Hyperkähler Metrics on Cotangent Bundles

by

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Except where references are given or explicitly stated otherwise, this dissertation is my own work and includes nothing which is the outcome of work done in collaboration.

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Contents

Introduction

Hyperkähler manifolds are, as the name suggests, generalisations of Kähler manifolds and in their geometry the quaternions as well as holomorphic-symplectic geometry play an interesting rôle.

Hyperkähler manifolds are Riemannian manifolds which are Kähler with respect to three complex structures I, J and K which satisfy the relations of the quaternions, namely

$$I^2 = J^2 = K^2 = IJK = -1.$$

In general, as $Sp(n) = GL(n, \mathbb{H}) \cap O(4n)$, hyperkähler manifolds have holonomy contained in $Sp(n)$ which is one of the groups on Berger's list of possible holonomy groups for Riemannian manifolds. As $Sp(n) \subset SU(2n)$, hyperkähler metrics are in particular Ricci-flat Kähler metrics.

If we fix one of the complex structures, I say, then

$$\omega_c(X, Y) = g(JX, Y) + ig(KX, Y)$$

is a complex symplectic form which is holomorphic with respect to I, hence $(\omega_c)^n$ is a nowhere vanishing section of the canonical bundle which gives an alternative proof for Ricci-flatness of hyperkähler manifolds.

The quaternionic vector space $\mathbb{H}^n$ with the flat metric is obviously an example of a hyperkähler manifold, but the hyperkähler metric found by Eguchi and Hanson in [18] on the cotangent bundle of the complex projective line $\mathbb{C}P^1$ was probably the first non-trivial complete example. The Eguchi-Hanson metric is a special case of the Calabi metrics on $T^*\mathbb{C}P^n$ constructed in [10].

The problem of existence of hyperkähler structures on compact manifolds has essentially been answered by Yau's proof of the Calabi conjecture together with an application of Bochner's vanishing theorem; for details see [5]. Nevertheless, not much explicit information about these metrics is known.

As the underlying differential equation, a complex Monge-Ampère equation, is highly nonlinear, a direct calculation of hyperkähler metrics is not a feasible approach, in particular in the non-compact case and in the absence of symmetries. The two most powerful tools for the construction of hyperkähler manifolds are the twistor method and the hyperkähler quotient construction, see [24].

The latter can be used to describe the Calabi metrics, see 1.7. An infinite-dimensional version of the hyperkähler quotient has been used by Kronheimer, Biquard and Kovalev (cf. [29, 30, 7, 28]) to find hyperkähler structures on coadjoint orbits. Their metrics are compatible with the Kostant-Kirillov symplectic form on complex orbits; if the orbit is semisimple, the metric is complete. Furthermore, if the compact orbit is itself a complex manifold, then we can identify the complex orbit with the cotangent bundle of the compact orbit by using one of the new complex structures on the complex orbit given by the hyperkähler structure.

In order to find new examples of non-compact hyperkähler manifolds it seems reasonable to look at manifolds which are already naturally holomorphic-symplectic. However, Guan [20] has given an example of a compact holomorphic-symplectic manifold which is not Kähler, hence not all holomorphic-symplectic manifolds carry a compatible hyperkähler structure.

In this thesis, we consider the general case of the cotangent bundle of a complex manifold with its canonical holomorphic-symplectic structure; all examples given above fall in this category. We are almost exclusively concerned with hyperkähler metrics which are compatible with this given holomorphic-symplectic structure.

We have made extensive use of the twistor method which proves the existence of a hyperkähler structure by a holomorphic existence theorem based on deformation theory, but the information obtained is essentially local in nature. By constructing the twistor space and showing the existence of twistor lines we shall prove

Theorem 2.1 *Let X be a real-analytic Kähler manifold. Then there exists a hyperkähler metric in a neighbourhood of the zero section of the cotangent bundle which is compatible with the canonical holomorphic-symplectic structure.*

Furthermore, the S^1-action given by scalar multiplication in the fibres is isometric and the restriction of the hyperkähler metric to the zero section induces the original Kähler metric.

Real analyticity of the Kähler manifold is necessary for such a hyperkähler extension to exist as can be seen from the twistor interpretation, so all complex manifolds which possess a hyperkähler metric on their cotangent bundle have a real-analytic Kähler structure.

In addition, any hyperkähler metric with the above properties is unique. More precisely, we shall show

Corollary 3.3 *For any hyperkähler metric on a neighbourhood of the zero section of the cotangent bundle of a real-analytic Kähler manifold which is compatible with the canonical holomorphic-symplectic structure, induces the given Kähler metric on the zero section and for which the S^1-action by scalar multiplication in the fibres is isometric, if we restrict to a suitable neighbourhood of the zero section, the metric can be obtained by our construction.*

The proof will involve analysing the structure of the twistor space for such a hyperkähler metric in detail using the S^1-action; in particular, the S^1-action allows us to find a standard coordinate system for each "half" of the twistor space.

Hyperholomorphic bundles can be regarded as a gauge-theoretical analogue of hyperkähler manifolds. A hyperholomorphic bundle over a hyperkähler manifold is a vector bundle with a connection which is compatible with all the complex structures of the hyperkähler manifold, and can therefore be thought of as a generalisation of a holomorphic vector bundle over a Kähler manifold.

Hyperholomorphic bundles over a hyperkähler manifold correspond to certain holomorphic vector bundles over its twistor space. This correspondence enables us to use our twistor construction for a hyperkähler manifold M in the cotangent bundle of a real-analytic Kähler manifold and prove

Theorem 4.2 *For any vector bundle E with real-analytic connection whose curvature is of type $(1,1)$ over a real-analytic Kähler manifold there exists an extension to a hyperholomorphic bundle over a neighbourhood U of the zero section contained in M.*

It is necessary for the existence of such a hyperholomorphic extension that E has curvature of type $(1,1)$, hence we show the result for the largest possible class of vector bundles.

We discuss the completeness question for our hyperkähler metrics by deriving necessary conditions from work of Yau, Cheeger and Gromoll and Anderson on complete Ricci-flat manifolds, see [43, 13, 1]. These conditions involve the (non)existence of certain harmonic functions and topological constraints on the fundamental group and the first Betti number of the base manifold. All three methods allow us to show (see also 5.9, 5.12)

Corollary 5.6 *There exist no complete hyperkähler metrics on the cotangent bundle $T^*\Sigma$ of a Riemann surface Σ of genus > 1.*

We also give examples of two types of incompleteness which occur: either a hyperkähler metric in a neighbourhood of the zero section blows up at finite radius or the metric can be extended to the full cotangent bundle, but is nevertheless incomplete. An example of the latter is the cotangent bundle of the moduli space of stable bundles on Riemann surfaces. Hitchin has shown in [23] that the moduli space of Higgs bundles is a complete hyperkähler manifold which contains the cotangent bundle of the moduli space of stable bundles as a proper subset. An example of the former type of incompleteness is the hyperkähler metric on a neighbourhood of the zero section of the cotangent bundle of a Riemann surface of genus greater than one (which exists by 2.1), see 5.14.

We conclude by briefly looking at our results in the context of Ricci-flat Kähler geometry. The fact that all necessary conditions given in our thesis only use Ricci-flatness, but not the full hyperkähler condition suggests that there may be an analogue in Ricci-flat Kähler geometry using special Lagrangian submanifolds.

The material is organised as follows:

In chapter 1 we provide the background material on hyperkähler manifolds, the twistor method, the quotient construction, hyperholomorphic bundles and foliations of symplectic manifolds. The facts presented here are well-known.

In chapter 2 we deal with the construction of the twistor space for a hyperkähler manifold in the cotangent bundle, and in chapter 3 we cover the uniqueness proof.

In chapter 4 we carry out a gauge-theoretical construction to obtain a hyperholomorphic vector bundle over a neighbourhood of the zero section in the cotangent bundle.

In chapter 5 we derive some necessary conditions for completeness of hyperkähler metrics on cotangent bundles, and in chapter 6 we indicate a possible generalisation to Ricci-flat Kähler geometry.

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Chapter 1

Background Material

In this chapter we gather the necessary information on hyperkähler manifolds, their twistor theory, group actions on hyperkähler manifolds, hyperholomorphic bundles and Lagrangian foliations of symplectic manifolds. The material presented here is not new, but is included for the convenience of the reader and dealt with from an angle which might be helpful for the constructions presented in the remaining chapters.

1.1 Definition and basic properties

In this section we give the definition of a hyperkähler manifold and mention some of its basic properties. We also introduce the notations we shall use in this thesis.

A Riemannian manifold (M, g) is called **hyperkähler** if it possesses three covariant constant (with respect to the Levi-Civita connection) skew-adjoint automorphisms I, J and K of its tangent bundle TM satisfying the relations of the quaternions

$$I^2 = J^2 = K^2 = IJK = -1.$$

It is obvious from the fact that they are covariant constant that the three almost complex structures I, J and K are integrable, and the manifold M is Kähler with respect to any of these complex structures, i. e. the forms $\omega_1(X,Y) = g(IX,Y)$, $\omega_2(X,Y) = g(JX,Y)$ and $\omega_3(X,Y) = g(KX,Y)$ are closed.

For any $\mathbf{u} = (a,b,c) \in S^2 \subset \mathbb{R}^3$, $I_{\mathbf{u}} = aI + bJ + cK$ is again a complex and indeed a Kähler structure on M. We think of I, J and K as the standard orthonormal basis of $\mathbb{R}^3$ and use stereographic projection $S^2 \to \mathbb{C}P^1$, i. e.

$$(a,b,c) \mapsto \frac{b+ic}{1+a}.$$

Then we can introduce the complex structures I_λ for all $\lambda \in \mathbb{C}P^1$ by

$$I_\lambda = \frac{1-|\lambda|^2}{1+|\lambda|^2} I + \frac{\lambda+\bar{\lambda}}{1+|\lambda|^2} J + \frac{\lambda-\bar{\lambda}}{i(1+|\lambda|^2)} K.$$

In this notation I_0 corresponds to I, I_1 to J and I_i to K.

Any hyperkähler manifold is in particular holomorphic-symplectic; fixing one of the complex structures, I say, $\omega_c = \omega_2 + i\omega_3$ is a holomorphic-symplectic form on M with respect to I because ω_c is a covariant constant, hence closed, $(2,0)$-form as for any vector fields X and Y of type $(0,1)$

$$\begin{aligned}\omega_c(X,Y) &= g(JX,Y) + ig(KX,Y) = g(IJX,IY) + ig(KX,Y) \\ &= g(KX,-iY) + ig(KX,Y) = 0.\end{aligned}$$

Hence $(\omega_c)^n$ is a non-vanishing holomorphic $2n$-form, i. e. the canonical bundle of M is trivial, and thus a hyperkähler manifold is in particular Ricci-flat.

We call a manifold **hypercomplex** if it possesses three complex structures I, J and K satisfying the relation $IJK = -1$.

1.2 Twistor space

In this section we give a survey of the concept of the twistor space of a hyperkähler manifold; our main references are [24] and [22].

The twistor space of a hyperkähler manifold is a generalisation of the Penrose-Ward transform for anti-self-dual metrics in dimension four. The basic idea is to encode the hyperkähler structure in holomorphic form which then allows, for example, the application of algebro-geometric methods. Regarding the construction of hyperkähler manifolds it is important to show that the hyperkähler manifold can be reconstructed from the twistor data.

As a smooth manifold the twistor space Z of a hyperkähler manifold M of real dimension $4n$ is just $Z = M \times S^2$ where S^2 is the standard 2-sphere. The key to defining a complex structure on Z is to recall that every hyperkähler manifold has in fact an S^2-worth of complex structures, namely $I_{\mathbf{u}} = aI + bJ + cK$ for any $\mathbf{u} = (a,b,c) \in S^2 \subset \mathbb{R}^3$. We define an almost complex structure on Z by setting

$$\mathbf{I} = (I_{\mathbf{u}}, \tilde{I}) \text{ on } T_zZ \cong T_mM \oplus T_{\mathbf{u}}S^2 \text{ for } z = (m,\mathbf{u}) \in Z \cong M \times S^2$$

where $\tilde{I}$ is the standard complex structure on the complex line $\mathbb{C}P^1 \cong S^2$.

Using the Newlander-Nirenberg theorem, it can be shown that $\mathbf{I}$ is integrable (see [24]), thus Z is a complex manifold of dimension $2n+1$ with a natural holomorphic projection p onto $\mathbb{C}P^1$ given by projecting onto the second factor.

Holomorphic sections of p are called **twistor lines**.

The twistor space Z has a real structure τ, i. e. an anti-holomorphic involution, given by

$$\tau(m,\lambda) = (m, -\bar{\lambda}^{-1})$$

for $(m,\lambda) \in Z$, covering the antipodal map on $\mathbb{C}P^1$.

A twistor line $s : \mathbb{C}P^1 \to Z$ is called **real** if $\tau s(\lambda) = s(-\bar{\lambda}^{-1})$, and each point $m \in M$ gives rise to a real twistor line $\{m\} \times S^2$. For the reconstruction of the hyperkähler manifold from its twistor space it is important to notice that the normal bundle of any of these twistor lines is $\mathbb{C}^{2n}(1) := \mathbb{C}^{2n} \otimes p^*\mathcal{O}(1)$, cf. [24].

We now turn to the encoding of the hyperkähler metric in the twistor picture. This will involve using fibrewise complex symplectic forms analogous to $\omega_2 + i\omega_3$ which is holomorphic with respect to I (see section 1.1).

For any $\lambda \in \mathbb{C}P^1 \setminus \{\infty\}$ we can choose $A_\lambda \in SO(3)$ such that $A_\lambda I = I_\lambda$, for example

$$A_\lambda = \begin{pmatrix} \frac{1-|\lambda|^2}{1+|\lambda|^2} & 0 & \frac{2|\lambda|}{1+|\lambda|^2} \\ \frac{\lambda+\bar{\lambda}}{1+|\lambda|^2} & \frac{\bar{\lambda}-\lambda}{2i|\lambda|} & -\frac{1-|\lambda|^2}{1+|\lambda|^2}\frac{\lambda+\bar{\lambda}}{2|\lambda|} \\ \frac{\lambda-\bar{\lambda}}{i(1+|\lambda|^2)} & \frac{\lambda+\bar{\lambda}}{2|\lambda|} & \frac{1-|\lambda|^2}{1+|\lambda|^2}\frac{\bar{\lambda}-\lambda}{2i|\lambda|} \end{pmatrix}$$

We set $\omega_\lambda(X,Y) = g((A_\lambda J + iA_\lambda K)X, Y)$ which is a symplectic form on $p^{-1}(\lambda) \subset Z$ which is holomorphic with respect to I_λ.

Obviously there is a choice involved here and fixing A_λ corresponds to fixing the second column of this matrix, i. e. fixing a unit length vector orthogonal to $\underline{x}(\lambda)$ where

$$\underline{x} : \mathbb{C}P^1 \setminus \{\infty\} \to S^2 \setminus (-1,0,0)$$

is the inverse of stereographic projection; hence the choice of A_λ involves fixing a unitary section of $T^*S^2 \cong T^*\mathbb{C}P^1$ (as S^2 with the induced metric from $\mathbb{R}^3$ is isometric to $\mathbb{C}P^1$ with the Fubini-Study metric). For the above choice of A_λ this corresponds to the section

$$\xi(\lambda) = -\frac{\lambda}{2i|\lambda|}(1+|\lambda|^2)\frac{\partial}{\partial\lambda}$$

and with respect to ξ

$$\omega_\lambda = \frac{2i|\lambda|}{1+|\lambda|^2}\omega_1 + i\frac{\lambda|\lambda|^2 - \bar{\lambda}}{|\lambda|(1+|\lambda|^2)}\omega_2 + \frac{\bar{\lambda} + \lambda|\lambda|^2}{|\lambda|(1+|\lambda|^2)}\omega_3.$$

As it is more natural to fix a holomorphic section ξ_0 of $T\mathbb{C}P^1 \cong \mathcal{O}(2)$ instead of the unitary section ξ, we fix $\xi_0(\lambda) = \frac{1}{2}\frac{\partial}{\partial\lambda}$ and rewrite ω_λ with respect to ξ_0 as

$$\omega_\lambda = (\omega_2 + i\omega_3) - 2\lambda\omega_1 - \lambda^2(\omega_2 - i\omega_3),$$

hence globally we have constructed a holomorphic section ω of $\Lambda^2 T_F^*(2)$ where $\Lambda^2 T_F^*(2) = \Lambda^2 T_F^* \otimes p^*\mathcal{O}(2)$ and $T_F = \ker(dp : TZ \to T\mathbb{C}P^1)$ is the tangent bundle along the fibres.

Example 1.1 Twistor space of flat space
The vector space $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}^* \cong \mathbb{C}^2$ with the standard flat metric is a hyperkähler manifold.

We use coordinates z, w on $\mathbb{C}^2$ and the standard holomorphic coordinate system on $\mathbb{C}P^1$ with $U_0 = \mathbb{C}P^1 \setminus \{\infty\}$, $U_1 = \mathbb{C}P^1 \setminus \{0\}$, and transition function $\lambda \mapsto \lambda^{-1}$.

Let $p : Z \to \mathbb{C}P^1$ be the twistor space of $\mathbb{H}$.

We introduce holomorphic coordinates η_1, η_2 in the fibres $p^{-1}(\lambda)$ of Z by using the $+i$-eigenvectors of I_λ. For each $(z, w) \in \mathbb{C}^2$, $\{(z, w)\} \times S^2$ is a real twistor line with normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$, hence η_1 and η_2 vary as in $\mathcal{O}(1)$ and $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$.

The three Kähler forms are given by

$$\begin{aligned}
\omega_1 &= \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w}) \\
\omega_2 &= \frac{1}{2}(dz \wedge dw + d\bar{z} \wedge d\bar{w}) \\
\omega_3 &= \frac{i}{2}(d\bar{z} \wedge d\bar{w} - dz \wedge dw)
\end{aligned}$$

hence the twisted symplectic form along the fibres is

$$\omega = dz \wedge dw - i\lambda(dz \wedge d\bar{z} + dw \wedge d\bar{w}) - \lambda^2 d\bar{z} \wedge d\bar{w}.$$

The real structure τ is given in these coordinates by

$$\tau(\lambda, \eta_1, \eta_2) = (-\bar{\lambda}^{-1}, \bar{\lambda}^{-1}\bar{\eta}_2, -\bar{\lambda}^{-1}\bar{\eta}_1).$$

Twistor lines are of the form

$$s(\lambda) = (\lambda, u_1\lambda + u_2, u_3\lambda + u_4),$$

hence real lines satisfy the conditions

$$u_4 = -\bar{u}_1 \text{ and } u_3 = \bar{u}_2.$$

Conversely, given the twistor space we can re-discover the hyperkähler manifold as the parameter space of real twistor lines. More precisely, the following theorem holds.

Theorem 1.2 (see Hitchin, Karlhede, Lindström, Roček)
Let Z^{2n+1} be a complex manifold such that

(i) Z is a holomorphic fibre bundle $p : Z \to \mathbb{C}P^1$ over the projective line,

(ii) the bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$,

(iii) there exists a holomorphic section ω of $\Lambda^2 T_F^*(2) = \Lambda^2 T_F^* \otimes p^*\mathcal{O}(2)$ (where $T_F = \ker(dp : TZ \to T\mathbb{C}P^1)$ is the tangent bundle along the fibre) defining a symplectic form on each fibre,

(iv) Z has a real structure τ compatible with (i), (ii) and (iii) and inducing the antipodal map on $\mathbb{C}P^1$.

Then the parameter space of real sections is a $4n$-dimensional manifold with a natural hyperkähler metric for which Z is the twistor space.

In condition (iv) "compatible" means that the metric constructed from the twistor data is positive definite.

A detailed proof can be found in [24], but we sketch some of the ideas involved in the proof.

Let $s : \mathbb{C}P^1 \to Z$ be one of the twistor lines in (i) with normal bundle $N = \mathbb{C}^{2n}(1)$. As $H^1(s(\mathbb{C}P^1), N) = 0$, using Kodaira-Spencer deformation theory we conclude that this rational curve is part of a complete family of deformations parametrised by M^c (which we can think of as the complexified hyperkähler manifold) of dimension $4n = \dim H^0(s(\mathbb{C}P^1), N)$ (cf. [27]). If s_m for $m \in M$ is a real twistor line its complexified tangent space is given by

$$T_m M \otimes \mathbb{C} = T_m M^c = H^0(s_m(\mathbb{C}P^1), N).$$

We now describe how to construct the hyperkähler metric.

For any $m \in M$ we have the splitting

$$T_m M \otimes \mathbb{C} = H^0(s_m(\mathbb{C}P^1), N(-1)) \otimes H^0(\mathbb{C}P^1, \mathcal{O}(1))$$

as $N(-1) = N \otimes p^*\mathcal{O}(-1) \cong \mathbb{C}^{2n}$ is trivial.

We choose sections s_0 and s_∞ of $\mathcal{O}(1)$ such that $s_0(0) = 0$, $s_\infty(\infty) = 0$ and $\langle s_0, s_\infty \rangle = 1$ with respect to the standard skew-symmetric form on $H^0(\mathbb{C}P^1, \mathcal{O}(1))$. With respect to the standard affine parameter λ on $\mathbb{C}P^1$, $s_0(\lambda) = \lambda$ and $s_\infty(\lambda) = 1$.

The twisted symplectic form ω in (iii) gives rise to a non-degenerate skew-symmetric form

$$\omega : N(-1) \times N(-1) \to H^0(\mathbb{C}P^1, \mathcal{O}) = \mathbb{C}$$

as for a section s, $T_F|_{s(\mathbb{C}P^1)} = N$.

We therefore obtain a non-degenerate symmetric form g -the complexified hyperkähler metric- by multiplying the two skew-symmetric forms. For $v = s \otimes s_0 + t \otimes s_\infty$ and $v' = s' \otimes s_0 + t' \otimes s_\infty$ in $T_m M \otimes \mathbb{C} = H^0(s_m(\mathbb{C}P^1), N(-1)) \otimes H^0(\mathbb{C}P^1, \mathcal{O}(1))$ we have

$$g(v, v') = \omega(s, t')\langle s_0, s_\infty\rangle + \omega(t, s')\langle s_\infty, s_0\rangle = \omega(s, t') + \omega(s', t).$$

1.3 Group actions on hyperkähler manifolds

In this section we shall summarise how group actions on hyperkähler manifolds which respect all or part of the hyperkähler structure can be used to either construct new hyperkähler manifolds or to gain more insights on the hyperkähler structure. Our main references are [24] and [22].

Definition 1.3 An action of a Lie group on a hyperkähler manifold is triholomorphic if it preserves the hyperkähler metric and all three complex structures I, J and K, i. e. for any vector field X generated by the action $\mathcal{L}_X g = 0$ and $\mathcal{L}_X I = \mathcal{L}_X J = \mathcal{L}_X K = 0$.

A triholomorphic action obviously preserves the Kähler forms ω_1, ω_2 and ω_3.

Before we explain the quotient construction for triholomorphic actions on hyperkähler manifolds we recall some details about symplectic quotients.

Let (M, ω) be a symplectic manifold and let the Lie group G act on M preserving the symplectic form ω. Then for any vector field X generated by the action

$$0 = \mathcal{L}_X \omega = \imath(X) d\omega + d(\imath(X)\omega) = d(\imath(X)\omega)$$

where $\imath$ denotes the contraction mapping. If $H^1(M; \mathbb{R}) = 0$ then the closed form $\imath(X)\omega$ is exact, hence there exists a function μ^X on M such that $d\mu^X = \imath(X)\omega$. Putting these together we obtain a map

$$\mu : M \to \mathfrak{g}^*$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G by

$$\langle \mu(m), \xi\rangle = \mu^X(m)$$

where X is the vector field generated by $\xi \in \mathfrak{g}$.

