Rozansky-Witten invariants of hyperkähler manifolds

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Contents

Abstract

We investigate invariants of compact hyperkähler manifolds introduced by Rozansky and Witten: they associate an invariant to each graph homology class. It is obtained by using the graph to perform contractions on a power of the curvature tensor and then integrating the resulting scalar-valued function over the manifold, arriving at a number. For certain graph homology classes, the invariants we get are Chern numbers, and in fact all characteristic numbers arise in this way.

We use relations in graph homology to study and compare these hyperkähler manifold invariants. For example, we show that the norm of the Riemann curvature can be expressed in terms of the volume and characteristic numbers of the hyperkähler manifold. We also investigate the question of whether the Rozansky-Witten invariants give us something more general than characteristic numbers. Finally, we introduce a generalization of these invariants which incorporates holomorphic vector bundles into the construction.

The work contained in this dissertation is my own original research carried out with the assistance of my PhD supervisor, Prof N. Hitchin, and others mentioned in the acknowledgements. Propositions 2, 5, 6, and Theorem 15 have appeared in a joint preprint **math.DG/9908114** *with Prof Hitchin entitled* "Curvature and characteristic numbers of hyperkähler manifolds". *Otherwise, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference has been made in the text.*

Justin Sawon
October, 1999

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0 Introduction

In this thesis we study new invariants of hyperkähler manifolds introduced by Rozansky and Witten in [44]. They described a three-dimensional sigma model with target space a hyperkähler manifold, and showed that the partition function of the theory is a three-manifold invariant of finite type. In a "perturbative" expansion the weights depend on the hyperkähler manifold and are indexed by trivalent graphs, and Rozansky and Witten give a purely differential geometric construction of these weights. Roughly speaking, we take a tensor power of the Riemann curvature tensor of the hyperkähler manifold; the trivalent graph tells us how to contract indices (using the holomorphic symplectic form and its dual) to get something we can integrate, resulting in a number.

$$\begin{array}{ccc} \text{hyperkähler manifold } X & & \text{trivalent graph } \Gamma \\ & \searrow \quad \swarrow & \\ & \text{number } b_\Gamma(X) & \end{array}$$

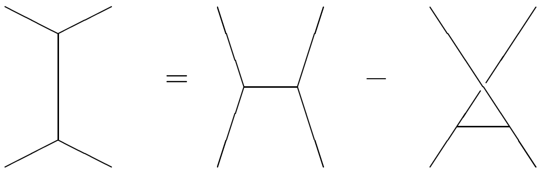
The weight $b_\Gamma(X)$ will be our main object of study. It satisfies the following properties (Propositions 1 to 4 respectively in Chapter 1):

- it is independent of the choice of complex structure on X,
- it is invariant under deformations of the hyperkähler metric on X,
- the dependence on Γ is only through its graph homology class,
- taking products of hyperkähler manifolds corresponds to taking coproducts in graph homology.

Proposition 1 follows from Rozansky and Witten's original definition. While we use a complex geometric description of X, their approach uses a description which is independent of choosing a complex structure (which is compatible with the hyperkähler metric) from the start. Proposition 2 was also known to Rozansky and Witten, though we shall follow the approach of Kapranov [33], which was inspired by the ideas of Kontsevich [39]. In this approach we pass from forms to cohomology classes: the Riemann curvature tensor is replaced by the Atiyah class [1], which is represented by the curvature in Dolbeault cohomology. We then work within the framework of complex algebraic geometry. As a cohomology class the Atiyah class can be described in many ways, most of which do not require us to choose a hyperkähler metric. This is useful for calculations as usually the existence of a hyperkähler metric is deduced from Yau's theorem [52] and there is no explicit knowledge of what it looks like.

Since the weights satisfy Proposition 2 it makes sense to call them *Rozansky-Witten invariants of hyperkähler manifolds*, and we adopt this terminology, though strictly speaking the Rozansky-Witten invariants are the three-manifold invariants constructed from these weights.

Graph homology is the space of linear combinations of oriented trivalent graphs modulo the IHX and AS (anti-symmetry) relations. The AS relations say that reversing the orientation of a graph gives us minus the original graph. The IHX relations say that three graphs which are identical except for small regions which look like I, H, and X respectively, are related by



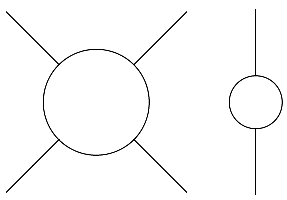
Graph homology is graded by degree, which is given by half the number of vertices of a graph. Proposition 3 says that the weights $b_\Gamma(X)$ descend to graph homology. In the "perturbative" expansion of the partition function it is the fact that the weights satisfy the IHX relations that ensures we get a topological invariant of the three-manifold. This does not rely at all on Proposition 2; indeed if the weights were not metric independent we would simply need to choose a hyperkähler manifold with a specified metric on it and this would still lead to a three-manifold invariant.

On the other hand, the fact that the weights are hyperkähler manifold invariants does not depend at all on Proposition 3. From this point of view the IHX relations are really just a fancy way of writing integration by parts. However, the power of this formalism should not be under-estimated; whenever two graphs are homologous their corresponding Rozansky-Witten invariants are identical.

It also follows from Proposition 3 that hyperkähler manifolds give rise to elements of the dual of graph homology, ie. they can be thought of as elements of graph cohomology. Conversely, we can regard elements of graph cohomology as being *virtual* hyperkähler manifolds. We check that this idea makes sense in Subsection 2.3, before making use of it in Chapter 3 by associating virtual manifolds to the $\mathfrak{su}(2)$ weight system arising in perturbative SU(2) Chern-Simons theory.

Many of the techniques we will use to evaluate the Rozansky-Witten invariants will rely on the hyperkähler manifold being irreducible. Proposition 4 allows us to extend these calculations to reducible hyperkähler manifolds.

Fundamental to most of our calculations is the fact that for certain choices of trivalent graphs Γ we get characteristic numbers of X. More precisely, we take Γ to be a *polywheel*, which is defined to be the disjoint union of a collection of wheels

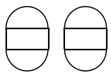


closed by summing over all ways of joining the spokes pairwise. Then the corresponding Rozansky-Witten invariant $b_\Gamma(X)$ is a Chern number (Proposition 5 in Subsection 2.2), and all characteristic numbers arise in this way. Hence the Rozansky-Witten invariants can be thought of as generalized characteristic numbers. The first question which springs to mind is: how much more general are they, if at all? The corresponding question in graph homology would be: how many graphs are there, if any, in the complement of the subspace spanned by polywheels? In degree four and higher this complementary subspace is always non-empty, and one might expect that the Rozansky-Witten invariants corresponding to these graphs would be more general than characteristic numbers, but this is by no means automatic.

We will return to this question shortly. First let us mention that it also pays to consider graphs which lie in the polywheel subspace. The Rozansky-Witten invariants corresponding to these graphs must necessarily be characteristic numbers, though this may not be at all obvious from their structure. Let Θ be the unique trivalent graph with two vertices (which looks just like theta); then the most important example for us is the graph Θ^k consisting of k disjoint copies of Θ. Using the Wheeling Theorem [8] we can show that Θ^k lies in the polywheel subspace for all k (Proposition 13 in Subsection 4.2). On the other hand, the Rozansky-Witten invariant corresponding to Θ^k can be expressed in terms of the $\mathcal{L}^2$-norm of the curvature and the volume of our hyperkähler manifold (Proposition 6 in Subsection 2.4). So our theory says that this expression must be a characteristic number, which is a surprising result. We can be more precise than this: it is given by the multiplicative polynomial $\hat{A}^{1/2}$, where $\hat{A}$ is the multiplicative polynomial corresponding to the $\hat{A}$-genus (Theorem 15 in Subsection 5.1). Using the positivity of the norm, we can also deduce some universal inequalities for the characteristic numbers of hyperkähler manifolds (Corollary 16 in Subsection 5.1).

We can evaluate the Rozansky-Witten invariants for those graphs lying in the polywheel subspace by writing then in terms of Chern numbers. There are two main families of compact hyperkähler manifolds, due to Beauville [10]: the Hilbert schemes of points on a K3 surface and the generalized Kummer varieties. For these examples, we begin with some limited information about the Chern numbers which we obtain using the Riemann-Roch formula and the known values of the Hirzebruch χ_y-genus. Combining this with some direct reasoning allows us to calculate the Rozansky-Witten invariants corresponding to Θ^k for all k (Propositions 19 and 21 in Chapter 5). This in turn helps us to evaluate more Chern numbers, and hence more Rozansky-Witten invariants, leading to complete tables of these values up to degree four, ie. real-dimension sixteen (see Appendices D and E.1). We can also compute the values of the Rozansky-Witten invariants in low degrees for products of these manifolds, by virtue of Proposition 4.

