.41}$$

$$[Y_2, Y_3] = -\frac{(e^t-2)}{2}Y_1 \qquad [Y_0, Y_3] = \frac{e^t}{(e^t-2)}Y_3 \tag{9.42}$$

From these relations we find that,

$$(\nabla_3 Y_1, Y_2) = -\frac{(e^t-2)}{2} + \frac{1}{(e^t-2)} \qquad (\nabla_1 Y_2, Y_3) = -\frac{1}{(e^t-2)} \tag{9.43}$$

$$(\nabla_2 Y_3, Y_1) = -\frac{1}{(e^t-2)} \qquad (\nabla_3 Y_0, Y_3) = -\frac{e^t}{(e^t-2)} \tag{9.44}$$

modulo doubly exponentially decreasing terms.

To calculate the defect term we will consider the bundle of forms Ω^* as a Clifford module. Denote the left Clifford action of Y_i^* by $c^i = \epsilon^i - \iota^i$ and another action by $b^i = \epsilon^i + \iota^i$. Here ϵ^i and ι^i are exterior and interior multiplication by Y_i^*. We have the commutation relations [BGV92, p.148]

$$\left\{c^i, c^j\right\} = -2\delta_{ij} \quad \{b^i, b^j\} = 2\delta_{ij} \quad \text{and} \quad \{b^i, c^j\} = 0. \tag{9.45}$$

The square of the signature operator can then be written with respect to the orthonormal basis $\{Y_i\}$ as [BGV92, p.130]

$$D^2 = -\sum_i \left(\nabla_i\nabla_i - \nabla_{\nabla_i i}\right) - \sum_{ijkl} R_{ijkl}\epsilon^k\iota^l\epsilon^i\iota^j. \tag{9.46}$$

It is a straightforward calculation to show that

$$D^2 = -\sum_i (\nabla_i\nabla_i) + \frac{e^t}{(e^t-2)}\nabla_0 + \frac{1}{2}\left(b^1b^2c^1c^2 - b^0b^3c^0c^3\right) \tag{9.47}$$

modulo terms of order $O(e^{-t})$.

We adopt the notational simplification $b^{ij} := b^i b^j$ and Fourier expand sections of $\Omega_\pm$ with respect to the circle action associated to Y_3 . Using the formula (9.47) and considering the m^{th} Fourier component another calculation shows that

$$D^2 = -\partial_t^2 - \nabla_1^2 - \nabla_2^2 + \frac{1}{2}\left(b^{12}c^{12} - b^{03}c^{03}\right) + \partial_t \\ - \left(\frac{e^t-2}{2}\right)^2 \left[im + \frac{e^t(b^{03}+c^{30})}{(e^t-2)^2} - \left(\frac{1}{(e^t-2)^2} - \frac{1}{2}\right)\left(b^{12}+c^{21}\right)\right]^2 \quad (9.48)$$

modulo terms of order $O(e^{-t})$.

For future reference we define the operator

$$K^2 = -\left(\frac{e^t-2}{2}\right)^2 \left[im - \left(\frac{1}{(e^t-2)^2} - \frac{1}{2}\right)\left(b^{12}+c^{21}\right)\right]^2 \quad (9.49)$$

and the subspace

$$Z = \ker\left(im + \frac{1}{2}(b^{12}+c^{21})\right)^2. \quad (9.50)$$

We also record that the signature operator may be written, up to terms of order $O(e^{-t})$, as

$$D = c^0\partial_t + c^1 X_1 + c^2 X_2 + c^3\left(\frac{e^t-2}{2}\right)\left[X_3 + \frac{1}{(e^t-2)}(b^{03}+c^{30}) - \frac{1}{2}(b^{21}+c^{12})\right]. \quad (9.51)$$

9.4 Parametrices

To calculate the defect term δ_D for the index we must construct a suitable approximation to the heat kernel e^{-sD^2}. In this section we construct such an approximation inductively on the subspace $Z^\perp$ following the techniques in [Ste89, p.257]. In Section 9.4.1 we show that very few terms in this approximation contribute to our calculation. Lastly, we argue in Section 9.4.2 that the subspace Z contributes nothing to δ_D.

