Symplectic Geometry and Isomonodromic Deformations

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To my father, Ray

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Abstract

In this thesis we study the natural symplectic geometry of moduli spaces of meromorphic connections (with arbitrary order poles) over Riemann surfaces. The aim is to understand the symplectic geometry of the monodromy data of such connections, involving Stokes matrices. This is motivated by the appearance of Stokes matrices in the theory of Frobenius manifolds due to Dubrovin [**31**], and in the derivation of the isomonodromic deformation equations of Jimbo, Miwa and Ueno [**60**].

The main results of this thesis are:

• An extension to the meromorphic case of the infinite dimensional description, due to Atiyah and Bott [**10**], of the symplectic structure on moduli spaces of flat connections. This involves using an appropriate notion of singular C^∞ connections and realises the natural moduli space of monodromy data as an infinite dimensional symplectic quotient.

• An explicit finite dimensional *symplectic* description of moduli spaces of meromorphic connections on trivial holomorphic vector bundles over the Riemann sphere. A similar description is given of certain extended moduli spaces involving a compatible framing at each pole; these are the phase spaces of the isomonodromic deformation equations.

• A proof that the monodromy map is a symplectic map. In other words the above two symplectic structures are related by the transcendental map taking meromorphic connections to their monodromy data. The analogue of this result in inverse scattering theory is well-known and was important in developing the quantum inverse scattering method.

• A symplectic description of the full family of Jimbo-Miwa-Ueno isomonodromic deformation equations. In modern language we prove that the isomonodromic deformation equations are equivalent to a flat *symplectic* Ehresmann connection on a symplectic fibre bundle. This fits together, into a uniform framework, all the previous results for the six Painlevé equations and Schlesinger's equations.

• Finally we look at the simplest case involving Stokes matrices in detail. We present a conjecture relating Stokes matrices to Poisson-Lie groups (which we prove in the simplest case) and also prove directly that in low-dimensional cases the Poisson structure on the local moduli space of semisimple Frobenius manifolds does arise from a Poisson-Lie group.

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Contents

CHAPTER 0

Introduction

1. Motivation

Moduli spaces of representations of fundamental groups of Riemann surfaces have been intensively studied in recent years and have been found to have an incredibly rich structure. For example, the space of irreducible unitary representations of the fundamental group of a compact Riemann surface is identified, by a theorem of Narasimhan and Seshadri [**86**], with the moduli space of stable holomorphic vector bundles on the surface. In particular, this description puts a Kähler structure on the space of fundamental group representations; i.e. it has a symplectic structure together with a compatible complex structure. A remarkable fact is that although the complex structure on the space of representations will depend on the complex structure of the surface, the *symplectic* structure only depends on the topology, a fact often referred to as 'the symplectic nature of the fundamental group' [**38**].

If, instead of unitary representations, we consider the moduli space of *complex* fundamental group representations, then the geometry is richer still. In particular, due to results of Hitchin, Donaldson and Corlette [**47, 29, 23**], the Kähler structure above now becomes a *hyper-Kähler* structure and the symplectic structure becomes a *complex symplectic* structure, which is still topological.

One of the main aims of this thesis is to generalise this complex symplectic structure. (Hyper-Kähler structures will not be considered here.)

The first step is to recall that, since we are over a Riemann surface, there is a one to one correspondence between complex fundamental group representations and *holomorphic* connections (obtained by taking a holomorphic connection to its monodromy/holonomy representation).

Then replace the word 'holomorphic' by 'meromorphic' above; in this thesis we will study the symplectic geometry of moduli spaces of meromorphic connections with arbitrary order poles.

In fact, as in the holomorphic case, these moduli spaces may be realised in a more topological way, by using a generalised notion of monodromy data (not just involving a fundamental group representation). Firstly, for meromorphic connections with only simple poles (logarithmic connections), the symplectic geometry has been previously studied: by restricting any meromorphic connection to the complement of its polar divisor and taking the corresponding monodromy representation, a map is obtained from the moduli space of meromorphic connections to the moduli space of representations of the fundamental group of the punctured Riemann surface. For logarithmic connections this map is generically a covering map and so we are essentially in the well-known case of representations of fundamental groups of punctured Riemann surfaces.

However in the general case of meromorphic connections with arbitrary order poles there are local moduli at the poles—it is not sufficient to restrict to the complement of the polar divisor and take the monodromy representation as above.

Fortunately this extra data—the local moduli of meromorphic connections—has been studied in the theory of differential equations for many years; it has a natural monodromy-type description in terms of 'Stokes matrices'[1].

Thus by extending the notion of monodromy data to incorporate not just a fundamental group representation but also the Stokes matrices at each pole, a monodromy-type description is obtained of moduli spaces of meromorphic connections with arbitrary order poles.

Then the main question we ask is: "What is the natural symplectic geometry of these moduli spaces of generalised monodromy data?"

In other words, we seek a uniform symplectic description of a vast family of moduli spaces, which specialises to the known cases when the poles are all of order at most one.

Recently, Martinet and Ramis [**78**] have constructed a huge group associated to any Riemann surface, the so-called 'wild fundamental group', whose set of finite dimensional representations naturally corresponds to the set of meromorphic connections on the surface. Although we will not directly use this perspective, the question above can then be provocatively rephrased as asking: "What is the symplectic nature of the *wild* fundamental group?"

Apart from the obvious desire to extend existing results, the motivation for studying the symplectic geometry of meromorphic connections arose from the following two sources.

1) Firstly Stokes matrices naturally appear in the geometry of moduli spaces of 2-dimensional topological quantum field theories. This is due to B. Dubrovin: in his fundamental paper [**31**], Dubrovin defined the notion of a Frobenius manifold to be the geometrical/coordinate-free manifestation of the so-called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations governing deformations of 2D topological field theories. One of the main results of [**31**] is the classification of semisimple Frobenius manifolds: the local moduli space of semisimple Frobenius manifolds is the same as a certain moduli space of meromorphic connections on the Riemann sphere. Moreover the geometry of the moduli space of meromorphic connections reflects the geometry of the moduli space of semisimple Frobenius manifolds. At first glance, the meromorphic connections which arise in this way appear to be quite simple: they just have two poles on $\mathbb{P}^1$, of orders one and two respectively. Moreover the moduli of such connections is essentially identified with the Stokes matrices arising at the order two pole. In fact in the Frobenius manifold case there is just one Stokes matrix and this is simply an upper triangular matrix with ones on the diagonal; the local moduli space of n-dimensional semisimple Frobenius manifolds is identified with the upper triangular unipotent subgroup of $GL_n(\mathbb{C})$ and so is isomorphic to a complex vector space of dimension $n(n-1)/2$. One of the intriguing aspects of [**31**]

[1]Whereas fundamental solutions of logarithmic connections have polynomial growth at the poles of the connection, the basic new feature of meromorphic connections with higher order poles is that their solutions will generally have exponential behaviour at the poles and this behaviour varies on different sectors at each pole. The Stokes matrices encode (in a way we will make precise later) the change in asymptotic behaviour of solutions on different sectors at a pole in a similar way to how a monodromy representation encodes the change in solutions when they are continued around a non-trivial loop.

was the explicit formula for a Poisson bracket on this space of matrices in the $n = 3$ case:

$$S := \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \{x, y\} = xy - 2z \\ \{y, z\} = yz - 2x \\ \{z, x\} = zx - 2y. \end{array} \tag{1}$$

This Poisson structure has 2-dimensional symplectic leaves parameterised by the values of the Markoff polynomial

$$x^2 + y^2 + z^2 - xyz.$$

For example, the quantum cohomology of the complex projective plane $\mathbb{P}^2(\mathbb{C})$ is a 3-dimensional semisimple Frobenius manifold and corresponds to the point $S = \left(\begin{smallmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. (The manifold is just the complex cohomology $H^*(\mathbb{P}^2)$ and the Frobenius structure comes from the 'quantum product', deforming the usual cup product.) This is an integer solution of the Markoff equation $x^2 + y^2 + z^2 - xyz = 0$, and quite surprisingly, it follows that the solution of the WDVV equations corresponding to the quantum cohomology of $\mathbb{P}^2$ is not an algebraic function, from the fact (proved by Markoff in the nineteenth century) that the Markoff equation has an infinite number of integer solutions[2].

The point I wish to make from this discussion above is simply that spaces of Stokes matrices do seem to have very interesting Poisson/symplectic structures. In this thesis we will give two new intrinsic descriptions of Dubrovin's Poisson structure (1), firstly from an infinite dimensional point of view and secondly in terms of Poisson-Lie groups.

2) Our second motivation is from the general theory of isomonodromic deformations. Whilst Frobenius manifolds suggest there is interesting symplectic geometry on moduli spaces of meromorphic connections, they do not really justify considering the general case of meromorphic connections with arbitrarily many poles of arbitrary order. The motivation for the general case is as follows.

The basic fact is that *families* of moduli spaces of meromorphic connections (on vector bundles over the Riemann sphere) are the natural arena for the isomonodromic deformation equations of Jimbo, Miwa and Ueno [**60**]. These form a large family of nonlinear ordinary differential equations each having the so-called 'Painlevé property'. Special cases include the six Painlevé equations and Schlesinger's equations. We will give a geometrical description of the isomonodromic deformation equations on page xvi and have included some background material on the Painlevé equations and isomonodromic deformations in Appendix A, since they may be unfamiliar to geometers. Here we just give some of the reasons why, in this thesis, we study the geometry of isomonodromic deformations.

Firstly in the theory of integrable systems there is the Painlevé test for integrability: this is based on the hypothesis that a nonlinear *partial* differential equation is 'integrable' if it admits some dimensional reduction to an ODE with the Painlevé property. (For example the KdV equation has a reduction to the first Painlevé equation and all six Painlevé equations appear as reductions of the anti-self-dual Yang-Mills equations; see [**1, 79**] for more discussion and examples.) Moreover isomonodromic deformations provide the largest family of ODEs which do have the Painlevé property, and so in some

[2] The solutions are an orbit of a natural (Poisson) braid group action on the upper triangular matrices and in turn this orbit is the set of branches of the WDVV solution. See [**31**] Appendix F.

sense isomonodromic deformations underly *all* integrable equations[3]. This explains why isomonodromic deformations have appeared recently in such a diverse range of nonlinear problems in geometry and theoretical physics (such as Frobenius manifolds [**31**] or the construction of Einstein metrics [**100, 49**]); they have a *universal* nature.

On the other hand general solutions of isomonodromic deformation equations cannot be given explicitly in terms of known special functions: general solutions (at least of the six Painlevé equations) are *new* transcendental functions (see [**103**]). This is the reason we turn to geometry to understand the isomonodromic deformation equations.

Recent work on isomonodromic deformations seems to have focused mainly on particular examples, in particular exploring the rich geometry of the six Painlevé equations and searching for the few, very special, explicit solutions that they do admit. Here however we go back to the original paper [**60**] of Jimbo, Miwa and Ueno and study the geometry of the general case. The question we address is simply "What is the symplectic geometry of the full family of isomonodromic deformation equations of Jimbo, Miwa and Ueno?".

In certain special cases, such as the Schlesinger or Painlevé equations the symplectic geometry is well known. However in general there are new features[4] which (I assume) is the reason this question has not been previously answered.

The relationship between the two questions we have underlined is that, geometrically, the isomonodromy equations constitute a flat Ehresmann connection on a *family* of moduli spaces of meromorphic connections. Our aim is thus to find natural symplectic structures on moduli spaces of (generic) meromorphic connections (on $\mathbb{P}^1$) and then prove that the isomonodromic deformation equations preserve these symplectic structures. The method of proof is also interesting: we transfer the problem from the moduli spaces of meromorphic connections over to the corresponding spaces of monodromy data. On the spaces of monodromy data, we give a different description of the symplectic structures and then prove that they are preserved by the isomonodromy flows. This is the generalisation of the fact that the symplectic structure on moduli spaces of representations of the fundamental group of a Riemann surface is topological.

In modern language our main result will be

Theorem. *The isomonodromic deformation equations of Jimbo-Miwa-Ueno are equivalent to a flat* symplectic *connection on a symplectic fibre bundle.*

For example, understanding the symplectic geometry of the isomonodromic deformation equations enables us to ask questions about their quantisation. This has been addressed in the simple pole case by Reshetikhin [**91**] and Harnad [**44**] and leads to the Knizhnik-Zamolodchikov equations.

Although apparently not mentioned in the literature, a useful perspective has been to interpret the paper [**60**] of Jimbo, Miwa and Ueno, as stating that the Gauss-Manin connection in non-Abelian cohomology (in the sense of Simpson [**99**]) generalises to the

[3]The uncertainty here is because there is no generally agreed definition of the word 'integrable'. There are just lots of examples. It is not clear which of the notions 'having the Painlevé property' or 'being an isomonodromic deformation' is the more fundamental, or whether they are in fact equivalent.

[4]For example, we do not know a canonical 'a priori' symplectic trivialisation of the bundle of extended moduli spaces in the general case (see p150).

case of meromorphic connections, at least over $\mathbb{P}^1$; the isomonodromy equations *are* the natural Gauss-Manin connection on relative moduli spaces of meromorphic connections[5].

This statement offers a fantastic guide for future generalisation.

2. Schlesinger's Equations

Perhaps the simplest way to explain the results in this thesis is to firstly describe the starting point. Thus here we will briefly explain the picture we wish to generalise; this is Hitchin's approach to Schlesinger's equations, which appeared in [**48**]. (Schlesinger's equations are the isomonodromic deformation equations for *logarithmic* connections on trivial vector bundles over the Riemann sphere).

Suppose we have m, $n\times n$ complex matrices $A_1,\dots,A_m\in\mathrm{End}_n(\mathbb{C})$ together with m distinct complex numbers $a_1,\dots,a_m\in\mathbb{C}$. Then consider the following meromorphic connection on the trivial rank n holomorphic vector bundle over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$:

$$\nabla := d-\left(A_1\frac{dz}{z-a_1}+\cdots+A_m\frac{dz}{z-a_m}\right). \tag{2}$$

Here d is the exterior derivative on $\mathbb{P}^1$, acting on sections of the trivial vector bundle (which are just column vectors of functions). This connection has simple poles at each a_i and will have no further pole at ∞ if and only if

$$A_1+\cdots+A_m=0. \tag{3}$$

Henceforth we will suppose that this equation holds (by changing the coordinate z any logarithmic connection on the trivial bundle can be written in this form).

Thus if we remove a small open disc D_i from around each a_i and restrict ∇ to the m-holed sphere

$$S:=\mathbb{P}^1\setminus(D_1\cup\cdots\cup D_m)$$

we obtain a (nonsingular) holomorphic connection. In particular it is flat and so by taking the monodromy we obtain a representation of the fundamental group of the m-holed sphere; just parallel translate bases of solutions around closed loops.

This procedure defines a holomorphic map, which we will call the *monodromy map*, from the set of connections to the set of complex representations of the fundamental group of S:

$$\big\{(A_1,\dots,A_m)\ \big|\ A_1+\cdots+A_m=0\big\}\xrightarrow{\nu_{\mathbf{a}}}\big\{(M_1,\dots,M_m)\ \big|\ M_1\cdots M_m=1\big\} \tag{4}$$

where we have chosen appropriate loops generating the fundamental group of S and the matrix $M_i\in GL_n(\mathbb{C})$ represents the automorphism obtained by parallel translating a basis of solutions around the ith loop.

This map is the key to the whole theory; it is defined by taking the m matrices A_i then building a logarithmic connection (2) (using the choice of points $\mathbf{a}=(a_1,\dots,a_m)$) and taking the monodromy representation of the restriction of ∇ to the m-holed sphere. Note that the additive relation on the left-hand side is converted into a multiplicative relation, so we may think of this monodromy map $\nu_{\mathbf{a}}$ as a kind of generalisation of the exponential function. However there is no explicit formula for the monodromy map in

[5]But note that one needs an explicit description of part of the moduli space to obtain explicit equations.

general; it depends on the points $a_1, \dots, a_m$ in a complicated way. Writing down $\nu_{\mathbf{a}}$ explicitly amounts to explicitly solving Painlevé-type equations which we know in general involves 'new' transcendental functions.

We can however study the geometry of the monodromy map, particularly the *symplectic* geometry. To put (4) in a more natural symplectic framework there are two modifications to be made. Firstly, as it stands, $\nu_{\mathbf{a}}$ depends on a choice of basepoint. To remove this dependence, we quotient both sides of (4) by the diagonal conjugation action of $GL_n(\mathbb{C})$.

Secondly restrict the matrices A_i to be in fixed adjoint orbits. (These are natural complex symplectic manifolds, since they may be identified with coadjoint orbits using the trace, and the coadjoint orbits are given their natural Kostant-Kirillov symplectic structures.) Thus we pick m generic (co)adjoint orbits $O_1, \dots, O_m$ and require $A_i \in O_i$. Also define $\mathcal{C}_i \subset GL_n(\mathbb{C})$ to be the conjugacy class containing $\exp(2\pi\sqrt{-1}A_i)$ for any $A_i \in O_i$. Fixing A_i to be in O_i implies that the local monodromy of ∇ around a_i will be in the conjugacy class $\mathcal{C}_i$.

The wonderful fact is that condition (3) is precisely the vanishing of the moment map for the diagonal conjugation action of $GL_n(\mathbb{C})$ on $O_1 \times \cdots \times O_m$. The line (4) thus becomes

$$O_1 \times \cdots \times O_m // GL_n(\mathbb{C}) \xrightarrow{\nu_{\mathbf{a}}} \operatorname{Hom}_{\boldsymbol{\mathcal{C}}}\big(\pi_1(S), GL_n(\mathbb{C})\big)/GL_n(\mathbb{C}) \tag{5}$$

where the subscript $\boldsymbol{\mathcal{C}} = (\mathcal{C}_1, \dots, \mathcal{C}_m)$ on the right means that we restrict to representations of the fundamental group having local monodromy around a_i in the conjugacy class $\mathcal{C}_i$. From the fact that the dimensions of the spaces on the left and the right are the same, and the fact that the monodromy map is holomorphic, it follows that $\nu_{\mathbf{a}}$ is a local holomorphic isomorphism.

The symplectic geometry of this set of representations on the right has been much studied recently. The primary description of its symplectic structure is due to Atiyah and Bott [**10, 9**] and involves interpreting the set of representations as an infinite dimensional symplectic quotient, starting with all C^∞ connections on the manifold-with-boundary S. On the other hand, in [**108**] Weil defined the notion of 'parabolic group cohomology' and proved that each tangent space to such a moduli space of representations of a discrete group is an H^1 in parabolic group cohomology. A purely finite dimensional description of the symplectic structure is then given simply by the cup product [**18**]. However, starting from this finite dimensional description it is hard to prove that the symplectic form is closed and many people have been involved in finding a simple, purely finite dimensional proof [**61, 36, 5, 42, 3**].

To return to the story here, on the left-hand side of (5) we also have a natural symplectic structure: by construction it is a finite dimensional symplectic quotient. One of Hitchin's key results in [**48**] was to prove, for any choice of pole positions $\mathbf{a}$, that the monodromy map $\nu_{\mathbf{a}}$ is *symplectic*; it pulls back the Atiyah-Bott symplectic structure on the right to the symplectic structure on the left, coming from the coadjoint orbits. This 'symplecticness of the monodromy map' is the key to understanding intrinsically why Schlesinger's equations are symplectic, as we will explain below, after first describing what Schlesinger's equations are.

Observe that if we now vary the positions $a_1, \dots, a_m$ of the poles slightly (so that they stay in the discs we removed from $\mathbb{P}^1$ to obtain S) then the domain and the range of the monodromy map do not change (i.e. the spaces on the left and the right of (5) do not change). However the monodromy map $\nu_{\mathbf{a}}$ does vary. The question Schlesinger asked was: "How should the matrices A_i vary with respect to the pole positions a_i such that the monodromy representation $\nu_{\mathbf{a}}(A_1, \dots, A_m)$ does not vary?". The answer he found in [**94**], was that they should satisfy the beautiful family of nonlinear equations which now bear his name:

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \qquad \text{if } i \neq j, \text{ and}$$
$$\frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}.$$

These are the equations for *isomonodromic* deformations of the logarithmic connections ∇ on $\mathbb{P}^1$ that we began with in (2). In the 2×2 case with just 4 poles $(n = 2, m = 4)$ Schlesinger's equations are equivalent to the sixth Painlevé equation.

Hitchin's observation now is that the local self-diffeomorphisms of the symplectic manifold $O_1 \times \cdots \times O_m // GL_n(\mathbb{C})$ induced by integrating Schlesinger's equations, are clearly symplectic diffeomorphisms, because they are of the form

$$\nu_{\mathbf{a}'}^{-1} \circ \nu_{\mathbf{a}}$$

for two sets of pole positions $\mathbf{a}$ and $\mathbf{a}'$, and the monodromy map is a local *symplectic* isomorphism for all $\mathbf{a}$.

This is the picture we want to generalise to the case of higher order poles. The generalisation of Schlesinger's equations for this case were written down by Jimbo, Miwa and Ueno in [**60**] as an off-shoot of the theory of 'holonomic quantum fields'. We return to their paper [**60**], find the relevant symplectic structures and prove they are symplectic by generalising Hitchin's argument. The main missing ingredient in the higher order pole case is the Atiyah-Bott construction of a symplectic structure on the generalised monodromy data; when the discs are removed any local moduli at the poles is lost.

Note that, in essence, once Hitchin has proved that the monodromy map is symplectic then the symplecticness of the isomonodromy equations is equivalent to the fact that symplectic structure on the representations of the fundamental group of the punctured/holed Riemann sphere is topological (which amounts to it being independent of the pole positions in the logarithmic case). To see this more clearly, the above picture can be rephrased in terms of symplectic fibrations, as we will explain in the next section.

3. Summary of Results

First, in Chapter 1 we quote the background results we use regarding the local moduli of meromorphic connections on Riemann surfaces, describing in particular how Stokes matrices arise. Note that for isomonodromic deformations in the sense of Jimbo, Miwa and Ueno, we only need study *generic* meromorphic connections on the Riemann sphere[6]. Thus we will focus on this case, but point out which of our results hold immediately in greater generality. Three important definitions in Chapter 1, related to germs of meromorphic connections, are the notions of *compatible framings* p2, of *formal equivalence* p3 and of *formal normal forms* p4.

Next, Chapters 2, 3 and 4 each give a different approach to the moduli of meromorphic connections on the Riemann sphere.

[6]The precise notion of 'generic' is given in Definition 1.2 and we will refer to such connections as *nice*.

• In Chapter 2 we give explicit finite dimensional complex symplectic descriptions of moduli spaces of meromorphic connections on *trivial* holomorphic vector bundles over $\mathbb{P}^1$, in terms of coadjoint orbits and cotangent bundles. On one hand these moduli spaces are the natural phase spaces for the isomonodromy equations. On the other, as in the logarithmic case described above, the general philosophy is that this 'additive' or 'linear' picture will give insight into the symplectic nature of the monodromy data (which is thought of as the corresponding 'multiplicative' or 'nonlinear' picture).

Fix m points $a_1, \ldots, a_m$ on $\mathbb{P}^1$ and choose a nice formal normal form ${}^iA^0$ at each a_i, having pole orders $k_i \geq 1$ respectively say. Denote this data (pole positions and formal normal forms) by $\mathbf{A}$. Let $\mathcal{M}^*(\mathbf{A})$ be the moduli space of meromorphic connections on trivial rank n holomorphic vector bundles over $\mathbb{P}^1$ which are formally equivalent to ${}^iA^0$ at a_i and have no other poles. Then the main result of Chapter 2 is

Theorem 2.35. *By choosing a local coordinate on $\mathbb{P}^1$ at each a_i, the moduli space $\mathcal{M}^*(\mathbf{A})$ may be identified with a complex symplectic quotient of a product of complex coadjoint orbits:*

$$\mathcal{M}^*(\mathbf{A}) \cong O_1 \times \cdots \times O_m // GL_n(\mathbb{C}), \tag{6}$$

where O_i is a coadjoint orbit of the complex Lie group

$$G_{k_i} := GL_n(\mathbb{C}[\zeta]/\zeta^{k_i})$$

of $(k_i - 1)$-jets of bundle automorphisms.
Moreover, the complex symplectic structure defined on $\mathcal{M}^*(\mathbf{A})$ in this way does not depend on the local coordinate choices.

Note that the case when all the poles are simple (all $k_i = 1$) reduces to Hitchin's case of $GL_n(\mathbb{C})$ coadjoint orbits (in that case coordinate choices are not needed to obtain such a description).

In fact in order to obtain *explicit* equations for isomonodromic deformations Jimbo, Miwa and Ueno use slightly larger moduli spaces of meromorphic connections. A compatible framing is included at each pole and only the irregular types (Definition 1.4), rather than the formal normal forms, are fixed at each pole. We will refer to these as *extended moduli spaces.*

By defining complex symplectic manifolds $\widetilde{O}_i$ ('extended orbits') which are slightly larger than the coadjoint orbits O_i, we also obtain a symplectic description of these extended moduli spaces (the result is as above, but with O_i replaced by $\widetilde{O}_i$; see Theorem 2.43). We show that these extended moduli spaces are *fine* moduli spaces (Proposition 2.52). The non-extended moduli spaces are obtained as complex symplectic quotients of the extended moduli spaces with respect to a large Hamiltonian torus action (Corollary 2.48). Due to this fact we will mainly focus on the extended spaces.

We also undertake a detailed study of the extended orbits $\widetilde{O}_i$ since they are the building blocks for the phase spaces of the isomonodromy equations. In particular we give three descriptions of them: as principal T-bundles over families of G_{k_i} coadjoint orbits (Corollary 2.15), as symplectic quotients of cotangent bundles of the groups G_{k_i} (Proposition 2.19), and symplectically as products of the cotangent bundle $T^*GL_n(\mathbb{C})$ with coadjoint orbits of certain unipotent subgroups of the groups G_{k_i} (Lemma 2.13). This last description, symplectically 'decoupling' $\widetilde{O}_i$, will turn out to be very useful in giving a symplectic description of the isomonodromic deformation equations.

• In Chapter 3 we give a C^∞ approach to meromorphic connections. The aim here is to generalise the Atiyah-Bott description of the symplectic structure on the moduli space of (non-singular) flat connections on a compact Riemann surface.

Firstly we give an entirely C^∞ description of the local moduli spaces of analytic equivalence classes of meromorphic connection germs. After defining a suitable notion of C^∞ singular connections ('C^∞ connections with poles') we discover that the notion of fixing the formal equivalence class of a meromorphic connection translates over nicely into the C^∞ world to become the notion of fixing the 'C^∞ Laurent expansion' of a C^∞ singular connection. (This suggests that, in order to get *symplectic* moduli spaces, we should look at spaces of flat C^∞ singular connections with fixed C^∞ Laurent expansions, modulo an appropriate gauge group.)

The main local result, Corollary 3.9, says that there is a canonical bijection between the local analytic equivalence classes and suitable gauge equivalence classes of germs of *flat* C^∞ singular connections with fixed C^∞ Laurent expansion.

Now define $\mathcal{M}(\mathbf{A})$ to be the set of isomorphism classes of meromorphic connections on arbitrary rank n, *degree zero* holomorphic vector bundles over $\mathbb{P}^1$ which are formally equivalent to ${}^iA^0$ at a_i and have no other poles. Then the local result above globalises to yield a C^∞ description of $\mathcal{M}(\mathbf{A})$:

Theorem 3.17. *There is a canonical bijection between the set of $\mathcal{G}_T$ orbits of flat C^∞ singular connections with fixed Laurent expansions and the set $\mathcal{M}(\mathbf{A})$ of isomorphism classes defined above:*

$$\mathcal{M}(\mathbf{A}) \cong \mathcal{A}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_T.$$

(A similar description holds immediately in the arbitrary genus case too.)

We also prove the analogous result in the extended version, giving C^∞ descriptions of the sets $\mathcal{M}_{\mathrm{ext}}(\mathbf{A})$ of isomorphism classes of compatibly framed meromorphic connections with fixed irregular types (Proposition 3.20).

The crucial point now is that we have set up this C^∞ approach such that the Atiyah-Bott symplectic structure generalises naturally. The new technical difficulty is that standard Banach/Sobolev methods cannot be used since we have fixed full infinite jets of derivatives at the poles. Instead we use Fréchet spaces, but not in a very deep way.

The main results are that the extended space $\mathcal{A}_{\mathrm{ext}}(\mathbf{A})$ of C^∞ singular connections is an infinite dimensional symplectic Fréchet manifold and that the corresponding gauge group acts in a Hamiltonian way, with moment map given by the curvature. This implies immediately that the extended moduli space $\mathcal{M}_{\mathrm{ext}}(\mathbf{A})$ arises, at least formally, as an infinite dimensional complex symplectic quotient. We will see below that this procedure does in fact induce a symplectic structure on (at least) a dense open subset of $\mathcal{M}_{\mathrm{ext}}(\mathbf{A})$ (and similarly on $\mathcal{M}(\mathbf{A})$ by taking the finite dimensional torus quotients).

• In Chapter 4 we describe the monodromy approach to meromorphic connections on $\mathbb{P}^1$ (the generalisation of this chapter to higher genus is straightforward). In the first section we explain what is meant by the monodromy data of a meromorphic connection and define *monodromy manifolds* $M_{\mathrm{ext}}(\mathbf{A})$ to house the monodromy data (Stokes matrices, connection matrices and exponents of formal monodromy). The procedure of passing from a meromorphic connection to its monodromy data defines the monodromy map

$$\nu : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \longrightarrow M_{\text{ext}}(\mathbf{A})$$

from the moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ to the monodromy manifold $M_{\text{ext}}(\mathbf{A})$. This is a holomorphic map between two explicit, finite dimensional manifolds of the same dimension. All of this material is known, but we emphasise how to describe the monodromy manifolds as 'multiplicative' versions of the symplectic quotients appearing in (6) above.

Then we define how to take the generalised monodromy data of a flat C^∞ singular connection from Chapter 3. The main consequence of this definition is that we obtain an *isomorphism* between the monodromy data and gauge orbits of flat C^∞ singular connections (Theorem 4.10), generalising the well-known correspondence in the non-singular case. In turn this gives a bijection between sets of isomorphism classes of meromorphic connections and the corresponding monodromy data (Corollary 4.11).

To summarise, in the extended version, all the spaces so far fit together in the following commutative diagram (which is described more fully on page 79):

$$\begin{array}{ccccc} & & \mathcal{M}_{\text{ext}}(\mathbf{A}) & \xrightarrow{\cong} & \mathcal{A}_{\text{ext,fl}}(\mathbf{A})/\mathcal{G}_1 \\ & & \bigcup & & \big\downarrow \cong \\ \widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!/ GL_n(\mathbb{C}) & \cong & \mathcal{M}^*_{\text{ext}}(\mathbf{A}) & \xrightarrow{\nu} & M_{\text{ext}}(\mathbf{A}). \end{array} \tag{7}$$

Basically the bottom line of (7) appears in the work [**60**] of Jimbo, Miwa and Ueno but the symplectic structures and the rest of the diagram do not.

• In Chapter 5 we prove one of the main results of this thesis:

Theorem 5.1. *The monodromy map ν is symplectic.*

In more detail, we show that the Atiyah-Bott approach defines a genuine symplectic structure on the dense open submanifold $\nu(\mathcal{M}^*_{\text{ext}}(\mathbf{A}))$ of the monodromy manifold $M_{\text{ext}}(\mathbf{A})$, and that this symplectic structure pulls back along ν to the explicit symplectic structure defined on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ in terms of the extended orbits $\widetilde{O}_i$.

In some sense this is the 'inverse monodromy theory' version of the result in inverse scattering theory, that the map from the set of initial potentials to scattering data is a symplectic map (see [**35**] Chapter III).

• In Chapter 6 we study the full family of isomonodromic deformation equations of Jimbo, Miwa and Ueno [**60**]. To start with we give a geometrical picture of the isomonodromy equations. See Figure 1. The base space X is the manifold of deformation parameters. A point of X gives a choice of m distinct points $a_1, \ldots, a_m$ on $\mathbb{P}^1$ together with a choice of irregular type at each a_i. (In the case of Schlesinger's equations X just parameterises the pole positions; all of the irregular types are zero.) Over X we construct two fibre bundles: the (extended) moduli bundle $\mathcal{M}^*_{\text{ext}}$ whose fibres are the extended moduli spaces $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$, and the extended monodromy bundle M_{ext} whose fibres are the extended monodromy manifolds $M_{\text{ext}}(\mathbf{A})$. The fibre-wise monodromy maps fit together to define a holomorphic bundle map (which we will still call the monodromy map and denote by ν).

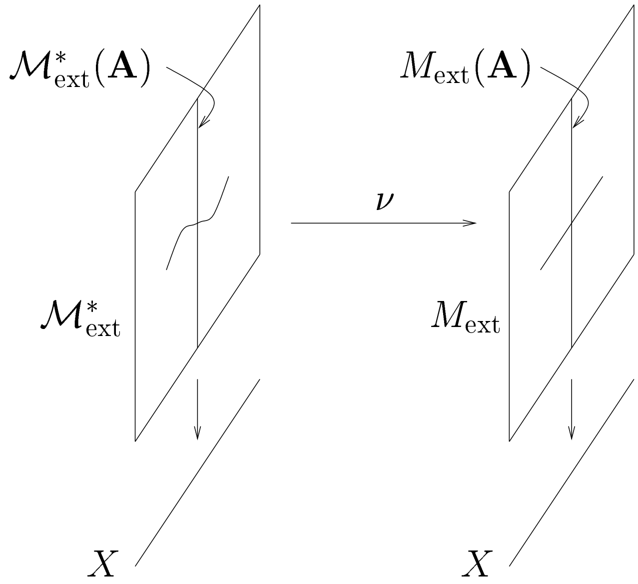

FIGURE 1. Isomonodromic Deformations

Now the point is that there is a canonical way to identify nearby fibres of the monodromy bundle M_{ext}; essentially just keep all the monodromy data constant. Geometrically this amounts to a natural flat (Ehresmann) connection on the fibre bundle M_{ext}, transverse to the fibres; we will call this the *isomonodromy connection* since nearby points of M_{ext} are on the same horizontal leaf iff they have the same monodromy data.

Then pull the isomonodromy connection back to $\mathcal{M}^*_{\text{ext}}$ along the monodromy map ν. This induced connection will be called the *isomonodromy connection on* $\mathcal{M}^*_{\text{ext}}$. The Jimbo-Miwa-Ueno isomonodromic deformation equations are precisely the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$, when the bundle is described explicitly in terms of the extended orbits $\widetilde{O}_i$.

Using all the results of previous chapters we prove the following

Theorem. *The bundle $\mathcal{M}^*_{\text{ext}}$ of moduli spaces is a* symplectic *fibre bundle and the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$ is a flat* symplectic *connection.*

The content of this is two-fold. Being a symplectic fibre bundle means that the fibres of $\mathcal{M}^*_{\text{ext}}$ are symplectic manifolds (as we proved in Chapter 2) and that they fit together such that $\mathcal{M}^*_{\text{ext}}$ is locally trivial as a bundle of symplectic manifolds. (The structure group of the fibration is contained in the group of symplectic diffeomorphisms of a standard fibre.) This is proved in Theorem 6.4. Secondly, that the isomonodromy connection is symplectic, means that the local analytic isomorphisms induced between the fibres of $\mathcal{M}^*_{\text{ext}}$ (by integrating the isomonodromy connection) are symplectic diffeomorphisms. This is proved in Theorem 6.18.

It is interesting to relate this fibre bundle picture to Hitchin's original viewpoint described above. The point to be made is that the symplecticness of the isomonodromy connection on M_{ext} is equivalent to the 'symplectic nature of the fundamental group of the punctured Riemann sphere' (i.e. we get symplectic isomorphisms between the monodromy manifolds if we move the pole positions). Therefore once we know that the monodromy map is symplectic, the symplecticness of the isomonodromic deformation equations (which are on $\mathcal{M}^*_{\text{ext}}$) is *equivalent* to the symplectic nature of the fundamental group.

In the general case, using the same argument, since we have proved that the monodromy map is symplectic, we deduce from the theorem above that the *wild* fundamental group also has a 'symplectic nature'.

• As in the case of the usual fundamental group, having found an infinite dimensional description of the symplectic structure on the monodromy manifolds, the natural next step is to look for a purely finite dimensional approach. This is the question we address in Chapter 7, for the simplest case involving Stokes matrices: the case with only two poles, of orders one and two respectively.

Our main observation is that the symplectic/Poisson geometry of the monodromy data appears to coincide with that of a well-known Poisson-Lie group: the dual group G^* of $G = GL_n(\mathbb{C})$ with its standard/canonical Poisson-Lie group structure. This observation is presented precisely in Conjecture 7.5. Basically when written down in a natural way, the monodromy data may be identified with the universal cover of G^* and the conjecture then says that the corresponding monodromy map is a Poisson map from the dual $\mathfrak{g}^*$ of the Lie algebra of G to the dual Poisson Lie group G^*:

$$\nu : \mathfrak{g}^* \longrightarrow G^*$$

for any value of the deformation parameters. We prove that the symplectic leaves certainly match up under the monodromy map and that the conjecture is true in the 2×2 case. More evidence to support the conjecture is given in the Frobenius manifold chapter below. The difficulty in general is that we do not know very much about the map ν; it involves 'new transcendental functions'.

Anyway this establishes a new relationship between Stokes matrices (the natural moduli of meromorphic connections) and Poisson-Lie groups, and leads in turn to intriguing questions about the quantisations which we will return to later.

• In Chapter 8 we return to our initial motivation related to Frobenius manifolds. Isomonodromic deformations occur in the theory of Frobenius manifolds in two equivalent ways and in Section 2 we explain the relationship between the two points of view and discuss a question raised by Hitchin in [**48**]. The picture we describe was essentially already known to Dubrovin in [**31**] but is included here since it took some time to understand.

However the main result of Chapter 8 is in Section 1. There, we establish that Dubrovin's explicit formula (1) above, for the Poisson structure on the moduli space of semisimple Frobenius manifolds arises from the dual Poisson-Lie group to $GL_3(\mathbb{C})$.

Recently M.Ugaglia [**102**] has extended Dubrovin's formula to the general case and we check also that the 4×4 case arises from Poisson-Lie groups[7]. Both of these results (and the general $n \times n$ case), would be an immediate corollary of the conjecture in Chapter 7 and we view these calculations as support for the conjecture.

• Finally some work in progress is described in Appendix F.

Note: most of the notation we use is listed, along with page references, in Appendix G.

[7]This is as far as we calculated; it is just a question of calculating the Poisson-Lie group Poisson structure explicitly.

CHAPTER 1

Background Material

In this chapter we give the basic definitions we will use regarding meromorphic connections on holomorphic vector bundles over Riemann surfaces and quote results describing how the local moduli of such connections at a singularity may be encoded in Stokes matrices. The main references used regarding Stokes matrices are [**14, 16, 67, 68, 78**]. The survey article [**105**] was useful too.

The last section of this chapter gives some brief definitions regarding symplectic geometry and some useful formulae relating to group cotangent bundles.

1. Meromorphic Connections and Linear Differential Systems

Let Σ be a Riemann surface and choose an effective divisor

$$D = k_1(a_1) + \cdots + k_m(a_m) > 0$$

on Σ, so that $a_1, \ldots, a_m$ are m distinct points of Σ and $k_1, \ldots, k_m > 0$ are positive integers. We will specialise to $\Sigma = \mathbb{P}^1$ later, since we wish to study isomonodromic deformations of connections on $\mathbb{P}^1$.

Let $V \to \Sigma$ be a rank n holomorphic vector bundle over Σ.

Definition 1.1. A *meromorphic connection* ∇ on V with poles on D is a map

$$\nabla : V \longrightarrow V \otimes K(D)$$

from the sheaf of holomorphic sections of V to the sheaf of sections of $V \otimes K(D)$, satisfying the Leibniz rule:

$$\nabla(fv) = (df) \otimes v + f\nabla v$$

where v is a local section of V, f is a local holomorphic function and K is the sheaf of holomorphic one-forms on Σ.

Concretely if we choose a local coordinate z on Σ vanishing at a_i then in terms of a local trivialisation of V, ∇ has the form:

$$\nabla = d - {}^i\!A \tag{8}$$

where ${}^i\!A$ is a matrix of meromorphic one-forms:[1]

$${}^i\!A = {}^i\!A_{k_i}\frac{dz}{z^{k_i}} + \cdots + {}^i\!A_1\frac{dz}{z} + {}^i\!A_0 dz + \cdots$$

for $n \times n$ matrices ${}^i\!A_j$ $(j \le k_i)$.

Definition 1.2. A meromorphic connection ∇ will be said to be *nice* if at each a_i the leading coefficient ${}^i\!A_{k_i}$ is
1) diagonalisable with distinct eigenvalues and $k_i \ge 2$, or
2) diagonalisable with distinct eigenvalues mod $\mathbb{Z}$ and $k_i = 1$.

[1] Pre-superscripts ${}^i\!A$, when used, will denote local information near $a_i \in \Sigma$.

This condition is independent of the trivialisation and coordinate choice. We will restrict to nice connections since they are simplest yet sufficient for our purposes (to describe the symplectic nature of isomonodromic deformations). The local moduli results in this section hold in much greater generality though and the interested reader should see the references for more details.

DEFINITION 1.3. A *compatible framing at* a_i of a vector bundle V with nice connection ∇ is a choice of isomorphism g between the fibre V_{a_i} and $\mathbb{C}^n$ which is compatible with ∇ in the sense that the leading term ${}^iA_{k_i}$ of ∇ is *diagonal* in any local trivialisation of V extending this isomorphism.

Concretely if we have already chosen a trivialisation in a neighbourhood of a_i so that $\nabla = d - {}^iA$ as above, then a compatible framing is represented by a constant matrix (which will still be denoted g) that diagonalises the leading term of ${}^iA$:

$$g \in GL_n(\mathbb{C}) \quad \text{such that} \quad g \cdot {}^iA_{k_i} \cdot g^{-1} \quad \text{is diagonal.}$$

Next we define an equivalence relation on the set of compatibly framed connections:

DEFINITION 1.4. We will say that two compatibly framed nice connections (V, ∇, g) and (V', ∇', g') have the same *irregular type at* a_i if there exists a neighbourhood U of a_i in Σ and a holomorphic isomorphism

$$\varphi : V|_U \longrightarrow V'|_U$$

between the vector bundles restricted to U such that:
- φ relates the framings: $g = g' \circ \varphi_{a_i}$, and
- the $\mathrm{End}(V)$ valued meromorphic one form which is the difference $\nabla - \varphi^*(\nabla')$ between ∇ and the pullback of ∇' to V along φ, has at most a first order pole at a_i (i.e. it has a logarithmic singularity).

This equivalence relation 'having the same irregular type' will be important in later chapters to obtain symplectic moduli spaces of compatibly framed connections but won't be discussed further in this section. See Section 4 of Chapter 2.

Now let $E = \mathbb{C}^n$ be a fixed complex vector space with preferred basis.

DEFINITION 1.5. A *germ of a meromorphic linear differential system* (of rank n), or just *system* from now on, is a germ of a meromorphic connection on the trivial vector bundle with fibre E.

REMARK 1.6. We use the terminology that *a* trivial vector bundle is just globally trivialisable, but *the* trivial vector bundle means we have chosen a trivialisation as well.

Let $\mathbb{C}[\![z]\!]$ be the ring of formal power series and $\mathbb{C}\{z\}$ the sub-ring of power series with radius of convergence greater than 0. Taylor expansion provides an isomorphism between $\mathbb{C}\{z\}$ and the germs of holomorphic functions at $z = 0$. The set of systems is isomorphic to $\mathrm{End}(E)\otimes\mathbb{C}\{z\}[z^{-1}]$; a matrix of germs of meromorphic functions $A' \in \mathrm{End}(E)\otimes\mathbb{C}\{z\}[z^{-1}]$ determines a system of equations for $v(z) \in E$:

$$\frac{dv}{dz} = A'v \tag{9}$$

which corresponds to the connection germ $\nabla = d_A = d - A$ on E where $A = A'dz$.

In particular, given a meromorphic connection ∇ on a holomorphic vector bundle V we can choose a local trivialisation of V to obtain a system (8). In general we will use the letter k to denote the order of the pole of a system, so that

$$A = A_k \frac{dz}{z^k} + \cdots + A_1 \frac{dz}{z} + A_0 dz + \cdots$$

with $A_k \neq 0$. The (infinite dimensional) vector space of systems with poles of order at most k will be denoted Syst_k.

Given two systems A, B on E then $dX = BX - XA$ defines a system on $\mathrm{End}(E)$ which will be denoted $\mathrm{Hom}(A, B)$.

In this section, to agree with the references used, we will work with systems rather than connections and also with a fixed choice of local coordinate; translation between the language of connections and systems is straightforward but we will try to emphasise which notions are coordinate-dependent or trivialisation-dependent here.

DEFINITION 1.7.

- The group of *local analytic gauge transformations* is

$$G\{z\} := GL_n(\mathbb{C}\{z\}).$$

- The group of *formal transformations* is

$$\widehat{G} := GL_n(\mathbb{C}[\![z]\!]).$$

(Recall for a ring R, that $GL_n(R)$ is the group of $n \times n$ matrices with entries in R whose determinant is a unit in R.)

The group $G\{z\}$ acts on the set of systems by gauge transformations: if $F \in G\{z\}$ then

$$F[A] := (dF)F^{-1} + FAF^{-1}$$

is its action on the system A. Observe such F is an invertible solution of $\mathrm{Hom}(A, B)$, where $B := F[A]$. The orbits of $G\{z\}$ give the *analytical classification* of systems. Isomorphic connections give rise to analytically equivalent systems.

To understand the analytic classes, a formal classification is used. Two systems A, B are said to be *formally equivalent* if there is a formal gauge transformation $\widehat{F} \in \widehat{G}$ such that $B = \widehat{F}[A]$. Note $\widehat{G}$ does not act on the set of systems we are considering: generally $\widehat{F}[A]$ will not have convergent entries. However any two analytically equivalent systems are formally equivalent, so the set of analytic classes is partitioned into classes containing formally equivalent systems. Let

$$\mathrm{Syst}(A) := \{\text{systems formally equivalent to } A\} \subset \mathrm{Syst}_k$$

and define

$${}^0C(A) := \mathrm{Syst}(A)/G\{z\}$$

to be the set of analytic classes which are formally equivalent to A. It is this set we wish to describe in this section.

In the logarithmic case ($k = 1$) the formal and analytical classifications coincide: the basic fact ([**46**] Theorem 11.3) is that any formal power series solution of a logarithmic system converges in a neighbourhood of 0. Thus if $\widehat{F} \in \widehat{G}$ is a formal transformation such that $\widehat{F}[A]$ is convergent then it is a power series solution of the logarithmic system

$\mathrm{Hom}(A, \widehat{F}[A])$ and so converges in a neighbourhood of 0. Hence $G\{z\}$ acts transitively on $\mathrm{Syst}(A)$ for logarithmic A and so ${}^0C(A)$ is just a point.

Similarly in the Abelian case ($n = 1$), if $\widehat{F}$ is a formal transformation such that $\widehat{F}[A]$ is convergent and $n = 1$ then $\mathrm{Hom}(A, \widehat{F}[A])$ is also rank one; if $B := \widehat{F}[A] - A$ then $d\widehat{F} = B\widehat{F}$. Hence B is nonsingular at 0 so $\widehat{F}$ is convergent and ${}^0C(A)$ is again a point.

In the non-Abelian, irregular case ($n, k \geq 2$) we will see how the set ${}^0C(A)$ precisely describes the difference between the formal and analytic pictures.

A key result is that within each nice formal class there is a normal form.

Definition 1.8. A *nice formal normal form* A^0 is a nice diagonal system with no holomorphic part. Such A^0 can be uniquely written as

$$A^0 := dQ + \Lambda\frac{dz}{z}, \quad Q := \mathrm{diag}(q_1, \dots, q_n) \tag{10}$$

where $q_1, \dots, q_n \in z^{-1}\mathbb{C}[z^{-1}]$ are diagonal polynomials of degree $k-1$ in z^{-1} with no constant term and $\Lambda = \mathrm{Res}_0(A^0)$ is a constant diagonal matrix.

Two nice formal normal forms are formally equivalent if and only if one is obtained from the other by the action of the symmetric group Sym_n permuting the diagonal entries.

Remark 1.9. Beware that this notion of formal normal form is coordinate-dependent. To remedy this, in general *any* nice diagonal system will be thought of as representing a nice formal normal form. Thus in later chapters, by 'a choice of formal normal form A^0' we will mean a choice of orbit of nice diagonal systems under diagonal analytic gauge transformations. Given any nice diagonal system A^0 and a choice of local coordinate it is easy to find a diagonal gauge transformation to remove the holomorphic part and put it in the form (10). For each coordinate choice, any such orbit contains a unique element of the form (10). In this chapter the coordinate z has been chosen so we use Definition 1.8.

Lemma 1.10. *If A is a nice system then it is formally equivalent to a nice formal normal form:*

$$A = \widehat{F}[A^0] \tag{11}$$

for some nice formal normal form A^0 and formal transformation $\widehat{F} \in \widehat{G}$.

Proof. This is quite well known: one diagonalises A term by term and then removes the holomorphic part. We will give an algorithm to find $\widehat{F}$ in Appendix B, in a version that works for parameter-dependent A □

A useful way to understand the relationship between $A, A^0, \widehat{F}, \Lambda$ and Q is in terms of *formal fundamental solutions*. A formal fundamental solution of A is an invertible solution of $\mathrm{Hom}(0, A)$ (namely an isomorphism with the trivial system on E, usually thought of as just a matrix whose columns make up a basis of solutions of A). The 'formal' part means that the solutions a priori have entries in some large differential ring such as

$$R := \mathbb{C}[\![z, \log(z), e^{q_1}, \dots, e^{q_n}]\!]$$

where $\log(z), e^{q_i}$ are regarded as symbols and d/dz acts in the obvious way. For any complex number λ an element z^λ of R is defined by the formula $z^\lambda = \exp(\lambda \cdot \log(z))$ using the power series formula for exp. Thus we can consider the elements $z^\Lambda e^Q$ and

$\widehat{F}(z)z^\Lambda e^Q$ of $\text{End}(E) \otimes R$. Then the relations between $A, A^0, \widehat{F}, \Lambda$ and Q above just say that $z^\Lambda e^Q$ and $\widehat{F}(z)z^\Lambda e^Q$ are formal fundamental solutions of A^0 and A respectively:

$$d(z^\Lambda e^Q) = A^0(z^\Lambda e^Q) \qquad \text{and} \qquad d(\widehat{F}z^\Lambda e^Q) = A(\widehat{F}z^\Lambda e^Q).$$

The relationship between these formal solutions and analytic solutions is subtle but leads to the main classification theorem below.

Definition 1.11.
- A *marked pair* is a pair $(A, \widehat{F})$ consisting of a nice system A and a choice of formal isomorphism $\widehat{F} \in \widehat{G}$ between A and some formal normal form A^0 ($A = \widehat{F}[A^0]$);
- We will say A^0 is the formal normal form *associated* to the marked pair $(A, \widehat{F})$.

Observe that if $(A, \widehat{F})$ is a marked pair and we set

$$g = \widehat{F}(0)^{-1} \in GL_n(\mathbb{C})$$

then (A, g) is a compatibly framed system (gAg^{-1} has diagonal leading term). Conversely we have

Lemma 1.12. *If (A, g) is a compatibly framed nice system then there is a unique normal form A^0 and unique formal transformation $\widehat{F} \in \widehat{G}$ such that*

$$A = \widehat{F}[A^0] \qquad \textit{and} \qquad g = \widehat{F}(0)^{-1}.$$

Proof. The uniqueness of A^0 is clear: in general A determines a Sym_n orbit of normal forms and we choose the one having the same leading term as gAg^{-1}. To see that g determines $\widehat{F}$ we use

Lemma 1.13. *The stabiliser in $\widehat{G}$ of a nice formal normal form A^0 is the subgroup of constant diagonal matrices:*

$$\textit{If } \widehat{F} \in \widehat{G} \textit{ and } \widehat{F}[A^0] = A^0 \textit{ then } \widehat{F} = \widehat{F}(0) \in T \cong (\mathbb{C}^*)^n.$$

Proof. Suppose $\widehat{F}[A^0] = A^0$, then $\widehat{F}$ is a solution of $\text{Hom}(A^0, A^0)$ so

$$d(z^{-\Lambda}e^{-Q}\widehat{F}z^\Lambda e^Q) = 0.$$

This is an equality in $\text{End}(E) \otimes Rdz$, where R is the differential ring used above. It follows that $\widehat{F} = z^\Lambda e^Q K z^{-\Lambda} e^{-Q}$ for some constant matrix K. In the irregular case ($k \geq 2$) K must be diagonal since otherwise $\widehat{F}$ would not have entries in $\mathbb{C}[\![z]\!]$, contradicting $\widehat{F} \in \widehat{G}$. It then follows that $\widehat{F} = K$ and so $\widehat{F}$ is a constant diagonal matrix. In the logarithmic case ($k = 1, Q = 0$) the argument is similar: K must commute with the diagonal matrix $M_0 := \exp(2\pi i\Lambda)$ which forces K to be diagonal (M_0 has distinct eigenvalues using the definition of 'nice'). As above it then follows that $\widehat{F} = K$ and so $\widehat{F}$ is a constant diagonal matrix. □

Lemma 1.12 now follows easily: if $A = \widehat{F}[A^0] = \widehat{H}[A^0]$ and $g^{-1} = \widehat{F}(0) = \widehat{H}(0)$ then $\widehat{F}^{-1}\widehat{H}$ stabilises A^0 and so using Lemma 1.13:

$$\widehat{F}^{-1}\widehat{H} = \widehat{F}(0)^{-1}\widehat{H}(0) = g \cdot g^{-1} = 1$$

□

DEFINITION 1.14.
• The *formal normal form associated to a compatibly framed system* (A, g) is the uniquely determined formal normal form A^0 in Lemma 1.12.
• The *exponent of formal monodromy* of (A, g) is the residue of the associated formal normal form:

$$\Lambda = \mathrm{Res}_0(A^0).$$

• The *formal monodromy* of (A, g) is $M_0 := \exp(2\pi i\Lambda)$. It is the local monodromy of the formal normal form A^0. Lemma 1.33 will describe the (complicated) relationship between M_0 and the local monodromy of A.

REMARK 1.15. It follows that a compatibly framed connection (V, ∇, g) has a canonically associated formal normal form. In fact, even if a coordinate had not been chosen, we can still canonically associate a formal normal form to (V, ∇, g) in the sense of Remark 1.9. It then follows in particular that the diagonal matrices Λ and M_0 are intrinsic; they are canonically associated to a compatibly framed connection.

Due to Lemma 1.10 the set ${}^0C(A)$ of analytic classes of systems formally equivalent to A is the same as ${}^0C(A^0)$ for any formal normal form A^0 of A. Thus we fix a nice formal normal form A^0 and study ${}^0C(A^0)$.

DEFINITION 1.16. The set of *applicable formal transformations* is the set

$$\widehat{G}(A^0) := \left\{ \widehat{F} \in \widehat{G} \mid \widehat{F}[A^0] \text{ is convergent } \right\}$$

of formal transformations which map A^0 to another system.

This is not (at least a priori) a group. Anyway, the set of systems formally equivalent to A^0, $\mathrm{Syst}(A^0)$, is just $(\widehat{G}(A^0))[A^0]$. In Lemma 1.13 we found that the stabiliser of A^0 is the torus T so we deduce:

$$\mathrm{Syst}(A^0) \cong \widehat{G}(A^0)/T.$$

The analytic classes of systems formally equivalent to A^0 are just the orbits of $G\{z\}$, that is:

$${}^0C(A^0) \cong G\{z\}\backslash\widehat{G}(A^0)/T.$$

The stabiliser group T is not very complicated and so understanding ${}^0C(A^0)$ is largely reduced to understanding the set

$$\mathcal{H}(A^0) := G\{z\}\backslash\widehat{G}(A^0).$$

Lemma 1.17. *The set $\mathcal{H}(A^0)$ is canonically isomorphic to the set of isomorphism classes of compatibly framed systems having associated formal normal form A^0.*

Proof. The set of applicable formal transformations is clearly isomorphic to the set of marked pairs having associated formal normal form A^0:

$$\widehat{F} \in \widehat{G}(A^0) \quad \Longleftrightarrow \quad \text{marked pair } (\widehat{F}[A^0], \widehat{F}).$$

In turn, by Lemma 1.12, such marked pairs correspond to compatibly framed systems having associated formal normal form A^0; the map

$$\widehat{F} \quad \longmapsto \quad (\widehat{F}[A^0], \widehat{F}(0)^{-1})$$

is a bijection from $\widehat{G}(A^0)$ onto the set of compatibly framed systems having associated formal normal form A^0. The orbits of the action of $G\{z\}$ on $\widehat{G}(A^0)$ correspond to the isomorphism classes: framed systems (A, g) and (B, g') are isomorphic if there exists an analytic transformation $h \in G\{z\}$ such that $B = h[A]$ and $g' = g \cdot h(0)^{-1}$ □

The remarkable fact now is that $\mathcal{H}(A^0)$ naturally has the structure of a finite dimensional unipotent Lie group and so in particular it is isomorphic to a vector space, namely its Lie algebra. The next section will be spent describing this result.

Remark 1.18. (Birkhoff Classes). Another related way of thinking of $\mathcal{H}(A^0)$ that is commonly used is as the set of systems formally equivalent to A^0 and having the same leading term as A^0, modulo the group of analytic transformations which have constant term 1. More precisely define the group of *formal Birkhoff transformations*

$$\widehat{B} := \left\{\widehat{F} \in \widehat{G} \mid \widehat{F}(0) = 1\right\}$$

to be the subgroup of $\widehat{G}$ of elements with constant term 1. Also by intersecting with $G\{z\}$ and $\widehat{G}(A^0)$ define $B\{z\}$ and $\widehat{B}(A^0)$, the *convergent* and *applicable* Birkhoff transformations respectively. Let $\mathrm{Syst}_B(A^0)$ be the set of systems which are formally equivalent to A^0 and have the same leading term as A^0. Now Lemma 1.13 implies that the map $\widehat{F} \mapsto \widehat{F}[A^0]$ from $\widehat{B}(A^0)$ to $\mathrm{Syst}_B(A^0)$ is bijective. Moreover it is clear that $B\{z\}\backslash\widehat{B}(A^0) \cong G\{z\}\backslash\widehat{G}(A^0)$ and so the stated result follows:

$$\mathcal{H}(A^0) \cong \mathrm{Syst}_B(A^0)/B\{z\}.$$

2. Stokes Factors, Torus Actions and Local Monodromy

We will explain how the set $\mathcal{H}(A^0)$ defined above may be described in terms of 'Stokes factors' which encode how certain fundamental solutions differ on sectors at 0. Fix a nice formal normal form

$$A^0 := dQ + \Lambda\frac{dz}{z}, \quad Q := \mathrm{diag}(q_1, \dots, q_n)$$

as above and define $q_{ij}(z)$ to be the leading term of $q_i - q_j$. Thus if $q_i - q_j = a/z^{k-1} + b/z^{k-2} + \cdots$ then $q_{ij} = a/z^{k-1}$.

Let the circle S^1 parameterise rays (directed lines) emanating from $0 \in \mathbb{C}$ and so open intervals (arcs) $U \subset S^1$ parameterise open sectors $\mathrm{Sect}(U) \subset \mathbb{C}$ with vertex 0. Intrinsically one can think of this circle as being the boundary circle of the real oriented blow up of $\mathbb{C}$ at the origin[2]. If $d_1, d_2 \in S^1$ then $\mathrm{Sect}(d_1, d_2)$ will denote the sector swept out by rays rotating in a positive sense from d_1 to d_2. The radius of these sectors will not be fixed, but will be taken sufficiently small when required later.

Definition 1.19. The *anti-Stokes directions* are the directions $d \in S^1$ such that for some $i \neq j$

$$q_{ij}(z) \in \mathbb{R}_{<0} \text{ on the ray specified by } d. \tag{12}$$

The set of anti-Stokes directions will be denoted by $\mathbb{A} \subset S^1$.

Remark 1.20. These are the directions along which $e^{q_i - q_j}$ decays most rapidly as z approaches 0. Due to this characterisation we see that anti-Stokes directions may be intrinsically associated to any nice meromorphic connection.

[2] In polar coordinates the real oriented blow up of $\mathbb{C}$ at $z = 0$ is $\mathbb{R}_{\geq 0} \times S^1$ and the projection onto $\mathbb{C}$ is $(r, e^{i\theta}) \mapsto z = re^{i\theta}$.

It is easy to see that $\mathbb{A}$ has $\pi/(k-1)$ rotational symmetry: if $q_{ij}(z) \in \mathbb{R}_{<0}$ then $q_{ji}(z\exp(\frac{\pi\sqrt{-1}}{k-1})) \in \mathbb{R}_{<0}$. Also in any arc $U \subset S^1$ subtending angle $\pi/(k-1)$ there are at most $n(n-1)/2$ anti-Stokes directions.

Definition 1.21. Let $d \in S^1$ be an anti-Stokes direction.

- The *roots* of d are the ordered pairs (ij) such that (12) holds:
$$\mathrm{Roots}(d) := \{(ij) \mid q_{ij}(z) \in \mathbb{R}_{<0} \text{ along } d\}.$$
- The *multiplicity* of d is the number of roots that d has:
$$\mathrm{Mult}(d) = \#\mathrm{Roots}(d).$$
- The *group of Stokes factors*[3] associated to d is the group
$$\mathbb{S}\mathrm{to}_d(A^0) := \{K \in GL_n(\mathbb{C}) \mid K_{ij} = \delta_{ij} \text{ unless } (ij) \text{ is a root of } d\}.$$

It is not hard to check that the group $\mathbb{S}\mathrm{to}_d(A^0)$ is a unipotent subgroup of $GL_n(\mathbb{C})$ of dimension $\mathrm{Mult}(d)$. These groups will be studied in more detail in Section 3; they arise as faithful representations of abstract *Stokes groups* $\mathrm{Sto}_d(A^0)$ which will be defined in Definition 1.25. We can now state the main result describing the set $\mathcal{H}(A^0)$ of isomorphism classes of compatibly framed systems having associated formal normal form A^0:

Theorem 1.22. *There is a natural isomorphism*
$$\mathcal{H}(A^0) \cong \prod_{d\in\mathbb{A}} \mathrm{Sto}_d(A^0) \tag{13}$$
and for each choice of $\log(z)$ *in the direction* d *the Stokes group* $\mathrm{Sto}_d(A^0)$ *has a faithful representation* ρ *on* $\mathbb{C}^n$ *inducing an isomorphism*
$$\rho : \mathrm{Sto}_d(A^0) \cong \mathbb{S}\mathrm{to}_d(A^0). \tag{14}$$
In particular each $\mathrm{Sto}_d(A^0)$ *and therefore* $\mathcal{H}(A^0)$ *is a unipotent Lie group and the complex dimension of* $\mathcal{H}(A^0)$ *is* $(k-1)n(n-1)$ *where* k *is the order of the pole of* A^0 *and* n *is the rank.*

This is a hard result and we will not give the proof. Proofs may be found in [**68**] and [**14**] although the 'nice' version was known earlier [**16**]. To indicate the complexities involved we remark that the proofs use the Malgrange-Sibuya Theorem which expresses $\mathcal{H}(A^0)$ as the first cohomology of a sheaf of non-Abelian unipotent groups over the circle. We do not need the Malgrange-Sibuya Theorem here but remark that its proof provided motivation for Chapter 3. Our notation $\mathcal{H}(A^0)$ is supposed to be reminiscent of the cohomological description of this set.

It is sufficient for us here to define the maps in (13) and (14) and thereby explain how to associate Stokes factors to compatibly framed systems. The map in (13) rests on the key Proposition 1.24 below. Firstly we will set up a labelling convention.

If we make a choice of a sector not containing any anti-Stokes directions then we will label all the anti-Stokes directions and the sectors between them as follows. Let d_1 be the

[3]Beware that the terms 'Stokes factors' and 'Stokes matrices' are used in a number of different senses in the literature. The notions used here are given in Definitions 1.27 and 1.36. Our terminology is closest to Balser, Jurkat and Lutz [**16**]. However our approach is perhaps closer to that of Martinet and Ramis [**78**] but what we call Stokes factors, they call Stokes matrices, and they do not use the things we call Stokes matrices.

first anti-Stokes direction encountered when moving in a positive sense from the chosen sector and label the others as $d_2, \dots, d_r$ where d_{i+1} is the next anti-Stokes direction when turning in a positive sense from d_i and r is the number of anti-Stokes directions. The indices will be taken modulo r. The sector $\mathrm{Sect}(d_i, d_{i+1})$ will be referred to as the ith sector at 0 and will be denoted Sect_i. The sector we originally chose is (a subsector of) $\mathrm{Sect}_0 = \mathrm{Sect}_r = \mathrm{Sect}(d_r, d_1)$; the *last* sector at 0.

DEFINITION 1.23. The *supersector* associated to the sector Sect_i is:

$$\widehat{\mathrm{Sect}}_i := \mathrm{Sect}\left(d_i - \frac{\pi}{2k-2}, d_{i+1} + \frac{\pi}{2k-2}\right).$$

Thus the ith supersector is a sector containing the ith sector symmetrically (that is, the same direction bisects both) and has opening greater than $\frac{\pi}{k-1}$. The directions that bound the supersectors are usually referred to as *Stokes directions.*

Proposition 1.24. *Suppose $\widehat{F} \in \widehat{G}(A^0)$ is an applicable formal transformation, let $A := \widehat{F}[A^0]$ be the associated system and take the radius of the sectors $\mathrm{Sect}_i, \widehat{\mathrm{Sect}}_i$ to be less than the radius of convergence of A. Then the following hold:*
1) *On each sector Sect_i there is a canonical way to choose an invertible (holomorphic) solution $\Sigma_i(\widehat{F})$ of $\mathrm{Hom}(A^0, A)$.*
2) *If $\Sigma_i(\widehat{F})$ is analytically continued to the supersector $\widehat{\mathrm{Sect}}_i$ then $\Sigma_i(\widehat{F})$ is asymptotic to $\widehat{F}$ at 0 within $\widehat{\mathrm{Sect}}_i$:*

$$Æ_{\widehat{\mathrm{Sect}}_i}(\Sigma_i(\widehat{F})) = \widehat{F}.$$

3) *If $\widehat{H} \in \widehat{G}$ is the Taylor series at 0 of an analytic gauge transformation $H \in G\{z\}$ then*

$$\Sigma_i(\widehat{H}\widehat{F}) = H\Sigma_i(\widehat{F}) \quad \text{and} \quad \Sigma_i(\widehat{F}\widehat{H}) = \Sigma_i(\widehat{F})H.$$

See Appendix C for details about asymptotic expansions on sectors. The point of this result is that on a small sector there are generally many solutions asymptotic to $\widehat{F}$ and one is being chosen in a canonical way. There are basically two equivalent ways to define $\Sigma_i(\widehat{F})$: algorithmic (start with some solution and modify it to obtain the canonical one which is in fact uniquely characterised by the property 2), see [**16, 68**]), or summation-theoretic (modern summation theory provides methods of summing the formal series $\widehat{F}$ on the sector $\mathrm{Sect}(d_i, d_{i+1})$ to give the analytic solution $\Sigma_i(\widehat{F})$; the series $\widehat{F}$ is '$(k-1)$-summable', see[4] [**15, 74, 78**]). The last statement in Proposition 1.24 follows from the fact that the summation operator $\Sigma_i(\cdot)$ is a differential algebra morphism (from $(k-1)$-summable series to analytic functions on sectors) extending the usual summation operator from convergent power series to germs at 0 of holomorphic functions.

The details of the construction of $\Sigma_i(\widehat{F})$ will not be needed.