Armed with these calculations, we can now return to the fundamental question posed above. To show that a given Rozansky-Witten invariant is not a characteristic number all we need to do is exhibit two hyperkähler manifolds with the same Chern numbers which are distinguished by the invariant. We can allow the more general situation of disconnected manifolds since the invariants are additive on disjoint sums. In degree four (real-dimension sixteen) we construct two such manifolds by taking disjoint sums of the examples mentioned above. This shows that the Rozansky-Witten invariant corresponding to the disconnected graph



is not a characteristic number (Theorem 22 in Subsection 5.5). The interesting thing about this result is that all our calculations were based upon knowledge of the characteristic numbers. In particular, if the hyperkähler manifold X is reducible then the invariant can be written in terms of the characteristic numbers of the irreducible factors of X, and if X is irreducible then it can be expressed as a *rational* function of the characteristic numbers of X. There exists a large family of graphs whose Rozansky-Witten invariants have this property, ie. are given by a rational function of characteristic numbers on irreducible manifolds (Theorem 17 in Subsection 5.1). Indeed, together with the polywheels, these graphs span graph homology up to (and including) degree five. On the other hand, there are certainly higher degree graphs which are neither in the polywheel subspace nor in this other family of graphs. The precise nature of their Rozansky-Witten invariants, and how they are related to the characteristic numbers (if at all) is still a mystery.

The Rozansky-Witten theory shares many of the properties of Chern-Simons theory. In the "perturbative" models, the weights $b_\Gamma(X)$ constructed from a hyperkähler manifold X are analogous to those constructed from a semi-simple Lie algebra in perturbative Chern-Simons theory. We can extend this analogy to weights on chord diagrams D, which can roughly be described as unitrivalent graphs whose univalent vertices lie on a collection of oriented circles. The construction requires some additional information, which in the Lie algebra case is given by attaching a finite-dimensional representation to each oriented circle, and in the Rozansky-Witten case by attaching to each circle a holomorphic vector bundle E_a over our hyperkähler manifold. This gives us weights $b_D(X; E_a)$ which in the simplest case (no vector bundles) reduce to the Rozansky-Witten invariants $b_\Gamma(X)$.

The weights $b_D(X; E_a)$ satisfy many properties similar to those of $b_\Gamma(X)$. For example, we can show (Propositions 23, 24, and 25 respectively in Chapter 6):

- they do not depend on the choice of Hermitian structures (part of the construction) on the vector bundles E_a,
- the dependence on D is only through its equivalence class in the space of chord diagrams,
- certain chord diagrams give rise to weights which are integrals of the Chern classes of E_a.

Proposition 24 implies that these weight systems lead to finite-type invariants of knots and links, just as the weight system $b_\Gamma(X)$ gives us a finite-type three-manifold invariant. Proposition 25 enables us to use the Wheeling Theorem to derive a formula for the integral of the top component of the Mukai vector of a vector bundle in terms of weights corresponding to simple chord diagrams. This generalizes our earlier result which expressed the Rozansky-Witten invariant corresponding to Θ^k as a characteristic number given by the $\hat{A}^{1/2}$-polynomial.

We give here a rough guide to the contents of Chapters 1 to 6. In the first chapter we give our definition of the Rozansky-Witten invariant $b_\Gamma(X)$, as well as reviewing Rozansky and Witten's original definition and Kapranov's cohomological interpretation (all equivalent). These alternative descriptions are useful in proving Propositions 1 to 4, which occupies us for the remainder of this chapter.

In Chapter 2 we prove the fundamental relation between Chern numbers and Rozansky-Witten invariants corresponding to polywheels. We then show that regarding elements of graph cohomology as virtual hyperkähler manifolds makes sense from the point of view of taking products and its effect on characteristic numbers. We also establish an expression for the invariant $b_{\Theta^k}(X)$ in terms of the norm of the curvature and the volume of X.

Chapters 3 and 4 contain the results in graph homology which we will later translate into results on hyperkähler manifold by taking the corresponding Rozansky-Witten invariants. The first of these two chapters involves the use of the $\mathfrak{su}(2)$ weight system which arises in perturbative Chern-Simons theory with gauge group $\mathrm{SU}(2)$; in the second we introduce some ideas and results from the theory of finite-type invariants of knots. Together with some direct fooling around with graph homology, these give us all the relations we need.

Chapter 5 begins by reinterpreting the graph homology relations of the previous two chapters in terms of Rozansky-Witten invariants. This gives us some general results which we combine with some direct reasoning to calculate the Rozansky-Witten invariants for some specific examples of compact hyperkähler manifolds, namely the Hilbert schemes of points on a K3 surface and the generalized Kummer varieties.

In the final chapter we introduce the weight systems $b_D(X; E_a)$ constructed from collections of holomorphic vector bundles E_a over a hyperkähler manifold, and prove that they satisfy the properties outlined above.

Appendices A to E contain lists of graph homology relations and tables of Chern numbers and Rozansky-Witten invariants, as well as tables of other relations and data required in the calculations of Chapter 5.

Remarks

We will make a few general comments here to save from repeating them many times throughout the text. Firstly, in the topological sigma-model introduced by Rozansky and Witten the hyperkähler manifold X is taken to be either compact or asymptotically flat. Certain non-compact examples are believed to give special three-manifold invariants, in particular the generalized Casson-Walker invariants. However, in the asymptotically flat case one still has to be careful with the decay conditions on the curvature in order for the integrals to converge. Instead we choose to avoid these difficulties by working exclusively with compact hyperkähler manifolds.

We wish to use the techniques of complex geometry but there is no natural choice of complex structure on a hyperkähler manifold, which has an entire S^2 family of compatible complex structures. So we pick one at random. We have already mentioned Proposition 1, which says that the Rozansky-Witten invariants are independent of this choice. Characteristic classes also play an important role in this theory. On a complex manifold we can define Chern classes, though in the hyperkähler case the odd Chern classes vanish and the even ones can be related to the Pontryagin classes. Therefore the Chern numbers are equivalent to the Pontryagin numbers. So although we will tend to write expressions in terms of Chern numbers, one should remember that these numbers are really topological invariants and hence independent of the complex structure. For example, the Todd genus is really the $\hat{A}$-genus, if we wished to write it in a manifestly complex structure independent form.

Finally, after we introduce graph homology we abuse notation by simply using the notation for a graph to denote the graph homology class it represents. Indeed, we will even use the terminology "graph" when referring to its class. Thus when we equate two different graphs we mean that they are homologous, ie. they represent the same graph homology class. We expect this should not cause any confusion.

1 Definitions and basic properties

1.1 Topological sigma-models

In [44] Rozansky and Witten introduced a 3-dimensional topological sigma-model whose target space is a (compact or asymptotically flat) hyperkähler manifold X. The partition function of this theory is a finite-type invariant of the three-manifold M. A Feynman diagram calculation shows that it can be written in the form

$$\sum_{\Gamma} b_{\Gamma}(X) I_{\Gamma}^{\mathrm{RW}}(M),$$

where we sum over all trivalent graphs Γ, the *weights* $b_{\Gamma}(X)$ depend on the hyperkähler manifold, and the terms $I_{\Gamma}^{\mathrm{RW}}(M)$ depend on the three-manifold. This is not really a perturbative expansion as for a given hyperkähler manifold X of real-dimension $4k$ the weights $b_{\Gamma}(X)$ vanish except when Γ has $2k$ vertices, ie. we only get degree k terms. There is evidence to suggest that $I_{\Gamma}^{\mathrm{RW}}(M)$ is the LMO invariant of Le, Murakami, and Ohtsuki [40]. More precisely, if we write the LMO invariant of our three-manifold M as

$$Z^{\mathrm{LMO}}(M) = \sum_{\Gamma} I_{\Gamma}^{\mathrm{LMO}}(M)\Gamma$$

then it is believed that the $I_{\Gamma}^{\mathrm{RW}}(M)$ agree with the coefficients $I_{\Gamma}^{\mathrm{LMO}}(M)$. This is shown by Habegger and Thompson [28] at a "physical level of rigour" when the first Betti number of M is greater than zero. The LMO invariant takes values in the space of graph homology, the space of linear combinations of oriented trivalent graphs moduli the IHX relations and anti-symmetry. An element of graph cohomology gives a weight system on graph homology, ie. a weight system on trivalent graphs compatible with the IHX relations and anti-symmetry. In general, composing the LMO invariant with a weight system gives us a (scalar-valued) finite-type invariant. Although this statement is true for all three-manifolds, only for rational homology spheres can we say that all finite-type invariants arise in this way, as the LMO invariant is only universal in this special case.