First we use the functional calculus to express the heat kernel e^{-sD^2} as

$$\frac{1}{2\pi i}\int_\gamma e^{-s\lambda}\left(D^2-\lambda\right)^{-1} d\lambda, \quad (9.52)$$

where γ is a contour surrounding the spectrum of D^2 in $\mathbb{C}$. Thus to approximate e^{-sD^2} it is sufficient to approximate $\left(D^2-\lambda\right)^{-1}$. We do this inductively by considering our rescaled manifold as a non-trivial S^1 bundle over $[\pi,\infty)\times S^2$.

Consider a contractible open set in the base manifold with coordinate x. Then for each section ϕ of the bundle Ω_+ consider the m^{th} component ϕ_m with respect to the decomposition induced by the circle action on the S^1 fibre. Express this m^{th} component with respect to the Fourier transform in the coordinate of the base space, $\widehat{\phi_m}(\nu)$. The action of D^2 on ϕ is then given via the Fourier inversion theorem as multiplication of $\widehat{\phi_m}$ by $\|2\pi\nu\|^2 + K^2$ plus some lower order operator L. For convenience, we have denoted $\|\nu\|^2 = \sum g^{ij}v_iv_j$ where the functions g^{ij} are determined by the coordinate x.

Consequently, to approximate $\left(D^2-\lambda\right)^{-1}$ it suffices to invert $(\|2\pi\nu\|^2 + K^2 + L - \lambda)$ approximately. Such an inverse will be constructed in the form $\sum_{l=0}^N h_l\left(\|2\pi\nu\|^2+K^2-\lambda\right)^{-(l+1)}$ on the subspace $Z^\perp$. Following [Ste89, p.257] the inductive construction is as follows. Set $h_0(x,x',\lambda,\nu,m)$ to be the identity, then write

$$\begin{aligned}(D^2-\lambda)&\left[e^{im.(\psi-\psi')}e^{2\pi i(x-x').\nu}\frac{h_l(x,x',\lambda,\nu,m)}{\left(\|2\pi\nu\|^2+K^2-\lambda\right)^{l+1}}\right]\\ &= e^{im.(\psi-\psi')}e^{2\pi i(x-x').\nu}\left[\frac{h_l(x,x',\lambda,\nu,m)}{\left(\|2\pi\nu\|^2+K^2-\lambda\right)^{l}}\right]\\ &\quad - 2\nabla\left(e^{im.(\psi-\psi')}e^{2\pi i(x-x').\nu}\right).\nabla\left[\frac{h_l(x,x',\lambda,\nu,m)}{\left(\|2\pi\nu\|^2+K^2-\lambda\right)^{l+1}}\right]\\ &\quad + \left(e^{im.(\psi-\psi')}e^{2\pi i(x-x').\nu}\right)\left(D^2-K^2\right)\left[\frac{h_l(x,x',\lambda,\nu,m)}{\left(\|2\pi\nu\|^2+K^2-\lambda\right)^{l+1}}\right]\\ &= I_{1,l}+I_{2,l}+I_{3,l}.\end{aligned} \quad (9.53)$$

Define $h_{l+1} = -\left(I_{2,l}+I_{3,l}\right)\left[e^{im.(\psi'-\psi)}e^{2\pi i(x'-x).\nu}\left(\|2\pi\nu\|^2+K^2-\lambda\right)^{l+1}\right]$. Take for the semilocal approximation to $\left(D^2-\lambda\right)^{-1}$ on $Z^\perp$ the operator,

$$H := \sum_m \sum_{l=0}^{N} H_l^m := \sum_m \sum_{l=0}^{N} \int_{\mathbb{R}^3} e^{im.(\psi-\psi')} e^{2\pi i(x-x').\nu} \frac{h_l\left(x,x',\lambda,\nu,m\right) d\nu}{2\pi \left(\|2\pi\nu\|^2+K^2-\lambda\right)^{l+1}}, \tag{9.54}$$

for some large N. Here for clarity we have suppressed the usual pre and post multiplication by bump functions, summation over a cover and projection onto the relevant subspace. We have by definition that

$$\left(D^2-\lambda\right) H = 1 - \sum_m \int_{\mathbb{R}^3} e^{im.(\psi-\psi')} e^{2\pi i(x-x').\nu} \frac{h_{N+1} d\nu}{2\pi \left(\|2\pi\nu\|^2+K^2-\lambda\right)^{N+1}}. \tag{9.55}$$