To define the map $\mathcal{H}(A^0) \to \prod_{d\in\mathbb{A}} \mathrm{Sto}_d(A^0)$ in Theorem 1.22, choose $\widehat{F} \in \widehat{G}(A^0)$ representing an element of $\mathcal{H}(A^0) \cong G\{z\}\backslash\widehat{G}(A^0)$ and an anti-Stokes direction $d \in \mathbb{A}$. The sums of $\widehat{F}$ on the two sectors adjacent to d may be analytically continued across d, and they will generally be different on the overlap. Thus to each anti-Stokes direction $d = d_i$ there is an associated automorphism

$$K_{d_i} := (\Sigma_i(\widehat{F}))^{-1}\Sigma_{i-1}(\widehat{F})$$

[4]The fact that the $(k-1)$-sum of $\widehat{F}$ in Sect_i has the property 2) appears to be well known and can be deduced from the more general 'multisum' results proved in [**15**].

describing how the sums of $\widehat{F}$ differ on both sides of d_i; it is a solution of $\mathrm{Hom}(A^0, A^0)$ asymptotic to 1 on a sectorial neighbourhood of d_i.

DEFINITION 1.25. The *Stokes group* $\mathrm{Sto}_d(A^0)$ is the set of such automorphisms that arise as we vary the choice of $\widehat{F}$:

$$\mathrm{Sto}_d(A^0) := \left\{ \kappa_d = (\Sigma_i(\widehat{F}))^{-1}\Sigma_{i-1}(\widehat{F}) \;\middle|\; \widehat{F} \in \widehat{G}(A^0) \right\}.$$

Taking all such automorphisms gives a map

$$\widehat{G}(A^0) \to \prod_{d\in\mathbb{A}} \mathrm{Sto}_d(A^0); \quad \widehat{F} \mapsto (\kappa_{d_1}, \ldots, \kappa_{d_r}). \tag{15}$$

The third part of Proposition 1.24 implies that each automorphism κ_d only depends on the $G\{z\}$ orbit of $\widehat{F}$ and so a well defined map $\mathcal{H}(A^0) \to \prod_{d\in\mathbb{A}} \mathrm{Sto}_d(A^0)$ is induced, as required. The weight of the first part of Theorem 1.22 is that the fibres of (15) are precisely the $G\{z\}$ orbits.

The faithful representation of $\mathrm{Sto}_d(A^0)$ arises since the solutions of $\mathrm{Hom}(A^0, A^0)$ are known explicitly: given a choice of $\log(z)$ in the direction d we get a genuine fundamental solution $z^\Lambda e^Q$ of A^0 there. The map

$$\rho : \kappa \mapsto K := e^{-Q} z^{-\Lambda} \kappa z^\Lambda e^Q$$

then relates solutions κ of $\mathrm{Hom}(A^0, A^0)$ to the constant matrices $K \in \mathrm{End}(E)$ (that is, to solutions of $\mathrm{Hom}(0, 0)$). From this perspective the weight of the second part of Theorem 1.22 is that the image of $\mathrm{Sto}_d(A^0)$ under ρ is as given in the definition of $\mathbb{S}\mathrm{to}_d(A^0)$. Thus the abstract automorphisms κ are represented by concrete Stokes factors but we emphasise that the isomorphism ρ depends on a choice of $\log(z)$.

REMARK 1.26. (Log. choices). Rather than choose independent branches of $\log(z)$ on each anti-Stokes direction we will choose a labelling of the sectors and anti-Stokes directions as above, then choose a branch of $\log(z)$ along d_1 and extend this choice in a positive sense across Sect_1 to $d_2, \ldots, d_r$ and finally across Sect_r. (In the $k = 1$ case there are no anti-Stokes directions so choose any direction $d_1 \in S^1$ and then choose a branch of $\log(z)$ as above starting on d_1.)

DEFINITION 1.27. Given such a choice of labelling and $\log(z)$, the *Stokes factors* of a compatibly framed system (A, g) with associated formal normal form A^0 are the r-tuple of matrices:

$$\mathbf{K} = (K_1, \ldots, K_r) \in \prod_{d\in\mathbb{A}} \mathbb{S}\mathrm{to}_d(A^0)$$

$$K_i := \rho(\kappa_{d_i}) = e^{-Q} z^{-\Lambda} \kappa_{d_i} z^\Lambda e^Q \qquad \text{using the choice of } \log(z) \text{ along } d_i.$$

REMARK 1.28. From the proof of Lemma 1.17 or from Remark 1.18 one may equivalently regard these as the Stokes factors of a marked pair $(A, \widehat{F})$ (and write $\mathbf{K}(\widehat{F})$) or of a system A with the same (diagonal) leading term as A^0 (and write $\mathbf{K}(A)$).

A useful way of thinking about these Stokes factors is facilitated by

DEFINITION 1.29. Fix a choice of sector labelling and $\log(z)$ as above. If (A, g) is a compatibly framed system with associated formal normal form A^0 then the *canonical fundamental solution of A* on Sect_i is

$$\Phi_i := \Sigma_i(\widehat{F}) z^\Lambda e^Q$$

where z^Λ uses the given choice of $\log(z)$ on Sect_i. (Note this implies $\Phi_{i+r} = \Phi_i$.)

REMARK 1.30. In general, given a compatibly framed system, we need to choose a local coordinate, a sector and a branch of $\log(z)$ on the sector, before obtaining a canonical solution on the sector.

Of course Φ_i may be continued (as a solution of $\mathrm{Hom}(0, A)$) to wherever A is defined and nonsingular. From Proposition 1.24 we know that Φ_i will be asymptotic to $\widehat{F}z^\Lambda e^Q$ at 0 within $\widehat{\mathrm{Sect}}_i$ (when continued without any winding).

Immediately we have:

Lemma 1.31. *If Φ_i is continued across the anti-Stokes ray d_{i+1} then on Sect_{i+1}:*

$$\Phi_i = \Phi_{i+1}K_{i+1} \quad \text{for } i = 1, \dots, r-1, \text{ and}$$

$$\Phi_i = \Phi_1 K_1 M_0 \quad \text{for } i = r,$$

where $M_0 = e^{2\pi\sqrt{-1}\Lambda}$ is the formal monodromy.

Proof. This follows straight from the definitions, taking care to use the appropriate branches of $\log(z)$ (the Stokes factors were defined using the choices of $\log(z)$ on the anti-Stokes directions and the canonical solutions use the choices on the sectors) □

Thus, in summary, a compatibly framed system (A, g) has canonical fundamental solutions Φ_i and the Stokes factors express how these differ (according to Lemma 1.31) and moreover they encode the moduli of (A, g) (according to Theorem 1.22).

2.1. Torus Actions. To return to the analytic classes ${}^0C(A^0) = \mathcal{H}(A^0)/T$ we will examine how the torus T acts. By definition $\mathcal{H}(A^0) = G\{z\}\backslash\widehat{G}(A^0)$ and T acts on the right of the set $\widehat{G}(A^0)$ of applicable formal transformations:

$$t(\widehat{F}) = \widehat{F} \cdot t^{-1}$$

where $t \in T, \widehat{F} \in \widehat{G}(A^0)$. Recall from the proof of Lemma 1.17 that $\widehat{G}(A^0)$ is isomorphic to the set of compatibly framed systems having associated formal normal form A^0:

$$\widehat{F} \mapsto (A, g)$$

where $g = \widehat{F}(0)^{-1} \in GL_n(\mathbb{C})$ and $A = \widehat{F}[A^0] \in \mathrm{Syst}(A^0)$. Thus T acts on the set of compatibly framed systems as:

$$t(A, g) = (A, tg).$$

The corresponding action on compatibly framed connections is as follows. If (V, ∇, g) is a compatibly framed connection germ at $0 \in \mathbb{C}$, then

$$g : V_0 \xrightarrow{\cong} \mathbb{C}^n$$

is an isomorphism such that the leading coefficient of ∇ at 0 is diagonal in any trivialisation extending g. Composing g on the left with a diagonal matrix $t \in T$ does not change this property so we have a well defined (intrinsic) torus action on compatibly framed connections:

$$t(V, \nabla, g) = (V, \nabla, t \circ g).$$

On the other hand we can see how this torus acts on the corresponding Stokes data:

Lemma 1.32. *The torus $T = (\mathbb{C}^*)^n$ acts by diagonal conjugation on the set of Stokes groups $\prod_{d\in\mathbb{A}} \mathrm{Sto}_d(A^0)$:*

$$t(\kappa_{d_1}, \dots, \kappa_{d_r}) := (t\kappa_{d_1}t^{-1}, \dots, t\kappa_{d_r}t^{-1})$$

and on r-tuples of Stokes factors:

$$t(K_1, \dots, K_r) := (tK_1t^{-1}, \dots, tK_rt^{-1})$$

where $t \in T$. These actions correspond via the isomorphisms (13) and (14) in Theorem 1.22 to the action of T on $\mathcal{H}(A^0)$.

Proof. Since the action on $\mathcal{H}(A^0)$ corresponds to the action of $t \in T$ on $\widehat{G}(A^0)$ which is given by $\widehat{F} \mapsto \widehat{F}t^{-1}$, the result follows by using the last part of Proposition 1.24 and the fact that Λ and Q are diagonal □

2.2. Local Monodromy. It will be useful later to know how the Stokes factors and the formal monodromy of a compatibly framed system (A, g) determine the conjugacy class in $GL_n(\mathbb{C})$ of the local monodromy of A (that is, the monodromy of A around a simple closed loop encircling the origin once in a positive direction). Observe that this conjugacy class is well defined since choosing a different representative germ defined on a smaller neighbourhood of 0 and/or using a smaller loop around 0 only conjugates the monodromy.

Lemma 1.33. *The local monodromy of (A, g) is conjugate to the product*

$$K_r \cdots K_1 M_0$$

where $K_1, \dots, K_r$ are the Stokes factors of (A, g) with respect to some labelling and $\log(z)$ choice, and M_0 is the formal monodromy.

Proof. Just extend the fundamental solution Φ_r of A on Sect_r around the origin in a positive sense over each anti-Stokes ray $d_1, \dots, d_r$ in turn. Lemma 1.31 implies that on Sect_i the continuation of Φ_r is equal to $\Phi_i K_i \cdots K_1 M_0$ for $i = 1, 2, \dots, r$. Thus on return to Sect_r, Φ_r has become $\Phi_r K_r \cdots K_1 M_0$ □

3. Stokes Matrices

The Stokes factors of a compatibly framed system (A, g) do not behave well under small perturbations of (A, g). For example if an anti-Stokes direction has multiplicity greater than one then it can break up into distinct anti-Stokes directions under arbitrarily small changes in the formal normal form A^0. The dimensions of the groups of Stokes factors jump accordingly.

To remedy this, the Stokes factors may be collected up into *Stokes matrices* which are more stable, as we will explain in this section[5].

[5]Two reasons for working with the Stokes factors, rather than entirely with the Stokes matrices are: 1) The Stokes factors can be used to explicitly describe the braid group action on monodromy data, at least in the Frobenius manifold case, see Dubrovin [**31**] Appendix F, and 2) The Stokes factors occur in the description, due to Ramis, of the local differential Galois group of the system (see [**68, 78**]).

DEFINITION 1.34. Let U_+ be the upper triangular unipotent subgroup of $GL_n(\mathbb{C})$:

$$U_+ := \left\{ C \in GL_n(\mathbb{C}) \;\middle|\; C_{ij} = \delta_{ij} \text{ unless } i < j \right\}.$$

Similarly let U_- be the lower triangular unipotent subgroup.

Recall that the set $\mathbb{A}$ of anti-Stokes directions was symmetric under rotation by $\pi/(k-1)$. Thus the number r of anti-Stokes directions is divisible by $2k-2$:

$$r = (2k-2)l \qquad \text{for some integer } l.$$

Observe that Sect_{i+l} is simply Sect_i rotated by $\pi/(k-1)$ (l is the number of anti-Stokes directions in each $\pi/(k-1)$ 'half-period').

The basic result is then

Proposition 1.35. *Fix an ordered labelling $d_1, \ldots, d_r$ of the anti-Stokes directions. Then there is a (unique) permutation matrix*[6]*, $P \in GL_n(\mathbb{C})$, such that for $i = 1, 2, \ldots$ the multiplication map*

$$\mathbb{S}\mathrm{to}_{il}(A^0) \times \cdots \times \mathbb{S}\mathrm{to}_{(i-1)l+1}(A^0) \longrightarrow GL_n(\mathbb{C});$$

$$(K_{il}, \ldots, K_{(i-1)l+1}) \longmapsto P^{-1} K_{il} \cdots K_{(i-1)l+1} P$$

is a diffeomorphism onto $\begin{cases} U_+ & \text{if } i \text{ is odd} \\ U_- & \text{if } i \text{ is even.} \end{cases}$

Before proving this we give the key definition

DEFINITION 1.36. The ith *Stokes matrix* (with respect to a fixed labelling and $\log(z)$ choice) is the following product of l Stokes factors

$$S_i := K_{il} \cdots K_{(i-1)l+1} \in PU_{\pm}P^{-1} \subset GL_n(\mathbb{C})$$

for $i = 1, \ldots, 2k-2$.

Thus from Theorem 1.22 we obtain an even simpler description of the set of isomorphism classes of compatibly framed systems with associated formal normal form A^0:

Corollary 1.37. $\mathcal{H}(A^0) \cong (U_- \times U_+)^{k-1}$

Proof. Just take $i = 1, 2, \ldots, 2k-2$ in Proposition 1.35 □

Proof (of Proposition 1.35). For any l-tuple (or 'half-period's worth') of consecutive anti-Stokes directions

$$\mathbf{d} = (d_a, \ldots, d_{a+l-1})$$

define a total ordering of the set $\{q_1, \ldots, q_n\}$ as follows

$$q_i \underset{\mathbf{d}}{<} q_j \quad \Longleftrightarrow \quad \mathrm{Re}(q_{ij}(z)) < 0 \text{ along } \arg(z) = \theta(\mathbf{d}) \tag{16}$$

where $\theta(\mathbf{d}) \in S^1$ is the direction of the midpoint of $\mathbf{d}$; it is the direction bisecting $\mathrm{Sect}(d_a, d_{a+l-1})$ (also recall that $q_{ij}(z) = (\mathrm{const.})/z^{k-1}$ is the leading term of $q_i - q_j$). This is the natural dominance ordering along $\theta(\mathbf{d})$ since

$$\mathrm{Re}(q_{ij}(z)) < 0 \text{ along } \arg(z) = \theta(\mathbf{d}) \quad \Longleftrightarrow \quad e^{q_i}/e^{q_j} \to 0 \text{ as } z \to 0 \text{ along } \arg(z) = \theta(\mathbf{d}).$$

[6]A permutation matrix is a matrix of the form $P_{ij} = \delta_{\pi(i)j}$ for some permutation π of $\{1, \ldots, n\}$. It follows that $(P^{-1})_{ij} = \delta_{\pi^{-1}(i)j} = \delta_{i\pi(j)}$ and $(PAP^{-1})_{ij} = A_{\pi(i)\pi(j)}$ for any $n \times n$ matrix A.

To see this we need only check that we do not have $\mathrm{Re}(q_{ij}(z)) = 0$ along $\arg(z) = \theta(\mathbf{d})$. Indeed if this were the case it would follow that $\theta(\mathbf{d}) - \pi/(2k-2)$ is an anti-Stokes direction, which is not true since $\theta(\mathbf{d}) - \pi/(2k-2)$ is midway between d_{a-1} and d_a.

Next, if we let $\mathbf{d}+l$ denote $(d_{a+l}, \dots, d_{a+2l-1})$ (the 'next half-period's worth' of anti-Stokes directions) then observe that the ordering $\underset{\mathbf{d}}{<}$ is opposite to $\underset{\mathbf{d}+l}{<}$ (and so is the same as $\underset{\mathbf{d}+2l}{<}$):

$$q_i \underset{\mathbf{d}}{<} q_j \qquad \Longleftrightarrow \qquad q_j \underset{\mathbf{d}+l}{<} q_i. \tag{17}$$

Define the *group of Stokes matrices* associated to $\mathbf{d}$ to be

$$\mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0) := \left\{ S \in GL_n(\mathbb{C}) \;\middle|\; S_{ij} = \delta_{ij} \text{ unless } q_i \underset{\mathbf{d}}{<} q_j \right\}.$$

It follows from (16) that for each anti-Stokes direction d in $\mathbf{d}$ the group of Stokes factors $\mathbb{S}\mathrm{to}_d(A^0)$ is a subgroup of $\mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0)$. In fact $\mathbb{S}\mathrm{to}_d(A^0)$ is the intersection of all the groups of Stokes matrices coming from consecutive l-tuples $\mathbf{d}$ that contain d:

$$\mathbb{S}\mathrm{to}_d(A^0) = \bigcap_{\mathbf{d} \ni d} \mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0). \tag{18}$$

To see this first observe that

$$q_i \underset{\mathbf{d}}{<} q_j \qquad \Longleftrightarrow \qquad (ij) \text{ is a root of some } d \in \mathbf{d} \tag{19}$$

since (ij) being a root of $d \in \mathbf{d}$ means that $q_{ij}(z) \in \mathbb{R}_{<0}$ along $\arg(z) = d$ and so $\mathrm{Re}(q_{ij}(z)) < 0$ along $\theta(\mathbf{d})$ because the angle between d and $\theta(\mathbf{d})$ is less then $\pi/(2k-2)$. Conversely if $\mathrm{Re}(q_{ij}(z)) < 0$ along $\theta(\mathbf{d})$ then $q_{ij}(z)$ will be real and negative along some direction within $\pi/(2k-2)$ of $\theta(\mathbf{d})$; such a direction will be an anti-Stokes direction occurring in $\mathbf{d}$.

The inclusion $\subset$ in (18) follows directly from (19). In the other direction, let $(d, \dots)$ and $(\dots, d)$ respectively denote the l-tuples of consecutive anti-Stokes directions beginning and ending with d. Observe that if a root (ij) arises in both $(d, \dots)$ and $(\dots, d)$ then it is a root of d; $q_{ij}(z)$ is real and negative at most once in any sector of opening less than $2\pi/(k-1)$. Thus if K is in the right-hand side of (18) then $K_{ij} = \delta_{ij}$ unless (ij) is a root of d, and so $K \in \mathbb{S}\mathrm{to}_d(A^0)$.

Now fix $a = 1$ so $\mathbf{d} = (d_1, \dots, d_l)$. Define π to be the permutation of $\{1, \dots, n\}$ corresponding to the ordering $\underset{\mathbf{d}}{<}$ and let P be the corresponding permutation matrix:

$$q_i \underset{\mathbf{d}}{<} q_j \qquad \Longleftrightarrow \qquad \pi(i) < \pi(j); \qquad P_{ij} := \delta_{\pi(i)j}.$$

For an $n \times n$ matrix S the condition

$$S_{ij} = \delta_{ij} \text{ unless } \pi(i) < \pi(j)$$

is easily seen to be equivalent to the condition

$$(P^{-1}SP)_{ij} = \delta_{ij} \text{ unless } i < j$$

and so we deduce that $\mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0) = PU_+P^{-1}$. From (17) we then deduce that the groups of Stokes matrices for the subsequent half-periods alternate as follows

$$\mathbb{S}\mathrm{to}_{\mathbf{d}+l}(A^0) = PU_-P^{-1}, \quad \mathbb{S}\mathrm{to}_{\mathbf{d}+2l}(A^0) = PU_+P^{-1}, \quad \dots$$

All that remains is to see that the multiplication map

$$\mathbb{S}\mathrm{to}_l(A^0) \times \cdots \times \mathbb{S}\mathrm{to}_1(A^0) \longrightarrow \mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0);$$
$$(K_l, \ldots, K_1) \longmapsto K_l \cdots K_1$$

is a (surjective) diffeomorphism. If we conjugate by P we see this is a question about the multiplication map to U_+ from a full set of 'complementary' subgroups of U_+. (In the generic case we have $n(n-1)/2$ one dimensional groups on the left-hand side.) Anyway the fact that the multiplication map is a diffeomorphism is well known in the theory of algebraic groups (see for example Chapter 14 of [**20**]). It is also proved directly in Lemma 2 on p75 of [**16**] □

In terms of the canonical solutions, the Stokes matrices arise as follows.

Lemma 1.38. *Fix a choice of labelling and* $\log(z)$ *as usual and for each* i *let* Φ_{il} *be the corresponding canonical fundamental solution of* A *on* Sect_{il} *from Lemma 1.29. Then if* Φ_{il} *is continued in a positive sense across the anti-Stokes rays* $d_{il+1}, \ldots, d_{(i+1)l}$ *and onto* $\mathrm{Sect}_{(i+1)l}$ *we have:*

$$\Phi_{il} = \Phi_{(i+1)l} S_{i+1} \qquad \text{for } i = 1, \ldots, 2k-3, \text{ and}$$
$$\Phi_{(2k-2)l} = \Phi_l S_1 M_0 \qquad \text{for } i = 2k-2 = r/l$$

where $M_0 = e^{2\pi\sqrt{-1}\Lambda}$ *is the formal monodromy.*

Proof. This follows from the version for Stokes factors (Lemma 1.31) and the definition of the Stokes matrices (Definition 1.36) □

Thus if we choose some direction $d \in S^1$ which is *not* an anti-Stokes direction together with a branch of $\log(z)$ on d then the Stokes matrices arise by comparing the $2k-2$ canonical solutions that arise on small sectorial neighbourhoods of the directions

$$d,\ d + \pi/(k-1),\ d + 2\pi/(k-1),\ \ldots$$

This procedure is stable in that if the compatibly framed system is perturbed slightly then d will still not be an anti-Stokes direction and so we can still define the Stokes matrices. In fact Sibuya and Hsieh [**98, 51, 97**] prove that if the compatibly framed system (A, g) varies holomorphically with respect to some parameters, then the canonical solutions, when defined, also depend holomorphically on the parameters (see also [**60**] Proposition 3.2 p325). In particular it then follows from Lemma 1.38 that the Stokes matrices vary holomorphically with the parameters.

The stability of the Stokes matrices is crucial in defining isomonodromic deformations, since locally we now have a well defined notion of the Stokes matrices being constant. It is the factorisation process passing from Stokes matrices to Stokes factors which is badly behaved.

Finally we will record the description in terms of Stokes matrices of the torus action and the local monodromy:

Lemma 1.39.
- *The torus* $T = (\mathbb{C}^*)^n$ *acts by diagonal conjugation on the Stokes matrices*

$$t(S_1, \ldots, S_{2k-2}) := (tS_1t^{-1}, \ldots, tS_{2k-2}t^{-1})$$

where $t \in T$ *and this action corresponds to the action of* T *on* $\mathcal{H}(A^0)$.

• *The local monodromy of* (A, g) *is conjugate to the product*

$$S_{2k-2} \cdots S_1 M_0 \in GL_n(\mathbb{C})$$

where $S_1, \dots, S_{2k-2}$ *are the Stokes matrices of* (A, g) *with respect to some labelling and* $\log(z)$ *choice, and* M_0 *is the formal monodromy.*

Proof. Both statements follow directly from the corresponding versions (Lemmas 1.32, 1.33) for Stokes factors together with the definition (Definition 1.36) of the Stokes matrices □

Remark 1.40. The definition of the groups of Stokes factors and Stokes matrices can be motivated as follows. Recall from (18) in the proof of Proposition 1.35 that the group of Stokes factors associated to an anti-Stokes direction d is the intersection of all the groups of Stokes matrices that are associated to l-tuples $\mathbf{d}$ (of consecutive anti-Stokes directions) that contain d.

Thus it suffices to motivate the definition of the groups of Stokes matrices which may be described as

$$\mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0) = \left\{ S \in GL_n(\mathbb{C}) \;\middle|\; S_{ij} = \delta_{ij} \text{ unless } e^{q_i - q_j} \text{ decays along } \arg(z) = \theta(\mathbf{d}) \right\}$$

(the groups of Stokes factors are then fixed by (18)).

Relabel the anti-Stokes directions so that $\mathbf{d} = (d_1, \dots, d_l)$ and choose a branch of $\log(z)$ on Sect_0 (preceding d_1) and extend this choice in a positive sense to Sect_l (which is after d_l)[7]. The Stokes matrices for $\mathbf{d}$ are then defined by extending the corresponding canonical solution $\Phi_r = \Phi_0$ in a positive sense to Sect_l and setting

$$S = S_{\mathbf{d}} = \Phi_l^{-1}\Phi_0 = e^{-Q} z^{-\Lambda} \Sigma_l(\widehat{F})^{-1} \Sigma_0(\widehat{F}) z^{\Lambda} e^{Q}. \tag{20}$$

Now we know the asymptotic expansions of the sums on the corresponding supersectors:

$$Æ_{\widehat{\mathrm{Sect}_0}}(\Sigma_0(\widehat{F})) = \widehat{F} = Æ_{\widehat{\mathrm{Sect}_l}}(\Sigma_l(\widehat{F}))$$

and so on the intersection we have

$$Æ_{\widehat{\mathrm{Sect}_0} \cap \widehat{\mathrm{Sect}_l}}(\Sigma_l(\widehat{F})^{-1} \Sigma_0(\widehat{F})) = 1.$$

This intersection of the supersectors is a small sectorial neighbourhood of the bisecting direction $\theta(\mathbf{d})$ of the l-tuple $\mathbf{d}$. (Exceptionally this intersection will have two components, in which case we just take the component containing $\theta(\mathbf{d})$.) Thus from (20) we deduce

$$Æ_{\widehat{\mathrm{Sect}_0} \cap \widehat{\mathrm{Sect}_l}}(z^{\Lambda} e^{Q} S e^{-Q} z^{-\Lambda}) = 1$$

and hence $S_{ij} = \delta_{ij}$ unless $e^{q_i - q_j}$ decays along $\arg(z) = \theta(\mathbf{d})$, which is just the triangularity condition required to be in the group of Stokes matrices.

Remark 1.41. Note that in most of the recent references we have used, Stokes matrices are used to classify meromorphic connections within fixed formal *meromorphic* classes, modulo *meromorphic* equivalence. Whereas here we classify meromorphic connections within fixed formal *analytic* classes, modulo *analytic* equivalence, as is done in the older literature. The fact is that the sets equivalence classes are the same in both cases (compare [**70**] and [**14**]). It is important for us to work with analytic, rather than meromorphic

[7]In our usual convention the $\log(z)$ choice on Sect_l is not obtained from that on Sect_0 in this way and this leads to the occurrence of M_0 in Lemma 1.38.

gauge transformations, because then the C^∞ viewpoint in Chapter 3 is cleaner. This distinction relates to the difference between 'regular singular' connections ([**25, 62**]) and 'logarithmic' connections which is nicely illustrated in [**88**].

4. Some Symplectic Geometry

This section gives a small amount of background material on symplectic geometry and Lie group actions. The aim is mainly to give the conventions we use in this thesis regarding Hamiltonian vector fields, fundamental vector fields and moment maps. At the end we derive the formulae we will need for moment maps and symplectic structures for the natural actions of Lie groups on their cotangent bundles.

Some general references for symplectic geometry are the works [**6, 41, 66, 81**]. This thesis deals exclusively with *complex* symplectic structures and for this the start of the paper [**28**] was useful. For symplectic fibrations, see [**40, 81**].

4.1. Fundamental Vector Fields. Firstly suppose M is a complex manifold and a complex Lie group G acts on M. That is, we have a map

$$\Phi : G \times M \longrightarrow M; \qquad (g,m) \longmapsto \Phi(g,m) = g\cdot m$$

such that $(gh)\cdot m = g\cdot(h\cdot m)$ and $1\cdot m = m$. Thus for any point $m \in M$ there is a map from G to M defined by

$$\phi_m : G \longrightarrow M; \qquad g \longmapsto g\cdot m.$$

If we differentiate ϕ_m at the identity of G we obtain a linear map from the Lie algebra $\mathfrak{g}$ of G to the tangent space T_mM to M at m.

DEFINITION 1.42. The *fundamental vector field* $V_F(X)$ on M associated to an element $X \in \mathfrak{g}$ and the action Φ is the vector field on M such that, for any point $m \in M$

$$V_F(X)_m = -(\phi_m)_*(X) \in T_m(M).$$

Thus we obtain a map

$$V_F(\cdot) : \mathfrak{g} \longrightarrow Vect_M$$

from $\mathfrak{g}$ to the vector fields $Vect_M$ on M. The sign in the definition above is chosen such that this map is a Lie algebra homomorphism, where the vector fields are given their usual Lie algebra structure, defined by the Lie bracket[8]. Also $V_F(\cdot)$ intertwines the adjoint action of G on $\mathfrak{g}$ with the action of G on $Vect_m$ induced from Φ. Note that, having made this definition, it turns out that the flow of the vector field $V_F(X)$ is given by $\exp(-tX)\cdot m$, where $\exp(tX)$ is the 1-parameter subgroup of G generated by $X \in \mathfrak{g}$.

4.2. Hamiltonian vector fields. Now suppose (M,ω) is a complex *symplectic* manifold. Thus M is a complex manifold and ω is a complex symplectic form: a closed nondegenerate *holomorphic* 2-form. In particular ω gives an isomorphism between the holomorphic tangent bundle and the holomorphic cotangent bundle. Given a holomorphic function $f : M \to \mathbb{C}$ on M we can differentiate it to obtain a holomorphic one-form, and then use ω to convert it into a holomorphic vector field:

[8]Since we are interested in the complex/holomorphic symplectic case, it is better to view V_F as a morphism of *sheaves* of Lie algebras over M, from the constant sheaf $\mathfrak{g}$ to the sheaf of holomorphic vector fields. However most of our complex manifolds will be affine varieties so this is not really necessary.

DEFINITION 1.43. The *Hamiltonian vector field* $V_H(f)$ on M associated to f is the vector field on M such that the following equality of one-forms holds:

$$df = -i_{V_H(f)}\omega = \omega(\cdot, V_H(f)).$$

It is a basic fact that the flows of Hamiltonian vector fields preserve the symplectic structure ω. If we give the set $\mathcal{O}_M$ of functions on M the natural Lie algebra structure coming from the Poisson bracket defined by ω:

$$\{f, g\} := \omega(V_H(f), V_H(g)),$$

then the sign in the above definition is chosen such that the map

$$V_H(\cdot) : \mathcal{O}_M \longrightarrow Vect_M$$

is a Lie algebra homomorphism.

4.3. Moment maps. Combining the last two sections leads directly to the concept of a moment map:

DEFINITION 1.44. A *moment map* for the action Φ of a Lie group G on a symplectic manifold (M, ω) is a map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

from M to the vector space dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, such that
1) For any $X \in \mathfrak{g}$:

$$d\langle X, \mu\rangle = -i_{V_F(X)}\omega, \qquad \text{and}$$

2) μ is equivariant: it intertwines the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$.

Thus 1) says that the Hamiltonian vector field of the function $\langle X, \mu\rangle$ on M is the same as the fundamental vector field of X, for any $X \in \mathfrak{g}$. A group action on a symplectic manifold is said to be *Hamiltonian* if it admits a moment map. Another way to view condition 1) is as follows. Clearly a moment map gives a map from $\mathfrak{g}$ into the functions on M:

$$\widetilde{\mu} : \mathfrak{g} \longrightarrow \mathcal{O}_M; \qquad X \longmapsto \langle X, \mu\rangle.$$

Then 1) says that $\widetilde{\mu}$ factorises V_F through $\mathcal{O}_M$; that is $\widetilde{\mu}$ fits into the sequence

$$\mathfrak{g} \xrightarrow{\widetilde{\mu}} \mathcal{O}_M \xrightarrow{V_H} Vect_M$$

such that the composition is V_F. Also (the infinitesimal version of) condition 2) says that $\widetilde{\mu}$ is also a Lie algebra homomorphism.

Moment maps lead to the notion of symplectic (or *Marsden-Weinstein*) quotient. Suppose that a group G acts on a symplectic manifold (M, ω) and has a moment map μ. In its simplest form the symplectic quotient construction says that (under suitable general conditions) the quotient

$$M//G := \mu^{-1}(0)/G$$

of the subset $\mu^{-1}(0) \subset M$ by the action of G is again a symplectic manifold with symplectic structure $\bar{\omega}$ determined by requiring

$$i^*(\omega) = p^*(\bar{\omega})$$

where $p : \mu^{-1}(0) \to M//G$ is the projection and $i : \mu^{-1}(0) \hookrightarrow M$ is the inclusion.

See the literature referred to above for more details.

4.4. Symplectic aspects of group cotangent bundles. Here we give some formulae for the canonical symplectic structure on the cotangent bundle of G and moment maps for the natural actions of G on T^*G induced from left and right multiplication in G.

To start with, use the left multiplications in G to trivialise the tangent bundle TG. For each $g \in G$ the left multiplication $L_g : G \to G; h \mapsto gh$ gives an isomorphism

$$(dL_g)_1 : \mathfrak{g} = T_1G \to T_gG$$

and so induces a trivialisation:

$$G \times \mathfrak{g} \cong TG; \qquad (g, X) \mapsto (g, (dL_g)_1 X)$$

which will be referred to as the *left* trivialisation of TG.

By taking duals the *left* trivialisation of the cotangent bundle is also obtained:

$$G \times \mathfrak{g}^* \cong T^*G; \qquad (g, A) \mapsto (g, (dL_{g^{-1}})_1^\vee A)$$

where $(dL_{g^{-1}})_1^\vee$ denotes inverse of the the dual linear map to $(dL_g)_1$. Similarly, starting from the right multiplications, the right trivialisations may be defined. Generally (unless otherwise stated) we will always use the left trivialisations when referring to elements of TG or T^*G from now on.

Now we can write down the natural symplectic structure on T^*G explicitly:

Lemma 1.45. *Using the left trivialisations throughout, the canonical symplectic structure on T^*G is given by the following two-form on $G \times \mathfrak{g}^*$:*

$$\omega_{(g,A)}((X_1, A_1), (X_2, A_2)) = \langle A_1, X_2\rangle - \langle A_2, X_1\rangle - \langle A, [X_1, X_2]\rangle$$

where $(g, A) \in G \times \mathfrak{g}^*$, $A_1, A_2 \in \mathfrak{g}^* \cong T_A\mathfrak{g}^*$ and $X_1, X_2 \in \mathfrak{g} \cong T_gG$.

Proof. The symplectic structure on T^*G is defined to be the exterior derivative of the canonical one-form θ on T^*G. In terms of the left invariant Maurer-Cartan form '$g^{-1}dg$' (which is a $\mathfrak{g}$ valued one-form on G that we pull back to T^*G) we have

$$\theta = \langle A, g^{-1}dg\rangle.$$

Using the left trivialisation, θ is therefore the one-form on $G \times \mathfrak{g}^*$ given by:

$$\theta_{(g,A)}(X_1, A_1) = \langle A, X_1\rangle$$

since by definition the value of $g^{-1}dg$ when evaluated on a tangent vector in T_gG (represented by $X_1 \in \mathfrak{g}$) is simply X_1.

Now $d(g^{-1}dg) = -(g^{-1}dg) \wedge (g^{-1}dg)$ (Maurer-Cartan equation) and so

$$d\theta = \langle dA \overset{\wedge}{,} g^{-1}dg\rangle - \langle A, (g^{-1}dg) \wedge (g^{-1}dg)\rangle.$$

This is easily evaluated to give the desired formula □

If the right trivialisations are used instead, the formula looks the same upto one sign:

Lemma 1.46. *Using the right trivialisations throughout, the canonical symplectic structure on T^*G is given by the following two-form on $G \times \mathfrak{g}^*$:*

$$\omega_{(g,A)}((X_1, A_1), (X_2, A_2)) = \langle A_1, X_2\rangle - \langle A_2, X_1\rangle + \langle A, [X_1, X_2]\rangle$$

where $(g, A) \in G \times \mathfrak{g}^*$, $A_1, A_2 \in \mathfrak{g}^* \cong T_A\mathfrak{g}^*$ and $X_1, X_2 \in \mathfrak{g} \cong T_gG$.

Proof. As in Lemma 1.45, except that the right invariant Maurer-Cartan form $(dg)g^{-1}$ satisfies $d((dg)g^{-1}) = +(dg)g^{-1} \wedge (dg)g^{-1}$ □

Next we will look at the G actions on T^*G.

DEFINITION 1.47. The *left action* of G on T^*G is (in terms of the left trivialisation):

$$h(g, A) = (hg, A).$$

The *right action*[9] of G on T^*G is (also in terms of the left trivialisation):

$$h(g, A) = (gh^{-1}, \mathrm{Ad}_h^* A).$$

These are both Hamiltonian actions and their moment maps are as follows:

Lemma 1.48. *The left action of G on T^*G is Hamiltonian with equivariant moment map given (in terms of the left trivialisation) by*

$$\mu_L : G \times \mathfrak{g}^* \to \mathfrak{g}^*; \qquad (g, A) \mapsto -\mathrm{Ad}_g^*(A).$$

The right action of G on T^*G is Hamiltonian with equivariant moment map given by

$$\mu_R : G \times \mathfrak{g}^* \to \mathfrak{g}^*; \qquad (g, A) \mapsto A.$$

Proof. We will explain how to prove this for the left action (the proof for the right action consists of the same steps).

Choose $X \in \mathfrak{g}$ and define f to be the function on T^*G which is the X component of μ_L:

$$f(g, A) = \langle \mu_L(g, A), X \rangle = -\langle A, \mathrm{Ad}_{g^{-1}}(X) \rangle.$$

We want to prove that the Hamiltonian vector field $V_H(f)$ associated to f is equal to the fundamental vector field $V_F(X)$ associated to X. A straightforward calculation gives

$$V_F(X)_{(g,A)} = -(\mathrm{Ad}_{g^{-1}}(X), 0).$$

It then follows, using the description of the canonical one-form θ on T^*G given in Lemma 1.45, that

$$f = \theta(V_F(X)) = i_{V_F(X)}\theta.$$

Hence using Cartan's formula (that on differential forms $\mathcal{L}_{V_F(X)} = di_{V_F(X)} + i_{V_F(X)}d$) we obtain

$$df = -i_{V_F(X)}d\theta + \mathcal{L}_{V_F(X)}\theta.$$

Now one may check that θ is preserved by the left action and so $\mathcal{L}_{V_F(X)}\theta = 0$. Thus $df = -i_{V_F(X)}\omega$ and so $V_H(f) = V_F(X)$ as required □

[9] Note this is a 'left action' in the usual terminology (as are all our actions). By 'right action' above, we mean 'the left action of G on T^*G induced from the left action $h(g) = gh^{-1}$ of G on itself by *right* multiplication'.

CHAPTER 2

Meromorphic Connections on Trivial Bundles

Choose m distinct points $a_1, \dots, a_m \in \mathbb{P}^1$ and a nice formal normal form ${}^iA^0$ at each a_i. Let $\mathbf{A}$ denote this m-tuple of formal normal forms. Define $\mathcal{M}^*(\mathbf{A})$ to be the set of isomorphism classes of pairs (V, ∇) where V is a *trivial* rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V with formal normal form ${}^iA^0$ at a_i for each i and no other poles. The aim of this chapter is to give an explicit symplectic description of these moduli spaces and the related 'extended' moduli spaces.

One of the main result of this chapter, Theorem 2.35, is the description of $\mathcal{M}^*(\mathbf{A})$ as a complex symplectic quotient of a product of complex coadjoint orbits by $GL_n(\mathbb{C})$. As usual for quotients of affine varieties by reductive groups, $\mathcal{M}^*(\mathbf{A})$ may not be Hausdorff but will have a dense open subset that is a genuine complex symplectic manifold. Moreover the symplectic structure obtained in this way (from that on the coadjoint orbits) is intrinsic; it is independent of the coordinate choices made in order to obtain this description.

A similar description is given of the extended moduli spaces $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ which are defined as sets of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a nice meromorphic connection ∇ on trivial V having compatible framings $\mathbf{g} = ({}^1g, \dots, {}^mg)$ (one at each a_i) and the same irregular type as ${}^iA^0$ at a_i (but with arbitrary exponent of formal monodromy). These extended moduli spaces provide a convenient level at which to study isomonodromic deformations. They are (genuine) complex symplectic manifolds, they have the moduli spaces $\mathcal{M}^*(\mathbf{A})$ as symplectic quotients and have the property that their symplectomorphism class is not dependent on the choice $\mathbf{A}$ of nice formal normal forms. Moreover $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ is a fine moduli space (see Section 5). The idea of using extended spaces has roots in the original work of Jimbo, Miwa and Ueno [**60**] and in work of L.Jeffrey and J.Huebschmann (see[1] [**57, 42**]).

To begin with, in the first two sections we will study the coadjoint orbits and 'extended orbits' out of which the moduli spaces $\mathcal{M}^*(\mathbf{A})$ and the extended moduli spaces $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ will be built. A detailed understanding of the symplectic geometry of these orbits will help to understand the symplectic geometry of the moduli spaces.

1. The Groups G_k and their Coadjoint Orbits

Let k be a positive integer and consider the ring

$$\mathbb{C}[\zeta]/(\zeta^k)$$

of polynomials in an indeterminate ζ, modulo terms of order k.

Definition 2.1.

[1] The term 'extended moduli space' is not used in exactly the same sense as in [**57**]; see Definition 2.42 or Proposition 3.20 here.

• G_k is the complex Lie group consisting of invertible $n \times n$ matrices with entries in $\mathbb{C}[\zeta]/(\zeta^k)$:

$$G_k := GL_n(\mathbb{C}[\zeta]/(\zeta^k)).$$

• Let $\mathfrak{g}_k$ denote the Lie algebra of G_k and $\mathfrak{g}_k^*$ the vector space dual of $\mathfrak{g}_k$.
• Let B_k be the subgroup of G_k consisting of elements with constant term 1:

$$B_k := \{ \ g \in G_k \ \big| \ g(0) = 1 \ \}.$$

• Let $\mathfrak{b}_k$ be the Lie algebra of B_k and $\mathfrak{b}_k^*$ the vector space dual of $\mathfrak{b}_k$.

For $k = 1$, the group G_k is just $GL_n(\mathbb{C})$ but for $k > 1$, G_k is not even reductive since B_k is then a nontrivial unipotent normal subgroup; there is an exact sequence of groups:

$$1 \longrightarrow B_k \longrightarrow G_k \longrightarrow GL_n(\mathbb{C}) \longrightarrow 1 \tag{21}$$

where the homomorphism onto $GL_n(\mathbb{C})$ is given by evaluation at $\zeta = 0$. This sequence splits because $GL_n(\mathbb{C})$ embeds in G_k as the subgroup of constant matrices. It follows that G_k is the semi-direct product $GL_n(\mathbb{C}) \ltimes B_k$ where $GL_n(\mathbb{C})$ acts on B_k by conjugation. Coadjoint orbits of the groups G_k will be the building blocks out of which the moduli spaces $\mathcal{M}^*(\mathbf{A})$ are formed, so they will be studied in some detail. An element $g \in G_k$ is of the form:

$$g = g_0 + g_1\zeta + \cdots + g_{k-1}\zeta^{k-1}$$

with $g_i \in \mathrm{End}(E)$ where $E = \mathbb{C}^n$. Such g_i's make up an element of G_k precisely if $\det(g_0) \neq 0$. The Lie algebra $\mathfrak{g}_k$ of G_k consists of elements

$$X = X_0 + X_1\zeta + \cdots + X_{k-1}\zeta^{k-1}$$

with $X_i \in \mathrm{End}(E)$ arbitrary. Occasionally the ring structure on $\mathfrak{g}_k$ coming from this matrix representation will be used (although not in an essential way); the product XY for $X, Y \in \mathfrak{g}_k$ is defined in the obvious manner. Elements of $\mathfrak{g}_k^*$, the vector space dual of $\mathfrak{g}_k$, will be written suggestively as:

$$A = \left(\frac{A_k}{\zeta^k} + \cdots + \frac{A_1}{\zeta} \right) d\zeta$$

for arbitrary $A_i \in \mathrm{End}(E)$. The matrix A_k will be referred to as the leading coefficient of A whilst $A_k d\zeta/\zeta^k$ is the leading *term* of A. The pairing between $\mathfrak{g}_k^*$ and $\mathfrak{g}_k$ is given by

$$\langle A, X \rangle = \mathrm{Res}_0(\mathrm{Tr}(AX)) = \sum_{i=1}^{k} \mathrm{Tr}(A_i X_{i-1})$$

where Res_0 is the residue map, picking out the coefficient of $d\zeta/\zeta$. Thus the residue of A pairs with the constant term of X and the leading term of A pairs with the coefficient of ζ^{k-1} in X. Observe that the product AX is a well defined element of $\mathfrak{g}_k^*$, where $A \in \mathfrak{g}_k^*$ and $X \in \mathfrak{g}_k$. Similarly XA is well-defined; $\mathfrak{g}_k^*$ is a 'bimodule' over the ring $\mathfrak{g}_k$.

If we pass to Lie algebras in the exact sequence (21) of groups and then dualise, the following exact sequence of vector spaces is obtained:

$$0 \longrightarrow \mathfrak{gl}_n(\mathbb{C})^* \longrightarrow \mathfrak{g}_k^* \xrightarrow{\pi} \mathfrak{b}_k^* \longrightarrow 0.$$

The splitting of (21) induces a natural splitting of this sequence:

$$\mathfrak{g}_k^* = \mathfrak{b}_k^* \oplus \mathfrak{gl}_n(\mathbb{C})^*.$$

The projection onto $\mathfrak{gl}_n(\mathbb{C})^*$ just picks out the residue term and will be denoted π_{Res}

$$\pi_{\mathrm{Res}} : \mathfrak{g}_k^* \to \mathfrak{gl}_n(\mathbb{C})^*; \qquad A \mapsto A_1 \frac{d\zeta}{\zeta}.$$

Henceforth $\mathfrak{b}_k^*$ will be identified with the kernel of π_{Res} in $\mathfrak{g}_k^*$, that is with the elements having zero residue. The projection π onto $\mathfrak{b}_k^*$ just removes the residue term:

$$\pi : \mathfrak{g}_k^* \to \mathfrak{b}_k^*; \qquad A \mapsto A - \pi_{\mathrm{Res}}(A).$$

The bimodule structure of $\mathfrak{g}_k^*$ can be used to give explicit descriptions of the coadjoint actions:

Lemma 2.2. *Suppose $A \in \mathfrak{g}_k^*, g \in G_k$ and $X \in \mathfrak{g}_k$, then:*
- *the coadjoint action of G_k on $\mathfrak{g}_k^*$ is: $\mathrm{Ad}_g^*(A) = gAg^{-1}$, and*
- *the coadjoint action of $\mathfrak{g}_k$ on $\mathfrak{g}_k^*$ is: $\mathrm{ad}_X^*(A) = [X, A] = XA - AX$.*

Proof. These are straightforward using the definitions: $\langle \mathrm{Ad}_g^*(A), X\rangle = \langle A, \mathrm{Ad}_{g^{-1}}X\rangle$ and $\langle \mathrm{ad}_X^*(A), Y\rangle = \langle A, \mathrm{ad}_{-X}Y\rangle$ together with the formulae: $\mathrm{Ad}_g X = gXg^{-1}$ and $\mathrm{ad}_X Y = XY - YX$ for the adjoint actions on $\mathfrak{g}_k$ □

DEFINITION 2.3.
- The G_k coadjoint orbit through $A \in \mathfrak{g}_k^*$ will be denoted

$$O(A) = \{\ gAg^{-1} \bigm| g \in G_k\ \} \subset \mathfrak{g}_k^*.$$

- The *nice* G_k coadjoint orbits are those whose elements have leading coefficients which are
1) diagonalisable with distinct eigenvalues, if $k \geq 2$, or
2) diagonalisable with distinct eigenvalues mod $\mathbb{Z}$, if $k = 1$.

These coadjoint orbits are homogeneous spaces for G_k and so are smooth complex manifolds. As is well known in symplectic geometry, coadjoint orbits have natural (Kostant-Kirillov) symplectic structures; they are the symplectic leaves of the (Lie) Poisson bracket on the dual of the Lie algebra. Since everything is complex here, $O(A)$ is naturally a complex symplectic manifold. Some useful facts about these orbits are as follows:

Lemma 2.4. *Let $O \subset \mathfrak{g}_k^*$ be a coadjoint orbit of G_k and suppose $A \in O$.*
1) As a subspace of $T_A\mathfrak{g}_k^* \cong \mathfrak{g}_k^*$ the tangent space to O at A is

$$T_A O = \{\ [X, A] \bigm| X \in \mathfrak{g}_k\ \} \subset \mathfrak{g}_k^*.$$

2) The Kostant-Kirillov symplectic form on O is given by the explicit formula

$$\omega_A([A, X], [A, Y]) = \langle A, [X, Y]\rangle.$$

This means that if $P, Q \in T_A O$ then $\omega_A(P, Q) = \langle A, [X, Y]\rangle$ for any $X, Y \in \mathfrak{g}_k$ such that $P = [A, X]$ and $Q = [A, Y]$.
3) The action of $GL_n(\mathbb{C})$ on O (via the inclusion $GL_n(\mathbb{C}) \hookrightarrow G_k$ and the coadjoint action) is Hamiltonian with moment map given by taking the residue term:

$$\pi_{\mathrm{Res}} : O \longrightarrow \mathfrak{gl}_n(\mathbb{C})^*; \quad A = \left(\frac{A_k}{\zeta^k} + \cdots + \frac{A_1}{\zeta}\right) d\zeta \longmapsto A_1 \frac{d\zeta}{\zeta}.$$

Proof. The first part follows from the definition of ad^* as the derivative of Ad^*. For 2), if $X, Y \in \mathfrak{g}_k$ are regarded as linear functions on $\mathfrak{g}_k^*$, their Poisson bracket is defined to be the function on $\mathfrak{g}_k^*$ taking the value:

$$\{X, Y\}(A) = \langle A, [X, Y]\rangle = \langle [A, X], Y\rangle$$

at $A \in \mathfrak{g}_k^*$. Thus the value at A of the Hamiltonian vector field of the function X is

$$\{X, \cdot\}(A) = [A, X] \in T_A\mathfrak{g}_k^* \cong \mathfrak{g}_k^*.$$

These vectors span T_AO, and the formula for the symplectic form follows since by definition

$$\omega_A([A, X], [A, Y]) = \{X, Y\}(A).$$

For the third part recall that the moment map for the coadjoint action is simply the inclusion $O \to \mathfrak{g}_k^*$. Then the moment map for the subgroup $GL_n(\mathbb{C}) \subset G_k$ is just the composition of this inclusion with the natural projection $\mathfrak{g}_k^* \to \mathfrak{gl}_n(\mathbb{C})^*$ (which is the transpose of the derivative at 1 of the inclusion $GL_n(\mathbb{C}) \hookrightarrow G_k$). This is the map we have denoted by π_{Res} □

Now we will use the Cayley-Hamilton Theorem for matrices over the ring $\mathbb{C}[\zeta]/(\zeta^k)$ to see that the nice coadjoint orbits are affine algebraic varieties cut out in $\mathfrak{g}_k^*$ by the characteristic polynomial map. The elements in a given orbit are characterised by their set of 'eigenvalues' in the ring $\mathbb{C}[\zeta]/(\zeta^k)$.

Let X be an element of the Lie algebra $\mathfrak{g}_k$ and define the characteristic polynomial

$$P_X(\lambda) := \det(\lambda 1 - X) \in \mathbb{C}[\lambda, \zeta]/(\zeta^k)$$

of X over the ring $\mathbb{C}[\zeta]/(\zeta^k)$; it is a degree n polynomial in λ with coefficients in $\mathbb{C}[\zeta]/(\zeta^k)$. The basic results we need are

Lemma 2.5. *Suppose $X \in \mathfrak{g}_k$ has constant term $X(0)$ with distinct eigenvalues.*
1) If $g \in G_k$ then $P_{gXg^{-1}} = P_X$
2) If $Y \in \mathfrak{g}_k$ and $P_Y = P_X$ then $Y = gXg^{-1}$ for some $g \in G_k$.

Proof. 1) is clear from the definition of P_X. For 2) we firstly observe that by setting $\zeta = 0$ in the equality $P_Y = P_X$ we obtain the equality of the usual characteristic polynomials of the constant terms $X(0)$ and $Y(0)$. In particular $Y(0)$ will therefore have distinct eigenvalues. Thus we can choose diagonalisations

$$X^0 = \mathrm{diag}(x_1, \ldots, x_n) \quad \text{and} \quad Y^0 = \mathrm{diag}(y_1, \ldots, y_n)$$

of X and Y respectively, where $x_i, y_j \in \mathbb{C}[\zeta]/(\zeta^k)$ (the algorithm in Appendix B can easily be adapted to do this). Moreover these diagonalisations can be chosen such that $x_i(0) = y_i(0)$ for all i since the constant terms of X and Y are conjugate. By assumption and from 1) we have $P_{X^0} = P_{Y^0}$. Now the Cayley-Hamilton theorem tells us that Y^0 satisfies its characteristic polynomial, that is

$$P_{Y^0}(Y^0) = 0 \in \mathrm{End}_n(\mathbb{C}[\zeta]/(\zeta^k)).$$

Thus $P_{X^0}(Y^0) = 0$. The ith diagonal entry of this equation reads

$$(y_i - x_1)(y_i - x_2) \cdots (y_i - x_n) = 0 \in \mathbb{C}[\zeta]/(\zeta^k). \tag{22}$$

Now if $i \neq j$ then $y_i - x_j$ has nonzero constant term and so is a unit in $\mathbb{C}[\zeta]/(\zeta^k)$. Therefore we can multiply the equation (22) through by the inverses of all of these elements to obtain, for each i:

$$(y_i - x_i) = 0 \in \mathbb{C}[\zeta]/(\zeta^k).$$

Thus $X^0 = Y^0$ and so X and Y are conjugate in $\mathfrak{g}_k$ as required □

Immediately we obtain two corollaries

Corollary 2.6. *The nice coadjoint orbits are affine algebraic varieties.*

Proof. We can translate between coadjoint orbits and adjoint orbits by multiplying through by $d\zeta/\zeta^k$. If A is an element in a nice coadjoint orbit write $A = Xd\zeta/\zeta^k$ so that $X \in \mathfrak{g}_k$ has constant term with distinct eigenvalues. Thus we have

$$O(A) = \{\, Yd\zeta/\zeta^k \;\big|\; Y \in \mathfrak{g}_k \text{ and } P_Y = P_X \}$$

and so the result follows □

Corollary 2.7. *Suppose $A \in \mathfrak{g}_k^*$ has leading coefficient with distinct eigenvalues and we have chosen an order of them $e_1, \dots, e_n$. Then this choice of diagonalisation*

$$\mathrm{diag}(e_1, \dots, e_n)$$

of the leading coefficient of A extends uniquely to a diagonalisation of A. That is there are unique elements $f_1, \dots, f_n \in \mathbb{C}[\zeta]/(\zeta^k)$ such that $f_i(0) = e_i$ for all i and the corresponding diagonal element of $\mathfrak{g}_k^*$ is in the same coadjoint orbit as A:

$$\mathrm{diag}(f_1, \dots, f_n)d\zeta/\zeta^k \in O(A)$$

□

Finally it will be useful later to know that these G_k coadjoint orbits behave well under automorphisms of the ring $\mathbb{C}[\zeta]/(\zeta^k)$ which will correspond below to local coordinate changes on $\mathbb{P}^1$:

Lemma 2.8. *Suppose $\zeta \mapsto \zeta' = f(\zeta) = \lambda_1\zeta + \cdots \lambda_{k-1}\zeta^{k-1}$ where $\lambda_i \in \mathbb{C}$ and $\lambda_1 \neq 0$. Then the induced linear automorphism*

$$\phi : \mathfrak{g}_k^* \to \mathfrak{g}_k^*; \quad A = \left(\frac{A_k}{\zeta^k} + \cdots + \frac{A_1}{\zeta}\right) d\zeta \mapsto A' = \left(\frac{A_k}{f^k} + \cdots + \frac{A_1}{f}\right) df$$

induces a symplectic isomorphism $O(A) \cong O(A')$ for any $A \in \mathfrak{g}_k^*$.
Moreover ϕ commutes with the $GL_n(\mathbb{C})$ action and does not affect the moment map π_{Res}.

Proof. Firstly observe that $\phi(gAg^{-1}) = g'A'(g')^{-1}$ for any $g \in G_k$ and so in particular orbits are mapped to orbits under ϕ. Also if $g \in GL_n(\mathbb{C})$ then $g' = g$ so ϕ commutes with the $GL_n(\mathbb{C})$ action. Next observe $\mathrm{Res}_0(A) = \mathrm{Res}_0(A')$ for any $A \in \mathfrak{g}_k^*$ (since, for example, Res_0 is a linear map and $\mathrm{Res}_0(df/f^i)$ is zero for $i > 1$ and $\mathrm{Res}_0(df/f) = 1$). Thus the moment map is not affected and also $\mathrm{Res}_0\mathrm{Tr}(A) = \mathrm{Res}_0\mathrm{Tr}(A')$. Since ϕ is linear it equals its derivative and so $\phi_*([A, X]) = [A', X']$ for any $X \in \mathfrak{g}_k$. Feeding these facts into the definition of the symplectic structures on the coadjoint orbits yields the result □

REMARK 2.9. This is an instance of a more general fact: any group automorphism $\phi : G_k \to G_k$ will induce an automorphism of $\mathfrak{g}_k^*$ which, on restriction, yields symplectic isomorphisms between coadjoint orbits. The inner automorphisms of G_k (which are obtained by letting G_k act on itself by conjugation) will just induce automorphisms of each coadjoint orbit whereas the outer automorphisms (such as the coordinate transformations above) will generally identify distinct orbits.

2. Extended Orbits

Now suppose that k is at least two and choose a *diagonal* element

$$A^0 := \left(\frac{A_k^0}{\zeta^k} + \cdots + \frac{A_2^0}{\zeta^2}\right) d\zeta \in \mathfrak{b}_k^*$$

of $\mathfrak{b}_k^*$ such that its leading coefficient A_k^0 has distinct diagonal entries.

Let $O_B = O_B(A^0)$ be the B_k coadjoint orbit through A^0 and observe that each element in this coadjoint orbit has the same leading coefficient A_k^0, since the elements of B_k have constant term 1.

DEFINITION 2.10. The *extension* or *extended orbit* associated to the B_k coadjoint orbit O_B is the set:

$$\widetilde{O} = \widetilde{O}(A^0) := \left\{(g_0, A) \in GL_n(\mathbb{C}) \times \mathfrak{g}_k^* \;\middle|\; \pi(g_0 A g_0^{-1}) \in O_B\right\}$$

where $\pi : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ is the natural projection removing the residue.

REMARK 2.11. The element $A \in \mathfrak{g}_k^*$ in a pair $(g_0, A) \in \widetilde{O}$ will eventually correspond to the polar part of a meromorphic connection and g_0 will correspond to a compatible framing.

This section aims to elucidate some of the geometry of these extended orbits. They are the building blocks out of which the extended moduli spaces will be built.

REMARK 2.12. At the end of this section (p34) we will define extended orbits also in the $k = 1$ case; our main interest is when $k \geq 2$ (the irregular case) but we will show that the $k = 1$ extended orbits have most of the same properties. We also describe the $k = 2$ case in detail on p33 as an illustrative example.

We will give a number of different descriptions of $\widetilde{O}$ including: as a principal T-bundle over a family of G_k coadjoint orbits parameterised by the Lie algebra $\mathfrak{t}$ (Corollary 2.15), and as a symplectic quotient of the cotangent bundle T^*G_k of the group G_k (Proposition 2.19). However perhaps the most interesting description of $\widetilde{O}$ will be given first; immediately we can prove that $\widetilde{O}$ is a complex manifold and is in fact simply the product of the cotangent bundle of $GL_n(\mathbb{C})$ with the original B_k coadjoint orbit O_B:

Lemma 2.13. *(Decoupling). The following map is a complex analytic isomorphism:*

$$\widetilde{O} \cong T^*GL_n(\mathbb{C}) \times O_B; \quad (g_0, A) \mapsto ((g_0, \pi_{\mathrm{Res}}(A)), \pi(g_0 A g_0^{-1}))$$

where $T^*GL_n(\mathbb{C}) \cong GL_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^*$ via the left trivialisation and π, π_{Res} are the projections from $\mathfrak{g}_k^*$ onto $\mathfrak{b}_k^*$, $\mathfrak{gl}_n(\mathbb{C})^*$ respectively.

Proof. The map is clearly well defined. One may check that the map:

$$((g_0, S), B) \mapsto (g_0, g_0^{-1} B g_0 + S) \in \widetilde{O}$$

where $((g_0, S), B) \in T^*GL_n(\mathbb{C}) \times O_B$, gives an inverse $\square$

In particular this description endows $\widetilde{O}$ with a complex symplectic structure, coming from the standard symplectic structures on the cotangent bundle $T^*GL_n(\mathbb{C})$ and on the coadjoint orbit $O_B \subset \mathfrak{b}_k^*$ respectively.

Decoupling $\widetilde{O}$ as $T^*GL_n(\mathbb{C}) \times O_B$ will turn out to be important when we study the symplectic geometry of irregular isomonodromic deformations since the irregular deformation parameters will correspond to the choice of the diagonal element A^0 of $\mathfrak{b}_k^*$. It will thus be sufficient to understand how the B_k coadjoint orbits $O_B(A^0)$ vary with A^0.

In some sense though this decoupled description of $\widetilde{O}$ is too simple and obscures some of the geometry. Consider for example the projection onto the second factor

$$\widetilde{O} \to \mathfrak{g}_k^*; \qquad (g_0, A) \mapsto A$$

and let Θ denote the image of this projection in $\mathfrak{g}_k^*$:

$$\Theta := \left\{ A \in \mathfrak{g}_k^* \;\middle|\; (g_0, A) \in \widetilde{O} \text{ for some } g_0 \in GL_n(\mathbb{C}) \right\}.$$

We then have

Lemma 2.14.

1) *There is a free action of the diagonal torus $T \cong (\mathbb{C}^*)^n \subset GL_n(\mathbb{C})$ on the extended orbit $\widetilde{O}$ given by:*

$$t(g_0, A) = (tg_0, A).$$

2) *The T orbits in $\widetilde{O}$ are precisely the fibres of the projection $\widetilde{O} \to \Theta$ onto the second factor.*

3) *For each $A \in \Theta$ there is a unique diagonal element $R = \Lambda d\zeta/\zeta \in \mathfrak{t}^*$ such that $A^0 + R$ is a diagonalisation of A:*

$$A = g^{-1}(A^0 + R)g \qquad \textit{for some } g \in G_k.$$

4) *If $A \in \mathfrak{g}_k^*$ is conjugate to $A^0 + R$ for some $R \in \mathfrak{t}^*$ then $A \in \Theta$.*

Before proving this, observe that this lemma gives our second description of $\widetilde{O}$:

Corollary 2.15.

- *Θ is a disjoint union of G_k coadjoint orbits parameterised by $\mathfrak{t}^*$:*

$$\Theta = \coprod_{R \in \mathfrak{t}^*} O(A^0 + R) \subset \mathfrak{g}_k^*.$$

- *The extended orbit $\widetilde{O}$ is a principal T bundle over Θ.*

$\square$

In the language of Poisson geometry we will see that $\widetilde{O}$ is a full symplectic realisation of the Poisson manifold Θ.

Proof (of Lemma 2.14).

1) To see that the T action is well defined it is sufficient to prove that the T action

$$O_B \to O_B; \qquad B \mapsto tBt^{-1}$$

on the B_k coadjoint orbit O_B is well defined. It is, since $B = bA^0b^{-1}$ for some $b \in B_k$ where A^0 is the diagonal element in O_B and thus

$$tBt^{-1} = (tbt^{-1})(tA^0t^{-1})(tb^{-1}t^{-1}) = (tbt^{-1})A^0(tbt^{-1})^{-1} \in O_B$$

since A_0 is diagonal and $tbt^{-1} \in B_k$. The action on $\widetilde{O}$ is clearly free.

2) Clearly the T orbits are contained in the fibres. Conversely we can just look at the leading coefficients: if $(g_0, A) \in \widetilde{O}$ then $g_0 A_k g_0^{-1} = A_k^0$. It follows that any other element (g_0', A) of $\widetilde{O}$ will have $g_0' = tg_0$ for some $t \in GL_n(\mathbb{C})$ commuting with A_k^0. Now A_k^0 is diagonal with distinct diagonal entries and so $t \in T$.

3) This follows from Corollary 2.7; Λ is the residue of the unique diagonalisation of A determined by the diagonalisation A_k^0 of the leading coefficient A_k.

4) If $gAg^{-1} = A^0 + R$ then observe that $(g(0), A)$ is in $\widetilde{O}$:

$$\pi(g(0)Ag(0)^{-1}) = \pi(b(A^0 + R)b^{-1}) = bA^0b^{-1} \in O_B$$

where $b = g(0)g^{-1} \in B_k$. Thus $A \in \Theta$ □

In particular we can make the following definition:

Definition 2.16. The *winding map* of $\widetilde{O}$ is the surjection:

$$w : G_k \times \mathfrak{t}^* \to \widetilde{O}$$
$$(g, R) \mapsto (g(0), g^{-1}(A^0 + R)g)$$

where $R = \Lambda d\zeta/\zeta$ for some $\Lambda \in \mathfrak{t}$. Any section of w is an *unwinding* of $\widetilde{O}$.

Then we can deduce

Corollary 2.17. *As a manifold, $\widetilde{O}$ is isomorphic to the product*

$$(G_k/\delta(B_k)) \times \mathfrak{t}^*$$

where $\delta(B_k) \subset B_k$ is the subgroup of diagonal elements, acting on G_k by left multiplication.

Proof. Just observe that the fibres of the winding map are the orbits of $\delta(B_k)$ [For any $R \in \mathfrak{t}^*$, the stabiliser in G_k of $A^0 + R$ under the coadjoint action is just the diagonal subgroup $\delta(G_k)$] □

A useful observation now is that the winding map may in fact be completely unwound:

Lemma 2.18. *The winding map $w : G_k \times \mathfrak{t}^* \to \widetilde{O}$ admits a section:*

$$u : \widetilde{O} \longrightarrow G_k \times \mathfrak{t}^*.$$

Proof. There are a number of ways to do this. The method here is based on an observation in [**60**]. Observe that the product of a diagonal matrix with an off-diagonal matrix is again off-diagonal (i.e. each diagonal entry is zero). Now if $b \in B_k$ then $b = \delta(b) + (\text{off diagonal})$ where $\delta(b) \in B_k$ is the diagonal part of b and so it follows that

$$\delta(b)^{-1} \cdot b = 1 + (\text{off diagonal}) \in B_k.$$

Therefore if we define B_k^{od} to be the subset

$$B_k^{\text{od}} = \{1 + (\text{off diagonal}) \in B_k\} \subset B_k$$

we obtain an isomorphism

$$B_k/\delta(B_k) \cong B_k^{\mathrm{od}}; \qquad b \mapsto \delta(b)^{-1} \cdot b$$

where the diagonal subgroup $\delta(B_k)$ acts on B_k by left multiplication. Also we have that $\delta(B_k)$ is the stabiliser in B_k of $A^0 + R \in \mathfrak{g}_k^*$ for any $R \in \mathfrak{t}^*$, and thus:

$$B_k/\delta(B_k) \cong \{b^{-1}(A^0 + R)b \mid b \in B_k\} \subset \mathfrak{g}_k^*.$$

Hence we have an isomorphism

$$\phi_R : B_k^{\mathrm{od}} \cong \{b^{-1}(A^0 + R)b \mid b \in B_k\}; \qquad b \mapsto b^{-1}(A^0 + R)b.$$

We will use the inverse of this isomorphism to produce a section of the winding map as follows. If $(g_0, A) \in \widetilde{O}$, set $R = \Lambda d\zeta/\zeta$ where Λ is the residue of the diagonalisation of A (as in the third part of Lemma 2.14). Then we have

$$g_0 A g_0^{-1} \in \{b^{-1}(A^0 + R)b \mid b \in B_k\}$$

since for example due to the surjectivity of w, $A = g^{-1}(A^0 + R)g$ for some $g \in G_k$ with $g(0) = g_0$ and so taking $b = g g_0^{-1}$ will do. It follows that the definition

$$u(g_0, A) := \big((\phi_R^{-1}(g_0 A g_0^{-1})) \cdot g_0, R\big) \in G_k \times \mathfrak{t}^*$$

gives a section of w □

Of course, it would be nice to understand the torus action on $\widetilde{O}$ in a symplectic context. A moment map for the torus action on $\widetilde{O}$ is not apparent in the decoupled description: it amounts to finding a moment map for the conjugation action of the torus on the coadjoint orbit O_B. We proceed therefore by giving another description of $\widetilde{O}$ which illuminates its symplectic geometry.