The most well-known weight systems are those arising from quadratic Lie algebras and super-algebras, which includes all semi-simple Lie algebras. Some of these weight systems arise naturally in perturbative Chern-Simons theory, first studied by Axelrod and Singer [2, 3] (see also Bar-Natan [5] and Kontsevich [37]). In fact, there is a close analogy with the Rozansky-Witten theory. Expanding around the trivial connection we get a finite-type invariant of three-manifolds which looks like

$$\sum_{\Gamma} c_{\Gamma}(\mathfrak{g}) I_{\Gamma}^{\mathrm{CS}}(M),$$

where now the weight system $c_{\Gamma}(\mathfrak{g})$ depends on the Lie algebra $\mathfrak{g}$ of the gauge group, and the terms $I_{\Gamma}^{\mathrm{CS}}(M)$ depend on the three-manifold. The latter have a configuration space interpretation (see Bott and Taubes [17]), and when M is a rational homology sphere they are believed to coincide with the coefficients of the Aarhus integral of Bar-Natan, Garoufalidis, Rozansky, and Thurston [6]. Thus we again recover the LMO invariant, as the Aarhus integral is an alternative definition in this case.

These perturbative expansions of the partition functions are perhaps only true at a "physical level of rigour". We can also say that the theory of the three-manifold terms $I_\Gamma(M)$ is not completely understood. However, we can make precise mathematical sense of the weight systems $b_\Gamma(X)$ and $c_\Gamma(\mathfrak{g})$. The main object of our study will be the weights $b_\Gamma(X)$ arising from a (compact) hyperkähler manifold X, and we investigate their basic properties in this chapter.

1.2 Hyperkähler geometry

Let X^{4k} be a compact hyperkähler manifold of real-dimension $4k$. This means that X has holonomy contained in $\mathrm{Sp}(k)$ (with holonomy equal to $\mathrm{Sp}(k)$ if and only if X is irreducible). Then X has the following structures:

- complex structures I, J, and K, acting like the quaternions on the tangent bundle $T = TX$,
- a metric g which is Kählerian with respect to each of I, J, and K,
- corresponding Kähler forms ω_1, ω_2, and ω_3, which are d-closed skew-symmetric two-forms.

More generally, if a, b, and c are real numbers satisfying $a^2 + b^2 + c^2 = 1$ then $aI + bJ + cK$ gives a complex structure on X with respect to which g is Kählerian, and hence we obtain a two-sphere of compatible complex structures.

Fix a specific complex structure I, and regard X as a complex manifold. Then $\omega = \omega_2 + i\omega_3$ is a complex symplectic form which is holomorphic with respect to I. Let T and T^* be the complex tangent and cotangent bundles respectively. We can use ω to identify them, and thus $T \cong T^*$.

Take the Levi-Civita connection associated to the metric g. The Riemann curvature tensor of this connection is a section

$$K \in \Omega^{1,1}(X, \mathrm{End}T) = \Omega^{0,1}(X, T^* \otimes \mathrm{End}T)$$

with components $K^i{}_{jk\bar{l}}$ relative to local complex coordinates (with respect to the complex structure I), where the ij indices refer to $\mathrm{End}T$ and the $k\bar{l}$ indices refer to $\Omega^{1,1}$. Using ω to identify T and T^*, we get a section

$$\Phi \in \Omega^{0,1}(X, T^* \otimes T^* \otimes T^*).$$

defined by

$$\Phi_{ijk\bar{l}} = \sum_m \omega_{im} K^m_{jk\bar{l}}.$$

Since X is hyperkähler, there is an $\mathrm{Sp}(2k,\mathbb{C})$ reduction of the frame bundle. Therefore the curvature takes values in the Lie algebra of $\mathrm{Sp}(2k,\mathbb{C})$ consisting of matrices of the form A^k_j such that $S_{ij}=\sum_k \omega_{ik}A^k_j$ is symmetric. In other words, $\Phi_{ijk\bar{l}}$ is symmetric in ij. It is also symmetric in jk as the Levi-Civita connection is torsion-free and preserves the complex structure. Thus

$$\Phi \in \Omega^{0,1}(X, \mathrm{Sym}^3 T^*).$$

This will be one of the main ingredients in the construction of the weights $b_\Gamma(X)$. The others will be the holomorphic symplectic form

$$\omega \in \mathrm{H}^0(X, \Lambda^2 T^*)$$

and its dual

$$\tilde{\omega} \in \mathrm{H}^0(X, \Lambda^2 T).$$

The latter is the skew form on T^* dual to ω. Note that if ω is represented by the matrix ω_{ij} relative to local complex coordinates, then in a dual basis the matrix ω^{ij} of $\tilde{\omega}$ is minus the inverse of ω_{ij}.

1.3 The Rozansky-Witten invariants

Let Γ be a (possibly disconnected) oriented trivalent graph with $2k$ vertices. The orientation is an equivalence class of orientations on the edges and an ordering of the vertices; if the orderings differ by a permutation π and n edges are oriented in the reverse manner, then we regard these as equivalent if $\mathrm{sgn}\pi=(-1)^n$. This definition differs from the usual notion of orientation on trivalent graphs, which is an equivalence class of cyclic orderings of the outgoing edges at each vertex, with two orderings being equivalent if they differ on an even number of vertices. However, we shall see later that these two definitions are in fact equivalent.

Let the vertices of Γ be $v_1,\ldots,v_{2k}$. Place a copy of Φ at each vertex and label the outgoing edges with the holomorphic indices of Φ. For example, at v_t we label the outgoing edges by i_t, j_t, and k_t. We can do this in any order, as $\Phi_{i_tj_tk_t\bar{l}_t}$ is symmetric in these indices. We then contract along each edge using the dual holomorphic symplectic form, $\tilde{\omega}$. Thus if an edge is labelled by i_t at one end and i_s at the other, then we contract with $\omega^{i_ti_s}$ if the edge is oriented from v_t to v_s, or with $\omega^{i_si_t}$ if it has the opposite orientation. This gives us a section of $(\bar{T}^*)^{\otimes 2k}$ with components

$$\Phi_{i_1j_1k_1\bar{l}_1}\cdots\Phi_{i_{2k}j_{2k}k_{2k}\bar{l}_{2k}}\omega^{i_1*}\omega^{j_1*}\omega^{k_1*}\cdots$$

where summation over repeated indices is assumed. Projecting this to the exterior product gives us an element

$$\Gamma(\Phi)\in\Omega^{0,2k}(X).$$

We multiply this form by the kth power of the holomorphic symplectic form ω^k in $\Omega^{2k,0}(X)$. This gives us an element in $\Omega^{2k,2k}(X)$ which we can integrate over the manifold.

Definition *The Rozansky-Witten invariant of the hyperkähler manifold X corresponding to the trivalent graph Γ is*

$$b_\Gamma(X) = \frac{1}{(8\pi^2)^k k!}\int_X \Gamma(\Phi)\omega^k.$$

Strictly speaking this is the weight corresponding to Γ occurring in the "perturbative" expansion of the Rozansky-Witten invariant of three-manifolds, but we shall adopt the above terminology. Roughly speaking, $b_\Gamma(X)$ is given by taking a product of curvature tensors, contracting according to the particular trivalent graph Γ, and then integrating over the manifold. We have chosen the additional factor for several reasons which will become evident in due course. For the time being, let us just state that dividing by π^{2k} makes the invariants integral for the examples of hyperkähler manifolds we will study, the $k!$ factor ensures that the invariants behave nicely with respect to products of manifolds, and the 8^k factor allows us to make simple comparisons with characteristic numbers. The overall normalization seems to give integral invariants on compact hyperkähler manifolds; at least this is true for the examples we will look at. In general, the Rozansky-Witten invariants of three-manifolds satisfy some integrality properties, but this still leaves some freedom in choosing the normalization of the weights. Apart from being the 'right one' for the various reasons outlined above, our choice also agrees with the one made by Rozansky and Witten [44].

In the rest of this chapter we shall look at some of the basic properties of $b_\Gamma(X)$; in particular, we shall prove the following results (which were at least known to Rozansky and Witten, even if they were not explicitly proved in [44]):

- it is independent of the choice of complex structure on X (ie. there is nothing special about using I in our definition),
- it is constant on connected components of the moduli space of hyperkähler metrics on X (we shall follow Kapranov's approach [33] in order to prove this),
- the dependence on the graph Γ is only through its graph homology class,
- products of hyperkähler manifolds correspond to coproducts in graph homology (which has the structure of a commutative cocommutative Hopf algebra).

1.4 The physicist's definition

The original 'physics' definition of $b_\Gamma(X)$ as given in [44] is slightly different to the one given above. Indeed, our definition is the same as that given in the Appendix of [44] for complex symplectic manifolds X. These generalize hyperkähler manifolds, and of course the definition reduces to one equivalent to the original definition in the case that X is hyperkähler. From Rozansky and Witten's original definition, however, it will be clear that there is no dependence on the complex structure, so we review it here.