If G is semi-simple or a torus (or, more generally, under assumptions on the cohomology of the Lie algebra), we can make μ into an equivariant map, the so-called **moment map** of the action. The definition of μ is ambiguous up to addition of a constant $\zeta \in \mathfrak{z} \subset \mathfrak{g}^*$ where $\mathfrak{z}$ is the space of elements of $\mathfrak{g}^*$ invariant under the coadjoint action of G.

Proposition 1.4 *If $\zeta \in \mathfrak{z}$ is a regular value of μ, then $\mu^{-1}(\zeta)/G$ is again a symplectic manifold if G acts freely and properly.*

Proof: see [24] $\square$

Remark 1.5 *If M is a Kähler manifold and the action of the Lie group G preserves both the metric and the Kähler form ω, then the quotient $\mu^{-1}(\zeta)/G$ for a regular value $\zeta \in \mathfrak{z}$ is again a Kähler manifold if G acts freely and properly. Moreover, if the action can be extended to an action of the complexified group $G^{\mathbb{C}}$, the quotient can be identified with the quotient $M^s/G^{\mathbb{C}}$ of stable points in M by $G^{\mathbb{C}}$ where a point in M is called stable if its $G^{\mathbb{C}}$-orbit intersects $\mu^{-1}(\zeta)$ (cf. [24, 22]).*

Now let M be a hyperkähler manifold with a triholomorphic action by a Lie group G.

Using the Kähler forms ω_i, $i = 1, 2, 3$, we obtain maps $\mu_i : M \to \mathfrak{g}^*$, and we assume we can choose them simultaneously equivariant, hence we can form the hyperkähler moment map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$.

Proposition 1.6 *If $\zeta \in \mathfrak{z} \otimes \mathbb{R}^3$ is a regular value of μ, the quotient $\mu^{-1}(\zeta)/G$ is again a hyperkähler manifold if G acts freely and properly. Furthermore, if M is complete and G is compact, the quotient is complete.*

Proof: see [24, 22] $\square$

For applications of the hyperkähler quotient construction we describe its connection to complex-symplectic quotients.

We write the hyperkähler moment map μ as $\mu = \begin{pmatrix} \mu_1 \\ \mu_c \end{pmatrix}$ where

$$\mu_c = \mu_2 + i\mu_3 : M \to \mathfrak{g}^* \otimes \mathbb{C} = (\mathfrak{g}^{\mathbb{C}})^*$$

is the complex moment map. If the action extends to an action of $G^{\mathbb{C}}$ this is the moment map for the $G^{\mathbb{C}}$-action with respect to the holomorphic-symplectic form $\omega_c = \omega_2 + i\omega_3$. If $\zeta = (a, \alpha)$, then the hyperkähler quotient can be identified with the quotient $\mu_c^{-1}(\alpha)^s/G^{\mathbb{C}}$ of the stable points in the complex-symplectic quotient by the complexified group $G^{\mathbb{C}}$ where a point in $\mu_c^{-1}(\alpha)$ is stable if its $G^{\mathbb{C}}$-orbit intersects $\mu_1^{-1}(a)$ because $\mu^{-1}(\zeta) = \mu_1^{-1}(a) \cap \mu_c^{-1}(\alpha)$.

We can relate the hyperkähler quotient to the twistor construction.

If Z is the twistor space of the hyperkähler manifold M and $\omega \in \Lambda^2 T_F^*(2)$ is the twisted symplectic form along the fibres (cf. section 1.2) then we can define a holomorphic section

$$\tilde{\mu} : Z \to \mathfrak{g}^{\mathbb{C}^*}(2) := (\mathfrak{g}^{\mathbb{C}})^* \otimes p^*\mathcal{O}(2)$$

by calculating the complex moment map fibrewise for each complex structure, thus on the fibre over λ the map $\tilde{\mu}$ is given by

$$\tilde{\mu} = \mu_c - 2\lambda\mu_1 - \lambda^2\bar{\mu}_c.$$

For a regular value $\zeta \in \mathfrak{z} \otimes \mathbb{R}^3$ the twistor space of the quotient $\mu^{-1}(\zeta)/G$ is given by $\tilde{\mu}^{-1}(\zeta)/G^{\mathbb{C}}$ because

$$\mathfrak{g}^* \otimes \mathbb{R}^3 \subset (\mathfrak{g}^{\mathbb{C}})^* \otimes \mathbb{C}^3 \cong (\mathfrak{g}^{\mathbb{C}})^* \otimes H^0(\mathbb{C}P^1, \mathcal{O}(2)) \subset H^0(Z, (\mathfrak{g}^{\mathbb{C}})^*(2))$$

where $\xi = (a, \alpha)$ corresponds to $\alpha - 2\lambda a - \lambda^2\alpha \in (\mathfrak{g}^{\mathbb{C}})^* \otimes H^0(\mathbb{C}P^1, \mathcal{O}(2))$.

Example 1.7 Calabi metrics
We illustrate the hyperkähler quotient by constructing the Calabi metric on the cotangent bundle of complex projective space $T^*\mathbb{C}P^{n-1}$. Calabi originally described these metrics in [10] by giving their Kähler potential.

We realise complex projective space $\mathbb{C}P^{n-1}$ as a Kähler quotient of $\mathbb{C}^n$. Starting with the S^1-action on $\mathbb{C}^n$ given by scalar multiplication and using the flat metric with Kähler form $\alpha = i\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ the moment map ν is given by

$$\begin{array}{rccc} \nu: & \mathbb{C}^n & \longrightarrow & i\mathbb{R} \\ & z & \mapsto & i\|z\|^2 \end{array}$$

where we identify the Lie algebra of S^1 with $i\mathbb{R}$. Taking the regular value i the Kähler quotient is

$$\nu^{-1}(i)/S^1 = \{z \in \mathbb{C}^n; \|z\|^2 = 1\}/S^1 = \{z \in \mathbb{C}^n; z \neq 0\}/\mathbb{C}^* = \mathbb{C}P^{n-1}$$

since $z \in \mathbb{C}^n$ is stable if and only if $z \neq 0$.

We now perform the analogous hyperkähler quotient on flat quaternionic vector space $\mathbb{H}^n = V \oplus V^*$ (where V is an n-dimensional complex vector space) with the induced S^1-action given by $u.(z,\zeta) = (uz, u^{-1}\zeta)$ where $(z,\zeta) \in V \oplus V^* = \mathbb{H}^n$. Fixing the natural complex structure on $V \oplus V^*$ the Kähler form is given by

$$\omega_1 = i\sum_{j=1}^n dz_j \wedge d\bar{z}_j + d\zeta_j \wedge d\bar{\zeta}_j$$

and the corresponding holomorphic-symplectic form by

$$\omega_c = \sum_{j=1}^n dz_j \wedge d\zeta_j$$

where $z = (z_1, \dots, z_n)$ and $\zeta = (\zeta_1, \dots, \zeta_n)$. The real moment map $\mu_1 : V \oplus V^* \to i\mathbb{R}$ is given by

$$\mu_1(z, \zeta) = i(\|z\|^2 - \|\zeta\|^2)$$

and the complex moment map $\mu_c : V \oplus V^* \to \mathbb{C}$ by

$$\mu_c(z, \zeta) = \zeta(z).$$

Taking the regular value $\begin{pmatrix} i \\ 0 \end{pmatrix}$, the hyperkähler quotient is

$$\begin{aligned} \mu^{-1}\begin{pmatrix} i \\ 0 \end{pmatrix}/S^1 &= \{(z,\zeta) \in V \oplus V^*; \|z\|^2 - \|\zeta\|^2 = 1, \zeta(z) = 0\}/S^1 \\ &= \{(z,\zeta) \in V \oplus V^*; z \neq 0, \zeta(z) = 0\}/\mathbb{C}^* \end{aligned}$$

because for any $(z, \zeta) \in V \oplus V^*$, $z \neq 0$ we can find $t > 0$ such that

$$t^2\|z\|^2 - t^{-2}\|\zeta\|^2 = 1,$$

hence the $\mathbb{C}^*$-orbit of (z, ζ) intersects $\mu_1^{-1}(1)$, i. e. $(z, \zeta) \in \mu_c^{-1}(0)$ with $z \neq 0$ is a stable point for the hyperkähler quotient.

As generally for a point $[z]$ in $\mathbb{P}(V)$ the tangent space is given by

$$T_{[z]}\mathbb{P}(V) = \text{Hom}\ ([z], V/[z]) = [z]^* \otimes V/[z],$$

the cotangent space is

$$T^*_{[z]}\mathbb{P}(V) = [z] \otimes (V/[z])^* = [z] \otimes [z]^0$$

where $[z]^0$ is the annihilator of the linear subspace $[z]$ in V.

Thus with respect to the complex structure I we can identify the quotient

$$\{(z,\zeta) \in V \oplus V^*; z \neq 0, \zeta(z) = 0\}/\mathbb{C}^*$$

with $T^*\mathbb{P}(V) = T^*\mathbb{C}P^{n-1}$.

As $\mathbb{H}^n$ with the flat metric is complete and S^1 is compact, the Calabi metric on $T^*\mathbb{C}P^{n-1}$ is complete.

We now look at the fibres of the twistor space over $\mathbb{C}^* = \mathbb{C}P^1 \setminus \{0, \infty\}$ to investigate the other complex structures. We use the description of the twistor space of a hyperkähler quotient given earlier in this section. If $\lambda = a + ib$ we consider the matrix

$$B(\lambda) = \begin{pmatrix} 1 - (a^2 - b^2) & 2ab \\ -2ab & 1 + (a^2 - b^2) \end{pmatrix}$$

and the expression $\tau - \lambda^2\bar{\tau}$ for $\tau \in \mathbb{C}$ corresponds to $B(\lambda) \cdot (\Re\tau, \Im\tau)^T$ in $\mathbb{R}^2$. We also notice that $\xi = (i, 0)$ corresponds to $-2\lambda i \in H^0(\mathcal{O}(2))$.

If $\det B(\lambda) \neq 0$ then

$$\tilde{\mu}^{-1}(\xi)|_{p^{-1}(\lambda)} = \mu_1^{-1}(0) \cap (\mu_c - \lambda^2\bar{\mu}_c)^{-1}(-2\lambda i) = \mu_1^{-1}(0) \cap \mu_c^{-1}(\tau)$$

where τ satisfies $\tau - \lambda^2\bar{\tau} = -2\lambda i$, hence $\tau \neq 0$ if $\lambda \neq 0$. As $z \neq 0$ and $\zeta \neq 0$ for $(z, \zeta) \in \mu_c^{-1}(\tau)$, every point in $\mu_c^{-1}(\tau)$ is stable.

If $\det B(\lambda) = 0$ there is a solution $\tau \neq 0$ to $\tau - \lambda^2\bar{\tau} = 0$, thus

$$\tilde{\mu}^{-1}(\xi)|_{p^{-1}(\lambda)} = \mu_1^{-1}(i) \cap \mu_c^{-1}(\tau)$$

and again every point in $\mu_c^{-1}(\tau)$ is stable.

Therefore all other complex structures correspond to the complex-symplectic quotient $\mu_c^{-1}(\tau)/\mathbb{C}^*$ for $\tau \neq 0$. These quotients can be identified with

$$\mathbb{P}(V) \times \mathbb{P}(V^*) \setminus \{([z], [\zeta]); \zeta(z) = 0\}$$

which is the product of the projective space and its dual projective space minus the incidence correspondence.

We now give a different description of these complex manifolds in terms of affine bundles.

Let L be the tautological line bundle over $\mathbb{P}(V)$, i. e.

$$L = \{([z], v) \in \mathbb{P}(V) \times V; v \in [z]\}.$$

We consider the vector bundle $E = L \otimes V^*$ over $\mathbb{P}(V)$ where V^* denotes the trivial bundle with fibre V^*. Then we can define a vector bundle epimorphism β by

$$\begin{array}{ccccc} E & \stackrel{\beta}{\rightarrow} & 1 & \rightarrow & 0 \\ ([z], v \otimes \zeta) & \mapsto & \zeta(v) & & \end{array}$$

and the kernel of β is

$$\{([z], v \otimes \zeta); v \in [z], \zeta(v) = 0\} = \{([z], \zeta) \in \mathbb{P}(V) \times V^*; \zeta(z) = 0\} \cong T^*\mathbb{P}(V),$$

thus we have a short exact sequence

$$0 \rightarrow T^*\mathbb{P}(V) \rightarrow E \rightarrow 1 \rightarrow 0$$

and

$$\mathbb{P}(E) \setminus \mathbb{P}(T^*\mathbb{P}(V)) \cong \mathbb{P}(V) \times \mathbb{P}(V^*) \setminus \{([z], [\zeta]); \zeta(z) = 0\}.$$

Hence for all other complex structures the hyperkähler manifold looks like an affine bundle over $\mathbb{C}P^{n-1}$ with bundle of translations $T^*\mathbb{C}P^{n-1}$.

Therefore we can think of the twistor space for the Calabi metric as a family of affine bundles degenerating into a vector bundle over 0 and ∞ which is exactly the picture we shall produce in our construction in section 2.1 for the more general case of a real-analytic Kähler manifold.

We conclude this section with a brief discussion of isometric circle actions which are not triholomorphic, but only holomorphic with respect to one of the complex structures.

An example is the circle action by scalar multiplication in the fibres of the cotangent bundle of complex projective space $\mathbb{C}P^n$ (or indeed any complex manifold which possesses a hyperkähler metric in a neighbourhood of the zero section of its cotangent bundle for which this action is isometric) which is holomorphic with respect to the induced complex structure from the base.

Proposition 1.8 *Let S^1 act on the hyperkähler manifold M isometrically, preserving ω_1 and rotating ω_2 and ω_3. Then the moment map μ of the action with respect to ω_1 is a Kähler potential for ω_2.*

Proof: (from [24])
Let X be the Killing field of the action. Then

$$\mathcal{L}_X\omega_1 = 0 \ , \ \mathcal{L}_X\omega_2 = \omega_3 \ , \ \mathcal{L}_X\omega_3 = -\omega_2.$$

Write $d = \partial + \bar{\partial}$ where ∂ and $\bar{\partial}$ are the $(1,0)$-part and $(0,1)$-part with respect to J. The for any vector field Y on M

$$d\mu(JY) = (\partial\mu + \bar{\partial}\mu)(JY) = i(\partial\mu - \bar{\partial}\mu)(Y).$$

On the other hand,

$$d\mu(JY) = \omega_1(X, JY) = g(IX, JY) = g(KX, Y) = \omega_3(X, Y),$$

hence $\imath(X)\omega_3 = i(\partial\mu - \bar{\partial}\mu)$ and therefore

$$-2i\partial\bar{\partial}\mu = d(\imath(X)\omega_3) = \mathcal{L}_X\omega_3 = -\omega_2.$$

$\square$

1.4 Hyperholomorphic bundles

Hyperholomorphic bundles can be regarded as the gauge-theoretical analogue of hyperkähler manifolds.

As the theory of hyperholomorphic bundles is not very well documented in the literature (Ward and Wells [40] and also Mason and Woodhouse [33] describe the four-dimensional situation very well, Capria and Salamon [11] deal with the quaternionic-Kähler case, see also Verbitsky [39]), we summarise some basic properties and describe the encoding of hyperholomorphic bundles into the twistor picture.

Throughout this section vector bundles are meant to be vector bundles with connection. We call a vector bundle holomorphic in this context if it possesses a holomorphic structure which is compatible with the connection.

Generally, a holomorphic bundle $\pi : E \to X$ possesses a $\bar{\partial}_E$-operator, mapping sections into $(0,1)$-forms with values in E which is defined in terms of a local holomorphic trivialisation

$$\begin{array}{ccccc} \pi^{-1}(U_\alpha) & \to & U_\alpha & \times & \mathbb{C}^k \\ v & \mapsto & (\pi(v) & , & \psi_\alpha(\pi(v)).v) \end{array}$$

(where $(U_\alpha)_{\alpha \in A}$ form a cover of X and rank $E = k$) by

$$(\bar{\partial}_E s)_\alpha = \bar{\partial} s_\alpha$$

where $s_\alpha = \psi_\alpha s$. This is well-defined because an easy calculation shows that $(\bar{\partial}_E s)_\beta = \psi_\beta \psi_\alpha^{-1} (\bar{\partial}_E s)_\alpha$.

We call a connection on E compatible with the holomorphic structure if $\bar{\partial}_E = \bar{\partial}_A$ where $\bar{\partial}_A$ is the $(0,1)$-part of the exterior covariant derivative d_A.

Definition 1.9 A vector bundle with connection E over a hyperkähler manifold M is hyperholomorphic if it is holomorphic for all complex structures given by the hyperkähler structure on M.

We recall from the theory of vector bundles over complex manifolds

Proposition 1.10 *A vector bundle with connection E over a complex manifold X is holomorphic if and only if the $(0,2)$-part of its curvature F vanishes.*

Proof: We can split the exterior covariant derivative d_A given by the connection into its $(1,0)$-part ∂_A and its $(0,1)$-part $\bar{\partial}_A$.

If E is holomorphic, then

$$F^{0,2} = \bar{\partial}_A^2 = \bar{\partial}_E^2 = 0.$$

Conversely, if $F^{0,2} = 0$ then it can be shown (see, for example, [17]) that there is a system of local trivialisations such that the connection form is of type $(1,0)$, hence locally $(\bar{\partial}_A s)_\alpha = \bar{\partial} s_\alpha$ defines a holomorphic structure on E. □

Remark 1.11 *If $E \to M$ is a hyperholomorphic bundle, then 1.10 implies that the curvature is of type $(1,1)$ with respect to any complex structure on M.*

An important tool for the construction of hyperholomorphic bundles is the fact that they can be encoded into the twistor picture as a holomorphic bundle over twistor space with additional properties. This will be used in chapter 4.

Proposition 1.12 *Hyperholomorphic bundles over a hyperkähler manifold are in one-to-one correspondence with holomorphic vector bundles over its twistor space which are trivial along each real twistor line.*

Proof: Let $E \to M$ be a hyperholomorphic bundle over a hyperkähler manifold M, and let $p : Z \to \mathbb{C}P^1$ be the twistor space of M. Denote by $q : Z \to M$ the projection onto its first factor (which is not a holomorphic map). The bundle $W = q^*E$ with its pulled back connection is a holomorphic bundle over Z because E is holomorphic with respect to all complex structures on M and W is trivial in the $\mathbb{C}P^1$-direction, therefore it is holomorphic with respect to the complex structure on the twistor space Z. It is obvious from the definition that W is trivial along each real twistor line.

Conversely, let $W \to Z$ be a holomorphic vector bundle which is trivial along each real twistor line. By a deformation argument it is trivial along each nearby twistor line. For any point m in the complexified hyperkähler manifold M^c (i. e. the parameter space of all nearby twistor lines) we define

$$E'_m = H^0(s_m(\mathbb{C}P^1), W)$$

where s_m is the twistor line corresponding to m. As W is trivial along each twistor line, E'_m is naturally isomorphic by evaluation to W_z for any $z \in s_m(\mathbb{C}P^1)$, and $E' \to M^c$ is a vector bundle of the same rank as W.

Let $z \in Z$ and

$$S_z = \{m \in M^c; z \in s_m(\mathbb{C}P^1)\}$$

be the submanifold of M^c parameterising all twistor lines through z. There is a natural trivialisation of E' on S_z given by evaluation at the point z, i. e.

$$\begin{array}{rcl} E'|_{S_z} & \to & S_z \times W_z \\ (m,s) & \mapsto & (m, s(p(z))) \end{array}$$

where $p : Z \to \mathbb{C}P^1$ is the twistor projection. Hence we obtain a flat connection on $E'|_{S_z}$ for each $z \in Z$ by choosing the standard flat connection on the product.

Locally, a connection on a vector bundle is given by a matrix of 1-forms, thus by a matrix of elements of $H^0(\mathbb{C}P^{4n-1}, \mathcal{O}(1))$ since $T_m^*M^c = H^0(\mathbb{P}(T_mM^c), \mathcal{O}(1))$ and $\dim M^c = 4n$. We can already define covariant differentiation on the cone C of vectors at m tangent to some S_z through m, hence recalling the splitting (see section 1.2) $T_mM^c = H^0(N) = H^0(N(-1)) \otimes H^0(\mathcal{O}(1))$, we can differentiate along the decomposable tangent vectors $v \otimes \psi$.

Consider the map

$$\begin{array}{cccc} f: & \mathbb{C}P^{2n-1} \times \mathbb{C}P^1 & \to & \mathbb{C}P^{4n-1} \\ & (v, \psi) & \mapsto & v \otimes \psi \end{array}.$$

The map f is linear in v for fixed ψ and linear in ψ for fixed v, thus of bidegree $(1,1)$, and $\mathcal{O}(1)$ pulls back to $\mathcal{O}(1,1)$. We consider the induced map

$$H^0(\mathbb{C}P^{4n-1}, \mathcal{O}(1)) \to H^0(\mathbb{C}P^{2n-1} \times \mathbb{C}P^1, \mathcal{O}(1,1)).$$

It is injective because $\mathbb{C}^{2n} \otimes \mathbb{C}^2$ is spanned by the decomposable elements, hence $v \otimes \psi$ do not lie in any linear subspace. By the Künneth formula,

$$\begin{aligned} \dim H^0(\mathbb{C}P^{2n-1} \times \mathbb{C}P^1, \mathcal{O}(1,1)) &= \dim H^0(\mathbb{C}P^{2n-1}, \mathcal{O}(1)) \cdot \dim H^0(\mathbb{C}P^1, \mathcal{O}(1)) \\ &= 4n, \end{aligned}$$

and as $\dim H^0(\mathbb{C}P^{4n-1}, \mathcal{O}(1)) = 4n$ the induced map is an isomorphism, hence there is a unique connection on E' which extends the given flat connection on $E'|_{S_z}$.

It remains to show that starting with a hyperholomorphic vector bundle $E \to M$ there is an identification of E' and E as vector bundles with connection.

There is an obvious natural identification of E'_m and E_m given by

$$\begin{array}{ccccc} E'_m = H^0(s_m(\mathbb{C}P^1), W) & \to & W_z & \to & E_m \\ s & \mapsto & s(\lambda) & \mapsto & q_*s(\lambda) \end{array}$$

where $z = s_m(\lambda)$, $\lambda \in \mathbb{C}P^1$.

In order to identify the connections we first show that the connection on E is flat on S_z.

Take $m \in S_z \cap M$, $z \in Z$, $p(z) = \lambda_0$, then

$$T_mS_z = \{s \in H^0(s_m(\mathbb{C}P^1), N); s(\lambda_0) = 0\} \subset T_mM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$$

where we split the complexified tangent bundle into its $(1,0)$-part and $(0,1)$-part with respect to the complex structure I_{λ_0}. Let $s_{\lambda_0} \in H^0(\mathbb{C}P^1, \mathcal{O}(1))$ be defined by $s_{\lambda_0}(\lambda) = \lambda - \lambda_0$, then the splitting $H^0(N) = H^0(N(-1)) \otimes H^0(\mathcal{O}(1))$ yields

$$T_m S_z = \{\tilde{s} \otimes s_{\lambda_0}; \tilde{s} \in H^0(s_m(\mathbb{C}P^1), N(-1)) \subset T^{1,0}.$$

As the curvature of E is of type $(1,1)$ with respect to all complex structures, in particular I_{λ_0}, the curvature vanishes on S_z.

Therefore, any covariant constant section of E on S_z is uniquely determined by its value at one point, hence we can identify $W_z = E_{q(z)}$ with the space of covariant constant sections of E on S_z.

By the very definition of the connection on E', the covariant constant sections of E' on S_z can be identified with W_z, hence E and E' have the same covariant constant sections on S_z. Since $\dim W_z = \text{rank } E$ the connections are equivalent on S_z for all $z \in Z$.

As we have seen above, the connections are uniquely determined by their values on S_z, hence E and E' are isomorphic as vector bundles with connection. $\square$

1.5 Foliations of symplectic manifolds

In this section we provide the material on the geometry of Lagrangian foliations of symplectic manifolds used in our construction in chapter 2 and also in the example in chapter 6.

The main reference for this section is [41, sections 4.6, 4.7].