Consider the left action (Definition 1.47) of G_k on its cotangent bundle T^*G_k. From the inclusion $B_k \hookrightarrow G_k$ a free Hamiltonian action of B_k on T^*G_k is induced. Also let B_k act on the B_k coadjoint orbit O_B by the coadjoint action. Combining these actions we obtain a free Hamiltonian B_k action on the product $T^*G_k \times O_B$. Then we have

Proposition 2.19. *The symplectic quotient of the product $T^*G_k \times O_B$ by B_k is isomorphic to the extension $\widetilde{O}$ of O_B as a complex symplectic manifold:*

$$\widetilde{O} \cong (T^*G_k \times O_B)/\!/B_k.$$

Proof. The moment map for the B_k action on O_B is just the inclusion $O_B \to \mathfrak{b}_k^*$ and the moment map for the B_k action on T^*G_k is the composition of the moment map for the left G_k action (see Lemma 1.48) with the projection $\pi : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$. Thus the moment map for the B_k action on the product is the sum of these moment maps:

$$\mu : T^*G_k \times O_B \to \mathfrak{b}_k^*; \qquad (g, A, B) \mapsto -\pi(\mathrm{Ad}_g^*(A)) + B.$$

The preimage of $0 \in \mathfrak{b}_k^*$ under μ is therefore:

$$\mu^{-1}(0) = \big\{(g, A, B) \bigm| \pi(gAg^{-1}) = B\big\}. \tag{23}$$

To identify the symplectic quotient $(T^*G_k \times O_B)/\!/B_k := \mu^{-1}(0)/B_k$ with $\widetilde{O}$ consider the map:

$$\chi : \mu^{-1}(0) \to \widetilde{O}; \qquad (g, A, B) \mapsto (g(0), A).$$

This map is well defined since the condition in (23) implies

$$\pi(g(0)Ag(0)^{-1}) = g(0)g^{-1}Bgg(0)^{-1}$$

and this is in the same B_k coadjoint orbit as B since $g(0)g^{-1} \in B_k$. We claim that χ is surjective and has precisely the B_k orbits in $\mu^{-1}(0)$ as fibres. Surjectivity is clear since we can write down a section of χ:

$$s : (g_0, A) \mapsto (g_0, A, \pi(g_0 A g_0^{-1})) \in \mu^{-1}(0)$$

where $(g_0, A) \in \widetilde{O}$. To examine the fibres of χ, suppose $\chi(g, A, B) = \chi(g', A', B')$. Thus $A = A'$ and $g(0) = g'(0)$ so $g' = \beta g$ where $\beta := g'g^{-1}$ is in B_k. Then from the condition in (23) we have:

$$B' = \pi(g'A'g'^{-1}) = \pi(\beta g A g^{-1}\beta^{-1}) = \beta B \beta^{-1}.$$

Hence $(g', A', B') = \beta(g, A, B)$ and so each fibre of χ is contained in a B_k orbit. Conversely it is clear that B_k acts within the fibres of χ. This identifies the quotient $\mu^{-1}(0)/B_k$ with $\widetilde{O}$ as manifolds.

To identify the symplectic structures we proceed as follows. Identify $\widetilde{O}$ with $T^*GL_n(\mathbb{C}) \times O_B$ as in Lemma 2.13 to give $\widetilde{O}$ its symplectic structure. The section s of χ (when also composed with the inclusion $\mu^{-1}(0) \to T^*G_k \times O_B$) is then given explicitly as the map

$$s' : T^*GL_n(\mathbb{C}) \times O_B \to T^*G_k \times O_B;$$

$$(g_0, S, B) \mapsto (g_0, \pi(g_0^{-1}Bg_0) + S, B).$$

It is now sufficient to prove that s' pulls the symplectic form on $T^*G_k \times O_B$ back to that on $T^*GL_n(\mathbb{C}) \times O_B$. To this end, for $i = 1, 2$, choose arbitrary $X_i \in \mathfrak{gl}_n(\mathbb{C})$, $A_i \in \mathfrak{gl}_n(\mathbb{C})^*$ and $Y_i \in \mathfrak{b}_k^*$ so that

$$(X_1, A_1, [B, Y_1]), \qquad (X_2, A_2, [B, Y_2])$$

represent arbitrary tangents to $T^*GL_n(\mathbb{C}) \times O_B$ at (g_0, S, B) (using left trivialisations where appropriate). We will evaluate the symplectic form on these tangent vectors and compare this value with that of the symplectic form on the push forward of these tangents along s'. The map s' pushes these tangents forward to

$$(X_i, \pi(N_i) + A_i, [B, Y_i]) \in T_{s'(g_0,S,B)}(T^*G_k \times O_B)$$

respectively for some elements $\pi(N_1), \pi(N_2) \in \mathfrak{b}_k^*$. The explicit formulae for the symplectic structures are used now to complete the proof. Firstly the O_B components are identical so clearly agree. For $T^*GL_n(\mathbb{C})$ the symplectic form at (g_0, S) evaluated on $(X_1, A_1), (X_2, A_2)$ is, from Lemma 1.45:

$$\langle A_1, X_2\rangle - \langle A_2, X_1\rangle - \langle S, [X_1, X_2]\rangle. \tag{24}$$

Similarly on T^*G_k the symplectic form at $(g_0, \pi(g_0^{-1}Bg_0) + S)$ evaluated on $(X_1, \pi(N_1) + A_1)$, $(X_2, \pi(N_2) + A_2)$ is:

$$\langle \pi(N_1) + A_1, X_2\rangle - \langle \pi(N_2) + A_2, X_1\rangle - \langle \pi(g_0^{-1}Bg_0) + S, [X_1, X_2]\rangle. \tag{25}$$

Now all the terms involving the projection π here drop out since they have no residue and are paired against elements of $\mathfrak{gl}_n(\mathbb{C})$ (namely X_1, X_2 or $[X_1, X_2]$) and so evaluate to zero. Thus the expressions (24) and (25) are equal and hence the symplectic structures on $\widetilde{O}$ and $(T^*G_k \times O_B)//B_k$ agree $\square$

REMARK 2.20. Equivalently this proposition says that $\widetilde{O}$ is symplectically isomorphic to the symplectic quotient of T^*G_k by B_k over the B_k coadjoint orbit $-O_B$.

Proposition 2.19 makes it straightforward to write down the explicit formula for the pullback of the symplectic structure on $\widetilde{O}$ to $G_k \times \mathfrak{t}^*$ along the winding map. This formula will be very useful in proving the monodromy map is symplectic.

Proposition 2.21. *Using the left trivialisation the symplectic form ω on the extended orbit $\widetilde{O}$ pulls back along the winding map w to the following two-form on $G_k \times \mathfrak{t}^*$:*

$$\Omega_{(g,R)}((X_1, R_1), (X_2, R_2)) = \langle R_1, \widetilde{X}_2\rangle - \langle R_2, \widetilde{X}_1\rangle + \langle A, [X_1, X_2]\rangle$$

$$\begin{aligned} \textit{where}\qquad & (g, R) \in G_k \times \mathfrak{t}^*,\\ & (X_i, R_i) \in \mathfrak{g}_k \times \mathfrak{t}^* \cong T_{(g,R)}(G_k \times \mathfrak{t}^*) \textit{ for } i = 1, 2,\\ & A = g^{-1}(A^0 + R)g \textit{ is the } \mathfrak{g}_k^* \textit{ component of } w(g, R) \textit{ and}\\ & \widetilde{X}_i = gX_ig^{-1} \in \mathfrak{g}_k \textit{ for } i = 1, 2. \end{aligned}$$

REMARK 2.22. One can understand this formula in terms of the description of $\widetilde{O}$ as a principal T bundle over the family Θ of G_k coadjoint orbits. The first two terms symplectically pair up the T orbit directions in $\widetilde{O}$ with the $\mathfrak{t}^*$ directions. The last term corresponds to the symplectic structure on the G_k coadjoint orbits. Note that $R_i = \Lambda_i d\zeta/\zeta$ for a diagonal matrix Λ_i and so only the diagonal part of the constant term of each $\widetilde{X}_i$ plays a role in the first two terms in the expression for Ω.

Proof (of Proposition 2.21). We claim that Ω is the pullback of the symplectic form on T^*G_k along the map:

$$\lambda : G_k \times \mathfrak{t}^* \to T^*G_k; \qquad (g, R) \mapsto (g, g^{-1}(A^0 + R)g)$$

where we use the left trivialisation to identify T^*G_k and $G_k \times \mathfrak{g}_k^*$. This is true since the winding map w is the composition of the following two maps. Firstly map $G_k \times \mathfrak{t}^*$ into the subset $\mu^{-1}(0)$ of $T^*G_k \times O_B$ (defined in Proposition 2.19):

$$\iota : G_k \times \mathfrak{t}^* \hookrightarrow \mu^{-1}(0); \qquad (g, R) \mapsto (g, g^{-1}(A^0 + R)g, A^0).$$

Notice that the O_B component is constant (equal to A^0) in the image of this map. Secondly project down to $\widetilde{O}$ as in the proof of Proposition 2.19:

$$\chi : \mu^{-1}(0) \to \widetilde{O}; \qquad (g, A, B) \mapsto (g(0), A).$$

Now the symplectic form on $\widetilde{O}$ can be calculated from any lift of tangent vectors to $\mu^{-1}(0)$ (and using the symplectic form on $T^*G_k \times O_B$). In particular we can use the composition of ι with the section $u : \widetilde{O} \to G_k \times \mathfrak{t}^*$ of the winding map to lift tangent vectors to $\mu^{-1}(0)$. Since the O_B component of ι is constant all of these lifted tangent vectors will have zero O_B component and so our claim holds.

To pull back the symplectic form along λ it is simplest to pass to the right trivialisation of T^*G_k. If we identify $G_k \times \mathfrak{g}_k^*$ with T^*G_k by the right trivialisation then the map λ is represented by the simpler map:

$$\rho : G_k \times \mathfrak{t}^* \to G_k \times \mathfrak{g}_k^*; \qquad (g, R) \mapsto (g, A^0 + R).$$

We can now use the formula for the symplectic structure on T^*G_k in terms of the right trivialisation (from Lemma 1.46) to compute Ω. If $(X_i, R_i) \in \mathfrak{g}_k \times \mathfrak{t}^*$ represents a tangent vector to $G_k \times \mathfrak{t}^*$ at (g, R) (using the *left* trivialisation of TG_k) then

$$d\rho_{(g,R)}(X_i, R_i) = (gX_ig^{-1}, R_i) \in \mathfrak{g}_k \times \mathfrak{g}_k^*$$

where gX_ig^{-1} is regarded as an element of T_gG_k by the *right* trivialisation. Thus Lemma 1.46 gives:

$$\Omega_{(g,R)}((X_1, R_1), (X_2, R_2)) = \langle R_1, gX_2g^{-1}\rangle - \langle R_2, gX_1g^{-1}\rangle + \langle A^0 + R, [gX_1g^{-1}, gX_2g^{-1}]\rangle$$

which rearranges to give the desired formula $\square$

We are now in a position to describe the relevant group actions on $\widetilde{O}$ symplectically. Firstly define some maps:

Definition 2.23.

$$\mu_{GL_n(\mathbb{C})} : \widetilde{O} \to \mathfrak{gl}_n(\mathbb{C})^*; \qquad (g_0, A) \mapsto \pi_{\mathrm{Res}}(A)$$

$$\mu_T : \widetilde{O} \to \mathfrak{t}^*; \qquad (g_0, A) \mapsto -R$$

where R is the residue term of the diagonalisation of A determined by g_0 as in the third part of Lemma 2.14 and π_{Res} is the projection onto the residue term.

Then we can deduce:

Corollary 2.24.

- *The free action of $GL_n(\mathbb{C})$ on $\widetilde{O}$ defined by*

$$h(g_0, A) = (g_0h^{-1}, hAh^{-1})$$

is Hamiltonian with moment map $\mu_{GL_n(\mathbb{C})}$. The symplectic quotient at the value 0 of $\mu_{GL_n(\mathbb{C})}$ is just O_B:

$$\widetilde{O}//GL_n(\mathbb{C}) = \mu^{-1}_{GL_n(\mathbb{C})}(0)/GL_n(\mathbb{C}) \cong O_B.$$

- *The free action of $T \cong (\mathbb{C}^*)^n$ on $\widetilde{O}$ defined (as before) by*

$$t(g_0, A) = (tg_0, A)$$

commutes with the $GL_n(\mathbb{C})$ action above and is Hamiltonian with moment map μ_T. The symplectic quotient at the value $-R$ of μ_T is the G_k coadjoint orbit through the element $A^0 + R$ of $\mathfrak{g}_k^*$:

$$\widetilde{O}\underset{-R}{//}T = \mu_T^{-1}(-R)/T \cong O(A^0 + R).$$

Proof. The first part is clear from the decoupled description of $\widetilde{O}$ in Lemma 2.13; $GL_n(\mathbb{C})$ only acts on the $T^*GL_n(\mathbb{C})$ factor and it does so by the natural right action. The moment map is obtained from Lemma 1.48. Also $(T^*GL_n(\mathbb{C}))//GL_n(\mathbb{C})$ is a point and so $\widetilde{O}//GL_n(\mathbb{C}) \cong O_B$.

For the second part, as in the proof of Proposition 2.21 we have the maps:

$$\begin{array}{ccc} G_k \times \mathfrak{t}^* & \overset{\iota}{\hookrightarrow} & T^*G_k \\ \big\downarrow{\scriptstyle w} & & \\ \widetilde{O} & & \end{array}$$

where we have dropped the constant O_B component of ι. We will identify $G_k \times \mathfrak{t}^*$ with its image in T^*G_k. The left G_k action on T^*G_k restricts to a T action which (from Lemma 1.48) will have moment map

$$\nu : T^*G_k \to \mathfrak{t}^*; \qquad (g, A) \mapsto -\delta(\pi_{\mathrm{Res}}(gAg^{-1}))$$

since $\delta \circ \pi_{\mathrm{Res}}$ is the dual of the derivative of the inclusion $T \hookrightarrow G_k$. This torus action restricts to an action on the submanifold $G_k \times \mathfrak{t}^*$ which goes down to the above T action on $\widetilde{O}$. Now pick any $X \in \mathfrak{t}$. It follows that the corresponding fundamental vector fields on $G_k \times \mathfrak{t}^*$ and $\widetilde{O}$ are w-related:

$$(dw)_{(g,R)}(\widetilde{X}_{(g,R)}) = \widetilde{X}_{w(g,R)} \tag{26}$$

for any point (g, R) of $G_k \times \mathfrak{t}^*$.

Observe now, from the definition of ι, that the composition $\nu \circ \iota : G_k \times \mathfrak{t}^* \to \mathfrak{t}^*$ is just minus the projection onto the second factor: $(g, R) \mapsto -R$. This map is constant on the fibres of the winding map w (c.f. Corollary 2.17) and the induced map on $\widetilde{O}$ is the map $\mu_T : \widetilde{O} \to \mathfrak{t}^*$ we defined above. Thus to get at the Hamiltonian vector fields associated to μ_T the following fact (from [**66**]) is very useful. Let $\widetilde{f} := \langle \nu, X \rangle$ and $f := \langle \mu_T, X \rangle$ be the Hamiltonian functions given by the components of the moment maps corresponding to X on T^*G_k and $\widetilde{O}$ respectively. Then for any $(g, R) \in G_k \times \mathfrak{t}^*$ the Hamiltonian vector fields are w-related:

$$(dw)_{(g,R)}(X_{\widetilde{f}}) = (X_f)_{w(g,R)}. \tag{27}$$

To see this, observe the following equality of one-forms on $G_k \times \mathfrak{t}^*$:

$$(dw)^*\big(\omega(\cdot, dw_*(X_{\widetilde{f}}))\big) = \Omega(\cdot, X_{\widetilde{f}}) = d\widetilde{f} = (dw)^*(df) = (dw)^*(\omega(\cdot, X_f))$$

where ω is the symplectic form on $\widetilde{O}$ and Ω is its pullback along w to $G_k \times \mathfrak{t}^*$ (which is the restriction of the symplectic form on T^*G_k, c.f. Proposition 2.21). Now w is surjective on tangent vectors and ω is nondegenerate and so (27) follows. Finally, since ν is the moment map for the T action on T^*G_k, we have $(X_{\widetilde{f}})_{(g,R)} = \widetilde{X}_{(g,R)}$ and so (26) and (27) give:

$$(X_f)_{w(g,R)} = \widetilde{X}_{w(g,R)}$$

and it follows that μ_T is indeed the moment map for the T action on $\widetilde{O}$.

It is easy to see that, as manifolds, the symplectic quotients are the G_k coadjoint orbits. To prove they are symplectically isomorphic, lift any two tangent vectors to $O(A^0 + R)$ all the way up to $G_k \times \mathfrak{t}^*$ and use the formula in Proposition 2.21. The $\mathfrak{t}^*$ components of these lifted tangents are zero and so the first two terms in the formula are zero. The third term tells us that the symplectic structure on the symplectic quotient agrees with the standard Kostant-Kirillov structure on the coadjoint orbit □

To give a feel for these extended orbits we will now look at the simplest irregular case ($k = 2$) in detail. This case occurs in many important problems, motivated a lot of this work and will be returned to a number of times. It is not however illustrative of the 'general case' since for $k = 2$ the B_k coadjoint orbit O_B is just a point.

EXAMPLE 2.25. (The $k = 2$ case). First observe that B_2 is Abelian:

$$B_2 = \left\{ \, 1 + X_1\zeta \;\middle|\; X_1 \in \mathrm{End}(E) \, \right\}$$

which is isomorphic as a group to the vector space $(\mathrm{End}(E), +)$. Thus if we choose a diagonal element $A^0 = A_2^0 d\zeta/\zeta^2$ of $\mathfrak{b}_k^*$ then the B_k coadjoint orbit O_B through A^0 is simply the point A^0 itself. The extended orbit associated to O_B is therefore, by definition:

$$\widetilde{O} := \left\{ \left(g_0, A_2\frac{d\zeta}{\zeta^2} + A_1\frac{d\zeta}{\zeta}\right) \in GL_n(\mathbb{C}) \times \mathfrak{g}_k^* \;\middle|\; g_0 A_2 g_0^{-1} = A_2^0 \right\}.$$

The decoupling isomorphism (Lemma 2.13) tells us that $\widetilde{O}$ is symplectically isomorphic to the cotangent bundle $T^*GL_n(\mathbb{C})$:

$$\left(g_0, A_2\frac{d\zeta}{\zeta^2} + A_1\frac{d\zeta}{\zeta}\right) \mapsto \left(g_0, A_1\frac{d\zeta}{\zeta}\right) \in GL_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^*$$

where $T^*GL_n(\mathbb{C}) \cong GL_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^*$ via the left trivialisation.

The moment map for the free Hamiltonian T action is minus the residue of the diagonalisation of $A_2 d\zeta/\zeta^2 + A_1 d\zeta/\zeta$ specified by g_0, but here this amounts to just taking the diagonal part of the partial diagonalisation:

$$\mu_T(g_0, A) = -\delta(g_0 A_1 g_0^{-1})\frac{d\zeta}{\zeta}.$$

The quotient of $\widetilde{O}$ by this T action is the image of the projection to $\mathfrak{g}_2^*$:

$$\Theta = \left\{ A_2 d\zeta/\zeta^2 + A_1 d\zeta/\zeta \in \mathfrak{g}_2^* \;\middle|\; A_2 \text{ is conjugate to } A_2^0 \right\}.$$

This is a $2n^2 - n$ dimensional Poisson submanifold of $\mathfrak{g}_2^*$. It is an n-parameter family of G_2 coadjoint orbits, which are its symplectic leaves.

Since $\widetilde{O}$ is isomorphic to $T^*GL_n(\mathbb{C})$, the quotient of $\widetilde{O}$ by the free Hamiltonian $GL_n(\mathbb{C})$ action is isomorphic as a Poisson manifold to the dual of the Lie algebra of $GL_n(\mathbb{C})$ (since this is true for $T^*GL_n(\mathbb{C})$):

$$\widetilde{O}/GL_n(\mathbb{C}) \cong \left\{ (A_2^0 d\zeta/\zeta^2 + A_1' d\zeta/\zeta \;\middle|\; A_1' \in \mathfrak{gl}_n(\mathbb{C}) \right\} \cong \mathfrak{gl}_n(\mathbb{C})^*.$$

The symplectic leaves are just the $GL_n(\mathbb{C})$ coadjoint orbits and the residual T action has moment map given by taking the diagonal part of A_1'. Thus the symplectic quotients are subsets of $GL_n(\mathbb{C})$ coadjoint orbits with fixed diagonal entries modulo conjugation by T; we will see later that these are the additive version of symplectic spaces of Stokes matrices.

2.1. Extended orbits in the $k = 1$ case. To end this section we will define extended orbits also in the $k = 1$ case and briefly run through which of the properties of the general extended orbits above still hold true.

DEFINITION 2.26.

- Let $\mathfrak{t}'$ be the subset of nice elements of $\mathfrak{t}^*$:

$$\mathfrak{t}' := \left\{ R = \Lambda\frac{d\zeta}{\zeta} \in \mathfrak{t}^* \;\middle|\; \Lambda \in \mathfrak{t} \text{ has distinct eigenvalues mod } \mathbb{Z} \right\}.$$

- The $k = 1$ extended orbit is the set

$$\widetilde{O} = \left\{ (g_0, A) \in GL_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^* \;\middle|\; g_0 A g_0^{-1} \in \mathfrak{t}' \right\}.$$

REMARK 2.27. The set $\mathfrak{t}' \subset \mathfrak{t}^*$ (and therefore $\widetilde{O}$) is not algebraic although this is the only lack of algebraicity in this chapter. The condition on $\Lambda\frac{d\zeta}{\zeta} \in \mathfrak{t}^*$ to be in $\mathfrak{t}'$ ensures that $\exp(2\pi i \Lambda)$ has distinct eigenvalues.

Thus $\widetilde{O}$ is the set of pairs consisting of an element A which is in a nice $GL_n(\mathbb{C})$ coadjoint orbit, together with the choice of a matrix g_0 diagonalising A. Note that $\widetilde{O}$ doesn't depend on any choice of initial diagonal element A^0.

Definition 2.28. The winding map in the $k = 1$ case is the map

$$w : GL_n(\mathbb{C}) \times \mathfrak{t}' \to \widetilde{O}; \qquad (g_0, R) \mapsto (g_0, g_0^{-1} R g_0).$$

It is immediate then that

Lemma 2.29. *In the $k = 1$ case, the winding map is an isomorphism.*

□

In particular $\widetilde{O}$ is a complex manifold. To put a symplectic structure on it we have

Lemma 2.30. *Under the natural embedding $\widetilde{O} \subset GL_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^*$ the $k = 1$ extended orbit $\widetilde{O}$ is a symplectic submanifold of the cotangent bundle $T^*GL_n(\mathbb{C})$, where, as usual, we identify $T^*GL_n(\mathbb{C})$ with $GL_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^*$ by the left trivialisation.*

Proof. Recall from Lemma 1.48 that the moment map for the left action of $GL_n(\mathbb{C})$ on its cotangent bundle is

$$\mu_L : GL_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^* \to \mathfrak{gl}_n(\mathbb{C})^*; \qquad (g_0, A) \mapsto -g_0 A g_0^{-1}.$$

Thus $\widetilde{O} = \mu_L^{-1}(\mathfrak{t}') \subset T^*GL_n(\mathbb{C})$. Now observe that $\mathfrak{t}'$ is transverse (as a submanifold of $\mathfrak{gl}_n(\mathbb{C})^*$) to any $GL_n(\mathbb{C})$ coadjoint orbit that it intersects. It then follows that $\widetilde{O}$ is a symplectic submanifold of $T^*GL_n(\mathbb{C})$ using Theorem 26.7 on p195 of the book by Guillemin and Sternberg [**41**] □

As in the irregular case, the $k = 1$ extended orbit may also be viewed as a principal T-bundle over a family of $GL_n(\mathbb{C})$ coadjoint orbits. The free T action is defined in the same way: $t(g_0, A) = (tg_0, A)$, but there is a slight subtlety now. The family of coadjoint orbits obtained by quotienting by T is not a subset of $\mathfrak{gl}_n(\mathbb{C})^*$, rather it is a covering of a subset of $\mathfrak{gl}_n(\mathbb{C})^*$.

In more detail the quotient $\widetilde{O}/T$ is the following family of coadjoint orbits parameterised by $\mathfrak{t}'$:

$$\Theta := \{(R, A) \in \mathfrak{t}' \times \mathfrak{gl}_n(\mathbb{C})^* \mid A \in O(R)\} \cong \bigsqcup_{R \in \mathfrak{t}'} O(R)$$

where $\bigsqcup$ denotes disjoint union. The extended orbit is a principal T-bundle over Θ. If we also define another family of coadjoint orbits:

$$\Theta' := \bigcup_{R \in \mathfrak{t}'} O(R) \subset \mathfrak{gl}_n(\mathbb{C})^*$$

then Θ is a Sym_n-covering of Θ' and Θ' is the image of $\widetilde{O}$ when projected onto its second factor into $\mathfrak{gl}_n(\mathbb{C})^*$. In the irregular case this projection expressed $\widetilde{O}$ as a principal T bundle; here $\widetilde{O}$ is a principal $N(T)$-bundle over Θ' where $N(T) \cong \mathrm{Sym}_n \ltimes T$ is the normaliser of T in $GL_n(\mathbb{C})$.

Next we observe that the same formula

$$\Omega_{(g,R)}((X_1, R_1), (X_2, R_2)) = \langle R_1, \widetilde{X}_2 \rangle - \langle R_2, \widetilde{X}_1 \rangle + \langle A, [X_1, X_2] \rangle$$

as in Proposition 2.21 for the pullback Ω of the symplectic form on $\widetilde{O}$ along the winding map, holds also in the $k = 1$ case. To see this recall from the first line of the proof of Proposition 2.21 that this formula for Ω is just the pullback to $G_k \times \mathfrak{t}^*$ of the symplectic form on T^*G_k along the map

$$\lambda : G_k \times \mathfrak{t}^* \to T^*G_k; \qquad (g, R) \mapsto (g, g^{-1}(A^0 + R)g).$$

But here ($k = 1$, $A^0 = 0$) the restriction $\lambda|_{G_k \times \mathfrak{t}'}$ is just the composition $i \circ w$ of the winding map w with the inclusion $i : \widetilde{O} \hookrightarrow T^*GL_n(\mathbb{C})$ from which the symplectic form was defined.

Finally, considering the actions of T and $GL_n(\mathbb{C})$ on $\widetilde{O}$, we find that Corollary 2.24 still holds with $O_B =$ (point); the moment maps μ_T and $\mu_{GL_n(\mathbb{C})}$ are defined in the same way as before.

3. Moduli Spaces and Polar Parts Manifolds

We are now in a position to give explicit finite dimensional symplectic descriptions of spaces of meromorphic connections on trivial holomorphic vector bundles over $\mathbb{P}^1$.

3.1. Firstly we will explain the connection between meromorphic connections and G_k coadjoint orbits. Suppose that $a \in \mathbb{P}^1$ and that A is a rank n system at a with a pole of order k (i.e. it is a meromorphic connection germ at a on the trivial vector bundle). Then if we choose a local coordinate z vanishing at a we have

$$A = \frac{A_k dz}{z^k} + \cdots + \frac{A_1 dz}{z} + A_0 dz + \cdots$$

for $n \times n$ matrices A_i, $(i \leq k)$.

Definition 2.31.
- The *polar* or *principal part* of the system A at a (with respect to the coordinate z) is the element

$$\mathrm{PP}_a(A) = \left(A_k \frac{d\zeta}{\zeta^k} + \cdots + A_1 \frac{d\zeta}{\zeta}\right) \in \mathfrak{g}_k^*$$

of the dual of the Lie algebra of G_k obtained by removing the nonsingular terms in the Laurent expansion of A at a and replacing the coordinate z by the indeterminate ζ.
- This procedure defines the *polar part map*

$$\mathrm{PP}_a : {}^a\mathrm{Syst}_k \longrightarrow \mathfrak{g}_k^*$$

from the set of systems at a with poles of order k to the dual of the Lie algebra of G_k.
- The *coadjoint orbit associated to A via the coordinate z* is the G_k coadjoint orbit

$$O(A) := O(\mathrm{PP}_a(A)) \subset \mathfrak{g}_k^*$$

containing the polar part of A at a.

It is clear from the definitions of the gauge action and the coadjoint action that if two systems at a are formally equivalent then their polar parts (with respect to any single coordinate choice) lie in the same coadjoint orbits (this is simply because the term

$(d\widehat{F})\widehat{F}^{-1}$ in the gauge action does not affect the polar parts)[2]. For *nice* systems the converse is also true:

Proposition 2.32. *Suppose $A, B \in {}^a\mathrm{Syst}_k$ are nice systems at a. Then A and B are formally equivalent if and only if their associated coadjoint orbits (with respect to some/any coordinate z) are the* same.

Proof. (This result is essentially well known.) We need to show that if the polar parts of A and B are in the same coadjoint orbit then they are formally equivalent. Assume (without loss of generality) that A is a normal form, that is

$$A = (A_k z^{-k} + \cdots + A_1 z^{-1})dz$$

with each A_i diagonal. By hypothesis $\mathrm{PP}_0(A) = g\mathrm{PP}_0(B)g^{-1}$ for some $g \in G_k$. Pick $F \in G\{z\}$ such that $F \equiv g(z) \mod z^k$, then

$$A = FBF^{-1} + (\text{holomorphic}) = F[B] + (\text{holomorphic}).$$

The result now follows by examining the algorithm given in Appendix B to put $F[B]$ into normal form: firstly we obtain a formal series

$$\widehat{H} = \prod_{i \geq k}(1 + z^i H_i) \quad \text{such that} \quad (\widehat{H}F)[B] = A + D$$

with D diagonal and nonsingular. Then by defining the diagonal formal series

$$\widetilde{F} := \exp(-\int^z D) \quad \text{so that} \quad d\widetilde{F}(\widetilde{F})^{-1} = d\log\widetilde{F} = -D$$

we find $(\widetilde{F}\widehat{H}F)[B] = A$ so A and B are formally equivalent □

3.2. Now we look at (global) connections on trivial bundles over $\mathbb{P}^1$.

Recall that we have chosen m distinct points $a_1, \ldots, a_m \in \mathbb{P}^1$ and nice formal normal forms $\mathbf{A} = ({}^1A^0, \ldots, {}^mA^0)$. Here we are using the coordinate independent notion of normal form since we have made no coordinate choices yet and so ${}^iA^0$ is just a nice diagonal system at a_i having a pole of order k_i (see Remark 1.9).

DEFINITION 2.33. The moduli space $\mathcal{M}^*(\mathbf{A})$ is the set of isomorphism classes of pairs (V, ∇) where V is a *trivial* rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V which is formally equivalent to ${}^iA^0$ at a_i for each i and has no other poles.

REMARK 2.34. These moduli spaces will be empty unless we impose the condition

$$\sum_{i=1}^{m} \mathrm{Res}_{a_i}\mathrm{Tr}({}^iA^0) = 0 \tag{28}$$

on the choice of formal normal forms. To see this suppose ∇ is a meromorphic connection on a trivial bundle V (or more generally on any degree zero bundle) which is formally equivalent to ${}^iA^0$ at a_i for each i and has no other poles. Then the top exterior power $\bigwedge^n V$ of V is a trivial line bundle and in terms of any trivialisation the connection induced

[2] In particular (for each coordinate choice) a coadjoint orbit may be canonically associated to a nice *connection* germ; different choices of trivialisation to go from connections to systems lead to the same coadjoint orbit.

from ∇ is of the form $d-\phi$ for a meromorphic one form ϕ on $\mathbb{P}^1$. Now the fact that ∇ is formally equivalent to ${}^iA^0$ at a_i implies that $\mathrm{Res}_{a_i}(\phi) = \mathrm{Res}_{a_i}\mathrm{Tr}({}^iA^0)$ for each i and then the residue theorem implies (28). Thus without further mention we will always assume that (28) holds[3].

The main result in this section is then

Theorem 2.35. *For $i = 1, \dots, m$ let z_i be a local coordinate near a_i on $\mathbb{P}^1$ vanishing at a_i and let $O_i \subset \mathfrak{g}^*_{k_i}$ be the coadjoint orbit associated to the normal form ${}^iA^0$ via the coordinate z_i. Then:*

- *The polar part maps induce an isomorphism between the set $\mathcal{M}^*(\mathbf{A})$ of isomorphism classes defined above and the symplectic quotient of $O_1 \times \cdots \times O_m$ by $GL_n(\mathbb{C})$ at the value 0 of the moment map:*

$$\boxed{\mathcal{M}^*(\mathbf{A}) \cong O_1 \times \cdots \times O_m /\!/ GL_n(\mathbb{C})}$$

- *In this way $\mathcal{M}^*(\mathbf{A})$ inherits an intrinsic complex symplectic structure. That is, the symplectic structure obtained is not dependent on the coordinate choices.*

Remark 2.36. The complex symplectic quotients

$$O_1 \times \cdots \times O_m /\!/ GL_n(\mathbb{C})$$

will occasionally be referred to as *polar parts manifolds.* They are a direct generalisation of the 'residue manifolds' used by Hitchin [**48**]. Observe that the polar parts manifolds are detached from the geometry of the curve; it is the choice of local coordinates which gives the isomorphism in Theorem 2.35 realising the moduli space $\mathcal{M}^*(\mathbf{A})$ concretely as a polar parts manifold.

Proof. To start with we choose local coordinates as follows. Since we are working on $\mathbb{P}^1$ its easy to find a single coordinate chart containing $a_1, \dots, a_m$. (Pick a point on $\mathbb{P}^1$ distinct from each a_i, label it '∞' and let z be a global coordinate on $\mathbb{C} = \mathbb{P}^1 \setminus \infty$ having a simple pole at ∞.) Each point a_i is identified with a complex number by z and we can define local coordinates $z_i := z - a_i$ vanishing at a_i for each i.

Now suppose ∇ is a meromorphic connection on a holomorphically trivial vector bundle V over $\mathbb{P}^1$ which is formally equivalent to ${}^iA^0$ at a_i for each i. Then in any global trivialisation of V, ∇ is of the form:

$$\nabla = d - A = d - \sum_{i=1}^{m}\left({}^iA_{k_i}\frac{dz}{(z-a_i)^{k_i}} + \cdots + {}^iA_1\frac{dz}{(z-a_i)}\right) \tag{29}$$

for some $n \times n$ matrices ${}^iA_j$ $(1 \le j \le k_i)$.

From Proposition 2.32 we see that each term in this sum lives in a fixed coadjoint orbit. More precisely, the polar parts do:

$${}^iA_{k_i}\frac{d\zeta}{\zeta^{k_i}} + \cdots + {}^iA_1\frac{d\zeta}{\zeta} \in O_i \subset \mathfrak{g}^*_{k_i}$$

where O_i is the coadjoint orbit associated to the normal form ${}^iA^0$ via the coordinate z_i.

[3]In general the left-hand side of (28) needs to be the (integer) degree of the bundles being used. Here we work only with degree zero bundles; the extension to arbitrary degree being essentially trivial once the degree zero case is understood.

Two other key observations to be made immediately from (29) are:

• Choosing a different trivialisation of V corresponds to conjugating all of the matrices ${}^iA_j$ by a single invertible matrix $g \in GL_n(\mathbb{C})$.

• Given any collection of matrices ${}^iA_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, k_i$ then the expression (29) defines a meromorphic connection on the trivial bundle over $\mathbb{P}^1$ with poles only at $a_1, \ldots, a_m$ if and only if there is no further pole at ∞, that is, iff the sum of the residues is zero:

$$ {}^1A_1 + \cdots + {}^mA_1 = 0. \tag{30} $$

These two facts fit together very naturally in symplectic geometry: the sum of the residues on the left-hand side of (30) is the moment map for the diagonal conjugation action of $GL_n(\mathbb{C})$ on the product $O_1 \times \cdots \times O_m$ of coadjoint orbits (see the third part of Lemma 2.4).

This gives the required isomorphism. All that remains is to see that the symplectic structure is indeed independent of the coordinate choices.

Suppose different local coordinates $z_1', \ldots, z_m'$ were chosen instead. Let $\mathrm{Syst}(\mathbf{A})$ denote the set of meromorphic connections on *the* trivial rank n vector bundle over $\mathbb{P}^1$ which are formally equivalent to ${}^iA^0$ at a_i for each i. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Syst}(\mathbf{A}) & \xrightarrow{\mathbf{PP}} & O(\mathrm{PP}_1({}^1A^0)) \times \cdots \times O(\mathrm{PP}_m({}^mA^0)) \\ \| & & \downarrow \phi \\ \mathrm{Syst}(\mathbf{A}) & \xrightarrow{\mathbf{PP}'} & O(\mathrm{PP}_1'({}^1A^0)) \times \cdots \times O(\mathrm{PP}_m'({}^mA^0)) \end{array}$$

where the top row uses the original choice of local coordinates to take polar parts and the bottom row uses the new choices. Lemma 2.8 implies that the right-hand map ϕ induced by the coordinate changes is a symplectic isomorphism. Moreover ϕ intertwines the Hamiltonian $GL_n(\mathbb{C})$ actions and does not affect the moment map, so the symplectic quotients agree □

In particular Theorem 2.35 shows that fixing the formal normal forms of the connections is the right thing to do to get a symplectic manifold; originally one fixed the formal type to get good local moduli spaces of meromorphic connections. We now see it also gives symplectic global moduli spaces.

REMARK 2.37. Theorem 2.35 describes $\mathcal{M}^*(\mathbf{A})$ as the quotient of an affine algebraic variety

$$ \mu^{-1}_{GL_n(\mathbb{C})}(0) \subset O_1 \times \cdots \times O_m \tag{31} $$

by the action of the reductive group $GL_n(\mathbb{C})$. General theory (see [**85, 87**]) then ensures that (31) has a dense open subset of stable points such that the quotient of the subset of stable points by $GL_n(\mathbb{C})$ is a complex manifold. As an invariant theory problem, our situation is just the quotient of certain tuples of $n \times n$ matrices by overall conjugation by $GL_n(\mathbb{C})$. The stability condition for such quotients has been worked out by Artin [**7**]. This will not be pursued here for two reasons. Firstly our primary interest here is to study certain differential equations on $\mathcal{M}^*(\mathbf{A})$ (or on bundles having $\mathcal{M}^*(\mathbf{A})$ as fibre) and for this it is sufficient to work on the dense open subset. Secondly in the next section we will see how $\mathcal{M}^*(\mathbf{A})$ arises from the extended moduli spaces $\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$; for these the corresponding $GL_n(\mathbb{C})$ action is free and so $\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$ itself is a manifold and moreover the whole isomonodromy picture lifts up (or indeed was first defined) on these extended spaces.

REMARK 2.38. The symplectic quotients

$$O_1 \times \cdots \times O_m // GL_n(\mathbb{C}) \tag{32}$$

appearing in Theorem 2.35 have been previously studied. They are algebraically completely integrable Hamiltonian systems. That is there are N independent Poisson commuting functions on (32) such that the fibres of the corresponding map to $\mathbb{C}^N$ are dense open subsets of Abelian varieties (where N is half the dimension of (32)). See Beauville [**17**], Adams-Harnad-Hurtubise [**2**] or the survey by Donagi and Markman at the start of [**28**]. Roughly, their perspective is to regard (32) as a space of Higgs fields on trivial bundles, rather than as a space of meromorphic connections. Since a trivial holomorphic vector bundle over a compact Riemann surface has a canonical flat connection (which is just d in any global trivialisation) there is a simple relationship between these two viewpoints: just add or subtract d.

4. Extended Moduli Spaces

In this section we give explicit finite dimensional symplectic descriptions of spaces of compatibly framed meromorphic connections with fixed irregular types on trivial vector bundles over $\mathbb{P}^1$. The story begins similarly to the last section: things have now been set up so that we can literally just replace the coadjoint orbits of Section 1 by the extended orbits of Section 2.

Having done that we find that our study of the geometry of the extended orbits tells us a lot about these 'extended' moduli spaces. Firstly we see they decouple into a product of symplectic manifolds and we then deduce that the symplectic isomorphism class of any extended moduli space is not dependent on the choice of nice irregular types, but just on the pole orders $k_1, \ldots, k_m$ and on the rank n (Corollary 2.44).

Secondly we see that for each pole there is a torus action which changes the choice of compatible framing at that pole. Moreover these actions are Hamiltonian (with respect to the symplectic structure we have defined) having moment maps given by the exponents of formal monodromy. It follows that the moduli spaces of the last section are obtained by taking symplectic quotients by these torus actions.

4.1. We start by explaining the relation between compatibly framed connections and extended orbits.

Suppose that $a \in \mathbb{P}^1$ and that (A, g) is a rank n compatibly framed system at a with a pole of order k. If we choose a local coordinate z vanishing at a we have

$$A = \frac{A_k dz}{z^k} + \cdots + \frac{A_1 dz}{z} + A_0 dz + \cdots$$

for $n \times n$ matrices A_i, $(i \leq k)$. The compatible framing is represented by a matrix $g \in GL_n(\mathbb{C})$ such that gA_kg^{-1} is diagonal.

Recall from Lemma 1.12 that (using the coordinate choice z) (A, g) has a unique associated formal normal form

$$A^0 := dQ + \Lambda \frac{dz}{z}$$

where Q is a diagonal matrix of polynomials of degree $k-1$ in z^{-1} with no constant term and Λ, the exponent of formal monodromy, is a constant diagonal matrix. We can identify A^0 with a diagonal element of $\mathfrak{g}_k^*$ by replacing z by the indeterminate ζ and so the irregular part dQ is identified with an element of $\mathfrak{b}_k^*$.

DEFINITION 2.39. The *extended orbit* associated to the compatibly framed system (A, g) (via the coordinate z) is the unique extended orbit containing $(g, \mathrm{PP}_a(A))$, where $\mathrm{PP}_a(A) \in \mathfrak{g}_k^*$ is the polar part of A (with respect to z). Equivalently it is the extended orbit associated to the B_k coadjoint orbit through dQ:

$$\widetilde{O}(A, g) := \{(h, B) \in GL_n(\mathbb{C}) \times \mathfrak{g}_k^* \mid \pi(hBh^{-1}) \in O_B\}$$

where $\pi : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ is the natural projection removing the residue and $O_B \subset \mathfrak{b}_k^*$ is the coadjoint orbit through dQ.

Immediately we have

Lemma 2.40. *If (A, g) and (A', g') are compatibly framed systems then the following three conditions on them are equivalent:*
1) *They have the same irregular type (see Definition 1.4).*
2) *The irregular parts dQ and dQ' of their associated formal normal forms are the same.*
3) *Their associated extended orbits are the same.*
(Each condition is independent of the choice of coordinate z.)

Proof. The last two conditions are clearly equivalent from the definition of associated extended orbits. The equivalence between the first two conditions is straightforward; the details are as follows.

Translating the definition of having the same irregular type into the language of systems, we find that (A, g) and (A', g') have the same irregular type iff there is an analytic transformation $H \in G\{z\}$ such that $H(0) = g'$ and

$$H[A'] = gAg^{-1} + \theta \tag{33}$$

for some matrix θ of meromorphic one-forms with only first order poles at $z = 0$.

On the other hand note that the irregular part dQ of the formal normal form associated to (A, g) is the irregular part of (the polar part of) $F[A]$ for any $F \in \widehat{G}$ (convergent or not) such that $F(0) = g$ and such that this irregular part is diagonal.

Thus if (A, g) and (A', g') have the same irregular type just choose any element $H' \in G\{z\}$ (or even polynomial in z) such that $H'(0) = 1$ and the irregular part of the polar part of $H'H[A']$ is dQ'. Then putting $F = H'g$ and applying H' to both sides of (33) we deduce that $dQ = dQ'$.

Conversely if $dQ = dQ'$ then we have F, F' such that $F(0) = g, F'(0) = g'$ and the irregular parts of $F[A]$ and $F'[A']$ are both dQ. Thus $F'[A'] = F[A] + \theta'$ for some matrix θ' of one forms with at most first order poles. Applying gF^{-1} to both sides of this yields an expression of the form (33) with $H = gF^{-1}F'$ □

REMARK 2.41. Thus using a choice of local coordinate z, we see that the set of compatibly framed systems[4] at a, modulo the equivalence relation 'same irregular type' is isomorphic to the set of irregular parts of formal normal forms (dQ's), and so can be described explicitly:

$$\{\text{order } k \text{ irregular types at } a\} \cong (\mathbb{C}^n \setminus \text{diagonals}) \times (\mathbb{C}^n)^{k-2}; \quad [(A, g)] \longmapsto dQ \tag{34}$$

[4]Replacing the word system by connection makes no difference here.

where the formal normal form associated to the compatibly framed system (A, g) is $dQ + \Lambda dz/z$ for some Λ. The left-hand side of (34) is intrinsic but the map depends on a coordinate choice. For $k = 1$ all systems have the same irregular type. We will return to this later when we discuss isomonodromic deformations; here we fix the irregular type and obtain symplectic moduli spaces.

4.2. Recall that we have chosen m distinct points $a_1, \ldots, a_m \in \mathbb{P}^1$ and (coordinate independent) nice formal normal forms $\mathbf{A} = ({}^1A^0, \ldots, {}^mA^0)$. As usual the polar divisor is $D = \sum_i k_i(a_i)$ where k_i is the order of the pole of ${}^iA^0$.

In this section it is only the irregular type represented by these formal normal forms that is significant. As above, in terms of any coordinate the irregular types are represented as the irregular parts of the formal normal forms.

Definition 2.42. The *extended moduli space* $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ is the set of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a nice meromorphic connection ∇ (with poles on D) on a *trivial* holomorphic vector bundle V over $\mathbb{P}^1$ with compatible framings

$$\mathbf{g} = ({}^1g, \ldots, {}^mg); \qquad {}^ig : V_{a_i} \longrightarrow \mathbb{C}^n$$

such that $(V, \nabla, \mathbf{g})$ has the same irregular type at a_i as ${}^iA^0$.

Note that this differs from the definition of the moduli space $\mathcal{M}^*(\mathbf{A})$ in two ways: firstly we now have compatible framings and secondly only the irregular types are fixed, rather than the full formal normal forms—the exponents of formal monodromy are still free.

The analogue of Theorem 2.35 is then

Theorem 2.43. *For $i = 1, \ldots, m$ let z_i be a local coordinate near a_i on $\mathbb{P}^1$ vanishing at a_i and let $\widetilde{O}_i$ be the extended orbit associated to the normal form ${}^iA^0$ via the coordinate z_i. Then:*

- *The polar part maps induce an isomorphism between the set $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ of isomorphism classes defined above and the symplectic quotient of $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m$ by $GL_n(\mathbb{C})$ at the value 0 of the moment map:*

$$\boxed{\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \cong \widetilde{O}_1 \times \cdots \times \widetilde{O}_m // GL_n(\mathbb{C})}$$

- *In this way $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ inherits an intrinsic complex symplectic structure. That is, the symplectic structure obtained is not dependent on the coordinate choices.*
- *The complex dimension of $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ is $(\sum k_i)n(n-1) + 2nm - 2n^2$.*

Proof. Start by choosing local coordinates $z_i = z - a_i$ as in the proof of Theorem 2.35.

Now suppose $(V, \nabla, \mathbf{g})$ represents an element of $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$. Then in any (global) trivialisation of V, ∇ is of the form:

$$\nabla = d - A = d - \sum_{i=1}^{m} \left({}^iA_{k_i} \frac{dz}{(z-a_i)^{k_i}} + \cdots + {}^iA_1 \frac{dz}{(z-a_i)} \right) \tag{35}$$

for some $n \times n$ matrices ${}^iA_j$ $(1 \le j \le k_i)$. The framings are represented by matrices ${}^ig \in GL_n(\mathbb{C})$ which diagonalise the leading coefficients (so that ${}^ig \cdot {}^iA_{k_i} \cdot {}^ig^{-1}$ is diagonal for each i).

From Lemma 2.40 and the definition of associated extended orbits, we see that for each i the polar part at a_i of $d-A$ together with the compatible framing ${}^ig$ make up an element of the extended orbit $\widetilde{O}_i$ (which is associated to ${}^iA^0$ via z_i):

$$\left({}^ig, \mathrm{PP}_i(A)\right) \in \widetilde{O}_i.$$

As in Theorem 2.35 two other observations are to be made:
- Choosing a different trivialisation of V corresponds to changing $({}^ig, \mathrm{PP}_i(A))$ to

$$\left({}^igh^{-1}, h(\mathrm{PP}_iA)h^{-1}\right)$$

where $h \in GL_n(\mathbb{C})$. This corresponds to the free diagonal action of $GL_n(\mathbb{C})$ on the product $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m$ coming from the separate $GL_n(\mathbb{C})$ actions on the extended orbits defined in Corollary 2.24 on p32.
- An element

$$\left(({}^1g, {}^1A), \ldots, ({}^mg, {}^mA)\right) \in \widetilde{O}_1 \times \cdots \times \widetilde{O}_m$$

of the product of the extended orbits defines a meromorphic connection on the trivial bundle over $\mathbb{P}^1$ (via the expression (35)) iff the sum of the residues is zero:

$$ {}^1A_1 + \cdots + {}^mA_1 = 0 \tag{36}$$

where

$${}^iA = \left({}^iA_{k_i}\frac{d\zeta}{\zeta^k} + \cdots + {}^iA_1\frac{d\zeta}{\zeta}\right) \in \mathfrak{g}^*_{k_i}.$$

Again these two facts fit together symplectically: Corollary 2.24 implies that the sum of the residues on the left-hand side of (36) is the moment map for the diagonal $GL_n(\mathbb{C})$ action on the product of extended orbits.

This gives the required isomorphism and the coordinate independence of the symplectic structure follows as in Theorem 2.35.

To compute the dimension, observe

$$\dim \mathcal{M}^*_{\mathrm{ext}}(\mathbf{A}) = \dim(\widetilde{O}_1 \times \cdots \times \widetilde{O}_m) - 2n^2$$

since the $GL_n(\mathbb{C})$ action is free and its moment map is submersive. Also recall that $\widetilde{O}_i$ is a principal T-bundle over an n-parameter family of G_{k_i} coadjoint orbits (see Corollary 2.15, p27 for the irregular case or Section 2.1 if $k_i = 1$). Thus

$$\dim(\widetilde{O}_i) = \dim(O_i) + 2n = k_in(n-1) + 2n$$

and summing these terms gives the stated result □

Thus the choice of a local coordinate at each a_i on $\mathbb{P}^1$ gives a realisation of the extended moduli space $\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$ as a concrete *extended polar parts manifold*:

$$\widetilde{O}_1 \times \cdots \times \widetilde{O}_m // GL_n(\mathbb{C}).$$

We can now capitalise on our detailed study of the extended orbits $\widetilde{O}_i$ to reveal more of the structure of the extended polar parts manifolds, and therefore of the extended moduli spaces.

4.3. Decoupling. Recall the decoupling lemma from p26, which says that if ${}^iA^0$ is irregular ($k_i \geq 2$) then the associated extended orbit decouples into a product

$$\widetilde{O}_i = {}^iO_B \times T^*GL_n(\mathbb{C}) \tag{37}$$

of the B_{k_i} coadjoint orbit ${}^iO_B$ (associated to ${}^iA^0$) and the cotangent bundle of $GL_n(\mathbb{C})$.

Moreover the free Hamiltonian $GL_n(\mathbb{C})$ action on $\widetilde{O}_i$ acts only on the $T^*GL_n(\mathbb{C})$ factor in (37) and it does so by the standard right action (see Corollary 2.24 p32, Lemma 2.13, p26 and Definition 1.47 p20).

Immediately we find that if each pole is irregular ($k_i \geq 2$) then the extended moduli space decouples into a product of B_k coadjoint orbits together with something which is independent of the choices of formal normal forms $\mathbf{A}$:

$$\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \cong \left({}^1O_B \times \cdots \times {}^mO_B\right) \times \left((T^*GL_n\mathbb{C})^m /\!/ GL_n(\mathbb{C})\right).$$

It is easy to see that the symplectic quotient $(T^*GL_n\mathbb{C})^m /\!/ GL_n(\mathbb{C})$ of m-copies of the cotangent bundle of $GL_n(\mathbb{C})$ by the free diagonal action of $GL_n(\mathbb{C})$ (coming from the standard right actions on each factor) is isomorphic (as a complex symplectic manifold) to $m-1$ copies of $T^*GL_n(\mathbb{C})$:

$$(T^*GL_n\mathbb{C})^m /\!/ GL_n(\mathbb{C}) \cong (T^*GL_n\mathbb{C})^{(m-1)}.$$

In the general case, when some of the poles are simple, the picture is the same:

Corollary 2.44. *The extended moduli space decouples into a product of complex symplectic manifolds:*

$$\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \cong \left({}^1O_B \times \cdots \times {}^mO_B\right) \times M_{\mathbf{k}} \tag{38}$$

where ${}^iO_B$ is the B_{k_i} coadjoint orbit associated to ${}^iA^0$ (so is a point if $k_i = 1$) and $M_{\mathbf{k}}$ is a complex symplectic manifold which only depends on the orders $\mathbf{k} = (k_1, \ldots, k_m)$ of the poles and the rank n.

Proof. It remains just to see what happens if some/all of the k_i's are 1. Suppose (without loss of generality) that $k_1, \ldots, k_p \geq 2$ and $k_{p+1} = \cdots = k_m = 1$ where $0 \leq p < m$. Recall from Section 2.1, p34 that there is just one $k = 1$ extended orbit (which we will call $\widetilde{O}_m$ here) and it is a symplectic submanifold of $T^*GL_n(\mathbb{C})$. Thus

$$\widetilde{O}_1 \times \cdots \times \widetilde{O}_m \cong \left({}^1O_B \times \cdots \times {}^mO_B\right) \times \left((T^*GL_n\mathbb{C})^p \times (\widetilde{O}_m)^{m-p}\right)$$

and so naturally we define $M_{\mathbf{k}}$ to be the symplectic quotient of the second factor (which is a submanifold of $(T^*GL_n\mathbb{C})^m$) by $GL_n(\mathbb{C})$:

$$M_{\mathbf{k}} := \left((T^*GL_n\mathbb{C})^p \times (\widetilde{O}_m)^{m-p}\right) /\!/ GL_n(\mathbb{C})$$

to obtain (38).

If there is at least one irregular singularity ($p \geq 1$), it is clear that $M_{\mathbf{k}}$ is a complex manifold since the symplectic quotient by $GL_n(\mathbb{C})$ just removes a copy of $T^*GL_n(\mathbb{C})$:

$$M_{\mathbf{k}} \cong \left((T^*GL_n\mathbb{C})^{p-1} \times (\widetilde{O}_m)^{m-p}\right).$$

In the remaining case (when all poles are simple; which is not really of interest in this thesis) $M_{\mathbf{k}}$ is still a complex manifold, essentially because the $GL_n(\mathbb{C})$ moment map

$$\widetilde{O}_m \longrightarrow \mathfrak{gl}_n(\mathbb{C})^*$$

is surjective on tangent vectors and the action is free □

Having obtained this decoupled description of the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$, the next observation to make is that, as a complex symplectic manifold, it is not dependent on the choice $\mathbf{A}$ of the nice formal normal forms, but only on the pole orders $k_1, \dots, k_m$ and the rank n:

Corollary 2.45. *Fix positive integers $k_1, \dots, k_m$ and n. If $\mathbf{A}$ and $\mathbf{B}$ are two m-tuples of nice formal normal forms having rank n and pole orders $k_1, \dots, k_m$ then there is a complex symplectic isomorphism between the corresponding extended moduli spaces:*

$$\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \cong \mathcal{M}^*_{\text{ext}}(\mathbf{B}).$$

REMARK 2.46. In contrast, the analogous result for the moduli spaces $\mathcal{M}^*(\mathbf{A})$ is not true; their symplectomorphism class depends on the exponents of formal monodromy.

Proof (of Corollary 2.45). Due to Corollary 2.44 it is sufficient to show that if A^0 and B^0 are nice diagonal elements of $\mathfrak{b}^*_k$ (where $k \geq 2$) then the B_k coadjoint orbits through A^0 and B^0 are symplectically isomorphic:

$$O_B(A^0) \cong O_B(B^0).$$

But this follows directly from the fact that the group B_k is unipotent (so in particular it is nilpotent) and from the Theorem of Michéle Vergne [**106**] which says that coadjoint orbits of nilpotent Lie groups have global Darboux coordinates. Concretely, if $2N = \dim O_B(A^0) = \dim O_B(B^0)$ then we have symplectic isomorphisms

$$O_B(A^0) \cong \left(\mathbb{C}^{2N}, \sum_{i=1}^{N} dx_i \wedge dx_{i+N}\right) \cong O_B(B^0)$$

□

4.4. Torus Actions on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$. Recall from Section 2.1, p11 that for each pole $a_1, \dots, a_m$ there is a natural action of the torus $T \cong (\mathbb{C}^*)^n$ on the set of compatibly framed connections, which changes the choice of compatible framing. This yields an action of m copies of T, i.e. of T^m, on the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$.

On the other hand if $(V, \nabla, \mathbf{g})$ represents an element of $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ then at each a_i it has a canonically associated exponent of formal monodromy ${}^i\Lambda \in \mathfrak{t} = \text{Lie}(T)$ (see Definition 1.14 and Remark 1.15 on p6). Moreover ${}^i\Lambda$ only depends on the isomorphism class of $(V, \nabla, \mathbf{g})$ so we can canonically define maps

$${}^i\mu_T : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \longrightarrow \mathfrak{t} \cong \mathfrak{t}^*$$
$$[(V, \nabla, \mathbf{g})] \longmapsto -{}^i\Lambda$$

for $i = 1, \dots, m$ where we identify $\mathfrak{t}$ with its dual using the trace. (Beware that here ${}^i\Lambda$ has nothing to do with the residue of the chosen formal normal form ${}^iA^0$; $(V, \nabla, \mathbf{g})$ needs to only have the same irregular type as ${}^iA^0$).

The key observation now is:

Proposition 2.47. *With respect to the complex symplectic structure defined in Theorem 2.43, the torus action on the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ changing the framings at a_i is Hamiltonian with moment map ${}^i\mu_T$ for $i = 1, \dots, m$.*

Proof. If we choose local coordinates z_i at a_i on $\mathbb{P}^1$ for each i, then we obtain a concrete realisation of the extended moduli space

$$\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \cong \widetilde{O}_1 \times \cdots \times \widetilde{O}_m // GL_n(\mathbb{C}) \tag{39}$$

from Theorem 2.43. Then the torus action on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ changing the framings at a_i corresponds to the torus action on $\widetilde{O}_i$ defined in Section 2 (see Lemma 2.14, p27 and Corollary 2.24, p32); this makes sense because the T and $GL_n(\mathbb{C})$ actions on the extended orbit $\widetilde{O}_i$ commute. (Beware though that the action of T^m on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ is not free even though the T^m action on the product $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m$ is free.)

A moment map for the torus action on $\widetilde{O}_i$ is given in Definition 2.23, p32. It is clear from the definitions that under the identification (39) this moment map is identified with the 'exponent of formal monodromy map' ${}^i\mu_T$ above □

Now we can deduce that the moduli space $\mathcal{M}^*(\mathbf{A})$ may be obtained as a symplectic quotient of the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ by these torus actions.

Let ${}^i\Lambda^0 \in \mathfrak{t}$ be the residue of the formal normal form ${}^iA^0$ for each i and let

$$\mathbf{\Lambda} = ({}^1\Lambda^0, \ldots, {}^m\Lambda^0) \in \mathfrak{t}^m \cong (\mathfrak{t}^*)^m$$

denote this m-tuple of diagonal matrices. Let $\boldsymbol{\mu}$ denote the moment map for the T^m action:

$$\boldsymbol{\mu} = ({}^1\mu_T, \ldots, {}^m\mu_T) : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \to (\mathfrak{t}^*)^m.$$

The result is then:

Corollary 2.48. *The moduli space $\mathcal{M}^*(\mathbf{A})$ is naturally isomorphic to the symplectic quotient of the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ by T^m at the value $-\mathbf{\Lambda}$ of the moment map $\boldsymbol{\mu}$:*

$$\mathcal{M}^*(\mathbf{A}) \;\cong\; \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \underset{-\mathbf{\Lambda}}{//} T^m \;=\; \boldsymbol{\mu}^{-1}(-\mathbf{\Lambda})/T^m.$$

Proof. Choose local coordinates z_i as usual on $\mathbb{P}^1$. For each i let O_i be the G_{k_i} coadjoint orbit associated to ${}^iA^0$ via z_i and let $\widetilde{O}_i$ be the extended orbit associated to ${}^iA^0$ via z_i.

From Corollary 2.24, p32 we have symplectic isomorphisms

$$O_i \;\cong\; \widetilde{O}_i \underset{-{}^i\Lambda^0}{//} T \tag{40}$$

for $i = 1, \ldots, m$. Thus to prove Corollary 2.48 it is enough to reorder the symplectic quotients as follows. Let $\mathbf{O} = O_1 \times \cdots \times O_m$ and $\widetilde{\mathbf{O}} = \widetilde{O}_1 \times \cdots \times \widetilde{O}_m$. From Theorem 2.35 and (40) we have

$$\mathcal{M}^*(\mathbf{A}) \;\cong\; \mathbf{O}//GL_n(\mathbb{C}) \;\cong\; \left(\widetilde{\mathbf{O}} \underset{-\mathbf{\Lambda}}{//} T^m\right)//GL_n(\mathbb{C}).$$

Now since the T^m and $GL_n(\mathbb{C})$ actions on the extended orbits $\widetilde{\mathbf{O}}$ commute with each other, we can reorder:

$$\left(\widetilde{\mathbf{O}} \underset{-\mathbf{\Lambda}}{//} T^m\right)//GL_n(\mathbb{C}) \;\cong\; \widetilde{\mathbf{O}} \underset{(-\mathbf{\Lambda},0)}{//} (T^m \times GL_n(\mathbb{C})) \;\cong\; \left(\widetilde{\mathbf{O}}//GL_n(\mathbb{C})\right) \underset{-\mathbf{\Lambda}}{//} T^m.$$

Finally using Theorem 2.43 we identify $\widetilde{\mathbf{O}}//GL_n(\mathbb{C})$ with $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ □

5. Universal Family over $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$

The final fact about the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ that we will need, is that it supports a universal family of compatibly framed connections; it is a fine moduli space.

Due to the explicit description of $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ given in Theorem 2.43 the construction of the universal family is straightforward and it is not hard to prove it indeed has the required universal property. Firstly we need to make precise what we mean by 'family' in this context, thereby properly setting up the moduli problem that $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ solves.

5.1. Families of Connections. Let Σ be any Riemann surface. If S is any complex manifold, consider the product $\Sigma \times S$ as a trivial Σ bundle over S via the projection onto the second factor:

$$\pi : \Sigma \times S \longrightarrow S.$$

Thus we regard the Σ directions in the product as being vertical; the vertical tangent bundle T_{Vert} is the subbundle of $T(\Sigma \times S)$ of directions in the kernel of the derivative π_* of the projection:

$$0 \longrightarrow T_{\text{Vert}} \longrightarrow T(\Sigma \times S) \xrightarrow{\pi_*} \pi^*(TS) \longrightarrow 0.$$

Thus by dualising this sequence we can define the vertical exterior derivative:

$$d_{\text{Vert}} : \mathcal{O}_{\Sigma\times S} \longrightarrow T^*_{\text{Vert}}$$

(from the sheaf of holomorphic functions on the product to the sheaf of 'vertical' holomorphic one forms) by composing the usual exterior derivative $d : \mathcal{O}_{\Sigma\times S} \to T^*(\Sigma \times S)$ with the projection $T^*(\Sigma \times S) \to T^*_{\text{Vert}}$. Occasionally, abusing notation, we write $d_{\text{Vert}} = d_\Sigma$.

Definition 2.49.

- A *vertical holomorphic connection* on a holomorphic vector bundle $V \to \Sigma \times S$ is a map

$$\nabla_{\text{Vert}} : V \longrightarrow V \otimes T^*_{\text{Vert}}$$

from the sheaf of sections of V to the sheaf of sections of $V \otimes T^*_{\text{Vert}}$, satisfying the Leibniz rule:

$$\nabla_{\text{Vert}}(fv) = (d_{\text{Vert}}f) \otimes v + f\nabla_{\text{Vert}}(v)$$

where v is a local section of V and f is a local holomorphic function.

- A *family of holomorphic connections on Σ parameterised by S* is a pair $(V, \nabla_{\text{Vert}})$ consisting of a holomorphic vector bundle V on the product $\Sigma \times S$ and a vertical holomorphic connection ∇_{Vert} on V.

Now fix an effective divisor $D = k_1(a_1) + \cdots + k_m(a_m)$ on $\mathbb{P}^1$ and a nice formal normal form ${}^iA^0$ at a_i for each i. In this section we are interested in families of compatibly framed nice meromorphic connections on trivial bundles over $\mathbb{P}^1$ with poles on D and irregular type $\mathbf{A}$, so will explain how to modify Definition 2.49 for this case.

1) Firstly we need to allow ∇_{Vert} to be a meromorphic (vertical) connection, with poles on the divisor

$$\widetilde{D} := D \times S = k_1(\{a_1\} \times S) \times \cdots \times k_m(\{a_m\} \times S)$$

on $\mathbb{P}_1 \times S$.

2) Secondly we require ∇_{Vert} to be nice and have a compatible framing at each pole; that is for $i = 1, \ldots, m$ we have a vector bundle isomorphism

$${}^ig : V|_{\{a_i\}\times S} \longrightarrow \mathbb{C}^n$$

(where $\mathbb{C}^n$ denotes the trivial vector bundle over $\{a_i\} \times S$) such that in any local trivialisation of V extending (part of) this framing, the leading coefficient of $\nabla_{\rm Vert}$ is diagonal. (This leading coefficient will be an $n \times n$ matrix whose entries are local holomorphic functions on S.)

3) Finally we require the restriction of $(V, \nabla_{\rm Vert})$ to each vertical $\mathbb{P}^1$ to have the desired properties; for each $s \in S$:

• $V|_{\mathbb{P}^1 \times \{s\}}$ is a trivial (holomorphic) vector bundle, and
• with respect to the compatible framing ${}^ig(s)$, the connection $\nabla_{\rm Vert}|_{\mathbb{P}^1 \times \{s\}}$ on $\mathbb{P}^1 \times \{s\}$ has irregular type ${}^iA^0$ at a_i.

Definition 2.50. A *family of compatibly framed meromorphic connections on trivial bundles over $\mathbb{P}^1$ with irregular type $\mathbf{A}$ parameterised by S* (or just *family* for the rest of this chapter) is a tuple $(V, \nabla_{\rm Vert}, \mathbf{g})$ as described in 1)-3) above.

Remark 2.51. Families are covariant objects; if $(V, \nabla_{\rm Vert}, \mathbf{g})$ is a family parameterised by S and we have a holomorphic map from a complex manifold S' to S

$$\varphi : S' \longrightarrow S$$

then we can pullback the family $(V, \nabla_{\rm Vert}, \mathbf{g})$ along φ in the obvious way to obtain a family

$$\varphi^*(V, \nabla_{\rm Vert}, \mathbf{g}) = (\varphi^*(V), \varphi^*(\nabla_{\rm Vert}), \varphi^*(\mathbf{g}))$$

parameterised by S'.

5.2. The Universal Family.

Proposition 2.52. *There is a family $(V, \nabla_{\rm Vert}, \mathbf{g})$ (of compatibly framed meromorphic connections on trivial bundles over $\mathbb{P}^1$ with irregular type $\mathbf{A}$) parameterised by the extended moduli space $\mathcal{M}^*_{\rm ext}(\mathbf{A})$ such that*

1) For all $u \in \mathcal{M}^*_{\rm ext}(\mathbf{A})$ the restriction of $(V, \nabla_{\rm Vert}, \mathbf{g})$ to the projective line, $\mathbb{P}^1 \times \{u\}$ is in the isomorphism class specified by u. (See Definition 2.42, p42.)