The holonomy of the Levi-Civita connection of an arbitrary Riemannian metric on X lies in $\mathrm{SO}(4k)$, but for a hyperkähler metric it lies in an $\mathrm{Sp}(k)$ subgroup. Let TX be the (real) tangent bundle of X. Then it is known (see Salamon's book [45]) that the complexified tangent bundle of X decomposes into the tensor product of a rank $2k$ complex vector bundle V with structure group $\mathrm{Sp}(k)$ and a trivializable rank 2 complex vector bundle S with structure group $\mathrm{Sp}(1)$,

$$TX \otimes_{\mathbb{R}} \mathbb{C} = V \otimes S$$

Using indices $I, J, \ldots \in \{1, \ldots, 2k\}$ on V and $A, B, \ldots \in \{1, 2\}$ on S, we note that both bundles possess non-degenerate skew-symmetric two-forms ϵ_{IJ} and ϵ_{AB} with inverses ϵ^{IJ} and ϵ^{AB} respectively. If $i, j, \ldots$ denote indices on TX coming from local real coordinates on X, then we use the covariantly constant tensors γ^i_{IA} and γ^{IA}_i to describe the maps

$$\gamma^i_{IA} : TX \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes S$$

and

$$\gamma^{IA}_i : V \otimes S \to TX \otimes_{\mathbb{R}} \mathbb{C}.$$

The Levi-Civita connection on $TX \otimes_{\mathbb{R}} \mathbb{C}$ reduces to an $\mathrm{Sp}(k)$ connection on V tensored with the trivial connection on S. The Riemann curvature tensor can be written

$$R_{ijkl} = -\gamma^{IA}_i \gamma^{JB}_j \gamma^{CK}_k \gamma^{DL}_l \epsilon_{AB} \epsilon_{CD} \Omega_{IJKL}$$

and after some manipulations involving the symmetries of R_{ijkl} we find that Ω_{IJKL} is completely symmetric (see Proposition 9.3 in Salamon [45]). Rozansky and Witten [44] essentially define $b_\Gamma(X)$ using the bundle V instead of TX. Specifically, they use Ω_{IJKL} and ϵ^{IJ} instead of $\Phi_{ijk\bar{l}}$ and ω^{ij}.

A choice of complex structure on X compatible with the hyperkähler structure corresponds to a choice of trivialization of the bundle S. This allows us to identify TX (which is now a complex vector bundle with the chosen complex structure) and V, and shows that our approach in Subsection 1.3 gives the same result as in Rozansky and Witten's original definition. Furthermore, in the latter we do not assume any particular trivialization of S, which means we do not need to specify a complex structure, and hence

Proposition 1 *The invariant $b_\Gamma(X)$ is independent of the choice of compatible complex structure on X.*

From Rozansky and Witten's original definition we can also show that $b_\Gamma(X)$ is a real number. This requires us to make use of the quaternionic structure on the bundle V.

1.5 Kapranov's approach

A third (equivalent) definition due to Kapranov [33] enables us to prove metric independence. The idea behind his construction is to work with cohomology classes instead of differential forms, and leads to a description which does not rely on specific knowledge of the hyperkähler metric. In most examples, the existence of a hyperkähler metric follows from Yau's theorem [52], but no explicit description is known. Kapranov's approach enables us to avoid this obstacle and leads to a method of calculation on specific examples of hyperkähler manifolds.

The form
$$\Phi \in \Omega^{0,1}(X, \mathrm{Sym}^3 T^*)$$
is $\bar{\partial}$-closed by the Bianchi identity for the Riemannian curvature. Therefore it represents a Dolbeault cohomology class
$$[\Phi] \in \mathrm{H}^{0,1}_{\bar{\partial}}(X, \mathrm{Sym}^3 T^*).$$
This is actually the Atiyah class α_T of the tangent bundle of X (see [1]). We shall say more about this shortly, but first we complete the construction. As before we construct
$$[\Gamma(\Phi)] \in \mathrm{H}^{0,2k}_{\bar{\partial}}(X)$$
which only depends on the cohomology class of Φ. Now $\mathrm{H}^{0,2k}_{\bar{\partial}}(X)$ is one-dimensional and generated by $[\bar{\omega}^k]$. Multiplying by $[\omega^k]$ as before gives an element of $\mathrm{H}^{2k,2k}_{\bar{\partial}}(X)$ which we can integrate to get
$$b_\Gamma(X) = \frac{1}{(8\pi^2)^k k!} \int_X [\Gamma(\Phi)][\omega^k].$$
Here, and throughout, the integral of a top degree Dolbeault cohomology class over the manifold means that we integrate some form which represents the class. This definition of $b_\Gamma(X)$ is clearly no different to before, since we can use the same forms as before to represent the Dolbeault cohomology classes. However, since Φ represents the Atiyah class we can use other representations of α_T in order to get a quite different looking description.

In general, the Atiyah class of a holomorphic vector bundle E over X is the obstruction to the existence of a global holomorphic connection on E (see Atiyah [1]). Suppose that we have an open set $U \in X$ over which E has a holomorphic connection ∇_U. We can think of ∇_U as a map from E to the first jet bundle $J_1(E)$ which gives the identity when composed with the projection $J_1(E) \to E$. In other words, it is a splitting of the short exact sequence
$$0 \to E \otimes T^* \to J_1(E) \to E \to 0.$$
Tensoring with E^* we get
$$0 \to \mathrm{Hom}(E, E \otimes T^*) \to \mathrm{Hom}(E, J_1(E)) \to \mathrm{Hom}(E, E) \to 0$$

and ∇_U is a section of $\mathrm{Hom}(E, J_1(E))$ over U which maps to Id in $\mathrm{Hom}(E, E)$. Consider the corresponding long exact sequence

$$0 \to \mathrm{H}^0(X, \mathrm{Hom}(E, E \otimes T^*)) \to \mathrm{H}^0(X, \mathrm{Hom}(E, J_1(E)))$$
$$\to \mathrm{H}^0(X, \mathrm{Hom}(E, E)) \to \mathrm{H}^1(X, \mathrm{Hom}(E, E \otimes T^*)) \to \dots$$

A global connection ∇ on E is a section in $\mathrm{H}^0(X, \mathrm{Hom}(E, J_1(E)))$ which maps to Id in $\mathrm{H}^0(X, \mathrm{Hom}(E, E))$. The image of Id

$$\alpha_E \in \mathrm{H}^1(X, \mathrm{Hom}(E, E \otimes T^*)) = \mathrm{H}^1(X, T^* \otimes \mathrm{End} E)$$

is the obstruction to the existence of such a ∇. Clearly if $\alpha_E \neq 0$ then ∇ cannot exist by exactness, whereas if α_E vanishes, then Id is the image of some ∇. We call α_E the Atiyah class of E, and it can be described in the following ways:

- Take an open cover $\{U_i\}$ of X and choose local holomorphic connections ∇_i on $E|_{U_i}$ over each open set U_i which look like $d + A_i$ in local holomorphic coordinates, where $A_i \in \Omega^{1,0}(U_i, \mathrm{End} E)$. Then

  $$\nabla_i - \nabla_j = A_i - A_j \in \Omega^{1,0}(U_i \cap U_j, \mathrm{End} E) = \mathrm{H}^0(U_i \cap U_j, T^* \otimes \mathrm{End} E)$$

  gives a Čech representative of α_E.

- Take a smooth global connection ∇ of type $(1, 0)$ on E. Then the $(1, 1)$ part of the curvature of ∇ gives a Dolbeault representative of α_E. If ∇ looks like $d+A$ over the open set U, with $A \in \Omega^{1,0}(U, \mathrm{End} E)$, then locally the $(1, 1)$ part of the curvature of ∇ will be $\bar{\partial} A$. Note that if E is equipped with an Hermitian metric h then the curvature of the compatible h-connection is automatically of type $(1, 1)$.

  In order to construct a smooth global connection of type $(1, 0)$, we could patch together local holomorphic connections ∇_i (as above) using a partition of unity $\{\psi_i\}$. The terms $\psi_i \nabla_i$ are only differential operators, but when we add them we get

  $$\nabla = \sum \psi_i \nabla_i$$

  which since $\sum \psi_i \equiv 1$ is a connection (as can be seen from looking at its symbol). Relative to a local trivialization ∇ looks like $d + A$ with

  $$A = \sum \psi_i A_i.$$

  Since the A_i are holomorphic, the $(1, 1)$ part of the curvature is

  $$\sum (\bar{\partial} \psi_i) A_i.$$

- Under the right conditions, we can also represent α_E as the residue of a meromorphic connection ∇ on E. We require ∇ to have a simple pole along a smooth (not necessarily connected) divisor D. Let $L = \mathcal{O}(D)$ be the line bundle corresponding to D and s the canonical section, which vanishes along D. A meromorphic section of a bundle with a simple pole along D is the same as a holomorphic section of the same bundle twisted by L; the corresponding map is given by multiplying by s. For example, the smooth differential operator $s\nabla$ on X is a (non-singular) global holomorphic section of $\mathrm{Hom}(E, J_1(E)) \otimes L$.