The operator H is a good approximation to $\left(D^2-\lambda\right)^{-1}$ on the subspace $Z^\perp$ in the sense that by setting

$$E_s = \frac{1}{2\pi i} \int_\gamma e^{-s\lambda} H d\lambda \tag{9.56}$$

we have

$$\lim_{L\to\infty} \int_{t=L} \int_0^\infty \mathrm{tr}\ c^0 \tau D \left(e^{-sD^2} - E_s\right) \pi_{Z^\perp} ds = 0 \tag{9.57}$$

where $\pi_{Z^\perp}$ denotes projection onto the subspace $Z^\perp$. This follows easily from the estimates of [Ste93, p.485] by noting that on the subspace $Z^\perp$ the inequality

$$\|2\pi\nu\|^2 + e^{2t}\left[im + b^{12} + c^{21}\right]^2 \leqslant \text{const.} \left(\|2\pi\nu\|^2 + K^2 - \lambda\right) \tag{9.58}$$

holds, so that all the error terms decay sufficiently fast.

9.4.1 Basic Estimates

In this subsection we show that very few of the terms of H will contribute to the defect term of the index. This is a simple consequence of the decay of the other terms for large t. To do this we note that by the iterative construction we can express (see [SSZ95, p.498])

$$DH_l^m = \sum_\sigma \left(\|2\pi\nu\|^2 + K^2 - \lambda\right)^{-\sigma-l-1} h_{l,\sigma}\left(x,\nu,m\right) \tag{9.59}$$

and

$$h_{l,\sigma}\left(x,\nu,m\right) = \sum_{\substack{0\leqslant|A|+B\leqslant 2\sigma+1 \\ 0\leqslant C}} h_{l,\sigma,A,B,C}(x)\nu^A(e^t m)^B e^{-tC} \tag{9.60}$$

where each $h_{l,\sigma,A,B,C}(x)$ is bounded. These bounded multi index functions will be written as $h_\bullet$ in formulas for brevity. In calculating the defect term to the index we are interested in the limit as L tends to infinity of

$$\int_0^\infty \frac{ds}{2\pi i} \int_{t=L} \mathrm{tr}\ c^0\tau D \int_\gamma e^{-s\lambda} \left(D^2-\lambda\right)^{-1} d\lambda d\sigma. \tag{9.61}$$

We use the parametrix H to estimate this limit as in (9.57), and see which terms decay fast enough not to contribute to δ_D. By definition and equations (9.59) and (9.60) we see that

$$\begin{aligned}
(9.61) &= \sum_m \sum_{l=0}^{N} \int_0^\infty \frac{ds}{2\pi i} \int_{t=L} \mathrm{tr}\ c^0\tau D \int_\gamma \frac{e^{-s\lambda} e^{im.(\psi-\psi')} e^{2\pi i(x-x').\nu} h_l}{2\pi\left(\|2\pi\nu\|^2+K^2-\lambda\right)^{l+1}} d\nu d\lambda d\sigma \\
&= \sum_{m,l,A,B,C.\sigma} \int_0^\infty \frac{ds}{4\pi^2 i} \int_{t=L} \mathrm{tr}\ c^0\tau \int_\gamma \int_{\mathbb{R}^3} \frac{e^{-s\lambda} h_\bullet \nu^A (e^t m)^B e^{-tC}}{\left(\|2\pi\nu\|^2+K^2-\lambda\right)^{l+\sigma+1}} d\nu d\lambda d\sigma \\
&= \sum \int \frac{ds d\sigma}{4\pi^2 i} \int_{\mathbb{R}^3} \frac{\mathrm{tr}\ c^0\tau}{(l+\sigma)!} \left(\frac{-d}{dt_0}\right)\Bigg|_{t_0=0}^{l+\sigma} e^{-s\left(\|2\pi\nu\|^2+K^2+t_o\right)} h_\bullet \nu^A (e^t m)^B e^{-tC} d\nu \\
&= \sum \int \frac{ds d\sigma}{4\pi^2 i} \int_{\mathbb{R}^3} \frac{\mathrm{tr}\ c^0\tau}{(l+\sigma)!} e^{-s\left(\|2\pi\nu\|^2+K^2\right)} s^{l+\sigma} h_\bullet \nu^A (e^t m)^B e^{-tC} d\nu.
\end{aligned}$$