Definition 1.13 A foliation of a manifold M is a subbundle P of the tangent bundle TM such that for all vector fields X, Y on M which are tangent to P, i. e. $X(m) \in P_m$, the fibre of P over m, for all $m \in M$, the vector field $[X, Y]$ is again tangent to P.

Remark 1.14 *If P is a foliation of M, rank $P = k$, $\dim M = n$, then in a neighbourhood of each point in M there exists a coordinate system $x_1, \dots, x_n$ of M such that the subspaces $A = \{x_i = const, k+1 \leq i \leq n\}$ are integral manifolds of P, i. e. $T_m A = P_m$ for all $m \in A$.*

A maximal connected integral manifold of a foliation is called a leaf.

Definition 1.15 A foliation of a symplectic manifold (M, ω) is called Lagrangian if for any $m \in M$, P_m is a Lagrangian subspace of $T_m M$, i. e. rank $P = \frac{1}{2} \dim M$ and $\omega|_{P_m} = 0$. This is sometimes called a polarisation.

Proposition 1.16 *Let P be a Lagrangian foliation of a symplectic manifold (M, ω) and let Λ be a leaf of P. Then there exists a natural flat torsion-free affine connection on $T\Lambda$.*

Proof: We first define a partial connection

$$\nabla : \Gamma_P(TM) \times \Gamma_P(TM) \to \Gamma_P(TM)$$

where $\Gamma_P(TM)$ is the space of vector fields on M tangent to P by

$$\imath(\nabla_X Y)\omega = \mathcal{L}_X(\imath(Y)\omega) = \imath(X)d(\imath(Y)\omega)$$

for any $X, Y \in \Gamma_P(TM)$ where $\imath$ denotes the contraction mapping.

Note that $\nabla_X Y \in \Gamma_P(TM)$ because for any $Z \in \Gamma_P(TM)$

$$\omega(\nabla_X Y, Z) = \imath(Z)\mathcal{L}_X(\imath(Y)\omega) = X(\omega(Y, Z)) - \omega(Y, [X, Z]) = 0$$

since $[X, Z]$ is also in $\Gamma_P(TM)$.

Any vector fields $\tilde{X}, \tilde{Y}$ on Λ can be extended locally to $X, Y \in \Gamma_P(TM)$ and we define

$$\nabla_{\tilde{X}}\tilde{Y} = \nabla_X Y|_\Lambda.$$

It is easy to check that this definition is independent of the choice of extensions. It immediately follows from the definition that

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y$$

for any function f, hence the above definition indeed gives an affine connection. Since

$$\begin{aligned}
\imath(\nabla_X Y - \nabla_Y X)\omega &= \imath(X)d(\imath(Y)\omega) - \imath(Y)d(\imath(X)\omega) \\
&= \imath(X)\mathcal{L}_Y\omega - \imath(Y)\mathcal{L}_X\omega \\
&= \imath([X, Y])\omega + 2d(\omega(X, Y)) = \imath([X, Y])\omega
\end{aligned}$$

the connection is torsion-free. The connection is flat because

$$\begin{aligned}
\imath(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z)\omega &= \mathcal{L}_X(\imath(\nabla_Y Z)\omega) - \mathcal{L}_Y(\imath(\nabla_X Z)\omega) \\
&= \mathcal{L}_X\mathcal{L}_Y(\imath(Z)\omega) - \mathcal{L}_Y\mathcal{L}_X(\imath(Z)\omega) \\
&= \mathcal{L}_{[X,Y]}(\imath(Z)\omega) \\
&= \imath(\nabla_{[X,Y]}Z)\omega
\end{aligned}$$

□

Example 1.17 Let $M = T^*Q$ be the cotangent bundle of a manifold Q and let ω be the canonical symplectic structure on T^*Q. Then the individual cotangent spaces form a Lagrangian foliation P of T^*Q called the vertical foliation. The flat connection on the leaves coincides with the one given by the vector space structure; this can be shown using local coordinates.

The zero section in T^*Q is a transverse Lagrangian section of P, i. e. it is a Lagrangian submanifold which intersects each leaf of P transversally in exactly one point.

We shall now show following [41] that the above example is indeed the model for any symplectic manifold with Lagrangian foliation if the foliation has a transverse Lagrangian section.

Proposition 1.18 *Let P be a Lagrangian foliation of a symplectic manifold (M, ω) and let Q be a Lagrangian submanifold of M which intersects each leaf of P transversally in exactly one point. If the leaves of P are simply connected and geodesically complete (with respect to the connection constructed in 1.16), M can be naturally identified with the cotangent bundle T^*Q. Under this identification ω coincides with the canonical symplectic form of the cotangent bundle, P coincides with the vertical foliation and Q coincides with the zero section. If the leaves are not geodesically complete, there exists a neighbourhood of Q in M which can be identified with a neighbourhood of the zero section in T^*Q.*

Proof: Let $q \in Q$ and denote by Λ_q the leaf of P through q.

If Λ_q possesses a complete flat affine connection and is simply connected, it is isomorphic to the standard affine space (cf. [26, Chapter V, Theorem 4.2]), and choosing q as the origin, Λ_q is a vector space. Therefore,

$$\Lambda_q \cong T_m \Lambda_q = P_m$$

for any $m \in \Lambda_q$. As Q is a transverse Lagrangian section of P,

$$T_q Q \oplus P_q = T_q M$$

for all $q \in Q$, and we can use the symplectic form ω to identify P_q and T_q^*Q, hence we obtain an identification of Λ_q with T_q^*Q. Varying $q \in Q$, this identifies M and T^*Q, and P coincides with the vertical foliation and $Q \subset M$ coincides with the zero section.

As there are no local invariants of symplectic manifolds by Darboux's theorem, ω coincides with the canonical symplectic form ω' of the cotangent bundle on vectors in $T_q M \cong T_q(T^*Q)$ for any $q \in Q$. It remains to show that $\omega = \omega'$ elsewhere.

Let f be a function on M such that df vanishes on $P \subset TM$, and we consider the Hamiltonian vector fields X_f and X'_f constructed using ω and ω' respectively. Then

$X_f, X'_f \in \Gamma_P(TM)$ and for any $Y \in \Gamma_P(TM)$

$$\imath(\nabla_Y X_f)\omega = \imath(Y)d(\imath(X_f)\omega) = \imath(Y)d^2 f = 0$$

and similarly, X'_f is covariant constant. As $X_f = X'_f$ on Q, they coincide everywhere (using parallel transport along the flat connection). Also

$$\mathcal{L}_{X_f}\omega = \imath(X_f)d\omega + d(\imath(X_f)\omega) = d^2 f = 0$$

and similarly $\mathcal{L}_{X'_f}\omega' = 0$, hence $\mathcal{L}_{X_f}(\omega - \omega') = 0$, i. e. $\omega = \omega'$ along flow lines of X_f starting in Q. As any flow line of X_f in a leaf Λ_q is a geodesic and as Λ_q is geodesically complete, we conclude that any point in M can be reached by such a flow line, thus $\omega = \omega'$ everywhere.

If the leaves are not geodesically complete, we can choose a simply connected and geodesically convex neighbourhood U_q of q in Λ_q and identify it with a neighbourhood of $0 \in T^*_q Q$ using the above process. As any point in a geodesically convex neighbourhood can be reached by a flow line of one the Hamiltonian vector fields X_f used above, the proof given in the complete case holds here as well. $\square$

Corollary 1.19 *Let (M, ω) be a symplectic manifold with Lagrangian foliation P and let $Q \subset M$ be a transverse Lagrangian section of P. Then there exists a coordinate system $p_1, \dots, p_n, q_1, \dots, q_n$ of M in a neighbourhood of each point in Q such that $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ and the leaves of P coincide locally with $q_i = const.$*

Proof: This is an immediate consequence of 1.18. $\square$

Remark 1.20 *We can carry out the same constructions if M is a holomorphic-symplectic manifold and P is a foliation of complex Lagrangian submanifolds of M. In particular, if Q is a complex Lagrangian section of P we can identify a neighbourhood of Q in M with a neighbourhood of the zero section in T^*Q as holomorphic-symplectic manifolds and the coordinate system in 1.19 will be a system of holomorphic coordinates.*

Chapter 2

The Existence Result

In this chapter we describe a way in which a Kähler metric on a complex manifold extends to a hyperkähler metric on a neighbourhood of the zero section of its cotangent bundle.

It is necessary for the Kähler metric to be real-analytic. This can be shown using the twistor space: the complexified metric is given in terms of the holomorphic twisted symplectic form, hence applying the real structure implies real-analyticity.

By constructing the twistor space and showing the existence of twistor lines we shall prove

Theorem 2.1 *Let X be a real-analytic Kähler manifold. Then there exists a hyperkähler metric in a neighbourhood of the zero section of the cotangent bundle which is compatible with the canonical holomorphic-symplectic structure.*

Furthermore, the S^1-action given by scalar multiplication in the fibres is isometric and the restriction of the hyperkähler metric to the zero section induces the original Kähler metric.

This result has been proved independently by D. Kaledin using different methods, cf. [25].

Kaledin uses the concept of what he calls Hodge manifolds X; these are manifolds with an action of the quaternions on their cotangent bundle and a compatible S^1-action (i. e. the map $\mathbb{H} \otimes_{\mathbb{R}} T^*X \to T^*X$ is equivariant where S^1 acts on $\mathbb{H}$ by conjugation using the "preferred" almost complex structure $I : \mathbb{C} \to \mathbb{H}$) such that the almost complex structure I and the "complementary" almost complex structure J are both integrable. Using the S^1-action, he proves that if J and J' are two almost complex structures anti-commuting with I then they are equivalent, and he calls this the "complementary" complex structure. A Hodge manifold is therefore hypercomplex (for a definition see section 1.1) with an S^1-action which is holomorphic for the preferred complex structure.

Notice that this is the same picture as obtained from our twistor construction, namely that all the complex structures of the hyperkähler manifold in the cotangent bundle which are not the "preferred" complex structure I induced from the complex structure on the base manifold (or $-I$) are equivalent since points in the fibres over $\mathbb{C}P^1 \setminus \{0, \infty\}$ of the twistor space can be identified using the twistor lines corresponding to the complexification X^c, see chapter 3.

The study of Hodge manifolds is equivalent to studying a certain kind of connection on a neighbourhood of the zero section of the tangent bundle TX of a complex manifold X. Kaledin considers formal neighbourhoods and extends all definitions to the formal setting. He shows that formal Hodge manifold structures on TX correspond to so-called linear flat extended connections on T^*X which in turn correspond to connections ∇ on T^*X with vanishing torsion and curvature of type $(1,1)$. He continues by proving that Kähler metrics on X compatible with ∇ correspond to formal germs of hyperkähler metrics on a formal neighbourhood of the zero section in TX compatible with the formal Hodge manifold structure. Using a Darboux-type theorem, he can identify a formal neighbourhood of the zero section in the tangent bundle TX with a formal neighbourhood of the zero section in the cotangent bundle T^*X. Hence the Kähler metric on X gives a formal germ of a hyperkähler metric on a neighbourhood of the zero section of T^*X which is S^1-invariant and extends the Kähler metric. If the Kähler metric is real-analytic, Kaledin proves convergence to a metric in an open neighbourhood of the zero section.

Kaledin's construction heavily relies on the S^1-action whereas we construct the hyperkähler metric (or rather its twistor space) not explicitly using the action; however, the circle action plays a crucial rôle in our proof of uniqueness given in chapter 3.

Kaledin needs the S^1-action to describe the problem in a holomorphic setting which is different from our holomorphic formulation using the concept of the twistor space of a hyperkähler manifold.

Our construction is partially motivated by work of LeBrun [31] and Burns [9].

LeBrun constructed in [31] a self-dual Einstein metric on $M^3 \times (a,b)$ where M^3 is a real-analytic Riemannian conformal manifold by considering the space of *null geodesics* in the complexification of M^3 which turns out to be the twistor space for the desired Einstein metric. The method produces a complex 3-manifold with a 3-parameter family of rational curves corresponding to points in M^3. Deformation arguments show that there is a 4-parameter family containing these, corresponding to the points of the 4-dimensional Einstein manifold.

Our case is similar: we construct a twistor space with a family of rational curves and allow them to deform. For the complexification of a real-analytic 2-manifold, we can find two families of null geodesics. In higher dimensions, for a Kähler manifold, these will be two families of totally geodesic null subspaces. Using the connection defined in 1.16 these null spaces carry an affine structure.

This fits into the picture of the construction by Burns in [9] of the twistor space of a hyperkähler metric in a neighbourhood of the zero section of the cotangent bundle of a compact simply-connected homogeneous Kähler manifold. Burns produces two halves of the twistor space over $\mathbb{C}$ by considering two families of affine bundles parametrised by $\mathbb{C}$ which degenerate into vector bundles over $0 \in \mathbb{C}$ and patches them together to obtain the twistor space.

We have also seen in 1.7 that the twistor space of the Calabi metrics on $T^*\mathbb{C}P^{n-1}$ consists of a family of affine bundles with bundle of translation equal to $T^*\mathbb{C}P^{n-1}$ parametrised by $\mathbb{C}P^1$; the affine bundle degenerates into a vector bundle (i. e. into $T^*\mathbb{C}P^{n-1}$) over 0 and ∞.

This situation is essentially the same as in our construction.

Affine bundles over a Kähler manifold X with bundle of translations T^*X correspond to extensions of the form

$$0 \to T^*X \to V \to 1 \to 0$$

where the affine bundle is $\mathbb{P}(V) \setminus \mathbb{P}(T^*X)$. These extensions are parametrised up to holomorphic equivalence by $H^1(X, T^*X)$. If X is compact, then $H^1(X, T^*X) \cong H^{1,1}(X)$, hence the Kähler form gives rise to a natural non-trivial extension class.

However, we point out that our construction also works in the non-compact case -it is a canonical local construction.

2.1 Construction

2.1.1 Real-analytic Kähler manifolds

A real-analytic Kähler manifold X is a Kähler manifold which has a real-analytic coordinate system $x_1, \dots, x_{2n}$ such that the metric is given by

$$g = \sum_{i,j=1}^{2n} g_{ij} dx_i dx_j$$

with real-analytic coefficient functions g_{ij}.

Proposition 2.2 *There exists a holomorphic coordinate system on X which is real-analytic and for which the Kähler metric is given by real-analytic coefficient functions.*

Proof: Take the above real-analytic coordinate system. We want to find real-analytic solutions z to the equation $\bar{\partial} z = 0$ where $\bar{\partial}$ is the operator given by the Cauchy-Riemann equations for the complex structure I on X.

As X is Kähler, $\nabla I = 0$, hence by standard theorems on the dependence of solutions of ordinary differential equations on coefficients and initial conditions I is real-analytic in the coordinate system $x_1, \dots, x_n$, and so is the $\bar{\partial}$-operator. Therefore we find real-analytic solutions to $\bar{\partial} z = 0$ by elliptic regularity.

As these holomorphic coordinates are obtained from the real-analytic coordinates by a real-analytic coordinate change, the Kähler metric is given by real-analytic coefficient functions in the holomorphic coordinate system. $\square$

2.1.2 Complexification of a real-analytic Kähler manifold

Let X be a real-analytic Kähler manifold. As explained in 2.2 , we can choose holomorphic coordinates $z_1, \dots, z_n$ such that the Kähler form is given by $\omega = \sum_{i,j=1}^n h_{ij} dz_i \wedge d\bar{z}_j$, where h_{ij} is real-analytic with respect to z_k and $\bar{z}_k$, $1 \leq k \leq n$. We can write h_{ij} as a convergent power series in $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$

$$h_{ij} = \sum_{I,J} a_{ij}^{IJ} z_1^{i_1} \cdots z_n^{i_n} \bar{z}_1^{j_1} \cdots \bar{z}_n^{j_n}$$

using the multi-indices $I = (i_1, \dots, i_n)$ and $J = (j_1, \dots, j_n)$.

Locally, the complexification X^c is given by holomorphic coordinates $z_1, \dots, z_n$, $\tilde{z}_1, \dots, \tilde{z}_n$, and the complexified Kähler form is

$$\omega_c = \sum_{i,j=1}^n h_{ij} dz_i \wedge d\tilde{z}_j$$

where we think of h_{ij} as an analytic continuation

$$h_{ij} = \sum_{I,J} a_{ij}^{IJ} z_1^{i_1} \cdots z_n^{i_n} \tilde{z}_1^{j_1} \cdots \tilde{z}_n^{j_n}$$

of the above coefficients. Then ω_c is a holomorphic-symplectic form on X^c.

On X^c there is a real structure $\sigma : X^c \to X^c$ given locally by

$$\sigma(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n) = (\bar{\tilde{z}}_1, \dots, \bar{\tilde{z}}_n, \bar{z}_1, \dots, \bar{z}_n).$$

We can then identify X with the set of real points, i. e. the fixed points of σ:

$$(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) \in X^c,$$

but under this identification X is not a complex submanifold of X^c.

The complexification X^c carries two natural foliations given locally by $z_i = const$ for $1 \leq i \leq n$ and $\tilde{z}_i = const$ respectively. We call these the +-foliation (respectively −-foliation). On a small open subset their leaves form submanifolds of X^c which are obviously Lagrangian with respect to ω_c.

The real structure interchanges the foliations, and $X \subset X^c$ intersects each leaf in exactly one point, namely in

$$(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n) = (\bar{\tilde{z}}_1, \ldots, \bar{\tilde{z}}_n, \tilde{z}_1, \ldots, \tilde{z}_n).$$

We can therefore identify the local space of leaves B_+ with X, and the space of leaves B_- with $\bar{X}$, i. e. X with the opposite complex structure.

Example 2.3 The standard affine quadric

$$Q^2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3; z_1^2 + z_2^2 + z_3^2 = 1\}$$

is the complexification of $\mathbb{C}P^1$.

We can extend Q^2 to a quadric $\tilde{Q}^2$ in $\mathbb{C}P^3$ given by $\lambda_0\lambda_1 = \lambda_2\lambda_3$ where $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are homogeneous coordinates for $\mathbb{C}P^3$ by setting $\lambda_0 = z_1 + iz_2$, $\lambda_1 = z_1 - iz_2$, $\lambda_2 = 1 + z_3$ and $\lambda_3 = 1 - z_3$. The quadric $\tilde{Q}^2$ is biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ by

$$\begin{array}{ccccc} \mathbb{C}P^1 & \times & \mathbb{C}P^1 & \rightarrow & \mathbb{C}P^3 \\ ((z_0, z_1) & & (w_0, w_1)) & \mapsto & (z_0w_0, z_1w_1, z_0w_1, z_1w_0) \end{array}$$

hence $\tilde{Q}^2$ possesses two natural families of projective lines. Q^2 has thus two families of affine lines parametrised by $\mathbb{C}P^1$ and can be thought of as a ruled surface in two different ways.

We identify $X = \mathbb{C}P^1$ with $S^2 \subset \mathbb{R}^3$ using stereographic projection

$$S^2 \longrightarrow \mathbb{C}P^1, (x_1, x_2, x_3) \mapsto \frac{x_1}{1 - x_3} + i\frac{x_2}{1 - x_3}.$$

This, in fact, gives an identification as manifolds with metric if we use the metric on S^2 induced from the flat metric on $\mathbb{R}^3$ and the Fubini-Study metric $\omega = 4i\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$ on $\mathbb{C}P^1$. The map

$$Q^2 \longrightarrow X^c, (z_1, z_2, z_3) \mapsto \left(z = \frac{z_1 + iz_2}{1 - z_3}, \tilde{z} = \frac{z_1 - iz_2}{1 - z_3}\right)$$

identifies Q^2 with X^c.

Furthermore, the null lines given by $z = const$ and $\tilde{z} = const$ respectively are exactly the two natural families of affine lines mentioned above, namely $\frac{\lambda_0}{\lambda_3} = const$ and $\frac{\lambda_1}{\lambda_3} = const$ respectively.

2.1.3 The bundle of affine linear functions

We consider one of the foliations, without loss of generality the $+$-foliation, with space of leaves $B = B_+$. Each leaf carries a flat torsion-free affine connection, see 1.16.

For each $b \in B$ let V_b be the vector space of affine linear functions on the leaf Λ_b determined by b, i. e. of functions $f : \Lambda_b \to \mathbb{C}$ such that $\nabla(df) = 0$.

If we choose an open set $U \subset X^c$ such that $\Lambda_b \cap U$ is simply-connected for all $b \in B$, then $\dim V_b = n + 1$ because there is a basis of covariant constant 1-forms $\theta_1, \dots, \theta_n$ on each leaf (as the connection is flat) which are closed (as the connection is torsion-free), and therefore $\theta_i = df_i$, $1 \leq i \leq n$, and $f_1, \dots, f_n$ together with $f_0 \equiv 1$ form a basis of V_b. Furthermore, they form a holomorphic vector bundle $V \to B$.

We consider the dual bundle $V^* \to B$ and define the maps

- $\pi : V^* \to \mathbb{C}$ by evaluating $\alpha \in V_b^*$ at the affine linear function $1 \in V_b$, i. e. $\pi(\alpha) = \alpha(1)$.
- $\phi : U \times \mathbb{C}^* \to V^*, (x, \lambda) \mapsto \lambda\delta_x$ where δ_x denotes the evaluation map at x of an affine linear function.

As $\pi\delta_x = \delta_x(1) = 1$,

$$\begin{array}{rccc} s_x : & \mathbb{C} & \to & V^* \\ & \lambda & \mapsto & \phi(x, \lambda) \end{array}$$

is a section of π for each $x \in U \subset X^c$.

Example 2.4 We consider the example of flat space as it is going to serve as a useful model for the general case.

Let $X = \mathbb{C}^n$ and $\omega = \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ be the Kähler form of the standard flat metric. Then $X^c = \mathbb{C}^n \oplus \mathbb{C}^n$ and $\omega_c = \sum_{j=1}^n dz_j \wedge d\tilde{z}_j$ is the standard 2-form. For the foliation $z_j = const$ the functions $1, \tilde{z}_1, \dots, \tilde{z}_n$ form a basis of affine linear functions, and V is trivial over $B \cong \mathbb{C}^n$.

If we choose coordinates $z_1, \dots, z_n, x_0, \dots, x_n$ for V^* with respect to the dual basis the map ϕ is given in these coordinates by

$$\phi(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n, \lambda) = (z_1, \dots, z_n, \lambda, \lambda\tilde{z}_1, \dots, \lambda\tilde{z}_n).$$

These calculations will be made again in more detail for the general case in the proof of 2.9.

Remark 2.5 *Locally, V^* can be identified with the standard flat model described in 2.4.*

Proof: The complexified manifold X^c is foliated by Lagrangian submanifolds and choosing a suitable section of the quotient map onto $B_+ \cong X$ we can identify X as a transverse Lagrangian section.

Using 1.19 we can find coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ in U such that the foliation is given by $q_i = const$ and $\omega_c = \sum_{i=1}^n dq_i \wedge dp_i$. In these coordinates $f_i = p_i$, $1 \le i \le n$, and if we choose coordinates $q_1, \dots, q_n, u_0, \dots, u_n$ in V^* with respect to the dual basis to $f_0, \dots, f_n$, the map ϕ is given by

$$\phi(p_1, \dots, p_n, q_1, \dots, q_n, \lambda) = (q_1, \dots, q_n, \lambda, \lambda p_1, \dots, \lambda p_n)$$

as in the flat case. □

Using the coordinates of the above proof illustrates that we can think of V^* as a family of affine bundles over $B = X$ parametrised by $\lambda \in \mathbb{C}$, degenerating into a vector bundle over $\lambda = 0$.

We also notice that the above argument allows us to identify a neighbourhood of the zero section in $\pi^{-1}(0) \subset V^*$ with a neighbourhood of the zero section of the cotangent bundle.

Proposition 2.6 *ϕ is a holomorphic diffeomorphism of $U \times \mathbb{C}^*$ onto its image.*

Proof: If $\phi(x_1, \lambda_1) = \phi(x_2, \lambda_2)$, then applying π gives $\lambda_1 = \lambda_2$ and therefore $\delta_{x_1} = \delta_{x_2}$, i. e. x_1 and x_2 determine the same fibre.