  In general, if we have a bundle F on X, then we get the short exact sequence

$$F \xrightarrow{s} F \otimes L \to F \otimes L|_D \to 0$$

  where the first map is given by multiplying by the section s and the second map by restricting to the divisor D (the zero set of s). The corresponding long exact sequence is

$$\mathrm{H}^0(X, F) \xrightarrow{s} \mathrm{H}^0(X, F \otimes L) \to \mathrm{H}^0(D, F \otimes L) \xrightarrow{\delta} \mathrm{H}^1(X, F) \to \dots$$

  The second group in this sequence is the space of meromorphic sections of F with simple poles along D, and restricting to D gives us the residue of the section. Let F_1, F_2, and F_3 be the bundles $\mathrm{Hom}(E, E \otimes T^*)$, $\mathrm{Hom}(E, J_1(E))$, and $\mathrm{Hom}(E, E)$ respectively. Then we get the following commutative double long exact sequence.

$$\begin{array}{ccccccccc}
\mathrm{H}^0(X, F_1) & \to & \mathrm{H}^0(X, F_1 \otimes L) & \to & \mathrm{H}^0(D, F_1 \otimes L) & \to & \mathrm{H}^1(X, F_1) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}^0(X, F_2) & \to & \mathrm{H}^0(X, F_2 \otimes L) & \to & \mathrm{H}^0(D, F_2 \otimes L) & \to & \mathrm{H}^1(X, F_2) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}^0(X, F_3) & \to & \mathrm{H}^0(X, F_3 \otimes L) & \to & \mathrm{H}^0(D, F_3 \otimes L) & \to & \mathrm{H}^1(X, F_3) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}^1(X, F_1) & \to & \mathrm{H}^1(X, F_1 \otimes L) & \to & \mathrm{H}^1(D, F_1 \otimes L) & \to & \mathrm{H}^2(X, F_1) & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

  Now $s\nabla$ in $\mathrm{H}^0(X, F_2 \otimes L)$ maps down to $s\mathrm{Id}$ in $\mathrm{H}^0(X, F_3 \otimes L)$, which in turn is the horizontal image of Id in $\mathrm{H}^0(X, F_3)$, which maps down to the Atiyah class α_E in $\mathrm{H}^1(X, F_1)$. In the other direction, since $s\mathrm{Id}$ must map to zero horizontally, it follows that the horizontal image of $s\nabla$, namely $s\nabla|_D$ in $\mathrm{H}^0(D, F_2 \otimes L)$, must map down to zero. Then by exactness, $s\nabla|_D$ must be the vertical image of some element β_E in $\mathrm{H}^0(D, F_1 \otimes L)$. Finally, the horizontal image of β_E, which lies in $\mathrm{H}^1(X, F_1)$, must again be the Atiyah class α_E. In summary, we have the following pattern of maps and elements.

$$\begin{array}{ccccccc}
 & & & & \beta_E & \xrightarrow{\delta} & \alpha_E & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & s\nabla & \to & s\nabla|_D & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
\mathrm{Id} & \xrightarrow{s} & s\mathrm{Id} & \to & 0 & & & & \\
\downarrow & & \downarrow & & & & & & \\
\alpha_E & \to & 0 & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

The point here is that β_E in $\mathrm{H}^0(D, T^* \otimes \mathrm{End}E \otimes L)$ is the residue of the meromorphic connection ∇, and α_E is completely determined by this section over the divisor D, as $\delta\beta_E = \alpha_E$.

Using any of these approaches on the tangent bundle T gives us

$$\alpha_T \in \mathrm{H}^1(X, T^* \otimes \mathrm{End}T).$$

We can use the holomorphic symplectic form to identify T and T^*, so that

$$\alpha_T \in \mathrm{H}^1(X, T^* \otimes T^* \otimes T^*).$$

In fact we get

$$\alpha_T \in \mathrm{H}^1(X, \mathrm{Sym}^3 T^*)$$

because as before, we have an $\mathrm{Sp}(2k, \mathbb{C})$ reduction of the frame bundle and the (local holomorphic or smooth global) connections can be chosen to be torsion-free and complex structure preserving. Taking the product of $2k$ copies of α_T, one for each vertex of the trivalent graph Γ, and using the graph to contract indices with $\tilde{\omega}$, as before, we get

$$\Gamma(\alpha_T) \in \mathrm{H}^{2k}(X, \mathcal{O}_X).$$

This cohomology space is one-dimensional, but we need to be careful as there is no natural basis. Instead, to produce a number in a canonical way we can pair $\Gamma(\alpha_T)$ with

$$\omega^k \in \mathrm{H}^0(X, \Lambda^{2k} T^*)$$

using Serre duality. Up to the factor $\frac{1}{(8\pi^2)^k k!}$ this is precisely $b_\Gamma(X)$.

Using the meromorphic connection approach, we can also write down a residue formula for $b_\Gamma(X)$. Firstly let us assume that D is connected. Since β_T maps to α_T, it can be taken to have the same symmetries, ie.

$$\beta_T \in \mathrm{H}^0(D, \mathrm{Sym}^3 T^* \otimes L).$$

Now instead of constructing $\Gamma(\alpha_T)$ as before, we can replace one of the copies of α_T with β_T and restrict the entire construction to D. This gives us

$$\Gamma(\alpha_T, \beta_T) \in \mathrm{H}^{2k-1}(D, L)$$

and the map

$$\delta : \mathrm{H}^{2k-1}(D, L) \to \mathrm{H}^{2k}(X, \mathcal{O}_X)$$

will take $\Gamma(\alpha_T, \beta_T)$ to $\Gamma(\alpha_T)$. By the adjunction formula the canonical line bundle of D is

$$\begin{aligned} \mathcal{K}_D &= \mathcal{K}_X \otimes \mathcal{O}_X(D)|_D \\ &= \mathcal{K}_X \otimes L|_D \end{aligned}$$

where $\mathcal{K}_X$ is the canonical line bundle of X, and therefore

$$\Gamma(\alpha_T, \beta_T) \in \mathrm{H}^{2k-1}(D, \mathcal{K}_D \otimes \mathcal{K}_X^*|_D).$$

Now as X is a Calabi-Yau manifold, the bundle $\mathcal{K}_X = \Lambda^{2k}T^*$ must be trivial, indeed ω^k is a non-vanishing section. Therefore $\Gamma(\alpha_T, \beta_T)$ lies in a one-dimensional cohomology group

$$\mathrm{H}^{2k-1}(D, \mathcal{K}_D \otimes \mathcal{K}_X^*|_D) \cong \mathrm{H}^{2k-1}(D, \mathcal{K}_D)$$

but again we must be careful as there is no natural isomorphism between these two spaces. To obtain a number in a canonical way we must pair $\Gamma(\alpha_T, \beta_T)$ with

$$\omega^k|_D \in \mathrm{H}^0(D, \mathcal{K}_X|_D)$$

using Serre duality. We claim that this gives the same result as the Serre duality pairing of $\Gamma(\alpha_T)$ with ω^k, up to a factor of $2\pi i$. This is essentially just Cauchy's residue formula, but we shall be a little more precise.

Consider the Poincaré residue map, which takes sections of $\mathcal{K}_X \otimes L$ to sections of $\mathcal{K}_D$. Suppose that D is given in local coordinates $z_1, \ldots, z_{2k}$ by $f(z) = 0$. Write a section of $\mathcal{K}_X \otimes L$ (ie. a meromorphic section of $\mathcal{K}_X$) as

$$\nu = \frac{g(z)dz_1 \wedge \ldots \wedge dz_{2k}}{f(z)} = \frac{df}{f} \wedge \nu'$$

then the map to sections of $\mathcal{K}_D$ takes ν to ν'. By the adjunction formula, this map gives an isomorphism for sections over D. Now consider

$$\Gamma(\alpha_T, \beta_T)\omega^k|_D \in \mathrm{H}^{2k-1}(D, \mathcal{K}_X \otimes L|_D).$$

Using Čech cohomology, observe that this element can be described as sections of $\mathcal{K}_X \otimes L$ on $2k$-fold intersections of open sets in D. The Poincaré residue map takes these sections isomorphically to sections of $\mathcal{K}_D$, and hence

$$\Gamma(\alpha_T, \beta_T)\omega^k|_D \in \mathrm{H}^{2k-1}(D, \mathcal{K}_D).$$

Under the map δ, this maps to

$$\Gamma(\alpha_T)\omega^k \in \mathrm{H}^{2k}(X, \mathcal{K}_X)$$

and both these cohomology spaces are one-dimensional and can be identified with $\mathbb{C}$ simply by integrating (either over contours in the case of Čech cohomology or over the entire space if we use Dolbeault cohomology). The difference between the two numbers we get from the above elements is precisely a factor of $2\pi i$, as we have 'cancelled' off df/f and the contour integral over this additional variable would give

$$\int_{|f|=1} \frac{df}{f} = 2\pi i$$

by Cauchy's residue formula. Lastly, in the case that D is disconnected we simply need to perform the above calculation on each of the connected components and sum the results. Thus we have another description of the Rozansky-Witten invariant

$$b_\Gamma(X) = \frac{2\pi i}{(8\pi^2)^k k!} \int_D \Gamma(\alpha_T, \beta_T)\omega^k|_D.$$

According to the description above, if locally we write

$$\frac{\omega^k}{f}|_D = \frac{df}{f} \wedge \nu'$$

then the section $\omega^k|_D$ of $\mathcal{K}_X|_D$ should be replaced by the section $f\nu'$ of $\mathcal{K}_D \otimes L^*|_D$ in the above integral.