An elementary calculation shows that $-K^2$ is dominated by $-e^{2t}(|m|+1)/24$ on $Z^\perp$. Changing variables, using this estimate on $-K^2$ and then applying the Poisson summation formula we obtain that equation (9.61) is

$$\begin{aligned}
&\leqslant \sum \int ds d\sigma \int_{\mathbb{R}^3} \left(\frac{\nu^A d\nu}{e^{\|2\pi\nu\|^2}}\right) \frac{\mathrm{tr}\ c^0\tau e^{-sK^2} s^{l+\sigma-3/2-|A|/2} m^B h_\bullet}{(l+\sigma)! e^{t(C-B)}} \\
&\leqslant \sum \int du d\sigma \frac{\mathrm{tr}\ c^0\tau e^{-u(|m|+1)} u^{l+\sigma-3/2-|A|/2} m^B h_\bullet}{e^{t(C-B+2l+2\sigma-1-|A|)}} \\
&\leqslant \sum \int du d\sigma \frac{\mathrm{tr}\ c^0\tau e^{-u} u^{l+\sigma-3/2-|A|/2} h_\bullet}{e^{t(C-B+2l+2\sigma-1-|A|)}} d_m^B \left[u\sqrt{\frac{2}{\pi}} \frac{1}{(u^2+m^2)}\right] \qquad (9.62) \\
&\leqslant \sum \int du d\sigma \frac{\mathrm{tr}\ c^0\tau e^{-u} u^{l+\sigma-1/2-|A|/2} h_\bullet}{e^{t(C-B+2l+2\sigma-1-|A|)} m^{2+B}} \\
&\leqslant O\left(e^{-t(C-B+2l+2\sigma-1-|A|)}\right) \leqslant O\left(e^{-t(C+2l-2)}\right).
\end{aligned}$$

Here the inequalities are up to certain constant factors. Note also that we require $l \geqslant 1$ for the above estimate to make sense. Consequently, we see that only the $l = 0$ and the $l = 1$ terms of H will contribute to the defect term of the trace.

9.4.2 The Subspace Z.

On the subspace Z the above iterative construction of the parametrix does not give good error estimates. This is because on this subspace we no longer have an inequality like (9.58) which ensures that the iterative construction gives a good approximation. In this subsection we construct a parametrix for $(D^2 - \lambda)^{-1}$ on Z and show that the trace of this does not contribute to the defect term in the index. The primary tool used to show this will be the trace identities of V K Patodi. These can be stated in this context as in [Ste89, p.252]

Lemma 9.4.1 Let $\{i_1, \ldots, i_r\}$ and $\{j_1, \ldots, j_s\}$ be two subsets of $\{1, \ldots, 4\}$, one of which is non-empty, then

$$\mathrm{tr}\ b^{j_1} \ldots b^{j_s} c^{i_1} \ldots c^{i_r} = 0. \tag{9.63}$$

On the subspace Z we construct a parametrix global on the whole SO_3 orbit of the manifold. This is in exactly the same spirit as the construction of the singular parametrix in [Ste93, § 6]. Unlike before the parametrix will be implicit. This will pose no difficulties however since we shall argue that it does not contribute sufficient Clifford factors to prevent cancellation by Lemma 9.4.1.