We choose a basis $f_0 \equiv 1, f_1, \dots, f_n$ for V, then the affine linear functions $g_j = f_j - f_j(x_1) f_0$ vanish at x_1 and thus at x_2, i. e. $f_j(x_1) = f_j(x_2)$ for $0 \le j \le n$. As $f_0, \dots, f_n$ form a basis, we conclude $x_1 = x_2$.

If we choose coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ in U as in the proof of 2.5, ϕ is given by

$$\phi(p_1, \dots p_n, q_1, \dots, q_n, \lambda) = (q_1, \dots, q_n, \lambda, \lambda p_1, \dots, \lambda p_n)$$

hence ϕ is a local diffeomorphism for $\lambda \neq 0$ by the inverse function theorem. Taking into account the injectivity of ϕ, the result follows. □

We notice that, defining

$$\begin{array}{cccc} \phi_\lambda : & U & \to & V^*|_{\pi^{-1}(\lambda)} \\ & x & \mapsto & \phi(x, \lambda) \end{array}$$

for each fibre $V_b^* = \pi^{-1}(b)$, $b \in B$, the inverse image $\phi_\lambda^{-1}(V_b^*)$ is a leaf of the foliation. This can be shown using the above mentioned local coordinates.

2.1.4 The attaching map and explicit twistor lines

Applying the above construction to both foliations and choosing $c \in]0,1[$ we define

$$Z_+ = V_+^*|_{\pi_+^{-1}(\{\lambda\in\mathbb{C}; |\lambda|<c\})} \cup \phi_+(U \times \{\lambda \in \mathbb{C}; c \leq |\lambda| < 1/c\})$$

and

$$Z_- = V_-^*|_{\pi_-^{-1}(\{\tilde\lambda\in\mathbb{C}; |\tilde\lambda|<c\})} \cup \phi_-(U \times \{\tilde\lambda \in \mathbb{C}; c \leq |\tilde\lambda| < 1/c\}).$$

We shall show that

$$Z = Z_+ \coprod_{\sim} Z_-$$

where $\phi_+(x,\lambda) \sim \phi_-(x,\lambda^{-1})$ for $c < |\lambda| < 1/c$ is the desired twistor space.

The **twistor projection** $p : Z \to \mathbb{C}P^1$ is obtained by using π_+ and π_- on Z_+ and Z_- respectively. As the sections s_x^+ and s_x^- patch together, we have a family of explicit **twistor lines** $s_x : \mathbb{C}P^1 \to Z$ parametrised by $x \in U$.

Example 2.7 We continue the example of $X = \mathbb{C}P^1$ from 2.3. We shall illustrate the different steps involved in the construction in this example using the same notation as in the general case.

We recall that the two foliations in $X^c \cong Q^2$ correspond to the following families of affine lines parametrised by $\lambda \in X = \mathbb{C}P^1$ and $\mu \in X = \mathbb{C}P^1$ respectively:

- for the $+$-foliation

$$z_1 + iz_2 = \lambda(1 - z_3) \text{ and } z_1 - iz_2 = \lambda^{-1}(1 + z_3)$$

- for the $-$-foliation

$$z_1 + iz_2 = \mu^{-1}(1 + z_3) \text{ and } z_1 - iz_2 = \mu(1 - z_3)$$

If $\lambda \neq 0$ we can choose $z_1 + iz_2$ as affine parameter, if $\lambda \neq \infty$ we can choose $z_1 - iz_2$. We use the standard coordinate patches $U_0 = X \setminus \{\infty\}$ and $U_1 = X \setminus \{0\}$ for $X = \mathbb{C}P^1$.

If $\lambda \neq 0, \infty$, the ansatz

$$\begin{aligned}
\tilde a + \tilde b(z_1 + iz_2) &= a + b(z_1 - iz_2) \\
&= a + b(\lambda^{-1}(1 + z_3)) \\
&= a + b\lambda^{-1}(1 + (1 - \lambda^{-1}(z_1 + iz_2))) \\
&= a + 2\lambda^{-1}b - \lambda^{-2}b(z_1 + iz_2)
\end{aligned}$$

yields the transition function for V_+, namely

$$(\lambda, a, b) \mapsto (\lambda^{-1}, a + 2\lambda^{-1}b, -\lambda^{-2}b)$$

hence with respect to the dual basis the transition function for V_+^* is

$$(\lambda, \alpha, \beta) \mapsto (\lambda^{-1}, \alpha, 2\lambda\alpha - \lambda^2\beta).$$

Similarly, the transition function for V_-^* is given by

$$(\mu, \gamma, \delta) \mapsto (\mu^{-1}, \gamma, 2\mu\gamma - \mu^2\delta).$$

The maps ϕ_+ and ϕ_- are

$$\begin{array}{rccc} \phi_+ : & Q^2 \times \mathbb{C} & \rightarrow & V_+^* \\ & (z_1, z_2, z_3, u) & \mapsto & (\lambda(z_1, z_2, z_3), \alpha(u), \beta(z_1, z_2, z_3, u)) \end{array}$$

$$\begin{array}{rccc} \phi_- : & Q^2 \times \mathbb{C} & \rightarrow & V_-^* \\ & (z_1, z_2, z_3, u) & \mapsto & (\mu(z_1, z_2, z_3), \gamma(u), \delta(z_1, z_2, z_3, u)) \end{array}$$

where

$$\lambda(z_1, z_2, z_3) = \frac{z_1 + iz_2}{1 - z_3}\ ,\ \mu(z_1, z_2, z_3) = \frac{z_1 - iz_2}{1 - z_3},$$

$$\alpha(u) = \gamma(u) = u$$

$$\beta(z_1, z_2, z_3, u) = u(z_1 - iz_2)\ ,\ \delta(z_1, z_2, z_3, u) = u(z_1 + iz_2).$$

Using the identification $\phi_+(z_1, z_2, z_3, u) \sim \phi_-(z_1, z_2, z_3, u^{-1})$ yields the following relations:

$$\gamma = \alpha^{-1}\ ,\ \alpha^{-1}\beta\lambda = \gamma^{-1}\mu\delta$$

$$2\lambda - \alpha^{-1}\lambda^2\beta = \gamma^{-1}\delta\ ,\ \alpha^{-1}\beta = 2\mu - \gamma^{-1}\mu^2\delta$$

and analogous expressions on $\{(z_1, z_2, z_3) \in Q^2; z_3 \neq -1\}$.

We shall show now how the twistor space Z constructed this way can be identified with the twistor space for the Eguchi-Hanson metric on $T^*\mathbb{C}P^1$ described in [21]. It is obtained by blowing-up the singularities in

$$\tilde{Z} = \{(x, y, z) \in \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2); xy = u^2 - z^2\}.$$

If we use the standard coordinate patches U_0 and U_1 for $\mathbb{C}P^1$ we can use coordinates u, x, y, z and $\tilde{u}, \tilde{x}, \tilde{y}, \tilde{z}$ for the total space of $\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$. In these coordinates we define the following maps

$$\begin{array}{cccc} \psi_+ : & V_+^* & \to & \tilde{Z}|_{U_0} \\ & (\lambda, \alpha, \beta) & \mapsto & (u = \alpha, x = \beta, y = 2\lambda\alpha - \beta\lambda^2, z = \beta\lambda - \alpha) \end{array}$$

$$\begin{array}{cccc} \psi_- : & V_-^* & \to & \tilde{Z}|_{U_1} \\ & (\mu, \gamma, \delta) & \mapsto & (\tilde{u} = \gamma, \tilde{x} = 2\mu\gamma - \mu^2\delta, \tilde{y} = \delta, \tilde{z} = \delta\mu - \gamma). \end{array}$$

Using the above identifications, it is easy to see that ψ_+ and ψ_- together give a well-defined map

$$\psi : Z \longrightarrow \tilde{Z}$$

because x, y and z transform as in $\mathcal{O}(2)$ and $xy = u^2 - z^2$ and $\tilde{x}\tilde{y} = \tilde{u}^2 - \tilde{z}^2$ respectively.

Starting with $\tilde{Z}$, away from its singularities, we can re-discover the space Z of our construction by setting

$$\lambda = \frac{y}{u - z}\ ,\ \alpha = u\ ,\ \beta = x$$

$$\mu = \frac{\tilde{x}}{\tilde{u} - \tilde{z}}\ ,\ \gamma = \tilde{u}\ ,\ \delta = \tilde{y}.$$

The equation $xy = u^2 - z^2$ ensures that the transformation laws for Z are satisfied.

As the space Z of our construction does not have singularities (as will be proved for the general case in the next section) it is a desingularisation of $\tilde{Z}$ and therefore the twistor space for the Eguchi-Hanson metric on $T^*\mathbb{C}P^1$.

2.2 Proof

We now have to show that Z is indeed a twistor space, and that the hyperkähler metric thus obtained satisfies all the properties claimed in 2.1.

We prove the first part using the theorem of Hitchin, Karlhede, Lindström, Roček outlined in section 1.2.

Proposition 2.8 *Z is a $(2n+1)$-dimensional complex manifold, and the projection $p : Z \to \mathbb{C}P^1$ is holomorphic.*

Proof: First we show that Z_+ is open in $V_+^*|_{\pi_+^{-1}(\{\lambda\in\mathbb{C}; |\lambda|<1/c\})}$.

Points in $V_+^*|_{\pi_+^{-1}(\{\lambda\in\mathbb{C}; |\lambda|<c\})}$ and points in $\phi_+(U \times \{\lambda \in \mathbb{C}; c < |\lambda| < 1/c\})$ obviously have open neighbourhoods in Z_+.

Now for a point $z = \phi_+(x, \lambda)$ with $|\lambda| = c$ there is a neighbourhood $W \times B_\epsilon(\lambda)$ of (x, λ) in $U \times \{\lambda \in \mathbb{C}; |\lambda| < 1/c\}$, hence $\phi_+(W \times B_\epsilon(\lambda))$ is an open neighbourhood of z as ϕ_+ is a diffeomorphism. Similarly, Z_- is open in $V_-^*|_{\pi_-^{-1}(\{\lambda \in \mathbb{C}; |\lambda| < 1/c\})}$.

It is obvious that we can cover Z by coordinate charts using the coordinate charts for V_+^* and V_-^* respectively.

In order to conclude that Z is a manifold it remains to show that it is Hausdorff. Using sections 8.3 and 5.2 in [8] it suffices to show that the graph of the equivalence relation is closed which is equivalent to the graph of the identification map being closed.

Let (x_n, λ_n) be a sequence in $U \times \{\lambda \in \mathbb{C}; c < |\lambda| < 1/c\}$ such that

$$\phi_+(x_n, \lambda_n) \to p_+ \in V_+^*|_{\pi_+^{-1}(\{\lambda \in \mathbb{C}; |\lambda| < 1/c\})}$$

and

$$\phi_-(x_n, \lambda_n^{-1}) \to p_- \in V_-^*|_{\pi_-^{-1}(\{\tilde{\lambda} \in \mathbb{C}; |\tilde{\lambda}| < 1/c\})}.$$

Then

$$\lambda_n = \pi_+ \phi_+(x_n, \lambda_n) \to \pi_+ p_+ =: \lambda_+$$

$$\lambda_n^{-1} = \pi_- \phi_-(x_n, \lambda_n) \to \pi_- p_- =: \lambda_-$$

and $c \leq |\lambda_+| < 1/c$, $c \leq |\lambda_-| < 1/c$. But $\lambda_- = \lambda_+^{-1}$, hence $c < |\lambda_+|$ and $c < |\lambda_-|$.

As ϕ_+ is a diffeomorphism, x_n converges in U to x. By continuity, $p_+ = \phi_+(x, \lambda_+)$ and $p_- = \phi_-(x, \lambda_+^{-1})$, hence $p_+ \sim p_-$ and the limit point lies on the graph.

As $p|_{Z_+} = \pi_+$ and $p|_{Z_-} = \pi_-$, p is holomorphic. □

2.2.1 Twistor lines and their normal bundle

In 2.1.4 we introduced sections $s_x : \mathbb{C}P^1 \to Z$ by using the maps ϕ_+ and ϕ_- on each half V_+^* and V_-^* respectively. We shall prove below that these sections are twistor lines by showing that they have the correct normal bundle. First we have to establish the following

Lemma 2.9 *The degree of the normal bundle N of any of the explicit twistor lines is $2n$.*

Proof: We prove the lemma by a calculation in local coordinates.

Let $z_1, \ldots, z_n, \tilde{z}_1, \ldots, \tilde{z}_n$ be local coordinates for X^c and let the holomorphic-symplectic form be given by $\omega_c = \sum_{i,j} \omega_{ij} dz_i \wedge d\tilde{z}_j$. The connection on the leaves $z_i = const.$ is defined by

$$i\left(\nabla_{\frac{\partial}{\partial \tilde{z}_j}} \frac{\partial}{\partial \tilde{z}_i}\right) \omega_c = i\left(\frac{\partial}{\partial \tilde{z}_j}\right) d\left(i\left(\frac{\partial}{\partial \tilde{z}_i}\right)\omega_c\right) = -\sum_l \frac{\partial \omega_{li}}{\partial \tilde{z}_j} dz_l.$$

Setting $\nabla_{\frac{\partial}{\partial \tilde{z}_j}} \frac{\partial}{\partial \tilde{z}_i} = \sum_k \Gamma_k^{ji} \frac{\partial}{\partial \tilde{z}_k}$ yields

$$\sum_k \Gamma_k^{ji} \omega_{lk} = \frac{\partial \omega_{li}}{\partial \tilde{z}_j}.$$

As $d\omega_c = 0$ we can choose $f_1, \ldots, f_n$ such that $\frac{\partial f_i}{\partial \tilde{z}_j} = \omega_{ij}$ for $1 \leq i, j \leq n$. For these f_i

$$\frac{\partial^2 f_i}{\partial \tilde{z}_r \partial \tilde{z}_s} - \sum_k \Gamma_k^{rs} \frac{\partial f_i}{\partial \tilde{z}_k} = 0$$

for all $1 \leq i \leq n$, $1 \leq r, s \leq n$, i. e. $\nabla(df_i) = 0$, and $f_0 \equiv 1, f_1, \ldots, f_n$ form a basis for V_+.

Denote by $\eta_0, \ldots, \eta_n$ the dual basis for V_+^*, i. e. $\eta_j(f_i) = \delta_{ij}$ for $0 \leq i, j \leq n$, and let $z_1, \ldots, z_n, x_0, \ldots, x_n$ be coordinates in V_+^* with respect to this basis. Then

$$\pi_+(z_1, \ldots, z_n, x_0, \ldots, x_n) = \sum_{j=0}^n x_j \eta_j(f_0) = x_0.$$

Since

$$\lambda \delta_{(\underline{z}, \underline{\tilde{z}})}(\sum_{j=0}^n a_j f_j) = \lambda \sum_{j=0}^n a_j f_j(\underline{z}, \underline{\tilde{z}})$$

where $\underline{z} = (z_1, \ldots, z_n)$ and $\underline{\tilde{z}} = (\tilde{z}_1, \ldots, \tilde{z}_n)$ and

$$\sum_{j=0}^n x_j \eta_j(\sum_{i=0}^n a_i f_i) = \sum_{j=0}^n x_j a_j$$

the map ϕ_+ is, in these coordinates, given by

$$\phi_+(z_1, \ldots, z_n, \tilde{z}_1, \ldots, \tilde{z}_n, \lambda) = (z_1, \ldots, z_n, \lambda, \lambda f_1(\underline{z}, \underline{\tilde{z}}), \ldots, \lambda f_n(\underline{z}, \underline{\tilde{z}}))$$

Since

$$dx_i = \lambda\left(\sum_j \omega_{ij} d\tilde{z}_j + \frac{\partial f_i}{\partial z_j} dz_j\right) + f_i d\lambda,$$

$$\begin{aligned} &dz_1 \wedge \ldots \wedge dz_n \wedge dx_1 \wedge \ldots \wedge dx_n \wedge dx_0 \\ &\qquad = \lambda^n \det(\omega) dz_1 \wedge \ldots \wedge dz_n \wedge d\tilde{z}_1 \wedge \ldots \wedge d\tilde{z}_n \wedge d\lambda. \end{aligned}$$

Similarly, for V_-^* we choose coordinates $\tilde{z}_1, \dots, \tilde{z}_n, \tilde{x}_0, \dots, \tilde{x}_n$ with respect to the dual basis of $\tilde{f}_0 \equiv 1, \tilde{f}_1, \dots, \tilde{f}_n$, a basis of V_- such that $\frac{\partial \tilde{f}_i}{\partial z_j} = \omega_{ji}$. Then

$$\phi_-(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n, \tilde{\lambda}) = (\tilde{z}_1, \dots, \tilde{z}_n, \tilde{\lambda}, \tilde{\lambda}\tilde{f}_1(\underline{z}, \underline{\tilde{z}}), \dots, \tilde{\lambda}\tilde{f}_n(\underline{z}, \underline{\tilde{z}}))$$

If $\tilde{\lambda} = \lambda^{-1}$ then

$$\begin{aligned} &d\tilde{z}_1 \wedge \dots \wedge d\tilde{z}_n \wedge d\tilde{x}_1 \wedge \dots \wedge d\tilde{x}_n \wedge d\tilde{x}_0 \\ &\qquad = (-1)^{n^2+1}\lambda^{-(2n+2)}(\lambda^n \det(\omega) dz_1 \wedge \dots \wedge dz_n \wedge d\tilde{z}_1 \wedge \dots \wedge d\tilde{z}_n \wedge d\lambda). \end{aligned}$$

Therefore $dz_1 \wedge \dots \wedge dz_n \wedge dx_1 \wedge \dots \wedge dx_n \wedge dx_0$ is a non-vanishing section of $K_Z|_{\mathbb{C}P^1} \otimes p^*\mathcal{O}(2n+2)$, i. e. $\deg K_Z|_{\mathbb{C}P^1} = -(2n+2)$.

From the exact sequence

$$0 \longrightarrow N^* \longrightarrow T^*Z|_{\mathbb{C}P^1} \longrightarrow T^*\mathbb{C}P^1 \longrightarrow 0$$

we derive

$$\deg N^* = \deg K_Z|_{\mathbb{C}P^1} - \deg T^*\mathbb{C}P^1 = -(2n+2) - (-2) = -2n.$$

$\square$

Proposition 2.10 *The normal bundle of any of the explicit twistor lines $s_x : \mathbb{C}P^1 \to Z$ in Z is isomorphic to $\mathbb{C}^{2n}(1)$.*

Proof: Take any of the lines $C = s_x(\mathbb{C}P^1)$ for $x \in X^c$.

Claim: There exist sections $s_1, \dots, s_n$ of the normal bundle N of C in Z such that $s_1(0) = \dots = s_n(0) = 0$ and $s_1(1), \dots, s_n(1)$ are linearly independent and tangential to the $+$-foliation (where we identify $p^{-1}(1)$ with X^c using ϕ_+).

Proof: Using 2.5 we can choose coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ of X^c such that the $+$-foliation is given by $q_i = const$ and

$$\phi_+(p_1, \dots, p_n, q_1, \dots, q_n, \lambda) = (q_1, \dots, q_n, \lambda, \lambda p_1, \dots, \lambda p_n).$$

We consider the 1-parameter family of sections $s_{t,j} : \mathbb{C}P^1 \to Z$ given by

$$s_{t,j}(\lambda) = (q_1, \dots, q_n, \lambda, \lambda p_1, \dots, \lambda(p_j + t), \dots, \lambda p_n).$$

Then $s_j(\lambda) = \frac{d}{dt}|_{t=0} s_{t,j}(\lambda)$ is section of the normal bundle such that $s_j(0) = 0$. Furthermore, $s_1(1), \dots, s_n(1)$ are linearly independent and tangential to the $+$-foliation which concludes the proof of the claim.

Similarly, we can find sections $t_1, \dots, t_n$ of N such that $t_1(\infty) = \dots = t_n(\infty) = 0$ and $t_1(1), \dots, t_n(1)$ are linearly independent and tangent to the $-$-foliation. As the foliations are transverse, $s_1(1), \dots, s_n(1), t_1(1), \dots, t_n(1)$ form a basis for $N_{s_x(1)}$.

As rank $N = 2n$, by the Birkhoff-Grothendieck theorem $N = \oplus_{i=1}^{2n} \mathcal{O}(m_i)$ where $m_i \in \mathbb{Z}$. Using 2.9, $\sum_{i=1}^{n} m_i = 2n$.

Suppose $m_i < 0$ for some i, then, as $\mathcal{O}(m_i)$ does not have sections, $s_1(1), \dots, s_n(1)$, $t_1(1), \dots, t_n(1)$ lie in a at most $(2n-1)$-dimensional space contradicting linear independence.

Suppose $m_i = 0$ for some i. Let $s_j^i, t_j^i : \mathbb{C}P^1 \to \mathcal{O}(m_i) = \mathcal{O}$ be the corresponding components of s_j, t_j. Then s_j^i and t_j^i are constant and, in fact, $s_j^i = t_j^i = 0$ since $s_j(0) = t_j(\infty) = 0$. Therefore $s_1(1), \dots, s_n(1)$,$t_1(1), \dots, t_n(1)$ lie in a subspace of dimension at most $2n-1$, giving a contradiction.

Hence $m_i > 0$ for $1 \leq i \leq 2n$, and therefore $m_i = 1$. □

2.2.2 The real structure

The natural real structure σ of the complexified manifold X^c (cf. 2.1.2) induces a real structure τ on Z in the following way:
A point $z \in Z|_{p^{-1}(\{\lambda \in \mathbb{C}; c < |\lambda| < 1/c\})}$ can be represented by $\phi_+(x, \lambda) \in V_+^*$ with $x \in X^c$; we define

$$\tau(z) = [\phi_-(\sigma(x), -\bar{\lambda})] = [\phi_+(\sigma(x), -\bar{\lambda}^{-1})]$$

where $[\alpha]$ denotes the equivalence class of α under the identification $\phi_+(x, \lambda) \sim \phi_-(x, \lambda^{-1})$. On the level of bundles τ interchanges V_+^* and V_-^*. Note that τ is well-defined since

$$\phi_+(\sigma(x), -\bar{\lambda}^{-1}) \sim \phi_-(\sigma(x), -\bar{\lambda}).$$

The map τ is anti-holomorphic (as σ is) and induces the antipodal map on $\mathbb{C}P^1$ (as $p\tau(z) = -\bar{\lambda}^{-1}$ if $p(z) = \lambda$). Furthermore,

$$\tau^2(z) = \tau[\phi_-(\sigma(x), -\bar{\lambda})] = [\phi_+(\sigma^2(x), -(\overline{-\bar{\lambda}}))] = [\phi_+(x, \lambda)] = z$$

hence τ is indeed a real structure.

2.2.3 The twisted symplectic form along the fibres

We describe in this section how the complexified Kähler form ω_c on X^c gives rise to a symplectic form on each fibre $p^{-1}(\lambda)$ of the twistor space.

For $\lambda \in \mathbb{C}$, $c < |\lambda| < 1/c$, denote by ϕ_λ the map

$$\begin{array}{rccc} \phi_\lambda : & U & \longrightarrow & p^{-1}(\lambda) \subset Z \\ & x & \mapsto & s_x(\lambda) \end{array}$$

where $s_x : \mathbb{C}P^1 \to Z$ are the explicit twistor lines given in the construction, and denote by ψ_λ its inverse.

Let s_0 and s_∞ be sections of $\mathcal{O}(1)$ such that $s_0(0) = s_\infty(\infty) = 0$ and $\langle s_0, s_\infty \rangle = 1$ with respect to the standard skew-symmetric form on $H^0(\mathbb{C}P^1, \mathcal{O}(1))$ as introduced in section 1.2. Then $\lambda = \frac{s_0}{s_\infty}$ is an affine parameter for $\mathbb{C}P^1$ and $s_0 s_\infty$ is a section of $\mathcal{O}(2)$. Define

$$\omega = \psi_\lambda^*(\omega_c) \otimes s_0 s_\infty$$

Then ω is a section of $\Lambda^2 T_F^*(2)$ on the complement of the fibres over 0 and ∞ of Z inducing a symplectic form on each fibre.