The beauty of these approaches is that we no longer need to know the hyperkähler metric explicitly in order to perform the construction. In fact, all we need is a compact complex manifold X with a holomorphic symplectic form ω, usually referred to as a complex symplectic space. In the Kähler case X will admit a hyperkähler metric, but there are non-Kähler examples as well, such as Kodaira surfaces [34, 35] (see also Barth, Peters, and Van de Ven [9]) and their Douady spaces (as observed by Beauville [10]). There are also simply-connected examples due to Guan [27] (see Bogomolov [14] for a clearer description of this construction).

Returning to the hyperkähler case, we can use Kapranov's approach to prove the following result.

Proposition 2 *The number $b_\Gamma(X)$ is invariant under a deformation of the hyperkähler metric. In particular, it is constant on connected components of the moduli space of hyperkähler metrics.*

Proof To begin with, consider what happens when we rescale the metric g by a factor λ. The holomorphic symplectic form ω will also rescale by λ, and its dual $\tilde{\omega}$ by λ^{-1}. The Riemann curvature tensor K does not change, but since Φ is defined by contracting with ω it will rescale by λ. Since $\Gamma(\Phi)$ is made from $2k$ copies of

Φ (one for each vertex) and $3k$ copies of $\tilde{\omega}$ (one for each edge) it will rescale by λ^{-k}. Finally, we multiply this by ω^k, which rescales by λ^k and hence the net result is that $b_\Gamma(X)$ is invariant under such rescalings of the metric. This corresponds to a volume-changing deformation of the metric on X. Now consider deformations in the transverse direction.

The hyperkähler metric is determined by the three closed skew-forms ω_1, ω_2, and ω_3 (see Hitchin, Karlhede, Lindström, and Roček [30]). Thus an arbitrary first order deformation of the metric will be given by a linear combination of the three variations $\delta\omega_1$, $\delta\omega_2$, and $\delta\omega_3$, subject to some additional algebraic conditions. Fixing the holomorphic symplectic form $\omega = \omega_2 + i\omega_3$ (with respect to I), we see from Kapranov's approach that $b_\Gamma(X)$ doesn't depend on the Kähler form ω_1, and hence is invariant under the deformation $\delta\omega_1$ (where it is implied that ω_2 and ω_3 are kept fixed as we vary ω_1). Of course, there is nothing special about the complex structure I (as we saw earlier), and thus $b_\Gamma(X)$ must also be invariant under the deformations $\delta\omega_2$ and $\delta\omega_3$, and hence under linear combinations of these deformations.

Note that by fixing two of ω_1, ω_2, and ω_3 and varying the third we do not get the deformation described by rescaling g, but we have already seen above that $b_\Gamma(X)$ is invariant under this volume-changing deformation, and hence $b_\Gamma(X)$ is invariant under arbitrary first-order deformations of the hyperkähler metric. Finally, it follows from the results of Tian [48] and Todorov [49] that the obstruction to deforming a Calabi-Yau manifold vanishes, and hence the moduli spaces of Calabi-Yau manifolds are smooth. In particular, the moduli space of hyperkähler metrics on X is smooth. Indeed it was shown earlier by Bogomolov [16] that the Kuranishi family of moduli of a compact hyperkähler manifold is smooth at the base point. If two hyperkähler metrics can be joined by a path, then integrating the variation of $b_\Gamma(X)$ (which we have shown to be zero) along this path proves that $b_\Gamma(X)$ is constant on connected components of the moduli space. □

1.6 Graph homology

We defined an orientation of Γ as being an equivalence class of orientations on the edges and an ordering of the vertices; if the orderings differ by a permutation π and n edges are oriented in the reverse manner, then we regard these as equivalent if $\mathrm{sgn}\pi = (-1)^n$. When associating $\tilde{\omega}$ to an edge, the orientation of the edge tells us whether to use $\omega^{i_t i_s}$ or $\omega^{i_s i_t} = -\omega^{i_t i_s}$; thus changing the orientation on one edge will reverse the sign of $\Gamma(\Phi)$. The ordering of the vertices tells us in which order to multiply the copies of Φ associated to the vertices; since we later project to the exterior product $\Omega^{0,2k}(X)$, changing the order of the vertices by an odd permutation will also change the sign of $\Gamma(\Phi)$ (whereas changing the order by an even permutation will have no effect). It follows that reversing the orientation of Γ will have the effect of changing the sign of $\Gamma(\Phi)$, and hence

$$b_{\bar{\Gamma}}(X) = -b_\Gamma(X)$$

where $\bar{\Gamma}$ is Γ with the opposite orientation. It is clear that this is the right notion of orientation of trivalent graphs to use when defining the Rozansky-Witten invariants (and hence we shall call it the Rozansky-Witten orientation).

The standard notion of orientation of a trivalent graph is given by an equivalence class of cyclic ordering of the outgoing edges at each vertex, with two orderings being equivalent if they differ on an even number of vertices. Thus a trivalent graph drawn in the plane inherits a canonical orientation given by taking the anticlockwise cyclic ordering at each vertex (whenever we draw a trivalent graph in the plane we shall assume this orientation). We promised to show that these two notions of orientation are equivalent; essentially we follow Kapranov's proof [33].

Given a trivalent graph Γ, let $V(\Gamma)$, $E(\Gamma)$, and $F(\Gamma)$ be the sets of its vertices, edges, and flags respectively (a *flag* being an edge together with a choice of vertex lying on it). For a finite set S, let $\det S$ be the highest exterior power of the vector space $\mathbb{R}^S$ (the real vector space whose basis is given by the elements of S). The standard notion of orientation on Γ is the same as an orientation on the one-dimensional vector space

$$\bigotimes_{v\in V(\Gamma)} \det F(v)$$

where $F(v)$ is the three-element set of flags whose chosen vertex is v. Now $\det F(\Gamma)$ is the highest exterior power of the vector space

$$\bigoplus_{f\in F(\Gamma)} \mathbb{R}^{\{f\}} = \bigoplus_{v\in V(\Gamma)} \mathbb{R}^{F(v)}.$$

The right hand side is a direct sum of three-dimensional spaces parametrized by $V(\Gamma)$. Since three-forms anticommute, this implies

$$\det F(\Gamma) \cong \det V(\Gamma) \otimes \Big(\bigotimes_{v\in V(\Gamma)} \det F(v)\Big).$$

We can also write

$$\bigoplus_{f\in F(\Gamma)} \mathbb{R}^{\{f\}} = \bigoplus_{e\in E(\Gamma)} \mathbb{R}^{F(e)}$$

where $F(e)$ is the two-element set of flags containing the edge e. The right hand side is a direct sum of two-dimensional spaces parametrized by $E(\Gamma)$. However, two-forms commute so

$$\det F(\Gamma) \cong \bigotimes_{e\in E(\Gamma)} \det F(e).$$

The Rozansky-Witten orientation on Γ is given by an equivalence class of orientations on the edges and ordering of the vertices, or in other words an orientation on the one-dimensional vector space

$$\det V(\Gamma) \otimes \Big(\bigotimes_{e\in E(\Gamma)} \det F(e)\Big)$$

since an orientation of the two-dimensional space $\mathbb{R}^{F(e)}$ is clearly the same as an orientation of the edge e. This space is isomorphic to

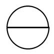
$$\det V(\Gamma) \otimes \det F(\Gamma) \cong \det V(\Gamma)^{\otimes 2} \otimes \Big(\bigotimes_{v \in V(\Gamma)} \det F(v) \Big).$$

Being the square of a real line, the one-dimensional space $\det V(\Gamma)^{\otimes 2}$ has a canonical orientation, and hence the Rozansky-Witten orientation is equivalent to an orientation on

$$\bigotimes_{v \in V(\Gamma)} \det F(v)$$

ie. equivalent to the standard notion of orientation on Γ, completing the proof.