Projection onto the subspace Z is given by

$$\pi_Z = \frac{1}{2}\left(1 + b^{12}c^{21}\right). \tag{9.64}$$

On this subspace we have $b^{12} = c^{12}$ and also that $m = 0$. A calculation shows that on the subspace Z the operators D and D^2 are equal to

$$D^2 = -\partial_t^2 + (d+\delta)^2_{S^2} + O(e^{-t}) \tag{9.65}$$

and

$$D = c^0\partial_t + (d+\delta)_{S^2} + \frac{1}{2}\left(c^3b^{03} - c^0\right) + O(e^{-t}). \tag{9.66}$$

Here $(d+\delta)_{S^2}$ denotes the signature operator of the two sphere which acts in the natural way upon $\Omega^*(M_{\mathrm{r}}) \cap Z$. Let $h\,(\lambda, x, y)$ be the Schwartz kernel of $(d+\delta)^2_{S^2} - \lambda$. For our parametrix set

$$H_2(y, y') = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(t-t').\nu} h\left(\lambda - |\nu|^2 - i\nu, y_0, y_0'\right) d\nu \tag{9.67}$$

where once again pre and post multiplication by cut off functions, projection onto the relevant subspace and summation over a suitable cover have been suppressed in the interests of clarity. That this construction provides a good estimate, in the sense of Section 9.4, of $(D^2 - \lambda)^{-1}$ on the subspace Z, follows in an identical manner to the estimates of [Ste89, p.267]. Here we note that since the operator D^2 is conjugate to

$$-\partial_t^2 + (d+\delta)^2_{S^2} + \frac{1}{4} + O(e^{-t}), \tag{9.68}$$

the spectrum of $(d+d^*)^2_{S^2} - |\nu|^2 + i\nu$ is bounded below by $1/4$. This is the technical condition which allows the estimates of M. Stern in [Ste89, p.267] to carry over with only minor changes.

The H_2 contributes only c^1, c^2, b^1 and b^2 Clifford factors. Consequently the Clifford factors in the expression

$$\frac{1}{2}\mathrm{tr}\ c^0\tau D H_0\left(1 + b^{12}c^{21}\right) \tag{9.69}$$

which are not of the order of $O(e^{-t})$ will all have a c^0, c^3 or b^3 factor that does not cancel. Lemma 9.4.1 therefore completes the proof that the subspace Z does not contribute to the defect term.

9.5 Index Defect Term

In Section 9.4.2 we showed that the trace in the defect term could be taken over the complement of Z. In Section 9.4.1 we showed that on this subspace the only terms in the iterative construction that contribute to the defect are h_0 and h_1. We now proceed to compute the contribution of DH_0^m to the defect term and remark that a similar computation of DH_1^m shows that this term contributes nothing to the trace. Firstly note that

$$\begin{aligned}\sum_m H_0^m &= \frac{1}{4\pi^2 i}\int_\gamma e^{-s\lambda}\sum_m \int_{\mathbb{R}^3} \frac{e^{im(\psi-\psi')}e^{2\pi i(x-x').\nu}d\nu d\lambda}{(\|2\pi\nu\|^2+K^2-\lambda)} \\ &= \sum_m \frac{1}{2\pi(4\pi s)^{3/2}}e^{-|x-x'|^2/4s}e^{-sK^2}e^{im(\psi-\psi')}.\end{aligned} \tag{9.70}$$

We are considering tr $c^0\tau DH_0^m$ in the limit as $L\to\infty$. The trace identities and the asymptotic form of D inequation (9.51) show that the only non-zero contribution to the defect term will originate from

$$\sum_m \mathrm{tr}\ c^{12}\left(\frac{e^t-2}{2}\right)\left[im+\left(\frac{1}{(e^t-2)^2}-\frac{1}{2}\right)(b^{21}+c^{12})\right]H_0^m\pi_{Z^\perp}. \tag{9.71}$$