2) The family $(V, \nabla_{\rm Vert}, \mathbf{g})$ has the following universal property: if $(V', \nabla'_{\rm Vert}, \mathbf{g}')$ is a family parameterised by some complex manifold S' then the canonical map

$$\varphi : S' \longrightarrow \mathcal{M}^*_{\rm ext}(\mathbf{A}); \qquad s \mapsto [(V, \nabla_{\rm Vert}, \mathbf{g})|_{\mathbb{P}^1 \times \{s\}}]$$

taking $s \in S'$ to the isomorphism class of $(V, \nabla_{\rm Vert}, \mathbf{g})|_{\mathbb{P}^1 \times \{s\}}$ is a holomorphic map and is such that the family over S' is the pullback of the family over $\mathcal{M}^*_{\rm ext}(\mathbf{A})$:

$$(V', \nabla'_{\rm Vert}, \mathbf{g}') \cong \varphi^*(V, \nabla_{\rm Vert}, \mathbf{g}).$$

Proof. Firstly we explicitly construct a family over $\mathcal{M}^*_{\rm ext}(\mathbf{A})$. Choose local coordinates $z_i = z - a_i$ on $\mathbb{P}^1$ as in the proof of Theorem 2.35. Then using Theorem 2.43 we can obtain an embedding

$$\iota : \mathcal{M}^*_{\rm ext}(\mathbf{A}) \hookrightarrow \widetilde{O}_1 \times \cdots \times \widetilde{O}_m \subset \big(GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_1}\big) \times \cdots \times \big(GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_m}\big). \tag{41}$$

For example using Theorem 2.43 and the description of the $GL_n(\mathbb{C})$ action we may identify $\mathcal{M}^*_{\rm ext}(\mathbf{A})$ with the set of elements

$$\big(({}^1g, {}^1A), \ldots, ({}^mg, {}^mA)\big) \in \mu^{-1}(0) \subset \widetilde{O}_1 \times \cdots \times \widetilde{O}_m$$

such that ${}^1g = 1$, where $\mu : \widetilde{O}_1 \times \cdots \times \widetilde{O}_m \to \mathfrak{gl}_n(\mathbb{C})^*$ is the moment map for the diagonal $GL_n(\mathbb{C})$ action (which is given by the sum of the residues).

Then from (41) for $i = 1, \dots, m$, we get tautological maps

$$ {}^iA : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \longrightarrow \mathfrak{g}^*_{k_i} $$

$$ {}^ig : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \longrightarrow GL_n(\mathbb{C}) $$

such that for any $u \in \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ the element ${}^ig(u) \cdot {}^iA(u) \cdot {}^ig^{-1}(u) \in \mathfrak{g}^*_{k_i}$ has diagonal leading term. By replacing the symbol ζ by the coordinate z_i the element ${}^iA(u)$ is identified with a matrix of meromorphic one forms on $\mathbb{P}^1$. As $u \in \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ varies we obtain a matrix ${}^iA$ of vertical meromorphic one forms on $\mathbb{P}^1 \times \mathcal{M}^*_{\text{ext}}(\mathbf{A})$. Thus the sum

$$ A := {}^1A + \cdots + {}^mA $$

is also a matrix of vertical one forms and we define a vertical meromorphic connection on the trivial rank n vector bundle over $\mathbb{P}^1 \times \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ simply as

$$ \nabla_{\text{Vert}} := d_{\mathbb{P}^1} - A. $$

The maps ${}^ig : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \to GL_n(\mathbb{C})$ give compatible framings. Thus we have defined a family $(V, \nabla_{\text{Vert}}, \mathbf{g})$ over $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ with V being the trivial bundle.

Property 1) is clear from the definition. To prove property 2), suppose that $(V', \nabla'_{\text{Vert}}, \mathbf{g}')$ is another family parameterised by some complex manifold S'.

Now we can use the framing along $a_1 \times S'$ to trivialise V'. (By assumption $V'|_{\mathbb{P}^1 \times \{s\}}$ is trivial for each $s \in S'$ so the framing at (a_1, s) extends to a unique trivialisation of $V'|_{\mathbb{P}^1 \times \{s\}}$ since $\mathbb{P}^1$ is compact. As s varies we get the required trivialisation.) In this trivialisation ∇'_{Vert} has the form

$$ \nabla'_{\text{Vert}} = d_{\mathbb{P}^1} - A' $$

for some matrix of vertical meromorphic one forms A' on $\mathbb{P}^1 \times S'$ with poles on the divisor $D \times S'$. If we define

$$ {}^iA' := \mathrm{PP}_i(A') $$

to be the polar part of A' at a_i (with respect to z_i) then ${}^iA'$ is a holomorphic map from S' to $\mathfrak{g}^*_{k_i}$. Also the compatible framings are represented as a holomorphic map

$$ \mathbf{g}' = ({}^1g', \dots, {}^mg') : S' \longrightarrow GL_n(\mathbb{C})^m $$

with ${}^1g' = 1$. It follows that for each $s \in S'$ we have a point

$$ \left(({}^1g', {}^1A'), \dots, ({}^mg', {}^mA')\right)(s) \in \mu^{-1}(0) \subset \widetilde{O}_1 \times \cdots \times \widetilde{O}_m \tag{42} $$

of the zero set of the $GL_n(\mathbb{C})$ moment map inside the product of extended orbits. As s varies we see this is a holomorphic map from S' to $\mu^{-1}(0)$. Moreover since ${}^1g' = 1$ we see S' maps into the subspace we identified above with $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$; the canonical map is holomorphic.

Finally it is manifest from (42) that for each i, the map $({}^ig', {}^iA') : S' \to \widetilde{O}_i$ coincides with the composition of the canonical map $S' \to \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ and the tautological map $({}^ig, {}^iA) : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \to \widetilde{O}_i$ and so the family $(V, \nabla_{\text{Vert}}, \mathbf{g})$ over $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ indeed pulls back along the canonical map to the original family over S' $\square$

CHAPTER 3

C^∞ Approach to Meromorphic Connections

In the previous chapter we thoroughly studied moduli spaces of meromorphic connections on trivial holomorphic vector bundles, capitalising on the observation that such connections are determined by their polar parts.

Now we move on to our second point of view on meromorphic connections: a C^∞ approach. We begin by defining a suitable notion of C^∞ singular connections. Then we find that the local fixed formal type moduli spaces ${}^0C(A^0)$ and $\mathcal{H}(A^0)$ of meromorphic connection germs and compatibly framed meromorphic connections germs from Chapter 1 are easily described in terms of *flat* singular connections. Fixing the formal type of a meromorphic connection corresponds to fixing the C^∞ Laurent expansion of a C^∞ singular connection.

Next we globalise to $\mathbb{P}^1$ and obtain C^∞ descriptions of spaces of meromorphic connections with fixed formal types on arbitrary degree zero holomorphic bundles (and the corresponding extended version—see Section 3.1).

Finally, and this is the key point, we observe that the well known Atiyah-Bott symplectic structure (on spaces of nonsingular connections over compact surfaces) naturally generalises to give symplectic structures on these spaces of singular connections with fixed Laurent expansions. See Section 4.

Although we work exclusively with 'nice' connections over $\mathbb{P}^1$ here (as we wish to study isomonodromic deformations of such connections) we remark that this C^∞ approach works over arbitrary compact Riemann surfaces (maybe with boundary) and the 'nice' hypothesis is also superfluous (see Remark 3.10).

1. Singular Connections: C^∞ Connections with Poles

Let $D = k_1(a_1) + \cdots + k_m(a_m)$ be an effective divisor on $\mathbb{P}^1$ as usual and choose local coordinates $z_i = z - a_i$ on $\mathbb{P}^1$. Define the sheaf of 'smooth functions with poles on D' to be the sheaf of C^∞ sections of the holomorphic line bundle associated to the divisor D:

$$C^\infty[D] := \mathcal{O}[D] \otimes_{\mathcal{O}} C^\infty$$

where $\mathcal{O}$ is the sheaf of holomorphic functions and C^∞ the infinitely differentiable complex functions. Any local section of $C^\infty[D]$ near a_i is of the form

$$\frac{f}{(z-a_i)^{k_i}}$$

for a C^∞ function f. Similarly define sheaves $\Omega^r[D]$ of C^∞ r-forms with poles on D (so in particular $\Omega^0[D] = C^\infty[D]$).

A basic feature is that 'C^∞-Laurent expansions' can be taken at each a_i. This gives a map

$$L_i : \Omega^*[D](\mathbb{P}^1) \to \mathbb{C}[\![z_i, \bar{z}_i]\!] z_i^{-k_i} \otimes \bigwedge\nolimits^* \mathbb{C}^2 \tag{43}$$

where $\mathbb{C}^2 = \mathbb{C}dz_i \oplus \mathbb{C}d\bar{z}_i$. For example if f is a C^∞ function defined in a neighbourhood of a_i then

$$L_i\left(\frac{f}{(z-a_i)^{k_i}}\right) = \frac{L_i(f)}{z_i^{k_i}}$$

where $L_i(f)$ is the Taylor expansion of f at a_i.

The Laurent map L_i has nice morphism properties, for example

$$L_i(\omega_1 \wedge \omega_2) = L_i(\omega_1) \wedge L_i(\omega_2)$$

and L_i commutes with the exterior derivative d, where d is defined on the right-hand side of (43) in the obvious way $(d(z_i^{-1}) = -dz_i/z_i^2)$.

We will repeatedly make use of the fact that the kernel of L_i consists of nonsingular forms, that is: if $L_i(\omega) = 0$ then ω is nonsingular at a_i. This apparently innocuous statement is surprisingly tricky to prove directly, but since it is crucial for us we remark it follows from the following:

Lemma 3.1. *(Division). Let $\Delta \subset \mathbb{C}$ be a disk containing the origin. Suppose $f \in C^\infty(\Delta)$ and that the Taylor expansion of f at 0 is in the ideal in $\mathbb{C}[\![z,\bar{z}]\!]$ generated by z. Then $f/z \in C^\infty(\Delta)$.*

That is, if we can divide the Taylor series by z then we can divide the function by z and it will still be infinitely differentiable at 0.

Proof. In the real case (for functions on an interval) this is easy but in the complex case the difficulty arises from the fact that the the ideal in $\mathbb{C}[\![z,\bar{z}]\!]$ generated by z is of infinite codimension. Anyway this lemma is a special case of a much more general result due to Malgrange (see [**69**]). The particular instance here is discussed by Martinet [**77**] p115□

Another fact we will use is that the C^∞ Laurent expansion map L_i in (43) is *surjective* for each i. This is a classical result of E.Borel which we discuss in Appendix D.

Now let $V \to \mathbb{P}^1$ be a rank n, C^∞ vector bundle. The main definition of this section is:

Definition 3.2. A C^∞ *singular connection* ∇ *on* V *with poles on* D is a map

$$\nabla : V \longrightarrow V \otimes \Omega^1[D]$$

from the sheaf of (C^∞) sections of V to the sheaf of sections of $V \otimes \Omega^1[D]$, satisfying the Leibniz rule:

$$\nabla(fv) = (df) \otimes v + f\nabla v$$

where v is a local section of V and f is a local C^∞ function.

Concretely in terms of the local coordinate z_i on $\mathbb{P}^1$ vanishing at a_i and a local trivialisation of V, ∇ has the form:

$$\nabla = d - \frac{{}^iA}{z_i^{k_i}} \tag{44}$$

where ${}^iA$ is an $n \times n$ matrix of C^∞ one forms.

In this thesis, to study isomonodromic deformations, we need only to consider the case when V is the trivial rank n, C^∞ vector bundle over $\mathbb{P}^1$. (Recall any degree zero vector bundle over $\mathbb{P}^1$ is C^∞ trivial.)

DEFINITION 3.3.
• Let $\mathcal{A}[D]$ denote the set of C^∞ singular connections with poles on D on the trivial C^∞ rank n vector bundle:

$$\mathcal{A}[D] := \left\{ d - \alpha \;\middle|\; \alpha \in \mathrm{End}_n\left(\Omega^1[D](\mathbb{P}^1)\right) \right\}$$

where $\Omega^1[D]$ is the sheaf of C^∞ one-forms with poles on D.
• The group of C^∞ bundle automorphisms (or 'gauge group') is

$$\mathcal{G} := GL_n(C^\infty(\mathbb{P}^1)).$$

Observe that the *meromorphic* connections on the trivial holomorphic vector bundle with poles on D (Definition 1.1) are a subset of the singular connections $\mathcal{A}[D]$, according to the above definition.

The action of a bundle automorphism $g \in \mathcal{G}$ on a singular connection $d - \alpha \in \mathcal{A}[D]$ is given explicitly by the formula

$$g[\alpha] = g\alpha g^{-1} + (dg)g^{-1}.$$

As in the nonsingular case, singular connections have curvature:

DEFINITION 3.4.
• The *curvature* of a singular connection $d-\alpha \in \mathcal{A}[D]$ is the following matrix of singular two-forms

$$\mathcal{F}(\alpha) := (d-\alpha)^2 = -d\alpha + \alpha^2 \in \mathrm{End}_n\left(\Omega^2[2D](\mathbb{P}^1)\right).$$

• The *flat* connections are those with zero curvature.
(The subset of flat singular connections will be denoted $\mathcal{A}_{\mathrm{fl}}[D]$; it is preserved under gauge transformations since $\mathcal{F}(g[\alpha]) = g(\mathcal{F}(\alpha))g^{-1}$ for $g \in \mathcal{G}$.)

REMARK 3.5. Occasionally one comes across notions of curvature of singular connections involving distributional derivatives. For example a meromorphic connection on a Riemann surface with a simple pole is sometimes said to have a δ-function singularity in its curvature at the pole, to account for the monodromy around the pole. The definition of curvature we use (Definition 3.4) does *not* involve distributional derivatives, and so, for us, *any* meromorphic connection over a Riemann surface is flat.

Now choose a nice formal normal form ${}^iA^0$ at a_i for each i and let $\mathbf{A}$ denote this m-tuple of normal forms as usual. Since $d - \alpha \in \mathcal{A}[D]$ is on the trivial vector bundle, and by definition ${}^iA^0$ is a germ of a connection on the trivial bundle, we can compare the Laurent expansion of α at a_i with ${}^iA^0$. In particular the following definition makes sense:

DEFINITION 3.6. The set of singular connections with fixed Laurent expansions $\mathbf{A}$ is

$$\mathcal{A}(\mathbf{A}) := \left\{ d - \alpha \in \mathcal{A}[D] \;\middle|\; L_i(\alpha) = {}^iA^0 \text{ for each } i \right\}.$$

Observe in particular that if $d - \alpha \in \mathcal{A}(\mathbf{A})$ then it follows from the division lemma (Lemma 3.1) that the $(0,1)$ part of α is *nonsingular* over all of $\mathbb{P}^1$. Also observe that $\mathcal{A}(\mathbf{A})$ is an affine space modelled on the set of matrices of C^∞ one forms on $\mathbb{P}^1$ having zero Taylor expansion at each marked point a_i.

The full motivation for this definition comes in the next section where we will explain the natural relationship between meromorphic connections with fixed formal type and singular connections with fixed Laurent expansions.

2. Smooth Local Picture

We will give a C^∞ description of the sets of local analytic classes ${}^0C(A^0)$ and $\mathcal{H}(A^0)$ defined in Chapter 1.

We begin with a straightforward observation. Define the group

$${}^0\mathcal{G} := \{\text{germs at 0 of } g \in \mathcal{G}\}$$

where as usual two elements of $\mathcal{G}$ define the same germ if they agree throughout some neighbourhood of 0. Taking Taylor expansion at the origin defines a surjective group homomorphism

$$L_0 : {}^0\mathcal{G} \to GL_n(\mathbb{C}[\![z,\bar{z}]\!]).$$

Let ${}^0\mathcal{G}_1$ be its kernel, consisting of elements taking the value 1 and having all derivatives vanish at the origin, so there is an exact sequence of groups:

$$1 \longrightarrow {}^0\mathcal{G}_1 \longrightarrow {}^0\mathcal{G} \xrightarrow{L_0} GL_n(\mathbb{C}[\![z,\bar{z}]\!]) \longrightarrow 1.$$

Also fix a nice formal normal form A^0 at 0 and define the lifts of the set of applicable formal transformations and of the stabiliser torus $T \cong (\mathbb{C}^*)^n$ as follows:

$${}^0\mathcal{G}(A^0) := L_0^{-1}(\widehat{G}(A^0)), \qquad {}^0\mathcal{G}_T := L_0^{-1}(T).$$

Immediately we have:

Lemma 3.7. *Taking Taylor series at* 0 *gives bijections of sets:*

$$G\{z\}\backslash {}^0\mathcal{G}(A^0)/{}^0\mathcal{G}_1 \xrightarrow{L_0} \mathcal{H}(A^0) \tag{45}$$

and

$$G\{z\}\backslash {}^0\mathcal{G}(A^0)/{}^0\mathcal{G}_T \xrightarrow{L_0} {}^0C(A^0) \tag{46}$$

where (46) may be obtained from (45) by the residual action of ${}^0\mathcal{G}_T/{}^0\mathcal{G}_1 \cong T$.

Proof. Observe that ${}^0\mathcal{G}_1$ acts on ${}^0\mathcal{G}(A^0)$ and on ${}^0\mathcal{G}_T$, and so restricting the isomorphism ${}^0\mathcal{G}/{}^0\mathcal{G}_1 \xrightarrow{L_0} GL_n(\mathbb{C}[\![z,\bar{z}]\!])$ gives bijections:

$${}^0\mathcal{G}(A^0)/{}^0\mathcal{G}_1 \xrightarrow{L_0} \widehat{G}(A^0) \quad \text{and} \quad {}^0\mathcal{G}_T/{}^0\mathcal{G}_1 \xrightarrow{L_0} T.$$

Now recall the definitions: $\mathcal{H}(A^0) = G\{z\}\backslash\widehat{G}(A^0)$ and ${}^0C(A^0) = \mathcal{H}(A^0)/T$ $\square$

Having lifted things up into a smooth context a new interpretation of the smooth quotients above will be given. In particular it is desirable to remove the groups $G\{z\}$ occurring on the left-hand sides in (45) and (46).

Let k be the order of the pole of the chosen normal form A^0 and let ${}^0\mathcal{A}[k] = {}^0\mathcal{A}[k(0)]$ denote the set of germs at 0 of C^∞ singular connections on the trivial bundle, with poles of order at most k.

Now given $g \in {}^0\mathcal{G}(A^0)$, so that $L_0(g) \in \widehat{G}(A^0)$ is an applicable formal transformation, let $A = L_0(g)[A^0]$ be the associated system and define a map σ from ${}^0\mathcal{G}(A^0)$ to the set of germs of singular connections ${}^0\mathcal{A}[k]$ as follows:

$$\sigma : {}^0\mathcal{G}(A^0) \to {}^0\mathcal{A}[k];$$

$$\sigma(g) := g^{-1}[A] = g^{-1}[L_0(g)[A^0]].$$

This is slightly subtle: the first gauge transformation $L_0(g)$ is just a formal series whereas the second, g^{-1}, is a C^∞ germ. The composition $g^{-1}\circ L_0(g)$ does not make sense; σ should be interpreted exactly as stated. There are three basic observations to make about this map σ:

- The singular connection $\sigma(g)$ has Laurent expansion A^0:

$$L_0(\sigma(g)) = A^0$$

—this follows from the morphism properties of L_0.

- The fibres of σ contain the orbits of the action of $G\{z\}$ on the left of ${}^0\mathcal{G}(A^0)$:

$$\sigma(hg) = \sigma(g) \qquad \text{for any holomorphic } h \in G\{z\}$$

—this is clear from the definition of σ.

- Finally: $\sigma(g)$ is a *flat* singular connection.

—indeed it is C^∞ gauge equivalent to the meromorphic connection A.

Thus σ gives a map into the flat connection germs with Laurent expansion A^0, i.e. into ${}^0\mathcal{A}_{\text{fl}}(A^0)$. In fact it is surjective and its fibres are precisely the $G\{z\}$ orbits:

Proposition 3.8. *If A^0 is a nice normal form then the map σ defined above induces a bijection of sets*

$$G\{z\}\backslash {}^0\mathcal{G}(A^0) \longrightarrow {}^0\mathcal{A}_{\text{fl}}(A^0).$$

Proof. We have seen the induced map is well defined and now show it is bijective.

For surjectivity, suppose $d-\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$ is a flat singular connection with Laurent expansion A^0. Thus the $d\bar{z}$ component $\alpha^{0,1}$ of α has zero Laurent expansion at 0 and so in particular is nonsingular. It follows (see [**10**] p555 or [**14**] p67) that there exists $g \in {}^0\mathcal{G}$ with $(\bar{\partial}g)g^{-1} = \alpha^{0,1}$ and so $A := g^{-1}[\alpha]$ is still flat and has no $(0,1)$ part. By writing $A = \gamma dz/z^k$ for smooth γ observe that flatness implies $\bar{\partial}\gamma = 0$ and so A is meromorphic.

We claim now that A is formally equivalent to A^0, and that $L_0(g)$ is a formal isomorphism between them (i.e. $L_0(g)[A] = A^0$, in other words $(A, L_0(g^{-1}))$ is a marked pair with associated formal normal form A^0). Firstly $L_0(g)$ is a formal transformation (i.e. has no terms containing $\bar{z}$) because $L_0(\bar{\partial}g) = L_0(\alpha^{0,1}g) = 0$ since $L_0(\alpha^{0,1}) = 0$. Secondly just observe

$$L_0(g^{-1})[A^0] = L_0(g^{-1})[L_0(\alpha)] = L_0(g^{-1}[\alpha]) = L_0(A) = A$$

and so the claim follows. In particular $g^{-1} \in {}^0\mathcal{G}(A^0)$ and by construction $\sigma(g^{-1}) = \alpha$ and so σ is onto.

Finally if $g_1[A] = g_2[B]$ with A, B meromorphic then $h[A] = B$ with $h := g_2^{-1}g_1$. Looking at $(0,1)$ parts gives $(\bar{\partial}h)h^{-1} = 0$ and so h is holomorphic. This proves injectivity □

Combining this with Lemma 3.7 immediately yields the main result of this section:

Corollary 3.9. *If A^0 is a nice formal normal form then there are canonical bijections:*

$$ {}^0\mathcal{A}_{\text{fl}}(A^0)/{}^0\mathcal{G}_1 \cong \mathcal{H}(A^0)$$

$$ {}^0\mathcal{A}_{\text{fl}}(A^0)/{}^0\mathcal{G}_T \cong {}^0C(A^0)$$

between the ${}^0\mathcal{G}_1$ orbits of flat singular connection germs with Laurent expansion A^0 and the set of analytic equivalence classes of compatibly framed systems with associated formal normal form A^0, and between the ${}^0\mathcal{G}_T$ orbits of flat singular connection germs with Laurent expansion A^0 and the set of analytic equivalence classes of systems formally equivalent to A^0.

Proof. This follows directly by substituting ${}^0\mathcal{A}_{\text{fl}}(A^0)$ for $G\{z\}\backslash {}^0\mathcal{G}(A^0)$ in Lemma 3.7. In summary: to go from a flat singular connection $d-\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$ to $\mathcal{H}(A^0)$ just solve $(\bar\partial g)g^{-1} = \alpha^{0,1}$ and take the $G\{z\}$ orbit of $L_0(g^{-1}) \in \widehat{G}(A^0)$ to give an element of $\mathcal{H}(A^0)$ (see the proof of Proposition 3.8). Conversely, given $\widehat{F} \in \widehat{G}(A^0)$, let $A = \widehat{F}[A^0]$ be the associated system and use E.Borel's theorem to find $g \in {}^0\mathcal{G}$ such that $L_0(g) = \widehat{F}$. Then set $\alpha = g^{-1}[A]$ to give $\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$ □

It is surprising that the analytic equivalence classes may be encoded in this entirely C^∞ way. These bijections can be thought of as relating the two distinguished types of elements (the meromorphic connections and the connections with fixed Laurent expansion) within the ${}^0\mathcal{G}$ orbits in ${}^0\mathcal{A}_{\text{fl}}[k]$. That is, they relate the conditions $\alpha \in \text{Syst}(A^0)$ and $\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$ on $\alpha \in {}^0\mathcal{A}_{\text{fl}}[k]$ by moving within α's ${}^0\mathcal{G}$ orbit.

Remark 3.10. This description of the analytic classes easily extends to the general (non-nice) case. Since it is unneeded here we just state the result. Let A be any system and let ${}^0\mathcal{G}_{\text{Stab}(A)}$ be the subgroup of ${}^0\mathcal{G}$ consisting of elements g whose Taylor expansion stabilises A (i.e. $L_0(g)[A] = A$). Then there is a canonical bijection between the set of analytic classes of systems formally equivalent to A and the set of ${}^0\mathcal{G}_{\text{Stab}(A)}$ orbits of flat singular connection germs with Laurent expansion A:

$$ {}^0C(A) \cong {}^0\mathcal{A}_{\text{fl}}(A)/{}^0\mathcal{G}_{\text{Stab}(A)}. $$

Similarly $\mathcal{H}(A) := G\{z\}\backslash\widehat{G}(A)$ is isomorphic to ${}^0\mathcal{A}_{\text{fl}}(A)/{}^0\mathcal{G}_1$, but in general $\mathcal{H}(A)$ cannot be interpreted as compatibly framed systems, only as marked pairs.

Now recall from Chapter 1 that the set $\mathcal{H}(A^0)$ maybe described in terms of Stokes factors, and so it follows from Corollary 3.9 that we can associate Stokes factors to any flat C^∞ connection with Laurent expansion A^0; there is a surjective map:

$$ \mathbf{K} : {}^0\mathcal{A}_{\text{fl}}(A^0) \longrightarrow \prod_{d\in\mathbb{A}} \mathbb{S}\text{to}_d(A^0) $$

whose fibres are the ${}^0\mathcal{G}_1$ orbits. After untangling the definitions we arrive at

Definition 3.11. The Stokes factors of $\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$ are:

$$ \mathbf{K}(\alpha) = \mathbf{K}(\widehat{F}) $$

where $\widehat{F} = L_0(g^{-1})$ for any $g \in {}^0\mathcal{G}$ solving $(\bar\partial g)g^{-1} = \alpha^{0,1}$ and the Stokes factors $\mathbf{K}(\widehat{F})$ are as defined in Remark 1.28.
(As usual the Stokes factors depend on choices of $\log(z)$ in the anti-Stokes directions.)

A more direct approach to these Stokes factors can be obtained since there are also canonical solutions on sectors in the C^∞ case:

Lemma 3.12. *Suppose $\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$. For each choice of $\log(z)$ on the sector Sect_i there is a canonical choice Φ_i of fundamental solution of α on Sect_i, given by:*

$$ \Phi_i := g\Sigma_i(L_0(g^{-1}))z^\Lambda e^Q $$

for any $g \in {}^0\mathcal{G}$ solving $(\bar\partial g)g^{-1} = \alpha^{0,1}$.

Proof. From the proof of Proposition 3.8, such g satisfies $L_0(g^{-1}) \in \widehat{G}(A^0)$ and is unique up to right multiplication by $h \in G\{z\}$. Let $A := L_0(g^{-1})[A^0] = g^{-1}[\alpha]$ be the associated system, determined by the choice of g. Proposition 1.24 then provides an invertible solution $\Sigma_i(L_0(g^{-1}))$ of $\mathrm{Hom}(A^0, A)$ on Sect_i. It follows that $g\Sigma_i(L_0(g^{-1}))$ is an invertible solution of $\mathrm{Hom}(A^0, \alpha)$ which is independent of the choice of g. Composing this with the fundamental solution $z^\Lambda e^Q$ of A^0 gives the result □

As in the holomorphic case the difference between these C^∞ fundamental solutions on various sectors is encoded in the Stokes factors:

Lemma 3.13. *Suppose $\alpha \in {}^0\mathcal{A}_{\mathrm{fl}}(A^0)$ and make a choice of sector labelling and* $\log(z)$ *branches as in Remark 1.26. Let Φ_i be the canonical fundamental solution of α on* Sect_i *from Lemma 3.12. If Φ_i is extended (as a solution of* $\mathrm{Hom}(0, \alpha)$*) across the anti-Stokes ray d_{i+1} then on the overlap:*

$$\Phi_i = \Phi_{i+1}K_{i+1}(\alpha) \quad \text{for } i = 1, \dots, r-1, \text{ and}$$

$$\Phi_i = \Phi_1 K_1(\alpha) M_0 \quad \text{for } i = r,$$

where $M_0 = e^{2\pi\sqrt{-1}\Lambda}$ is the formal monodromy.

Proof. When the expression $\Phi_{i+1}^{-1}\Phi_i$ is formed the C^∞ factors 'g' cancel out. The remaining expression is that for the corresponding holomorphic fundamental solutions and so Lemma 3.13 follows from the holomorphic version (Lemma 1.31) and from the definition of $\mathbf{K}(\alpha)$ □

Moreover, as before, the Stokes factors encode the local monodromy in the C^∞ case too:

Corollary 3.14. *The local monodromy of $\alpha \in {}^0\mathcal{A}_{\mathrm{fl}}(A^0)$ is conjugate to*

$$K_r(\alpha) \cdots K_1(\alpha) M_0$$

Proof. See Lemma 1.33 □

Following Chapter 1 we can define the ith Stokes *matrix* $S_i(\alpha)$ of $\alpha \in {}^0\mathcal{A}_{\mathrm{fl}}(A^0)$ in terms of the Stokes factors $\mathbf{K}(\alpha)$ as in Definition 1.36. Then the subsequent results of Chapter 1 regarding Stokes matrices all translate naturally into this C^∞ context.

3. Globalisation

Recall we have fixed a divisor $D = \sum k_i(a_i)$ on $\mathbb{P}^1$, chosen nice formal normal forms $\mathbf{A} = (\dots, {}^iA^0, \dots)$ and defined $\mathcal{A}(\mathbf{A})$ to be the set of singular connections on the trivial rank n vector bundle on $\mathbb{P}^1$ having Laurent expansion ${}^iA^0$ at a_i for each i.

Following the results of the last section we are led to consider such connections which are *flat*.

Firstly we make a definition:

DEFINITION 3.15. Let $\mathcal{M}(\mathbf{A})$ be the set of isomorphism classes of pairs (V, ∇) where V is a *degree zero* rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V which is formally equivalent to ${}^iA^0$ at a_i for each i and has no other poles.

REMARK 3.16. The only difference between $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}^*(\mathbf{A})$ is that the bundles are required to be holomorphically trivial in $\mathcal{M}^*(\mathbf{A})$ but only degree zero in $\mathcal{M}(\mathbf{A})$; there is a natural inclusion $\mathcal{M}^*(\mathbf{A}) \subset \mathcal{M}(\mathbf{A})$.

The main result we will prove in this section is then:

Theorem 3.17. *There is a canonical bijection between the set of $\mathcal{G}_T$ orbits of flat C^∞ singular connections with fixed Laurent expansions and the set $\mathcal{M}(\mathbf{A})$ of isomorphism classes defined above:*

$$\mathcal{M}(\mathbf{A}) \cong \mathcal{A}_{\text{fl}}(\mathbf{A})/\mathcal{G}_T.$$

($\mathcal{G}_T$ is the subgroup of $\mathcal{G}$ of elements with Taylor expansion equal to a constant diagonal matrix at each a_i.)

Proof. Suppose (V, ∇) represents an isomorphism class in $\mathcal{M}(\mathbf{A})$. The meromorphic connection ∇ is in particular a C^∞ singular connection, according to Definition 3.2. Since V is degree zero it is C^∞ trivial so, by choosing a trivialisation, (V, ∇) determines a singular connection $d-\alpha$ on the trivial bundle over $\mathbb{P}^1$. (We will denote the trivial bundle E here.)

From the local picture in Section 2 above, since ∇ is formally equivalent to ${}^iA^0$ at a_i, we can choose $g \in \mathcal{G}$ such that $g[\alpha]$ has Laurent expansion ${}^iA^0$ at a_i for all i. This gives an element $g[\alpha]$ of $\mathcal{A}_{\text{fl}}(\mathbf{A})$ and we take the $\mathcal{G}_T$ orbit through it to define the required map. We need to check this $\mathcal{G}_T$ orbit only depends on the isomorphism class of (V, ∇) and that the map is bijective.

Suppose we have two such pairs (V, ∇) and (V', ∇') and we choose C^∞ trivialisations of V and V' so that ∇, ∇' give singular connections $d-\alpha_1$, $d-\alpha_2$ on E respectively. Then the meromorphic connections are isomorphic if and only if α_1 and α_2 are in the same $\mathcal{G}$ orbit:

$$(V, \nabla) \cong (V', \nabla') \qquad \text{iff} \qquad \alpha_1 = g[\alpha_2] \text{ for some } g \in \mathcal{G}. \tag{47}$$

Assuming this statement is true we see that an isomorphism class $[(V, \nabla)]$ of meromorphic connections determines (and is determined by) a $\mathcal{G}$ orbit of singular connections on the trivial bundle. This $\mathcal{G}$ orbit has a subset of singular connections having Laurent expansion ${}^iA^0$ at a_i for each i. This subset is a $\mathcal{G}_T$ orbit of singular connections (since T is the stabiliser of ${}^iA^0$) and is the element of $\mathcal{A}_{\text{fl}}(\mathbf{A})/\mathcal{G}_T$ corresponding to $[(V, \nabla)]$. Theorem proved.

The statement (47) follows from a standard $\bar{\partial}$-operator argument. Consider the following diagram, where the horizontal maps are the chosen C^∞ trivialisations of V and V':

$$\begin{array}{ccc} V & \xrightarrow{\cong} & E \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle g} \\ V' & \xrightarrow{\cong} & E. \end{array} \tag{48}$$

Given an isomorphism $(V, \nabla) \cong (V', \nabla')$, (that is, an isomorphism $\varphi : V \to V'$ of holomorphic vector bundles such that $\varphi^*(\nabla') = \nabla$), then $g \in \mathcal{G}$ arises from the commutativity of the diagram and it is clear that $\alpha_1 = g[\alpha_2]$.

Conversely, given $g \in \mathcal{G}$ such that $\alpha_1 = g[\alpha_2]$ observe that the holomorphic sections of V correspond to the C^∞ sections v of E that solve

$$\bar{\partial}_1 v = 0$$

where $\bar\partial_1 = \bar\partial - \alpha_1^{0,1}$ (v is just a column vector of functions on $\mathbb{P}^1$).

Similarly holomorphic sections of V' correspond to C^∞ sections v' of E that solve $\bar\partial_2 v' = 0$ where $\bar\partial_2 = \bar\partial - \alpha_2^{0,1}$. Now the $(0,1)$ component of the equation $\alpha_1 = g[\alpha_2]$ implies that

$$\bar\partial_2(g(v)) = 0 \quad \Longleftrightarrow \quad \bar\partial_1 v = 0$$

and so the map φ (defined from g to make (48) commute), takes holomorphic sections of V to holomorphic sections of V': it is an isomorphism of holomorphic vector bundles. The rest of the equation $\alpha_1 = g[\alpha_2]$ implies that φ pulls ∇' back to ∇ □

3.1. Extended Version. We may incorporate compatible framings into the C^∞ picture simply by replacing the group $\mathcal{G}_T$ by $\mathcal{G}_1$ in Theorem 3.17: $\mathcal{A}_{\text{fl}}(\mathbf{A})/\mathcal{G}_1$ corresponds to the set of isomorphism classes of compatibly framed meromorphic connections with fixed formal types.

However, as in Chapter 2, we expect it is better (from a symplectic point of view) to only fix the *irregular* type of compatibly framed connections (leaving the residue of the formal normal form free). Thus in this section we define extended spaces of C^∞ singular connections and thereby obtain a C^∞ description of the set (which we will denote $\mathcal{M}_{\text{ext}}(\mathbf{A})$) of isomorphism classes of compatibly framed meromorphic connections on degree zero bundles having fixed irregular types.

The main definition is:

Definition 3.18. The extended set of C^∞ singular connections determined by $\mathbf{A}$ on the trivial rank n vector bundle over $\mathbb{P}^1$ is

$$\mathcal{A}_{\text{ext}}(\mathbf{A}) := \left\{ d - \alpha \in \mathcal{A}[D] \;\middle|\; \begin{array}{l} \text{For each } i,\ L_i(\alpha) \text{ is a nice formal normal} \\ \text{form with the same irregular type as } {}^i\!A^0 \end{array} \right\}. \tag{49}$$

What does this mean? We have fixed local coordinates $z_i = z - a_i$ on $\mathbb{P}^1$ and are using the usual (coordinate dependent) notion of formal normal form. Thus the condition on α in (49) is that its Laurent expansion at a_i is of the form

$$L_i(\alpha) = {}^i\!A^0 + \Lambda \frac{dz}{z - a_i}$$

for some diagonal matrix $\Lambda \in \mathfrak{t}$. Moreover if ${}^i\!A^0$ is logarithmic ($k_i = 1$) then we also require the residue $\text{Res}_i(L_i(\alpha))$ to have distinct eigenvalues mod $\mathbb{Z}$.

Remark 3.19. Due to this extra condition in the logarithmic case, $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is generally not an affine space (whereas $\mathcal{A}(\mathbf{A})$ is). Nonetheless we can identify the tangent space to $\mathcal{A}_{\text{ext}}(\mathbf{A})$ with the vector space of $n \times n$ matrices of C^∞ one forms on $\mathbb{P}^1$ with poles on D and having Laurent expansion $\Lambda_i dz/(z - a_i)$ at a_i for each i, for any diagonal matrices $\Lambda_1, \ldots, \Lambda_m$. See Section 4.

Proposition 3.20. *The set $\mathcal{M}_{\text{ext}}(\mathbf{A})$ of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a nice meromorphic connection ∇ (with poles on D) on a* degree zero *holomorphic vector bundle V over $\mathbb{P}^1$ with compatible framings $\mathbf{g}$ such that $(V, \nabla, \mathbf{g})$ has the same irregular type at a_i as ${}^i\!A^0$ is canonically isomorphic to the set of $\mathcal{G}_1$ orbits of flat connections in $\mathcal{A}_{\text{ext}}(\mathbf{A})$:*

$$\mathcal{M}_{\text{ext}}(\mathbf{A}) \cong \mathcal{A}_{\text{ext,fl}}(\mathbf{A})/\mathcal{G}_1.$$

Proof. We have observed for any m-tuple $\mathbf{A}'$ of nice formal normal forms that the quotient $\mathcal{A}_{\mathrm{fl}}(\mathbf{A}')/\mathcal{G}_1$ corresponds to compatibly framed connections having associated formal normal forms $\mathbf{A}'$. Repeating this statement for all $\mathbf{A}'$ having the same irregular type (at each a_i) as $\mathbf{A}$ yields Proposition 3.20 $\square$

4. Symplectic Structure and Moment Map

Moduli spaces of flat (nonsingular) connections over compact surfaces (possibly with boundary) have been intensively studied recently. In particular they have natural symplectic or Poisson structures which give deep geometrical insight (see the lecture notes [**11**] of M. Audin for a very readable overview).

In this section we observe that the well known Atiyah-Bott symplectic structure on nonsingular connections naturally generalises to the singular case we have been studying. Moreover, as in the nonsingular case we find that the curvature is a moment map for the action of the gauge group. Thus the moduli spaces of *flat* connections, which were identified with $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}_{\text{ext}}(\mathbf{A})$ in the previous section, arise as infinite dimensional symplectic quotients. We will concentrate on the (better behaved) extended case here and find, as in the previous chapter, that $\mathcal{M}(\mathbf{A})$ is a finite dimensional (symplectic) quotient of $\mathcal{M}_{\text{ext}}(\mathbf{A})$ by a torus.

The main technical difficulty here is that standard Sobolev/Banach space methods cannot be used since we want to fix infinite-jets of derivatives at the singular points $a_i \in \mathbb{P}^1$. Instead the infinite dimensional spaces here are naturally Fréchet manifolds. We will not use any deep properties of Fréchet spaces but do need a topology and differential structure (the explicitness of our situation means we can get by without using an implicit function theorem[1]). The reference used for Fréchet spaces is Treves [**101**] and for Fréchet manifolds or Lie groups see Hamilton [**43**] and Milnor [**83**]; we will give direct references to these works rather than full details here.

4.1. The Atiyah-Bott Symplectic Structure on $\mathcal{A}_{\text{ext}}(\mathbf{A})$.

Consider the complex vector space $\Omega^1[D](\mathbb{P}^1, \text{End}(E))$ of $n \times n$ matrices of global C^∞ singular one forms on $\mathbb{P}^1$ with poles on D (see p51). This is the space of sections of a C^∞ vector bundle and so can be given a Fréchet topology in a standard way ([**43**] p68).

Now define W to be the vector subspace

$$W := \left\{ \phi \in \Omega^1[D](\mathbb{P}^1, \text{End}(E)) \;\middle|\; L_i(\phi) \in \mathfrak{t}\frac{dz}{z - a_i} \text{ for } i = 1, \dots, m \right\}$$

of $\Omega^1[D](\mathbb{P}^1, \text{End}(E))$ of elements having Laurent expansion zero at each i, except for a possibly nonzero, diagonal residue term. This is a closed subspace[2] and so inherits a Fréchet topology. (Closed subspaces of Fréchet spaces are Fréchet.)

Lemma 3.21. *The extended space $\mathcal{A}_{\text{ext}}(\mathbf{A})$ of singular connections is a complex Fréchet manifold and if $\alpha \in \mathcal{A}_{\text{ext}}(\mathbf{A})$ then the tangent space to $\mathcal{A}_{\text{ext}}(\mathbf{A})$ at α is canonically isomorphic to the complex Fréchet space W defined above:*

$$T_\alpha \mathcal{A}_{\text{ext}}(\mathbf{A}) \cong W.$$

[1] We will give $\mathcal{M}_{\text{ext}}(\mathbf{A})$ the structure of a complex manifold directly using the monodromy description in Chapter 4 and will explicitly construct local slices for the $\mathcal{G}_1$ action.

[2] since the Laurent expansion maps L_i are *continuous* (if we put the topology of simple convergence of coefficients on the formal power series ring which is the image of the Laurent expansion map L_i); see [**101**] p390, where this continuity is used to give another proof of E.Borel's theorem on the surjectivity of the Taylor expansion map.

Proof. If there are no logarithmic singularities (all $k_i \geq 2$) then $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is an affine space modelled on W; if $\alpha_0 \in \mathcal{A}_{\text{ext}}(\mathbf{A})$ then

$$\mathcal{A}_{\text{ext}}(\mathbf{A}) = \{\alpha_0 + \phi \mid \phi \in W\}.$$

Thus by choosing a basepoint α_0, $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is identified with the Fréchet space W and the result follows. In general (some $k_i = 1$), $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is identified in this way with an open subset of W (recall the residues must be regular mod $\mathbb{Z}$): if $\alpha_0 \in \mathcal{A}_{\text{ext}}(\mathbf{A})$ then the map

$$\{\alpha_0 + \phi \mid \phi \in W\} \to \mathfrak{t}^m; \qquad \alpha \mapsto \big(\mathrm{Res}_i L_i(\alpha)\big)_{i=1}^m$$

taking the residues is continuous and $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is the inverse image of an open subset of $\mathfrak{t}^m$. Thus $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is identified with an open subset of W; it is thus a Fréchet manifold (with just one coordinate chart) and the tangent spaces are canonically identified with W as in the finite dimensional case (see discussion [**83**] p1030) □

Thus following Atiyah-Bott [**10**] p587 we can define a two form on $\mathcal{A}_{\text{ext}}(\mathbf{A})$ by the formula:

$$\omega_\alpha(\phi, \psi) = \int_{\mathbb{P}^1} \mathrm{Tr}(\phi \wedge \psi) \tag{50}$$

where $\alpha \in \mathcal{A}_{\text{ext}}(\mathbf{A})$ and $\phi, \psi \in T_\alpha \mathcal{A}_{\text{ext}}(\mathbf{A})$. This integral is well defined since the two form $\mathrm{Tr}(\phi \wedge \psi)$ on $\mathbb{P}^1$ is nonsingular; its Laurent expansion at a_i is a $(2,0)$ form and so zero. Then the division lemma implies $\mathrm{Tr}(\phi \wedge \psi)$ is nonsingular.

Thus ω_α is a skew symmetric complex bilinear form on the tangent space $T_\alpha \mathcal{A}_{\text{ext}}(\mathbf{A})$. It is nondegenerate in the sense that if $\omega_\alpha(\phi, \psi) = 0$ for all ψ then $\phi = 0$ (if $\phi \neq 0$ then ϕ is nonzero at some point $p \neq a_1, \ldots, a_m$ and it is easy then to construct ψ vanishing outside a neighbourhood of p and such that $\omega_\alpha(\phi, \psi) \neq 0$). Also ω_α is continuous as a map $W \times W \to \mathbb{C}$, since it is continuous in each factor, and (for Fréchet spaces) such 'separately continuous' bilinear maps are continuous ([**101**] p354).

Finally the right-hand side of (50) is independent of α, so ω is a constant two form on $\mathcal{A}_{\text{ext}}(\mathbf{A})$ and in particular it is closed.

Owing to these properties we will say ω is a complex *symplectic* form on $\mathcal{A}_{\text{ext}}(\mathbf{A})$. (See for example Kobayashi [**63**] for a discussion of the more well known theory of symplectic *Banach* manifolds.)

The next step is to see how this symplectic structure interacts with the group actions on $\mathcal{A}_{\text{ext}}(\mathbf{A})$. Before doing this we prove the following lemma which will be useful:

Lemma 3.22. *The curvature is an infinitely differentiable (even holomorphic) map*

$$\mathcal{F} : \mathcal{A}_{\text{ext}}(\mathbf{A}) \longrightarrow \Omega^2(\mathbb{P}^1, \mathrm{End}(E))$$

to the Fréchet space of $\mathrm{End}(E)$ *valued nonsingular two forms on* $\mathbb{P}^1$*. The derivative of* $\mathcal{F}$ *at* $\alpha \in \mathcal{A}_{\text{ext}}(\mathbf{A})$ *is*

$$(d\mathcal{F})_\alpha : T_\alpha \mathcal{A}_{\text{ext}}(\mathbf{A}) \to \Omega^2(\mathbb{P}^1, \mathrm{End}(E));$$
$$(d\mathcal{F})_\alpha(\phi) = -d_\alpha \phi = -d\phi + (\alpha \wedge \phi + \phi \wedge \alpha)$$

where $\phi \in T_\alpha \mathcal{A}_{\text{ext}}(\mathbf{A}) = W$ *and* $d_\alpha : \Omega^1[D](\mathbb{P}^1, \mathrm{End}(E)) \to \Omega^2[2D](\mathbb{P}^1, \mathrm{End}(E))$ *is the operator naturally induced from the singular connection* α.

Proof. Recall the curvature is given explicitly by

$$\mathcal{F}(\alpha) = -d\alpha + \alpha \wedge \alpha$$

and observe (by looking at Laurent expansions and using the division lemma) that this is a matrix of *nonsingular* two forms. That $\mathcal{F}$ is C^∞ with the stated derivative follows from basic facts about calculus on Fréchet spaces (see [**43**] Part I). It is holomorphic because its derivative has no anti-holomorphic part □

4.2. Group Actions. Firstly, the full gauge group

$$\mathcal{G} := GL_n(C^\infty(\mathbb{P}^1)) = C^\infty(\mathbb{P}^1, GL_n(\mathbb{C}))$$

is a Fréchet Lie group; that is, it is a Fréchet manifold such that the group operations $g, h \mapsto g \cdot h$ and $g \mapsto g^{-1}$ are C^∞ maps (see Milnor [**83**] Example 1.3). $\mathcal{G}$ is locally modelled on the Fréchet space

$$\mathrm{Lie}(\mathcal{G}) := C^\infty(\mathbb{P}^1, \mathfrak{gl}_n(\mathbb{C}))$$

of $n \times n$ matrices of smooth functions on $\mathbb{P}^1$. Moreover $\mathcal{G}$ has a complex analytic structure coming from the exponential map

$$\exp : \mathrm{Lie}(\mathcal{G}) \longrightarrow \mathcal{G}; \qquad x \longmapsto \exp(x)$$

which is defined pointwise in terms of the exponential map for $GL_n(\mathbb{C})$. This implies $\mathrm{Lie}(\mathcal{G})$ is a canonical coordinate chart for $\mathcal{G}$ in a neighbourhood of the identity since exp has a local inverse $g \mapsto \log(g)$ (also defined pointwise). In particular $\mathrm{Lie}(\mathcal{G})$ is so identified with the tangent space to $\mathcal{G}$ at the identity; the Lie algebra of $\mathcal{G}$.

The group we are really interested in here is $\mathcal{G}_1$, the subgroup of $\mathcal{G}$ consisting of elements $g \in \mathcal{G}$ having Taylor expansion 1 at each $a_i \in \mathbb{P}^1$. As above, the Taylor expansion maps are continuous and so $\mathcal{G}_1$ (the intersection of their kernels) is a closed subgroup of $\mathcal{G}$. It follows that $\mathcal{G}_1$ is a complex Fréchet Lie group with Lie algebra

$$\mathrm{Lie}(\mathcal{G}_1) := \{x \in \mathrm{Lie}(\mathcal{G}) \bigm| L_i(x) = 0 \text{ for } i = 1, \dots, m\}$$

where L_i is the Taylor expansion map at a_i. (The same statements also hold for $\mathcal{G}_T$ except now $\mathrm{Lie}(\mathcal{G}_T) := \{x \in \mathrm{Lie}(\mathcal{G}) \bigm| L_i(x) \in \mathfrak{t} \text{ for } i = 1, \dots, m\}$.)

Lemma 3.23. *The groups $\mathcal{G}_1$ and $\mathcal{G}_T$ act holomorphically on $\mathcal{A}_{\mathrm{ext}}(\mathbf{A})$ by gauge transformations and the fundamental vector field of $x \in \mathrm{Lie}(\mathcal{G}_T)$ takes the value*

$$-d_\alpha x \in T_\alpha \mathcal{A}_{\mathrm{ext}}(\mathbf{A})$$

at $\alpha \in \mathcal{A}_{\mathrm{ext}}(\mathbf{A})$, where $d_\alpha : C^\infty(\mathbb{P}^1, \mathrm{End}(E)) \to \Omega^1[D](\mathbb{P}^1, \mathrm{End}(E))$ is the singular connection naturally induced from α.

Proof. The action map

$$\mathcal{G}_T \times \mathcal{A}_{\mathrm{ext}}(\mathbf{A}) \longrightarrow \mathcal{A}_{\mathrm{ext}}(\mathbf{A})$$
$$(g, \alpha) \longmapsto g\alpha g^{-1} + (dg)g^{-1}$$

can be factored into simpler maps each of which is holomorphic (see [**43**]).

To calculate the fundamental vector field we use the exponential map for the group. Given $x \in \mathrm{Lie}(\mathcal{G}_T)$ we obtain a one parameter subgroup

$$\mathbb{C} \longrightarrow \mathcal{G}_T; \qquad t \longmapsto \exp(tx)$$

of $\mathcal{G}_T$ where $t \in \mathbb{C}$. Thus the flow through $\alpha \in \mathcal{A}_{\mathrm{ext}}(\mathbf{A})$ generated by x is given by the map

$$\mathbb{C} \longrightarrow \mathcal{A}_{\mathrm{ext}}(\mathbf{A}); \qquad t \longmapsto e^{tx}\alpha e^{-tx} + t dx$$

where d is the exterior derivative on $\mathbb{P}^1$. The derivative of this flow with respect to t at $t = 0$ is thus

$$x\alpha - \alpha x + dx = d_\alpha x.$$

The result follows by recalling our convention that the fundamental vector field is minus the tangent to the flow □

4.3. The Curvature is a Moment Map. It is clear that the action of $\mathcal{G}_T$ on $\mathcal{A}_{\text{ext}}(\mathbf{A})$ preserves the symplectic form ω: if $g \in \mathcal{G}_T$ and $\alpha \in \mathcal{A}_{\text{ext}}(\mathbf{A})$ then the derivative of the action of g is simply conjugation:

$$(g[\cdot])_* : T_\alpha \mathcal{A}_{\text{ext}}(\mathbf{A}) \to T_{g[\alpha]}\mathcal{A}_{\text{ext}}(\mathbf{A}); \quad \phi \mapsto g\phi g^{-1}.$$

(Recall $g[\alpha] = g\alpha g^{-1} + (dg)g^{-1}$.) Thus ω is preserved because

$$\operatorname{Tr}(\phi \wedge \psi) = \operatorname{Tr}(g\phi g^{-1} \wedge g\psi g^{-1})$$

for any $\phi, \psi \in T_\alpha \mathcal{A}_{\text{ext}}(\mathbf{A})$.

More interestingly, this action is Hamiltonian. If we firstly look at the smaller group $\mathcal{G}_1$, then, as observed by Atiyah and Bott in the nonsingular case, the curvature is a moment map (as we will now explain).

Recall from Lemma 3.22 that the curvature gives a holomorphic map

$$\mathcal{F} : \mathcal{A}_{\text{ext}}(\mathbf{A}) \longrightarrow \Omega^2(\mathbb{P}^1, \operatorname{End}(E))$$

from singular connections to matrices of two forms on $\mathbb{P}^1$. There is a natural inclusion from $\Omega^2(\mathbb{P}^1, \operatorname{End}(E))$ to the dual of the Lie algebra of $\mathcal{G}_1$ given by taking the trace and integrating:

$$i : \Omega^2(\mathbb{P}^1, \operatorname{End}(E)) \longrightarrow \operatorname{Lie}(\mathcal{G}_1)^*;$$
$$\mathcal{F}(\alpha) \longmapsto \left(x \mapsto \int_{\mathbb{P}^1} \operatorname{Tr}(\mathcal{F}(\alpha)x)\right)$$

where $x \in \operatorname{Lie}(\mathcal{G}_1)$ is a matrix of functions on $\mathbb{P}^1$. Using this inclusion we will regard $\mathcal{F}$ as a map into the dual of the Lie algebra of the group. We then have

Theorem 3.24. *The curvature*

$$\mathcal{F} : \mathcal{A}_{\text{ext}}(\mathbf{A}) \longrightarrow \operatorname{Lie}(\mathcal{G}_1)^*$$

is an equivariant moment map for the $\mathcal{G}_1$ action on the extended space $\mathcal{A}_{\text{ext}}(\mathbf{A})$ of singular connections determined by $\mathbf{A}$.

Proof. We just check that the arguments from the nonsingular case still work. Given $x \in \operatorname{Lie}(\mathcal{G}_1)$, define a (Hamiltonian) function H_x on $\mathcal{A}_{\text{ext}}(\mathbf{A})$ to be the x component of $\mathcal{F}$:

$$H_x := \langle \mathcal{F}, x\rangle : \mathcal{A}_{\text{ext}}(\mathbf{A}) \to \mathbb{C}; \qquad H_x(\alpha) = \int_{\mathbb{P}^1} \operatorname{Tr}(\mathcal{F}(\alpha)x)$$

where the angled brackets denote the natural pairing between $\operatorname{Lie}(\mathcal{G}_1)$ and its dual. We need to show that the fundamental vector field of x is the Hamiltonian vector field of H_x, or equivalently that

$$(dH_x)_\alpha = \omega_\alpha(\cdot, -d_\alpha x) \tag{51}$$

as elements of $T^*_\alpha\mathcal{A}_{\rm ext}(\mathbf{A})$, since (from Lemma 3.23) $-d_\alpha x$ is the fundamental vector field of x. Now if $\phi\in T_\alpha\mathcal{A}_{\rm ext}(\mathbf{A})$ then

$$(dH_x)_\alpha(\phi) = -\int_{\mathbb{P}^1}\mathrm{Tr}((d_\alpha\phi)x) \tag{52}$$

from Lemma 3.22 and the chain rule. Now observe that $\mathrm{Tr}(\phi x)$ is a *nonsingular* one form on $\mathbb{P}^1$ (as $L_i(x)=0$ for all i) and that

$$d\mathrm{Tr}(\phi x) = \mathrm{Tr}((d_\alpha\phi)x) - \mathrm{Tr}(\phi\wedge d_\alpha x).$$

Thus the left-hand side integrates to zero over $\mathbb{P}^1$ by Stokes' theorem. Hence (52) implies

$$(dH_x)_\alpha(\phi) = -\int_{\mathbb{P}^1}\mathrm{Tr}(\phi\wedge d_\alpha x) = \omega_\alpha(\phi,-d_\alpha x)$$

so that (51) holds and we see the curvature is indeed a moment map.

The equivariance follows directly from the definition of the coadjoint action of $\mathcal{G}_1$ and the fact that $\mathcal{F}(g[\alpha]) = g\mathcal{F}(\alpha)g^{-1}$: for any $x\in\mathrm{Lie}(\mathcal{G}_1)$ we have

$$\begin{aligned}\langle \mathrm{Ad}^*_g(\mathcal{F}(\alpha)),x\rangle &= \langle\mathcal{F}(\alpha),\mathrm{Ad}_{g^{-1}}(x)\rangle = \\ &\int_{\mathbb{P}^1}\mathrm{Tr}(\mathcal{F}(\alpha)g^{-1}xg) = \int_{\mathbb{P}^1}\mathrm{Tr}(\mathcal{F}(g[\alpha])x) = \langle\mathcal{F}(g[\alpha]),x\rangle\end{aligned}$$

and so $\mathrm{Ad}^*_g(\mathcal{F}(\alpha)) = \mathcal{F}(g[\alpha])$ □

Remark 3.25. Recall that the infinitesimal version of the equivariance condition says

$$\{H_x,H_y\} = H_{[x,y]}$$

for any $x,y\in\mathrm{Lie}(\mathcal{G}_1)$. This can be proved directly as follows. From the definition of the Poisson bracket and the formula for the Hamiltonian vector fields:

$$\{H_x,H_y\}(\alpha) = \int_{\mathbb{P}^1}\mathrm{Tr}(d_\alpha x\wedge d_\alpha y).$$

Since $\mathrm{Tr}(x d_\alpha y)$ is nonsingular we can use Stokes' theorem to equate this with

$$-\int_{\mathbb{P}^1}\mathrm{Tr}(x d_\alpha^2 y) = -\int_{\mathbb{P}^1}\mathrm{Tr}(x[\mathcal{F}(\alpha),y]) = \int_{\mathbb{P}^1}\mathrm{Tr}(\mathcal{F}(\alpha)[x,y]) = H_{[x,y]}(\alpha)$$

as required.

Thus the subset of flat connections is the preimage of zero under the moment map $\mathcal{F}$:

$$\mathcal{A}_{\rm ext,fl}(\mathbf{A}) = \mathcal{F}^{-1}(0).$$

Therefore, at least in a formal sense, the extended moduli space is a symplectic quotient:

$$\mathcal{M}_{\rm ext}(\mathbf{A}) \cong \mathcal{A}_{\rm ext,fl}(\mathbf{A})/\mathcal{G}_1 = \mathcal{F}^{-1}(0)/\mathcal{G}_1.$$

(The first isomorphism is from Proposition 3.20.)

Is $\mathcal{M}_{\rm ext}(\mathbf{A})$ symplectic? In the next chapter we will see that $\mathcal{M}_{\rm ext}(\mathbf{A})$ naturally has the structure of a finite dimensional complex manifold. Moreover the subset $\mathcal{M}^*_{\rm ext}(\mathbf{A})$ of compatibly framed connections on trivial holomorphic bundles is a (dense) open submanifold. Then (in Proposition 5.3) we will construct slices for the $\mathcal{G}_1$ action over $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$. That is, if we restrict $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ to $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$:

$$\begin{array}{ccc} \mathcal{A}_{\text{ext,fl}}(\mathbf{A})|_{\mathcal{M}^*_{\text{ext}}(\mathbf{A})} & \hookrightarrow & \mathcal{A}_{\text{ext,fl}}(\mathbf{A}) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathcal{M}^*_{\text{ext}}(\mathbf{A}) & \hookrightarrow & \mathcal{M}_{\text{ext}}(\mathbf{A}) \end{array} \tag{53}$$

(where π is the natural projection passing to the $\mathcal{G}_1$ orbits), then for each $u \in \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ there is an open neighbourhood U of u and a holomorphic map

$$s : U \longrightarrow \mathcal{A}_{\text{ext}}(\mathbf{A})$$

such that
1) The image of s is in the subset $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})|_U$ of $\mathcal{A}_{\text{ext}}(\mathbf{A})$, and
2) s is a section of π in that $\pi(s(u)) = u$ for all $u \in U$.

Thus we can pull back the symplectic form ω on $\mathcal{A}_{\text{ext}}(\mathbf{A})$ along s to give a closed two form Ω_U on $U \subset \mathcal{M}^*_{\text{ext}}(\mathbf{A})$:

$$\Omega_U := s^*(\omega).$$

One of the main results of this thesis is then:

Theorem 3.26. *Each such two form Ω_U is nondegenerate and as U varies they fit together so that the symplectic form ω on $\mathcal{A}_{\text{ext}}(\mathbf{A})$ induces a symplectic form Ω on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$.*

Moreover, this symplectic form Ω is the same *as the symplectic form given explicitly on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ in Chapter 2 in terms of cotangent bundles and coadjoint orbits.*

This will be proved in the subsequent chapters, culminating in Theorem 5.8.

REMARK 3.27. We have not addressed the extent to which Ω extends to all of the manifold $\mathcal{M}_{\text{ext}}(\mathbf{A})$ (since this extension is not needed here). It seems straightforward (using the monodromy picture) to construct local slices around any $u \in \mathcal{M}_{\text{ext}}(\mathbf{A})$ and thereby obtain a closed (holomorphic) two form Ω' on $\mathcal{M}_{\text{ext}}(\mathbf{A})$ which restricts to Ω on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ (such extension is clearly unique). It may then be possible to adapt the usual nondegeneracy proof (for Banach symplectic quotients, see [**63**] Lemma 5.9) to prove that Ω' is nondegenerate. Alternatively there are two finite dimensional approaches that may do the trick[3].

4.4. Torus Actions. To end this chapter we consider the action of the larger group $\mathcal{G}_T$ also on the extended space of singular connections $\mathcal{A}_{\text{ext}}(\mathbf{A})$. This action is also Hamiltonian:

Proposition 3.28. *Let μ be the map*

$$\mu : \mathcal{A}_{\text{ext}}(\mathbf{A}) \longrightarrow \text{Lie}(\mathcal{G}_T)^*$$

given by taking the curvature together with the residue at each a_i: if $x \in \text{Lie}(\mathcal{G}_t)$

$$\langle \mu(\alpha), x\rangle := \int_{\mathbb{P}^1} \text{Tr}\,(\mathcal{F}(\alpha)x) - (2\pi\sqrt{-1})\sum_{i=1}^m \text{Res}_i L_i(\text{Tr}(\alpha x))$$

[3] 1) Modify the work in the meromorphic Higgs bundle case (see [**28**] Section 5.4) to the case of meromorphic connections to construct $\mathcal{M}_{(\text{ext})}(\mathbf{A})$ algebraically. This should just involve changing the map in the deformation complex and the symplectic structure should still arise from Grothendieck duality, as in the Higgs case. 2) See Appendix F.

for each $\alpha \in \mathcal{A}_{\mathrm{ext}}(\mathbf{A})$.

Then μ *is an equivariant moment map for the* $\mathcal{G}_T$ *action on* $\mathcal{A}_{\mathrm{ext}}(\mathbf{A})$.

Proof. This is similar to Theorem 3.24. For any $x \in \mathrm{Lie}(\mathcal{G}_T)$ define the function $H_x : \mathcal{A}_{\mathrm{ext}}(\mathbf{A}) \to \mathbb{C}$ to be the x component of μ:

$$H_x(\alpha) := \langle \mu(\alpha), x\rangle.$$

As before for any $\phi \in T_\alpha\mathcal{A}_{\mathrm{ext}}(\mathbf{A})$ we have

$$\omega_\alpha(\phi, -d_\alpha x) = -\int_{\mathbb{P}^1} \mathrm{Tr}(\phi \wedge d_\alpha x) \tag{54}$$

and we now find

$$(dH_x)_\alpha(\phi) = -\int_{\mathbb{P}^1} \mathrm{Tr}((d_\alpha\phi)x) - (2\pi\sqrt{-1})\sum_i \mathrm{Res}_i L_i(\mathrm{Tr}(\phi x)) \tag{55}$$

and our task is to show (54) and (55) are equal. We do this by using the C^∞ Cauchy integral theorem (see Lemma 5.9).

Recall ϕ is a matrix of C^∞ one forms on $\mathbb{P}^1$ with (at worst) first order poles in its $(1,0)$ part at each a_i. Also $x \in \mathrm{Lie}(\mathcal{G}_T)$ is a matrix of functions on $\mathbb{P}^1$ and has Taylor expansion equal to a constant diagonal matrix at each a_i. Thus we can choose disjoint open discs D_i around the a_i's and C^∞ functions $f_i : \mathbb{P}^1 \to \mathbb{C}$ vanishing outside the corresponding D_i, such that

$$\mathrm{Tr}(\phi x) = \theta + f_1\frac{dx}{z-a_1} + \cdots f_m\frac{dx}{z-a_m}$$

for some *nonsingular* one form θ on $\mathbb{P}^1$. Thus on one hand we have

$$d\mathrm{Tr}(\phi x) = d\theta - \sum_i \frac{\partial f_i}{\partial \bar z}\frac{dz\wedge d\bar z}{z-a_i}$$

and so by Stokes' theorem and Cauchy's integral theorem:

$$\int_{\mathbb{P}^1} d\mathrm{Tr}(\phi x) = 0 - \sum_i \frac{\partial f_i}{\partial \bar z}\frac{dz\wedge d\bar z}{z-a_i}$$

$$= -(2\pi\sqrt{-1})\sum_i f_i(a_i) = (2\pi\sqrt{-1})\sum_i \mathrm{Res}_i L_i(\mathrm{Tr}(\phi x)).$$

On the other hand

$$d\mathrm{Tr}(\phi x) = \mathrm{Tr}(d_\alpha(\phi x)) = \mathrm{Tr}((d_\alpha\phi)x) - \mathrm{Tr}(\phi\wedge d_\alpha x)$$

and so the equality of (54) and (55) follows.

The equivariance follows exactly as in Theorem 3.24 since the quotient $\mathcal{G}_T/\mathcal{G}_1 \cong T^m$ is Abelian

□

Thus, at least formally, the moduli space $\mathcal{M}(\mathbf{A})$ of meromorphic connections on degree zero bundles with fixed formal types arises as a symplectic quotient of $\mathcal{A}_{\mathrm{ext}}(\mathbf{A})$ by $\mathcal{G}_T$. Alternatively we can do this reduction in two stages. Firstly the symplectic quotient by $\mathcal{G}_1$ gives the finite dimensional manifold $\mathcal{M}_{\mathrm{ext}}(\mathbf{A})$. The residual action of $\mathcal{G}_T/\mathcal{G}_1 \cong T^m$ on $\mathcal{M}_{\mathrm{ext}}(\mathbf{A})$ should then be Hamiltonian with moment map given by taking the residues:

$$[\alpha] \longmapsto (-\mathrm{Res}_i L_i(\alpha))_{i=1}^m \in \mathfrak{t}^m \cong (\mathfrak{t}^*)^m.$$

Then

$$\mathcal{M}_{\text{ext}}(\mathbf{A}) \underset{-\mathbf{\Lambda}}{/\!/} T^m \cong \mathcal{M}(\mathbf{A}) \tag{56}$$

where $\mathbf{\Lambda}$ is the m-tuple of residues of the formal normal forms $\mathbf{A}$. Since we haven't proved we have a symplectic structure on all $\mathcal{M}_{\text{ext}}(\mathbf{A})$ this is just a formal picture. However, on restriction to the dense open subset $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$, this T^m action coincides with that studied in Section 4.4 of Chapter 2 and the analogue of (56) was proved in Corollary 2.48.

CHAPTER 4

Monodromy

Now we come to the third (and final!) approach to meromorphic connections that we will use: monodromy.

To start with, in Section 1, we explain what is meant by the monodromy data of a meromorphic connection: as one might expect from the nonsingular case this data involves a representation of the fundamental group, however, to encode the local moduli at the singularities one also stores the Stokes matrices and the (exponents of) formal monodromy. Thus in the first section we define (extended) monodromy manifolds to store the monodromy data, emphasising that they may be described as 'multiplicative' versions of the (extended) polar parts manifolds (recall from Chapter 2 that the polar parts manifolds were the concrete realisations of the moduli spaces $\mathcal{M}^*_{(\mathrm{ext})}(\mathbf{A})$).