Proposition 2.11 *ω extends to $p^{-1}(0)$ and $p^{-1}(\infty)$, and coincides with the canonical symplectic structure of the cotangent bundle.*

Proof: Choosing the usual affine parameter $\lambda = \frac{s_0}{s_\infty}$ of $\mathbb{C}P^1$ we can re-write $\omega_\lambda = \omega|_{p^{-1}(\lambda)}$ for $c < |\lambda| < 1/c$ as

$$\omega_\lambda = \psi_\lambda^*(\lambda \omega_c).$$

We choose flat coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ for X^c as in the proof of 2.5. Then $\omega_c = \sum_{i=1}^n dq_i \wedge dp_i$ and

$$\phi_\lambda : U \to p^{-1}(\lambda), (p_1, \dots, p_n, q_1, \dots, q_n) \mapsto (q_1, \dots, q_n, \lambda, \lambda p_1, \dots, \lambda p_n)$$

hence

$$\psi_\lambda(q_1, \dots, q_n, \lambda, \alpha_1, \dots, \alpha_n) = \left(\frac{\alpha_1}{\lambda}, \dots, \frac{\alpha_n}{\lambda}, q_1, \dots, q_n \right)$$

and

$$\omega_\lambda = \sum_i dq_i \wedge d\alpha_i$$

which obviously extends to the canonical symplectic form on $p^{-1}(0)$. Similarly, ω extends to $p^{-1}(\infty)$ and coincides with the canonical holomorphic-symplectic form of $T^*\bar{X}$. $\square$

So far, we have constructed a $(2n+1)$-dimensional complex manifold Z with a holomorphic projection $p : Z \to \mathbb{C}P^1$ (see 2.8) which has a family of holomorphic sections each with normal bundle $\mathbb{C}^{2n}(1)$ (see 2.10). We also have a real structure (see 2.2.2) and a holomorphic section ω of $\Lambda^2 T_F^*(2)$ giving a holomorphic-symplectic form in each fibre $p^{-1}(\lambda)$, $\lambda \in \mathbb{C}P^1$. Hence applying 1.2 we conclude that we have obtained a (possibly indefinite) hyperkähler metric in a neighbourhood of the zero section of the cotangent bundle which by 2.11 is compatible with the canonical holomorphic-symplectic structure.

2.2.4 The properties of the metric

It remains to show that our hyperkähler metric is positive definite and has the properties claimed in 2.1, namely that it restricts to the given Kähler metric on the zero section and that the circle action in the fibres is isometric.

We recall from section 1.2 that under the splitting of the complexified tangent space

$$H^0(N) \cong H^0(N(-1)) \otimes H^0(\mathcal{O}(1))$$

and using the sections s_0 and s_∞ of $\mathcal{O}(1)$, the complexified hyperkähler metric is given by

$$g(s \otimes s_0 + t \otimes s_\infty, s' \otimes s_0 + t' \otimes s_\infty) = \omega(s, t') + \omega(s', t).$$

In the proof of 2.10 we saw that tangent vectors of the $+$-foliation give rise to sections of the normal bundle of the corresponding explicit twistor line which vanish at 0.

Similarly, tangent vectors to the $-$-foliation give rise to sections of the normal bundle vanishing at ∞.

Proposition 2.12 *The hyperkähler metric restricts to the given Kähler metric on the zero section.*

Proof: In the fibres $p^{-1}(\lambda)$, $c < |\lambda| < 1/c$, the foliations of X^c give rise to foliations in the fibres of the twistor space.

Using the identification of $p^{-1}(1)$ with $U \subset X^c$ by ψ_1 where ψ_1 is defined in 2.2.3, tangent vectors to the $+$-foliation correspond to sections of the normal bundle of the form $s \otimes s_0$, and as $\langle s_0, s_0 \rangle = 0$, the $+$-foliation gives rise to null subspaces with respect to the complexified hyperkähler metric.

Similarly, tangent vectors to the $-$-foliation correspond to sections of the form $t \otimes s_\infty$, and hence form a null space.

Now take $v, w \in T_x p^{-1}(1)$ such that v is tangential to the $+$-foliation and w tangential to the $-$-foliation respectively. Then there exist sections σ and τ of N such that $\sigma(0) = 0$, $\sigma(1) = v$ (respectively $\tau(\infty) = 0$, $\tau(1) = w$), and $s = \frac{\sigma}{s_0}$ (respectively $t = \frac{\tau}{s_\infty}$) are well-defined sections of $N(-1) \cong \mathbb{C}^{2n}$. Therefore,

$$g(v, w) = \omega(s, t)\langle s_0, s_\infty \rangle = \omega(s(1), t(1)) = \omega_c(v, w)$$

using again the identification of $p^{-1}(1)$ with X^c. As the real structures of Z and X^c are compatible with the identifications used, the result follows. $\square$

Proposition 2.13 *The S^1-action by scalar multiplication in the fibres is isometric.*

Proof: The S^1-action is holomorphic with respect to the complex structure I which is the induced complex structure of the cotangent bundle, i. e. $\mathcal{L}_\xi I = 0$ where ξ is the Killing field of the action.

Claim: $\mathcal{L}_\xi \omega_1 = 0$ where ω_1 is the Kähler form with respect to the complex structure I.

Proof: As in the proof of 2.11 we can choose coordinates $q_1, \dots, q_n, \lambda, \alpha_1, \dots, \alpha_n$ of Z_+ such that S^1 acts by multiplication on $\alpha_1, \dots, \alpha_n$ and the twisted symplectic form is given in these coordinates by $\omega_\lambda = \sum_{j=1}^n dq_j \wedge d\alpha_j$. Therefore,

$$\mathcal{L}_\xi \omega_\lambda = i\omega_\lambda.$$

On the other hand we know from section 1.2 that

$$\omega_\lambda = \omega_2 + i\omega_3 - 2\lambda\omega_1 - \lambda^2(\omega_2 - i\omega_3)$$

hence

$$\mathcal{L}_\xi \omega_\lambda = \mathcal{L}_\xi(\omega_2 + i\omega_3) - 2\lambda\mathcal{L}_\xi\omega_1 - 2i\lambda\omega_1 - \lambda^2\mathcal{L}_\xi(\omega_2 - i\omega_3) - 2\lambda^2(\omega_3 + i\omega_2)$$

because S^1 acts on the affine parameter λ of $\mathbb{C}P^1$ by multiplication and thus $\xi(\lambda) = i\lambda$ and $\xi(\lambda^2) = 2i\lambda^2$. Comparing the coefficients of λ yields $2\mathcal{L}_\xi\omega_1 + 2i\omega_1 = 2i\omega_1$ hence $\mathcal{L}_\xi\omega_1 = 0$.

As $\omega_1(v, w) = g(Iv, w)$, we conclude $\mathcal{L}_\xi g = 0$. □

Proposition 2.14 *The constructed hyperkähler metric on $U \subset T^*X$ is positive definite.*

Proof: As $g|_{TX}$ coincides with the given Kähler metric, it is positive definite.

The splitting $TU|_X = TX \oplus (TX)^\perp$ must be a direct sum because every vector in $TX \cap (TX)^\perp$ is null.

As $\omega_2 + i\omega_3$ coincides with the canonical symplectic form, X is Lagrangian with respect to it, hence $\omega_2(v, w) = g(Jv, w) = 0$ for all $v, w \in TX$.

As J is invertible and $\dim(T_pX)^\perp = \dim J(T_pX)$, $(TX)^\perp = J(TX)$. But $g(Jv, Jw) = g(v, w)$, and therefore $g|_{J(TX)}$ is positive definite.

Taking into account that the sum is direct implies that g is positive definite on $TU|_X$. Observing that positive definiteness is an open condition completes the proof. □

Chapter 3

The Inverse Construction

In this chapter the question of uniqueness of the constructed hyperkähler metric is addressed.

If we start with a hyperkähler metric on a neighbourhood of the zero section of the cotangent bundle such that the circle action by scalar multiplication in the fibres is isometric, the holomorphic-symplectic structure coincides with the canonical one and the hyperkähler metric induces a given Kähler metric on the zero section, we shall prove that the twistor space and the twistor lines of any such metric are isomorphic to the ones constructed in section 2.1, thus a hyperkähler metric with these additional properties is unique.

We deal with the following slightly more general situation:

Let M be a hyperkähler manifold of real dimension $4n$ with an isometric S^1-action. We assume that the action is holomorphic with respect to one of the complex structures, I say, and acts on the corresponding holomorphic-symplectic form ω_c by scaling with $u \in S^1$.

Let $X \subset M$ be the fixed point set of this action, and we assume that the complex dimension of X is n.

Lemma 3.1 *(i) $X \subset M$ is Lagrangian with respect to ω_c.*

(ii) S^1 acts on the normal bundle $N_{X/M}$ of X in M by scalar multiplication.

Proof: For $u = e^{i\theta} \in S^1$ denote by $u.x$ the group action and by $e^{i\theta}x$ multiplication. Since X is the fixed point set of the action, $u.v = v$ for all $v \in TX$. Thus, for $v_1, v_2 \in TX$ and all $u \in S^1$

$$\omega_c(v_1, v_2) = \omega_c(u.v_1, u.v_2) = u.\omega_c(v_1, v_2) = e^{i\theta}\omega_c(v_1, v_2),$$

hence $\omega_c(v_1, v_2) = 0$, and (i) is shown.

For all $v \in TX$, $w \in N_{X/M}$ and $u = e^{i\theta} \in S^1$

$$\omega_c(v, u.w) = \omega_c(u.v, u.w) = u.\omega_c(v, w) = e^{i\theta}\omega_c(v, w) = \omega_c(v, e^{i\theta}w),$$

thus, as X is Lagrangian, $u.w = e^{i\theta}w$, and (ii) is proved. □

The main result is the following

Theorem 3.2 *Let $Z \xrightarrow{p} \mathbb{C}P^1$ be the twistor space of M. Then there exists $c \in]0,1[$ such that $Z|_{p^{-1}(\{\lambda\in\mathbb{C};|\lambda|<c\})}$ and $Z|_{p^{-1}(\{\lambda^{-1}\in\mathbb{C};|\lambda|<c\})}$ are standard, and can be identified with the flat model.*

Furthermore, each fibre over $\mathbb{C}^* = \mathbb{C}P^1 \setminus \{0, \infty\}$ carries two transverse Lagrangian foliations and can be identified with a neighbourhood of X in its complexification X^c respecting the transverse foliations of the fibre and of X^c. Each point in X^c gives rise to a twistor line joining a point in $X \subset p^{-1}(0)$ and a point in $\bar{X} \subset p^{-1}(\infty)$.

We note that X must be real-analytic and therefore its complexification is well-defined (cf. the argument given in section 2.1).

An immediate consequence of 3.2 is

Corollary 3.3 *For any hyperkähler metric on a neighbourhood of the zero section of the cotangent bundle of a real-analytic Kähler manifold which is compatible with the canonical holomorphic-symplectic structure, induces the given Kähler metric on the zero section and for which the S^1-action by scalar multiplication in the fibres is isometric, if we restrict to a suitable neighbourhood of the zero section, the metric can be obtained by our construction.*

3.1 Standardisation of parts of the twistor space

The aim of this paragraph is to prove the first of the two statements in 3.2, namely that the twistor space Z for a hyperkähler manifold M with an S^1-action as described can be identified with the standard flat model.

To achieve this we construct a holomorphic projection of a neighbourhood of X in Z onto X by first finding a particular coordinate system of Z near X. Then we show that the inverse images of points in X under this projection give rise to Lagrangian foliations in the fibres near 0 of the twistor space. An analogue of this process can be done near $\bar{X}$.

3.1.1 A normal coordinate system

The S^1-action on M induces an S^1-action on Z by taking the diagonal action on $M \times \mathbb{C}P^1$ where S^1 acts on $\mathbb{C}P^1$ keeping 0 and ∞ fixed, i. e. if λ is the usual affine parameter on $\mathbb{C}P^1 \setminus \{\infty\}$, $u.\lambda = e^{i\theta}\lambda$.

This action is holomorphic and its fixed point set contains two components, namely $X \subset p^{-1}(0)$ and $\bar{X} \subset p^{-1}(\infty)$.

Lemma 3.4 *The induced action on the normal bundle $N_{X/Z}$ of X in Z is given by scalar multiplication.*

Proof: Note that at a point $x \in X$

$$N_{X/Z_x} = N_{X/M_x} \oplus p^* T_{p(x)} \mathbb{C}P^1,$$

and the action in the additional normal direction has also weight 1. □

The main aim of this section is to prove

Proposition 3.5 *There exist holomorphic coordinates $w_0, \ldots, w_n, x_1, \ldots, x_n$ of Z near X such that $x_1, \ldots, x_n$ are S^1-invariant and coordinates for X for $w_0 = w_1 = \cdots = w_n = 0$, S^1 acts on $w_0, \ldots, w_n$ by multiplication with $e^{i\theta}$ and the twistor projection is given by $p(w_0, \ldots, w_n, x_1, \ldots, x_n) = w_0$.*

Furthermore, the twisted symplectic form along the fibres is $\sum_{i=1}^n dx_i \wedge dw_i$ in these coordinates.

For the proof we need to find eigenfunctions of the Killing field ξ of the circle action on Z. We achieve this by first producing an S^1-invariant holomorphic submersion onto an open subset of $\mathbb{C}^n$ and then applying Poincaré's theorem (cf. [2]) to the vector field obtained from the Killing field, but now acting along the fibres of this map.

The Killing field ξ is a holomorphic vector field, i. e. $\mathcal{L}_\xi I = 0$, hence it is the real part of a complex vector field with holomorphic coefficients. As we are only interested in holomorphic data we use this complex vector field instead of ξ itself in our calculations.

Lemma 3.6 *There exist coordinates $(z_1, \ldots, z_{2n+1})$ of Z near X such that $z_1, \ldots, z_n$ are S^1-invariant and X corresponds to $z_{n+1} = \cdots = z_{2n+1} = 0$.*

Proof: We can find coordinates $v_1, \ldots, v_n, z_{n+1}, \ldots, z_{2n+1}$ of Z such that $dv_1, \ldots, dv_n$ are invariant under the induced action on the cotangent bundle at points in X which correspond to $z_{n+1} = \cdots = z_{2n+1} = 0$. Then we average over the compact group S^1 to obtain

$$z_j = \frac{1}{2\pi} \int_{S^1} u.v_j \; du$$

for $1 \leq j \leq n$. At points in X, $dz_j = dv_j$, thus $z_1, \ldots, z_n$ locally define an S^1-invariant holomorphic submersion onto an open set of $\mathbb{C}^n$ giving the desired coordinate system.

□

The Killing field now acts along the fibres of the projection map

$$(z_1, \dots, z_{2n+1}) \mapsto (z_1, \dots, z_n).$$

As the induced action on the normal bundle is given by scalar multiplication (see 3.4) we can assume without loss of generality that the Killing field along the fibres is given by

$$\xi_F = i \sum_{j=n+1}^{2n+1} z_j \frac{\partial}{\partial z_j} + \text{ higher order terms.}$$

Next we establish the main lemma.

Lemma 3.7 *If ξ denotes the Killing field of the S^1-action on Z, then there exists a coordinate system $(z_1, \dots z_n, y_1, \dots, y_{n+1})$ of Z near X such that $\xi(z_i) = 0$ for all $1 \leq i \leq n$ and $\xi(y_j) = iy_j$ for all $1 \leq j \leq n+1$.*

Proof: Choosing a coordinate system as in 3.6, $\xi(z_i) = 0$ for $1 \leq i \leq n$.

Now we consider the vector field ξ_F for any fibre determined by $(z_1, \dots, z_n)$. The linear part of ξ_F at the singular point $z_{n+1} = \dots = z_{2n+1} = 0$ has (multiple) eigenvalue $+i$. We now apply Poincaré 's theorem in the following form (cf. [2], p. 190) which also holds in the case of multiple eigenvalues.

Theorem 3.8 *If the eigenvalues of the linear part of a holomorphic vector field at a singular point belong to the Poincaré domain (i. e. the convex hull of the eigenvalues does not contain zero) and are nonresonant, then the field is biholomorphically equivalent to its linear part in a neighbourhood of the singular point.*

Therefore we can find coordinate functions $y_1, \dots, y_{n+1}$ (which depend holomorphically on $z_1, \dots, z_{2n+1}$) such that

$$\xi_F = i \sum_{j=1}^{n+1} y_j \frac{\partial}{\partial y_j},$$

hence $\xi(y_j) = iy_j$ for $1 \leq j \leq n+1$, and $z_1, \dots, z_n, y_1, \dots, y_{n+1}$ is the desired coordinate system. □

We are now ready to show 3.5.

Proof: We start with a local coordinate system $z_1, \dots, z_n, y_1, \dots, y_{n+1}$ as in 3.7.

All other coordinate systems $\tilde{z}_1, \dots, \tilde{z}_n, \tilde{y}_1, \dots, \tilde{y}_{n+1}$ such that $\tilde{z}_1, \dots, \tilde{z}_n$ are S^1-invariant coordinates and the action is linear on $\tilde{y}_1, \dots, \tilde{y}_{n+1}$, can be obtained from

$z_1, \dots, z_n, y_1, \dots, y_{n+1}$ by a transformation of the form

$$\begin{aligned} \tilde{z}_i &= f_i(z_1, \dots, z_n) \\ \tilde{y}_j &= \sum_{l=1}^{n+1} g_{jl}(z_1, \dots, z_n) y_l \end{aligned}$$

for $1 \leq i \leq n$ and $1 \leq j \leq n+1$ because starting with the general ansatz for transition functions

$$\begin{aligned} \tilde{z}_i &= f_i(z_1, \dots z_n, y_1, \dots, y_{n+1}) \\ \tilde{y}_j &= g_j(z_1, \dots, z_n, y_1, \dots, y_{n+1}) \end{aligned}$$

for $1 \leq i \leq n$ and $1 \leq j \leq n+1$ we have to impose the conditions

$$\begin{aligned} f_i(u.(z_1, \dots, z_n, y_1, \dots, y_{n+1})) &= f_i(z_1, \dots, z_n, y_1, \dots, y_{n+1}) \\ g_j(u.(z_1, \dots, z_n, y_1, \dots, y_{n+1})) &= e^{i\theta} g_j(z_1, \dots, z_n, y_1, \dots, y_{n+1}) \end{aligned}$$

for all $u = e^{i\theta} \in S^1$ which enforces the above form of the coordinate changes.

As the twistor projection p is equivariant and S^1 acts by multiplication on the standard affine parameter on $\mathbb{C}P^1 \setminus \{\infty\}$, p is of the form

$$p(z_1, \dots, z_n, y_1, \dots, y_{n+1}) = \sum_{i=1}^{n+1} h_i(z_1, \dots, z_n) y_i,$$

hence by defining $\tilde{y}_{n+1} = p(z_1, \dots, z_n, y_1, \dots, y_{n+1})$ we have found coordinates $\tilde{z}_1, \dots, \tilde{z}_n$, $\tilde{y}_1, \dots, \tilde{y}_{n+1}$ such that the twistor projection is given by $\tilde{y}_{n+1}$ and $\tilde{z}_1, \dots, \tilde{z}_n$, $\tilde{y}_1, \dots, \tilde{y}_n$ are coordinates in the fibres.

As S^1 acts by scaling, the twisted symplectic form along the fibres is of the form

$$\omega = \sum_{i,j=1}^{n} h_{ij}(\tilde{z}_1, \dots, \tilde{z}_n) d\tilde{z}_i \wedge d\tilde{y}_j + \sum_{i,j=1}^{n} \tilde{y}_i g_{ij}(\tilde{z}_1, \dots, \tilde{z}_n) d\tilde{z}_i \wedge d\tilde{z}_j.$$

But since $X \subset p^{-1}(0)$ is Lagrangian, the second term vanishes, thus

$$\omega = \sum_{i=1}^{n} d\tilde{y}_i \wedge \alpha_i$$

where α_i are 1-forms in $\tilde{z}_1, \dots, \tilde{z}_n$. Since ω is closed, $d\alpha_i = 0$, hence locally $\alpha_i = dx_i$, and these are S^1-invariant coordinates for X.

Setting $w_i = \tilde{y}_i$ for $1 \leq i \leq n$ and $w_0 = \tilde{y}_{n+1}$ gives the desired coordinate system. $\square$

3.1.2 Identification with the flat model

The idea for producing a holomorphic projection of a neighbourhood U of X in Z onto X is to consider limit points of integral curves of $I\xi$ where I is the complex structure of the twistor space Z. The extended S^1-action is holomorphic with respect to I, and let ξ be the Killing field.

These curves can be thought of as parts of $\mathbb{C}^*$-orbits because the S^1-action extends to an action of $\lambda \in \mathbb{C}^*$ with λ near 1 as $[I\xi, \xi] = 0$.

To make this more precise we take a coordinate system as in 3.5. In these coordinates $I\xi$ is given by

$$I\xi = -\sum_{j=0}^{n} w_j \frac{\partial}{\partial w_j},$$

hence the integral curves $\phi_t(\underline{x}, \underline{w})$ with $\phi_0(\underline{x}, \underline{w}) = (\underline{x}, \underline{w})$ are

$$\phi_t(\underline{x}, \underline{w}) = (\underline{x}, e^{-t}\underline{w}),$$

and we define the projection $q : U \to X$ by

$$q(\underline{x}, \underline{w}) = \lim_{t\to\infty} \phi_t(\underline{x}, \underline{w}) = (\underline{x}, 0) \in X.$$

Obviously, for all $\underline{x} \in X$ and all $\lambda \in \mathbb{C}P^1$ such that $p^{-1}(\lambda) \cap U \neq \emptyset$, $q^{-1}(\underline{x}) \cap p^{-1}(\lambda)$ is a submanifold of $p^{-1}(\lambda)$ which is Lagrangian with respect to the symplectic form along the fibres given in coordinates by $\sum_{i=1}^{n} dx_i \wedge dw_i$.

As $p(U) \supset \{\lambda \in \mathbb{C}; |\lambda| < c\}$ for some $0 < c < 1$ we can identify $Z|_{p^{-1}(\{\lambda\in\mathbb{C};|\lambda|<c\})}$ with the standard flat model using these special coordinates.

This identification respects the foliations in the fibres and the symplectic structure along the fibres.

3.2 The identification of the fibres over $\mathbb{C}^*$

We describe how the fibres of the twistor space over $\mathbb{C}^*$ can be identified with the complexification X^c of the fixed point set X of the S^1-action and how the obvious foliations of X^c give rise to Lagrangian foliations with respect to the twisted symplectic form along the fibres.

These considerations will prove the second part of Theorem 3.2.

Lemma 3.9 *X^c can be identified with the set of all twistor lines $s : \mathbb{C}P^1 \to Z$ such that $s(0) \in X$ and $s(\infty) \in \bar{X}$.*

Proof: The complexified hyperkähler manifold M^c corresponds to the set of all twistor lines.

As $X \subset M$ is the fixed point set of the S^1-action, $X^c \subset M^c$ is the fixed point set of the complexified action and therefore consists of all twistor lines which are $\mathbb{C}^*$-invariant.

Hence $s(0)$ and $s(\infty)$ must be fixed points, i. e. $s(0) \in X$ and $s(\infty) \in \bar{X}$. Twistor lines with this property are $\mathbb{C}^*$-invariant because otherwise the action would give rise to a 1-parameter deformation of the original twistor line keeping $s(0)$ and $s(\infty)$ fixed and thus to a section of the normal bundle N vanishing at 0 and ∞; but as $N = \mathbb{C}^{2n}(1)$ and any non-zero section of $\mathcal{O}(1)$ vanishes only at one point, this section must be identically zero. □

Proposition 3.10 *The map*

$$\begin{array}{rccc} F: & X^c \times \mathbb{C}^* & \longrightarrow & Z \\ & (x, \lambda) & \mapsto & s_x(\lambda) \end{array}$$

where s_x is the twistor line corresponding to x defines an isomorphism from $U \times \mathbb{C}^*$ onto its image for a neighbourhood U of X in X^c.