As an example, let Γ be the two-vertex graph



with orientation induced from the planar embedding. Label the flags $1, \ldots, 6$ and the vertices $1, 2$ as in Figure 1. The corresponding orientation of $\det F(\Gamma)$ is given

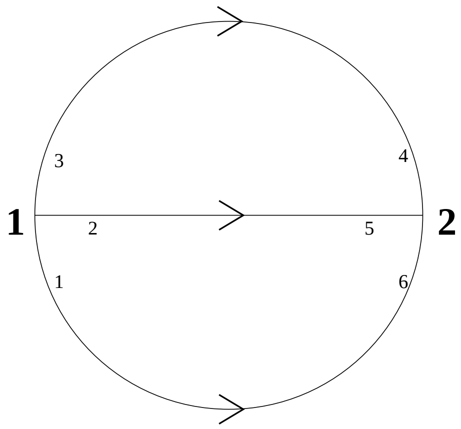


Figure 1: Compatibility of orientations

by the element

$$f_1 \wedge f_2 \wedge f_3 \wedge f_4 \wedge f_5 \wedge f_6$$

where the f_i's are a basis for $\mathbb{R}^{F(\Gamma)}$. We can rewrite this element as

$$f_1 \wedge f_6 \wedge f_2 \wedge f_5 \wedge f_3 \wedge f_4$$

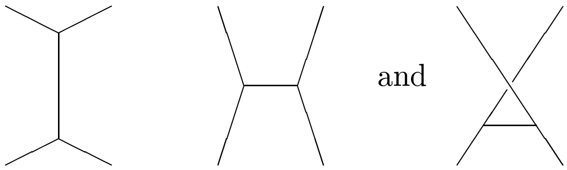
and this shows that the edges should be oriented from 1 to 6, from 2 to 5, and from 3 to 4, as shown. More precisely, the edges should be oriented in a manner *equivalent* to the one shown; for example, the above element could also be rewritten

$$f_1 \wedge f_6 \wedge f_5 \wedge f_2 \wedge f_4 \wedge f_3$$

which would correspond to orienting the edges from 1 to 6, from 5 to 2, and from 4 to 3. This is equivalent as it differs by reversing the orientation on an even number of edges.

We have shown that reversing the orientation of Γ changes the sign of $b_\Gamma(X)$. This is the first step in showing that the Rozansky-Witten invariants descend to graph homology (which we shall define shortly). The next step is to show compatibility with the IHX relation.

Suppose that we are given three trivalent graphs Γ_I, Γ_H and Γ_X which are identical except inside some ball where they look like



respectively. The orientations are induced from the planar embedding, but let us just say that the corresponding Rozansky-Witten orientations are given as follows: if the two vertices in each diagram are labelled t and s then the connecting edge should be oriented from t to s in Γ_I and Γ_X and from s to t in Γ_H. We will show that the Rozansky-Witten invariants corresponding to Γ_I, Γ_H, and Γ_X are related by

$$b_{\Gamma_I}(X) = b_{\Gamma_H}(X) - b_{\Gamma_X}(X).$$

Reversing the orientation on Γ_H, this can be rewritten

$$b_{\Gamma_I}(X) + b_{\bar{\Gamma}_H}(X) + b_{\Gamma_X}(X) = 0$$

where now the orientation of the edge joining the vertices t and s is the same in all three trivalent graphs.

The idea behind the proof is to show that

$$\Gamma_I(\Phi) + \bar{\Gamma}_H(\Phi) + \Gamma_X(\Phi) \in \Omega^{0,2k}(X)$$

is a $\bar{\partial}$-exact form. Thus it is cohomologous to zero in Dolbeault cohomology and hence using Kapranov's approach we obtain the desired result. Consider $d_{LC}^2\Phi$, where d_{LC} is the exterior derivative corresponding to the Levi-Civita connection. Using the facts that d_{LC}^2 gives the curvature and Φ is $\bar{\partial}$-closed (by the Bianchi identity) we find

$$\begin{aligned} \bar{\partial}\partial_{LC}\Phi &= d_{LC}^2\Phi \\ &= K\Phi \in \Omega^{1,2}(X, \mathrm{Sym}^3T^*) = \Omega^{0,2}(X, T^*\otimes\mathrm{Sym}^3T^*). \end{aligned}$$

Note that ∂_{LC} also depends on the Levi-Civita connection, whereas $\bar{\partial}$ does not. The product $K\Phi$ between $K \in \Omega^{1,1}(X, \mathrm{End}T)$ and $\Phi \in \Omega^{0,1}(X, \mathrm{Sym}^3T^*)$ is given by the induced action of $\mathrm{End}T = \mathrm{End}T^*$ on Sym^3T^* and wedging of forms. Note also that

$$\partial_{LC}\Phi \in \Omega^{1,1}(X, \mathrm{Sym}^3T^*) = \Omega^{0,1}(X, T^*\otimes\mathrm{Sym}^3T^*)$$

and so we can totally symmetrize $T^* \otimes \mathrm{Sym}^3 T^*$ in these terms to get forms taking values in $\mathrm{Sym}^4 T^*$. Denoting this symmetrizing operation by S, we obtain

$$S(\partial_{LC}\Phi) \in \Omega^{0,1}(X, \mathrm{Sym}^4 T^*)$$

and

$$S(K\Phi) \in \Omega^{0,2}(X, \mathrm{Sym}^4 T^*)$$

such that $\bar{\partial} S(\partial_{LC}\Phi) = S(K\Phi)$.

Consider the I part of the graph Γ_I. Assume that the vertices are labelled by t and s with $t < s$. We have seen that the edge joining these two vertices is oriented from t to s, and thus when calculating $\Gamma_I(\Phi)$ this part contributes two copies of Φ "joined" by a copy of $\tilde{\omega}$, namely the section

$$I(\Phi) \in C^\infty(X, (T^*)^{\otimes 4} \otimes (\bar{T}^*)^{\otimes 2})$$

with components

$$\begin{aligned} I(\Phi)_{j_t k_t j_s k_s \bar{l}_t \bar{l}_s} &= \Phi_{i_t j_t k_t \bar{l}_t} \omega^{i_t i_s} \Phi_{i_s j_s k_s \bar{l}_s} \\ &= \omega_{i_t m} K^m{}_{j_t k_t \bar{l}_t} \omega^{i_t i_s} \Phi_{i_s j_s k_s \bar{l}_s} \\ &= K^m{}_{j_t k_t \bar{l}_t} \delta^{i_s}_m \Phi_{i_s j_s k_s \bar{l}_s} \\ &= K^m{}_{j_t k_t \bar{l}_t} \Phi_{m j_s k_s \bar{l}_s}. \end{aligned}$$

The indices $j_t k_t j_s k_s$ refer to $(T^*)^{\otimes 4}$ and note that this term is symmetric in $j_t k_t$ and $j_s k_s$. Similarly, the $\bar{H}$ part of $\bar{\Gamma}_H$ contributes

$$\bar{H}(\Phi) \in C^\infty(X, (T^*)^{\otimes 4} \otimes (\bar{T}^*)^{\otimes 2})$$

with components

$$\bar{H}(\Phi)_{j_t k_t j_s k_s \bar{l}_t \bar{l}_s} = K^m{}_{j_t j_s \bar{l}_t} \Phi_{m k_t k_s \bar{l}_s}$$

to $\bar{\Gamma}_H(\Phi)$. This term is symmetric in $j_t j_s$ and $k_t k_s$. Finally, the X part of Γ_X contributes the section

$$X(\Phi) \in C^\infty(X, (T^*)^{\otimes 4} \otimes (\bar{T}^*)^{\otimes 2})$$

with components

$$X(\Phi)_{j_t k_t j_s k_s \bar{l}_t \bar{l}_s} = K^m{}_{j_t k_s \bar{l}_t} \Phi_{m j_s k_t \bar{l}_s}$$

to $\Gamma_X(\Phi)$, and this term is symmetric in $j_t k_s$ and $j_s k_t$. The sum of these three terms is totally symmetric in all four holomorphic indices, ie.

$$I(\Phi) + \bar{H}(\Phi) + X(\Phi) \in C^\infty(X, \mathrm{Sym}^4 T^* \otimes (\bar{T}^*)^{\otimes 2}).$$

In fact, projecting to the exterior product $\Omega^{0,2}(X, \mathrm{Sym}^4 T^*)$ the above sum gives $S(K\Phi)$, up to a factor. This is because the symmetrization $S(K\Phi)$ of $K\Phi$ is given by the sum of 24 terms, but due to the symmetries already present in K and Φ

there are actually just three distinct terms which correspond to the $I(\Phi)$, $\bar{H}(\Phi)$, and $X(\Phi)$ terms above. So we get eight copies of each of these three terms, and therefore

$$\Gamma_I(\Phi) + \bar{\Gamma}_H(\Phi) + \Gamma_X(\Phi) = \frac{1}{8}\Gamma_*(\Phi, S(K\Phi)) \in \Omega^{0,2k}(X)$$

where Γ_* is the graph which is identical to Γ_I away from the I part (and hence identical to $\bar{\Gamma}_H$ away from the $\bar{H}$ part, and Γ_X away from the X part), but contains one tetravalent vertex instead of the I part. The meaning of the right hand side should be obvious; we use Φ at trivalent vertices as before but now we substitute $S(K\Phi)$ for the tetravalent vertex. Note that we can orient Γ_* with the Rozansky-Witten orientation, ie. an equivalence class of orientations of the edges and ordering of the $2k-2$ trivalent vertices (the tetravalent vertex is assumed to be labelled by $2k-1$). In this situation the orientation is not so important because we only wish to show that $\Gamma_*(\Phi, S(K\Phi))$ is exact, and as before, reversing the orientation will merely change the sign of this term.