The procedure of taking the monodromy data then gives a holomorphic map, the *monodromy map*, from the polar parts manifolds to the monodromy manifolds. Much of this thesis arose through trying to understand this map: the point is that the monodromy map 'solves' the isomonodromic deformation equations, as we will see in Chapter 6. Apart from the presentation there is nothing in this first section that isn't in the work [**60**] of Jimbo, Miwa and Ueno.

In the second section we define the monodromy data of a flat C^∞ singular connection in $\mathcal{A}_{\mathrm{ext}}(\mathbf{A})$. This enables us to prove there is a one to one correspondence between the set of monodromy data and the set of gauge orbits of flat C^∞ singular connections (Theorem 4.10). This generalises the well-known correspondence between flat C^∞ *nonsingular* connections and representations of the fundamental group.

In turn we deduce (Corollary 4.11) that the monodromy map gives a bijection from meromorphic connections on degree zero bundles onto the monodromy data. In particular this enables us to define a complex manifold structure on the set $\mathcal{M}_{\mathrm{ext}}(\mathbf{A})$ of isomorphism classes of compatibly framed meromorphic connections on degree zero bundles with fixed irregular types.

1. Monodromy Manifolds and Monodromy Maps

The basic idea is this: a compatibly framed meromorphic connection gives rise to canonical fundamental solutions on each sector at each singularity a_i. All of these fundamental solutions extend to (multivalued) fundamental solutions on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$. The (generalised) monodromy data is simply the collection of all the constant $n \times n$ matrices that occur as the 'ratios' of any of these fundamental solutions, together with the exponents of formal monodromy. To make all this more precise and to remove some of the redundancy here we will fix some auxiliary data (which will remain fixed throughout this chapter).

As in previous chapters, choose an effective divisor $D = k_1(a_1) + \cdots + k_m(a_m)$ on $\mathbb{P}^1$, a local coordinate z_i vanishing at a_i and nice formal normal forms $\mathbf{A}$. For $i = 1, \dots, m$ choose disjoint closed discs $\bar{D}_i \subset \mathbb{P}^1$ centred at a_i with interior D_i. Label the anti-Stokes rays at a_i as ${}^i\!d_1, \dots, {}^i\!d_{r_i}$ and fix the radius of the open sector ${}^i\mathrm{Sect}_j = \mathrm{Sect}({}^i\!d_j, {}^i\!d_{j+1})$ to be equal to that of D_i. Thus D_i is a disjoint union of the point $\{a_i\}$, the rays ${}^i\!d_1, \dots, {}^i\!d_{r_i}$ and the sectors ${}^i\mathrm{Sect}_1, \dots, {}^i\mathrm{Sect}_{r_i}$. Choose a branch of $\log(z_i)$ on ${}^i\!d_1$ and extend it in a positive sense to all of $D_i \setminus a_i$ as usual. Pick a base-point p_i in the *last* sector ${}^i\mathrm{Sect}_{r_i}$ at a_i for each i and choose disjoint paths $\gamma_i : [0,1] \to \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ joining p_1 to p_i for $i = 2, \dots, m$ (and not intersecting $\bar{D}_j$ for $j \neq 1, i$). Write $[\gamma_i]$ for the track $\gamma_i([0,1])$ of the ith path. Let l_i be a simple closed loop in D_i based at p_i and going once around a_i in a positive sense $(i = 1, \dots, m)$. Without loss of generality we will assume the paths γ_i have been chosen such that the loop:

$$(\gamma_m^{-1} \cdot l_m \cdot \gamma_m) \cdots (\gamma_3^{-1} \cdot l_3 \cdot \gamma_3) \cdot (\gamma_2^{-1} \cdot l_2 \cdot \gamma_2) \cdot l_1 \tag{57}$$

based at p_1 is contractible in $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$. Let $\Gamma_i : [0,1] \times [0,1] \to \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ be a 'thickening' of the path γ_i (i.e. a ribbon following γ_i so that its track $[\Gamma_i]$ is a tubular neighbourhood of $[\gamma_i]$). Take the width of Γ_i small enough to ensure that if $i \neq j$ then $[\Gamma_i]$ and $[\Gamma_j]$ only intersect in the last sector at a_1. Finally define the set of *tentacles*:

$$\mathcal{T} := \bar{D}_1 \cup \bigcup_{i=2}^{m} (\bar{D}_i \cup [\Gamma_i]) \subset \mathbb{P}^1$$

made up of the central body $\bar{D}_1$ with legs $[\Gamma_i]$ and feet $\bar{D}_i$ $(i = 2, \dots, m)$. See Figure 2. Both $\mathcal{T}$ and its complement in $\mathbb{P}^1$ are homeomorphic to discs.

Now, given a compatibly framed meromorphic connection $(V, \nabla, \mathbf{g})$ representing an element of $\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$, use the compatible framing at a_1 to give a global holomorphic trivialisation of V, so that the germ of $(V, \nabla, \mathbf{g})$ at a_i is identified with a compatibly framed system at a_i for each i. Then using the choices of logarithm made above, ∇ has a canonical fundamental solution ${}^i\Phi_j$ on the jth sector at a_i (see Definition 1.29) for $j = 1, \dots, r_i$ and $i = 1, \dots, m$. Note that a priori ${}^i\Phi_j$ is only defined near a_i on ${}^i\mathrm{Sect}_j$ but it may be extended uniquely, as a fundamental solution of ∇, to all of ${}^i\mathrm{Sect}_j$ (which is simply connected). We will write ${}^i\Phi_0$ rather than ${}^i\Phi_{r_i}$ for the solution on the last sector at a_i. If ${}^1\Phi_0$ and ${}^i\Phi_0$ are further extended along the ribbon $[\Gamma_i]$ then

$${}^1\Phi_0 = {}^i\Phi_0 C_i \tag{58}$$

on $[\Gamma_i]$ for some constant matrix C_i; the ith *connection matrix* for ∇ $(i = 1, 2 \dots, m)$, where we have set $C_1 = 1$. There are also the Stokes matrices

$${}^i\mathbf{S} := ({}^iS_1, \dots, {}^iS_{2k_i-2})$$

of ∇ at each a_i giving the transitions between certain fundamental solutions at a_i (see Lemma 1.38).

Definition 4.1. The *generalised monodromy data* of $(V, \nabla, \mathbf{g})$ is the m-tuple of connection matrices $\mathbf{C} = (C_1, C_2, \dots, C_m)$ together with the Stokes matrices ${}^i\mathbf{S}$ for $i = 1, \dots, m$ and the exponents of formal monodromy (see Definition 1.14):

$$\mathbf{\Lambda} = ({}^1\Lambda, \dots, {}^m\Lambda).$$

One obvious restriction on this data is given by the fact that the monodromy of ∇ around the contractible loop (57) is the identity. If the fundamental solution ${}^1\Phi_0$ is

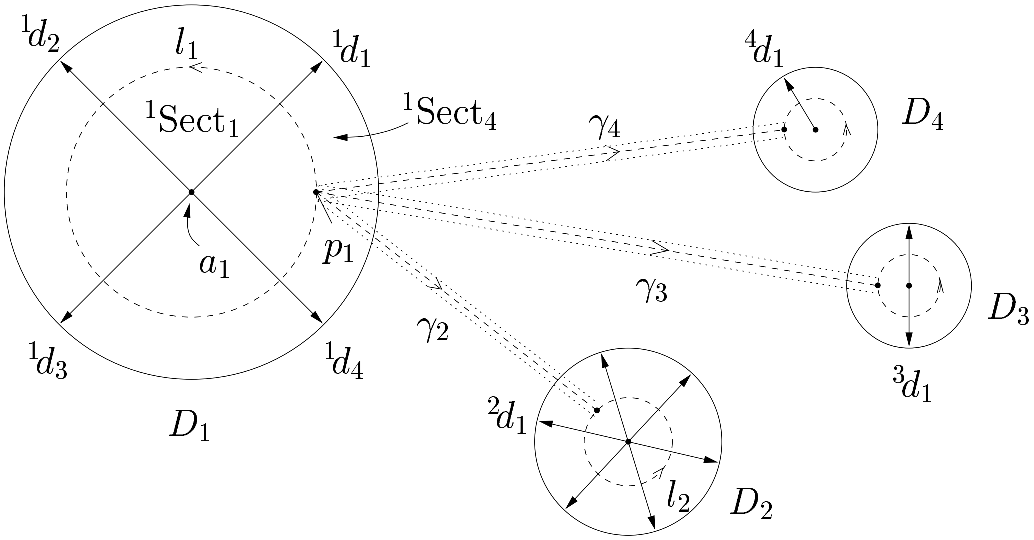

FIGURE 2. Tentacles

continued around this loop, then, from Lemma 1.39 and the definition of the connection matrices, the following relation is obtained:

$$\rho_m \cdot \rho_{m-1} \cdots \rho_3 \cdot \rho_2 \cdot \rho_1 = 1 \tag{59}$$

where ρ_i is the following $n \times n$ invertible matrix

$$\rho_i := C_i^{-1} \cdot {}^iS_{2k_i-2} \cdots {}^iS_2 \cdot {}^iS_1 \cdot \exp\big((2\pi\sqrt{-1})\,{}^i\Lambda\big) \cdot C_i$$

for $i = 1, \dots, m$. Another (less important) relation is that since V is degree zero, the sum of the traces of the exponents of formal monodromy is zero (see Remark 2.34):

$$\mathrm{Tr}({}^1\Lambda) + \cdots + \mathrm{Tr}({}^m\Lambda) = 0. \tag{60}$$

To house all this monodromy data we will use the following 'multiplicative version' of the extended orbits.

DEFINITION 4.2. Let $\widetilde{\mathcal{C}}_i$ be the product of $GL_n(\mathbb{C})$, the set $\mathcal{H}({}^iA^0)$ and the set $\mathfrak{t}$ of diagonal matrices:

$$\widetilde{\mathcal{C}}_i := GL_n(\mathbb{C}) \times \mathcal{H}({}^iA^0) \times \mathfrak{t}.$$

(If $k_i = 1$ we replace $\mathfrak{t}$ by the 'nice' elements $\mathfrak{t}'$ with distinct eigenvalues mod $\mathbb{Z}$. Also $\mathcal{H}({}^iA^0)$ is a point in this case so that $\widetilde{\mathcal{C}}_i = GL_n(\mathbb{C}) \times \mathfrak{t}'$.)

Here we think of $\mathcal{H}({}^iA^0)$ in terms of Stokes matrices (see p13), so write a point of $\mathcal{H}({}^iA^0)$ as ${}^i\mathbf{S} = ({}^iS_1, \dots, {}^iS_{2k_i-2})$. Observe immediately that $\dim \widetilde{\mathcal{C}}_i = k_i n(n-1) + 2n = \dim \widetilde{O}_i$; we will gradually see that much of the structure of $\widetilde{O}_i$ is shared by $\widetilde{\mathcal{C}}_i$.

Thus $(V, \nabla, \mathbf{g})$ determines a point:

$$(C_i, {}^i\mathbf{S}, {}^i\Lambda) \in \widetilde{\mathcal{C}}_i$$

for each i, and therefore a point of the product $\widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m$. The elements ρ_i above can be thought of as maps from $\widetilde{\mathcal{C}}_i$ to $GL_n(\mathbb{C})$:

$$\rho_i : \widetilde{\mathcal{C}}_i \longrightarrow GL_n(\mathbb{C});$$

$$(C_i, {}^i\mathbf{S}, {}^i\Lambda) \longmapsto C_i^{-1} \cdot {}^iS_{2k_i-2} \cdots {}^iS_2 \cdot {}^iS_1 \cdot \exp\big((2\pi\sqrt{-1}){}^i\Lambda\big) \cdot C_i.$$

These extend trivially to the product and we can then multiply them together to get the map

$$\boldsymbol{\rho} : \widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m \longrightarrow GL_n(\mathbb{C}); \qquad \boldsymbol{\rho} = \rho_m \cdots \rho_1$$

which appears in the relation (59).

Definition 4.3. The *extended monodromy manifold* is the set

$$M_{\text{ext}}(\mathbf{A}) := \big\{(\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}) \bigm| \boldsymbol{\rho}(\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}) = 1,\ \mathrm{Tr}({}^1\Lambda) + \cdots + \mathrm{Tr}({}^m\Lambda) = 0\big\}/GL_n(\mathbb{C}) \tag{61}$$

where $(\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}) \in \widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m$ is an m-tuple consisting of elements $(C_i, {}^i\mathbf{S}, {}^i\Lambda) \in \widetilde{\mathcal{C}}_i$ for each i. Also $GL_n(\mathbb{C})$ acts freely on $\widetilde{\mathcal{C}}_i$ via

$$g(C_i, {}^i\mathbf{S}, {}^i\Lambda) = (C_i g^{-1}, {}^i\mathbf{S}, {}^i\Lambda)$$

(where $g \in GL_n(\mathbb{C})$) and the action of $GL_n(\mathbb{C})$ on $\widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m$ in (61) is the diagonal combination of these actions. (Clearly in each such $GL_n(\mathbb{C})$ orbit there is a unique element with $C_1 = 1$.)

Lemma 4.4. *$M_{\text{ext}}(\mathbf{A})$ is a complex manifold and has the same dimension as the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$.*

Proof. Let $\widetilde{\mathcal{C}}_1'$ denote the subset of $\widetilde{\mathcal{C}}_1$ having $C_1 = 1$. Thus $M_{\text{ext}}(\mathbf{A})$ is identified with the subset of the product

$$\widetilde{\mathcal{C}}_1' \times \widetilde{\mathcal{C}}_2 \times \cdots \times \widetilde{\mathcal{C}}_m \tag{62}$$

such that $\boldsymbol{\rho}(\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}) = 1$ and $\sum \mathrm{Tr}({}^i\Lambda) = 0$. In fact we can forget the second of these equations because if $\boldsymbol{\rho}(\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}) = 1$ then, by taking the determinant we see that $\sum \mathrm{Tr}({}^i\Lambda)$ is an integer and so the subset of (62) satisfying just $\boldsymbol{\rho}(\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}) = 1$ breaks up into $\mathbb{Z}$ copies of $M_{\text{ext}}(\mathbf{A})$.

Now the result follows from the implicit function theorem since the map

$$\boldsymbol{\rho} : \widetilde{\mathcal{C}}_1' \times \widetilde{\mathcal{C}}_2 \times \cdots \times \widetilde{\mathcal{C}}_m \to GL_n(\mathbb{C})$$

is surjective on tangent vectors (submersive). (Here we exclude the case with just one simple pole ($m = 1, k_1 = 1$) since $M_{\text{ext}}(\mathbf{A})$ is then just a point.) If $m > 1$ this can be deduced from the fact that

$$\rho_m : \widetilde{\mathcal{C}}_m \to GL_n(\mathbb{C})$$

is surjective on tangent vectors, or if $m = 1, k_1 \geq 2$ from the fact that

$$\rho_1|_{\widetilde{\mathcal{C}}_1'} : \widetilde{\mathcal{C}}_1' \to GL_n(\mathbb{C})$$

is surjective on tangent vectors. (This can be deduced from the fact that the map $U_- \times U_+ \times T \to GL_n(\mathbb{C});\ (l, u, t) \mapsto lut$ is a local isomorphism.)

It follows that the dimension of $M_{\text{ext}}(\mathbf{A})$ is the sum of the dimensions of the $\widetilde{\mathcal{C}}_i$'s minus $2n^2$, and this is the same as that for $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ (see Theorem 2.43) □

DEFINITION 4.5. The *monodromy map*[1] is the map from the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ to the extended monodromy manifold $M_{\text{ext}}(\mathbf{A})$ obtained by taking all the monodromy data as above:

$$\nu : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \longrightarrow M_{\text{ext}}(\mathbf{A})$$
$$[(V, \nabla, \mathbf{g})] \longmapsto (\mathbf{C}, \mathbf{S}, \mathbf{\Lambda}).$$

Note that the monodromy map is *holomorphic* because the canonical fundamental solutions vary holomorphically with parameters (see remarks p15). Also recall that the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ was given the structure of a complex symplectic manifold by identifying it with an extended polar part manifold

$$\widetilde{O}_1 \times \cdots \times \widetilde{O}_m // GL_n(\mathbb{C})$$

in Theorem 2.43. Thus we have a holomorphic map between two *explicit* complex manifolds of the same dimension, yet it is notoriously difficult to calculate ν explicitly (the point is that a formula for ν, or rather its inverse ν^{-1}, will give explicit solutions to Painlevé equations and we know, in general, that this will involve new transcendental functions).

Nonetheless, we can examine the general structure of the monodromy map (and this is one of the main themes in this thesis). In the way it is set up here, it is known that the monodromy map is a biholomorphism onto its image in the monodromy manifold (see Corollary 4.13), and that the complement of this image is a divisor. Observe (if the reader hasn't already) that the description of $M_{\text{ext}}(\mathbf{A})$ is a 'multiplicative' version of the description of $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ as the symplectic quotient $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m // GL_n(\mathbb{C})$: the extended orbits $\widetilde{O}_i$ are replaced by the $\widetilde{\mathcal{C}}_i$'s and the additive moment map (the sum of the residues) is replaced by the product of the ρ_i's. This suggests we should think of ν as some kind of generalisation of the exponential function. However it is not true in general that ν comes from separate maps $\widetilde{O}_i \to \widetilde{\mathcal{C}}_i$; the additive identity $\sum \text{Res}_i = 0$ needs to be converted into the multiplicative identity $\rho_m \cdots \rho_1 = 1$, which is not at all easy. As an example the case with only two poles, of orders one and two respectively, will be studied in detail in Chapter 7. Until then we will continue to study the general case.

1.1. Torus Actions. As for the extended orbits, the torus $T \cong (\mathbb{C}^*)^n$ acts on $\widetilde{\mathcal{C}}_i$:

$$t(C_i, {}^i\mathbf{S}, {}^i\Lambda) = (t \cdot C_i,\ t \cdot {}^i\mathbf{S} \cdot t^{-1},\ {}^i\Lambda)$$

where $(C_i, {}^i\mathbf{S}, {}^i\Lambda) \in \widetilde{\mathcal{C}}_i$ and $t \in T$. These actions induce an action of T^m on the extended monodromy manifold $M_{\text{ext}}(\mathbf{A})$ since the relation $\rho_m \cdots \rho_1 = 1$ is preserved. It follows from Lemma 1.32 and Lemma 1.39 that the monodromy map ν intertwines this T^m action and the T^m action on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ described on p45.

The multiplicative version of the G_{k_i}-coadjoint orbit O_i is as follows

DEFINITION 4.6. Let $\mathcal{C}_i$ be the quotient of $GL_n(\mathbb{C}) \times \mathcal{H}({}^iA^0)$ by the free T action:

$$\mathcal{C}_i := \big(GL_n(\mathbb{C}) \times \mathcal{H}({}^iA^0)\big)/T$$

where the action is given by

$$t(C_i, {}^i\mathbf{S}) = (t \cdot C_i,\ t \cdot {}^i\mathbf{S} \cdot t^{-1})$$

for $t \in T$ and $(C_i, {}^i\mathbf{S}) \in GL_n(\mathbb{C}) \times \mathcal{H}({}^iA^0)$.

[1] Names like 'de Rham morphism' or 'Riemann-Hilbert map' could be used instead.

It is immediate that $\dim(\mathcal{C}_i) = \dim O_i$. Moreover if $k_i = 1$ then we can identify $\mathcal{C}_i$ with the conjugacy class in $GL_n(\mathbb{C})$ through the formal monodromy ${}^iM_0 := \exp\big((2\pi\sqrt{-1})^i\Lambda\big)$:

$$\mathcal{C}_i \ni [C_i] \longmapsto C_i^{-1} \cdot {}^iM_0 \cdot C_i.$$

Also there are well-defined maps from each $\mathcal{C}_i$ to $GL_n(\mathbb{C})$:

$$\rho_i : \mathcal{C}_i \longrightarrow GL_n(\mathbb{C}); \qquad [(C_i, {}^i\mathbf{S})] \longmapsto C_i^{-1} \cdot {}^iS_{2k_i-2} \cdots {}^iS_2 \cdot {}^iS_1 \cdot \exp\big((2\pi\sqrt{-1})^i\Lambda\big) \cdot C_i.$$

We can then fit these $\mathcal{C}_i$'s together to define the multiplicative analogue of the polar part manifold $O_1 \times \cdots \times O_m // GL_n(\mathbb{C})$:

Definition 4.7. The (non-extended) monodromy manifold is the set

$$M(\mathbf{A}) := \big\{[(\mathbf{C}, \mathbf{S})] \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \;\big|\; \boldsymbol{\rho}[(\mathbf{C}, \mathbf{S})] = 1\big\} / GL_n(\mathbb{C})$$

where $\boldsymbol{\rho}$ is the product of the ρ_i's:

$$\boldsymbol{\rho} := \rho_m \cdots \rho_1 : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to GL_n(\mathbb{C}).$$

As for the extended moduli spaces, we can consider the map:

$$\boldsymbol{\mu} : M_{\text{ext}}(\mathbf{A}) \longrightarrow (\mathfrak{t}^*)^m; \qquad (\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}) \longmapsto -\boldsymbol{\Lambda}$$

on the extended monodromy manifold taking all of the exponents of formal monodromy and find that:

$$M(\mathbf{A}) = \boldsymbol{\mu}^{-1}(-\boldsymbol{\Lambda}')/T^m$$

where $\boldsymbol{\Lambda}'$ is the m-tuple of residues of the formal normal forms $\mathbf{A}$. Since the monodromy map intertwines the T^m actions, we deduce that there is an induced monodromy map

$$\nu : \mathcal{M}^*(\mathbf{A}) \longrightarrow M(\mathbf{A}).$$

However (following [**60**]) we will work mainly with the extended version since the quotients $\mathcal{M}^*(\mathbf{A}), M(\mathbf{A})$ may not be manifolds.

Remark 4.8. If all of the poles are simple (all $k_i = 1$) then it is easy to identify the monodromy manifold $M(\mathbf{A})$ with the moduli space of representations of the fundamental group of the punctured sphere, having local monodromy around a_i in the conjugacy class $\mathcal{C}_i$:

$$M(\mathbf{A}) \cong \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}\big(\pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}), GL_n(\mathbb{C})\big)/GL_n(\mathbb{C}) \tag{63}$$

where $\boldsymbol{\mathcal{C}}$ denotes the m-tuple of $GL_n(\mathbb{C})$ conjugacy classes $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$.

Remark 4.9. By using a suitable sub-groupoid of the fundamental groupoid of the punctured sphere (for example with one basepoint in each sector at each pole) it is easy to realise *all* of the monodromy manifolds $M(\mathbf{A})$ as spaces of groupoid representations, generalising (63), and we hope to return to this approach at a later date.

Note that the wild fundamental group of Martinet and Ramis [**78**] involves a natural way to encode all of the restrictions on such representations that arise from the requirement that certain groupoid generators end up in certain groups of Stokes factors.

2. Monodromy of Flat Singular Connections

If instead of a meromorphic connection, we are given a flat C^∞ singular connection $\alpha \in \mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ (see p59) then we still have canonical fundamental solutions of α in each sector at each singularity (from Lemma 3.12). Thus, exactly as in the previous section we can define the monodromy data of α. This gives a map (which will be referred to as the C^∞ monodromy map):

$$\nu : \mathcal{A}_{\text{ext,fl}}(\mathbf{A}) \longrightarrow M_{\text{ext}}(\mathbf{A}); \qquad \alpha \longmapsto (\mathbf{C}, \mathbf{S}, \mathbf{\Lambda})$$

where $\mathbf{\Lambda}$ is the m-tuple of residues of α, $\mathbf{S}$ is all of the Stokes matrices of α (see Definition 3.11 and forwards), and $\mathbf{C}$ is the m-tuple of connection matrices for α defined as above by extending the canonical fundamental solutions ${}^1\Phi_0$ and ${}^i\Phi_0$ of α along Γ_i and setting $C_i := ({}^i\Phi_0)^{-1} \cdot {}^1\Phi_0$.

Recall that the group $\mathcal{G}_T$ (of C^∞ bundle automorphisms having Taylor expansion in T at each a_i) acts on $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ by gauge transformations. $\mathcal{G}_T$ also acts on $M_{\text{ext}}(\mathbf{A})$ via the surjective group homomorphism $\mathcal{G}_T \to T^m$ (given by evaluating at each a_i), and via the action of T^m on $M_{\text{ext}}(\mathbf{A})$ described explicitly above.

The main result relating the monodromy data to flat connections is then:

Theorem 4.10. *The C^∞ monodromy map*

$$\nu : \mathcal{A}_{\text{ext,fl}}(\mathbf{A}) \longrightarrow M_{\text{ext}}(\mathbf{A})$$

is surjective, has precisely the $\mathcal{G}_1$ orbits in $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ as fibres and intertwines the $\mathcal{G}_T$ actions on $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ and $M_{\text{ext}}(\mathbf{A})$.

Before proving this we deduce what the monodromy data corresponds to in the meromorphic world:

Corollary 4.11. *The monodromy map induces bijections:*

$$\mathcal{M}_{\text{ext}}(\mathbf{A}) \cong M_{\text{ext}}(\mathbf{A}) \quad \text{and} \quad \mathcal{M}(\mathbf{A}) \cong M(\mathbf{A})$$

between the spaces of meromorphic connections on degree zero bundles and the corresponding spaces of monodromy data. In particular $\mathcal{M}_{\text{ext}}(\mathbf{A})$ inherits the structure of a complex manifold from $M_{\text{ext}}(\mathbf{A})$.

Proof. The first bijection follows directly from Theorem 4.10 since we proved that $\mathcal{M}_{\text{ext}}(\mathbf{A}) \cong \mathcal{A}_{\text{ext,fl}}(\mathbf{A})/\mathcal{G}_1$ in Proposition 3.20. The second bijection follows from the first by fixing the exponents of formal monodromy and quotienting by the T^m action (using the intertwining property of ν) □

Remark 4.12. The fact that the monodromy map is injective was proved in [**60**] and we use essentially their argument to determine the fibres in the proof of Theorem 4.10 below.

We also deduce the following, since it is important (but is well known):

Corollary 4.13. *The monodromy map*

$$\nu : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \longrightarrow M_{\text{ext}}(\mathbf{A})$$

is surjective on tangent vectors and is a biholomorphism onto its image $\nu(\mathcal{M}^*_{\text{ext}}(\mathbf{A})) \subset M_{\text{ext}}(\mathbf{A})$.

Proof. This holds for any injective holomorphic map between manifolds of the same dimension (see for example [**90**] Theorem 2.14, Chapter 1) □

Proof (of Theorem 4.10). For surjectivity, recall that the Stokes matrices classify germs of singular connections up to C^∞ gauge transformations with Taylor expansion 1: by combining Theorem 1.22 with Corollary 3.9 germs ${}^i\alpha \in {}^i\mathcal{A}_{\text{ext,fl}}({}^iA^0)$ may be obtained having any given Stokes matrices and residue for each $i = 1, \dots, m$. It is straightforward to extend ${}^i\alpha$ to $\bar{D}_i$ (any smooth flat connection on the punctured disk $\bar{D}_i \setminus \{a_i\}$ with the same conjugacy class of local monodromy as ${}^i\alpha$ may be smoothly modified to agree with ${}^i\alpha$ near a_i). Next the ${}^i\alpha$'s are patched together along the ribbons $[\Gamma_i]$. Let ${}^i\Phi_0$ be the canonical solution of ${}^i\alpha$ on ${}^i\text{Sect}_{r_i}$ from Lemma 3.12. Since $GL_n(\mathbb{C})$ is path connected it is possible to choose a smooth map $\chi_i : [\Gamma_i] \to GL_n(\mathbb{C})$ such that $\chi_i = {}^1\Phi_0$ on ${}^1\text{Sect}_{r_1} \cap [\Gamma_1]$ and $\chi_i = {}^i\Phi_0 C_i$ on ${}^i\text{Sect}_{r_i} \cap [\Gamma_i]$ for $i = 2, \dots, m$. Define α over the tentacles $\mathcal{T}$ as follows:

$$\begin{aligned} \alpha|_{\bar{D}_i} &= {}^i\alpha && \text{for } i = 1, \dots, m, \text{ and} \\ \alpha|_{[\Gamma_i]} &= (d\chi_i)\chi_i^{-1} && \text{for } i = 2, \dots, m. \end{aligned}$$

It is easy to check these agree on the overlaps and that when ${}^1\Phi_0$ and ${}^i\Phi_0$ are extended over $[\Gamma_i]$ as fundamental solutions of α then ${}^1\Phi_0 = {}^i\Phi_0 C_i$. Finally the two relations in the definition of $M_{\text{ext}}(\mathbf{A})$ enable us to extend α over the rest of $\mathbb{P}^1$. Firstly the product relation $\rho_m\rho_{m-1}\cdots\rho_1 = 1$ ensures that α has no monodromy around the boundary $\partial\mathcal{T}$, so that any local fundamental solution Φ extends around the boundary to give a map

$$\Phi : \partial\mathcal{T} \longrightarrow GL_n(\mathbb{C})$$

from the boundary circle $\partial\mathcal{T} \cong S^1$ to $GL_n(\mathbb{C})$. Then the second relation $\sum \text{Tr}({}^i\Lambda) = 0$ ensures that this loop Φ in $GL_n(\mathbb{C})$ is contractible. To see this, firstly recall that the determinant map $\det : GL_n(\mathbb{C}) \to \mathbb{C}^*$ expresses $GL_n(\mathbb{C})$ as a fibre bundle over $\mathbb{C}^*$, with fibres diffeomorphic to $SL_n(\mathbb{C})$, and that $SL_n(\mathbb{C})$ is simply connected. Then, from the homotopy long exact sequence for fibrations, it follows that det induces an isomorphism of fundamental groups: $\pi_1(GL_n(\mathbb{C})) \cong \pi_1(\mathbb{C}^*) \cong \mathbb{Z}$. Thus we need to see that the loop

$$\varphi := \det(\Phi) : \partial\mathcal{T} \longrightarrow \mathbb{C}^*$$

in the punctured complex plane does not wind around zero. This follows since the winding number of φ is

$$\frac{1}{(2\pi i)} \int_{\partial\mathcal{T}} \frac{d\varphi}{\varphi} = \frac{1}{(2\pi i)} \int_{\partial\mathcal{T}} \text{Tr}(\alpha)$$

and we can use the C^∞ version of Cauchy's integral theorem (see Lemma 5.9) to see that this value is equal to $\sum \text{Tr}({}^i\Lambda)$ (using the fact that α is flat to deduce $d\text{Tr}(\alpha) = 0$).

Thus the loop Φ in $GL_n(\mathbb{C})$ may be extended over the complement of $\mathcal{T}$ in $\mathbb{P}^1$ to a smooth map

$$\Phi : \mathbb{P}^1 \setminus \mathring{\mathcal{T}} \longrightarrow GL_n(\mathbb{C}).$$

We then define $\alpha = (d\Phi)\Phi^{-1}$ on $\mathbb{P}^1 \setminus \mathring{\mathcal{T}}$ and thereby obtain $\alpha \in \mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ having the desired monodromy data and we see that the C^∞ monodromy map is indeed surjective.

To prove the intertwining property, observe (from Proposition 1.24 and Lemma 3.12) that if $h \in \mathcal{G}_T$ and $\alpha' = h[\alpha]$ then the canonical fundamental solutions of α and α' are related by:

$${}^i\Phi'_j = h \cdot {}^i\Phi_j \cdot t_i^{-1}$$

where $t_i = h(a_i) \in T$. The intertwining property and the fact that the $\mathcal{G}_1$ orbits are contained in the fibres of ν are then immediate from the definition of the Stokes matrices in terms of the canonical solutions (see Lemma 3.13 for example) together with the definition of the connection matrices.

To prove that the fibres are precisely the $\mathcal{G}_1$ orbits, suppose $\alpha, \alpha' \in \mathcal{A}_{\mathrm{ext,fl}}(\mathbf{A})$ have the same monodromy data. Let

$$Y := {}^1\Phi_0'({}^1\Phi_0)^{-1}$$

be the induced (invertible) solution of $\mathrm{Hom}(\alpha, \alpha')$ on ${}^1\mathrm{Sect}_{r_1}$. Then Y is single valued when extended to $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$ as a solution of $\mathrm{Hom}(\alpha, \alpha')$. For example when Y is extended around the loop $\gamma_i^{-1} \cdot l_i \cdot \gamma_i$ it has no monodromy since, when extended around this loop, ${}^1\Phi_0'$ and ${}^1\Phi_0$ are both multiplied on the right by the same constant matrix $\rho_i(\mathbf{C}, \mathbf{S}, \mathbf{\Lambda})$. (Such loops generate the fundamental group of $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$.) Also, since the monodromy data encodes the transitions between the various canonical fundamental solutions it follows that $Y = {}^i\Phi_j'({}^i\Phi_j)^{-1}$ for any i, j. Now observe (from Proposition 1.24 and Lemma 3.12) that ${}^i\Phi_j'({}^i\Phi_j)^{-1}$ is asymptotic to 1 at a_i on some sectorial neighbourhood of ${}^i\mathrm{Sect}_j$ ($j = 1, \ldots, r_i,\ i = 1, \ldots, m$). It follows that Y extends smoothly to $\mathbb{P}^1$ and has Taylor expansion 1 at each a_i and so is an element of $\mathcal{G}_1$. By construction $\alpha' = Y[\alpha]$ so α and α' are in the same $\mathcal{G}_1$ orbit $\square$

CHAPTER 5

The Monodromy Map is Symplectic

Much of the story so far can be summarised in the following commutative diagram:

$$\begin{array}{ccccc} & & \mathcal{M}_{\rm ext}(\mathbf{A}) & \xrightarrow{\cong} & \mathcal{A}_{\rm ext,fl}(\mathbf{A})/\mathcal{G}_1 \\ & & \bigcup & & \big\downarrow\cong \\ \widetilde{O}_1\times\cdots\times\widetilde{O}_m/\!/GL_n(\mathbb{C}) & \cong & \mathcal{M}^*_{\rm ext}(\mathbf{A}) & \xrightarrow{\nu} & M_{\rm ext}(\mathbf{A}). \end{array} \tag{64}$$

The extended moduli space $\mathcal{M}^*_{\rm ext}(\mathbf{A})$ was defined in Chapter 2 to be the set of isomorphism classes of compatibly framed meromorphic connections on trivial rank n vector bundles with irregular type $\mathbf{A}$. It was shown to be a fine moduli space and have an intrinsic complex symplectic structure given explicitly in terms of (finite dimensional) coadjoint orbits and cotangent bundles.

$\mathcal{M}_{\rm ext}(\mathbf{A})$ has the same definition as $\mathcal{M}^*_{\rm ext}(\mathbf{A})$ except with the word 'trivial' replaced by 'degree zero'. It was identified with the set of $\mathcal{G}_1$ orbits in the extended space $\mathcal{A}_{\rm ext,fl}(\mathbf{A})$ of flat C^∞ singular connections in Chapter 3. Moreover the curvature was shown to be a moment map for the action of the gauge group $\mathcal{G}_1$ on the symplectic Fréchet manifold $\mathcal{A}_{\rm ext}(\mathbf{A})$, so that (formally) $\mathcal{A}_{\rm ext,fl}(\mathbf{A})/\mathcal{G}_1$ is a complex symplectic quotient.

The extended monodromy manifold $M_{\rm ext}(\mathbf{A})$ was defined in Chapter 4 and looks like a multiplicative version of $\mathcal{M}^*_{\rm ext}(\mathbf{A})$ (when both are described explicitly). The act of taking monodromy data defines both the monodromy map ν and the right-hand isomorphism in the diagram. ν is a biholomorphic map onto its image, which is a dense open submanifold of $M_{\rm ext}(\mathbf{A})$.

Basically the bottom line of (64) appears in the work [**60**] of Jimbo, Miwa and Ueno but the symplectic structures and the rest of the diagram do not.

The torus $T^m \cong (\mathbb{C}^*)^{nm}$ acts on each space in (64) and these actions are intertwined by all the maps. The non-extended picture arises by taking a (symplectic) quotient by T^m: we obtain another commutative diagram as above but with the subscripts 'ext' and the tildes removed.

Now, the aim of this chapter is to see that the (Atiyah-Bott) symplectic structure on $\mathcal{A}_{\rm ext}(\mathbf{A})$ does indeed induce a symplectic structure on the open submanifold $\nu(\mathcal{M}^*_{\rm ext}(\mathbf{A}))$ of $M_{\rm ext}(\mathbf{A})$ that is the image of the monodromy map, and moreover that this symplectic structure pulls back along ν to the explicit symplectic structure on $\mathcal{M}^*_{\rm ext}(\mathbf{A})$. In other words we will prove:

Theorem 5.1. *The monodromy map ν is symplectic.*

In some sense this is the 'inverse monodromy theory' version of the well-known result in inverse scattering theory, that the map from the set of initial potentials to scattering data is a symplectic map (see [**35**] Chapter III).

REMARK 5.2. Analogous results have been proved in the logarithmic case (all $k_i = 1$) by Hitchin [**48**] and by Iwasaki [**54, 55**]. Our approach is closest to Hitchin's but note that we have had to significantly generalise the Atiyah-Bott symplectic structure to handle the arbitrary order pole case. On the other hand Iwasaki works exclusively in the rank 2 case (and with Fuchsian equations rather than systems) but he does this over arbitrary genus Riemann surfaces rather than just $\mathbb{P}^1$. Having obtained these results both Iwasaki and Hitchin use them to see intrinsically that the corresponding isomonodromy equations are symplectic. In the next chapter we will show that the same deduction is still possible in the arbitrary order pole case.

1. Factorising the Monodromy Map

Recall (from Theorem 3.17) how the top isomorphism in the above diagram arose: a meromorphic connection gives rise to a $\mathcal{G}$ orbit of C^∞ singular connections and we consider the subset whose Laurent expansion is a formal normal form at each a_i to define the map. In other words we can choose $g \in \mathcal{G}$ to 'straighten' a meromorphic connection to have fixed C^∞ Laurent expansions at each a_i and thereby specify an element of $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$.

In this section we show, at least for meromorphic connections on trivial bundles, that this straightening procedure can be carried out for a family of connections all at the same time. This can be easily rephrased as choosing a factorisation through $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ of the monodromy map ν, or as choosing a slice over $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ for the $\mathcal{G}_1$ action on $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ in the sense of Section 4 of Chapter 3. More precisely we have:

Proposition 5.3. *The monodromy map ν factorises through $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$. That is, it is possible to choose a map $\widetilde{\nu}$ from the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ to the extended space of flat singular connections determined by $\mathbf{A}$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{M}^*_{\text{ext}}(\mathbf{A}) & \xrightarrow{\widetilde{\nu}} & \mathcal{A}_{\text{ext,fl}}(\mathbf{A}) & \overset{i}{\hookrightarrow} & \mathcal{A}_{\text{ext}}(\mathbf{A}) \\ \| & & \big\downarrow{\scriptstyle /\mathcal{G}_1} & & \\ \mathcal{M}^*_{\text{ext}}(\mathbf{A}) & \xrightarrow{\nu} & M_{\text{ext}}(\mathbf{A}) & & \end{array}$$

and such that the composition $i \circ \widetilde{\nu}$ into the Fréchet manifold $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is holomorphic.

This will be deduced from the following

Proposition 5.4. *Let $U \subset \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ be an open subset. Let $d_{\mathbb{P}^1} - A$ be the corresponding universal family of meromorphic connections on the trivial bundle on $\mathbb{P}^1$ (with compatible framings $\mathbf{g_0} = ({}^1g_0, \ldots, {}^mg_0)$) parameterised by $u \in U$ (see Proposition 2.52). Then there exists a family of smooth bundle automorphisms*

$$g \in GL_n(C^\infty(U \times \mathbb{P}^1))$$

such that for each $u \in U$ and each $i = 1, 2, \ldots, m$:

- *$g(u, a_i) \in GL_n(\mathbb{C})$ is the compatible framing ${}^ig_0(u)$ at a_i specified by $u \in \mathcal{M}^*_{\text{ext}}(\mathbf{A})$,*
- *The singular connection $\alpha(u) := g(u)[A(u)]$ on $\mathbb{P}^1$ has Laurent expansion ${}^iA^0 + {}^iR(u)$ at $a_i \in \mathbb{P}^1$, where ${}^iR(u) \in \mathfrak{t}^*$ is the residue term of the diagonalisation of $A(u)$ at i specified by ${}^ig_0(u)$ and the square brackets denote the gauge action in the $\mathbb{P}^1$ direction.*

Moreover if $D_1, \ldots, D_m$ are disjoint open discs in $\mathbb{P}^1$ with $a_i \in D_i$ then we may assume that $g(u, z) = 1$ for all u if $z \in \mathbb{P}^1$ is not in any of the disks D_i.

Proof. Recall that A is a matrix of meromorphic one forms on $\mathbb{P}^1$ with coefficients which are holomorphic functions on U. Consider the Laurent expansion of $A(u)$ at $a_i \in \mathbb{P}^1$ for each i:

$$L_i(A) \in \mathrm{End}(\mathcal{R})\frac{dz}{z_i^{k_i}}$$

where $\mathcal{R}$ is the ring $\mathbb{C}\{z_i\} \otimes \mathcal{O}(U)$ of convergent power series in z_i with holomorphic functions on U as coefficients. For any $u \in U$ we have $L_i(A)(u) = L_i(A(u))$ as elements of $\mathrm{End}_n(\mathbb{C}\{z_i\})dz/z_i^{k_i}$.

By definition, for each i, the universal framing ${}^ig_0$ diagonalises the leading coefficient of $L_i(A)$. The argument in Appendix B then gives a *unique* formal power series (with values in $\mathcal{O}(U)$) providing a formal isomorphism with the corresponding formal normal form and having constant term ${}^ig_0$. That is, we have unique

$${}^i\widehat{g} \in GL_n(\widehat{\mathcal{R}})$$

(where $\widehat{\mathcal{R}}$ is the ring $\mathbb{C}[\![z_i]\!] \otimes \mathcal{O}(U)$) such that for each $u \in U$

$$ {}^i\widehat{g}(u)[A(u)] = {}^iA^0 + {}^iR(u) \in \mathrm{End}_n(\mathbb{C}[\![z_i]\!])\frac{dz}{z_i^{k_i}} \tag{65}$$

and ${}^i\widehat{g}$ agrees with the given compatible framing at $z = a_i$:

$${}^i\widehat{g}(u, a_i) = {}^ig_0(u)$$

where ${}^iR(u) \in \mathfrak{t}^*$ is the residue term of the diagonalisation of $A(u)$ at a_i.

Now the crucial step is to use the theorem of E. Borel on the surjectivity of the Taylor expansion map onto formal power series to find a smooth gauge automorphism over all of $U \times \mathbb{P}^1$ which attains the Taylor series ${}^i\widehat{g}$ at each a_i. That is, there exists

$$g \in GL_n(C^\infty(U \times \mathbb{P}^1))$$

such that for each $u \in U$ the Taylor expansion of $g(u)$ at a_i is ${}^i\widehat{g}(u)$:

$$L_i(g(u)) = {}^i\widehat{g}(u) \in GL_n(\mathbb{C}[\![z_i]\!]).$$

The result we need here is a parameter dependent version of Borel's theorem:

Lemma 5.5. *(E. Borel). Given $\widehat{f} \in \mathbb{C}[\![x, y]\!] \otimes C^\infty(U)$ (where x, y are real coordinates on $\mathbb{C} \cong \mathbb{R}^2$) and a compact neighbourhood I of the origin in $\mathbb{R}$ then there exists a smooth function*

$$f \in C^\infty(U \times I \times I)$$

such that the Taylor expansion of f at $x = y = 0$ is given by $\widehat{f}$:

$$L_0(f(u)) = \widehat{f}(u) \qquad \text{for all } u \in U.$$

Proof. See Appendix D □

In our situation we note that $\mathbb{C}[\![z]\!]$ is a sub-ring of $\mathbb{C}[\![z, \bar{z}]\!] = \mathbb{C}[\![x, y]\!]$ and apply Lemma 5.5 to each matrix entry of each ${}^i\widehat{g}$ in turn for $i = 1, \ldots, m$. This gives matrices of smooth functions:

$${}^ig \in \mathrm{End}_n(C^\infty(U \times \bar{D}_i))$$

for each i (where D_i is the disk containing a_i in $\mathbb{P}^1$) such that ${}^ig$ has the desired Taylor expansions at a_i:

$$L_i({}^ig(u)) = {}^i\widehat{g}(u) \qquad \text{for all } u \in U.$$

In particular $\det {}^i g(u, a_i) = \det {}^i g_0(u)$ is nonzero for all $u \in U$ and so there is a neighbourhood of $U \times \{a_i\} \subset U \times \mathbb{P}^1$ throughout which $\det({}^i g)$ is nonzero. It follows (since $GL_n(\mathbb{C})$ is connected) that we can find a smooth bundle automorphism

$$g \in GL_n(C^\infty(U \times \mathbb{P}^1))$$

that agrees with ${}^i g$ in some neighbourhood of $U \times \{a_i\} \subset U \times \mathbb{P}^1$ for each i. In particular g has the desired Taylor expansions at each a_i so that $\alpha = g[A]$ has the desired C^∞ Laurent expansions by construction □

Immediately we can factorise the monodromy map:

Proof (of Proposition 5.3). Construct g as in Proposition 5.4 with $U = \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ and then define $\widetilde{\nu}(u) = g(u)[A(u)]$ for all $u \in \mathcal{M}^*_{\text{ext}}(\mathbf{A})$.

All that remains is to see that the composition $\bar{\nu} := i \circ \widetilde{\nu}$ is holomorphic. Recall (from Lemma 3.21) that by choosing a basepoint $\mathcal{A}_{\text{ext}}(\mathbf{A})$ is identified with a Fréchet submanifold of the Fréchet space $\Omega^1[D](\mathbb{P}^1, \operatorname{End}(E))$ of matrices of C^∞ one forms with poles on the divisor D. Thus it is sufficient to prove that the map

$$\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \longrightarrow \Omega^1[D](\mathbb{P}^1, \operatorname{End}(E)); \qquad u \longmapsto \alpha(u) := g(u)[A(u)]$$

is holomorphic. This is a local statement so we will pick a point $u_0 \in \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ and an open ball $U \subset \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ containing u_0. Now if $W_0 \in T_{u_0}\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ is a tangent vector at u_0, then we will denote the partial derivative of α along W_0 by

$$W_0(\alpha) \in \Omega^1[D](\mathbb{P}^1, \operatorname{End}(E)).$$

Here we think of α as a section of the C^∞ vector bundle $\pi^*(\operatorname{End}_n(\Omega^1[D]))$ over $\mathbb{P}^1 \times U$ (where $\pi : \mathbb{P}^1 \times U \to \mathbb{P}^1$ is the obvious projection). This vector bundle is trivial in the U directions so the partial derivative makes sense[1].

Now it follows from basic facts about calculus on Fréchet spaces that the map $\bar{\nu}$ is holomorphic and has derivative $W_0(\alpha)$ along W_0 at u_0:

$$d\bar{\nu}_{u_0}(W_0) = W_0(\alpha) \in \Omega^1[D](\mathbb{P}^1, \operatorname{End}(E)).$$

This can be deduced from Examples 3.1.6 and 3.1.7 in [**43**] □

2. Symplecticness of Lifted Monodromy Maps

Let $\widetilde{\nu} : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \to \mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ be any map, as constructed in Proposition 5.3, from the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ to the extended space of singular flat connections determined by $\mathbf{A}$. Let $\bar{\nu}$ be the composition of $\widetilde{\nu}$ with the natural inclusion $i : \mathcal{A}_{\text{ext,fl}}(\mathbf{A}) \to \mathcal{A}_{\text{ext}}(\mathbf{A})$ into the Fréchet manifold $\mathcal{A}_{\text{ext}}(\mathbf{A})$:

$$\bar{\nu} := i \circ \widetilde{\nu} : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \to \mathcal{A}_{\text{ext}}(\mathbf{A}); \qquad u \mapsto \alpha(u) := g(u)[A(u)]$$

where $g \in GL_n(C^\infty(U \times \mathbb{P}^1))$ is from Proposition 5.4. Now, we have defined symplectic structures on both $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ (using cotangent bundles and coadjoint orbits, see Chapter 2) and on the space $\mathcal{A}_{\text{ext}}(\mathbf{A})$ of C^∞ singular connections (following Atiyah-Bott, see Chapter 3).

[1]Concretely, local sections are of the form $\sum h_i\theta_i$ for C^∞ functions h_i on U and sections θ_i of $\operatorname{End}_n(\Omega^1[D])$. Then W_0 differentiates just the h_i's: $W_0(\sum h_i\theta_i) = \sum W_0(h_i)\theta_i$.

Thus the natural question to ask is whether such a map $\bar{\nu}$ is symplectic i.e. whether it pulls back the symplectic structure on $\mathcal{A}_{\text{ext}}(\mathbf{A})$ to that on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$. The aim of this section is to prove that this is indeed the case:

Theorem 5.6. *Let $\widetilde{\nu}$ be a choice of factorisation of the monodromy map as constructed in Proposition 5.3. Then the corresponding map*

$$\bar{\nu} : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \to \mathcal{A}_{\text{ext}}(\mathbf{A})$$

from the extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ to the Fréchet manifold $\mathcal{A}_{\text{ext}}(\mathbf{A})$ of singular connections is a symplectic map.

REMARK 5.7. This is just a restatement of Theorem 5.1 since $\widetilde{\nu}$ is a slice for the $\mathcal{G}_1$ action on $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ over $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$.

The scheme of the proof will be straightforward: take two tangent vectors at a point of $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$, push them forward along $\bar{\nu}$ and compare what the symplectic forms evaluate to. The key steps are contained in the following theorem which will be needed later and doesn't involve Fréchet manifolds.

Theorem 5.8. *Choose a point $u_0 \in \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ of the extended moduli space and tangent vectors $W_1, W_2 \in T_{u_0}\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ at u_0. Choose $g \in GL_n(C^\infty(U \times \mathbb{P}^1))$ as in Proposition 5.4 with $U = \mathcal{M}^*_{\text{ext}}(\mathbf{A})$ and let $\alpha(u) = g(u)[A(u)]$ be the corresponding family of 'straightened' singular connections. Define matrices of singular one forms on $\mathbb{P}^1$, $\phi_1, \phi_2 \in \text{End}_n(\Omega^1[D](\mathbb{P}^1))$, to be the corresponding partial derivatives of $\alpha(u)$:*

$$\phi_1 = W_1(\alpha) \qquad \phi_2 = W_2(\alpha).$$

Then $\text{Tr}(\phi_1 \wedge \phi_2)$ is a nonsingular two-form on $\mathbb{P}^1$, and moreover

$$\frac{1}{2\pi i}\int_{\mathbb{P}^1} \text{Tr}(\phi_1 \wedge \phi_2) = \omega_{\mathcal{M}^*_{\text{ext}}(\mathbf{A})}(W_1, W_2)$$

where $\omega_{\mathcal{M}^*_{\text{ext}}(\mathbf{A})}$ is the symplectic form we have defined on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ in Chapter 2.

Proof. Recall that the full Laurent expansion of $\alpha(u)$ at the ith singularity a_i is fixed to be

$$(66) \qquad {}^iA^0 + {}^iR(u) = {}^iA^0_{k_i}\frac{dz}{(z-a_i)^{k_i}} + \cdots + {}^iA^0_2\frac{dz}{(z-a_i)^2} + {}^i\Lambda(u)\frac{dz}{(z-a_i)}$$

where ${}^iR(u) = {}^i\Lambda(u)\frac{dz}{(z-a_i)}$ is the residue term of the diagonalisation of $A(u)$ at a_i specified by ${}^ig_0(u)$ and the ${}^iA^0_j$'s are the constant diagonal matrices occurring in the chosen formal normal forms $\mathbf{A}$.

There are two observations to be made. Firstly the Taylor expansions of $g(u)$ induce a lifting

$$l : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \hookrightarrow (G_{k_1} \times \mathfrak{t}^*) \times \cdots \times (G_{k_m} \times \mathfrak{t}^*)$$

covering the natural projection from

$$\mu^{-1}_{GL_n(\mathbb{C})}(0) \subset \widetilde{O}_1 \times \cdots \times \widetilde{O}_m$$

onto $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ (recall from Chapter 2 that $\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \cong \widetilde{O}_1 \times \cdots \times \widetilde{O}_m // GL_n(\mathbb{C})$ and that we have surjective winding maps $w_i : G_{k_i} \times \mathfrak{t}^* \to \widetilde{O}_i$). This will give an expression for the symplectic form on $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$. The lifting l arises as follows. Let

$${}^i\widehat{g}(u) = L_i(g(u)) \in GL_n(\mathbb{C}[\![z_i, \bar{z}_i]\!])$$

be the Taylor expansion of $g(u)$ at $z = a_i$ (where $z_i = z - a_i$). Then ${}^i\widehat{g}(u)$ has no $\bar{z}_i$ terms, i.e. ${}^i\widehat{g}(u) \in GL_n(\mathbb{C}[\![z_i]\!])$; to see this consider the $(0,1)$ part of $L_i(g(u)[A(u)])$. Now truncate ${}^i\widehat{g}(u)$ after k_i terms, that is consider its image

$$ {}^i\overline{g}(u) \in GL_n(\mathbb{C}[\![z_i]\!]/(z_i)^{k_i}) \cong G_{k_i} $$

in the group G_{k_i} where the isomorphism is obtained by replacing the coordinate z_i by the symbol ζ to give the group G_{k_i} as studied in Chapter 2. Then for any u we have the following equality of elements of $\mathfrak{g}^*_{k_i}$:

$$ {}^i\overline{g}(u) \cdot {}^iA(u) \cdot {}^i\overline{g}(u)^{-1} = {}^iA^0 + {}^iR(u) $$

where

$$ {}^iA(u) := \mathrm{PP}_i(A(u)) \in \mathfrak{g}^*_{k_i} $$

is the polar part of $A(u)$ at a_i. This means that the element $(g(u,a_i), {}^iA(u))$ of the extended orbit $\widetilde{O}_i$ is the image under the winding map w_i of the element $({}^i\overline{g}(u), {}^iR(u))$ of $G_{k_i} \times \mathfrak{t}^*$. Hence as u varies we see $g(u)$ canonically induces a lifting

$$ l : \mathcal{M}^*_{\mathrm{ext}}(\mathbf{A}) \to (G_{k_1} \times \mathfrak{t}^*) \times \cdots \times (G_{k_m} \times \mathfrak{t}^*); \qquad u \mapsto \big({}^i\overline{g}(u), {}^iR(u)\big)_{i=1}^m $$

as stated.

In particular the tangents W_1, W_2 to $\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$ at u_0 may be lifted (pushed forward along l) to $(G_{k_1} \times \mathfrak{t}^*) \times \cdots \times (G_{k_m} \times \mathfrak{t}^*)$. Using the left trivialisations of the tangent bundles TG_{k_i} we define elements

$$ {}^iX_j \in \mathfrak{g}_{k_i} \qquad \text{and} \qquad {}^iR_j = {}^i\Lambda_j \frac{dz}{z - a_i} \in \mathfrak{t}^* $$

to be such that the pair $({}^iX_j, {}^iR_j)$ is the ith component of $dl_{u_0}(W_j)$ for $j = 1,2$ and $i = 1, \ldots, m$. That is, the lift of W_j to $T_{l(u_0)}\big((G_{k_1} \times \mathfrak{t}^*) \times \cdots \times (G_{k_m} \times \mathfrak{t}^*)\big)$ is

$$ dl_{u_0}(W_j) = (({}^1X_j, {}^1R_j), \ldots, ({}^mX_j, {}^mR_j)) $$

for $j = 1, 2$. In particular, using the formula for l, since we are using the left trivialisation, we have

$$ {}^iX_j = {}^i\overline{g}(u)^{-1} \cdot W_i({}^i\overline{g}(u)) \in \mathfrak{g}_{k_i}. \tag{67} $$

It follows that if we define $\dot{g}_j$ to be the derivative of $g(u)$ along W_j at u_0:

$$ \dot{g}_j := W_j(g(u)) \in \mathrm{End}_n(C^\infty(\mathbb{P}^1)) $$

then ${}^iX_j$ is the first k_i terms of the Taylor expansion of $g(u_0)^{-1}\dot{g}_j$ at a_i for $j = 1, 2$ and $i = 1, \ldots, m$.

Thus, using the formula in Proposition 2.21 for the pullback of the symplectic structure on $\widetilde{O}_i$ to $G_{k_i} \times \mathfrak{t}^*$ along the winding map w_i, we obtain the following expression for the symplectic form on $\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$:

$$ \omega_{\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})}(W_1, W_2) = \sum_{i=1}^m \Big(\langle {}^iR_1, {}^i\widetilde{X}_2 \rangle - \langle {}^iR_2, {}^i\widetilde{X}_1 \rangle + \langle {}^iA(u_0), [{}^iX_1, {}^iX_2] \rangle \Big) \tag{68} $$

where ${}^i\widetilde{X}_j := {}^i\overline{g}(u_0) \cdot {}^iX_j \cdot {}^i\overline{g}(u_0)^{-1} \in \mathfrak{g}_{k_i}$ for $j = 1, 2$ and $i = 1, \ldots, m$.

The second observation to be made from the explicit Laurent expansions of $\alpha(u)$ in (66) is that the two form $\mathrm{Tr}(\phi_1 \wedge \phi_2)$ on $\mathbb{P}^1$ is nonsingular. This follows since the Laurent expansion of $\phi_j = W_j(\alpha)$ at a_i is meromorphic (it is just ${}^iR_j$ in the notation above). Hence

$\mathrm{Tr}(\phi_1 \wedge \phi_2)$ has zero Laurent expansion at each a_i and so by the division lemma (Lemma 3.1) it is nonsingular. For later use we will define ${}^i\psi_j$ to be the matrix of nonsingular one forms on the disk $\bar{D}_i$ such that

$$\phi_j = {}^i\psi_j + {}^iR_j \qquad \text{on } \bar{D}_i \tag{69}$$

for $i = 1, \dots, m$ and $j = 1, 2$.

To actually calculate the integral of $\mathrm{Tr}(\phi_1 \wedge \phi_2)$ over $\mathbb{P}^1$ we need two preliminary results. Firstly a slightly modified version of the C^∞ version of Cauchy's integral theorem:

Lemma 5.9. *(Modified C^∞ Cauchy Integral Theorem). Let k be a nonnegative integer, $a \in \mathbb{C}$ a complex number and D_a an open disk in $\mathbb{C}$ containing the point a. If $f \in C^\infty(\bar{D}_a)$ and $\left(\frac{\partial f}{\partial \bar{z}}\right)/(z-a)^k \in C^\infty(\bar{D}_a)$ then $\left(\frac{\partial f}{\partial \bar{z}}\right)\frac{dz\wedge d\bar{z}}{(z-a)^{k+1}}$ is absolutely integrable over $\bar{D}_a$ and*

$$\frac{(2\pi i)}{k!}\frac{\partial^k f}{\partial z^k}(a) = \int_{\partial \bar{D}_a} \frac{f(z)dz}{(z-a)^{k+1}} + \int_{\bar{D}_a} \frac{\partial f}{\partial \bar{z}}\frac{dz \wedge d\bar{z}}{(z-a)^{k+1}}$$

where the line integral is taken in an anti-clockwise direction.

Proof. The $k = 0$ case is the usual C^∞ Cauchy integral theorem, see [**39**] p2, where it is deduced from Stokes' theorem. Then differentiating with respect to a gives the above result: the absolute integrability ensures that we may reorder the integration and differentiation. Note that the extra condition on f is needed to ensure that the area integral exists: the formula is not true for arbitrary smooth functions f □

Secondly, the basic fact enabling us to calculate the derivatives ϕ_1 and ϕ_2 of α with respect to u is

Lemma 5.10. *Let $\Delta \subset \mathbb{C}$ be a disk containing the origin. Fix a C^∞ global trivialisation of a trivial C^∞ vector bundle E over $\mathbb{P}^1$. Suppose $d_{A(t)}$ is a C^∞ singular connection on E for each $t \in \Delta$ and that it depends smoothly on t. Let $g(t,z) : \Delta \times \mathbb{P}^1 \to GL_n(\mathbb{C})$ be a smooth family of automorphisms of E. Then we have the following equality of* $\mathrm{End}(E)$*-valued singular one-forms on $\mathbb{P}^1$:*

$$\frac{\partial}{\partial t}\left(g(t)[A(t)]\right)\Big|_{t=0} = g(0)\left(\dot{A} + d_{A(0)}(g(0)^{-1}\dot{g})\right)g(0)^{-1}$$

where $g(t) = g(t,\cdot) : \mathbb{P}^1 \to GL_n(\mathbb{C})$ and the dot $(\dot{})$ denotes $\frac{\partial}{\partial t}\Big|_{t=0}$.

Proof. This is a straightforward computation. Firstly recall that in full generality:

$$d_{g[A]}B = g\left(d_A(g^{-1}Bg)\right)g^{-1}.$$

Using the trivialisation of E we have $d_{A(t)} = d - A(t)$ for a matrix of C^∞ one-forms-with-poles $A(t)$ on $\mathbb{P}^1$ depending smoothly on t. By definition

$$g(t)[A(t)] = g(t)A(t)g(t)^{-1} + d(g(t)) \cdot g(t)^{-1}$$

where d differentiates only in the $\mathbb{P}^1$ direction. Thus

$$\frac{\partial}{\partial t}(g(t)[A(t)])\Big|_{t=0} = g(0)\dot{A}g(0)^{-1} + \frac{\partial}{\partial t}(g(t)[A(0)])\Big|_{t=0}. \tag{70}$$

The last term here may be evaluated as follows. Let $h(t) = g(t) \cdot g(0)^{-1}$ so that $h(0)$ is identically $1 \in GL_n(\mathbb{C})$ on $\mathbb{P}^1$. Then $g(t)[A(0)] = h(t)[B]$ where $B := g(0)[A(0)]$ (which is independent of t). Whence

$$\begin{aligned}
\frac{\partial}{\partial t}(g(t)[A(0)])\Big|_{t=0} = \frac{\partial}{\partial t}(h(t)[B])\Big|_{t=0} &= \frac{\partial}{\partial t}\big(h(t)Bh(t)^{-1} + (dh(t))h(t)^{-1}\big)\Big|_{t=0} \\
&= \dot{h}B - B\dot{h} + \frac{\partial}{\partial t}(dh(t))\Big|_{t=0} - (dh(0))\dot{h} \qquad \text{(using } h(0) = 1) \\
&= \dot{h}B - B\dot{h} + d\dot{h} \\
&= d_B\dot{h} = d_{g(0)[A(0)]}(\dot{g}g(0)^{-1}) \qquad \text{(by definition)} \\
&= g(0)d_{A(0)}(g(0)^{-1}\dot{g})g(0)^{-1}.
\end{aligned}$$

Substituting this into (70) gives the result □

Thus by defining small curves in $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ tangent to W_1 and W_2 we deduce that we can write the derivatives ϕ_1, ϕ_2 in the following form. For $i = 1, 2$, define $\dot{A}_i = W_i(A(u))$ to be the derivatives of $A(u)$ at u_0 along W_i so that $\dot{A}_i$ is a matrix valued *meromorphic* one-form on $\mathbb{P}^1$. Then from Lemma 5.10:

$$\phi_i = g(u_0) \cdot \widetilde{\phi}_i \cdot g(u_0)^{-1}$$

where $\widetilde{\phi}_i$ is the matrix of singular one-forms on $\mathbb{P}^1$ defined by:

$$\widetilde{\phi}_i := \dot{A}_i + d_{A(u_0)}\left(g(u_0)^{-1}\dot{g}_i\right) \tag{71}$$

for $i = 1, 2$. In particular we have the following equality of two forms on $\mathbb{P}^1$:

$$\operatorname{Tr}(\phi_1 \wedge \phi_2) = \operatorname{Tr}(\widetilde{\phi}_1 \wedge \widetilde{\phi}_2).$$

We can now observe that this integrand is zero outside of the disks D_i. Recall we have arranged for the $g(u, z)$ to be identically $1 \in GL_n(\mathbb{C})$ if $z \notin \bigcup_i D_i$. Thus $\dot{g}_j$ is zero in this region of $\mathbb{P}^1$ and so $\widetilde{\phi}_j = \dot{A}_j$ there. But $\dot{A}_j$ is a $(1, 0)$-form and so $\operatorname{Tr}(\dot{A}_1 \wedge \dot{A}_2) = 0$. It follows that the integral splits up into integrals over the closed disks:

$$\int_{\mathbb{P}^1} \operatorname{Tr}(\phi_1 \wedge \phi_2) = \sum_{i=1}^{m} \int_{\bar{D}_i} \operatorname{Tr}(\widetilde{\phi}_1 \wedge \widetilde{\phi}_2). \tag{72}$$

Our task is to evaluate each term in this sum, which we will now do.

From the definition (71) of $\widetilde{\phi}_2$ we have

$$\int_{\bar{D}_i} \operatorname{Tr}(\widetilde{\phi}_1 \wedge \widetilde{\phi}_2) = \int_{\bar{D}_i} \operatorname{Tr}(\widetilde{\phi}_1 \wedge \dot{A}_2) + \int_{\bar{D}_i} \operatorname{Tr}\left(\widetilde{\phi}_1 \wedge d_{A(u_0)}(g(u_0)^{-1}\dot{g}_2)\right). \tag{73}$$

We will see both integrals on the right are well defined. The first term on the right-hand side can be evaluated as follows. Since $\dot{A}_2$ is a matrix of meromorphic one forms we have

$$\operatorname{Tr}(\widetilde{\phi}_1 \wedge \dot{A}_2) = \operatorname{Tr}(\widetilde{\phi}_1^{(0,1)} \wedge \dot{A}_2).$$

Now from the definition (71) the $(0, 1)$ part of $\widetilde{\phi}_1$ is

$$\widetilde{\phi}_1^{(0,1)} = \bar{\partial}(g(u_0)^{-1}\dot{g}_1) = \frac{\partial(g(u_0)^{-1}\dot{g}_1)}{\partial \bar{z}} d\bar{z}.$$

Also $\dot{A}_2$ has a pole of order at most k_i at $z = a_i$ and so we can define a smooth function on $\bar{D}_i$, $f \in C^\infty(\bar{D}_i)$, by the prescription

$$f dz = (z - a_i)^{k_i} \cdot \mathrm{Tr}\big(g(u_0)^{-1}\dot{g}_1 \dot{A}_2\big) \qquad \text{on } \bar{D}_i.$$

By taking the exterior derivative of both sides of this equality and dividing through by $(z - a_i)^{k_i}$ we deduce

$$\mathrm{Tr}(\widetilde{\phi}_1 \wedge \dot{A}_2) = -\frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{(z - a_i)^{k_i}} \qquad \text{on } \bar{D}_i,$$

where the minus sign occurs since we have reversed the order of dz and $d\bar{z}$. We may evaluate the integral of this over $\bar{D}_i$ using Lemma 5.9, the modified version of Cauchy's integral theorem. Observe that the Taylor expansion of $f dz$ at $z = a_i$ has no terms containing $\bar{z}_i$, where $z_i = z - a_i$. Thus $\partial f/\partial \bar{z}$ has zero Taylor expansion at a_i and in particular using the division lemma we deduce $(\partial f/\partial \bar{z})/(z - a_i)^k$ is infinitely differentiable throughout $\bar{D}_i$ for any k. Hence f satisfies the conditions in Lemma 5.9. Also f is zero on the boundary $\partial \bar{D}_i$ since $\dot{g}_1$ is zero there. Therefore Cauchy's integral theorem gives

$$\int_{\bar{D}_i} \mathrm{Tr}(\widetilde{\phi}_1 \wedge \dot{A}_2) = -\frac{(2\pi\sqrt{-1})}{k!} \frac{\partial^k f}{\partial z^k}(a_i) \qquad \text{with } k = k_i - 1. \tag{74}$$

This value is just $-(2\pi\sqrt{-1})$ times the coefficient of $(z - a_i)^k$ in the Taylor expansion of f at a_i, or equivalently $-(2\pi\sqrt{-1})$ times the residue of the Laurent expansion at a_i of

$$\frac{f dz}{(z - a_i)^{k_i}} = \mathrm{Tr}\big(g(u_0)^{-1}\dot{g}_1 \dot{A}_2\big). \tag{75}$$

This only involves the polar part ${}^i\!A_2$ of $\dot{A}_2$ at a_i and the first k_i terms of the Taylor expansion of $g(u_0)^{-1}\dot{g}_1$. But, from (67) we have that these first k_i terms are given by ${}^iX_1$. Also, since

$${}^i\!A(u) = \mathrm{PP}_i\Big(g(u)^{-1}\big({}^i\!A^0 + {}^iR(u)\big)g(u)\Big)$$

we have

$${}^i\!A_2 := W_2({}^i\!A(u)) = [{}^i\!A(u_0), {}^iX_2] + g(u_0, a_i)^{-1} \cdot {}^iR_2 \cdot g(u_0, a_i).$$

Therefore from (74) and (75) we deduce

$$\begin{aligned} \frac{1}{(2\pi\sqrt{-1})} \int_{\bar{D}_i} \mathrm{Tr}(\widetilde{\phi}_1 \wedge \dot{A}_2) &= -\langle {}^iX_1, [{}^i\!A(u_0), {}^iX_2] \rangle - \langle {}^iX_1, g(u_0, a_i)^{-1} \cdot {}^iR_2 \cdot g(u_0, a_i) \rangle \\ &= \langle {}^i\!A(u_0), [{}^iX_1, {}^iX_2] \rangle - \langle {}^iR_2, {}^i\widetilde{X}_1 \rangle. \end{aligned} \tag{76}$$

Now we move on to the last term

$$\int_{\bar{D}_i} \mathrm{Tr}(\widetilde{\phi}_1 \wedge d_{A(u_0)}(g(u_0)^{-1}\dot{g}_2))$$

in (73). First we claim that $d_{A(u_0)}\widetilde{\phi}_1 = 0$ as a matrix of two-forms on $\mathbb{P}^1$. This is equivalent to showing

$$d_{\alpha(u_0)}\phi_1 = 0 \tag{77}$$

since $\widetilde{\phi}_1 = g(u_0)^{-1}\phi_1 g(u_0)$ and $\alpha(u) = g(u)[A(u)]$. Now the fact that $\alpha(u)$ is a *flat* singular connection on $\mathbb{P}^1$ for all u means that

$$d(\alpha(u)) = \alpha(u) \wedge \alpha(u)$$

is an equality between matrices of singular two forms on $\mathbb{P}^1$ for each u (and both sides depend smoothly on u). Differentiating both sides along W_1 at u_0 yields (77).