Before we prove Proposition 3.10 we establish the following

Lemma 3.11 *For each $\lambda \in \mathbb{C}^*$ and any $x \in X$ the derivative of the map*

$$\begin{array}{rccc} F_\lambda: & X^c & \longrightarrow & p^{-1}(\lambda) \\ & x & \mapsto & F(x, \lambda) \end{array}$$

at x is an isomorphism.

Proof: Since the tangent space at a point in M^c can be identified with the sections of the normal bundle of the twistor line, we can identify $T_x X^c$ with those sections s of the normal bundle which satisfy $s(0) \in T_{s_x(0)} X$ and $s(\infty) \in T_{s_x(\infty)} \bar{X}$.

A point $x \in X \subset X^c$ corresponds to the real twistor line s_x with $s_x(0) = x$ and $s_x(\infty) = \bar{x}$.

We consider the blow-up $\hat{Z}$ of Z along X and $\bar{X}$ and let $\tilde{X}^c$ be the set of all sections $s : \mathbb{C}P^1 \to \hat{Z}$ obtained by deforming the proper transform of $s_x(\mathbb{C}P^1)$. For these sections $s(0) \in E_X$ and $s(\infty) \in E_{\bar{X}}$ where E_X and $E_{\bar{X}}$ are the exceptional divisors. Define

$$\tilde{F}_\lambda : \tilde{X}^c \longrightarrow p^{-1}(\lambda), s \mapsto s(\lambda)$$

then for $\lambda \in \mathbb{C}^*$, $\tilde{F}_\lambda$ is an isomorphism if and only if F_λ is because the blowing-down map is an isomorphism outside the exceptional divisors.

But using 3.13 we conclude that the normal bundle $\hat{N}$ in the blow-up is trivial: $\hat{N} = \mathbb{C}^{2n}$, hence we can prescribe any given vector in $T_{s(\lambda)}p^{-1}(\lambda)$ at λ, and $F'_\lambda(x)$ is an isomorphism. □

Proof of 3.10:
Since $\frac{\partial F}{\partial \lambda}(x,\lambda) \in T_{s_x(\lambda)}(s_x(\mathbb{C}P^1))$ is non-zero and

$$\begin{aligned} T_{s_x(\lambda)}Z &= N_{s_x(\lambda)} \oplus T_{s_x(\lambda)}(s_x(\mathbb{C}P^1)) \\ &= T_{F\,s_x(\lambda)} \oplus T_{s_x(\lambda)}(s_x(\mathbb{C}P^1)), \end{aligned}$$

using 3.11 and the inverse function theorem we deduce that F is a local isomorphism of $U \times \mathbb{C}^*$ where U is a neighbourhood of X in X^c.

Furthermore, F is injective because, for any $x \in X^c$, $s_x(\mathbb{C}^*)$ is a $\mathbb{C}^*$-orbit, and this completes the proof. □

Now a description of the procedure is given by which the standard foliations of X^c give rise to Lagrangian foliations in the fibres $p^{-1}(\lambda)$, $\lambda \in \mathbb{C}^*$, of Z.

Let s_x be the real twistor line with $s(0) = z \in X$ and $s(\infty) = \bar{z} \in \bar{X}$. We consider deformations s of s_x such that $s(0) = z$ is fixed and $s(\infty) \in \bar{X}$. These lines correspond to the foliation in X^c locally given by $z_i = const.$ if we assume that the complexification X^c has coordinates $(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n)$.

Similarly, the foliation $\tilde{z}_i = const.$ corresponds to twistor lines with $s(0) \in X$ and $s(\infty) = \tilde{z} \in \bar{X}$ fixed.

Proposition 3.12 *The foliations of X^c give rise to Lagrangian foliations in each fibre over $\mathbb{C}^*$ of the twistor space.*

Proof: Without loss of generality we consider the case $z_i = const.$, and the real twistor line C joining z and $\bar{z}$.

We blow-up Z along z and $\bar{X}$ and look at the proper transform $\hat{C}$ of C in the blow-up $\hat{Z}$. By 3.13 the normal bundle $\hat{N}$ of $\hat{C}$ in $\hat{Z}$ will be

$$\hat{N} = \mathbb{C}^n \oplus \mathbb{C}^n(-1).$$

Therefore, $H^1(\hat{N}) = 0$ and $H^0(\hat{N}) \cong \mathbb{C}^n$, hence the deformations of this line sweep out n-dimensional subspaces in each $p^{-1}(\lambda)$, $\lambda \in \mathbb{C}^*$.

If ω denotes the symplectic form along $p^{-1}(0)$ then the symplectic form H along $p^{-1}(\lambda)$ evaluated at two tangent vectors $X = v\lambda + w$ and $Y = v'\lambda + w'$ (here we use the identification of tangent vectors with sections of the normal bundle) is given by

$$H(X,Y) = \lambda^2\omega(v,v') + \lambda(\omega(v,w') + \omega(v',w)) + \omega(w,w').$$

Tangent vectors to a leaf of the foliation correspond to sections s of the normal bundle vanishing at $\lambda = 0$, i. e. $w = w' = 0$ and $s(\infty) \in \bar{X}$, i. e. $v, v' \in X$, and hence $\omega(v, v') = 0$ as $X \subset p^{-1}(0)$ is Lagrangian.

Therefore $H(X, Y) = 0$ if X and Y are tangent to a leaf, and thus the leaves are isotropic. □

Appendix: The blow-up problem

Let Z be a complex manifold of dimension $2n + 1$ and Y_1 and Y_2 complex submanifolds of dimension r and s respectively with $r + s \leq 2n$ and $Y_1 \cap Y_2 = \emptyset$.

Let C be a rational curve in Z intersecting Y_i transversally in exactly one point y_i, $i = 1, 2$. We assume that the normal bundle N of C in Z is $\mathbb{C}^{2n}(1)$.

We blow up Y_1 and Y_2 and consider the proper transform $\hat{C}$ of C in the blow-up $\hat{Z}$.

Proposition 3.13 *The normal bundle $\hat{N}$ of $\hat{C}$ in $\hat{Z}$ is*

$$\hat{N} = \mathbb{C}^{r+s} \oplus \mathbb{C}^{2n-(r+s)}(-1).$$

Proof: We use the following lemma which will be shown below.

Lemma 3.14 *There is an exact sequence of sheaves*

$$0 \longrightarrow \hat{N} \longrightarrow N \xrightarrow{q} N_{C \cup Y_1/Z, y_1} \oplus N_{C \cup Y_2/Z, y_2} \longrightarrow 0$$

where the map $\hat{N} \to N$ is induced by the blowing-down map, $N_{C \cup Y_i/Z, y_i}$ is the skyscraper sheaf supported at y_i spanned by the directions normal to C and Y_i, $i = 1, 2$, and the map q is the evaluation at y_1 and y_2 respectively followed by projection onto the directions normal to C and Y_1, respectively C and Y_2.

As $N = \mathbb{C}^{2n}(1)$ the above sequence gives rise to a long exact sequence in cohomology

$$0 \to H^0(\hat{N}) \to \mathbb{C}^{4n} \xrightarrow{q} \mathbb{C}^{2n-r} \oplus \mathbb{C}^{2n-s} \to H^1(\hat{N}) \to 0$$

q is onto because any section of N is of the form $a\lambda + b$ with $a, b \in \mathbb{C}^{2n}$, thus we can prescribe any value in any direction at any two given points of the curve.

Therefore $H^1(\hat{N}) = 0$ and $\dim H^0(\hat{N}) = r + s$, and by Riemann-Roch $\dim H^0(\hat{N}) - \dim H^1(\hat{N}) = \deg \hat{N} + \operatorname{rank} \hat{N} \cdot (1 - g)$, i. e. $\deg \hat{N} = r + s - 2n$. By Birkhoff-Grothendieck,

$$\hat{N} = \oplus_{i=1}^{2n} \mathcal{O}(m_i) \ , \ m_i \in \mathbb{Z} \ , \ \sum_{i=1}^{2n} m_i = r + s - 2n.$$

As $H^1(\hat{N}) = 0$, $m_i \geq -1$ for all $1 \leq i \leq 2n$.

Claim: $m_i \leq 0$ for all $1 \leq i \leq 2n$.

Suppose $m_i =: m \geq 2$ for some i. Then the map $\hat{N} \to N$ induces a homomorphism $\mathcal{O}(m) \to \mathbb{C}^{2n}(1)$ which must be zero as $H^0(\mathbb{C}^{2n}(1-m)) = 0$ for $m \geq 2$. Therefore $\mathcal{O}(m)$ would lie in the kernel of $\hat{N} \to N$ contradicting injectivity.

Suppose $m_i = 1$ for some i. Any map $\mathcal{O}(1) \to \mathbb{C}^{2n}(1)$ is, on the level of sections, of the form $s \mapsto s \otimes v$ for some fixed $v \in \mathbb{C}^{2n}$. $s \otimes v$ must vanish at two points for all s, hence $v = 0$ and $\mathcal{O}(1)$ lies in the kernel of $\hat{N} \to N$ giving a contradiction, and the claim is established.

Since $m_i \in \{0, -1\}$ and $\sum_{i=1}^{2n} m_i = r + s - 2n$,

$$\hat{N} = \mathbb{C}^{r+s} \oplus \mathbb{C}^{2n-r+s}(-1).$$

□

To conclude this section we have to prove Lemma 3.14.

Proof: We consider the local situation near $y := y_1$. A similar argument can be done near y_2. Outside y_1 and y_2 the blowing-down map p gives rise to an isomorphism $\hat{N} \to N$.

Let $z_1, \ldots, z_{2n+1}$ be coordinates for Z near y such that $Y := Y_1$ is given by $z_{r+1} = \cdots = z_{2n+1} = 0$, and C is given by $z_1 = \cdots = z_{2n} = 0$, hence y corresponds to $\underline{z} = 0$. On the blow-up $\hat{Z}$ of Z along Y we construct a coordinate system

$$(z_1, \ldots, z_r, \mu_1, \ldots, \mu_{2n-r}, z_{2n+1})$$

near the intersection of $\hat{C}$ with the exceptional divisor where $(\mu_1, \ldots, \mu_{2n-r})$ are affine coordinates for $\mathbb{C}P^{2n-r}$ and the blowing-down map is given by

$$p(z_1, \ldots, z_r, \mu_1, \ldots, \mu_{2n-r}, z_{2n+1}) = (z_1, \ldots, z_r, z_{2n+1}\mu_1, \ldots, z_{2n+1}\mu_{2n-r}, z_{2n+1}).$$

Therefore

$$p_* \frac{\partial}{\partial z_i} = \frac{\partial}{\partial z_i} \text{ for } 1 \leq i \leq r$$

$$p_* \frac{\partial}{\partial \mu_j} = z_{2n+1} \frac{\partial}{\partial z_{r+j}} \text{ for } 1 \leq j \leq 2n - r$$

and

$$p_* \frac{\partial}{\partial z_{2n+1}} = \frac{\partial}{\partial z_{2n+1}} + \sum_{l=0}^{2n-r} \mu_l \frac{\partial}{\partial z_{r+l}}.$$

p_* is an isomorphism except when $z_{2n+1} = 0$, thus p_* is an isomorphism of the normal bundles except at y.

Therefore, $\hat{N} \to N$ is injective as map of sheaves, and the quotient is a sheaf supported at y and spanned by $\frac{\partial}{\partial z_{r+1}}, \ldots, \frac{\partial}{\partial z_{2n}}$, i. e. the directions normal to C and Y and the map q is as claimed. □

Chapter 4

Hyperholomorphic Bundles

In this chapter we consider a gauge-theoretical analogue of our twistor construction for a hyperkähler metric in the cotangent bundle of a real-analytic Kähler manifold which extends the given Kähler metric.

Let $M \subset T^*X$ be the hyperkähler manifold of Theorem 2.1 for a real-analytic-Kähler manifold X. We ask the question which vector bundles $E \to X$ with connection and a compatible holomorphic structure can be extended to a hyperholomorphic bundle $E' \to U$ where U is a neighbourhood of the zero section of the cotangent bundle T^*X contained in M. For a definition of hyperholomorphic bundles see section 1.4.

Remark 4.1 *As one of the complex structures, I say, of the hyperkähler manifold in the cotangent bundle T^*X is the complex structure induced from X, the hyperholomorphic extension E' must be holomorphic with respect to I and $-I$. Therefore, if such an extension with $E'|_X = E$ as vector bundles with connections exists, it is necessary for E to have curvature of type $(1,1)$.*

We shall prove

Theorem 4.2 *For any vector bundle E with real-analytic connection whose curvature is of type $(1,1)$ over a real-analytic Kähler manifold there exists an extension to a hyperholomorphic bundle over a neighbourhood U of the zero section contained in M.*

A connection on a vector bundle over a real-analytic manifold is called real-analytic if there is a trivialisation in which its connection form with respect to real-analytic coordinates on the manifold has real-analytic coefficient functions.

The proof uses the equivalence of hyperholomorphic bundles over hyperkähler manifolds and holomorphic bundles over their twistor space which are trivial along each real twistor line (as outlined in section 1.4).

4.1 Construction

In this section we shall construct a holomorphic bundle $W \to Z$ where Z is the twistor space for $M \subset T^*X$.

We do this by first producing a holomorphic bundle over either half of the twistor space which are described in 2.1.3 and then patching these together respecting the gluing map for the twistor space given in 2.1.4.

4.1.1 A holomorphic bundle over each half

We start with a vector bundle E of rank k over a real-analytic Kähler manifold X with a real-analytic connection whose curvature is of type $(1,1)$.

Lemma 4.3 *There exist a holomorphic coordinate system $\underline{z} = (z_1, \ldots, z_n)$ of X such that z_j and $\bar{z}_j$ are real-analytic, $j = 1, \ldots, n$, and a local trivialisation*

$$\begin{array}{rcl} E|_U & \to & U \times \mathbb{C}^k \\ (\underline{z}, v) & \mapsto & (\underline{z}, \phi(\underline{z})v) \end{array}$$

where U are open sets in X such that $\phi : U \to \mathbb{C}^{k\times k}$ is holomorphic and the connection form a of E is locally given by

$$a = \sum_{i=1}^{n} a_i(\underline{z}, \underline{\bar{z}}) dz_i$$

where $a_1, \ldots, a_n$ are real-analytic.

Proof: The proof is essentially the same as the proof of 1.10 given by Donaldson and Kronheimer, see [17, pp. 50-53].

First we choose (as in 2.2) a holomorphic coordinate system $z_1, \ldots, z_n$ on X such that the coordinates and their conjugates are real-analytic. With respect to this coordinate system there is a trivialisation of E such that locally the $(0,1)$-part of the exterior covariant derivative is given by $\bar{\partial}_A = \bar{\partial} + \alpha$ where

$$\alpha = \sum_{i=1}^{n} \alpha_i(z) d\bar{z}_i$$

where $\alpha_1, \ldots, \alpha_n$ are real-analytic. We have to find locally a real-analytic gauge transformation g such that

$$g\alpha g^{-1} - (\bar{\partial} g) g^{-1} = 0.$$

The gauge transformation can be found inductively by solving successively the one-dimensional equations

$$\frac{\partial g}{\partial \bar{z}_j} = g\alpha_j$$

for $j = 1, \dots, n$ where we regard $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$ as parameters. Since α_j is real-analytic, so is g by elliptic regularity. As α_j depends real-analytically on the parameters, so does g which gives the desired gauge transformation. □

We complexify to $E^c \xrightarrow{q} X^c$ where X^c is the complexified Kähler manifold of our twistor construction introduced in 2.1.2: locally X^c has coordinates $(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n) = (\underline{z}, \underline{\tilde{z}})$.

The complexified bundle can be given in terms of the local trivialisation

$$\begin{array}{ccc} E^c|_{U^c} & \to & U^c \times \mathbb{C}^k \\ ((\underline{z}, \underline{\tilde{z}}), v) & \mapsto & ((\underline{z}, \underline{\tilde{z}}), \phi(\underline{z})v) \end{array}$$

where $U^c = \{(\underline{z}, \underline{\tilde{z}}) \in X^c; \underline{z} \in U\}$.

The functions $a_1, \dots, a_n$ can be extended to holomorphic functions in $\underline{z}, \underline{\tilde{z}}$ for $\underline{\tilde{z}}$ near $\underline{\bar{z}}$. This gives rise to a connection form a^c on E^c over an open subset of X^c which we -in abuse of notation- again call X^c.

We recall that X^c is equipped with two Lagrangian foliations, namely $z_i = const$ and $\tilde{z}_i = const$ respectively, see 2.1.2. The connection on E^c is flat along each leaf of either of the two foliations as E has curvature of type $(1,1)$.

For any $b \in B_+ = X$ we consider a simply-connected subspace Λ_b^+ of the leaf of the $+$-foliation parameterised by b. We define $(W_+)_b$ to be the space of covariant constant sections of E^c on Λ_b^+, i. e.

$$(W_+)_b = \{\sigma_+ \in \Gamma(\Lambda_b^+, E^c); \nabla\sigma_+ = 0\}.$$

As the connection is flat along each leaf and Λ_b^+ is simply-connected, $\dim(W_+)_b = k$ for all $b \in B_+$, and we obtain a holomorphic vector bundle $W_+ \to B_+$. Similarly, we construct a holomorphic vector bundle $W_- \to B_-$ by setting

$$(W_-)_b = \{\sigma_- \in \Gamma(\Lambda_b^-, E^c); \nabla\sigma_- = 0\}$$

where Λ_b^- is a simply-conncected subspace of the leaf of the $-$-foliation parametrised by $b \in B_- = \bar{X}$.

Each half of our twistor space is part of the total space of V_+^* and V_-^* respectively where $q_+ : V_+^* \to B_+$ (respectively $q_- : V_-^* \to B_-$) is the dual bundle to the bundle of affine-linear functions on each leaf of the $+$-foliation (respectively $-$-foliation), cf. 2.1.3. By pulling back W_+ and W_- to V_+^* and V_-^* respectively we obtain holomorphic vector bundles on each half of the twistor space.

4.1.2 The gluing procedure

We identify points in $q_+^* W_+^*$ and $q_-^* W_-^*$ using parallel transport.

More precisely, if s_x for $x = (z, \tilde{z}) \in X^c$ is one of the explicit twistor lines in 2.1.4 then $\Lambda_z^+ \cap \Lambda_{\tilde{z}}^- = \{x\}$, and

$$q_+ s_x^+(\lambda) = z \ , \ q_- s_x^-(\tilde{\lambda}) = \tilde{z}$$

for all $\lambda, \tilde{\lambda} \in \mathbb{C}$.

We identify $\alpha_+ \in (q_+^* W_+)_{s_x^+(\lambda)}$ and $\alpha_- \in (q_-^* W_-)_{s_x^-(\lambda^{-1})}$ if and only if $\sigma_+(x) = \sigma_-(x) \in E_x^c$ where $\sigma_+ = q_{+*}\alpha_+ \in (W_+)_z$ and $\sigma_- = q_{-*}\alpha_- \in (W_-)_{\tilde{z}}$.

This procedure yields a holomorphic vector bundle $W \to Z$ over the twistor space Z of rank k.

Before we shall prove in the next section that the bundle W constructed above has indeed the desired properties, we shall explore our construction for a simple example.

Example 4.4 We consider the case of $X = \mathbb{R}^2 = \mathbb{C}$ with line bundle L with (global) connection form $a = x_1 dx_2$ where x_1, x_2 are the usual coordinates for $\mathbb{R}^2$.

We rewrite a in terms of the complex variable $z = x_1 + ix_2$, i. e.

$$a = \frac{1}{4i}(zdz + \bar{z}dz - zd\bar{z} - \bar{z}d\bar{z}).$$

On the complexification $X^c = \mathbb{C}^2$, the complexified line bundle L^c has connection form

$$a^c = \frac{1}{4i}(zdz + \tilde{z}dz - zd\tilde{z} - \tilde{z}d\tilde{z}).$$

We calculate the covariant constant sections s on the leaf Λ_z^+ of the +-foliation as

$$\begin{aligned}
\frac{\partial s}{\partial \tilde{z}} - \frac{1}{4i}(z + \tilde{z})s = 0 \\
\frac{\partial}{\partial \tilde{z}}(\log s) = \frac{1}{4i}(z + \tilde{z}) \\
s(z, \tilde{z}) = c(z)e^{\frac{1}{4i}(z\tilde{z} + \frac{\tilde{z}^2}{2})}
\end{aligned}$$

for any holomorphic function $c(z)$. Similarly, the covariant constant sections $\tilde{s}$ on the leaf $\Lambda_{\tilde{z}}^-$ of the −-foliation are

$$\tilde{s}(z, \tilde{z}) = \tilde{c}(\tilde{z})e^{-\frac{1}{4i}(z\tilde{z} + \frac{z^2}{2})}$$

for any holomorphic function $\tilde{c}(\tilde{z})$.

We identify two sections s and $\tilde{s}$ using their values at the point $(z, \tilde{z})$ and obtain

$$\tilde{c}(\tilde{z}) = c(z)e^{\frac{1}{4i}(2z\tilde{z} + \frac{z^2 + \tilde{z}^2}{2})}.$$

Using the standard coordinate patches $V_0 = p^{-1}(U_0)$ and $V_1 = p^{-1}(U_1)$ where $U_0 = \mathbb{C}P^1 \setminus \{\infty\}$ and $U_1 = \mathbb{C}P^1 \setminus \{0\}$ of the twistor space $Z = \mathcal{O}(1) \oplus \mathcal{O}(1) \xrightarrow{p} \mathbb{C}P^1$ of flat space we calculate the transition function for the line bundle $W \to Z$ in terms of the usual coordinates λ, η_1, η_2 and $\tilde{\lambda}, \tilde{\eta_1}, \tilde{\eta_2}$ respectively, see 1.1. Recall that $\eta_1 = z$ and $\eta_2 = \lambda\tilde{z}$ (cf. 2.4), thus

$$e^{\frac{1}{4i}(2z\tilde{z}+\frac{z^2+\tilde{z}^2}{2})} = e^{\frac{1}{4i}(2\lambda^{-1}\eta_1\eta_2+\frac{\eta_1^2}{2}+\frac{\tilde{\eta_2}^2}{2})}.$$

As $e^{\frac{1}{4i}\frac{\eta_1^2}{2}}$ is holomorphic on V_0 and $e^{\frac{1}{4i}\frac{\tilde{\eta}_2^2}{2}}$ is holomorphic on V_1, after a change of trivialisation the line bundle W is given by the transition function

$$\begin{array}{rrcl} h_{10}: & V_0 \cap V_1 & \to & \mathbb{C}^* \\ & (\lambda, \eta_1, \eta_2) & \mapsto & e^{\frac{\eta_1\eta_2}{2i\lambda}}. \end{array}$$

We trivialise W along each twistor line $\eta_1 = u_1\lambda + u_2$, $\eta_2 = u_3\lambda + u_4$ for $(u_1, u_2, u_3, u_4) \in \mathbb{C}^4$ by factorising the transition function, namely

$$e^{\frac{(u_1\lambda+u_2)(u_3\lambda+u_4)}{2i\lambda}} = s_+ \cdot s_-$$

where

$$s_+ = e^{\frac{u_1u_3\lambda}{2i}} \text{ and } s_- = e^{\frac{1}{2i}(u_2u_3+u_1u_4+\frac{u_2u_4}{\lambda})}.$$

This defines a (local) trivialisation of a line bundle L' over $\mathbb{C}^4$ where each fibre is given by $L'_{\underline{u}} = \mathbb{C}(s_+, s_-)$.