Therefore we need to compute the trace, along $\psi=\psi'$ $x=x'$, of

$$\begin{aligned}&\sum_m \frac{c^{12}}{2\pi(4\pi s)^{3/2}}\left(\frac{e^t-2}{2}\right)\left[im+\left(\frac{1}{(e^t-2)^2}-\frac{1}{2}\right)(b^{21}+c^{12})\right] \\ &\qquad\qquad \times e^{-|x-x'|^2/4s}e^{-sK^2}e^{im(\psi-\psi')}\pi_{Z^\perp}.\end{aligned} \tag{9.72}$$

9.5.1 Evaluation of the Defect Term

Sections 9.3 and 9.2.2 have shown that

$$L^2(M_2^0)\ \text{Index}\ (d+\delta)_+ = -\frac{4}{3}-\delta_D \tag{9.73}$$

where equation (9.39) provides the numerical factor. The term δ_D originally defined in equation (9.40) is equal, by Section 9.4, to the limit as $L\to\infty$ of

$$\begin{aligned}&\int_0^\infty ds\int_{t=L}\mathrm{tr}\ c^0\tau De^{-sD^2}\pi_{Z^\perp}d\sigma \\ &=\int_0^\infty ds\int_{t=L}d\sigma'\sum_m\frac{\mathrm{tr}\ c^{12}(e^t-2)}{2\,(4\pi s)^{3/2}}\left[\left(\frac{1}{(e^t-2)^2}-\frac{1}{2}\right)(b^{21}+c^{12})+im\right]e^{-sK^2}\pi_{Z^\perp} \\ &=\int_0^\infty ds\int_{t=L}d\sigma'\sum_m\frac{\mathrm{tr}\ c^{12}U^{-1}}{(4\pi s)^{3/2}}\left[im+\left(b^{21}+c^{12}\right)\left(\frac{U^2}{4}-\frac{1}{2}\right)\right] \\ &\qquad\qquad \times e^{sU^{-2}\left[im+\left(b^{21}+c^{12}\right)\left(\frac{U^2}{4}-\frac{1}{2}\right)\right]^2}\pi_{Z^\perp}.\end{aligned} \tag{9.74}$$

Here $U^{-1}=(e^t-2)/2$ and $d\sigma'=\sigma^{12}$. Applying the Poisson summation formula shows that this is equal to

$$\begin{aligned}&\int_0^\infty ds\int_{t=L}d\sigma'\sum_m\frac{\mathrm{tr}\ c^{12}U^{-1}}{(4\pi s)^{3/2}}\int_{\mathbb{R}}\left[iy+\left(b^{21}+c^{12}\right)\left(\frac{U^2}{4}-\frac{1}{2}\right)\right] \\ &\qquad\qquad \times e^{sU^{-2}\left[iy+\left(b^{21}+c^{12}\right)\left(\frac{U^2}{4}-\frac{1}{2}\right)\right]^2}e^{-2\pi imy}\pi_{Z^\perp}dy.\end{aligned} \tag{9.75}$$

By changing variables and then integrating, firstly with respect to the new variable and then, with respect to s we see that the above equals

$$\begin{aligned}&\int_0^\infty ds\int_{t=L}d\sigma'\sum_m\frac{\mathrm{tr}\ c^{12}U^{-1}}{(4\pi s)^{3/2}}\left[\int_{\mathbb{R}}ixe^{-sU^{-2}x^2}e^{-2\pi imx}dx\right]e^{2\pi m(b^{21}+c^{12})(U^2-2)/4}\pi_{Z^\perp} \\ &=\int_0^\infty ds\int_{t=L}d\sigma'\sum_m\frac{\mathrm{tr}\ c^{12}U^2m}{8s^3}e^{-\pi^2m^2U^2/s}e^{2\pi m(b^{21}+c^{12})(U^2-2)/4}\pi_{Z^\perp} \\ &=\int_0^\infty ds\int_{t=L}d\sigma'\sum_{m\neq0}\frac{\mathrm{tr}\ c^{21}}{16\pi^2s^2}\partial_m\left(e^{-\pi^2m^2U^2/s}\right)e^{2\pi m(b^{21}+c^{12})(U^2-2)/4}\pi_{Z^\perp} \\ &=\int_{t=L}d\sigma'\sum_{m\neq0}\frac{\mathrm{tr}\ c^{12}}{8\pi^4U^2m^3}e^{2\pi m(b^{21}+c^{12})(U^2-2)/4}\pi_{Z^\perp}\end{aligned} \tag{9.76}$$