Next recall from (69) that $\phi_1 = {}^i\psi_1 + {}^iR_1$ on $\bar{D}_i$ with ${}^i\psi_1$ nonsingular. Thus

$$\widetilde{\phi}_1 = g(u_0)^{-1}({}^i\psi_1 + {}^iR_1)g(u_0)$$

on $\bar{D}_i$ and so Leibniz gives

$$\begin{aligned}\mathrm{Tr}(\widetilde{\phi}_1 \wedge d_{A(u_0)}(g(u_0)^{-1}\dot{g}_i)) &= -d\mathrm{Tr}(\widetilde{\phi}_1 g(u_0)^{-1}\dot{g}_2)\\ &= -d\mathrm{Tr}(g(u_0)^{-1}\cdot {}^i\psi_1 \cdot \dot{g}_2) - d\mathrm{Tr}(g(u_0)^{-1}\cdot {}^iR_1\cdot \dot{g}_2) \qquad \text{on } \bar{D}_i.\end{aligned}$$

The term involving ${}^i\psi_1$ is an exact nonsingular two form on $\bar{D}_i$ and so integrates to zero over $\bar{D}_i$ (the boundary integral vanishes as $\dot{g}_2 = 0$ there). For the last term, recall that ${}^iR_1 = {}^i\Lambda_1 \frac{dz}{z-a_i}$ for a constant diagonal matrix ${}^i\Lambda_1$, so that

$$-d\mathrm{Tr}(g(u_0)^{-1}\cdot {}^iR_1\cdot \dot{g}_2) = -d\left(\frac{\mathrm{Tr}\big(g(u_0)^{-1}\cdot {}^i\Lambda_1\cdot \dot{g}_2\big)}{z-a_i}dz\right) = \frac{\partial f}{\partial \bar{z}}\frac{dz\wedge d\bar{z}}{z-a_i}$$

where $f := \mathrm{Tr}(g(u_0)^{-1}{}^i\Lambda_1\dot{g}_2)$. This smooth function f vanishes on $\partial\bar{D}_i$ since $\dot{g}_2$ is zero there and so using Cauchy's integral theorem we find

$$\begin{aligned}\int_{\bar{D}_i}\mathrm{Tr}(\widetilde{\phi}_1 \wedge d_{A(u_0)}(g(u_0)^{-1}\dot{g}_i)) &= \int_{\bar{D}_i}\frac{\partial f}{\partial \bar{z}}\frac{dz\wedge d\bar{z}}{z-a_i} = (2\pi\sqrt{-1})f(a_i)\\ &= (2\pi\sqrt{-1})\mathrm{Tr}(g(u_0,a_i)^{-1}\cdot {}^i\Lambda_1\cdot \dot{g}_2(a_i))\\ &= (2\pi\sqrt{-1})\mathrm{Tr}(g(u_0,a_i)^{-1}\cdot {}^i\Lambda_1\cdot g(u_0,a_i)\cdot {}^iX_2)\\ &= (2\pi\sqrt{-1})\langle {}^iR_1, {}^i\widetilde{X}_2\rangle\end{aligned} \tag{78}$$

where we have used the identity (67) for ${}^iX_2$.

Thus combining (76) and (78) we have, from (73):

$$\frac{1}{(2\pi\sqrt{-1})}\int_{\bar{D}_i}\mathrm{Tr}(\widetilde{\phi}_1\wedge\widetilde{\phi}_2) = \langle {}^iA(u_0), [{}^iX_1, {}^iX_2]\rangle - \langle {}^iR_2, {}^i\widetilde{X}_1\rangle + \langle {}^iR_1, {}^i\widetilde{X}_2\rangle.$$

Now if we sum these terms for $i = 1,\dots,m$ we obtain on the left-hand side

$$\frac{1}{(2\pi\sqrt{-1})}\int_{\mathbb{P}^1}\mathrm{Tr}(\phi_1\wedge\phi_2)$$

from (72) and on the right-hand side, we obtain

$$\omega_{\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})}(W_1, W_2)$$

from (68), thereby completing the proof of Theorem 5.8 □

We can now deduce Theorem 5.6 that the lifted monodromy map

$$\bar{\nu} : \mathcal{M}^*_{\mathrm{ext}}(\mathbf{A}) \to \mathcal{A}_{\mathrm{ext}}(\mathbf{A})$$

is symplectic:

Proof (of Theorem 5.6). From Proposition 5.3 we know that $\bar{\nu}$ is holomorphic and that for any $u_0 \in \mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$ and tangent vector $W \in T_{u_0}\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$ we have

$$(d\bar{\nu})_{u_0}(W) = W(\alpha(u)) \in T_{\bar{\nu}(u_0)}\mathcal{A}_{\mathrm{ext}}(\mathbf{A}) \tag{79}$$

where $W(\alpha(u))$ is the matrix of singular one-forms on $\mathbb{P}^1$ obtained by partially differentiating $\alpha(u)$ along W at u_0. Thus from the definition of the (Atiyah-Bott type) symplectic form on $\mathcal{A}_{\text{ext}}(\mathbf{A})$ we see immediately that Theorem 5.8 gives

$$\begin{aligned}\omega_{\mathcal{M}^*_{\text{ext}}(\mathbf{A})}(W_1, W_2) =& \frac{1}{2\pi i}\int_{\mathbb{P}^1} \operatorname{Tr}(d\bar{\nu}(W_1) \wedge d\bar{\nu}(W_2))\\ =& \frac{1}{2\pi i}\omega_{\mathcal{A}_{\text{ext}}(\mathbf{A})}(d\bar{\nu}(W_1), d\bar{\nu}(W_2))\end{aligned}$$

for all $W_1, W_2 \in T_{u_0}\mathcal{M}^*_{\text{ext}}(\mathbf{A})$. Hence we are done:

$$\omega_{\mathcal{M}^*_{\text{ext}}(\mathbf{A})} = \frac{1}{2\pi i}(d\bar{\nu})^*(\omega_{\mathcal{A}_{\text{ext}}(\mathbf{A})}).$$

(The constant $2\pi i$ factor is harmless since we could rescale $\omega_{\mathcal{A}_{\text{ext}}(\mathbf{A})}$ throughout) $\square$

Finally Theorem 5.1 is now immediate since $\bar{\nu}$ is a slice for the $\mathcal{G}_1$ action on $\mathcal{A}_{\text{ext,fl}}(\mathbf{A})$ over $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$: the monodromy map *is* symplectic $\square$

CHAPTER 6

Isomonodromic Deformations

In this chapter we examine how the picture described so far (which was summarised at the beginning of Chapter 5) varies as we smoothly vary the positions of the poles and the choice of irregular types, of the meromorphic connections on $\mathbb{P}^1$. This leads naturally to the notion of 'isomonodromic deformations' of meromorphic connections with arbitrary order poles, in the sense of Jimbo, Miwa and Ueno [**60**]. The only data which remains fixed throughout this chapter is the rank n of the bundles, the number m of distinct poles and the multiplicities $k_1, \ldots, k_m$ of the poles.

Our main aim is to reveal the symplectic nature of the full family of Jimbo-Miwa-Ueno isomonodromic deformation equations (in Theorem 6.18). However we also make some effort to explain the fundamental results of [**60**] giving three different viewpoints on isomonodromic deformations: 1) in terms of families of meromorphic connections on $\mathbb{P}^1$ with constant monodromy data, 2) in terms of (full) *flat* meromorphic connections over families of $\mathbb{P}^1$'s, and 3) as solutions of explicit nonlinear differential equations (the 'deformation equations'). Due to this expository nature of part of this chapter we will give a brief overview below and point the expert reader to the new results.

In Section 1 we define (following [**60**]) a manifold X of deformation parameters encoding choices of pole positions and irregular types. Then we define two fibre bundles $\mathcal{M}^*_{\text{ext}}$ and M_{ext} over X, the fibres of which are the extended moduli spaces $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ and the extended monodromy manifolds $M_{\text{ext}}(\mathbf{A})$ respectively. We will call $\mathcal{M}^*_{\text{ext}}$ the *extended moduli bundle* and M_{ext} the *extended monodromy bundle*. M_{ext} appears in [**60**] (as the 'manifold of monodromy data') as does a concrete version of $\mathcal{M}^*_{\text{ext}}$ (as the 'manifold of singularity data'). The first new result we prove is then:

Theorem 6.4. *The extended moduli bundle $\mathcal{M}^*_{\text{ext}}$ is a* symplectic *fibre bundle.*

In other words the structure group of the fibration is a subgroup of the symplectic diffeomorphisms of a standard fibre; most of the work to prove this has already been done in Chapter 2.

The next step is to observe that there is a natural way to identify nearby fibres of the monodromy bundle: essentially just keep the monodromy data constant. Geometrically this amounts to a natural flat (Ehresmann) connection on the fibre bundle M_{ext}; we will call this the *isomonodromy connection*. Locally, points in M_{ext} on the same horizontal leaf of the isomonodromy connection have the same monodromy data.

Now the fibre-wise monodromy maps of Chapter 4 fit together to give a holomorphic bundle map $\nu : \mathcal{M}^*_{\text{ext}} \longrightarrow M_{\text{ext}}$. If we pull back the isomonodromy connection along ν to $\mathcal{M}^*_{\text{ext}}$ we obtain a flat Ehresmann connection on $\mathcal{M}^*_{\text{ext}}$; the *isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$*. (See Figure 3, p96.) The point is that the monodromy map ν is a highly nontrivial map, so the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$, when written out explicitly, gives a complicated system of nonlinear differential equations, such as the Schlesinger or Painlevé equations.

In Section 2 we explain the fundamental results of [**60**] relating horizontal sections of the isomonodromy connection to (full) flat meromorphic connections on vector bundles over families of $\mathbb{P}^1$'s. This gives both another geometrical point of view on isomonodromy as well as an explicit expression for the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$; the Jimbo-Miwa-Ueno deformation equations. (We will write down these equations, but will not use them.) At the end of Section 2 we will discuss the coordinate dependence of the isomonodromy connection/deformation equations and describe an intrinsic point of view.

Then in Section 3 we prove the main result of this thesis:

Theorem 6.18. *The isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$ is a* symplectic *connection.*

In other words, the local analytic diffeomorphisms induced between the fibres of $\mathcal{M}^*_{\text{ext}}$ by solving the deformation equations, are symplectic diffeomorphisms. This result is now a quite straightforward consequence of Theorem 5.8, in which we proved that the symplectic forms on the extended moduli spaces may be described in terms of integrals in the style of Atiyah-Bott. This generalises previous results for the Schlesinger equations (the simple pole case; see for example [**48**]) and for the six Painlevé equations (see [**89**], [**45**]). Note that the somewhat ad-hoc methods used for the six Painlevé equations are replaced by a more geometrical argument; we see intrinsically that the isomonodromy connection is symplectic *because* the monodromy map is symplectic.

1. The Isomonodromy Connection

1.1. Deformation Parameters. Firstly we discuss the notion of an 'irregular type' of a nice meromorphic connection on $\mathbb{P}^1$ at a pole.

Definition 6.1. Given a point $a \in \mathbb{P}^1$ and a positive integer k let $X_k(a)$ be the set of order k irregular types at a on $\mathbb{P}^1$.

Recall (from Remark 2.41, p41) that if we choose a local coordinate z on $\mathbb{P}^1$ vanishing at a then an irregular type at a is identified with the irregular part of a nice formal normal form:

$$dQ = A^0_k \frac{dz}{z^k} + \cdots + A^0_2 \frac{dz}{z^2}$$

for $n \times n$ diagonal matrices $A^0_k, \dots A^0_2$ which are subject only to the requirement that A^0_k has pairwise distinct eigenvalues. Thus the coordinate choice z allows us to identify $X_k(a)$ with

$$(\mathbb{C}^n \setminus \text{diagonals}) \times (\mathbb{C}^n)^{k-2} \ni (A^0_k, \dots, A_2).$$

If $k = 1$ (the logarithmic case) then $X_k(a) = \text{(point)}$.

Now let z be a standard coordinate on $\mathbb{P}^1$ (this will remain fixed throughout most of this chapter; see Remark 6.17 for a discussion of coordinate dependence).

Definition 6.2. The manifold $X = X_{\text{JMU}}$ of deformation parameters is

$$X := \left\{ (a_1 = \infty, a_2, \dots, a_m, {}^1A^0, \dots, {}^mA^0) \;\middle|\; a_i \in \mathbb{P}^1, {}^iA^0 \in X_{k_i}(a_i) \text{ and } a_i \neq a_j \text{ if } i \neq j \right\}.$$

Thus a point $(\mathbf{a}, \mathbf{A}) \in X$ specifies m distinct points of $\mathbb{P}^1$, the first of which is ∞, together with an irregular type at each marked point[1]. A notable feature of this choice of space of deformation parameters is that we have a canonical choice of local coordinate z_i on $\mathbb{P}^1$ vanishing at each a_i:

$$z_1 := 1/z, \qquad \text{and} \qquad z_i := z - a_i \quad \text{for } i \geq 2.$$

Thus using these local coordinates z_i, the set $X_{k_i}(a_i)$ of irregular types at a_i is canonically identified with $(\mathbb{C}^n \setminus \text{diagonals}) \times (\mathbb{C}^n)^{k_i-2}$ as above (unless $k_i = 1$ when it is a point). In this way the manifold X of deformation parameters is identified with

$$(\mathbb{C}^{m-1} \setminus \text{diagonals}) \times (\mathbb{C}^n \setminus \text{diagonals})^{m-l} \times (\mathbb{C}^n)^{l+\sum(k_i-2)}$$

where $l = \#\{i \mid k_i = 1\}$ is the number of simple poles. Observe that the fundamental group of X is a product of braid groups.

1.2. The Extended Moduli Bundle. As a set $\mathcal{M}^*_{\text{ext}}$ is defined simply by relaxing the conditions in the definition of the extended moduli spaces $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$:

DEFINITION 6.3. The *extended moduli bundle* $\mathcal{M}^*_{\text{ext}}$ is the set of isomorphism classes of data $(V, \nabla, \mathbf{g}, \mathbf{a})$ consisting of a nice meromorphic connection ∇ on a *trivial* rank n holomorphic vector bundle V over $\mathbb{P}^1$ (with compatible framings $\mathbf{g}$) such that ∇ has m poles on $\mathbb{P}^1$ which are labelled $a_1, \ldots, a_m$, the order of the pole at a_i is k_i and $a_1 = \infty$ in terms of the coordinate z.

Clearly we have a projection onto the manifold X of deformation parameters:

$$\mathcal{M}^*_{\text{ext}} \longrightarrow X \tag{80}$$

given by taking the pole positions and the irregular types (see Remark 2.41). The fibres of this projection are the corresponding extended moduli spaces $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ (see Definition 2.42).

Theorem 6.4. *The extended moduli bundle $\mathcal{M}^*_{\text{ext}}$ is a complex manifold. The projection above expresses it as a locally trivial* symplectic *fibre bundle over the manifold X of deformation parameters.*

(In particular $\mathcal{M}^*_{\text{ext}}$ has an intrinsic complex Poisson structure, its foliation by symplectic leaves is fibrating and the space of leaves is the complex manifold X of deformation parameters. The dimension of $\mathcal{M}^*_{\text{ext}}$ is $n^2(\sum k_i) - 2n^2 + m(n+1) - 1$.)

Proof. Recall from Corollary 2.44 that, using the local coordinates z_i, each extended moduli space $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ decouples canonically into a product of complex symplectic manifolds

$$\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \cong \left({}^1O_B \times \cdots \times {}^mO_B\right) \times M_{\mathbf{k}}$$

where ${}^iO_B$ is the B_{k_i} coadjoint orbit associated to ${}^iA^0$ and $M_{\mathbf{k}}$ is a complex symplectic manifold which only depends on the orders $\mathbf{k} = (k_1, \ldots, k_m)$ of the poles and the rank n. As we move around in X, ${}^iA^0$ varies and the coadjoint orbit ${}^iO_B$ moves around in $\mathfrak{b}^*_{k_i}$. Now the result of M.Vergne [**106**] we used earlier says more precisely that there are $2N := \dim({}^iO_B)$ functions x_j on $\mathfrak{b}^*_{k_i}$ which restrict to global Darboux coordinates on any of the orbits ${}^iO_B$ that arise as we vary ${}^iA^0$. (A key point here is that ${}^iO_B$ is always a

[1] A point $(\mathbf{a}, \mathbf{A})$ of X will sometimes be denoted as just $\mathbf{A}$ (since the irregular types are rooted at the a_i's) or by the letter t (since X will be the space of 'times' in the isomonodromic deformation equations).

generic coadjoint orbit in $\mathfrak{b}^*_{k_i}$.) Thus by choosing such functions x_j (for each i in turn) we obtain a (global) symplectic trivialisation of $\mathcal{M}^*_{\text{ext}}$ over X as required □

REMARK 6.5. That we obtain a *global* trivialisation arises here from the fact that we have a consistent choice, over all of X, of a local coordinate z_i at a_i on $\mathbb{P}^1$.

Thus decoupling the extended moduli spaces gives a symplectic trivialisation of $\mathcal{M}^*_{\text{ext}}$. However we will usually think of $\mathcal{M}^*_{\text{ext}}$ in a different way, as follows.

For each i we can use the coordinate z_i to associate to any point $t = (\mathbf{a}, \mathbf{A}) \in X$ in the space of deformation parameters, an extended orbit

$$\widetilde{O}_i(t) \subset GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_i};$$

—the extended orbit associated to the irregular type ${}^iA^0$ at a_i (see Definition 2.39). These extended orbits fit together to form a fibre bundle $\widetilde{\mathbf{O}}$ over X; the fibre over $t \in X$ is

$$\widetilde{\mathbf{O}}_t := \widetilde{O}_1(t) \times \cdots \times \widetilde{O}_m(t).$$

This bundle $\widetilde{\mathbf{O}}$ is a subbundle of the trivial fibre bundle over X with fibre

$$(GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_1}) \times \cdots \times (GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_m}). \tag{81}$$

Now in Theorem 2.43 we showed how, using the coordinates z_i, to identify the fibre $\mathcal{M}^*_{\text{ext}}(\mathbf{A})$ of $\mathcal{M}^*_{\text{ext}}$ over $t = (\mathbf{a}, \mathbf{A})$ with the symplectic quotient

$$\widetilde{O}_1(t) \times \cdots \times \widetilde{O}_m(t) /\!/ GL_n(\mathbb{C}) = \mu^{-1}(0)/GL_n(\mathbb{C}).$$

(Where the moment map μ for this $GL_n(\mathbb{C})$ action was the sum of the residues.) However we may identify this symplectic quotient with the submanifold of $\mu^{-1}(0)$ having ${}^1g_0 = 1$ (as in the proof of Proposition 2.52). More explicitly:

$$\mathcal{M}^*_{\text{ext}}(\mathbf{A}) \cong \left\{ (({}^1g_0, {}^1A), \ldots, ({}^mg_0, {}^mA)) \;\middle|\; ({}^ig_0, {}^iA) \in \widetilde{O}_i(t), {}^1g_0 = 1, \sum \mathrm{Res}_i({}^iA) = 0 \right\}.$$

In this way the bundle $\mathcal{M}^*_{\text{ext}}$ is also identified with a subbundle of the trivial bundle over X with fibre (81). (It is this concrete realisation of $\mathcal{M}^*_{\text{ext}}$ which appears as the 'manifold of singularity data' in [**60**].)

For example a section of the extended moduli bundle $\mathcal{M}^*_{\text{ext}}$ is identified with a map

$$s : X \longrightarrow (GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_1}) \times \cdots \times (GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_m})$$

$$t \longmapsto \left(({}^1g_0(t), {}^1A(t)), \ldots, ({}^mg_0(t), {}^mA(t))\right).$$

where ${}^ig_0 : X \to GL_n(\mathbb{C})$, ${}^iA : X \to \mathfrak{g}^*_{k_i}$ are maps such that for all t we have $({}^ig_0(t), {}^iA(t)) \in \widetilde{O}_i(t)$, ${}^1g_0(t) = 1$, and $\sum \mathrm{Res}_i({}^iA(t)) = 0$.

In fact the proof of Proposition 2.52 generalises directly to give a universal family of compatibly framed meromorphic connections on $\mathbb{P}^1$ over all of $\mathcal{M}^*_{\text{ext}}$. The embedding

$$\mathcal{M}^*_{\text{ext}} \hookrightarrow X \times GL_n(\mathbb{C})^m \times \mathfrak{g}^*_{k_1} \times \cdots \times \mathfrak{g}^*_{k_m}$$

gives the existence of a family over $\mathcal{M}^*_{\text{ext}}$, and the proof that it is universal is the same as in Proposition 2.52 (any family of data $(V, \nabla, \mathbf{g}, \mathbf{a})$ as specified in the definition of $\mathcal{M}^*_{\text{ext}}$ arises from a holomorphic map into $\mathcal{M}^*_{\text{ext}}$).

1.3. The Monodromy Bundle. Similarly we can fit the extended monodromy manifolds $M_{\text{ext}}(\mathbf{A})$ together to define the *extended monodromy bundle* M_{ext}, also fibring over X.

The local description of M_{ext} over a neighbourhood of some point $(\mathbf{a}, \mathbf{A}) \in X$ is as follows. The irregular types $\mathbf{A}$ determine the anti-Stokes directions at each a_i and we will choose a labelling of the anti-Stokes directions at each a_i as well as all the data going into the 'tentacles' described in Chapter 4. Thus in particular we have a point p_i in the last sector at a_i for all i, paths γ_i from p_1 to each p_i and a loop l_i based at p_i encircling a_i once in a positive direction.

Now observe that $(\mathbf{a}, \mathbf{A})$ may be varied slightly in X such that, for each i, none of the anti-Stokes directions at a_i cross over p_i: the same p_i, γ_i, l_i may be used for any point $(\mathbf{a}', \mathbf{A}')$ in some small open neighbourhood Δ of $(\mathbf{a}, \mathbf{A})$ in X. (See remarks p15.)

It follows that the monodromy manifolds $M_{\text{ext}}(\mathbf{A})$ and $M_{\text{ext}}(\mathbf{A}')$ are canonically isomorphic (for $\mathbf{A}' \in \Delta$): more concretely the monodromy data $(\mathbf{C}, \mathbf{S}, \mathbf{\Lambda})$ in $M_{\text{ext}}(\mathbf{A})$ is identified directly with monodromy data in $M_{\text{ext}}(\mathbf{A}')$; $\mathbf{C}, \mathbf{S}$ and $\mathbf{\Lambda}$ are held constant. (Here we remove the free $GL_n(\mathbb{C})$ action in the definition of $M_{\text{ext}}(\mathbf{A})$ by setting $C_1 = 1$.)

Thus there is a locally trivial fibre bundle M_{ext} over X with fibres $M_{\text{ext}}(\mathbf{A})$. The fact that we have canonical[2] isomorphisms between nearby fibres means we have a flat (complete) Ehresmann connection on the fibre bundle M_{ext}: we will call this the *isomonodromy connection* (since points of nearby fibres are on the same horizontal leaf of the isomonodromy connection if they have the same monodromy data).

REMARK 6.6. It is straightforward to check that a different initial choice of tentacles leads to the same notion of isomonodromy. This is clear if one thinks of the monodromy data $(\mathbf{C}, \mathbf{S})$ as a minimal way of encoding *all* possible 'ratios' of all the canonical fundamental solutions as first stated in Chapter 4: observe $(\mathbf{C}, \mathbf{S})$ stays constant iff all such ratios stay constant. (It is clear that keeping $\mathbf{\Lambda}$ constant is an intrinsic notion.)

1.4. Isomonodromic Deformations. The next step is to observe that the monodromy maps $\nu : \mathcal{M}^*_{\text{ext}}(\mathbf{A}) \to M_{\text{ext}}(\mathbf{A})$ (defined in Chapter 4) for each $\mathbf{A}$ fit together to define a holomorphic bundle map:

$$\nu : \mathcal{M}^*_{\text{ext}} \longrightarrow M_{\text{ext}}.$$

This will also be referred to as the monodromy map. (As for the fibre-wise monodromy maps, it is holomorphic since the canonical solutions depend holomorphically on any deformation parameters.)

DEFINITION 6.7. The *isomonodromy connection on* $\mathcal{M}^*_{\text{ext}}$ is the pull-back of the isomonodromy connection on M_{ext} along ν.

The situation is illustrated in Figure 3. The isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$ is a flat Ehresmann connection on the fibre bundle $\mathcal{M}^*_{\text{ext}}$ and is characterised by the fact that points in nearby fibres of $\mathcal{M}^*_{\text{ext}}$ are on the same horizontal leaf if the corresponding compatibly framed connections (on the trivial bundle over $\mathbb{P}^1$) have the same monodromy data.

[2]As always in this thesis 'canonical' means 'preferred' or 'without making extra choices' and is not synonymous to 'symplectic'.

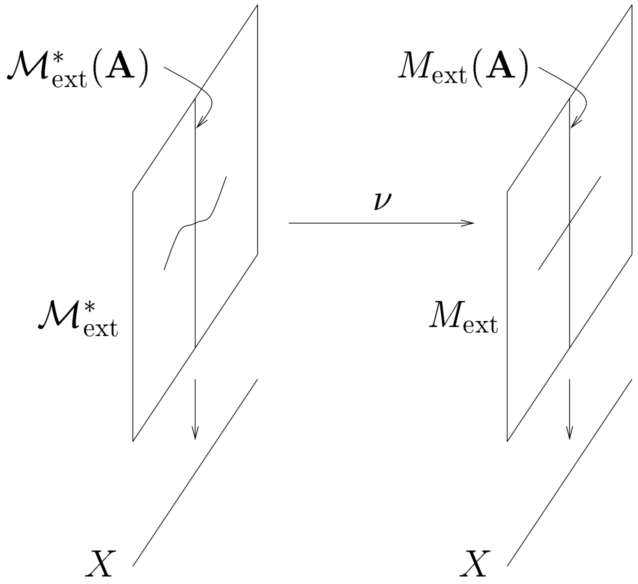

FIGURE 3. Isomonodromic Deformations

The point is that ν is a highly nonlinear map with respect to the explicit descriptions of the bundles $\mathcal{M}^*_{\rm ext}$ and $M_{\rm ext}$; whilst being trivial on $M_{\rm ext}$, the isomonodromy connection defines interesting nonlinear differential equations on $\mathcal{M}^*_{\rm ext}$, such as the Painlevé or Schlesinger equations (indicated by a wiggly line in the figure).

Equivalently one may view ν as a kind of nonlinear Fourier-Laplace transform (the 'monodromy transform'), converting hard nonlinear equations on the left-hand side into trivial equations on the right. (As in Corollary 4.13, the image of ν is dense in $M_{\rm ext}$ and ν is biholomorphic onto its image.)

2. Full Flat Connections and the Deformation Equations

Now we will explain the fundamental results of Jimbo, Miwa and Ueno [**60**] that give two other characterisations of the isomonodromy connection on $\mathcal{M}^*_{\rm ext}$; in terms of full[3] flat meromorphic connections, and in terms of explicit equations. To start with we will explain two quite general facts about full flat meromorphic connections; they will turn out to give us a good insight into isomonodromic deformations.

2.1. Induced Connections along Polar Divisors. Suppose ∇ is a (full) flat meromorphic connection on the trivial rank n vector bundle over a family $D_0\times\Delta$ of discs (where $D_0\subset\mathbb{P}^1$ is a disc and Δ is some space (polydisk) of parameters), such that the restriction

$$\nabla|_{D_0\times\{t\}} = d_{\mathbb{P}^1} - A(t)$$

of ∇ to any of the discs in the family is 'nice' and has only one pole (of order k_0 say), at some point $a_0(t)\in D_0$. Let $\Delta_0\cong\Delta$ be the polar divisor of ∇:

$$\Delta_0 := \{(a_0(t),t)\;\big|\; t\in\Delta\}\subset D_0\times\Delta$$

[3]The adjective 'full' will be used to emphasise the difference between a connection on a vector bundle over a product $\mathbb{P}^1\times\Delta$, and a 'vertical' connection over the product; a full connection differentiates in the Δ directions too.

which we assume is smooth and transverse to the fibres D_0. Suppose also that we have chosen some coordinate z on D_0 (so that $a_0(t)$ is identified with complex number via z) and let $z_0 := z - a_0(t)$. (Thus $z_0 : D_0 \times \Delta \to \mathbb{C}$ is a function such that the restriction of z_0 to any of the fibres $D_0 \times \{t\} \cong D_0$ is a coordinate on D_0 vanishing at $a_0(t)$.)

Our aim here is to explain how to associate to the data (∇, z_0), a flat connection ∇_0 on the trivial bundle over the polar divisor Δ_0. Note that this procedure *will* in general depend on the coordinate choice z_0. (See Remark 6.17 however.)

Firstly, ∇_0 is largely determined by requiring some compatible framing $g_0 : \Delta_0 \to GL_n(\mathbb{C})$ of ∇ along Δ_0 to be (the inverse matrix of) a fundamental solution of ∇_0. This does not determine ∇_0 completely because the compatible framing g_0 is not unique ($g_0(t)$ is only required to diagonalise the leading term of $A(t)$ at $a_0(t)$, so g_0 may be changed to $P \cdot h \cdot g_0$ for some map $h : \Delta_0 \to T$ and constant permutation matrix P).

The remaining indeterminacy in ∇_0 is removed as follows. Any choice g_0 of compatible framing extends in a unique way to a family of formal isomorphisms to formal normal forms. (This is just a relative version of Lemma 1.12, see also Appendix B.) That is we obtain a unique family of nice formal normal forms

$$A^0(t) = d_{\mathbb{P}^1}(Q(t)) + \Lambda(t)\frac{d_{\mathbb{P}^1} z_0}{z_0}$$

where $Q(t)$ is a diagonal matrix of degree $k_0 - 1$ polynomials in z_0^{-1} (with coefficients in $\mathcal{O}(\Delta_0)$) and $\Lambda : \Delta_0 \to \mathfrak{t}$ is the family of exponents of formal monodromy, together with a unique formal series

$$\widehat{g} \in GL_n\big(\mathbb{C}[\![z_0]\!] \otimes \mathcal{O}(\Delta_0)\big)$$

in z_0 with coefficients which are functions on Δ_0, such that for all $t \in \Delta_0$

$$\widehat{g}(t)|_{z_0=0} = g_0(t) \quad \text{and} \quad \widehat{g}(t)[A(t)]_{\mathbb{P}^1} = A^0(t). \tag{82}$$

Here $[\cdot]_{\mathbb{P}^1}$ means taking the gauge transformation in the $\mathbb{P}^1$ direction. We think of $\widehat{g}$ as a limit of local holomorphic functions on $D_0 \times \Delta$. Note that the coordinate choice z_0 has been used to determine the formal normal forms: A^0 has no nonsingular terms with respect to the coordinate z_0.

Next we define a way of extending the family of formal normal forms A^0 to a full connection on $D_0 \times \Delta$.

DEFINITION 6.8. The *full connection* associated to A^0, (or the *standard full connection near* Δ_0) is the diagonal meromorphic connection

$$d_{\mathbb{P}^1\times\Delta} - \widetilde{A}^0$$

on the trivial rank n vector bundle over $D_0 \times \Delta$ where

$$\widetilde{A}^0 := d_{\mathbb{P}^1\times\Delta}(Q(t)) + \Lambda(t)\frac{d_{\mathbb{P}^1\times\Delta}(z_0)}{z_0}.$$

Observe that this is a *flat* connection iff $\Lambda(t)$ is constant.

Now, the required restriction on the compatible framing g_0 arises as follows. Consider the full formal gauge transformation

$$\widehat{g}[\nabla]_{\mathbb{P}^1\times\Delta} \tag{83}$$

of the Laurent expansion of ∇ along Δ_0 (rather than just the gauge transformation in the $\mathbb{P}^1$ direction). By definition the vertical part ($\mathbb{P}^1$ component) of (83) will be the family

A^0 of formal normal forms. Moreover the fact that ∇ is flat restricts the rest of (83) quite substantially:

Lemma 6.9. *In the situation above, Λ is constant (independent of $t \in \Delta_0$), and*

$$\widehat{g}[\nabla]_{\mathbb{P}^1\times\Delta} = d_{\mathbb{P}^1\times\Delta} - (\widetilde{A}^0 + \pi^*(d_{\Delta_0}F))$$

for some diagonal matrix valued holomorphic function $F \in \mathrm{End}_n(\mathcal{O}(\Delta_0))$ which is unique upto the addition of a constant diagonal matrix, where $\pi : D_0 \times \Delta \to \Delta_0$ is the projection along the D_0 direction.

Proof. This is more or less a direct calculation, which we have put in Appendix E $\square$

It follows that if we replace the compatible framing g_0 by $e^{-F}g_0$ then, upto a constant matrix, g_0 is the unique compatible framing of ∇ along Δ_0 such that

$$\boxed{\widehat{g}[\nabla]_{\mathbb{P}^1\times\Delta} = d_{\mathbb{P}^1\times\Delta} - \widetilde{A}^0} \tag{84}$$

where $\widehat{g}$ is the formal series associated to g_0 satisfying (82). We will refer to such compatible framings g_0 as *good* compatible framings.

Definition 6.10. The *induced connection* ∇_0 along Δ_0 (associated to ∇ via z_0) is the unique connection on the trivial rank n vector bundle over Δ_0 such that the inverse matrix g_0^{-1} of any *good* compatible framing g_0 is a global fundamental solution. In other words

$$\nabla_0 := d_{\Delta_0} - \theta_0; \qquad \text{with} \qquad \theta_0 := -g_0^{-1}d_{\Delta_0}(g_0)$$

where the formal series $\widehat{g}$ associated to the compatible framing g_0 satisfies (84).

An equivalent, more direct, way to define the induced connection ∇_0 is as follows:

Lemma 6.11. *Given (∇, z_0) as above, let g_0 be* any *compatible framing of ∇ along Δ_0 and let A^0 be the associated family of formal normal forms*[4]. *Then the connection ∇_0 induced from ∇ along the polar divisor Δ_0 via z_0 is the restriction of the constant term in the Laurent expansion of $\nabla - (\widehat{g}^{-1}\cdot\widetilde{A}^0\cdot\widehat{g})$ along Δ_0 with respect to z_0:*

$$\nabla_0 = \mathrm{Const}_{z_0}(\nabla + \widehat{g}^{-1}\cdot\widetilde{A}^0\cdot\widehat{g})\big|_{\Delta_0}, \tag{85}$$

where $\widetilde{A}^0$ is the 'standard' full connection associated to A^0 in Definition 6.8 and $\widehat{g}$ is the formal series associated to g_0.

Proof. Observe that the right-hand side of (85) is independent of the choice of compatible framing g_0. Then choose g_0 to be a *good* compatible framing and observe that the equation (84) implies the result. In more detail, since ∇ is on the trivial bundle, we may write $\nabla = d_{\mathbb{P}^1\times\Delta} - \widetilde{A}$ for some meromorphic matrix $\widetilde{A}$ of one forms. Then (84) is equivalent to the equality

$$\widetilde{A} - (\widehat{g}^{-1}\cdot\widetilde{A}^0\cdot\widehat{g}) = -\widehat{g}^{-1}d_{\mathbb{P}^1\times\Delta}(\widehat{g})$$

of formal series in z_0 (with coefficients which are matrices of one-forms on $D_0\times\Delta$). The constant term on the right-hand side restricts to $\theta_0 = -g_0^{-1}d_{\Delta_0}(g_0)$ on Δ_0 and the left-hand side appears in the formula (85) $\square$

[4]Immediately (∇, z_0) determine a Sym_n orbit of formal normal forms; the choice of g_0 picks out some A^0 in this Sym_n orbit.

In particular the formula (85) makes sense in more generality: the original connection ∇ does not need to be flat in order to define ∇_0 using (85). (Although in general then ∇_0 will also not be flat.)

The final description we require of ∇_0 is given by pulling it down to the base Δ. (This will be useful to write down explicit isomonodromic deformation equations.) Recall that the induced connection $\nabla_0 = d_{\Delta_0} - \theta_0$ is defined on the submanifold Δ_0 of $D_0 \times \Delta$. (Note that d_{Δ_0} is the exterior derivative on Δ_0.) The projection $D_0 \times \Delta \to \Delta$ onto the second factor restricts to an isomorphism between Δ_0 and Δ. We will give a formula for the connection on Δ corresponding to ∇_0 on Δ_0 under this isomorphism. In other words, if $\varphi : \Delta \to \Delta_0$ is the inverse of the isomorphism coming from the projection then we want an expression for a matrix Θ_0 of one-forms on Δ such that $\nabla_0 = d_{\Delta_0} - \theta_0$ pulls back to $d_\Delta - \Theta_0$ along φ; i.e.

$$\Theta_0 = \varphi^*(\theta_0).$$

(Here d_Δ is the exterior derivative on Δ; below we will also use d_Δ to denote the Δ component of the (full) exterior derivative on product $D_0 \times \Delta$.) If we write $\nabla = d_{D_0\times\Delta} - \widetilde{A}$ and denote the D_0 and Δ components of $\widetilde{A}$ as A and Ω respectively (so $\widetilde{A} = A + \Omega$), then we find:

Lemma 6.12. [**60**]. *The induced connection $\varphi^*(\nabla_0) = d_\Delta - \Theta_0$ on the base Δ is given by the formula:*

$$\Theta_0 = g_0^{-1}(d_\Delta a_0)g_1 + \mathrm{Const}_{z_0}(\widehat{g}^{-1} \cdot \widetilde{A}^0_\Delta \cdot \widehat{g}) - \mathrm{Const}_{z_0}(\Omega) \tag{86}$$

where $\widehat{g} = g_0 + g_1 \cdot z_0 + g_2 \cdot z_0^2 + \cdots \in GL_n\big(\mathbb{C}[\![z_0]\!] \otimes \mathcal{O}(\Delta_0)\big)$ is the formal series associated to any compatible framing g_0, a_0 is regarded as a function on Δ via the coordinate z and the matrices g_i of functions on Δ_0 are pulled back to Δ along φ.

Proof. This is a straightforward calculation, once we check that the formula is independent of the choice of compatible framing g_0 and then take g_0 to be a 'good' compatible framing. We give more details in Appendix E □

2.2. Canonical Solutions are Horizontal. The other basic fact about full flat meromorphic connections that we need is that the canonical solutions at a pole (defined a priori as fundamental solutions only in the vertical direction) vary in the Δ directions to give full fundamental solutions (i.e. they are fundamental solutions in the Δ directions too). More precisely we have

Proposition 6.13. [**60**]. *Let ∇ be a full flat connection over $D_0 \times \Delta$ (as in the previous section), let g_0 be a 'good' compatible framing of ∇ along Δ_0 (with respect to z_0; see p98) and let $\widehat{g}$ be the corresponding formal series. Fix any point $t_0 \in \Delta$, choose a labelling of the sectors between the anti-Stokes directions at $a_0(t_0) \in D_0 \times \{t_0\}$, and choose $\log(z_0)$ branches on $D_0 \times \{t_0\}$. Let Δ' be a neighbourhood of $t_0 \in \Delta$ such that the last sector at $a_0(t_0)$ deforms into a unique sector at $a_0(t)$ for all $t \in \Delta'$ (the last sector at $a_0(t)$).*

Then the canonical fundamental solution (from Definition 1.29)

$$\Phi_0 := \Sigma_0(\widehat{g}^{-1})z_0^\Lambda e^Q$$

of $\nabla|_{\mathrm{Vert}}$ on the last sector at $a_0(t) \in D_0 \times \{t\}$ varies holomorphically with $t \in \Delta'$ and $\Phi_0(z,t)$ is a local fundamental solution of the original full connection ∇.

(The analogous statement holds on all the other sectors: just relabel.)

Proof. Morally this is the converse part of Theorem 3.3 of [**60**], and we will give the slight modification required here, in Appendix E □

2.3. Between Flat Connections and Isomonodromic Families. Now we come to the key result of Jimbo, Miwa and Ueno, relating horizontal sections of the isomonodromy connection (i.e. families of connections on $\mathbb{P}^1$ with the same monodromy) to *flat* (full) meromorphic connections over families of $\mathbb{P}^1$'s.

Theorem 6.14. [**60**]. *Let $\Delta \subset X$ be a simply connected open subset of the space X of deformation parameters, and let $\Delta_i \subset \mathbb{P}^1 \times \Delta$ be the ith polar divisor specified by $\Delta \subset X$.*
1) Suppose we have a flat meromorphic connection ∇ on the trivial rank n vector bundle over $\mathbb{P}^1 \times \Delta$, having pole positions and irregular types as specified by $\Delta \subset X$, together with an arbitrary compatible framing ${}^ig_0 : \Delta_i \to GL_n(\mathbb{C})$ of ∇ along Δ_i for each $i = 1, \dots, m$. Then, by solving only linear equations, we can construct a section

$$s : \Delta \longrightarrow \mathcal{M}^*_{\text{ext}}$$

of the bundle of extended moduli spaces over Δ, which is horizontal for the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$.
2) Conversely, given a horizontal section $s : \Delta \to \mathcal{M}^*_{\text{ext}}$ of the isomonodromy connection then the corresponding vertical connection $\nabla|_{\text{Vert}}$ over $\mathbb{P}^1 \times \Delta$ (obtained by pulling back the universal connection over $\mathbb{P}^1 \times \mathcal{M}^*_{\text{ext}}$) is the vertical part of a full flat meromorphic connection ∇ on the trivial bundle over $\mathbb{P}^1 \times \Delta$. Moreover, ∇ may be obtained algebraically from s, and the family of compatible framings ${}^ig_0$ are 'good' compatible framings of ∇ (with respect to the coordinates z_i).

Proof. 1) Since we are working over the space X of deformation parameters we automatically have local coordinates z_i vanishing at a_i. Let ∇_i be the induced flat connection on Δ_i defined algebraically from ∇ and ${}^ig_0$ using z_i in Lemma 6.11. Thus if we choose a base point $t_0 \in \Delta$ then we can integrate the matrix ${}^ig_0(t_0) \in GL_n(\mathbb{C})$ over Δ_i using ∇_i (i.e. solve linear equations). Replace the original compatible framing ${}^ig_0$ by the fundamental solution obtained in this way. Thus ${}^ig_0 \in GL_n(C^\infty(\Delta_i))$ is now a fundamental solution of ∇_i which is a compatible framing of ∇ at $(a_i(t_0), t_0) \in \mathbb{P}^1 \times \Delta$. From the description in Section 2.1 we see that ${}^ig_0$ is in fact still a compatible framing of ∇ along all of Δ_i. (It is now a 'good' compatible framing.) Repeat this procedure for each $i = 1, \dots, m$.

Now define ${}^iA(t) = \mathrm{PP}_i(A(t)) \in \mathfrak{g}^*_{k_i}$ to be the polar part of $A(t)$ with respect to the coordinate z_i, where $\nabla|_{\text{Vert}} = d_{\mathbb{P}^1} - A$ is the vertical part of ∇ (it is on the trivial bundle by assumption). Then we claim that the map

$$s : \Delta \longrightarrow (GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_1}) \times \cdots \times (GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_m})$$

$$t \longmapsto \big(({}^1g_0(t), {}^1A(t)), \dots, ({}^mg_0(t), {}^mA(t))\big)$$

is a horizontal section for the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$ (using the identification described in Section 1.2 above).

In other words we need to show that the monodromy data of the family $\nabla|_{\text{Vert}}$ of compatibly framed meromorphic connections on $\mathbb{P}^1$ is t-independent. But this now follows easily from Proposition 6.13: if Φ, Φ' are two canonical solutions to $\nabla|_{\mathbb{P}^1 \times \{t_0\}}$ on some sector at some pole respectively, and we have chosen a path between these two sectors, then the corresponding monodromy matrix is

$$\Phi^{-1}\Phi', \tag{87}$$

where Φ and Φ' are extended along the chosen path as fundamental solutions of $\nabla|_{\mathbb{P}^1\times\{t_0\}}$. (Recall that all the Stokes matrices and the connection matrices arose in this way for appropriate choices of sectors and paths.) This expression (87) is independent of z since Φ and Φ' are fundamental solutions of $\nabla|_{\mathbb{P}^1\times\{t_0\}}$; (87) is a fundamental solution of $\mathrm{Hom}(0,0)$ (where 0 denotes the trivial rank n system on $\mathbb{P}^1\times\{t_0\}$), so is constant in the $\mathbb{P}^1$ direction.

But Proposition 6.13 says that Φ and Φ' are also both fundamental solutions of the full connection ∇ when varied in the Δ directions too (if we use the good compatible framings). Thus the monodromy matrix (87) is also constant in the Δ directions (whenever Φ and Φ' are well defined). Explicitly

$$d(\Phi^{-1}\Phi') = -\Phi^{-1}(d\Phi)\Phi^{-1}\Phi' + \Phi^{-1}d\Phi' = -\Phi^{-1}\widetilde{A}\Phi' + \Phi^{-1}\widetilde{A}\Phi' = 0$$

where $\nabla = d - \widetilde{A}$ and d denotes the full exterior derivative $d_{\mathbb{P}^1\times\Delta}$.

2) Now for the converse, suppose that we have a horizontal section $s:\Delta\to\mathcal{M}^*_{\mathrm{ext}}$ of the isomonodromy connection over $\Delta\subset X$. Using the coordinate choices z_i we identify $\mathcal{M}^*_{\mathrm{ext}}|_\Delta$ with a subbundle of the product

$$\Delta\times(GL_n(\mathbb{C})\times\mathfrak{g}^*_{k_1})\times\cdots\times(GL_n(\mathbb{C})\times\mathfrak{g}^*_{k_m}),$$

as in Section 1.2, so that s is written explicitly in the form

$$t\longmapsto\big(({}^1g_0(t),{}^1A(t)),\ldots,({}^mg_0(t),{}^mA(t))\big),$$

where ${}^1g_0(t)=1$. The corresponding universal family of meromorphic connections on $\mathbb{P}^1\times\Delta$ is then

$$\nabla_{\mathrm{Vert}} := d_{\mathbb{P}^1} - A(t)$$

where, for each $t\in\Delta$, $A(t)$ is the matrix of meromorphic one-forms on $\mathbb{P}^1$ determined explicitly by requiring the polar parts of $A(t)$ to be as specified by s:

$$\mathrm{PP}_i(A(t)) = {}^iA(t)\in\mathfrak{g}^*_{k_i}.$$

(Such $A(t)$ exists since $\sum\mathrm{Res}_i({}^iA(t))=0$.)

Now s being horizontal for the isomonodromy connection means precisely that this family of compatibly framed meromorphic connections has constant monodromy data. We will now show geometrically how this implies that ∇_{Vert} is the vertical part of a *flat* full connection on the product $\mathbb{P}^1\times\Delta$.

Given any base-point $t_0\in\Delta$, consider the meromorphic connection $\nabla_{\mathrm{Vert}}|_{\mathbb{P}^1\times\{t_0\}}$ on the projective line $\mathbb{P}^1\times\{t_0\}$. Choose a pole of $\nabla_{\mathrm{Vert}}|_{\mathbb{P}^1\times\{t_0\}}$ and a sector between two anti-Stokes directions at this pole. Let $\Phi(z,t_0)$ be the canonical solution of $\nabla_{\mathrm{Vert}}|_{\mathbb{P}^1\times\{t_0\}}$ on this sector (using the given compatible framing and any logarithm choice). Now let Δ' be a small neighbourhood of $t_0\in\Delta$ such that the original sector deforms into a unique sector in $\mathbb{P}^1\times\{t\}$ for all $t\in\Delta'$ (i.e. such that the angle subtended by the chosen sector does not go to zero). Then if we define, for each $t\in\Delta'$, $\Phi(z,t)$ to be the canonical solution of $\nabla_{\mathrm{Vert}}|_{\mathbb{P}^1\times\{t\}}$ on the deformed sector, we obtain a matrix valued holomorphic function $\Phi(z,t)$ on the family of sectors (an open subset of $\mathbb{P}^1\times\Delta'$). (It is holomorphic with respect to t by results of Sibuya and Hsieh [**51, 97, 98**].)

We give a local definition of the full connection ∇ by saying that Φ is a fundamental solution of it:

$$\nabla(\Phi) = 0 \qquad \text{so} \qquad \nabla := d_{\mathbb{P}^1\times\Delta} - (d_{\mathbb{P}^1\times\Delta}\Phi)\Phi^{-1}.$$

Such ∇ is clearly flat. The fact that the monodromy data of ∇_{Vert} is constant implies that we can globalise this local definition of ∇ over all of $(\mathbb{P}^1 \times \Delta) \setminus (\Delta_1 \cup \cdots \cup \Delta_m)$, and that this definition is independent of the original choice of sector used to define Φ. To see this, suppose Φ' is another solution (defined either by starting at a different sector or by continuing Φ around a nontrivial loop). Then the matrix $\Phi^{-1}\Phi'$ is constant with respect to both z and t (this is just the assumption that the monodromy data of ∇_{Vert} is constant). Hence Φ and Φ' define the same connection ∇. Explicitly:

$$0 = d(\Phi^{-1}\Phi') = -\Phi^{-1}d\Phi\Phi^{-1}\Phi' + \Phi^{-1}d\Phi$$

so $(d\Phi)\Phi^{-1} = (d\Phi')(\Phi')^{-1}$ where $d = d_{\mathbb{P}^1\times\Delta}$ is the full exterior derivative.

Thus geometrically we see how a full flat connection ∇ arises from the horizontal section s of the isomonodromy connection. The final step is to see how, by using the fact that the coordinates z_i are global (i.e. they are meromorphic function on $\mathbb{P}^1$), we can write down an algebraic formula to give ∇ in terms of s.

Since ∇ is on the trivial bundle we have $\nabla = d_{\mathbb{P}^1\times\Delta} - \widetilde{A}$ for some meromorphic matrix $\widetilde{A}$ of one-forms on $\mathbb{P}^1 \times \Delta$. By decomposing $\widetilde{A}$ into components along $\mathbb{P}^1$ (its dz component) and along Δ we have

$$\widetilde{A} = A + \Omega$$

where A is the vertical part of $\widetilde{A}$ (so that $\nabla|_{\mathrm{Vert}} = \nabla_{\mathrm{Vert}} = d_{\mathbb{P}^1} - A$) and Ω is the Δ component[5]. Using the definition of ∇ given above we see that

$$\Omega = (d_\Delta\Phi)\Phi^{-1} \tag{88}$$

for any local fundamental solution Φ (as defined above), where d_Δ is the Δ component of the exterior derivative on[6] $\mathbb{P}^1 \times \Delta$. For fixed $t \in \Delta$, suppose Φ was first defined on the jth sector at a_i in $\mathbb{P}^1 \times \{t\}$ (with respect to some labelling choice). Then on this sector the canonical solution $\Phi(z,t)$ is defined as

$$\Phi(z,t) := {}^i\Sigma_j({}^i\widehat{g}^{-1})z_i^{{}^i\Lambda}e^{{}^iQ} \tag{89}$$

where ${}^i\widehat{g}$ is the formal series associated to the ith compatible framing ${}^ig_0(t)$. Now recall (from Proposition 1.24) that ${}^i\Sigma_j({}^i\widehat{g}^{-1})$ has asymptotic expansion ${}^i\widehat{g}^{-1}$ on a sectorial neighbourhood of the jth sector at a_i in $\mathbb{P}^1 \times \{t\}$. In fact (see [**98**]) this asymptotic expansion is valid *uniformly in* t, if we shrink Δ' sufficiently. Moreover for uniform asymptotic expansions we may reorder the two procedures of differentiation (with respect to the parameter t) and of taking the asymptotic expansion (see [**107**]). What this means is that we can use the formulae (88) and (89) to compute the asymptotic expansion of Ω. We deduce immediately that in a sectorial neighbourhood of the jth sector at a_i in $\mathbb{P}^1 \times \{t\}$ the asymptotic expansion of Ω at a_i is

$$Æ(\Omega) = \left({}^i\widehat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\widehat{g}\right) - {}^i\widehat{g}^{-1} \cdot d_\Delta({}^i\widehat{g}) \tag{90}$$

[5]Beware that in general each coordinate z_i defines a different description of the product $\mathbb{P}^1 \times \Delta$; $\mathbb{P}^1$ is always the vertical direction, but one could define the horizontal leaves to be the sets on which z_i =(constant). The definition of Ω uses the obvious splitting of the product $\mathbb{P}^1 \times \Delta$ such that z (or $z_1 = 1/z$) is constant on the horizontal leaves.

[6]do not confuse d_Δ with d_{Δ_i}, which is the usual exterior derivative on the manifold $\Delta_i \subset \mathbb{P}^1 \times \Delta$.

where ${}^i\widetilde{A}^0_\Delta$ is the Δ component of the full connection associated to the formal normal form ${}^iA^0$ in Definition 6.8; it is such that $d_\Delta(z_i^{i\Lambda}e^{iQ}) = {}^i\widetilde{A}^0_\Delta \cdot z_i^{i\Lambda}e^{iQ}$.

Now observe that we obtain the same conclusion (90) if we started on *any* sector at a_i. In particular Ω has the same asymptotic expansion on any sector at a_i and so is *meromorphic* and has *Laurent* expansion given by the right-hand side of (90):

$$L_i(\Omega) = \left({}^i\widehat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\widehat{g}\right) - {}^i\widehat{g}^{-1} \cdot d_\Delta({}^i\widehat{g}). \tag{91}$$

There are two conclusions to be drawn from this formula. Firstly observe that (91) is the Δ component of the key equation (84) characterising 'good' compatible framings. This implies that the compatible framing ${}^ig_0$ of ∇ is indeed good (the vertical part of (84) holds by definition of ${}^i\widehat{g}$).

Secondly the term ${}^i\widehat{g}^{-1} \cdot d_\Delta({}^i\widehat{g})$ in (91) involving the derivative of ${}^i\widehat{g}$ is clearly nonsingular at a_i, and so the polar part of Ω at a_i (with respect to z_i) is given entirely algebraically by the formula

$$\mathrm{PP}_i(\Omega) = \mathrm{PP}_i({}^i\widehat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\widehat{g}). \tag{92}$$

Thus Ω is determined upto a constant by these polar parts over each projective line $\mathbb{P}^1 \times \{t\}$, since it is meromorphic. This last constant is also determined algebraically since we are working in the trivialisation determined by the compatible framing along Δ_1 (i.e. ${}^1g_0 \equiv 1$): the induced connection along Δ_1 is the trivial connection and so, from Lemma 6.11, we deduce the value of the constant term of Ω along Δ_1 with respect to the coordinate z_1:

$$\mathrm{Const}_{z_1}(\Omega) = \mathrm{Const}_{z_1}({}^1\widehat{g}^{-1} \cdot {}^1\widetilde{A}^0_\Delta \cdot {}^1\widehat{g}).$$

Now using the fact that the coordinates z_i are global on $\mathbb{P}^1$ we can put together all of these polar parts at each a_i and the constant term at a_1, to give an explicit formula for Ω:

$$\Omega = \mathrm{Const}_{z_1}({}^1\widehat{g}^{-1} \cdot {}^1\widetilde{A}^0_\Delta \cdot {}^1\widehat{g}) + \sum_{i=1}^m \mathrm{PP}_{z_i}\left({}^i\widehat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\widehat{g}\right) \tag{93}$$

where PP_{z_i} is defined to be the map that removes all the nonsingular terms of a formal Laurent series in z_i and leaves the rest[7]. Clearly the right-hand side of (93) has the desired polar parts at each a_i, and using the specific choices of coordinates we have made ($z_i = z - a_i$ except $z_1 = 1/z$), one may check that it also has the desired constant term at a_1 □

2.4. The Deformation Equations. Now we can write down quite explicitly the nonlinear equations for horizontal sections of the isomonodromy connection on $\mathcal{M}^*_{\mathrm{ext}}$; the Jimbo-Miwa-Ueno deformation equations. Although we will not use these equations at all, preferring a more geometrical point of view, it is remarkable that one can write down such a vast family of equations, each of which has the Painlevé property.

Suppose we have a local horizontal section

$$s : \Delta \longrightarrow (GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_1}) \times \cdots \times (GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_m})$$

[7]This differs slightly from the map PP_i to $\mathfrak{g}^*_{k_i}$ previously defined. Here we do not replace the coordinate z_i with the symbol ζ to end up in $\mathfrak{g}^*_{k_i}$, but use the fact that z_i is globally defined to end up with a matrix of meromorphic one-forms.

of the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$ over Δ as above (where we still use the convention ${}^1g_0 = 1$). Then the flatness of the corresponding full connection ∇ over $\Delta \times \mathbb{P}^1$ (from Theorem 6.14) translates into the two equations:

$$d_\Delta \Omega = \Omega \wedge \Omega \tag{94}$$

and

$$d_\Delta A = -d_{\mathbb{P}^1}\Omega + A \wedge \Omega + \Omega \wedge A. \tag{95}$$

These are the Δ-Δ component and the z-Δ component of the equation $d\widetilde{A} = \widetilde{A} \wedge \widetilde{A}$ where $\nabla = d - \widetilde{A}$, $\widetilde{A} = A + \Omega$ and d is the full exterior derivative on $\mathbb{P}^1 \times \Delta$. (The z-z component is vacuous for dimensional reasons; meromorphic connections on curves are flat.) Also the 'goodness' of the compatible framings ${}^ig_0$ implies that

$$d_\Delta({}^ig_0) = -({}^ig_0)\Theta_i. \tag{96}$$

Note that Equation (94) says that Ω is a family of flat connections on Δ depending rationally on the 'spectral parameter' z; a situation that often arises in soliton theory. Our interest here is in the other two equations (95) and (96) (for each i) however. Recall that Ω and each Θ_i are determined algebraically from the section s. Explicitly we derived the formulae

$$\Omega = \text{Const}_{z_1}({}^1\widehat{g}^{-1} \cdot {}^1\widetilde{A}^0_\Delta \cdot {}^1\widehat{g}) + \sum_{i=1}^m \text{PP}_{z_i}({}^i\widehat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\widehat{g}) \tag{97}$$

and

$$\Theta_i = {}^ig_0^{-1}(d_\Delta a_i){}^ig_1 + \text{Const}_{z_i}({}^i\widehat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\widehat{g}) - \text{Const}_{z_i}(\Omega) \tag{98}$$

where ${}^i\widehat{g}$ is the formal series determined algebraically from ${}^ig_0$ (using the procedure in Appendix B) and ${}^i\widetilde{A}^0$ is the full connection associated to the family of formal normal forms parameterised by Δ (from Definition 6.8). Note that these formulae for Ω and Θ_i make sense for an arbitrary section of the bundle $\mathcal{M}^*_{\text{ext}}$ so that the equations (95) and (96) amount to a coupled system of nonlinear equations for horizontal sections $s = (\mathbf{g}, {}^1A, \ldots, {}^mA)$ of the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$:

Definition 6.15. The *Jimbo-Miwa-Ueno (isomonodromic) deformation equations* are the nonlinear (algebraic) differential equations (95) and (96) for sections $s : X \to \mathcal{M}^*_{\text{ext}}$ of the extended moduli bundle, where Ω and Θ_i are defined by (97) and (98) respectively.

Note that the rank n of the vector bundles, the number m of distinct poles on $\mathbb{P}^1$ and the multiplicities $k_1, \ldots, k_m$ of the poles are all arbitrary positive integers here. The Frobenius integrability of the deformation equations was proved directly in [**60**]. Slightly later Miwa [**84**] proved that any member of the family of deformation equations has the Painlevé property: any local solution of the deformation equations extends to a (multi-valued) meromorphic solution over the space

$$X \cong (\mathbb{C}^{m-1} \setminus \text{diagonals}) \times (\mathbb{C}^n \setminus \text{diagonals})^{m-l} \times (\mathbb{C}^n)^{l+\sum(k_i-2)}$$

of deformation parameters (where $l = \#\{i \mid k_i = 1\}$ is the number of simple poles). The fixed critical varieties (where solutions have essential singularities or branch points) are the diagonals that have been removed here: i.e. where two poles coalesce or where two eigenvalues of a leading term at an irregular singularity come together.

A number of examples of the deformation equations are given in the papers [**58, 60**] of Jimbo et al. In particular the cases of the Schlesinger equations and the six Painlevé equations are explained. Rather than study specific examples, our aim now is more to study the geometry of the general situation. In the next section we will show that the flows of the isomonodromy equations preserve the symplectic structures we have defined on the fibres of the bundle $\mathcal{M}^*_{\text{ext}}$ of extended moduli spaces; isomonodromic deformations are symplectic.

REMARK 6.16. Geometrically, to return to the first viewpoint on isomonodromic deformations (illustrated in Figure 3), the poles of solutions to the deformation equations over X arise since the image of $\mathcal{M}^*_{\text{ext}}$ in the monodromy bundle M_{ext} is only a dense open subset (it is the complement of a divisor). The isomonodromy connection on M_{ext} is complete and the poles occur when a solution leaf in M_{ext} intersects a divisor: the solution goes off the edge of $\nu(\mathcal{M}^*_{\text{ext}}) \subset M_{\text{ext}}$ but is perfectly well behaved in terms of M_{ext}. We are just keeping the monodromy data constant.

In terms of full flat connections the poles arise as follows. Each local solution of the deformation equations corresponds to a (full) flat meromorphic connection on a rank n vector bundle V over a family of $\mathbb{P}^1$'s parameterised by $\widetilde{X}$ (the universal cover of X). (See the papers [**71, 72**] of B.Malgrange.) The bundle V is such that it restricts to a degree zero bundle over each $\mathbb{P}^1$; the poles of the solution occur at the points $t \in \widetilde{X}$ where the restriction $V|_{\mathbb{P}^1\times\{t\}}$ of V to the corresponding $\mathbb{P}^1$ is *nontrivial*. This fits in with the above description of the pole positions since by Theorem 4.10 the monodromy data corresponds to compatibly framed meromorphic connections on arbitrary degree zero bundles; the divisor $M_{\text{ext}} \setminus \nu(\mathcal{M}^*_{\text{ext}})$ contains those on nontrivial degree zero bundles.

REMARK 6.17. (Intrinsic Version). We will briefly discuss the coordinate dependence of the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$. Recall that a global coordinate z was fixed on $\mathbb{P}^1$ at the start, and that one of the poles was required to be at ∞. Suppose now instead that the coordinate z has not been chosen and we just have a fixed, abstract copy of $\mathbb{P}^1$ (e.g. the space of one dimensional subspaces of some fixed two dimensional complex vector space.) Then a natural manifold of deformation parameters is

$$\bar{X} := \left\{(a_1, a_2, \ldots, a_m, {}^1A^0, \ldots, {}^mA^0) \mid a_i \in \mathbb{P}^1, {}^iA^0 \in X_{k_i}(a_i) \text{ and } a_i \neq a_j \text{ if } i \neq j\right\}.$$

A point $(\mathbf{a}, \mathbf{A}) \in \bar{X}$ specifies m arbitrary distinct points of $\mathbb{P}^1$ together with an (abstract) irregular type at each marked point (see Definition 6.1 and subsequent comments). An extended monodromy bundle M_{ext} may be constructed over $\bar{X}$ exactly as above and it will still have an isomonodromy connection on it. Also an extended moduli bundle $\mathcal{M}^*_{\text{ext}}$ may be constructed over $\bar{X}$ more or less as above (construct it locally over $\bar{X}$ by choosing local coordinates on $\mathbb{P}^1$, and then glue these pieces together). $\mathcal{M}^*_{\text{ext}}$ is still a symplectic fibre bundle. However, now the monodromy map is *not* well defined; its definition depends on (local) coordinate choices at each pole and so we cannot define the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$.

There are three ways around this problem. 1) Use a slightly smaller space of deformation parameters and specify a coordinate choice at each pole throughout (as has been done above, following [**60**]), 2) Use a larger space of deformation parameters encoding all possible choices of coordinates. Since we are on $\mathbb{P}^1$ we could require these to be global coordinates and still obtain a finite dimensional space of deformation parameters. But this is in some sense unnatural because the monodromy map only depends on local coordinate choices; this idea naturally leads to some infinite dimensional space of deformations parameters (maps into which would correspond to families of m-pointed $\mathbb{P}^1$'s together with irregular types and local coordinate choices). 3) Observe that the problem is entirely to do with the compatible framings; on quotienting by the torus actions (which change the choices of compatible framings) all the different possible definitions of the monodromy map coincide. In particular if we choose an exponent of formal monodromy ${}^i\Lambda \in \mathfrak{t}$ for each pole and perform the fibre-wise torus symplectic quotients, we get bundles $\mathcal{M}^*$ and M over $\bar{X}$ (with fibres of the form $\mathcal{M}^*(\mathbf{A})$ and $M(\mathbf{A})$ respectively). Then the induced monodromy map

$$\nu : \mathcal{M}^* \to M$$

is well defined. (We should restrict to the dense open subsets of 'stable' points of the moduli bundle $\mathcal{M}^*$ and the monodromy bundle M for them to be manifolds.) Moreover this non-extended picture is invariant under the natural action of the Möbius group $PSL_2(\mathbb{C})$ of automorphisms of $\mathbb{P}^1$. Tangents to the orbits of the $PSL_2(\mathbb{C})$ action on $\bar{X}$ correspond to 'trivial' isomonodromic deformations; we should quotient $\bar{X}$ by $PSL_2(\mathbb{C})$ to get only nontrivial deformations. Also note that the difference between finding an isomonodromic section of the extended or non-extended moduli bundles is essentially trivial; it is just a question of choosing a 'good' compatible framing (which just involves diagonalising some matrices and integrating a linear connection). Owing to the previous two remarks we see that there is no loss of generality in using the Jimbo-Miwa-Ueno space of deformation parameters $X = X_{\mathrm{JMU}}$. Moreover in doing so we get *explicit* deformation equations globally.

3. Isomonodromic Deformations are Symplectic

Now we will prove the main result of this thesis:

Theorem 6.18. *The isomonodromy connection on the extended moduli bundle $\mathcal{M}^*_{\mathrm{ext}} \to X$ is a symplectic connection. That is, the local analytic diffeomorphisms induced by the isomonodromy connection between the fibres of $\mathcal{M}^*_{\mathrm{ext}}$ are symplectic diffeomorphisms.*

Proof. We will show that arbitrary, small, isomonodromic deformations induce symplectomorphisms.