Now we give a covariant constant trivialisation in order to recover the connection on L'. We consider all twistor lines through $(b, a_1, a_2) \in Z$, i. e. all $\underline{u} \in \mathbb{C}^4$ such that

$$u_1b + u_2 = a_1 \text{ and } u_3b + u_4 = a_2$$

which is a plane $\Pi_{(b,a_1,a_2)}$ in $\mathbb{C}^4$. In terms of the other trivialisation (if we think of the plane as being parametrised by u_1, u_3), the covariant constant trivialisation is given by

$$\gamma(u_1, u_3)e^{-\frac{u_1u_3b}{2i}} = 1$$

for a function γ and $\gamma(u_1, u_3)(s_+, s_-)|_{\Pi_{(b,a_1,a_2)}}$ is covariant constant relative to the desired connection. Therefore the connection form A is determined on $\Pi_{(b,a_1,a_2)}$ by $d\gamma + A\gamma = 0$, hence

$$A = -d(\log\gamma) = -d(\frac{u_1u_3b}{2i}).$$

Writing $A = A_1du_1 + A_2du_2 + A_3du_3 + A_4du_4$, on the null plane $\Pi_{(b,a_1,a_2)}$ in $\mathbb{C}^4$ the equations

$$\begin{aligned} A_1 - bA_2 &= -\frac{1}{2i}bu_3 \\ A_3 - bA_4 &= -\frac{1}{2i}bu_1 \end{aligned}$$

are satisfied. Using the first equation and varying (b, a_1, a_2), we get

$$A_1(u_1, a_1 - u_1 b, u_3, a_2 - u_3 b) - bA_2(u_1, a_1 - u_1 b, u_3, a_2 - u_3 b) = -\frac{1}{2i} u_3 b.$$

Thus, $A_1 = A_1(u_1, u_3)$ and $A_2 = A_2(u_1, u_3)$, and $A_1 = 0$ and $A_2 = \frac{1}{2i} u_3$. Similarly, using the second equation we derive $A_3 = 0$ and $A_4 = \frac{1}{2i} u_1$, thus

$$A = \frac{1}{2i}(u_3 du_2 + u_1 du_4).$$

The real twistor lines correspond to $u_4 = -\bar{u}_1$ and $u_3 = \bar{u}_2$, hence over $\mathbb{R}^4 = \mathbb{C}^2$ the connection form is

$$A = \frac{1}{2i}(\bar{u}_2 du_2 - \bar{u}_4 du_4).$$

Setting $u_4 = x_3 - ix_4$ and $u_2 = x_1 + ix_2$ for $x_1, x_2, x_3, x_4 \in \mathbb{R}$, we obtain the curvature F as

$$F = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

which is self-dual with respect to the standard flat metric on $\mathbb{R}^4$ and its induced $*$-operator.

4.2 Proof

In order to prove that the bundle $W \to Z$ indeed encodes a hyperholomorphic bundle E' over the hyperkähler manifold in the cotangent bundle, we have to show

Proposition 4.5 *The vector bundle $W \to Z$ is trivial along each real twistor line.*

Proof: We show that W is trivial along each of the explicit twistor lines s_x, $x \in X^c$, defined in 2.1.4, hence by a deformation argument W will be trivial along each twistor line.

Now let $x = (z, \tilde{z}) \in X^c$ and pick $v \in E^c_x$, $v \neq 0$. We define $\sigma_+ \in (W_+)_z$ and $\sigma_- \in (W_-)_{\tilde{z}}$ by $\sigma_+(x) = \sigma_-(x) = v$ and parallel transport on the leaf Λ^+_z and $\Lambda^-_{\tilde{z}}$ respectively. Then the maps

$$\begin{array}{rcll} U_0 = \mathbb{C}P^1 \setminus \{0\} & \to & (W_+)_z \times \mathbb{C} & \cong (q_+^* W_+)_{s_x^+(\mathbb{C})} \\ \lambda & \mapsto & \sigma_+ \times \lambda & \end{array}$$

and

$$\begin{array}{rcll} U_1 = \mathbb{C}P^1 \setminus \{\infty\} & \to & (W_-)_{\tilde{z}} \times \mathbb{C} & \cong (q_-^* W_-)_{s_x^-(\mathbb{C})} \\ \tilde{\lambda} & \mapsto & \sigma_- \times \tilde{\lambda} & \end{array}$$

give a nowhere vanishing section of $W|_{s_x(\mathbb{C}P^1)} \to \mathbb{C}P^1$, hence $W|_{s_x(\mathbb{C}P^1)}$ is trivial. $\square$

The bundle $W \to Z$ gives rise to a bundle $q' : E' \to M^c$ over the complexified hyperkähler manifold M^c where $E'_m = H^0(s_m(\mathbb{C}P^1), W)$ for any $m \in M^c$ where s_m is the twistor line parametrised by m, see section 1.4.

To complete the proof of 4.2 we have to show

Proposition 4.6 *The hyperholomorphic bundle E' restricted to X is isomorphic to the original bundle $E \to X$ as bundle with connection.*

The proposition follows from two lemmas of which the first shows the equivalence as vector bundles and the second the equivalence of the connections in the complexified situation.

We recall that the complexified Kähler manifold X^c corresponds to all twistor lines s such that $s(0) \in X$ and $s(\infty) \in \bar{X}$, see 3.9.

Lemma 4.7 *There is a natural isomorphism of $E' \to X^c$ and $E^c \to X^c$ as vector bundles.*

Proof: Let $(x, s) \in E'$, i. e. $s \in H^0(s_x(\mathbb{C}P^1), W)$ and $x = (z, \tilde{z}) \in X^c$. The map

$$\begin{array}{rccl} E'_x = & H^0(s_x(\mathbb{C}P^1), W) & \to & (W_+)_z = \{\sigma_+ \in \Gamma(\Lambda_z^+, E^c); \nabla(\sigma_+) = 0\} \\ & s & \mapsto & s(0) \end{array}$$

is a canonical isomorphism since for $\sigma_+ \in (W_+)_z$ we define s by $s(\lambda) = \sigma_+$ for $\lambda \in U_0$ and $s(\tilde{\lambda}) = \sigma_-$ for $\tilde{\lambda} \in U_1$ where $\sigma_- \in (W_-)_{\tilde{z}}$ is defined by the requirement $\sigma_+(x) = \sigma_-(x) \in E^c_x$ and parallel transport.

Furthermore, the map

$$\begin{array}{rccc} (W_+)_z = & \{\sigma_+ \in \Gamma(\Lambda_z^+, E^c); \nabla(\sigma_+) = 0\} & \to & E^c_x \\ & \sigma_+ & \mapsto & \sigma_+(x) \end{array}$$

is a canonical isomorphism since for $v \in E^c_x$ we define $\sigma_+ \in (W_+)_z$ by $\sigma_+(x) = v$ and parallel transport. Then the following map is a canonical identification of E'_x and E^c_x

$$\begin{array}{ccccc} E'_x & \to & (W_+)_z & \to & E^c_x \\ s & \mapsto & s(0) & \mapsto & s(0)(x) \end{array}$$

and hence the bundles are naturally isomorphic. $\square$

Lemma 4.8 *The connections on $E' \to X^c$ and $E^c \to X^c$ correspond under the isomorphism of 4.7.*

Proof: By the general theory (cf. section 1.4) there is a unique connection on $E' \to M^c$ such that for any $y \in Z$

$$W_y = \{\sigma \in \Gamma(S_y, E'); \nabla(\sigma) = 0\}$$

where $S_y = \{m \in M^c; y \in s_m(\mathbb{C}P^1)\}$ is the set of all twistor lines passing through the point $y \in Z$. Now let $y = z \in X \subset p^{-1}(0)$. Then

$$\Lambda_z^+ = \{x \in X^c; s_x(0) = z, s_x(\infty) \in \bar{X}\} \subset S_z$$

(for a description of the foliations in terms of twistor lines see section 3.2), and by our construction

$$W_z = (W_+)_z = \{\sigma_+ \in \Gamma(\Lambda_z^+, E^c); \nabla(\sigma_+) = 0\}.$$

Taking 4.7 into account, we conclude that E' and E^c have the same covariant constant sections on Λ_z^+. But as $\dim W_z = \operatorname{rank} E^c$ the connection forms are the same on Λ_z^+. Similarly, taking $y = \tilde{z} \in \bar{X} \subset p^{-1}(\infty)$ and using $W_{\tilde{z}} = (W_-)_{\tilde{z}}$, the connections are equivalent on $\Lambda_{\tilde{z}}^-$.

We recall from the proof of 2.12 that tangent vectors to Λ_z^+ are of the form $v \otimes s_0$ and tangent vectors to $\Lambda_{\tilde{z}}^-$ of the form $v \otimes s_\infty$ (under the decomposition $H^0(N) = H^0(N(-1)) \otimes H^0(\mathbb{C}P^1, \mathcal{O}(1))$), hence the connections are the same for all directions of the form $v \otimes \psi$ where $\psi = \alpha s_0 + \beta s_\infty$.

As a tensor product is spanned by its decomposable elements $v \otimes \psi$, a connection is uniquely determined by its values on vectors $v \otimes \psi$, hence the connections must correspond under the isomorphism of E' and E. $\square$

We recall from 2.2.2 that the real structure on the twistor space Z is induced by the real structure on X^c. The construction of $W \to Z$ respects the real structure (as we can identify the relevant covariant constant sections using parallel transport), and thus the hyperholomorphic bundle $E' \to M^c$ is compatible with the real structure. Therefore, $E \to X$ and $E' \to X$ are isomorphic as vector bundles with connection which concludes the proof of 4.2.

Chapter 5

Global Questions

In this chapter we shall discuss some aspects of, especially some necessary conditions for, the global existence and completeness of hyperkähler metrics on cotangent bundles. Nearly all of these considerations, however, only make use of the fact that hyperkähler manifolds are Ricci-flat.

5.1 Connections to work and conjectures by Yau

5.1.1 Calabi-Yau-type theorems

One of the questions of geometry still unsolved is the proof of a generalisation of the Calabi-Yau theorem for non-compact manifolds. Yau proposed at the International Congress 1978 in Helsinki the conjecture that, for a compact projective manifold M with non-negative anticanonical divisor D, $M \setminus D$ admits a complete Ricci-flat metric, cf. [42].

As we can write T^*X as $\mathbb{P}(T^*X \oplus 1) \setminus \mathbb{P}(T^*X)$ the solution of this conjecture would shed considerable light on the question of complete hyperkähler metrics on cotangent bundles since again Bochner-type arguments might imply that these metrics were actually hyperkähler .

Remark 5.1 *Current work on Yau's conjecture by Tian and Yau (cf. [38]) only establishes the well-known case of complete metrics on $T^*\mathbb{C}P^n$, thus gives us little information.*

Proof: Tian and Yau have in [38] established Yau's conjecture under the additional assumption that there exists a Kähler-Einstein metric of positive scalar curvature on D. But under their assumption, $D = \mathbb{P}(T^*X)$ will be a Fano manifold which possesses a natural contact structure.

By the Leray-Hirsch theorem, $H^2(D) = H^2(X) \oplus \mathbb{Z}$, and since the Kähler form provides a non-trivial class in $H^2(X)$, the second Betti number $b_2 \geq 2$. Applying a theorem by LeBrun (Theorem 2.3. in [32]) we conclude $D = \mathbb{P}(T^*\mathbb{C}P^n)$. □

Nevertheless we may explore necessary conditions on X for $\mathbb{P}(T^*X)$ to be non-negative as this can give us some ideas on good potential candidates for complete hyperkähler metrics.

Before we can state and prove these conditions, we have to establish the following

Lemma 5.2 *(i) If $E \to X$ is a holomorphic vector bundle of rank k over a complex manifold X, then for $P = \mathbb{P}(E)$*

$$K_P = p^*K_X \otimes p^*\Lambda^k E^* \otimes L^k$$

where $p : P \to X$ and L is the tautological line bundle $L \to P$.

(ii) (a) If $E = T^*X \oplus 1$, then $K_P = L^{n+1}$ where $n = \dim X$.

(b) If $D = \mathbb{P}(T^*X)$ and L_D the tautological line bundle $L_D \to D$ then $K_D = L_D^n$ where $n = \dim X$.

Proof: ad (i): We start with using the Euler sequence fibrewise to get

$$0 \longrightarrow 1 \longrightarrow p^*E \otimes L^* \longrightarrow T_F \longrightarrow 0$$

where T_F denotes the tangent bundle along the fibres of p. Dualising and tensoring with L^* we obtain

$$0 \longrightarrow T_F^* \otimes L^* \longrightarrow p^*E^* \longrightarrow L^* \longrightarrow 0$$

and taking the top exterior power gives

$$p^*\Lambda^k E^* = \Lambda^{k-1}T_F^* \otimes L^{-k}.$$

We now consider the exact sequence

$$0 \longrightarrow p^*T^*X \longrightarrow T^*P \longrightarrow T_F^* \longrightarrow 0$$

and, again, take the top exterior power to get

$$K_P = p^*K_X \otimes \Lambda^{k-1}T_F^*.$$

Putting these together,

$$K_P = p^*K_X \otimes p^*\Lambda^k E^* \otimes L^k.$$

ad (ii)a: In this case, $k = n + 1$ and

$$K_P = p^*K_X \otimes p^*K_X^* \otimes L^{n+1}.$$

(ii)b follows by the same argument. □

Remark 5.3 *In our case, we can replace the positivity condition $K_P^*|_D \geq 0$ in Yau's conjecture by $K_D^* \geq 0$ where $P = \mathbb{P}(T^*X \oplus 1)$ and $D = \mathbb{P}(T^*X)$.*

Proof: By the adjunction formula,

$$K_P^*|_D = K_D^* \otimes N$$

where N is the normal bundle.

Claim: $N = L|_D$ where L is the tautological line bundle $L \to P$.

Proof of Claim:
Let $L \to P$ be the tautological line bundle. Then the divisor D corresponds to the zero set of the section $s : P \to L$ sending a point in P to the line $(0, *)$.

Now $ds|_{s=0}$ is a section of $L \otimes T^*$, and since it vanishes in the directions tangent to D, we can think of it as a section of $L \otimes N^*$, i. e. as a homomorphism from N to L.

As $ds|_{s=0}$ does not vanish identically (since then s would be identically zero), and the fibres of N and L are 1-dimensional, the homomorphism is in fact an isomorphism.

Using (ii)a of 5.2, we can write $N = (K_P|_D)^{\frac{1}{n+1}}$, and therefore

$$K_D^* = (K_P^*|_D)^{1-\frac{1}{n+1}}$$

which concludes the proof. □

We notice it is important that the condition on $D = \mathbb{P}(T^*X)$ is $K_D^* \geq 0$. If we assumed $K_D^* > 0$, then D would be a Fano manifold with a canonical contact structure and second Betti number $b_2 \geq 2$, hence as in the proof of 5.1 using LeBrun's work [32], $X = \mathbb{C}P^n$.

In the case of X being a complex surface, we can even say more, namely

Proposition 5.4 *If X is a complex surface and $D = \mathbb{P}(T^*X)$, then $K_D^* \geq 0$ implies $C \cdot C \geq 0$ for any curve C in X where $\cdot$ denotes the intersection pairing.*

Proof: If K_D^* is non-negative, K_D^* is in particular numerically effective, i. e. $K_D^* \cdot C' \geq 0$ for all curves C' in D.

Using (ii)b of 5.2, this is equivalent to $C' \cdot L|_D \leq 0$.

For a curve $C \overset{i}{\hookrightarrow} X$ let C' be the canonically lifted curve in D. Consider the diagram

$$\begin{array}{ccc} Q = \mathbb{P}(i^*T^*X) & \xrightarrow{j} & \mathbb{P}(T^*X) = D \\ \downarrow & & \downarrow \\ C & \xrightarrow{i} & X \end{array}$$

Since C' is a lift, there exists a curve C'' in Q such that $C' = j_*C''$. If we denote by L' the tautological line bundle over Q, $L' = j^*L|_D$, and the projection formula yields

$$C' \cdot L|_D = (j_*C'') \cdot L|_D = j_*(C'' \cdot j^*L) = j_*(C'' \cdot L').$$

As we want to describe the lifting of C to C'' as $C'' = s_*C$, we need to find sections of $\mathbb{P}(i^*T^*X) \to C$. Generally, such a section corresponds to a surjective vector bundle morphism $i^*(T^*X)^* = i^*(TX) \to U$ where U is any line bundle over C. In this case, $U^* = s^*L'$, and we obtain

$$C'' \cdot L' = (s_*C) \cdot L' = s_*(C \cdot s^*L') = s_*(C \cdot U^*).$$

Putting the above arguments together,

$$C' \cdot L|_D = j_*s_*(C \cdot U^*) = -\deg U.$$

In particular, we can use the surjective vector bundle morphism

$$i^*(TX) \longrightarrow N_{C/X} \longrightarrow 0$$

where $N_{C/X}$ is the normal bundle of C in X (which is a line bundle if X is a surface), and since $\deg N_{C/X} = C \cdot C$, the result follows. □

This is only a serious constraint in the case of rational surfaces. By the classification of rational surfaces (see, for example, [4]), the possible relatively minimal models are $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the Hirzebruch surfaces $F_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ for $n \geq 2$. Since F_n contains a curve of self-intersection $-n$, the only possible candidates among rational surfaces are $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$.

In fact, in both cases there exists a complete hyperkähler metric on the cotangent bundle, see [6].

5.1.2 Harmonic functions

One way of finding necessary conditions for the existence of complete hyperkähler metrics lies in the work of Yau on harmonic functions on complete manifolds of non-negative Ricci curvature, cf. [43].

Proposition 5.5 *If there exists a complete hyperkähler metric on T^*X where X is a complex manifold then every bounded harmonic function on T^*X is constant.*

Proof: This is an immediate consequence of a theorem in [43]. □

In the case of Riemann surfaces we can prove

Corollary 5.6 *There exist no complete hyperkähler metrics on the cotangent bundle $T^*\Sigma$ of a Riemann surface Σ of genus > 1.*

Proof: (communicated by Yau)
The assumption of existence of a complete metric would yield a complete Ricci-flat metric on $T^*\tilde{\Sigma}$ where $\tilde{\Sigma}$ is the universal cover of Σ. As $\tilde{\Sigma}$ can be identified with the unit disc we can certainly find non-constant bounded holomorphic functions on $\tilde{\Sigma}$, hence non-constant bounded harmonic functions on $T^*\tilde{\Sigma}$ giving a contradiction to 5.5. $\square$

Example 5.7 Bounded domains in $\mathbb{C}^n$ with the Bergmann metric
Let $U \subset \mathbb{C}^n$ be a bounded domain, i. e. a bounded open connected subset.

The set of all holomorphic L^2-integrable functions on U forms a separable Hilbert space H with respect to the L^2-inner product. Let $\phi_1, \phi_2, \ldots$ be an orthonormal basis of H. Then

$$k(z,\bar{z}) = \sum_{l=1}^{\infty} \phi_l(z)\overline{\phi_l(\bar{z})}$$

converges uniformly, hence defines a smooth function on U. The bilinear form

$$g = \sum_{i,j=1}^{n} \frac{\partial^2(\log k)}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

where $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ defines a Kähler metric on U, the so-called Bergmann metric.

If U is a pseudo-convex domain with smooth C^1-boundary, then the Bergmann metric on U is complete, see [16].

Proposition 5.5 implies that there are no complete hyperkähler metrics on T^*U because

$$\begin{array}{ccccc} T^*U & \to & U & \to & \mathbb{C} \\ (z,v) & \mapsto & z & \mapsto & z_i \end{array}$$

(where $z_i : \mathbb{C}^n \to \mathbb{C}$ is one of the coordinate functions for $1 \leq i \leq n$) is a non-constant bounded holomorphic function on T^*U since U is bounded, thus a non-constant bounded harmonic function.

5.2 Topological constraints

5.2.1 Fundamental group restrictions

By Cheeger and Gromoll's splitting theorem (cf. [13]), if N is a complete Riemannian manifold of non-negative Ricci curvature, every finitely generated subgroup of the fundamental group $\pi_1 N$ must have polynomial growth of order at most $\dim N$.

We recall that we can assign a growth function $\gamma(k)$ to any finitely generated group G by counting the number of different elements which can be obtained by words of length k from a given set of generators. It can be shown (cf. [19]) that the asymptotic behaviour of $\gamma(k)$ does not depend on the chosen set of generators. Now G is said to have polynomial growth of order at most p if there exists a constant $C > 0$ such that $\gamma(k) \leq Ck^p$; G has exponential growth if $\gamma(k) \geq ca^k$ for some constants $c > 0$, $a > 1$.

Applying these considerations to our problem we deduce

Proposition 5.8 *Let X be a compact complex manifold of dimension n. If there is a complete hyperkähler metric on the cotangent bundle T^*X, then the fundamental group $\pi_1 X$ has polynomial growth of order at most $4n$.*

Proof: As X is compact, $\pi_1(T^*X) = \pi_1 X$ is finitely generated. Since $\dim_{\mathbb{R}} T^*X = 4n$ the result follows from the splitting theorem. □

We now give an alternative proof for the case of Riemann surfaces of genus > 1.

Corollary 5.9 *There exist no complete hyperkähler metrics on the cotangent bundle of a Riemann surface of genus > 1.*

Proof: The constant negative scalar curvature metric on hyperbolic space provides any Riemann surface of genus > 1 with a metric of negative sectional curvature by taking quotients.

Now suppose there exists a complete hyperkähler metric on the cotangent bundle of such a Riemann surface. Applying the result by Milnor [34] that for a compact Riemannian manifold such that all sectional curvatures are less than zero, the growth of the fundamental group is exponential, yields a contradiction. □

Gromov has proved in [19] that a finitely generated group G has polynomial growth if and only if it contains a nilpotent subgroup of finite index, hence, after possibly taking finite covers, we might restrict our attention to manifolds with nilpotent fundamental groups.

In the case of complex surfaces we can even deduce

Proposition 5.10 *If X is a compact complex surface admitting a complete hyperkähler metric on its cotangent bundle then there exists a finite cover $\hat{X}$ of X with abelian fundamental group $\pi_1 \hat{X}$.*

Proof: Without loss of generality by the above considerations we can assume that $\pi_1 X$ is nilpotent.

Toledo has shown in [12] that the order of growth of the fundamental group is at least 10 if the group is nilpotent, but not almost abelian, i. e. does not have an abelian subgroup of finite index. But by Cheeger and Gromoll, the order of growth of $\pi_1 X$ is at most 8, so the result follows. $\square$

5.2.2 Restrictions on the first Betti number

Using a result of Anderson in [1] that for a complete Riemannian k-manifold N with non-negative Ricci curvature the first Betti number satisfies $b_1(N) = \dim H_1(N, \mathbb{Q}) \leq k$ and equality holds if and only if N is compact and flat, we have

Proposition 5.11 *If X is a compact complex manifold of dimension n admitting a complete hyperkähler metric on its cotangent bundle then $b_1(X) \leq 4n-2$.*

Proof: By Anderson's result, $b_1(X) \leq 4n-1$. But if there is a hyperkähler metric on T^*X, this induces a Kähler metric on X, hence $b_1(X)$ must be even, so the result follows. $\square$

Using this proposition we can now give a third proof for the non-existence of complete metrics in the case of Riemann surfaces of higher genus.

Corollary 5.12 *There exist no complete hyperkähler metrics on the cotangent bundle of a Riemann surface of genus > 1.*

Proof: If Σ is a Riemann surface of genus g, its first Betti number is $b_1(\Sigma) = 2g$, hence if $g > 1$ then $b_1(\Sigma) > 2 = 4 \cdot 1 - 2$ contradicting 5.11. $\square$

5.3 Types of incompleteness

In this section we shall present, by looking at various examples, two possible types of incompleteness of the hyperkähler metric in the cotangent bundle given by theorem 2.1.

As all our conditions for completeness given in this chapter are only necessary, they do not give any insight into the geometric nature of obstructions. Although general necessary and sufficient conditions for completeness are still unknown, we observe from the examples that the following two phenomena can occur:

- the hyperkähler metric defined in a neighbourhood of the zero section of the cotangent bundle blows up at finite radius (see 5.13 and 5.14) or

- the hyperkähler metric can indeed be extended to the full cotangent bundle, but is nevertheless incomplete (see 5.15).