Thus by expanding,

$$e^{U^2\pi mb^{21}/2}=1+\frac{U^2}{2}\pi mb^{21}+O(U^4) \tag{9.77}$$

and

$$e^{U^2\pi mc^{12}/2}=1+\frac{U^2}{2}\pi mc^{12}+O(U^4), \tag{9.78}$$

we see that δ_D is equal to

$$\begin{aligned}
&\lim_{L\to\infty}\int_{t=L} d\sigma' \sum_{m\neq 0} \frac{\operatorname{tr} c^{12}}{8\pi^4 U^2 m^3} e^{\pi m(b^{21}+c^{12})\frac{U^2}{2}} e^{-\pi m(b^{21}+c^{12})}\pi_{Z^\perp} \\
&= \lim \int_{t=L} d\sigma' \sum_{m\neq 0} \frac{\operatorname{tr} c^{12}}{8\pi^4 U^2 m^3}\left(1+\frac{U^2\pi m b^{21}}{2}\right)\left(1+\frac{U^2\pi m c^{12}}{2}\right) e^{-\pi m(b^{21}+c^{12})}\pi_{Z^\perp} \\
&= \lim \sum_{m\neq 0} \frac{\operatorname{tr} c^{12}}{2\pi^3 U^2 m^3}\left(1+\frac{U^2\pi m b^{21}}{2}\right)\left(1+\frac{U^2\pi m c^{12}}{2}\right) e^{-\pi m(b^{21}+c^{12})}\pi_{Z^\perp}.
\end{aligned} \tag{9.79}$$

Finally, recalling that

$$\pi_{Z^\perp} = \frac{1}{2}\left(1+b^{12}c^{12}\right), \tag{9.80}$$

we use the trace identities to evaluate this expression.

$$\begin{aligned}
\delta_D &= \lim_{L\to\infty}\sum_{m\neq 0}\frac{\operatorname{tr} c^{12}(-1)^{2m}}{4\pi^3 U^2 m^3}\left[1+\frac{U^2\pi m(b^{21}+c^{12})}{2}\right](1+b^{12}c^{12}) \\
&= \sum_{m\neq 0}\frac{\operatorname{tr} c^{1212}(b^{2112+1}+1)}{8\pi^2 m^2} = -\frac{\operatorname{tr}\mathrm{Id}}{2\pi^2}\sum_{m\geqslant 1}\frac{1}{m^2} = -\frac{2}{\pi^2}\sum_{m\geqslant 1}\frac{1}{m^2} \\
&= -\frac{2}{\pi^2}\zeta(2) = -\frac{1}{3}.
\end{aligned} \tag{9.81}$$

In this expression we have subsumed into the trace the fact that the boundary, $\{t=L\}\subset M_2^0$, is $SO_3/\mathbb{Z}_2 = S^3/\mathbb{Z}_4$ under the identification (9.3). Thus we conclude, by equation (9.73) and Corollary 9.2.8, that

$$L^2(M_2^0)\ \text{index}\ (d+\delta)_+ = -\frac{4}{3}+\frac{1}{3} = -1. \tag{9.82}$$

Consequently the Sen form [Sen94] is the unique harmonic form on the reduced two monopole moduli space.

Remark 9.5.1 We observed before that the L^2 index was a conformal invariant. Thus any complete manifold conformally equivalent to M_2^0 will possess a unique L^2 harmonic form. This harmonic form will be self dual or anti self dual depending upon the orientation of the manifold.

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