Let u_0 be any point in the extended moduli bundle $\mathcal{M}^*_{\mathrm{ext}}$ and let x_0 be the image of u_0 in the space X of deformation parameters. Let γ be any holomorphic map from the open unit disk $\widetilde{\Delta} \subset \mathbb{C}$ into X such that $\gamma(0) = x_0$. Let t denote a point of $\widetilde{\Delta}$ and let $\mathbf{A}_t$ denote the m-tuple of nice irregular types specified by $\gamma(t) \in X$. To simplify the notation, let $\mathcal{M}^*_t = \mathcal{M}^*_{\mathrm{ext}}(\mathbf{A}_t)$ denote the (symplectic) extended moduli space which is the fibre of $\mathcal{M}^*_{\mathrm{ext}}$ over $\gamma(t)$.

The standard vector field $\partial/\partial t$ on $\widetilde{\Delta} \subset \mathbb{C}$ gives a vector field on $\gamma(\widetilde{\Delta}) \subset X$ which we lift to a vector field V on $\mathcal{M}^*_{\mathrm{ext}}|_{\gamma(\widetilde{\Delta})}$, transverse to the fibres $\mathcal{M}^*_t$, using the isomonodromy connection.

We can integrate this lifted vector field throughout a neighbourhood of u_0 in $\mathcal{M}^*_{\mathrm{ext}}|_{\gamma(\widetilde{\Delta})}$. Concretely, this means that there is a contractible neighbourhood U of u_0 in $\mathcal{M}^*_0$, a neighbourhood $\Delta \subset \widetilde{\Delta}$ of 0 in $\mathbb{C}$ and a holomorphic map

$$F : U \times \Delta \to \mathcal{M}^*_{\mathrm{ext}}|_{\gamma(\Delta)}$$

such that for all $u \in U$ and $t \in \Delta$:

$$F(u,t) \in \mathcal{M}_t^*, \qquad F(u,0) = u \in \mathcal{M}_0^* \qquad \text{and}$$

$$\frac{\partial F}{\partial t}(u,t) = V_{F(u,t)}.$$

In particular for each $t \in \Delta$ we get a symplectic form on U:

$$\omega_t := (F|_t)^*(\omega_{\mathcal{M}_t^*})$$

where $\omega_{\mathcal{M}_t^*}$ is the symplectic form we have defined on the extended moduli space $\mathcal{M}_t^*$, and $F|_t = F(\cdot, t) : U \to \mathcal{M}_t^*$.

Now if we choose any two tangent vectors W_1, W_2 to U at u_0, it is sufficient for us to show that the function on Δ:

$$\omega_t(W_1, W_2), \qquad t \in \Delta$$

is *constant* in some neighbourhood of $0 \in \Delta$. We break this up into six steps.

1) Firstly the map F gives us a family of meromorphic connections on the trivial bundle over $\mathbb{P}^1$ parameterised by $U \times \Delta$. For each fixed $u \in U$ we get an isomonodromic family parameterised by Δ, that is, a vertical meromorphic connection on the trivial bundle over $\Delta \times \mathbb{P}^1$ (where $\mathbb{P}^1$ is the vertical direction), such that each connection on $\mathbb{P}^1$ has the same monodromy data. The result of Jimbo, Miwa and Ueno (Theorem 6.14 here) then tells us how to extend this vertical connection to a full *flat* connection over $\Delta \times \mathbb{P}^1$. From the algebraic formula (97) for the Δ-component Ω of the full connection we see this process of extending a vertical connection to a full connection will behave well when we vary $u \in U$; for each $u \in U$ we obtain flat meromorphic connection

$$\nabla_u$$

on the trivial bundle over $\Delta \times \mathbb{P}^1$ that depends holomorphically on u. As before we will denote the poles of ∇_u by $a_1(t), \ldots, a_m(t)$ and the polar divisor in $\Delta \times \mathbb{P}^1$ of ∇_u by $\widetilde{D} = \sum k_i \Delta_i$ (these are all independent of $u \in U$). Suppose Δ is sufficiently small so that we can choose disjoint open discs D_i in $\mathbb{P}^1$ such that $a_i(t) \in D_i$ for all $t \in \Delta$.

2) The next step is to push everything over to the C^∞ picture where the symplectic forms are expressed simply as integrals. To do this we choose a smooth bundle automorphism:

$$g \in GL_n(C^\infty(U \times \Delta \times \mathbb{P}^1))$$

which 'straightens' the whole family of connections ∇_u at the same time. The map F into $\mathcal{M}_{\text{ext}}^*$ specifies a family of good compatible framings

$${}^i g_0 : U \times \Delta_i \to GL_n(\mathbb{C})$$

of ∇_u along Δ_i for each i and all $u \in U$. In turn these framings extend uniquely to formal isomorphisms to the full connections associated to families of formal normal forms (see Definition 6.8). That is we obtain

$${}^i\widehat{g} \in GL_n\big(\mathbb{C}[\![z_i]\!] \otimes \mathcal{O}(U \times \Delta_i)\big)$$

which transforms the Laurent expansion of ∇_u along Δ_i into the full connection associated to the formal normal forms for each u:

$${}^i\widehat{g}[L_i(\nabla_u)] = d - d({}^iQ(t)) - {}^i\Lambda \frac{d(z_i)}{z_i},$$

where d denotes the exterior derivative on $\Delta \times \mathbb{P}^1$. The automorphism g is constructed as in Proposition 5.4 on p80 to have Taylor expansion at $z_i = 0$ equal to ${}^i\widehat{g}$ for all $t \in \Delta$ and for all $u \in U$. A minor modification of Proposition 5.4 is required: just replace the open set U there by $U \times \Delta$ and note that a_i and ${}^iA^0$ will now also vary; the same procedure will work provided Δ is small enough so that $a_i(t)$ remains in the disk D_i for all $t \in \Delta$.

Thus we can use g to straighten the whole family ∇_u at the same time. That is, if we define two families of C^∞ singular connections

$$\widetilde{\nabla}_u = g[\nabla_u]$$

on $\Delta \times \mathbb{P}^1$ parameterised by U, and

$$d_\alpha = d_{\mathbb{P}^1} - \alpha = \widetilde{\nabla}_u|_{\mathbb{P}^1} = g\big[\nabla_u|_{\mathbb{P}^1}\big]_{\mathbb{P}^1}$$

on $\mathbb{P}^1$ parameterised by $U \times \Delta$, then the Laurent expansion of $\widetilde{\nabla}_u$ at a_i agrees with the full connection associated to the formal normal forms:

$$L_i(\widetilde{\nabla}_u) = d - d({}^iQ(t)) - {}^i\Lambda \frac{d(z_i)}{z_i} \tag{99}$$

for all u and t where d is the exterior derivative on $\Delta \times \mathbb{P}^1$. It follows that the matrix α of singular one forms on $\mathbb{P}^1$ (with coefficients dependent on u, t) has the desired fixed Laurent expansions:

$$d_{\mathbb{P}^1} - \alpha(u,t) \in \mathcal{A}_{\mathrm{ext,fl}}(\mathbf{A}_t) \subset \mathcal{A}_{\mathrm{ext}}(\mathbf{A}_t)$$

for all $u \in U$ and $t \in \Delta$.

3) Now differentiate $\widetilde{\nabla}_u$ and d_α with respect to u along both W_1 and W_2 at $u = u_0$. Define Ψ_j and ψ_j to be these derivatives:

$$\Psi_j := W_j(\widetilde{\nabla}_u)$$

$$\psi_j := W_j(d_\alpha) = \Psi_j|_{\mathbb{P}^1}$$

for $j = 1, 2$. Each Ψ_j is a matrix of singular one forms on $\Delta \times \mathbb{P}^1$ and each ψ_j is a matrix of singular one forms on $\mathbb{P}^1$ parameterised by Δ. Clearly

$$\mathrm{Tr}(\psi_1 \wedge \psi_2) = \mathrm{Tr}(\Psi_1 \wedge \Psi_2)|_{\mathbb{P}^1}.$$

Also since the Laurent expansion of $\widetilde{\nabla}_u$ is given by (99) at each a_i we can deduce what the Laurent expansions of Ψ_1 and Ψ_2 are:

$$L_i(\Psi_j) = W_j({}^i\Lambda(u)) \frac{d_{\Delta \times \mathbb{P}^1}(z_i)}{z_i}$$

for $j = 1, 2$ and $i = 1, \ldots, m$. It follows that $\mathrm{Tr}(\Psi_1 \wedge \Psi_2)$ is a *nonsingular* two form on $\Delta \times \mathbb{P}^1$ since

$$L_i(\Psi_1 \wedge \Psi_2) = L_i(\Psi_1) \wedge L_i(\Psi_2) = 0$$

for each i.

4) Observe that for each $u \in U$ the flatness of ∇_u implies the flatness of $\widetilde{\nabla}_u$, and so the equation

$$\widetilde{\nabla}_u \circ \widetilde{\nabla}_u = 0$$

holds for all u. By differentiating this equation with respect to u along W_1 and W_2 we find

$$\widetilde{\nabla}_{u_0} \Psi_1 = 0 \qquad \text{and} \qquad \widetilde{\nabla}_{u_0} \Psi_2 = 0.$$

In particular the two form $\mathrm{Tr}(\Psi_1 \wedge \Psi_2)$ on $\Delta \times \mathbb{P}^1$ is *closed*:

$$d\mathrm{Tr}(\Psi_1 \wedge \Psi_2) = \mathrm{Tr}(\widetilde{\nabla}_{u_0}(\Psi_1) \wedge \Psi_2) - \mathrm{Tr}(\Psi_1 \wedge \widetilde{\nabla}_{u_0}(\Psi_2)) = 0.$$

5) Thus if we do the fibre integral over $\mathbb{P}^1$ we obtain a zero form on Δ (i.e. a function of t):

$$\int_{\mathbb{P}^1} \mathrm{Tr}(\Psi_1 \wedge \Psi_2) = \int_{\mathbb{P}^1} \mathrm{Tr}(\psi_1 \wedge \psi_2).$$

This is a *closed* 0-form (i.e. a *constant* function) since integration over the fibre ($\int_{\mathbb{P}^1}$) commutes with exterior differentiation d. See for example Bott and Tu [**21**] Proposition 6.14.1 (it is important here that $\mathrm{Tr}(\Psi_1 \wedge \Psi_2)$ is nonsingular).

6) Finally we appeal to Theorem 5.8 to see that for all $t \in \Delta$:

$$\frac{1}{2\pi i} \int_{\mathbb{P}^1} \mathrm{Tr}(\psi_1 \wedge \psi_2) = \omega_t(W_1, W_2)$$

and so the symplectic form is independent of t. $\square$

CHAPTER 7

One Plus Two Systems

In this chapter we study in detail the simplest case involving irregular singularities and Stokes matrices: that of meromorphic connections on $\mathbb{P}^1$ with only two poles, of orders one and two respectively. This case is of interest for numerous reasons; in particular it occurs in the theory of Frobenius manifolds (this will be discussed in Chapter 8).

The aim here is to discuss the symplectic/Poisson geometry of the one plus two case in detail. Our main input is the observation that the Poisson geometry of the space of monodromy data here appears to be the same as that of a certain Poisson-Lie group. This is made precise in Conjecture 7.5, which we prove in the two by two case. This gives a purely finite dimensional description of the symplectic/Poisson structure on the monodromy data on one hand, and a new way to think of Poisson-Lie groups on the other.

1. General Set-Up

1.1. The connections we wish to consider are on the trivial rank n vector bundle over $\mathbb{P}^1$ and of the form

$$\nabla := d - \left(U\frac{dz}{z^2} + V\frac{dz}{z}\right) \tag{100}$$

where $U, V \in \mathrm{End}_n(\mathbb{C})$ are $n \times n$ matrices. In this chapter we will allow V to be an *arbitrary* matrix[1], but require U to be diagonal with distinct eigenvalues:

$$U = \mathrm{diag}(u_1, \ldots, u_n)$$

where $u_i \neq u_j$ if $i \neq j$. Thus the meromorphic connection (100) has an order two pole at $z = 0$ and (unless $V = 0$) has a simple pole at ∞.

Since there are only two poles, moving them will give trivial isomonodromic deformations (i.e. we can always perform an automorphism of $\mathbb{P}^1$ to put the poles back to $z = 0$ and $z = \infty$). Thus we naturally only consider the deformations coming from the irregular singularity; namely in this chapter we define the set X of deformation parameters to be the set of order two irregular types at $z = 0$, i.e. to be the set of U's:

$$X := \left\{(u_1, \ldots, u_n) \in \mathbb{C}^n \;\middle|\; u_i \neq u_j \text{ if } i \neq j\right\} = \mathbb{C}^n \setminus \text{diagonals}.$$

Notice that the fundamental group of X is a braid group.

Once U is fixed the set of polar part data is just the set of V's:

$$V \in \mathrm{End}_n(\mathbb{C}).$$

[1] We will specialise to skew-symmetric V in the next chapter: that is the case of relevance to Frobenius manifolds.

This space is given its standard (Lie) Poisson structure by using the trace to identify $\mathrm{End}_n(\mathbb{C})$ with the dual of the Lie algebra of $GL_n(\mathbb{C})$ as usual. More interestingly the corresponding space of monodromy data may be identified with the n^2-dimensional manifold

$$U_- \times U_+ \times \mathfrak{t}$$

containing the Stokes matrices[2] and the exponent of formal monodromy at $z = 0$, where $U_\pm$ are the upper and lower triangular unipotent subgroups of $GL_n(\mathbb{C})$ and $\mathfrak{t} \cong \mathbb{C}^n$ is the set of diagonal matrices. (Don't confuse the groups $U_\pm$ with the diagonal matrix U.)

Thus for each fixed value of U we obtain a monodromy map:

$$\begin{aligned} \nu_U : \mathrm{End}_n(\mathbb{C}) &\longrightarrow U_- \times U_+ \times \mathfrak{t} \\ V &\longmapsto (S_-, S_+, \Lambda) \end{aligned} \tag{101}$$

taking $V \in \mathrm{End}_n(\mathbb{C})$ to the Stokes matrices and exponent of formal monodromy of the connection (100) at $z = 0$. (For any fixed value of U we make a choice of sector labelling and branch of $\log(z)$ to define the map ν_U.) This is a holomorphic map between two manifolds of the same dimension, which is a local analytic isomorphism about a generic point V. The exponent of formal monodromy may be obtained easily: it is just the diagonal part of V

$$\Lambda = \delta(V),$$

but the Stokes matrices will in general depend on U and V in a very complicated way (since the monodromy map solves Painlevé type equations).

The aim of this chapter is to examine more concretely the Poisson structure on the monodromy space $U_- \times U_+ \times \mathfrak{t}$. Currently we have two descriptions of it:

1) We can push forward the Poisson structure on $\mathrm{End}_n(\mathbb{C})$ along the monodromy map ν_U. This is hard because ν_U is in general not at all explicit.

2) Use the infinite dimensional symplectic quotient description of the monodromy manifolds given in Chapter 3. This is useful for seeing that the isomonodromy equations are symplectic but is also not very explicit.

Remark 7.1. This example fits in with the picture described in previous chapters as follows. We have a fixed coordinate z on $\mathbb{P}^1$ and the positions of the poles do not move: $a_1 := 0, a_2 := \infty$. We use the local coordinate z at a_1 and $w := 1/z$ at a_2. The orders of the poles are $k_1 = 2, k_2 = 1$. Choosing U corresponds to choosing an irregular type at $z = 0$; namely the irregular type of the formal normal form

$$A^0 := U\frac{dz}{z^2}.$$

Thus for each U we obtain the extended moduli space

$$\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A}) \cong \widetilde{O}_1 \times \widetilde{O}_2 // GL_n(\mathbb{C})$$

of compatibly framed connections on the trivial rank n vector bundle over $\mathbb{P}^1$ with irregular type A^0 at a_1 and with a simple pole at a_2 (see Section 4 of Chapter 2). This is a complex symplectic manifold of dimension $n^2 + n$. We now write elements of $\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$ in terms of the compatible framings at $z = 0$ (this puts them in the form (100)) and we forget the framing at $a_2 = \infty$. This procedure of forgetting the framing corresponds to quotienting $\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$ by the normaliser $N(T)$ of the torus T (see discussion p35). This yields an n^2-dimensional Poisson manifold which is easily identified (as a Poisson manifold) with

[2] see Corollary 1.37; more correctly the elements of $U_\pm$ are the *permuted* Stokes matrices.

the dense open subset of $\mathrm{End}_n(\mathbb{C})$ of matrices (V's) with distinct eigenvalues modulo integers (see the last paragraph of the explicit description of the $k = 2$ extended orbits given on p33). Similarly the quotient of the extended monodromy manifold $M_{\mathrm{ext}}(\mathbf{A})$ by $N(T)$ yields a dense open subset of $U_- \times U_+ \times \mathfrak{t}$ (see Definitions 4.2 and 4.3).

Thus to agree with the previous chapters we should (but will not) require V to have distinct eigenvalues mod $\mathbb{Z}$ (or equivalently require the local monodromy $S_-S_+\exp(2\pi i\Lambda) \in GL_n(\mathbb{C})$ to have distinct eigenvalues).

1.2. Symplectic Leaves. The first step in understanding the Poisson structure on the monodromy space $U_- \times U_+ \times \mathfrak{t}$ is to determine the symplectic leaves. This is straightforward using the geometry of the situation:

Lemma 7.2. *Define a map $\pi : U_- \times U_+ \times \mathfrak{t} \to GL_n(\mathbb{C})$ to be the product:*

$$\pi(S_-, S_+, \Lambda) = S_-S_+e^{2\pi i\Lambda} \in GL_n(\mathbb{C})$$

of the lower triangular Stokes matrix with the upper triangular Stokes matrix and the formal monodromy. Then the symplectic leaves of $U_- \times U_+ \times \mathfrak{t}$ are of the form $\pi^{-1}(C)$ for conjugacy classes $C \subset GL_n(\mathbb{C})$.

Proof. Firstly, the symplectic leaves of $\mathrm{End}_n(\mathbb{C})$ are the (co)adjoint orbits. Secondly, given a (generic) matrix $V \in \mathrm{End}_n(\mathbb{C})$ then the local monodromy of the meromorphic connection (100) around the simple pole at ∞ is conjugate to $\exp(2\pi iV)$. Now a simple loop around ∞ in $\mathbb{P}^1 \setminus \{0, \infty\}$ is also a simple loop around 0, so the local monodromy at 0 is also conjugate to $\exp(2\pi iV)$. Thus Lemma 1.39 implies that

$$S_-S_+\exp(2\pi i\Lambda) \qquad \text{is conjugate to} \qquad \exp(2\pi iV).$$

The result now follows because if we vary V slightly, then V moves in a fixed coadjoint orbit iff $\exp(2\pi iV)$ moves in a fixed conjugacy class $\square$

Thus the symplectic leaves of the monodromy space $U_- \times U_+ \times \mathfrak{t}$ are given by fixing the conjugacy class of the product $S_-S_+\exp(2\pi i\Lambda)$ of the lower triangular Stokes matrix with the upper triangular Stokes matrix and the diagonal formal monodromy. The main observation here is that a remarkably similar situation arises naturally in the theory of Poisson-Lie groups.

2. Poisson-Lie Groups

A Poisson-Lie group is a Lie group G together with a Poisson structure on G such that the multiplication map

$$G \times G \longrightarrow G$$

is a Poisson map. The references we have used for Poisson-Lie groups are [**4, 30, 65, 95, 96**]. Here we are just interested in one example of a Poisson-Lie group: the dual Poisson-Lie group to $GL_n(\mathbb{C})$ with its standard (complex) Poisson-Lie group structure. Let $G := GL_n(\mathbb{C})$, $\mathfrak{g} := \mathrm{Lie}(GL_n(\mathbb{C}))$ and define a group G^* explicitly as:

$$G^* := \{(b_-, b_+) \in B_- \times B_+ \bigm| \delta(b_-)\cdot\delta(b_+) = 1\}$$

where $\delta : G \to T$ takes the diagonal part of a matrix and where $B_\pm$ are the upper and lower triangular Borel subgroups of G (i.e. the triangular matrices with arbitrary diagonal entries). G^* is an n^2-dimensional subgroup of $G\times G$, having product induced from that on $G\times G$ in the obvious way. The Lie algebra of G^* is

$$\mathrm{Lie}(G^*)=\{(X_-,X_+)\in\mathfrak{b}_-\times\mathfrak{b}_+\ \big|\ \delta(X_-)+\delta(X_+)=0\}$$

where $\mathfrak{b}_\pm := \mathrm{Lie}(B_\pm)$. A Poisson-Lie group structure on G^* is determined by a Lie bialgebra structure on $\mathrm{Lie}(G^*)$. This is given by a Lie bracket on the dual of the Lie algebra, which we specify here by identifying $\mathrm{Lie}(G^*)^*$ with $\mathfrak{g}$ via the nondegenerate pairing:

$$\mathrm{Lie}(G^*)\times\mathfrak{g}\longrightarrow\mathbb{C};\qquad \langle(X_-,X_+),Y\rangle=\mathrm{Tr}\big((X_+-X_-)Y\big).$$

Thus $\mathrm{Lie}(G^*)\cong\mathfrak{g}^*$, explaining the notation. This does indeed specify a Lie bialgebra structure on $\mathrm{Lie}(G^*)$ (i.e. the corresponding Lie algebra 1-cochain is closed), and integrates to give a Poisson-Lie group structure on G^*. An alternative approach is to see that triple of Lie algebras of the groups $(G\times G, G_\Delta, G^*)$ is a Manin triple (where $G_\Delta\cong G$ is the diagonal subgroup of $G\times G$ and $\mathfrak{g}\times\mathfrak{g}$ is given the invariant nondegenerate bilinear form $\langle(X_1,X_2),(Y_1,Y_2)\rangle=\mathrm{Tr}(X_1Y_1)-\mathrm{Tr}(X_2Y_2)$). This Lie bialgebra structure appears for example in Drinfel'd's paper [**30**]. (See e.g. [**65**] for more details about the relationship between Poisson-Lie groups, Lie bialgebras and Manin triples.)

Thus there is a standard Poisson structure on the group G^*. What is perhaps slightly less well-known is that the symplectic leaves arise as follows:

Lemma 7.3. *Define a map $\pi' : G^*\to G$ to be the product:*

$$\pi'(b_-,b_+)=(b_-)^{-1}b_+\in GL_n(\mathbb{C})$$

of the inverse of the lower triangular matrix with the upper triangular matrix. Then the symplectic leaves of G^* are of the form $(\pi')^{-1}(C)$ for conjugacy classes $C\subset GL_n(\mathbb{C})$.

Proof. See Semenov-Tian-Shansky [**96**] Propositions 8 and 9 and Alekseev and Malkin [**4**] Example 2 p169. The basic idea is as follows. The group $G\times G$ naturally has a Poisson structure which is *symplectic* on a dense open subset containing the identity. When $G\times G$ is given this Poisson structure it is called the 'Heisenberg double' and will be denoted by D_+. This is not a Poisson-Lie group structure, but is useful for understanding the Poisson structure on the subgroup G^* of $G\times G$. The actions of the diagonal subgroup G_Δ on the left and the right of D_+ are Poisson actions. After perhaps restricing to dense open subsets, the quotients D_+/G_Δ and $G_\Delta\backslash D_+$ may then be identified as Poisson manifolds with a dense open subset of G^* such that the following diagram commutes:

(102)
$$\begin{array}{ccccc}
G_\Delta\backslash D_+ & \longleftarrow & D_+ & \longrightarrow & D_+/G_\Delta\\
\downarrow{\scriptstyle\pi_1} & & \bigcup & & \downarrow{\scriptstyle\pi_2}\\
G & \xleftarrow{\pi'} & G^* & \xrightarrow{\pi''} & G
\end{array}$$

where the maps on the top line are the natural projections, $\pi_1[(g,h)]=g^{-1}h$, $\pi_2[(g,h)]=gh^{-1}$ and $\pi''(b_-,b_+)=b_-b_+^{-1}$. The point is that the top row of the diagram is a full dual pair of symplectic realisations in the sense of Weinstein [**109**] (again we may need to restrict to dense open subsets). This implies in particular that the symplectic leaves of $G_\Delta\backslash D_+$ are the orbits of the induced action of $G\cong G_\Delta$ coming from the G_Δ action on the right of D_+. Under the projection π_1 this G action on $G_\Delta\backslash D_+$ becomes the standard conjugation action of G on itself. Thus the symplectic leaves of G^* are the inverse images under π' of conjugacy classes in G. (The induced action of G on G^*, from the action on the right of D_+, is called the (right) *dressing action*. Note that this situation is symmetric since $b_-^{-1}b_+$ is conjugate to $(b_-b_+^{-1})^{-1}$.) $\square$

Notice the similarities with Lemma 7.2: again the symplectic leaves are given by fixing the conjugacy class of the product of a lower triangular matrix and an upper triangular matrix.

3. From Stokes Matrices to Poisson-Lie Groups

By comparing Lemma 7.2 with Lemma 7.3 it is easy to write down a map from the space $U_- \times U_+ \times \mathfrak{t}$ of monodromy data to the Poisson-Lie group G^* such that the symplectic leaves match up:

DEFINITION 7.4. Let $\varphi : U_- \times U_+ \times \mathfrak{t} \to G^*$ be the map defined by

$$b_- = e^{-\pi i\Lambda}S_-^{-1}, \qquad b_+ = e^{-\pi i\Lambda}S_+e^{2\pi i\Lambda}.$$

This is defined precisely so that the relation

$$b_-^{-1}b_+ = S_-S_+\exp(2\pi i\Lambda)$$

holds, and so then the symplectic leaves match up under φ. Observe also that φ is a covering map; any fibre may be naturally identified with the fibre of the map $\mathfrak{t} \to T; \Lambda \mapsto \exp(\pi i\Lambda)$. (The space $U_- \times U_+ \times \mathfrak{t}$ is identified in this way with the universal cover of G^*.) The situation may be summarised in the commutative diagram:

$$\begin{array}{ccc} U_- \times U_+ \times \mathfrak{t} & \xrightarrow{\varphi} & G^* \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ G & = & G \end{array} \tag{103}$$

where π, π' and φ are as defined above. The images of π and π' in G are the same and equal to the dense open subset consisting of matrices which admit an 'LU' decomposition. Both π and π' are coverings of this image; π is the universal covering and π' is a 2^n-fold covering.

Now we claim that φ is a Poisson map. More precisely we have

Conjecture 7.5. *The composition*

$$\varphi \circ \nu_U : \mathfrak{g}^* \longrightarrow G^*$$

of the monodromy map $\nu_U : \mathfrak{g}^* \cong \mathrm{End}_n(\mathbb{C}) \to U_- \times U_+ \times \mathfrak{t}$ and the map $\varphi : U_- \times U_+ \times \mathfrak{t} \to G^*$ defined above, is a Poisson map for any choice of U, where $\mathfrak{g}^*$ is given its standard Lie Poisson structure and G^* its standard Poisson-Lie group structure.

(Note that both φ and ν_U are local isomorphisms so we are essentially identifying the two Poisson structures.)

At the moment there are three concrete reasons to believe this conjecture. Firstly the symplectic leaves match up by construction. Secondly the conjecture is true in the 2×2 case, as we will prove in the next section. Perhaps the most compelling evidence relates to Frobenius manifolds however: if we restrict to skew-symmetric matrices V then there is an explicit formula (due to B.Dubrovin [**31**] in the 3×3 case and M.Ugaglia [**102**] in general) for the Poisson structure on the corresponding space of monodromy data (which may be identified with U_+; there is just one independent Stokes matrix). Then their formulae agree with the Poisson structure on U_+ induced from that on G^* in all the cases we have checked; see Chapter 8. (I believe Conjecture 7.5 can be proved in general by using the formula for the Poisson bracket on G^* in [**24**] together with a direct calculation of the natural braid group action on $U_- \times U_+ \times \mathfrak{t}$.)

4. The Two by Two Case

In this section we examine what happens in the $SL_2(\mathbb{C})$ case when the matrices U and V are 2×2 and trace-free. We are able to deduce precisely what the Poisson structure is on the space of monodromy data due to the low dimensionality and knowledge of the torus action changing the compatible framing at $z=0$. This enables us to confirm Conjecture 7.5 in this case.

Label the matrix entries of U and V as follows:

$$U\frac{dz}{z^2}+V\frac{dz}{z}=\begin{pmatrix} u & 0 \\ 0 & -u\end{pmatrix}\frac{dz}{z^2}+\begin{pmatrix} \lambda & v_+ \\ v_- & -\lambda\end{pmatrix}\frac{dz}{z}.$$

Thus λ, v_+, v_- are three (linear) functions on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We identify $\mathfrak{sl}_2(\mathbb{C})$ with its dual using the trace pairing $(V,W)\mapsto \mathrm{Tr}(VW)$ (which is a constant multiple of the Killing form), so that we have a Poisson structure on $\mathfrak{sl}_2(\mathbb{C})$. Explicitly this Poisson structure is given by the formulae:

$$\{v_\pm,\lambda\}=\pm v_\pm \qquad \text{and} \qquad \{v_-,v_+\}=2\lambda. \tag{104}$$

Now define coordinates s_+, s_-, λ on the space of monodromy data:

$$S_-=\begin{pmatrix} 1 & 0 \\ s_- & 1\end{pmatrix},\qquad S_+=\begin{pmatrix} 1 & s_+ \\ 0 & 1\end{pmatrix},\qquad \Lambda=\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda\end{pmatrix}.$$

The monodromy map ν_U in (101) then expresses s_+ and s_- as (complicated) functions of u,λ and $v_\pm$.

Proposition 7.6. *In the $SL_2(\mathbb{C})$ case the Lie Poisson structure on $\mathfrak{g}^*$ pushes forward along the monodromy map ν_U to give the Poisson structure:*

$$\{s_\pm, l\}=\pm(2\pi i)s_\pm l \tag{105}$$

$$\{s_-,s_+\}=(2\pi i)\big(l^2-(1+s_-s_+)\big) \tag{106}$$

on the space $U_-\times U_+\times\mathfrak{t}$ of monodromy data, where $l:=\exp(2\pi i\lambda)$ and $\mathfrak{t}$ is the diagonal subalgebra of $\mathfrak{g}=\mathrm{Lie}(SL_2(\mathbb{C}))$.

Proof. Firstly (105) may be deduced from the fact that 2λ is the moment map for the torus action

$$t(S_-,S_+,\Lambda)=(tS_-t^{-1},tS_+t^{-1},\Lambda)$$

on the monodromy data, where t is a diagonal element of $SL_2(\mathbb{C})$. (This holds since 2λ is the moment map for the torus action on $\mathfrak{sl}_2(\mathbb{C})$ defined by restricting the adjoint action and we have proved that the monodromy map intertwines these two actions in Lemma 1.39). It follows that $\{s_\pm,\lambda\}=\pm s_\pm$ similarly to (104) and then (105) is immediate.

To deduce equation (106) we use

Lemma 7.7. *The function $l+l^{-1}(1+s_-s_+)$ on the set of monodromy data, pulls back along the monodromy map to a Casimir function on $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. It is sufficient to work over the dense open subset of $\mathfrak{sl}_2(\mathbb{C})$ where V has distinct eigenvalues mod $\mathbb{Z}$. From the proof of Lemma 7.2 we know that $S_-S_+\exp(2\pi i\Lambda)$ is conjugate to $\exp(2\pi iV)$ and so:

$$l + l^{-1}(1+s_-s_+) := \mathrm{Tr}(S_-S_+e^{2\pi i\Lambda}) = \mathrm{Tr}(e^{2\pi iV}).$$

This is a Casimir function since the right-hand side is clearly invariant under the adjoint action on $\mathfrak{sl}_2(\mathbb{C})$ □

Thus to conclude the proof we just calculate

$$0 = \{s_-, l + l^{-1}(1+s_-s_+)\} = -(2\pi i)ls_- + \{s_-, l^{-1}\}(1+s_-s_+) + l^{-1}s_-\{s_-, s_+\} \tag{107}$$

using the derivation property of the Poisson bracket. Now

$$\{s_-, l^{-1}\} = -\{s_-, l\}/l^2 = (2\pi i)s_-/l$$

so that (107) becomes

$$0 = -(2\pi i)s_-l + (2\pi i)(1+s_-s_+)s_-/l + \{s_-, s_+\}s_-/l$$

and this rearranges to give the required formula (106) for $\{s_-, s_+\}$ □

Remark 7.8. In fact, in this 2×2 case there are explicit formulae for the functions $s_\pm(u, \lambda, v_+, v_-)$ involving Γ functions (see Section 5 of the paper [**16**] by Balser, Jurkat and Lutz). Such formulae exist because the corresponding isomonodromic deformations in this 2×2 case are trivial. Anyway it is not too hard (using some standard identities for Γ functions) to use these formulae to give a self-contained direct proof of Proposition 7.6.

Next we want to compare the formulae in Proposition 7.6 with the Poisson structure on G^* using the map φ (here $G = SL_2(\mathbb{C})$). The Poisson structure on G^* seems hard to get at directly; one way to calculate it explicitly is as follows. Recall that map $\pi' : G^* \to G$; $(b_-, b_+) \mapsto b_-^{-1}b_+$ is a covering of a dense open subset. The Poisson structure on G^* pushes along π' to induce a Poisson structure on G (this Poisson structure on G is not a Poisson-Lie group structure and in particular it is different to the standard Poisson-Lie group structure on G). However using 'tensor notation' (see [**65**]) we have the following formula for this Poisson structure on G (this is formula (235) in [**4**]):

$$\{L^1, L^2\} = r_+L^1L^2 + L^1L^2r_- - L^1r_+L^2 - L^2r_-L^1 \tag{108}$$

where $r_\pm \in \mathfrak{g}\otimes\mathfrak{g}$ correspond to the elements $\pm((1/2)\delta+\pi_\pm)$ of $\mathrm{End}(\mathfrak{g}) \cong \mathfrak{g}\otimes\mathfrak{g}^*$ respectively, under the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ using the trace (where $\pi_\pm \in \mathrm{End}(\mathfrak{g})$ are the projections onto the strictly upper/lower triangular subalgebras of $\mathfrak{g}$ and, as usual, δ is the projection onto the diagonal matrices).

Thus in the 2×2 case here, we write

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_n(\mathbb{C})$$

and find that the formula (108) gives:

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{,}{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\} = \begin{pmatrix} 0 & -ab & ab & 0 \\ ac & 0 & ad-a^2 & -ab \\ -ac & a^2-ad & 0 & ab \\ 0 & ac & -ac & 0 \end{pmatrix}. \tag{109}$$

This notation means, for example, that the top left 2×2 submatrix on the right-hand side equals $\{a, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\} = \left(\begin{smallmatrix} \{a,a\} & \{a,b\} \\ \{a,c\} & \{a,d\} \end{smallmatrix}\right)$.

To compare this with the $\mathfrak{sl}_2(\mathbb{C})^*$ Poisson structure, we push down the Poisson structure on $U_-\times U_+\times\mathfrak{t}$ in Proposition 7.6 (coming from $\mathfrak{sl}_2(\mathbb{C})^*$) along π to G (see Diagram (103)). That is, we write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = S_-S_+e^{2\pi i\Lambda} = \begin{pmatrix} l & s_+/l \\ s_-l & (1+s_-s_+)/l \end{pmatrix} \in G$$

where $l := e^{2\pi i\lambda}$. Now using the formulae from Proposition 7.6 and the fact that $l + l^{-1}(1+s_-s_+) = a+d$ is a Casimir, it is straightforward to calculate all the Poisson brackets between a,b,c,d and find that they agree with the all the brackets in (109) (upto an overall $(2\pi i)$ factor). For example

$$\{a,b\} = \{l, l^{-1}s_+\} = l^{-1}\{l,s_+\} = -(2\pi i)s_+ = -(2\pi i)ab$$

or

$$\begin{aligned}\{c,b\} &= \{s_-l, s_+l^{-1}\} = s_-\{l,s_+\}/l + l\big(\{s_-,s_+\}/l - s_+\{s_-,l\}l^{-2}\big)\\ &= (2\pi i)\big(-s_-s_+ + l^2 - (1+s_-s_+) + s_+s_-\big) = (2\pi i)\big(a^2-ad\big).\end{aligned}$$

The overall factor of $(2\pi i)$ can be removed by rescaling the Poisson structure on G^*. Thus in summary we have proved:

Proposition 7.9. *Conjecture 7.5 is true in the 2×2 case.*

Remark 7.10. A less computational proof could be obtained by establishing simply that the (locally defined) map $\Lambda : G^*\to\mathfrak{t}$ is a moment map for the torus action $t(b_-,b_+) = (t^{-1}b_-t, tb_+t^{-1})$ on G^* where $t\in T$. (This action corresponds to the usual torus action on $U_-\times U_+\times\mathfrak{t}$ under φ.) In the 2×2 case the Poisson structure is then uniquely determined using the argument in Proposition 7.6. (The fact that Λ is a local moment map is a corollary of the conjecture but must surely be known generally; it gives a nice interpretation of the finite dimensional τ functions on $GL_n(\mathbb{C})$ as moment maps.)

CHAPTER 8

Frobenius Manifolds

The aim of this chapter is two-fold: 1) to explain how the moduli space of semisimple Frobenius manifolds is related to Poisson-Lie groups, and 2) to answer (a modified version of) a question raised by Hitchin in [**48**] on the relation between the local moduli space of semisimple Frobenius manifolds and representations of the fundamental group of a punctured $\mathbb{P}^1$.

I believe 1) is new but 2) is essentially in Dubrovin's seminal paper [**31**] (it is included here because the solution is quite attractive and it took some time to understand how the answer arises).

1. Frobenius Manifolds and Poisson-Lie Groups

In this section we specialise the study of the 'one plus two' systems in Chapter 7 to the case where V is a skew-symmetric matrix. This is the case that arises in the theory of Frobenius manifolds due to Boris Dubrovin [**31**] (see also [**32, 33**] and references therein). Our aim here is to show that the Poisson structure on the local moduli space of semisimple Frobenius manifolds (which is identified with a space of Stokes matrices in [**31**]) arises from the dual Poisson Lie group $GL_n(\mathbb{C})^*$ studied in Chapter 7.

To set this work in context we will say a few words about Frobenius manifolds (see Dubrovin's papers cited above for more details, or any of the papers [**12, 48, 75, 92**] by other authors.)

Firstly, using Atiyah's axioms [**8**], one finds that a two-dimensional topological quantum field theory (TFT) is equivalent to a Frobenius *algebra*. (If A is a finite dimensional commutative algebra (with identity) over $\mathbb{R}$ or $\mathbb{C}$, then A is a 'Frobenius algebra' if there is a linear form $\theta \in A^*$ such that $(a, b) = \theta(ab)$ is a nondegenerate inner product.)

However too much information is lost in the passage from a physically interesting field theory to the corresponding Frobenius algebra. Now, it was observed that many interesting 2D TFT's have a natural finite dimensional family of deformations; they sit in a canonical moduli space (see [**26, 27**]). Dubrovin's idea was to strengthen Atiyah's axioms to encode also this family of deformations. Thus Dubrovin defines a *Frobenius manifold* to be a manifold X such that each tangent space T_tX has a Frobenius algebra structure, together with some important requirements on how this family of algebras varies over X. In particular the structure constants of the algebras should be the third derivatives of some function F on X. However an arbitrary function F will not do, since the associativity (and certain scaling properties) of the Frobenius algebras mean that F must satisfy a very complicated system of overdetermined nonlinear PDEs: the so-called WDVV equations (named after Witten-Dijkgraaf-E.Verlinde-H.Verlinde).

The three main families of examples of Frobenius manifolds are: 1) Quantum cohomology (or topological sigma models of A-type), 2) Unfolding spaces of singularities

(K. Saito's theory, topological Landau-Ginzburg models) and recently 3) Barannikov-Kontsevich's construction, conjecturally producing the B-side of the 'Mirror conjecture' in arbitrary dimensions. See Manin's paper [**76**].

After defining Frobenius manifolds, Dubrovin then proceeded to study their moduli, i.e. the classification of Frobenius manifolds. One of the main results of [**31**] is that for a *semisimple*[1] Frobenius manifold the WDVV equations are equivalent to the isomonodromic deformation equations of the operator

$$\nabla = d - \left(U\frac{dz}{z^2} + V\frac{dz}{z}\right) \tag{110}$$

that we have been studying in Chapter 7, but with skew-symmetric V. Here the space of deformation parameters X is (locally) identified with an open patch in the Frobenius manifold (the u_i's are the so-called 'canonical coordinates' on the Frobenius manifold[2]). In particular the local moduli space of Frobenius manifolds is given by the $n(n-1)/2$ dimensional vector space of initial values of V at some base point $t_0 \in X$. This depends on the choice of basepoint: more naturally the moduli space of semisimple Frobenius manifolds is identified with the corresponding space of monodromy data of the connection (110) (this is independent of the choice of basepoint simply because the equations for V are *iso*-monodromic). This space of monodromy data is naturally identified with the space U_+ of unipotent upper triangular matrices (see below; there is just one independent Stokes matrix). Another good reason to think of the moduli of semisimple Frobenius manifolds in terms of Stokes matrices is that the tensor product of two Frobenius manifolds (for instance coming from the quantum cohomology of a product [**64**]) corresponds to the tensor product of their Stokes matrices (see [**33**] pp83-87).

For example in the case of three-dimensional Frobenius manifolds write

$$S := \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in U_+$$

so the moduli space is identified with $\mathbb{C}^3$ with coordinates x, y, z. In particular the quantum cohomology of the complex projective plane $\mathbb{P}^2(\mathbb{C})$ is a 3-dimensional semisimple Frobenius manifold and corresponds to the point $\left(\begin{smallmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in U_+$. (The manifold is just the complex cohomology $H^*(\mathbb{P}^2)$ and the Frobenius structure comes from the 'quantum product', deforming the usual cup product.) Anyway, Dubrovin was able to calculate explicitly the Poisson brackets between x, y, z coming from the usual Lie Poisson structure on the skew-symmetric matrices V ([**31**] Appendix F):

$$\begin{aligned} \{x, y\} &= xy - 2z \\ \{y, z\} &= yz - 2x \\ \{z, x\} &= zx - 2y. \end{aligned} \tag{111}$$

This Poisson structure has 2-dimensional symplectic leaves parameterised by the values of the Markoff polynomial

$$x^2 + y^2 + z^2 - xyz.$$

[1]Such that the Frobenius algebras are semisimple algebras at a generic point.

[2]To agree precisely with the notation of [**31**] we would need to negate V and replace z by $-1/z$.

(To see this is a Casimir function, observe that $\mathrm{Tr}(S^{-T}S) = 2 + xyz - (x^2+y^2+z^2)$ and we will see below (and from Lemma 7.2) that the local monodromy of (110) around zero is conjugate both to $S^{-T}S$ and to $\exp(2\pi i V)$.)

It is not too far from the truth to say that all of the work in this thesis arose through trying to understand the Poisson structure (111). In the remainder of this section we will prove:

Theorem 8.1. *Dubrovin's Poisson structure (111) arises from the Poisson-Lie group $GL_3(\mathbb{C})^*$. More precisely: the involution*

$$\iota : \mathrm{End}_n(\mathbb{C}) \to \mathrm{End}_n(\mathbb{C}); \qquad V \mapsto -V^T$$

fixing the skew-symmetric matrices induces (via the map $\varphi \circ \nu_U$ in Conjecture 7.5) the following involution of $G^* = GL_3(\mathbb{C})^*$:

$$\iota_{G^*} : G^* \to G^*; \qquad (b_-, b_+) \mapsto (b_+^T, b_-^T).$$

Then the fixed point set of this involution is

$$U_+ \cong \{(S^T, S) \mid S \in U_+\} \subset G^*$$

and the Poisson structure on G^* pushes down[3] onto U_+ and agrees with (111).

(Note that the group U_+ is not embedded as a subgroup of G^* so in particular it is not a Poisson-Lie subgroup[4].)

Remark 8.2. Very recently M. Ugaglia has extended Dubrovin's formula to the $n \times n$ case in [**102**]. The analogue of Theorem 8.1 holds also in all the cases I have checked; see below (the difficulty here is purely in terms of calculating the Poisson-Lie group Poisson bracket).

On one hand Theorem 8.1 establishes a connection between Poisson-Lie groups and Frobenius manifolds (which may for example be a useful way to understand the natural braid group actions) and on the other hand it supports Conjecture 7.5 (since the conjecture would imply that (111) comes from $GL_3(\mathbb{C})^*$ immediately; Dubrovin's formula is for the pushforward of the Poisson structure on the skew matrices along the monodromy map).

Proof (of Theorem 8.1). Recall from Chapter 7 that, for each fixed diagonal matrix U with distinct eigenvalues, we have defined maps

$$\mathrm{End}_n(\mathbb{C}) \xrightarrow{\nu_U} U_- \times U_+ \times \mathfrak{t} \xrightarrow{\varphi} G^*$$

$$V \longmapsto (S_-, S_+, \Lambda) \longmapsto (b_-, b_+)$$

where ν_U is the monodromy map taking the monodromy data of the meromorphic connection $d - \left(U\frac{dz}{z^2} + V\frac{dz}{z}\right)$ on $\mathbb{P}^1$ and φ is given explicitly by the formulae

$$b_- = e^{-\pi i \Lambda} S_-^{-1}, \qquad b_+ = e^{-\pi i \Lambda} S_+ e^{2\pi i \Lambda}.$$

Our first task is to determine what the involution ι of $\mathrm{End}_n(\mathbb{C})$ (fixing the skew-symmetric matrices) induces on the other two spaces when ν_U and φ are applied:

[3] by projecting the Poisson bivector onto the $(+1)$-eigenspace of the derivative of ι_{G^*}, along the (-1)-eigenspace; see p124

[4] In fact one may easily check that the Poisson structure (111) on the group U_+ does not give U_+ the structure of a Poisson-Lie group.

Lemma 8.3. *The involution of* $U_- \times U_+ \times \mathfrak{t}$ *corresponding to* ι *is given by the formula*

$$(S_-, S_+, \Lambda) \longmapsto (e^{i\pi\Lambda} S_+^{-T} e^{-i\pi\Lambda}, e^{-i\pi\Lambda} S_-^{-T} e^{i\pi\Lambda}, -\Lambda) \tag{112}$$

where $S_\pm^{-T}$ *denotes the inverse of the transpose of* $S_\pm$. *The corresponding involution of* G^* *is given by*

$$(b_-, b_+) \longmapsto (b_+^T, b_-^T). \tag{113}$$

Remark 8.4. The fact that (113) is simpler than (112) suggests that G^* gives a more natural way of storing the monodromy data.

Proof (of Lemma 8.3). Using the formula for φ it is easy to deduce (113) from (112), so we need to prove only (112). Thus we fix U and a (not necessarily skew-symmetric) matrix $V \in \mathrm{End}_n(\mathbb{C})$ and consider the two meromorphic connections

$$\nabla := d - \left(U\frac{dz}{z^2} + V\frac{dz}{z}\right) \quad \text{and} \quad \nabla' := d - \left(U\frac{dz}{z^2} - V^T\frac{dz}{z}\right)$$

on the trivial bundle over $\mathbb{P}^1$. Define diagonal matrices Λ and Λ' to be the respective diagonal parts of the residues:

$$\Lambda := \delta(V) \qquad \Lambda' := \delta(-V^T) = -\Lambda.$$

Then the formal normal forms at 0 of ∇ and ∇' are:

$$d - \left(U\frac{dz}{z^2} + \Lambda\frac{dz}{z}\right) \quad \text{and} \quad \nabla' := d - \left(U\frac{dz}{z^2} + \Lambda'\frac{dz}{z}\right)$$

respectively; from Appendix B we have unique formal power series $\widehat{F}, \widehat{H} \in GL_n(\mathbb{C}[\![z]\!])$ such that $\widehat{F}(0) = \widehat{H}(0) = 1$ and

$$\widehat{F}\left[U\frac{dz}{z^2} + \Lambda\frac{dz}{z}\right] = U\frac{dz}{z^2} + V\frac{dz}{z}, \tag{114}$$

$$\widehat{H}\left[U\frac{dz}{z^2} + \Lambda'\frac{dz}{z}\right] = U\frac{dz}{z^2} - V^T\frac{dz}{z}. \tag{115}$$

Now we claim that

$$\widehat{H}(z) \;=\; \widehat{F}^{-T}(-z) \tag{116}$$

as formal power series. To see this, transpose equation (114) and rewrite it in terms of $\widehat{K} := \widehat{F}^{-T}$ to obtain

$$\widehat{K}\left(U\frac{dz}{z^2} + \Lambda\frac{dz}{z}\right)\widehat{K}^{-1} - (d\widehat{K})\widehat{K}^{-1} = U\frac{dz}{z^2} + V^T\frac{dz}{z}. \tag{117}$$

Now replacing z by $-z$ in this equality of formal series and negating both sides yields (115) with $\widehat{H}(z) = \widehat{K}(-z) = \widehat{F}^{-T}(-z)$. Thus the claim follows by uniqueness.

Next, both ∇ and ∇' have the same irregular type $(U dz/z^2)$ at 0 so have the same set of anti-Stokes directions. Choose some labelling of these anti-Stokes directions and branches of $\log(z)$ following the usual convention from Remark 1.26, p10 (the same choices are used for both ∇ and ∇'). Here the pole at 0 is of order two so the set of anti-Stokes directions is symmetric under rotation by angle π (there are just two independent Stokes matrices). We will suppose without loss of generality that the real axis in the z-plane is in the *last* sector at 0, and that the permutation matrix P occurring in Proposition 1.35 (putting the Stokes matrices in triangular form), is equal to 1. Let p be the integer such that the sector Sect_i becomes the sector Sect_{i+p} when rotated by π (this integer was denoted by l in Chapter 1). Now the Stokes matrices of ∇ and ∇' are expressed in terms of the canonical solutions on sectors in Lemma 1.38. In turn these canonical solution come from 'summing' the series $\widehat{F}$ and $\widehat{H}$ on the various sectors. What we must do is convert the equation (116) into a relation between the Stokes matrices of ∇ and those of ∇'.

Lift the 'sums' $\Sigma_p(\widehat{F})$ and $\Sigma_{2p}(\widehat{F})$ (defined in Proposition 1.24) up to the universal cover $\widetilde{\mathbb{C}^*} \cong \mathbb{C}$ of $\mathbb{C}^*$ using the choices of branches of logarithm that we have made, and denote the corresponding matrix valued functions on $\widetilde{\mathbb{C}^*}$ by F_p and F_{2p} respectively. Define further functions F_{kp} on $\widetilde{\mathbb{C}^*}$ for any integer k by the prescription

$$F_{(k+2)p}(ze^{2\pi i}) = F_{kp}(z).$$

(Here we abuse notation and write functions on the universal cover in terms of z, by also specifying $\arg(z)$.) Similarly lift the sums of $\widehat{H}$ to define H_{kp} for any integer k.

The fundamental solutions of the formal normal forms of ∇ and ∇' are

$$z^{\Lambda}e^{-U/z} \qquad \text{and} \qquad z^{-\Lambda}e^{-U/z}$$

respectively. If we lift these up to $\widetilde{\mathbb{C}^*}$ using the chosen branches of logarithm, they become single valued functions on the universal cover. Then define the (lifted) canonical fundamental solutions to be

$$\widetilde{\Phi}_{kp} := F_{kp}z^{\Lambda}e^{-U/z} \quad \text{and} \quad \widetilde{\Psi}_{kp} := H_{kp}z^{-\Lambda}e^{-U/z}$$

respectively. For $k = 1, 2$ these are just the natural lifts of the usual canonical fundamental solutions on Sect_p and Sect_{2p}.

In terms of these lifted fundamental solutions, the Stokes matrices of ∇ are expressed simply as follows:

$$S_+ = S_1 = \widetilde{\Phi}_p^{-1}\widetilde{\Phi}_0 \qquad \text{and} \qquad S_- = S_2 = \widetilde{\Phi}_{2p}^{-1}\widetilde{\Phi}_p.$$

By replacing S by S' and Φ by Ψ, formulae for the Stokes matrices $S'_\pm$ of ∇' are also obtained.

Now in terms of the analytic matrix valued functions F_{kp}, H_{kp} on the universal cover, the key formula (116) translates to:

$$F_p(z) = H^{-T}_{p+kp}(ze^{\pi ik}) \qquad \text{for any } \textit{odd} \text{ integer } k. \tag{118}$$

This follows since F_p is characterised by having asymptotic expansion $\widehat{F}$ on a sector of opening $> \pi$ bisected by $\arg(z) = \pi$. The right-hand side of (118) has the same property due to (116).

This enables us firstly to relate S_+ and S'_-. The formulae for the Stokes matrices may be rewritten as:

$$S_+ = e^{U/z}z^{-\Lambda}F_p^{-1}F_0z^{\Lambda}e^{-U/z} \qquad \text{along } \arg(z) = \pi \tag{119}$$

$$S'_- = e^{U/w}w^{\Lambda}H_{2p}^{-1}H_pw^{-\Lambda}e^{-U/w} \qquad \text{along } \arg(w) = 2\pi \tag{120}$$

where w is simply another copy of the coordinate z on $\mathbb{P}^1$. (We wish to evaluate (119) and (120) at different arguments, so it is convenient to label the coordinate as w in (120).) Thus if we fix a value of z with $\arg(z) = \pi$ and set the value of w to be $w = ze^{i\pi}$ then (119) and (120) hold.

Now if we put $k = 1$ in equation (118) we find the following equality of matrices

$$F_p(z) = H_{2p}^{-T}(w).$$

Similarly we have $F_0(z) = H_p^{-T}(w)$. Substituting these into $(119)^{-T}$ we find

$$S_+^{-T} = e^{-U/z} z^{\Lambda} H_{2p}^{-1}(w) H_p(w) z^{-\Lambda} e^{U/z}.$$

Comparing this with (120) we see

$$S_+^{-T} = e^{-\pi i \Lambda} S'_- e^{\pi i \Lambda}$$

so that $S'_- = e^{i\pi\Lambda} S_+^{-T} e^{-i\pi\Lambda}$ as required. Similarly we find $S'_+ = e^{-i\pi\Lambda} S_-^{-T} e^{i\pi\Lambda}$ □

Thus under the composition $\varphi \circ \nu_U : \mathrm{End}_n(\mathbb{C}) \to G^*$ of the monodromy map ν_U and the explicit map φ, the skew-symmetric matrices correspond to the subset

$$U_+ \cong \{(S^T, S) \mid S \in U_+\} \subset G^*$$

which is the set of fixed points of the involution $\iota_{G^*} : (b_-, b_+) \mapsto (b_+^T, b_-^T)$. For the rest of this proof we will identify U_+ with this set of fixed points in G^*.

Therefore to prove Theorem 8.1 we just need to solve the following, self-contained problem in Poisson-Lie group theory:

- Calculate explicitly the Poisson structure on U_+ induced from the Poisson-Lie group structure on G^* via the involution ι_{G^*}.

The meaning of the word 'induced' here is as follows. Given a point $S \in U_+ \subset G^*$ then the derivative $(d\iota_{G^*})_S$ at S of ι_{G^*} is a linear involution of the tangent space $T_S G^*$. Thus the vector space $T_S G^*$ decomposes into two pieces; the $(+1)$ and the (-1) eigenspaces of $(d\iota_{G^*})_S$:

$$T_S G^* = (T_S G^*)_{(+1)} \oplus (T_S G^*)_{(-1)}. \tag{121}$$

(Vector space direct sum.) The $(+1)$-eigenspace is equal to the tangent space of the fixed point set U_+ of ι_{G^*} at S:

$$(T_S G^*)_{(+1)} = T_S U_+$$

and so we have a canonical projection map onto the first factor in (121):

$$\mathrm{pr}_S : T_S G^* \longrightarrow T_S U_+; \qquad \mathrm{pr}_S = \frac{1}{2}\big(1 + (d\iota_{G^*})_S\big).$$

Now if we think of the Poisson structure on G^* as a bivector on G^*

$$\mathcal{P}_{G^*} \in \Gamma(\Lambda^2 T, G^*)$$

(i.e. as the bivector on G^* such that $\{f, g\} = \langle df \otimes dg, \mathcal{P}_{G^*} \rangle)$, then by applying pr_S we obtain a bivector on U_+:

$$\mathcal{P}_{U_+} := \mathrm{pr}_S(\mathcal{P}_{G^*}). \tag{122}$$

This is the 'induced' Poisson structure on U_+. (It is not immediately clear that the Jacobi identity holds—i.e. that the bivector $\mathcal{P}_{U_+}$ is Poisson. However we will see directly that $\mathcal{P}_{G^*}$ agrees with Dubrovin's Poisson structure, so is itself Poisson.) One may check that the Poisson structure on the skew-symmetric matrices[5] arises in this way from the Poisson structure on $\mathrm{End}_n(\mathbb{C})$; this motivates the definition (122).

Now we explain how one may calculate $\mathcal{P}_{U_+}$. Here the tangent bundles of the groups U_+, G^* and G will be trivialised using their matrix representations, rather than using the group actions to identify the tangent spaces with the tangent space at the identity. Thus if $S \in U_+$ then

$$T_S U_+ = \{X_+ \in \mathrm{End}_n(\mathbb{C}) \bigm| (X_+)_{ij} = 0 \text{ if } i \geq j\} = \mathfrak{u}_+$$

is the vector space of strictly upper triangular matrices and $X_+ \in T_S U_+$ represents the derivative at $t = 0$ of the curve $S + tX_+$ through S, where $t \in \mathbb{C}$.

The projection pr_S from $T_S G^*$ onto $T_S U_+$ is given explicitly by the formula

$$(X_-, X_+) \longmapsto \frac{1}{2}(X_+ + X_-^T) \tag{123}$$

where $(X_-, X_+) \in T_{(S^T,S)} G^* = \mathfrak{u}_- \times \mathfrak{u}_+$.

To get the Poisson structure on G^* explicitly, we use the formula (108). This calculates the Poisson structure on G induced from the Poisson-Lie group structure on G^* via the map

$$\pi' : G^* \longrightarrow G; \qquad (b_-, b_+) \longmapsto b_-^{-1} b_+.$$

This map is a 2^n-fold covering of its image. We do not have (even locally) an explicit formula for its inverse. The useful observation now is that we can however explicitly invert the *derivative* of π', and this is all we need to calculate the Poisson structure on G^*. At $(S^T, S) \in G^*$ the derivative of π' is

$$d\pi'_{(S^T,S)}(X_-, X_+) = S^{-T} X_+ - S^{-T} X_- S^{-T} S \in T_{S^{-T}S} G = \mathrm{End}_n(\mathbb{C}).$$

The inverse of this is as follows. If $Y = S^{-T} X_+ - S^{-T} X_- S^{-T} S$ then

$$\begin{cases} X_- &= -\big(P_-(S^T Y S^{-1})\big) S^T \\ X_+ &= \big(P_+(S^T Y S^{-1})\big) S \end{cases} \tag{124}$$

where $P_\pm := (1/2)\delta + \pi_\pm : \mathrm{End}_n(\mathbb{C}) \to \mathrm{End}_n(\mathbb{C})$, $\pi_\pm$ are the projections onto the strictly upper/lower triangular matrices and δ projects onto the diagonal matrices.

Thus by composing (124) with the projection (123) we get an explicit linear map

$$T_{S^{-T}S} G \longrightarrow T_S U_+; \qquad Y \longmapsto \frac{1}{2}\big(P_+(S^T Y S^{-1}) S - S(P_-(S^T Y S^{-1}))^T\big) \tag{125}$$

taking tangents of G to tangents of U_+, for any $S \in U_+$ and $Y \in \mathrm{End}_n(\mathbb{C}) = T_{S^{-T}S} G$.

Hence, given a function on U_+ (e.g. x, y or z in the 3-dimensional case), its derivative at S is an element of $T_S^* U_+$ and the formula (125) identifies this with a linear form on $T_{S^{-T}S} G$. This is just a linear combination of the matrix entries of Y (e.g. x corresponds to the $(1,2)$ matrix entry of the expression on the right of (125)). Now the formula (108) explicitly gives the value of the Poisson bivector on G evaluated on the matrix entries of Y. Using a computer algebra package (Mathematica) we find that the formulae (125) and (108) do indeed yield Dubrovin's Poisson structure (111) (upto an overall constant

[5] The skew-symmetric matrices are the Lie algebra $\mathfrak{o}_n(\mathbb{C})$ of $O_n(\mathbb{C})$ (or $SO_n(\mathbb{C})$) in the standard representation, so the dual $\mathfrak{o}_n(\mathbb{C})^*$ has a standard (Lie) Poisson structure. Then $\mathfrak{o}_n(\mathbb{C})^*$ is identified with $\mathfrak{o}_n(\mathbb{C})$ using the trace pairing $\mathrm{Tr}(AB)$.

factor) $\square$

REMARK 8.5. The same calculation has been repeated in the 4×4 case and agrees with the formula found by M.Ugaglia [**102**]:

$$S := \begin{pmatrix} 1 & u & v & w \\ 0 & 1 & x & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\{u, z\} = 0 \qquad \{v, y\} = 2uz - 2xw \qquad \{w, x\} = 0$$

$$\begin{array}{llll} \{u, v\} = 2x - uv & \{u, w\} = 2y - uw & \{x, u\} = 2v - xu & \{y, u\} = 2w - yu \\ \{v, w\} = 2z - vw & \{v, x\} = 2u - vx & \{z, v\} = 2w - zv & \{w, y\} = 2u - wy \\ \{w, z\} = 2v - wz & \{x, y\} = 2z - xy & \{z, x\} = 2y - zx & \{y, z\} = 2x - yz \end{array}$$

by pushing forward the Poisson structure on the skew-symmetric matrices along the monodromy map. Thus the Poisson structure in the 4×4 case also comes from Poisson-Lie groups. In the $n \times n$ case the formula coming from Poisson-Lie groups has not been calculated[6], but is expected to agree with that found in [**102**]; this would follow immediately from Conjecture 7.5. (The aim of these calculations was to check the plausibility of the conjecture.)

2. Explicit Local Frobenius Manifolds

In this section our perspective changes. The aim here is to explain the answer to a modified version of the following question raised by Hitchin in [**48**]. (The work in this section was done before the rest of this thesis, and first appeared in [**19**].)

In the first instance, in [**31**] Dubrovin translates the WDVV equations for a semisimple Frobenius manifold into the isomonodromy equations for the operator (110) with an irregular singularity. Explicitly these (nonlinear) equations for the skew-symmetric matrix $V \in \mathfrak{o}_n(\mathbb{C})$ are

$$dV + [A, V] = 0, \qquad A := \text{ad}C(\text{ad}U)^{-1}(V) \tag{126}$$

where d is the exterior derivative on the space $X \cong \mathbb{C}^n \setminus$ (diagonals) of deformation parameters, $U = \text{diag}(u_1, \ldots, u_n)$ ($u_1, \ldots, u_n$ are the natural coordinates functions on $\mathbb{C}^n$), $C := dU$ (a matrix of one-forms on X) and ad denotes the usual bracket operation on matrices. A local solution of (126) determines (upto a discrete choice) a Frobenius manifold structure on a neighbourhood in X, in particular the rank n of the matrices is the dimension of the Frobenius manifold. (The path between local solutions of (126) and a semisimple Frobenius manifold is explained succinctly in Section 4 of [**48**].) Thus the set of solutions to (126)—and hence the space of semisimple Frobenius manifolds—is naturally given by the monodromy data of the meromorphic connection (110) on $\mathbb{P}^1$, i.e. with the set of upper triangular Stokes matrices U_+, as explained in Section 1 above.

However Dubrovin also observed (Remark 3.9 p219 [**31**]) that equation (126) arises also as the isomonodromy equations for a *logarithmic* connection on $\mathbb{P}^1$ (i.e. with only simple poles). Hitchin [**48**] picked up this thread and studied the geometry of such isomonodromic

[6]Note that in [**102**] the $n \times n$ case is essentially deduced from the 4×4 case.

deformations coming from Frobenius manifolds; the idea being that the monodromy data of a connection on $\mathbb{P}^1$ with simple poles is essentially just a representation of the fundamental group of the corresponding punctured $\mathbb{P}^1$ (and so is much more familiar geometrically than a Stokes matrix). In particular the local moduli space space of semisimple Frobenius manifolds should appear as some set of fundamental group representations.

The question Hitchin asked was to describe/characterise the set of such representations of the fundamental group of the punctured sphere that come from Frobenius manifolds.

With hindsight we see that the difficulty occurs in some sense because [**31**] Remark 3.9 and [**48**] use the 'wrong' choice of logarithmic connection. Fortunately the 'right' choice also occurs naturally in the theory of Frobenius manifolds and is in Appendix H of Dubrovin's paper [**31**]. The fortuitous introduction of a constant term 1/2 into the original logarithmic connection implies that it preserves a certain complex bilinear form. Thus the corresponding monodromy data (the fundamental group representation) is also naturally restricted: the local monodromy around each finite pole in $\mathbb{P}^1$ will be an $O_n(\mathbb{C})$-reflection.

It follows that the monodromy data of the new logarithmic connection—the moduli point of the semisimple Frobenius manifold—is an n-tuple of reflections in $O_n(\mathbb{C})$. A simple calculation shows that the set of equivalence classes of such n-tuples (modulo $O_n(\mathbb{C})$ conjugation) generically has dimension $n(n-1)/2$. Thus we do indeed have a good characterisation of the monodromy data corresponding to $V \in \mathfrak{o}_n(\mathbb{C})$, and therefore another description of the moduli space of semisimple Frobenius manifolds.

Thus there are two parallel ways of thinking of the moduli of semisimple Frobenius manifolds: in terms of Stokes matrices or in terms of reflections. Of course this story was essentially already known to Dubrovin in [**31**] but is included here since it took some time to understand. Note that Dubrovin finds many examples of Frobenius manifolds by starting with a nice subgroup of $O_n(\mathbb{C})$ generated by reflections and working backwards: see [**31**] Lecture 4 for example.

To summarise, in the three sections below we will:

1) Prove directly that Equation (126) arises as the isomonodromy equations of a logarithmic connection on $\mathbb{P}^1$. In fact we show this for any member of a family of connections parameterised by $\varepsilon \in \mathbb{C}$.

2) Show that a certain bilinear form is preserved iff $\varepsilon = 1/2$, and then deduce that *reflections* occur in the corresponding monodromy. This gives the required characterisation, answering the analogue of Hitchin's question for the modified logarithmic connection.

3) Explain how to relate this logarithmic-singular $O_n(\mathbb{C})$ isomonodromy problem to Schlesinger's equations (for $GL_n(\mathbb{C})$ logarithmic isomonodromic deformations); the key point here is to relate two natural choices of trivialisation of the trivial vector bundle over $X \times \mathbb{P}^1$.

2.1. Logarithmic connections.

Let X be an open ball in $\mathbb{C}^n$ not intersecting any of the diagonals $u_i = u_j$, where $u_1, \ldots, u_n$ are the usual coordinates on $\mathbb{C}^n$. We will study the nonlinear equation (126) for a skew symmetric matrix $V(u) \in \mathfrak{o}_n(\mathbb{C})$ dependent on $u \in X$.

Lemma 8.6. *If $V(x)$ satisfies Equation (126) then*

$$\nabla_A := d + A$$

is a flat connection on the trivial rank n complex vector bundle over X.