An interesting open question may be to develop criteria to distinguish between these two types of incompleteness.

Example 5.13 The cotangent bundle of hyperbolic space H
We consider the upper half-plane model of hyperbolic space **H** with coordinates x, y and metric $g_H = \frac{dx^2+dy^2}{y^2}$.

We define vector fields on $\mathbf{H} \times \{x_1^2 + x_2^2 + x_3^2 = 1, x_1 > 0\}$ (i. e. $x_2^2 + x_3^2 < 1$ and $x_1 = \sqrt{1-(x_2^2+x_3^2)}$)

$$X_0 = \frac{\partial}{\partial y}$$

$$X_1 = \frac{\partial}{\partial x} + \frac{x_2}{y}\frac{\partial}{\partial x_3} - \frac{x_3}{y}\frac{\partial}{\partial x_2}$$

$$X_2 = \frac{x_1}{y}\frac{\partial}{\partial x_3}$$

$$X_3 = -\frac{x_1}{y}\frac{\partial}{\partial x_2}$$

These vector fields satisfy the following version of Nahm's equations:

$$\frac{\partial X_1}{\partial y} = [X_2, X_3]$$

and cyclic permutations of the indices. We can define a hypercomplex structure by setting

$$IX_1 = X_0 \ , \ IX_2 = X_3$$

$$JX_0 = X_3 \ , \ JX_2 = X_1$$

and a conformal structure by requiring X_0, X_1, X_2 and X_3 to be orthogonal. We define a dual basis to X_0, X_1, X_2, X_3 by the 1-forms

$$\theta_0 = dy$$

$$\theta_1 = dx$$

$$\theta_2 = \frac{y}{x_1}dx_3 - \frac{x_2}{x_1}dx$$

$$\theta_3 = -\frac{y}{x_1}dx_2 - \frac{x_3}{x_1}dx.$$

The hyperkähler metric will be of the form

$$g = f(\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2)$$

for a function f. The conditions that $g|_{\mathbf{H}} = g_H$ and that the three Kähler forms are closed yields $f = \frac{x_1}{y^2}$, hence

$$g = \frac{x_1}{y^2}(dx^2 + dy^2) + \frac{1}{x_1 y^2}((ydx_3 - x_2 dx)^2 + (ydx_2 + x_3 dx)^2)$$

If we set $z = x + iy$ and $w = \frac{ix_2 - x_3}{y}$ then

$$\omega_2 + i\omega_3 = dz \wedge dw$$

is the symplectic structure of the cotangent bundle.

If ξ is the Killing field for the S^1-action given by scalar multiplication in the fibres, i. e. $\mathcal{L}_\xi z = 0$ and $\mathcal{L}_\xi w = iw$, then $\xi = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$. It is easy to check that ξ is an infinitesimal automorphism for the complex structure I (i. e. $\mathcal{L}_\xi I = 0$) and that $\mathcal{L}_\xi \omega_1 = 0$, hence $\mathcal{L}_\xi g = 0$ and the action is isometric, hence uniqueness (see Corollary 3.3) implies that we have obtained the same hyperkähler metric as in our twistor construction.

Example 5.14 The cotangent bundle of Riemann surfaces of genus > 1
As we have seen above (cf. 5.6, 5.9, 5.12) there are no complete hyperkähler metrics on the cotangent bundle of a Riemann surface Σ of genus > 1. In fact, the hyperkähler metric in a neighbourhood of the zero section which exists by our twistor construction cannot be extended to the full cotangent bundle.
Proof: The standard Riemannian metric on Σ is obtained from the hyperbolic metric on the universal cover by quotienting. If there were an extension of the hyperkähler metric to the full cotangent bundle this would yield (as the hyperkähler metric of our construction is unique) an extension of the hyperkähler metric on the open unit disc bundle in the cotangent bundle of hyperbolic space constructed in Example 5.13 to the full cotangent bundle, but we have seen in 5.13 that this is impossible. □

Example 5.15 Cotangent bundles of Kähler quotients by tori
Here we summarise work by Bielawski and Dancer on hyperkähler manifolds with torus actions, see [6].

Atiyah and others have shown (see, for example, [3]) that, given a $2n$-dimensional compact connected symplectic manifold with effective action of the torus T^n, the image of the moment map is a convex polytope in $\mathbb{R}^n$. Delzant has shown in [15] that this polytope determines the symplectic manifold completely.

Let X be such a symplectic manifold and Δ the image of the moment map, then the polytope Δ is defined by

$$\langle x, u_i \rangle \geq \lambda_i \text{ for } i = 1, \dots, d$$

where u_i is the inward-pointing normal vector of the ith $(n-1)$-dimensional face of Δ and d is the number of $(n-1)$-dimensional faces. Then $u_1, \dots, u_d$ form a basis of $\mathbb{Z}^n$, and we can think of them as describing a subtorus N of T^d; on the level of Lie algebras we have the short exact sequence

$$0 \to \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \to 0$$

where $\beta(e_i) = u_i$, $i = 1, \dots, d$, and $e_1, \dots, e_d$ is the standard basis of $\mathbb{R}^d$.

Then N acts on $\mathbb{C}^d$ preserving the Kähler form of the flat metric, and the moment map for this action is given by

$$\nu(z) = \sum_{k=1}^{d} (|z_k|^2 + \lambda_k) i^*(e_k).$$

The manifold X can be identified with the Kähler quotient $\nu^{-1}(0)/N$.

We can now consider the analogous hyperkähler quotient of $\mathbb{H}^d$ by N where we identify $\mathbb{H}^d = \mathbb{C}^d \times \mathbb{C}^d$ and take the dual action on the second factor. We can compare this hyperkähler quotient with the cotangent bundle T^*X. The real moment map is

$$\mu_1(z, w) = \sum_{k=1}^{d} (|z_k|^2 - |w_k|^2 + \lambda_k) i^*(e_k)$$

and the complex moment map

$$\mu_c(z, w) = \sum_{k=1}^{d} (z_k w_k) i^*(e_k).$$

Bielawski and Dancer show that the cotangent bundle T^*X can be identified (as a holomorphic-symplectic manifold) T^n-equivariantly with an open subset U of the (usually singular) hyperkähler quotient $M = \mu^{-1}(0)/N$ with the complex structure I.

Under this identification the hyperkähler metric on M restricted to the zero section of T^*X is the original Kähler metric on X.

The proof involves investigation of the different notions of stability for the Kähler and hyperkähler quotient.

Let $\mathbb{C}^d_\Delta$ be the set of stable points for the Kähler quotient, i. e. $X \cong \mathbb{C}^d_\Delta/N^{\mathbb{C}}$. Now consider the subset $Y = \mathbb{C}^d_\Delta \times \mathbb{C}^d$ of $\mathbb{H}^d$. As N acts freely on $\mathbb{C}^d_\Delta$, it acts freely on Y, thus the hyperkähler quotient of Y by N is a smooth manifold $U \subset M$. Bielawski and Dancer then showed that U with respect to the complex structure I is the complex-symplectic quotient of Y by $N^{\mathbb{C}}$ and so is

$$\mu_c^{-1}(0) \cap (\mathbb{C}^d_\Delta \times \mathbb{C}^d)/N^{\mathbb{C}}$$

which is biholomorphic to T^*X.

Bielawski and Dancer also showed that the metric is complete (i. e. $U = M$) if and only if X is a product of projective spaces. In all other cases, these Kähler quotients provide examples of manifolds where there is a hyperkähler metric on the full cotangent bundle which is nevertheless incomplete.

The above example gives essentially the same picture for finite-dimensional quotients as presented in [23] for infinite-dimensional quotients. Hitchin [23] showed that the moduli space of Higgs bundles over a Riemann surface is a complete hyperkähler manifold which contains the cotangent bundle of the moduli space of stable bundles as a dense open subset.

Chapter 6

Ricci-flat Kähler Geometry

In this chapter we outline a possible generalisation to Ricci-flat Kähler geometry of the problems discussed in previous chapters. This is partially motivated by the fact that hyperkähler manifolds are Ricci-flat Kähler (cf. section 1.1), and in particular the obstructions to completeness presented in chapter 5 only depend on the Ricci curvature rather than the full hyperkähler condition.

6.1 An analogous problem to the existence of hyperkähler structures on cotangent bundles

Let (M, ω) be a Ricci-flat Kähler manifold, and let Ω be a non-vanishing holomorphic n-form trivialising the canonical bundle where n is the complex dimension of M.

A submanifold X of M is called **Lagrangian** if $\dim X = \frac{1}{2} \dim M$ and $\omega|_X$ vanishes. A submanifold X is called **special** if $\Im \Omega|_X$ vanishes.

Now if M is hyperkähler, then by fixing one of the complex structures, I say, $(\omega_c)^n = (\omega_2 + i\omega_3)^n$ is a non-vanishing holomorphic $2n$-form on M, see section 1.1. If $X \subset M$ is a complex Lagrangian submanifold with respect to I, i. e. $\omega_c|_X = 0$, then X is special Lagrangian with respect to the complex structure $aJ + bK$ for any $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, because $(a\omega_2 + b\omega_3)|_X = 0$ as $\omega_2|_X = \omega_3|_X = 0$ and $\Omega|_X = 0$ for $\Omega = (-b\omega_2 + (a+i)\omega_3)^n$ which is a non-vanishing holomorphic $2n$-form with respect to $aJ + bK$.

In the case of $M = T^*X$ being a hyperkähler manifold compatible with the canonical holomorphic-symplectic structure, the zero section and the fibres are I-complex Lagrangian, hence special Lagrangian with respect to any of the complex structures orthogonal to I. We also note that the zero section in T^*X is not a complex submanifold with respect to any complex structure orthogonal to I.

Proposition 6.1 *If $M = T^*X$ is a hyperkähler manifold compatible with the canonical holomorphic-symplectic structure, then the zero section is real with respect to the complex structure J and (M, ω_2) is isomorphic (as a real symplectic manifold) to the real cotangent bundle $T^*_{\mathbb{R}}X$ with the canonical symplectic form.*

Proof: In order to prove that the zero section is real we show that the map

$$\begin{array}{rccc} f: & T^*X & \rightarrow & T^*X \\ & (z,\zeta) & \mapsto & (z,-\zeta) \end{array}$$

is anti-holomorphic with respect to J.

We can work locally in a coordinate neighbourhood, then the complex structures I and J correspond to

$$I = \begin{pmatrix} i1_n & 0 \\ 0 & -i1_n \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

If $z_1, \dots, z_n, w_1, \dots, w_n$ are holomorphic coordinates with respect to I then

$$f(z,w) = (z,-w).$$

We define

$$\begin{array}{rcl} \alpha_j & = & \frac{1}{2}((z_j + \bar{z}_j) + i(w_j + \bar{w}_j)) \\ \beta_j & = & \frac{1}{2}((i(\bar{z}_j - z_j) + (w_j - \bar{w}_j)), \end{array}$$

then $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are holomorphic coordinates for J. In these coordinates, the map f is given by

$$f(\alpha,\beta) = (\bar{\alpha}, -\bar{\beta})$$

which is obviously anti-holomorphic.

In analogy to example 1.1, we have

$$\omega_2 = \frac{1}{2}\sum_{j=1}^{n}(dz_j \wedge dw_j + d\bar{z}_j \wedge d\bar{w}_j)$$

hence for $z_j = x_j + iy_j$ and $w_j = u_j - iv_j$ where x, y, u, z are real coordinates,

$$\omega_2 = \sum_{j=1}^{n}(dx_j \wedge du_j + dy_j \wedge dv_j)$$

is the canonical symplectic form of the real cotangent bundle $T^*_{\mathbb{R}}X$. $\square$

The analogous question therefore is: which manifolds possess a Ricci-flat Kähler metric on their cotangent bundle such that the zero section and the fibres are special Lagrangian submanifolds? Is this structure compatible with the canonical symplectic structure of the cotangent bundle? Is the zero section real in this situation?

The Ricci-flat Kähler metric on T^*S^n constructed in [36] gives an example of the structures described above.

Example 6.2 The cotangent bundle of the sphere S^n
Stenzel has constructed a complete Ricci-flat Kähler metric on T^*S^n by giving its Kähler potential, see [36].

The cotangent bundle

$$T^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}; \|x\| = 1, \langle x, \xi \rangle = 0\}$$

can be given a complex structure by identifying it with the affine quadric

$$Q^n = \{z \in \mathbb{C}^{n+1}; \sum_{i=1}^{n+1} z_i^2 = 1\}$$

by

$$\begin{array}{rccc} F: & T^*S^n & \to & Q^n \\ & (x, \xi) & \mapsto & \sqrt{1 + \|\xi\|^2}\, x + i\xi. \end{array}$$

The zero section in T^*S^n can be identified with the fixed point set of the real structure $\sigma : Q^n \to Q^n$ with $\sigma(z_1, \ldots, z_{n+1}) = (\bar{z}_1, \ldots, \bar{z}_{n+1})$.

This corresponds to the map $\xi \mapsto -\xi$ on $T_x^*S^n$ for any $x \in S^n$, thus the zero section in T^*S^n is a real submanifold with respect to the complex structure given by Q^n.

Now consider the real structure $\sigma' : Q^n \to Q^n$ where

$$\sigma'(z_1, \ldots, z_n, z_{n+1}) = (-\bar{z}_1, \ldots, -\bar{z}_n, \bar{z}_{n+1}).$$

Its fixed point set corresponds to $T_{x_0}^*S^n$ for $x_0 = (0, \ldots, 0, 1) \in S^n$. Strictly speaking, the fixed point set of σ' is the disjoint union of $T_{x_0}^*S^n$ and $T_{-x_0}^*S^n$, but we restrict our attention to $T_{x_0}^*S^n$ for the rest of this example.

All other cotangent spaces $T_x^*S^n$ can be obtained from $T_{x_0}^*S^n$ using the $SO(n+1)$-action on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ given by $A.(x, \xi) = (Ax, A\xi)$. As $\|Ax\| = \|x\|$ and $\langle Ax, A\xi \rangle = \langle A^T Ax, \xi \rangle = \langle x, \xi \rangle$, this induces an action on T^*S^n. On Q^n this action is just given by multiplication as $SO(n+1) \subset GL(n+1, \mathbb{C})$.

Let

$$\begin{array}{rccc} \tau: & Q^n & \to & \mathbb{R} \\ & z & \mapsto & \sum_{i=1}^{n+1} z_i \bar{z}_i. \end{array}$$

and let f be a real-analytic solution of the ordinary differential equation

$$x(f')^n + f''(f')^{n-1}(x^2-1) = c > 0.$$

Then $\phi = f \circ \tau$ is a Kähler potential for a complete Ricci-flat metric, see [36, section 7].

To show that the zero section and the fibres are special Lagrangian we use the following two facts:

(i) If X is the fixed point set of a real structure σ on a Ricci-flat Kähler manifold M then X is special.

(ii) If X is the fixed point set of an isometric real structure σ on a Kähler manifold M then $\omega|_X = 0$.

Proof: To show (i) let Ω be the holomorphic $(n,0)$-form which exists as M is Ricci-flat, then $\sigma^*(\Im\Omega) = -\Im(\sigma^*\Omega)$, and as $\sigma|_X = id$ it follows $\Im\Omega|_X = 0$.

To show (ii), for $v, w \in TX$ we have $\sigma_* v = v$ and $\sigma_* w = w$, thus

$$\begin{aligned}\omega(v,w) &= \omega(\sigma_* v, \sigma_* w) = g(I\sigma_* v, \sigma_* w) = -g(\sigma_* Iv, \sigma_* w) = -g(Iv, w)\\ &= -\omega(v,w),\end{aligned}$$

hence $\omega|_X = 0$.

We deduce from these two facts that the zero section and the fibre $T^*_{x_0}S^n$ are special Lagrangian. The map τ is invariant under the action of $SO(n+1)$ since

$$\langle Az, \overline{Az}\rangle = \langle Az, A\bar{z}\rangle = \langle z, \bar{z}\rangle$$

for any $A \in SO(n+1)$ and $z \in \mathbb{C}^{n+1}$. As all other tangent spaces $T^*_x S^n$ can be obtained from $T^*_{x_0}S^n$ by this action, we conclude that all fibres are special Lagrangian.

In order to prove that Q^n with the Kähler form $\omega = i\partial\bar{\partial}\phi$ is symplectically diffeomorphic to T^*S^n with the canonical symplectic form we shall show that all radial geodesics (with respect to the connection defined by 1.16) in the leaf $L = T^*_{x_0}S^n \subset Q^n$ are complete.

As L is isomorphic to $\mathbb{R}^n$ the flat torsion-free affine connection on L given by 1.16 can be identified with the Levi-Civita connection of a Riemannian metric. If all radial geodesics are complete, then geodesic balls are compact, hence by the Hopf-Rinow theorem the leaf L is geodesically complete. By symmetry, all leaves are geodesically complete and obviously simply-connected, thus by 1.18 (Q^n, ω) can be identified with T^*S^n as symplectic manifolds.

It remains to show that radial geodesics in L are complete.

We choose $z_1, \dots, z_n$ as coordinates of Q^n near L, $z_j = x_j + iy_j$ for $1 \le j \le n$, then L has coordinates $y_1, \dots, y_n$, and radial geodesics γ are of the form

$$\dot\gamma = g(\|y\|^2) \sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j}$$

where g is a function such that $\nabla_{\dot\gamma}\dot\gamma = 0$. Assuming the Kähler form is given by

$$\omega = i \sum_{i,j=1}^{n} b_{ij} dz_i \wedge d\bar z_j = 2 \sum_{i,j=1}^{n} b_{ij} dx_i \wedge dy_j$$

the connection ∇ can be calculated according to 1.16 by

$$\imath(\nabla_Y Y)\omega = 2 \sum_{i,j,k=1}^{n} \alpha_k \frac{\partial(\alpha_j b_{ij})}{\partial y_k} dx_i$$

where $Y = \sum_{l=1}^{n} \alpha_l \partial/\partial y_l$ and $\imath$ is the contraction mapping. Hence γ is a radial geodesic if and only if

$$\sum_{j,k=1}^{n} g(\|y\|^2) y_k \frac{\partial(b_{ij} y_j g(\|y\|^2))}{\partial y_k} = 0$$

for all $i = 1, \dots, n$, thus if and only if

$$\frac{d}{d(\|y\|^2)}(\log g) = -\frac{1}{2\|y\|^2} - \frac{\sum_{j,k=1}^{n} y_j y_k \frac{\partial b_{ij}}{\partial y_k}}{2\|y\|^2 \sum_{j=1}^{n} b_{ij} y_j}.$$

Next we have to compute the coefficients b_{ij} from the Kähler potential $\phi = f \circ \tau$. We get

$$\frac{\partial^2 \phi}{\partial z_i \partial \bar z_j} = f'' \left(\bar z_i - \frac{\bar z_{n+1}}{z_{n+1}} z_i \right) \left(z_j - \frac{z_{n+1}}{\bar z_{n+1}} \bar z_j \right) + f' \left(\delta_{ij} + \frac{z_i \bar z_j}{|z_{n+1}|^2} \right)$$

where f' denotes differentiation with respect to τ. Hence on L

$$b_{ij} = 4 f'' y_i y_j + f' \left(\delta_{ij} + \frac{y_i y_j}{1 + \|y\|^2} \right).$$

Therefore the geodesic equation is

$$\begin{aligned} \frac{d}{d(\|y\|^2)}(\log g) = &- \frac{8f'''\|y\|^2(1+\|y\|^2)^2 + f' + f''(6 + 8\|y\|^2)(1+\|y\|^2)}{4f''\|y\|^2(1+\|y\|^2)^2 + f'(1+2\|y\|^2)(1+\|y\|^2)} \\ &- \frac{1}{2\|y\|^2}. \end{aligned}$$

Re-writing the equation in terms of $x = \tau$ yields

$$\begin{aligned} \frac{1}{2}(\log g)' = &- 2\frac{f'''(f')^{n-1}(x^2-1)(x+1) + (f')^n + f''(f')^{n-1}(2x+1)(x+1)}{f''(f')^{n-1}(x^2-1)(x+1) + (f')^n x(x+1)} \\ &- \frac{1}{x-1} \end{aligned}$$

thus

$$\frac{1}{2}(\log g)' = \frac{2(n-1)c}{x^2-1}\frac{1}{(f')^n} - \frac{(2n-3)x+3}{x^2-1}$$

if we use the differential equation for f, namely

$$x(f')^n + f''(f')^{n-1}(x^2-1) = c > 0$$

which yields an expression for $f'''(f')^{n-1}$ by differentiating again. Making the change of variable $w = \cosh^{-1} x$, the differential equation turns into

$$\frac{d}{dw}(f'(w))^n = nc(\sinh w)^{n-1}$$

(see [36, section 7]), hence $1/(f')^n$ tends to zero as $x \to \infty$, i. e. for large x the function g is asymptotically like e^x, hence the radial geodesics are complete.

In attacking the more general problem we have to define a suitable complex structure on the cotangent bundle of a Riemannian manifold.

If the manifold we start with is real-analytic we can consider its complexification and re-discover the original manifold as the fixed point set of a real structure. The problem is to identify the fibres of the cotangent bundle, i. e. a suitable foliation, in the complexification picture.

Szöke has constructed in [37] adapted complex structures on the tangent bundle of a real-analytic Riemannian manifold X, at least in a neighbourhood of the zero section of TX. A complex structure on a neighbourhood of the zero section of TX is called adapted if the leaves of the Riemann foliation are complex submanifolds. The leaves of the Riemann foliation are $\psi_\gamma(\mathbb{C} \setminus \mathbb{R})$ where γ is a geodesic in X (with respect to the Levi-Civita connection of the Riemannian metric) and ψ_γ is defined by

$$\psi_\gamma(x+iy) = N_y\dot\gamma(x)$$

where $N_y : TM \to TM$ is the map given by scalar multiplication by y in the fibres.

Szöke has shown in [37] that for any compact real-analytic manifold with a real-analytic Riemannian metric there exists an adapted complex structure in a neighbourhood of the zero section of the tangent bundle; in the case of compact symmetric spaces this adapted complex structure exists on the full tangent bundle.

If the manifold we start with is itself complex, the above approach might even yield a hypercomplex structure: in this case the cotangent bundle inherits a complex structure from the manifold and we can also use the adapted complex structure, but, generally, these will not anti-commute.

Dancer and Szöke proved in [14] that for hermitian symmetric spaces there exists a suitable real-analytic diffeomorphism such that the pullback of the adapted complex structure by this diffeomorphism and the natural complex structure anti-commute, thus they obtain a hypercomplex structure which can be shown to come from an underlying hyperkähler structure.

6.2 Necessary conditions for completeness

We can easily adapt the necessary conditions for the existence of a complete hyperkähler metric on the cotangent bundle to the Ricci-flat Kähler case. Then the following holds.

Proposition 6.3 *Let X be a manifold of dimension n which has a complete Ricci-flat Kähler metric on its cotangent bundle T^*X. Then*

(i) Every bounded harmonic function on T^*X is constant.

(ii) If X is compact, the fundamental group of X has polynomial growth of order at most $2n$.

(iii) If X is compact, the first Betti number satisfies $b_1(X) \leq 2n - 1$.

Proof: The results follow exactly as in 5.5, 5.8 and 5.11. □

In contrast to the hyperkähler case, the necessary condition (i) of 6.3 can not be easily applied in the more general case as the projection $T^*X \to X$ is not holomorphic and we thus cannot make use of holomorphic functions on X. The corresponding notion which makes sense is that of a harmonic morphism.

We finish by applying (ii) of 6.3 to hyperbolic 3-manifolds.

Proposition 6.4 *Let X be a closed hyperbolic 3-manifold. Then there are no complete Ricci-flat Kähler metrics on T^*X.*

Proof: By theorem 4.2 in [35], the fundamental group $\pi_1 X$ has uniformly exponential growth. □

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