Proof. Let E_i be the matrix whose only non-zero entry is a 1 in its (i,i) position, so the du_i component of A is

$$A_i := \mathrm{ad}E_i(\mathrm{ad}U)^{-1}V.$$

Firstly $\frac{\partial(\mathrm{ad}U)^{-1}}{\partial u_j} = -(\mathrm{ad}U)^{-1}\mathrm{ad}E_j(\mathrm{ad}U)^{-1} = -\mathrm{ad}E_j(\mathrm{ad}U)^{-2}$ since $\mathrm{ad}P$ and $\mathrm{ad}Q$ commute for any diagonal matrices P and Q. Thus

$$\begin{aligned}\frac{\partial A_i}{\partial u_j} &= -\mathrm{ad}E_i\mathrm{ad}E_j(\mathrm{ad}U)^{-2}V + \mathrm{ad}E_i(\mathrm{ad}U)^{-1}(-A_jV + VA_j)\\ &= -\mathrm{ad}E_i\mathrm{ad}E_j(\mathrm{ad}U)^{-2}V - \mathrm{ad}E_i(\mathrm{ad}U)^{-1}\mathrm{ad}E_j(\mathrm{ad}U)^{-1}V^2 + A_iA_j\\ &= (\text{terms symmetric in } i \text{ and } j) + A_iA_j\end{aligned}$$

Hence $\partial A_j/\partial u_i - \partial A_i/\partial u_j = [A_j, A_i]$ and we see the connection is flat □

Note that the matrix A is skew-symmetric and so ∇_A preserves the $O_n(\mathbb{C})$ metric induced from the standard bilinear form on $\mathbb{C}^n$.

Given any map $V : X \to \mathfrak{o}_n(\mathbb{C})$ we define $A := \mathrm{ad}C(\mathrm{ad}U)^{-1}V$ as above and then define a meromorphic connection $\widetilde{\nabla}$ on the trivial rank n vector bundle over $X \times \mathbb{P}^1$ with only logarithmic singularities by the formula:

$$\widetilde{\nabla} := d + A + (z.I - U)^{-1}(C - I.dz)(V - \varepsilon.I)$$

where z is the usual coordinate on $\mathbb{C} \subset \mathbb{P}^1$, d is the full exterior derivative on $X \times \mathbb{P}^1$ and $\varepsilon \in \mathbb{C}$ is an arbitrary constant. This definition is motivated by [**31**] Proposition H.2 and [**48**] Proposition 5.1. Below we will suppress the identity matrices (I's) multiplying the scalars. Thus $\widetilde{\nabla} := d + A + B$ where

$$B := (z - U)^{-1}(C - dz)(V - \varepsilon)$$

is a matrix of meromorphic one-forms on $X \times \mathbb{P}^1$. The curvature of $\widetilde{\nabla}$ is the matrix of two-forms:

$$(dA + A^2) + (dB + AB + BA) + B^2 = (dA + A^2) + d_AB + B^2$$

where d_A is the exterior covariant derivative of ∇_A pulled back to $X \times \mathbb{P}^1$.

Lemma 8.7. *If $V : X \to \mathfrak{o}_n(\mathbb{C})$ is any smooth map and we define $A := \mathrm{ad}C(\mathrm{ad}U)^{-1}V$ and $B := (z-U)^{-1}(C - dz)(V - \varepsilon)$ as above, then*

$$d_AB + B^2 = -(z - U)^{-1}(C - dz)(\nabla_AV)$$

(This is similar to [**48**] Proposition 5.1.)

Proof. First observe $(C - dz)$ squares to zero and commutes with $(z-U)^{-1}$. Now

$$\nabla_A(z-U)^{-1} = -(z-U)^{-1}(dz - C - [A,U])(z-U)^{-1}$$

and

$$\begin{aligned}d_A(C - dz) &= dC + AC - Adz + CA - dzA = AC + CA\\ &= C(\mathrm{ad}U)^{-1}VC - (\mathrm{ad}U)^{-1}VC^2 + C^2(\mathrm{ad}U)^{-1}V - C(\mathrm{ad}U)^{-1}VC\\ &= 0 \qquad (\text{since } C^2 = 0 \text{ and the other terms cancel}).\end{aligned}$$

Thus

$$\begin{aligned} d_A B &= (\nabla_A (z-U)^{-1})(C-dz)(V-\varepsilon) - (z-U)^{-1}(C-dz)(\nabla_A V) \\ &= -(z-U)^{-1}[U,A](z-U)^{-1}(C-dz)(V-\varepsilon) - (z-U)^{-1}(C-dz)(\nabla_A V) \end{aligned}$$

(two $(C-dz)$'s have annihilated). On the other hand:

$$\begin{aligned} B^2 &= (z-U)^{-1}(C-dz)(V-\varepsilon)(z-U)^{-1}(C-dz)(V-\varepsilon) \\ &= (z-U)^{-1}(C-dz)V(z-U)^{-1}(C-dz)(V-\varepsilon) \\ &\qquad \text{(the } (C-dz)\text{'s collide through the } \varepsilon) \\ &= (z-U)^{-1}[U,A](z-U)^{-1}(C-dz)(V-\varepsilon) \\ &\qquad \text{(since } (C-dz)V = [C,V] + VC - Vdz = [U,A] + V(C-dz)). \end{aligned}$$

The result now follows immediately by adding up the above expressions for $d_A B$ and B^2 $\square$

The main result of this section is now easy to prove:

Proposition 8.8. *Fix any constant $\varepsilon \in \mathbb{C}$. Then a map $V : X \to \mathfrak{o}_n(\mathbb{C})$ satisfies Equation (126) if and only if the connection $\widetilde{\nabla}$ constructed from V is flat.*

Proof. Recall the curvature of $\widetilde{\nabla}$ is $(dA+A^2)+d_A B+B^2$. Thus if V satisfies Equation (126), i.e. $\nabla_A V = 0$, then Lemmas 8.6 and 8.7 imply that $\widetilde{\nabla}$ is flat.

Conversely if V is such that $\widetilde{\nabla}$ is flat, using Lemma 8.7 we obtain:

$$(dA + A^2) - (z-U)^{-1}(C-dz)(\nabla_A V) = 0$$

Examining the $dz \wedge du_i$ component of this for each i we see that $\nabla_A V = 0$ $\square$

2.2. Orthogonality Properties and the Monodromy Map.

We make the trivial rank n vector bundle over $X \times \mathbb{P}^1$ into a degenerate $O_n(\mathbb{C})$ bundle by equipping it with the complex bilinear form $(,)$ defined by:

$$(s,t) := s^T(z-U)t = \sum_{i=1}^{n} (z-u_i)s_i t_i$$

where s, t are local sections regarded as column vectors. Taking the value of ε to be $1/2$ is special due to the following result:

Lemma 8.9. *If $V : X \to \mathfrak{o}_n(\mathbb{C})$ is any map then $\widetilde{\nabla}$ preserves the bilinear form $(,)$ if and only if $\varepsilon = 1/2$.*

Proof. We just calculate:

$$\begin{aligned} &d(z-U) - (A+B)^T(z-U) - (z-U)(A+B) \\ &\quad = dz - C - A^T(z-U) - (V-\varepsilon)^T(C-dz) - (z-U)A - (C-dz)(V-\varepsilon) \\ &\quad = [U,A] - [C,V] + (2\varepsilon - 1)(C-dz) \qquad \text{(since } A, V \text{ are skew, all else cancels)} \\ &\quad = (2\varepsilon-1)(C-dz) \qquad \text{(by definition of } A). \end{aligned}$$

But $\widetilde{\nabla}$ preserves $(,)$ iff this vanishes $\square$

For the rest of this section we will suppose that $\varepsilon = 1/2$ and that V satisfies Equation (126). More concretely Lemma 8.9 means that if $\Phi(u,z)$ is a local fundamental solution of $\widetilde{\nabla}$ then then $d(\Phi^T(z-U)\Phi) = 0$.

Observe $\widetilde{\nabla}$ has logarithmic singularities on the $n+1$ subvarieties:

$$Y_i := \{(u,z) \in X \times \mathbb{P}^1 | u_i = z\}, \quad Y_\infty := \{(u,z) \in X \times \mathbb{P}^1 | z = \infty\}$$

of $X \times \mathbb{P}^1$. We see this clearly by rewriting $\widetilde{\nabla}$ in the following way:

$$\widetilde{\nabla} = d + A - \sum_{i=1}^n \frac{d(z-u_i)}{z-u_i} E_i(V - 1/2).$$

In particular $\widetilde{\nabla}$ restricts to a genuine flat connection over the complement

$$Z := X \times \mathbb{P}^1 \backslash (Y_1 \cup \ldots \cup Y_n \cup Y_\infty)$$

and gives a Fuchsian isomonodromic deformation. Observe also that $(,)$ is nondegenerate over Z, and that the resulting $O_n(\mathbb{C})$ bundle is not (even topologically) isomorphic to the trivial $O_n(\mathbb{C})$ bundle over Z.

The projection $\mathrm{pr} : Z \to X$ expresses Z as a fibration over X; the fibres being $\mathrm{pr}^{-1}(u) = \mathbb{P}^1 \backslash \{u_1, \ldots, u_n, \infty\}$. Assuming X is small enough we choose n loops

$$\gamma_i : [0,1] \to \mathbb{P}^1$$

all based at some $p \in \mathbb{P}^1$ such that for any $u = (u_1, \ldots, u_n) \in X$, γ_i is a simple closed loop dividing $\mathbb{P}^1$ into two pieces, one containing u_i and the other containing $\{u_j \big| j \neq i\} \cup \{\infty\}$.

The homotopy classes of these loops freely generate the fundamental group $\pi_1(\mathrm{pr}^{-1}(u), p)$ of the punctured sphere $\mathrm{pr}^{-1}(u)$ for any $u \in X$. Moreover, since X is contractible, for each $u \in X$ the loops in $\mathrm{pr}^{-1}(u)$ generate $\pi_1(Z, (u,p))$ freely too (this follows from the homotopy long exact sequence for fibrations).

Now define $C_1 \subset O_n(\mathbb{C})$ to be the conjugacy class of the reflections, i.e. all the elements in $O_n(\mathbb{C})$ which are conjugate to $\mathrm{diag}(-1, 1, \ldots, 1)$. The space of monodromy data we are interested in is the set of $O_n(\mathbb{C})$ orbits of n-tuples of reflections: define the *monodromy manifold* to be

$$M := (C_1)^n / O_n(\mathbb{C})$$

where $O_n(\mathbb{C})$ acts on the product via diagonal conjugation:

$$g : (r_1, \ldots, r_n) \mapsto (g r_1 g^{-1}, \ldots, g r_n g^{-1}).$$

As usual for affine quotients, M has a dense open subset which is a smooth manifold. In fact we can see that the dimension of M is $n(n-1)/2$ since C_1 is of dimension $n-1$ ($=\dim \mathbb{P}^{n-1}$) and the action of $O_n(\mathbb{C})$ is generically free.

If we fix some point $u \in X$ and restrict $\widetilde{\nabla}$ to the projective line $\{u\} \times \mathbb{P}^1$ then it is given by the expression

$$d - \sum_{i=1}^n \frac{E_i(V - 1/2)}{z - u_i} dz. \tag{127}$$

This preserves the bilinear form restricted to $\{u\} \times \mathbb{P}^1$ and so we obtain:

Proposition 8.10. *The monodromy map induces a map*

$$\nu_u : \mathfrak{o}_n(\mathbb{C}) \longrightarrow M$$

from the set of skew-symmetric matrices to the monodromy manifold M, by taking V to the monodromy of the connection (127) on $\mathbb{P}^1 \setminus \{u_1, \dots u_n, \infty\}$.

Proof. We need to check that the local monodromy of (127) around a finite pole is a reflection. Thus pick a base point $p \in \mathbb{P}^1 \setminus \{u_1, \dots u_n, \infty\}$ and generators of the fundamental group as above. Let M_i be the monodromy matrix obtained by parallel translating a frame around the loop γ_i using (127) (i.e. $\Phi(\gamma_i(1)) = M_i$ where the columns of Φ are parallel and $\Phi(\gamma_i(0)) = I$). Let $N_i := SM_iS^{-1}$ for some choice of (diagonal) square root S of the matrix $(p.I - U(u))$. We take the element of the monodromy manifold corresponding to V to be the orbit of these matrices N_i

$$\nu_u(V) := [(N_1, \dots, N_n)] \in M.$$

This is well defined: M_i (and therefore N_i) is conjugate in $GL_n(\mathbb{C})$ to $\mathrm{diag}(-1, 1, \dots, 1)$ since the residue at u_i of the connection (127) is conjugate to $\mathrm{diag}(1/2, 0, \dots, 0)$. (This follows essentially from Appendix B: note that we only really need the eigenvalues of the residue to be distinct modulo *non-zero* integers. Also in the simple pole case any formal isomorphism is convergent: see [**46**].) Thus there is some basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ such that $N_ie_1 = -e_1$ and $N_ie_j = e_j$ for all $j \neq 1$. Also by Lemma 8.9: $(M_iv, M_iv) = (v, v)$ for all $v \in \mathbb{C}^n$, i.e $M_i^T(p.I - U(u))M_i = (p.I - U(u))$ and this immediately implies $N_i \in O_n(\mathbb{C})$. In particular if $j \neq 1$ then

$$e_1^T e_j = e_1^T N_i^T N_i e_j = (N_ie_1)^T N_i e_j = -e_1^T e_j.$$

It follows that e_1 is not isotropic ($e_1^T e_1 \neq 0$) and that e_1 is orthogonal to the hyperplane $\langle e_2, \dots, e_n\rangle$. Thus N_i is a reflection in the direction e_1 fixing the orthogonal hyperplane $\langle e_2, \dots, e_n\rangle$, i.e. $N_i \in C_1$. The image in M is independent of the choice of square root S since this choice corresponds to diagonal conjugation by $\mathrm{diag}(\pm 1, \dots, \pm 1) \in O_n(\mathbb{C})$. Also changing the choice of basepoint p just conjugates the monodromy representation and so the image in M is the same. □

Note: The map ν_u does depend on the choice of generators $[\gamma_i]$.

2.3. Relation with Schlesinger's Equations.

We will identify Equation (126) with a form of Schlesinger's equations.

If $\widetilde{\nabla}$ is flat (i.e. V satisfies (126)) then ∇_A is flat and we can obtain a gauge transformation $g : X \to O_n(\mathbb{C})$ such that $A = g^{-1}dg$ and thus changing $\widetilde{\nabla}$ to

$$\widetilde{\nabla}' := d + \sum_{i=1}^{n} S_i \frac{d(z - u_i)}{z - u_i}$$

where $S_i := -gE_i(V - 1/2)g^{-1}$ and g is regarded as constant in the $\mathbb{P}^1$ direction. Schlesinger's equations are precisely the condition on the matrices $S_i(u)$ such that the connection $\widetilde{\nabla}'$ is flat:

$$dS_i = \sum_{j \neq i} [S_i, S_j] \frac{d(u_i - u_j)}{u_i - u_j}. \tag{128}$$

We may rewrite this as

$$dS_i + [B_i, S_i] = 0, \quad \text{where } B_i := \sum_{j\neq i} S_j \frac{d(u_i - u_j)}{u_i - u_j}.$$

($d + B_i$ is the flat connection induced from $\widetilde{\nabla}'$ on Y_i, which we may identify with X.) Observe that each S_i is a rank 1, trace 1/2 matrix, $(\sum_{i=1}^n S_i) - 1/2 \in \mathfrak{o}_n(\mathbb{C})$ and that the images of the S_i are orthogonal. The key idea is that this works backwards too; given $(S_1, \dots, S_n)$ satisfying Schlesinger's equations (128) and these conditions then we can determine g (from the images of the S_i) and more or less recover V.

To make this precise it is convenient to introduce the space

$$Q := \{(v_1, \dots, v_n, \alpha_1, \dots, \alpha_n) | v_i, \alpha_i^T \in \mathbb{C}^n, \alpha_i(v_j) + \alpha_j(v_i) = \delta_{ij} = v_i^T v_j\}$$

i.e. the v_i are the columns of an orthogonal matrix and the α_i are row vectors such that the matrix $(\alpha_i(v_j) - \delta_{ij}/2)$ is skew-symmetric. There is a free action of $O_n(\mathbb{C})$ on Q:

$$g : (v_1, \dots, v_n, \alpha_1, \dots, \alpha_n) \mapsto (gv_1, \dots, gv_n, \alpha_1 g^T, \dots, \alpha_n g^T).$$

The quotient $Q/O_n(\mathbb{C})$ is isomorphic to $\mathfrak{o}_n(\mathbb{C})$, since there is a unique $g \in O_n(\mathbb{C})$ taking each v_i to the ith standard basis vector. An explicit projection onto the quotient is given by

$$\pi : Q \to \mathfrak{o}_n(\mathbb{C}); \quad (\pi(v, \alpha))_{ij} := (1/2)\delta_{ij} - \alpha_i(v_j).$$

Proposition 8.11. *The solutions to Equation (126) are exactly the maps*

$$\pi \circ (v, \alpha) : X \to \mathfrak{o}_n(\mathbb{C})$$

for $(v, \alpha) = (v_1, \dots, v_n, \alpha_1, \dots, \alpha_n) : X \to Q$ *satisfying the nonlinear equations:*

$$dv_i + B_i v_i = 0 = d\alpha_i - \alpha_i B_i, \quad B_i := \sum_{j\neq i} v_j \otimes \alpha_j \frac{d(u_i - u_j)}{u_i - u_j}. \tag{129}$$

(These equations appear in [**59**].)

Proof. The idea is simply to identify flows in $(Q/O_n(\mathbb{C}))$ with $O_n(\mathbb{C})$ orbits of flows in Q.

Firstly a simple check will show that the equations (129) are well-defined and invariant under the $O_n(\mathbb{C})$ action on Q. Now if (v, α) satisfies (129) and we define $S_i := v_i \otimes \alpha_i$ then

$$dS_i = (dv_i) \otimes \alpha_i + v_i \otimes d\alpha_i = -B_i v_i \otimes \alpha_i + v_i \otimes \alpha_i B_i = -[B_i, S_i]$$

and so the connection $\widetilde{\nabla}' := d + \sum_{i=1}^n S_i \frac{d(z-u_i)}{z-u_i}$ is flat. Now define a gauge transformation $g : X \to O_n(\mathbb{C})$ by taking v_i to be the ith column of g. Also set $V := \pi \circ (v, \alpha) : X \to \mathfrak{o}_n(\mathbb{C})$ and $A := \mathrm{ad}C(\mathrm{ad}U)^{-1}V$. It follows that $g^{-1}S_i g = E_i(-V + 1/2)$ and so the meromorphic connection

$$d + g^{-1}dg + (z - U)^{-1}(C - dz)(V - 1/2) \tag{130}$$

on $X \times \mathbb{P}^1$ is flat. The crucial result now is that $g^{-1}dg = A$ and so (130) is the connection $\widetilde{\nabla}$ associated to V. To see this we just calculate:

$$\begin{aligned}(g^{-1}dg)_{ij} &= (\text{row } i \text{ of } g^T)d(\text{column } j \text{ of } g)\\ &= -v_i^T B_j v_j = -\sum_{k=1}^{n} v_i^T v_k \alpha_k(v_j)\frac{d(u_k-u_j)}{u_k-u_j}\\ &= -\alpha_i(v_j)\frac{d(u_i-u_j)}{u_i-u_j} = (\mathrm{ad}C(\mathrm{ad}U)^{-1}V)_{ij}\\ &= (A)_{ij}.\end{aligned}$$

Thus (130) is the connection $\widetilde{\nabla}$ associated to V and so by Proposition 8.8, V satisfies Equation (126).

Conversely suppose V satisfies Equation (126). Pick a base-point $u^0 \in X$. By Lemma 8.6, ∇_A is flat so we can solve the equation

$$dg + gA = 0$$

over X (which is simply connected) to obtain a gauge transformation $g : X \to O_n(\mathbb{C})$ such that $g(u^0) = I$. Then define

$$S_i := gE_i(-V + 1/2)g^{-1}, \quad \text{and} \quad B_i := \sum_{j \neq i} S_j \frac{d(u_i-u_j)}{u_i-u_j}.$$

It follows that $d + \sum_{i=1}^{n} S_i \frac{d(z-u_i)}{z-u_i}$ is flat and so each $d + B_i$ is a flat connection over X. Let $v_i(u^0)$ be the ith standard basis vector and $\alpha_i(u^0)$ be the ith row of $(1/2 - V)$. Then parallel translate v_i, α_i around X using the flat connection $d + B_i$ to give the required solution $(v, \alpha) : X \to Q$ satisfying (129). (Choosing a different base-point is equivalent to a different choice of initial condition $g(u^0) \in O_n(\mathbb{C})$, which corresponds to the $O_n(\mathbb{C})$ action on Q) $\square$

Now let

$$\Lambda := \{\lambda = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) | \epsilon_i = \pm 1\} \subset O_n(\mathbb{C})$$

be the subgroup of diagonal orthogonal matrices. The adjoint action restricts to an action of Λ on $\mathfrak{o}_n(\mathbb{C})$ ($\lambda : V \mapsto \lambda V \lambda$). Also define $\mathcal{R} := \{(S_1, \ldots, S_n) | S_i$ is a rank 1,trace 1/2 matrix, $(\sum_{i=1}^{n} S_i) - 1/2 \in \mathfrak{o}_n(\mathbb{C})$ and the images of the S_i are orthogonal$\}$. Schlesinger's equations (128) are well defined on $\mathcal{R}$ and invariant under diagonal $O_n(\mathbb{C})$ conjugation.

The main result relating Schlesinger's equations (128) to the basic equation (126) for V is then

Corollary 8.12. *The $O_n(\mathbb{C})$ orbits of solutions $(S_1, \ldots, S_n)$ to Schlesinger's equations in $\mathcal{R}$ coincide with the Λ orbits of solutions to Equation (126).*

Proof. Just look at the point-wise map:

$$(v, \alpha) \mapsto (S_1, \ldots, S_n) := (v_1 \otimes \alpha_1, \ldots, v_n \otimes \alpha_n)$$

$\square$

APPENDIX A

Painlevé Equations and Isomonodromy

We give a brief account of the Painlevé equations and isomonodromic deformations, upto the work [**60**] of Jimbo, Miwa and Ueno. The main references used here are [**103, 104, 56**].

The story of isomonodromic deformations began with Riemann's work on the hypergeometric function. He was the first to consider the monodromy map (or *Riemann*-Hilbert map) and he also raised the question of the deformation theory, which led to Schlesinger's work which we described in the introduction.

The story of the Painlevé equations starts as follows. At the end of the nineteenth century many people were trying to discover new transcendental functions. One way to determine functions is in terms of differential equations: write down a differential equation and hope its solutions are 'new' functions. Thus one seeks 'good' differential equations.

Even if we restrict to *algebraic* differential equations of the form

$$F(t, y, y', \dots, y^{(r)}) = 0 \tag{131}$$

(where F is a polynomial, y is a function of $t \in \mathbb{C}$ and $y^{(r)} = \frac{d^r y}{dt^r}$) then the solutions $y(t)$ can still have nasty properties. For example if

$$y' = \frac{1}{2y} \tag{132}$$

then $y = \pm\sqrt{t-c}$ for some constant $c \in \mathbb{C}$; the position of the branch point of the solution depends on the integration constant: equation (132) has a *movable branch point*. Also if we differentiate the function

$$y := c_1 \exp\left(\frac{1}{t - c_2}\right) \tag{133}$$

twice with respect to t, it is not hard to eliminate the constants c_1 and c_2 and write down a second order equation whose solutions (133) have essential singularities depending on the integration constants, i.e. having *movable essential singularities*.

The differential equation (131) is said to have the 'Painlevé property' if its solutions do *not* have movable branch points or movable essential singularities. Solutions may have poles which depend on the integration constants, but any branch points or essential singularities must be fixed. The key motivational result is then:

Theorem. *(Poincaré, L.Fuchs)*

If a first order algebraic differential equation has the Painlevé property then it can be reduced to either

- *A linear equation, or*
- *The equation:* $(y')^2 = y^3 - g_2 y - g_3$ *for some constants* $g_2, g_3 \in \mathbb{C}$

and that's all. □

(See for example [**53**] or [**56**] for more details/precision.)

The second equation here is of course the equation satisfied by the Weierstrass $\wp$ function; the Painlevé property leads to a very good pedigree of functions.

In the light of this theorem many mathematicians, notably Poincaré and Picard, sought higher order equations with the Painlevé property which did not reduce to known equations (it is well known that any equation linear in y and its derivatives has the Painlevé property; the singularities of the solutions occur at the singularities of the coefficients). Eventually they became pessimistic about the existence of such new equations and gave up. Indeed Picard wrote to Mittag-Leffler in 1893 expressing his doubts; he had observed that movable essential singularities are possible if $r \geq 2$ but not for $r = 1$. It is also a very algebraically complex problem.

At this point Painlevé enters the story. In spite of the general pessimism he attacked the problem of finding second order equations with the Painlevé property and (heroically) together with his student Gambier, found a list of fifty such equations (see [**53**]). It turned out that forty-four of these were reducible, leaving six new equations with the Painlevé property. These are now commonly referred to as the 'Painlevé equations' and are given in Table 1.

$$
\begin{array}{ll}
\text{PI:} & y'' = 6y^2 + t \\
\text{PII:} & y'' = 2y^3 + ty + \alpha \\
\text{PIII:} & y'' = \dfrac{(y')^2}{y} - \dfrac{y'}{t} + \dfrac{\alpha y^2 + \beta}{t} + \gamma y^3 + \dfrac{\delta}{y} \\
\text{PIV:} & y'' = \dfrac{(y')^2}{2y} + \dfrac{3y^3}{2} + 4ty^2 + 2(t^2 - \alpha)y + \dfrac{\beta}{y} \\
\text{PV:} & y'' = \left(\dfrac{1}{2y} + \dfrac{1}{y-1}\right)(y')^2 - \dfrac{y'}{t} + \dfrac{(y-1)^2}{t^2}\left(\alpha y + \dfrac{\beta}{y}\right) + \dfrac{\gamma y}{t} + \dfrac{\delta y(y+1)}{y-1} \\
\text{PVI:} & y'' = \left(\dfrac{1}{y} + \dfrac{1}{y-1} + \dfrac{1}{y-t}\right)\dfrac{(y')^2}{2} - \left(\dfrac{1}{t} + \dfrac{1}{t-1} + \dfrac{1}{y-t}\right)y' \\
& \qquad\qquad + \dfrac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \dfrac{\beta t}{y^2} + \dfrac{\gamma(t-1)}{(y-1)^2} + \dfrac{\delta t(t-1)}{(y-t)^2}\right)
\end{array}
$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are parameters.

TABLE 1. The Painlevé Equations (Painlevé/Gambier $\sim$ 1906).

Three questions now spring immediately to mind:

- What is the geometry underlying these equations?
- Do they have any applications?
- Are there more new equations having even higher order?

Painlevé knew of no applications; they seemed 'cut-off on a separate island from the continent of analysis'. In 1906 Painlevé gave up mathematics and went on to become the Prime Minister of France (twice). Two of his students, Chazy and Garnier, pursued higher order equations with some success but no general pattern emerged due to the increasing complexity.

Some indication of the underlying geometry was soon found by R.Fuchs [**37**]: in modern language he found that the sixth Painlevé equation arises as the equation governing isomonodromic deformations of certain meromorphic connections over $\mathbb{P}^1$ with simple poles. In effect Fuchs discovered that in the case with only four poles ($m = 4$) and with the matrices A_i being 2×2 and trace free then Schlesinger's equations

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \qquad \text{if } i \neq j, \text{ and}$$
$$\frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}$$

are equivalent to the sixth Painlevé equation. The time variable t in PVI is the cross-ratio of the four points a_1, a_2, a_3, a_4 and the the parameters $\alpha, \beta, \gamma, \delta$ correspond to the choice of eigenvalues of the A_i's (which are constant throughout the deformation).

Thus the most complicated looking of the Painlevé equations arises in a natural geometrical problem and is the simplest nontrivial member of the family of Schlesinger equations. Nonetheless no physical applications were known and the Painlevé equations were then more-or-less forgotten for sixty years, being regarded somewhat as a mere mathematical curiosity. That is, until the late 1970's.

It was then that mathematical physicists Wu, McCoy, Tracy and Barouch discovered an unexpected link with quantum field theory [**110, 80**]. They found that the 'correlation functions' in certain quantum field theories satisfied Painlevé equations! Subsequently Jimbo, Miwa, Môri and Sato showed that this was a special case of a more general phenomenon in a series of papers developing the theory of 'holonomic' quantum fields [**93, 59**]. We don't want to delve into the physics here; the main upshot coming out of this of interest to us here is the paper [**60**] of Jimbo, Miwa and Ueno in which, as well as a number of other things, they:

- Defined the notion of 'monodromy preserving deformation' of a meromorphic connection on $\mathbb{P}^1$ with *arbitrary order* poles, and
- Wrote down explicit algebraic nonlinear differential equations ('deformation equations') governing these isomonodromic deformations, thereby generalising Schlesinger's equations.

One of the key points in their work was to understand the rôle of the Stokes matrices as the natural generalisation of the fundamental group representation (and therefore being the things to keep fixed throughout the deformation).

Subsequently Miwa [**84**] proved that *all* of these deformation equations have the Painlevé property (almost simultaneously Malgrange [**71**] independently proved that all of Schlesinger's equations have the Painlevé property). Thus not only do the Painlevé equations arise naturally in quantum field theory but we now have a vast family of equations with the Painlevé property: take a (generic) meromorphic connection with arbitrarily many poles (each of arbitrary order) on an arbitrary rank vector bundle over $\mathbb{P}^1$ and Jimbo-Miwa-Ueno give us nonlinear differential equations for (monodromy preserving) deformations of this connection, and these equations will have the Painlevé property. In [**58**] it is shown, extending R.Fuchs' result above, that all six Painlevé equations arise as the simplest nontrivial cases of the deformation equations. However to obtain the first five Painlevé equations it seems one must allow higher order poles.....

APPENDIX B

Formal Isomorphisms

We will give an algorithm to put a family of nice systems into normal form. This is taken from [**60**] Proposition 2.2, [**92**] Theorem B.1.3 and [**13**] Lemma 1 p42.

Let $\mathcal{R}$ be a commutative ring with a unit (for example the ring of holomorphic functions on a complex manifold). Fix positive integers $n, k \geq 1$ and suppose, for each integer $i \leq k$, that

$$A_i \in \mathrm{End}_n(\mathcal{R})$$

is an arbitrary $n \times n$ matrix with entries in $\mathcal{R}$ except for A_k, which is diagonal:

$$A_k = \mathrm{diag}(\alpha_1, \dots, \alpha_n) \qquad \alpha_i \in \mathcal{R}$$

and such that $\begin{cases} \text{if } k \geq 2 \text{ and } i \neq j : \alpha_i - \alpha_j \text{ is invertible in } \mathcal{R}, \text{ or} \\ \text{if } k = 1 \text{ and } i \neq j : \alpha_i - \alpha_j - p(\mathrm{id}_{\mathcal{R}}) \text{ is invertible in } \mathcal{R} \text{ for any } p \in \mathbb{Z}_{\neq 0}. \end{cases}$

Write

$$A = A_k \frac{dz}{z^k} + \dots + A_1 \frac{dz}{z} + A_0 dz + \dots \in \mathrm{End}_n(\mathcal{R}[\![z]\!]) \frac{dz}{z^k}.$$

Remark B.1. For example a family of nice compatibly framed connections gives rise to such data when suitable local coordinates are chosen and a compatible local trivialisation.

Proposition B.2. *There is a formal transformation*

$$\widehat{F} \in GL_n(\mathcal{R}[\![z]\!])$$

and diagonal matrices $A_k^0, \dots, A_1^0 \in End_n(\mathcal{R})$ such that

$$A = \widehat{F}[A^0]$$

where $A^0 = A_k^0 \frac{dz}{z^k} + \dots + A_1^0 \frac{dz}{z}$, $A_k^0 = A_k$, $\widehat{F}|_{z=0} = 1$ and the square brackets denote the gauge action in the z direction:

$$\widehat{F}[A^0] := \widehat{F} A^0 \widehat{F}^{-1} + \left(\frac{d\widehat{F}}{dz} \widehat{F}^{-1} \right) dz.$$

Proof. Firstly we diagonalise A and then we remove the nonsingular part.
1) 'Spectral Splitting'. The key fact is that if $\mathrm{End}_n^{\mathrm{od}}(\mathcal{R})$ denotes the set of off diagonal matrices (i.e. with only zeros on the diagonal) then bracketing with A_k gives an *isomorphism* (if $k \geq 2$):

$$\mathrm{ad}_{A_k} : \mathrm{End}_n^{\mathrm{od}}(\mathcal{R}) \longrightarrow \mathrm{End}_n^{\mathrm{od}}(\mathcal{R}).$$

Suppose inductively that the first p coefficients $A_k, A_{k-1}, \dots, A_{k-p+1}$ of A are diagonal (so the $p = 1$ case holds by assumption). By applying the gauge transformation $(1 + z^p H_p)$ to A (where $H_p \in \mathrm{End}_n(\mathcal{R})$), we find

$$(1 + z^p H_p)[A] = A + [H_p, A_k] z^{p-k} dz + p H_p z^{p-1} dz + O(z^{p-k+1}) dz.$$

Thus if $k \geq 2$, the first not necessarily diagonal coefficient is that of $z^{p-k}dz$ which is

$$A_{k-p} + [H_p, A_k] \tag{134}$$

and so by defining H_p to be the off diagonal matrix

$$H_p := (\mathrm{ad}_{A_k})^{-1}\big(A^{\mathrm{od}}_{k-p}\big)$$

we ensure that the first $p+1$ coefficients of A are now diagonal, completing the inductive step.

If instead $k = 1$ the term $pH_pz^{p-1}dz$ also enters into (134) but the conditions on the diagonal entries of A_k then ensure that the map

$$\mathrm{ad}_{A_k} - p : \mathrm{End}^{\mathrm{od}}_n(\mathcal{R}) \longrightarrow \mathrm{End}^{\mathrm{od}}_n(\mathcal{R})$$

(where p just means multiplication by p) is an isomorphism and so by setting

$$H_p := (\mathrm{ad}_{A_k} - p)^{-1}\big(A^{\mathrm{od}}_{k-p}\big)$$

we complete the inductive step also in this case.

Hence if we define a formal transformation $\widehat{H} \in GL_n(\mathcal{R}[\![z]\!])$ to be the infinite product

$$\widehat{H} := \quad \cdots(1 + z^pH_p)(1 + z^{p-1}H_{p-1})\cdots(1 + zH_1)$$

then $\widehat{H}[A]$ is diagonal.

2) Now define A^0 to be the polar part of $\widehat{H}[A]$ so that

$$\widehat{H}[A] = A^0_k\frac{dz}{z^k} + \cdots + A^0_1\frac{dz}{z} + D$$

with D diagonal and nonsingular. We can then formally integrate D (replace z^pdz by $z^{p+1}/(p+1)$ for each $p \geq 0$) to obtain a diagonal formal series with no constant term which we will denote $\int_0^z D \in \mathrm{End}_n(\mathcal{R}[\![z]\!])$. Then negate and exponentiate to define the diagonal formal series

$$\widetilde{F} := e^{\left(-\int_0^z D\right)}$$

so that $d\widetilde{F}(\widetilde{F})^{-1} = d\log\widetilde{F} = -D$.

Thus $(\widetilde{F}\widehat{H})[A] = A^0$ and so $\widehat{F} := (\widetilde{F}\widehat{H})^{-1}$ is the desired formal transformation $\square$

REMARK B.3. Such formal transformation $\widehat{F}$ is in fact unique. This follows directly from the construction used in [**60**] or can be proved in the same way as Lemma 1.13 here.

APPENDIX C

Asymptotic Expansions

We give an extremely brief introduction to the theory of asymptotic expansions for holomorphic functions on sectors. Let S be some open sector at the origin of the complex plane. For example $S = \{z \mid \alpha < \arg(z) < \beta, 0 < |z| < r\}$ for some constants α, β and r. Let f be a holomorphic function on S and let $\widehat{f} = \sum_{n \geq 0} a_n z^n \in \mathbb{C}[\![z]\!]$ be an arbitrary formal power series.

Definition C.1. The function f is said to *have asymptotic expansion* $\widehat{f}$ *on* S if, for each closed subsector $S' \subset S$ and integer p, there is a constant $c \in \mathbb{C}$ such that

$$|f(z) - \sum_{n=0}^{p} a_n z^n| \leq c|z|^{p+1} \quad \text{for all } z \in S'. \tag{135}$$

One way to think of this condition is in terms of convergence of the derivatives of $f(z)$: if f is holomorphic on S then it turns out that f has asymptotic expansion $\widehat{f}$ on S iff for each closed subsector $S' \subset S$ and integer n we have

$$\lim_{\substack{z \to 0 \\ z \in S'}} f^{(n)}(z) = n! a_n.$$

In particular if a function has an asymptotic expansion then it is unique.

Not every holomorphic function on S will posses an asymptotic expansion but if we let $\text{Æ}(S)$ denote the set of those that do, then $\text{Æ}(S)$ is a differential $\mathbb{C}$-algebra (i.e. it is closed under addition, multiplication and differentiation). Moreover by associating to each $f \in \text{Æ}(S)$ its uniquely determined asymptotic expansion we get a map:

$$\text{Æ}_S : \text{Æ}(S) \longmapsto \mathbb{C}[\![z]\!]$$

to the ring of formal power series. The basic fact is that this is a morphism of differential $\mathbb{C}$-algebras; in particular the asymptotic expansion of a derivative is the derivative of the asymptotic expansion.

These results are proved in [**73**]. See also [**107, 78, 14**] and references therein.

Remark C.2. The sums $\Sigma_i(\widehat{F})$ occurring in Proposition 1.24 in fact satisfy a stronger condition than simply having asymptotic expansion $\widehat{F}$ on the supersector $\widehat{\mathrm{Sect}}_i$. They admit an 'asymptotic expansion with Gevrey order $1/(k-1)$'; the growth with p of the constants c in (135) is precisely controlled. We do not need to use Gevrey asymptotics here, but they are a crucial part of the summation theory (see [**15**] or [**74**] for more details and references).

APPENDIX D

Borel's Theorem

We will discuss the version of the theorem of E.Borel, on the surjectivity of the Taylor expansion map, that was used to factorise the monodromy map in Chapter 5:

Theorem D.1. *(E. Borel). Given $\widehat{f} \in \mathbb{C}[\![x, y]\!] \otimes C^{\infty}(U)$ (where x, y are real coordinates on $\mathbb{C} \cong \mathbb{R}^2$) and a compact neighbourhood I of the origin in $\mathbb{R}$ then there exists a smooth function*

$$f \in C^{\infty}(U \times I \times I)$$

such that the Taylor expansion of f at $x = y = 0$ is given by $\widehat{f}$:

$$L_0(f(u)) = \widehat{f}(u) \qquad \text{for all } u \in U.$$

(Here U can be any smooth manifold.)

Proof. This will be deduced from the following version of Borel's Theorem given on p16 of Hörmander's book [**50**]:

Theorem D.2. *For $j = 0, 1, \dots$ let $f_j \in C^{\infty}(K)$ where K is a compact subset of $\mathbb{R}^n$, and let I be a compact neighbourhood of 0 in $\mathbb{R}$. Then one can find $f \in C^{\infty}(K \times I)$ such that*

$$\frac{\partial^j f(u, t)}{\partial t^j} = f_j(u), \qquad t = 0, j = 0, 1, \dots$$

□

By writing $\widehat{f} = \sum_j f_j t^j/(j)!$ we see this says that: given any formal power series

$$\widehat{f} \in \mathbb{C}[\![t]\!] \otimes C^{\infty}(K)$$

with coefficients which are smooth functions on a compact set $K \subset \mathbb{R}^n$, then there is a smooth function $f \in C^{\infty}(K \times I)$ having Taylor expansion $\widehat{f}$ on K (in other words: $L_0(f) = \widehat{f}$).

Now, in our situation we we start with a formal series in two variables $\widehat{f} \in \mathbb{C}[\![x, y]\!] \otimes C^{\infty}(U)$. By using Theorem D.2 twice we deduce:

Corollary D.3. *For any compact subset $K \subset U$ one can find $f \in C^{\infty}(K \times I \times I)$ such that*

$$L_{x=y=0}(f) = \widehat{f}.$$

Proof. Define smooth functions $f_{ij} \in C^{\infty}(U)$ to be the coefficients of $\widehat{f}$:

$$\widehat{f} = \sum_{i,j} f_{ij} \frac{x^i y^j}{(i)!(j)!}.$$

Then for each i define

$$\widehat{f_i} = \sum_j f_{ij} \frac{y^j}{(j)!} \in \mathbb{C}[\![y]\!] \otimes C^{\infty}(U)$$

so that $\widehat{f} = \sum_i \widehat{f}_i x^i/(i)!$. Then by applying Theorem D.2 to $\widehat{f}_i$ for each i we obtain $f_i \in C^\infty(K \times I)$ such that for each j:

$$\frac{\partial^j f_i(u,y)}{\partial y^j} = f_{ij}(u), \qquad \text{at } y = 0.$$

Then apply Theorem D.2 again to $\sum f_i x^i/(i)!$ to obtain the required function $f \in C^\infty(K \times I \times I)$ $\square$

Now we deduce Theorem D.1 by choosing a partition of unity on U. That is we choose smooth functions $\phi_\lambda \in C^\infty(U)$ indexed by λ in some set $\mathcal{I}$, such that ϕ_λ has compact support, the set of supports is locally finite, and for any $u \in U$ the finite sum $\sum_{\lambda \in \mathcal{I}} \phi_\lambda(u)$ is equal to 1.

Thus from Corollary D.3, for each $\lambda \in \mathcal{I}$ we obtain a smooth function

$$f_\lambda \in C^\infty(\mathrm{supp}(\phi_\lambda) \times I \times I)$$

such that $L_{x=y=0}(f) = \widehat{f}$. Then we define

$$f := \sum_{\lambda \in \mathcal{I}} \phi_\lambda \cdot f_\lambda.$$

Observe that $\phi_\lambda \cdot f_\lambda$ is a well defined smooth function on $U \times I^2$ and so by the local finiteness property $f \in C^\infty(U \times I \times I)$. Finally since each ϕ_λ only depends on u we have

$$L_{x=y=0} f = \sum_{\lambda \in \mathcal{I}} \phi_\lambda L_{x=y=0}(f_\lambda) = \sum_{\lambda \in \mathcal{I}} \phi_\lambda \widehat{f} = \widehat{f}$$

$\square$

APPENDIX E

Miscellaneous Proofs

Firstly we will prove Lemma 6.9 from page 98.

Recall we have a full flat meromorphic connection ∇ on $D_0 \times \Delta$ together with a compatible framing g_0 along the polar divisor Δ_0, with associated formal transformation $\widehat{g}$ and family of formal normal forms A^0, so that $\widehat{g}$ transforms the vertical part of ∇ into A^0.

Lemma 6.9 then claims that Λ (the family of exponents of formal monodromy) is *constant*, and that

$$\widehat{g}[\nabla]_{\mathbb{P}^1\times\Delta} = d_{\mathbb{P}^1\times\Delta} - (\widetilde{A}^0 + \pi^*(d_{\Delta_0}F))$$

for some diagonal matrix valued holomorphic function $F \in \mathrm{End}_n(\mathcal{O}(\Delta_0))$ which is unique upto the addition of a constant diagonal matrix, where $\pi : D_0 \times \Delta \to \Delta_0$ is the projection along the D_0 direction.

Proof (of Lemma 6.9). Retrivialise the product $D_0 \times \Delta$ with respect to the coordinate z_0, i.e. take the subsets on which z_0 is constant to be horizontal, rather that the subsets on which z is constant. Thus in this proof d_{Δ_0} denotes the horizontal part of the full exterior derivative in this new splitting (previously d_{Δ_0} was only defined on $\Delta_0 \subset D_0 \times \Delta$).

Let $d_{\Delta_0} - B$ be the Δ_0 component of $\widehat{g}[\nabla]_{\mathbb{P}^1\times\Delta}$ so that

$$\widehat{g}[\nabla]_{\mathbb{P}^1\times\Delta} = d_{\mathbb{P}^1\times\Delta} - (A^0 + B).$$

This is formally flat because ∇ is flat. The (D_0-Δ_0) part of the equation for this flatness is:

$$d_{\mathbb{P}^1}B + d_{\Delta_0}A^0 = A^0 \wedge B + B \wedge A^0. \tag{136}$$

Since A^0 is diagonal this equation splits into two independent pieces, the diagonal part and the off-diagonal part. Firstly we deduce that the off-diagonal part B^{od} of B is zero. Suppose $B^{\mathrm{od}} \neq 0$ and let M/z_0^r be its leading term, $M \in \mathrm{End}_n^{\mathrm{od}}(\Omega^1_{\mathrm{hol}}(\Delta_0))$. Equation (136) implies

$$d_{\mathbb{P}^1}B^{\mathrm{od}} = A^0 \wedge B^{\mathrm{od}} + B^{\mathrm{od}} \wedge A^0. \tag{137}$$

The right-hand side of this has a pole of order $r + k_0$ (since the leading term of A^0 has distinct eigenvalues; it is 'nice') whereas the left-hand side has a pole of order at most $r + 1$. Thus $B^{\mathrm{od}} = 0$ unless $k_0 = 1$. If $k_0 = 1$, say $A^0 = A^0_1 dz/z_0$, then looking at the coefficient of dz/z_0^{r+1} in (137) we see

$$(-r)M = [A^0_1, M]$$

which implies $M = 0$ (and therefore $B^{\mathrm{od}} = 0$) since A^0 is nice; the difference between any two eigenvalues of A^0_1 is never the integer $-r$.

Thus B is diagonal, and so (136) now reads

$$d_{\mathbb{P}^1}B + d_{\Delta_0}A^0 = 0$$

and therefore

$$d_{\mathbb{P}^1}B = -d_{\Delta_0}d_{\mathbb{P}^1}Q(t) - d_{\Delta_0}\left(\Lambda(t)\frac{d_{\mathbb{P}^1}z_0}{z_0}\right) = -d_{\Delta_0}d_{\mathbb{P}^1}Q(t) - d_{\Delta_0}(\Lambda(t))\frac{d_{\mathbb{P}^1}z_0}{z_0}.$$

Observe that this implies $d_{\Delta_0}\Lambda(t) = 0$ since $d_{\mathbb{P}^1}B$ will have no residue term: write $B = \sum_j B_j(t)z_0^j$ for $B_j \in \mathrm{End}_n(\Omega^1_{\mathrm{hol}}(\Delta_0))$ and apply $d_{\mathbb{P}^1}$.

Thus

$$d_{\mathbb{P}^1}B = d_{\mathbb{P}^1}d_{\Delta_0}Q(t)$$

and so by thinking about the kernel of $d_{\mathbb{P}^1}$ we see

$$B = d_{\Delta_0}Q(t) + \phi(t)$$

for some diagonal matrix of one forms $\phi \in \mathrm{End}_n(\Omega^1_{\mathrm{hol}}(\Delta_0))$ on Δ_0.

Finally the (Δ_0-Δ_0) part of the equation for the flatness of $d - A^0 - B$ says that $d_{\Delta_0}B = 0$ (since B is diagonal). It follows that $d_{\Delta_0}(\phi(t)) = 0$ and so, since Δ_0 is a polydisk, $\phi = d_{\Delta_0}F$ for some diagonal $F \in \mathrm{End}_n(\mathcal{O}(\Delta_0))$ □

Next we will prove Lemma 6.12, in which we claimed the following formula holds:

$$\Theta_0 = g_0^{-1}(d_\Delta a_0)g_1 + \mathrm{Const}_{z_0}(\widehat{g}^{-1}\cdot \widetilde{A}^0_\Delta \cdot \widehat{g}) - \mathrm{Const}_{z_0}(\Omega) \tag{138}$$

where $\widehat{g} = g_0 + g_1\cdot z_0 + g_2\cdot z_0^2 + \cdots \in GL_n\big(\mathbb{C}[\![z_0]\!]\otimes\mathcal{O}(\Delta_0)\big)$ is the formal series associated to any compatible framing g_0, a_0 is regarded as a function on Δ via the coordinate z and the matrices g_i of functions on Δ_0 are pulled back to Δ along φ.

Proof (of Lemma 6.12). Firstly check that the formula is independent of the choice g_0 of compatible framing. If g_0 is replaced by $P\cdot h\cdot g_0$ for an arbitrary map $h : \Delta_0 \to T$ and constant permutation matrix P, then the formal series $\widehat{g}$ changes to $P\cdot h\cdot\widehat{g}$ and $\widetilde{A}^0_\Delta$ changes to $P\cdot\widetilde{A}^0_\Delta\cdot P^{-1}$. It is then clear that the expression on the right of (138) is unchanged.

Thus we can without loss of generality take g_0 to be a 'good' compatible framing. In particular we then have that the key equation (84) holds. The Δ component of the constant term in the Laurent expansion of (84) with respect to z_0 reads:

$$\mathrm{Const}_{z_0}(\Omega) - \mathrm{Const}_{z_0}(\widehat{g}^{-1}\cdot\widetilde{A}^0_\Delta\cdot\widehat{g}) = \mathrm{Const}_{z_0}(\widehat{g}^{-1}d_\Delta\widehat{g}). \tag{139}$$

Now g_0 is (the inverse matrix of) a fundamental solution of ∇_0 and so g_0^{-1} becomes a fundamental solution of $\varphi^*(\nabla_0) = d_\Delta - \Theta_0$ when pulled back to Δ along φ. It follows that

$$\Theta_0 = -g_0^{-1}d_\Delta(g_0) \tag{140}$$

where g_0 is thought of as a function on Δ using φ. In particular Θ_0 will appear when we expand the right-hand side of (139): we find

$$\widehat{g}^{-1}d_\Delta\widehat{g} = g_0^{-1}(d_\Delta g_0) - g_0^{-1}(d_\Delta a_0)g_1 + O(z_0)$$

and so by taking the constant term in this and using the formulae (139) and (140) we deduce the desired equation (138) □

Finally we give proof of the key Proposition 6.13 enabling us to see that flat connections have constant monodromy data.

Proposition 6.13. [**60**]. *Let ∇ be a full flat connection over $D_0 \times \Delta$ (as in the Section 2.1 of Chapter 6), let g_0 be a 'good' compatible framing of ∇ along Δ_0 (with respect to z_0; see p98) and let $\widehat{g}$ be the corresponding formal series. Fix any point $t_0 \in \Delta$, choose a labelling of the sectors between the anti-Stokes directions at $a_0(t_0) \in D_0 \times \{t_0\}$, and choose $\log(z_0)$ branches on $D_0 \times \{t_0\}$. Let Δ' be a neighbourhood of $t_0 \in \Delta$ such that the last sector at $a_0(t_0)$ deforms into a unique sector at $a_0(t)$ for all $t \in \Delta'$ (the last sector at $a_0(t)$).*

Then the canonical fundamental solution (from Definition 1.29)

$$\Phi_0 := \Sigma_0(\widehat{g}^{-1}) z_0^{\Lambda} e^{Q}$$

of $\nabla|_{\mathrm{Vert}}$ on the last sector at $a_0(t) \in D_0 \times \{t\}$ varies holomorphically with $t \in \Delta'$ and $\Phi_0(z, t)$ is a local fundamental solution of the original full connection ∇.

Proof. This is essentially the converse part of Theorem 3.3 of [**60**]; write $\nabla = d - \widetilde{A}$ and let Ω be the Δ component of $\widetilde{A}$ so that $\widetilde{A} = A + \Omega$. The aim is to show that $d_\Delta \Phi_0 = \Omega \Phi_0$. From the definition of $\widehat{g}$ we have

$$A + \Omega = \widehat{g}^{-1}[\widetilde{A}^0]_{\mathbb{P}^1 \times \Delta}$$

and this has Δ component

$$\Omega = \widehat{g}^{-1} \cdot \widehat{A}^0_\Delta \cdot \widehat{g} - \widehat{g}^{-1} d_\Delta \widehat{g}. \tag{141}$$

Observe that the flatness of ∇ implies

$$d_\Delta A = -d_{\mathbb{P}^1}\Omega + A \wedge \Omega + \Omega \wedge A.$$

Now the key observation is that this equation implies that the matrix of one forms $d_\Delta \Phi_0 - \Omega \Phi_0$ satisfies the equation

$$d_{\mathbb{P}^1}(d_\Delta \Phi_0 - \Omega \Phi_0) = A(d_\Delta \Phi_0 - \Omega \Phi_0)$$

(also using the fact that $d_{\mathbb{P}^1}\Phi_0 = A\Phi_0$). Then if we define a matrix of one forms

$$K := \Phi_0^{-1}(d_\Delta \Phi_0 - \Omega \Phi_0) \tag{142}$$

it follows that $d_{\mathbb{P}^1} K = 0$ so that K is constant in the D_0 direction. Then using our precise knowledge of the asymptotics of Φ_0 in a sector at $a_0(t)$ it follows from (142) that K has zero asymptotic expansion in the same sector at $a_0(t)$. (Here we use the fact that the asymptotic expansions are uniform in t to see that d_Δ commutes with the operation of taking the asymptotic expansion; the point is, due to (141), that the asymptotic expansion of Φ_0 is a formal fundamental solution in the Δ directions.) This is slightly simpler than in [**60**] because here Ω is precisely the Δ component of the full flat connection, rather than just the polar part. Anyway it follows immediately that $K = 0$ because K is constant in the vertical direction, and so $d_\Delta \Phi_0 = \Omega \Phi_0$ □

APPENDIX F

Work in Progress

• **Symplectic structures on monodromy manifolds**

The aim here is to replace the infinite dimensional Atiyah-Bott type approach of Chapter 3 by a finite-dimensional/direct construction. The most promising approach seems to be via the theory of 'quasi-Hamiltonian G-spaces, with Lie group valued moment maps' due to Alekseev, Malkin and Meinrenken [**3**].

I believe all of the manifolds $\widetilde{\mathcal{C}}_i$ and $\mathcal{C}_i$ of Chapter 4 are naturally *complex* quasi-Hamiltonian G-spaces with ($G = GL_n(\mathbb{C})$), generalising the conjugacy class example. The G actions and the G-valued maps (ρ) are given in Chapter 4. All that is left is to find explicitly the invariant two-form (which is the q-Hamiltonian analogue of the symplectic form).

Then the monodromy manifolds defined in Chapter 4 arise as the q-Hamiltonian reduction of the q-Hamiltonian 'fusion' of the $\mathcal{C}_i$'s or the $\widetilde{\mathcal{C}}_i$'s. For example the extended monodromy manifold $M_{\text{ext}}(\mathbf{A})$ (see Definition 4.3) is (a connected component of)

$$\left(\widetilde{\mathcal{C}}_1 \circledast \widetilde{\mathcal{C}}_2 \circledast \cdots \circledast \widetilde{\mathcal{C}}_m\right) /\!/ G$$

where $\circledast$ denotes the 'fusion product' in the category of q-Hamiltonian G-spaces and $/\!/$ denotes q-Hamiltonian reduction at the value 1 of the q-moment map. As a manifold this is the same as in the definition of $M_{\text{ext}}(\mathbf{A})$ (but with the condition on the Λ's omitted). However, from the q-Hamiltonian theory, this space would naturally obtain a symplectic structure.

At least heuristically the above discussion can be motivated from the infinite dimensional viewpoint. The category of q-Hamiltonian G-spaces is equivalent to the category of Hamiltonian spaces for the loop group LG of G, at least for compact G (see [**3**] Section 8). From this perspective $\mathcal{C}$ is the 'holonomy/monodromy manifold' (in the sense of [**3**]) associated to the infinite dimensional space of flat C^∞ singular $GL_n(\mathbb{C})$-connections on the closed unit disc in $\mathbb{C}$ with just one pole (at 0) and with C^∞ Laurent expansion fixed to be a formal normal form at the pole, modulo smooth gauge transformations which are the identity on the boundary of the disc and have Taylor expansion in the constant diagonal subgroup $T \subset GL_n([\![z, \bar{z}]\!])$ at 0. (The extended version $\widetilde{\mathcal{C}}$ can be obtained similarly, by only fixing the irregular type of the connections at 0 and requiring the gauge transformation to have Taylor expansion 1 at 0—in other words we incorporate a compatible framing at 0.)

The symplectic structures on these infinite dimensional spaces are defined (at least formally) via the infinite dimensional viewpoint in Chapter 3, analogously to the non-singular cases. The Hamiltonian LG action arises as the residual action of bundle automorphisms over the boundary circle of the disc. The fusion of the spaces $\widetilde{\mathcal{C}}_1, \dots, \widetilde{\mathcal{C}}_m$ corresponds to gluing the m corresponding discs into an $m+1$-holed $\mathbb{P}^1$ and the quotient corresponds to capping off the final hole.

In summary, the spaces $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ are viewed as new pieces which can be glued together to build moduli spaces, and the above description is of the corresponding Hamiltonian LG spaces.

The close relationship between q-Hamiltonian spaces and Poisson G-spaces suggests that, in the order two pole case, the Poisson structure on the dual Poisson Lie group G^* should relate to the required two-form on $\widetilde{\mathcal{C}}$.

• Time-dependent Hamiltonians and τ functions

In [**60**], Jimbo, Miwa and Ueno explicitly define a one-form ω on the extended moduli bundle $\mathcal{M}^*_{\text{ext}}$. They prove ω restricts to a *closed* one-form on each solution leaf of the isomonodromic deformation equations (horizontal section of the isomonodromy connection on $\mathcal{M}^*_{\text{ext}}$). In particular, for each choice of solution leaf, a function τ may be defined locally on the base-space X of deformation parameters by the formula

$$d\log\tau = \omega.$$

Subsequently Miwa [**84**] proved that such a function τ extends to a *holomorphic* function on the universal cover of X. These 'isomonodromy τ functions' are interesting for numerous reasons; they first arose as correlation functions in certain exactly solvable models and are analogous to theta functions in the theory of Abelian varieties. Such a τ function recently made a spectacular appearance in the formula for certain generating functions of genus one Gromov-Witten invariants [**34**].

In the simplest cases (for example for Schlesinger's equations) it is known how to interpret the one-form ω as giving time-dependent Hamiltonians for the isomonodromic deformation equations, and the main question here is to find such an interpretation in the general case.

In the known cases there is an 'a priori symplectic trivialisation' of the extended moduli bundle $\mathcal{M}^*_{\text{ext}}$; it comes as a product of the base-space X with a fixed symplectic manifold:[1]

$$X \times \left(\widetilde{O}_1 \times \cdots \times \widetilde{O}_m // GL_n(\mathbb{C})\right). \tag{143}$$

In other words there are *two* flat symplectic connections on the fibre-bundle $\mathcal{M}^*_{\text{ext}}$: the isomonodromy connection and the *complete* connection determining the trivialisation (143).

The Hamiltonians specify how to modify the complete connection to obtain the isomonodromy connection. To explain concretely what this means, make a choice of local coordinates $\{t_1, \ldots, t_r\}$ on X. Then the one-form ω is of the form

$$\omega = H_1\pi^*(dt_1) + \cdots + H_r\pi^*(dt_r)$$

for functions H_i on $\mathcal{M}^*_{\text{ext}}$, where $\pi : \mathcal{M}^*_{\text{ext}} \to X$. These functions H_i are then the time-dependent Hamiltonians: the isomonodromy equations arise by adding the corresponding (vertical) Hamiltonian vector fields to the horizontal vector fields specified by the decomposition (143).

[1] Recall that in the general case we used the theorem of M.Vergne [**106**] to give local symplectic trivialisations of $\mathcal{M}^*_{\text{ext}}$ (Theorem 6.4), and this involves making arbitrary choices; there doesn't appear to be a preferred trivialisation from this point of view. No choices are needed if the coadjoint orbits O_B do not vary, for example if the poles are all of order at most two.

More abstractly this can be rephrased naturally in the language of symplectic connections. The standard way to encode a symplectic connection on a symplectic fibration is in terms of a 'fundamental' two-form Ω (on the total space of the fibration) that restricts to the symplectic form on each fibre [**40, 81**]. (The horizontal directions of the connection determined by Ω are those which are Ω-orthogonal to all the vertical(=fibre) directions.) If Ω is closed then the connection it determines is a symplectic connection.

The complete connection above is determined by a closed two-form Ω such that $\Omega|_{\mathrm{Vert}}$ is the symplectic form on the fibre, and

$$\Omega(\partial/\partial t_i, \cdot) = 0$$

(where $\partial/\partial t_i$ is regarded as a vector field on $\mathcal{M}^*_{\mathrm{ext}}$ via the splitting (143)). Then the one-form ω determines a fundamental two-form for the isomonodromy connection by the formula:

$$\Omega_{\mathrm{IMD}} = \Omega + d\omega \tag{144}$$

where d is the exterior derivative on $\mathcal{M}^*_{\mathrm{ext}}$. (Clearly Ω_{IMD} is closed and it restricts to the symplectic form on the fibres since $d\omega|_{\mathrm{Vert}} = 0$.)

Thus we can abstract the notion of 'time-dependent Hamiltonians' as specifying the difference $d\omega$ between the fundamental two-forms of two flat symplectic connections.

If we return now to the general case, the one form ω defined in [**60**] is the natural candidate to give the Hamiltonians and we can obtain a closed two-form Ω_{IMD} in a natural way from the isomonodromy connection. However we do not have a natural, a priori, symplectic trivialisation of the extended moduli bundle $\mathcal{M}^*_{\mathrm{ext}}$ at the moment. Thus the question is to find a natural (complete) symplectic connection on $\mathcal{M}^*_{\mathrm{ext}}$ (with fundamental two-form Ω), such that (144) holds. (There are several seemingly natural candidates for Ω but the general proof is still missing.) Turning this question around, we claim that $\Omega_{\mathrm{IMD}} - d\omega$ defines a (complete) flat symplectic connection on $\mathcal{M}^*_{\mathrm{ext}}$ and want to describe the corresponding trivialisation of $\mathcal{M}^*_{\mathrm{ext}}$ more directly.

APPENDIX G

Notation

Whenever used, pre-superscripts (${}^iA$) denote local information near $a_i \in \mathbb{P}^1$.

A^0	$p4$	A formal normal form (generally)
$\mathbb{A}$	$p7$	Set of anti-Stokes directions
$\mathbf{A}$		The m-tuple $({}^1A^0, \dots, {}^mA^0)$ of formal normal forms
$\mathcal{A}[D]$	$p53$	C^∞ singular connections on $\mathbb{P}^1$ with poles on D
$\mathcal{A}_{\mathrm{fl}}[D]$	$p53$	Flat connections in $\mathcal{A}[D]$
$\mathcal{A}(\mathbf{A})$	$p53$	C^∞ singular connections with fixed Laurent expansions
${}^0\mathcal{A}[k]$	$p54$	C^∞ singular connection germs with poles of order $\leq k$
$\mathcal{A}_{\mathrm{ext}}(\mathbf{A})$	$p59$	Extended space of C^∞ singular connections
a_i		ith marked point on $\mathbb{P}^1$
Æ	$p141$	Asymptotic expansion
B_k	$p21$	$\{g \in G_k \mid g(0) = 1\}$
$\widehat{B}$	$p7$	Formal Birkhoff transformations
$\widehat{B}(A^0)$	$p7$	Applicable Birkhoff transformations
$B\{z\}$	$p7$	Convergent Birkhoff transformations
$\mathfrak{b}_k$	$p21$	$\mathrm{Lie}(B_k)$
$\mathbb{C}[\![z]\!]$		Formal power series ring
$\mathbb{C}\{z\}$		Ring of power series with radius of convergence > 0
${}^0C(A)$	$p3$	$\mathrm{Syst}(A)/G\{z\}$
$C^\infty[D]$	$p51$	C^∞ complex functions with poles on D
C_i	$p70$	ith connection matrix
$\mathcal{C}_i$	$p73$	Multiplicative version of O_i
$\widetilde{\mathcal{C}}_i$	$p71$	Multiplicative version of $\widetilde{O}_i$
D	$p1$	Effective divisor on $\mathbb{P}^1$
d_i	$p9$	ith anti-Stokes direction
$\mathbf{d}$	$p13$	Half-period's worth of consecutive anti-Stokes directions
E		$\mathbb{C}^n$
$\mathcal{F}(\alpha)$	$p53$	Curvature of singular connection α

G_k	$p21$	$GL_n(\mathbb{C}[\zeta]/(\zeta^k))$
$G\{z\}$	$p3$	$GL_n(\mathbb{C}\{z\})$
$\widehat{G}$	$p3$	$GL_n(\mathbb{C}[\![z]\!])$
$\widehat{G}(A)$	$p6$	Applicable formal transformations
$\mathcal{G}$	$p53$	$C^\infty(\mathbb{P}^1, GL_n(\mathbb{C}))$
${}^0\mathcal{G}$	$p54$	Germs at 0 of $g \in \mathcal{G}$
${}^0\mathcal{G}_1$	$p54$	$\ker\big(L_0 : {}^0\mathcal{G} \to GL_n(\mathbb{C}[\![z, \bar{z}]\!])\big)$
${}^0\mathcal{G}_T$	$p54$	$L_0^{-1}(T)$
${}^0\mathcal{G}(A^0)$	$p54$	
$\mathfrak{g}_k$	$p21$	$\mathrm{Lie}(G_k)$
γ_i	$p70$	A path in $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$
Γ_i	$p70$	Thickening of γ_i
$\mathcal{H}(A^0)$	$p6$	
$\mathrm{Hom}(A, B)$	$p3$	
k_i	$p1$	Order of ith pole
K_j	$p10$	A Stokes factor
κ_i	$p9$	
$\mathbf{K}$	$p10$	Stokes factors
$\mathbf{K}(A)$	$p10$	
$\mathbf{K}(\widehat{F})$	$p10$	
$\mathbf{K}(\alpha)$	$p56$	Stokes factors of flat singular connection α
L_i	$p51$	Laurent or Taylor expansion at a_i
Λ	$pp4, 6$	Exponent of formal monodromy
M_0	$p6$	Formal monodromy
m		Number of distinct marked points
$\mathrm{Mult}(d)$	$p8$	The multiplicity of anti-Stokes direction d
$\mathcal{M}^*(\mathbf{A})$	$p37$	
$\mathcal{M}^*_{\mathrm{ext}}(\mathbf{A})$	$p42$	
$\mathcal{M}^*_{\mathrm{ext}}$	$p93$	
$\mathcal{M}(\mathbf{A})$	$p57$	
$\mathcal{M}_{\mathrm{ext}}(\mathbf{A})$	$p59$	
$M(\mathbf{A})$	$p74$	Monodromy manifold
$M_{\mathrm{ext}}(\mathbf{A})$	$p72$	Extended monodromy manifold
M_{ext}	$p95$	Extended monodromy bundle
n		The rank of the bundles
ν	$p73$	Monodromy map

$O(A)$	$p23$	G_k coadjoint orbit
O_i	$p38$	ith coadjoint orbit
O_B	$p26$	B_k coadjoint orbit
$\widetilde{O}$	$pp26, 34$	An extended orbit
$\mathcal{O}$		Sheaf of holomorphic functions
PP_i	$p36$	Map taking the polar part at a_i
Φ_i	$pp10, 56$	Canonical fundamental solutions
π_{Res}	$p23$	
Q	$p4$	
q_i	$p4$	ith diagonal entry of Q
q_{ij}	$p7$	Leading *term* of $q_i - q_j$
$\mathrm{Roots}(d)$	$p8$	The roots of anti-Stokes direction d
ρ	$p10$	Representation of Stokes group
ρ_i	$p72$	
S_i	$p13$	A Stokes matrix
S^{-T}		The transpose of the inverse of the matrix S
$\mathrm{Sect}(U)$	$p7$	Sector associated to interval $U \subset S^1$
Sect_i	$p9$	ith sector between anti-Stokes rays
$\widehat{\mathrm{Sect}}_i$	$p9$	ith supersector
$\mathbb{S}\mathrm{to}_d(A^0)$	$p8$	Group of Stokes factors
$\mathrm{Sto}_d(A^0)$	$p10$	Stokes group
$\mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0)$	$p14$	Group of Stokes matrices
Syst_k	$p3$	Systems with poles of order $\le k$
$\mathrm{Syst}(A)$	$p3$	Systems formally equivalent to A
$\mathrm{Syst}_B(A^0)$	$p7$	Systems formally equivalent to A^0 with same leading term
$\Sigma_i(\cdot)$	$p9$	
T	$p11$	A torus $\cong (\mathbb{C}^*)^n$
$\mathfrak{t}$		Lie algebra of the torus T
$\mathfrak{t}'$	$p34$	‘Nice’ elements of $\mathfrak{t}^*$
$\mathcal{T}$	$p70$	‘Tentacles’
$\Omega^r[D]$	$p51$	C^∞ complex r-forms with poles on D
$\underset{\mathbf{d}}{<}$	$p13$	Dominance ordering of $\{q_1, \ldots, q_n\}$ associated to $\mathbf{d}$

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