We have seen that $S(K\Phi) = \bar{\partial}S(\partial_{LC}\Phi)$, and so we can substitute this into the above formula. Since Φ and $\tilde{\omega}$ are $\bar{\partial}$-closed the right hand side becomes

$$\Gamma_*(\Phi, \bar{\partial}S(\partial_{LC}\Phi)) = \bar{\partial}\Gamma_*(\Phi, S(\partial_{LC}\Phi))$$

where

$$\Gamma_*(\Phi, S(\partial_{LC}\Phi)) \in \Omega^{0,2k-1}(X).$$

Therefore in Dolbeault cohomology

$$[\Gamma_I(\Phi)] + [\bar{\Gamma}_H(\Phi)] + [\Gamma_X(\Phi)] = 0 \in \mathrm{H}^{0,2k}_{\bar{\partial}}(X),$$

and therefore

$$b_{\Gamma_I}(X) + b_{\bar{\Gamma}_H}(X) + b_{\Gamma_X}(X) = 0$$

as claimed.

In the above argument we used a graph Γ_* with one tetravalent vertex and the other vertices trivalent. More generally, let Γ' be a graph whose vertices are all of valency three or greater. We can place copies of Φ at the trivalent vertices and copies of $S(\partial_{LC}\Phi)$ at the tetravalent vertices. At b-valent vertices we place copies of

$$\underbrace{S(\partial_{LC}\cdots S(\partial_{LC}}_{b-3}\Phi)\cdots) \in \Omega^{0,1}(X, \mathrm{Sym}^b T^*).$$

Following the usual construction we arrive at a form

$$\Gamma'(\Phi, S(\partial_{LC}\Phi), \ldots) \in \Omega^{0,V}(X)$$

where V is the total number of vertices of Γ'. Unlike $\Gamma(\Phi) \in \Omega^{0,2k}(X)$, these more general forms are not necessarily $\bar{\partial}$-closed, hence they do not represent Dolbeault cohomology classes and we cannot obtain scalar-valued invariants from them by integrating. However, they do enable us to construct strong homotopy Lie algebra and operadic structures on the spaces of forms and cohomology of hyperkähler manifolds. For more details see Kapranov [33]. Let us return to our discussion of trivalent graphs.

Graph homology is the space of rational linear combinations of oriented trivalent graphs modulo the AS and IHX relations. The AS relations say that a graph with its orientation reversed is equivalent to minus the graph, ie.

$$\bar{\Gamma} \equiv -\Gamma.$$

The IHX relations say that

$$\Gamma_I \equiv \Gamma_H - \Gamma_X.$$

We will continue to denote the equivalence class of a graph Γ (its graph homology class) simply by Γ. The space of graph homology is denoted $\mathcal{A}(\emptyset)$. It admits a grading given by half the number of vertices; note that this grading is preserved by the AS and IHX relations. We denote the kth graded component (ie. graph homology classes with $2k$ vertices) by $\mathcal{A}(\emptyset)^k$. The graphs in Figure 2 can be taken as a basis for graph homology in low degree.

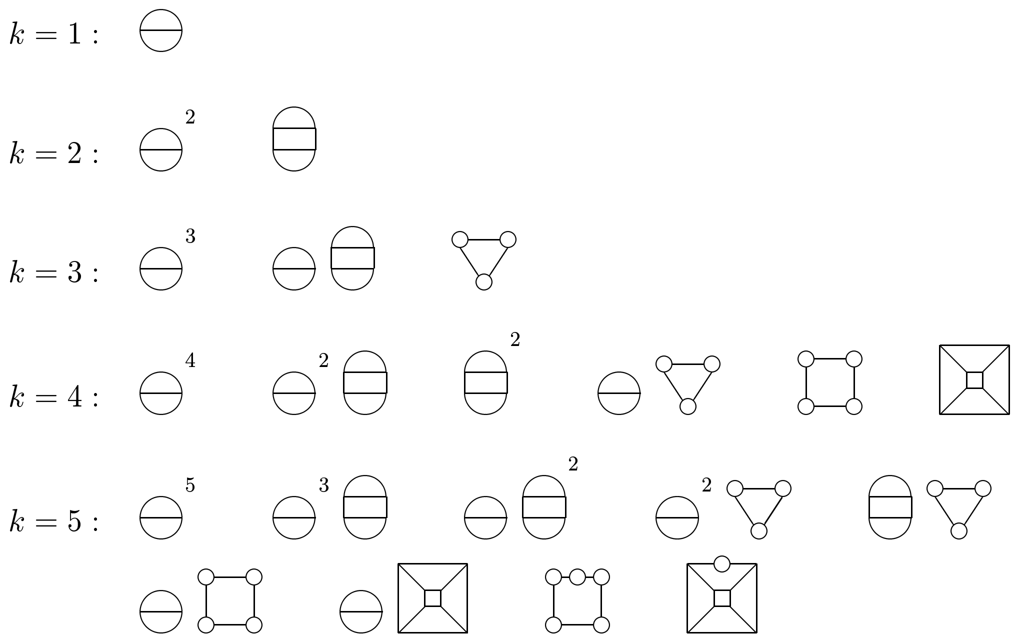
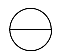
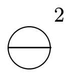


Figure 2: Basis for graph homology in low degree

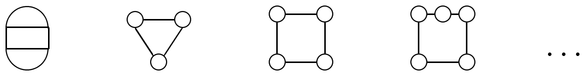
We will denote the two-vertex graph



by Θ and call it simply theta. Note that the graphs need not be connected, and



means to take the disjoint union of two copies of theta. We will denote the graphs



by Θ_2, Θ_3, Θ_4, Θ_5, etc. respectively, and call them *necklace* graphs.

We can linearly extend the Rozansky-Witten invariants to arbitrary rational linear combinations of trivalent graphs. Then from what we have shown in this subsection the Rozansky-Witten invariants are compatible with the AS and IHX relations (the graph homology equivalence relations). Thus we have shown

Proposition 3 *The invariant $b_\Gamma(X)$ only depends on the trivalent graph Γ through its graph homology class. If Γ_1 and Γ_2 are homologous then they define the same invariant $b_{\Gamma_1}(X) = b_{\Gamma_2}(X)$.*

It follows that the Rozansky-Witten invariants allow us to associate to a hyperkähler manifold an element of graph cohomology $\mathcal{A}(\emptyset)^*$, which is the space dual to the space of graph homology. Conversely, we may at times choose to regard an element of $\mathcal{A}(\emptyset)^*$ as a "virtual" hyperkähler manifold. In the next chapter we shall relate certain Rozansky-Witten invariants to characteristic numbers. For our virtual manifolds to make sense we need to show that these characteristic numbers behave as they should. For example, there will be certain integrality restraints, though these can possibly be overcome by rescaling the graph cohomology element. In the next chapter we will also check the behaviour under taking products. The product on graph cohomology is the dual of the coproduct on graph homology, which is given by the formula

$$\Delta(\Gamma) = \sum_{\gamma \sqcup \gamma' = \Gamma} \gamma \otimes \gamma'$$

where the sum is over all decompositions of Γ into two disjoint subgraphs. Graph homology also has a product given by disjoint union of graphs. Together these structures make $\mathcal{A}(\emptyset)$ into a commutative cocommutative Hopf algebra.

Taking the coproduct of a graph is in some sense dual to taking the product of hyperkähler manifolds, as we shall now explain. Suppose we have a reducible hyperkähler manifold $X \times Y$, with X and Y of real-dimensions $4k$ and $4l$ respectively. Let p_1 and p_2 be the projections from $X \times Y$ onto X and Y respectively. We wish to calculate the invariant $b_\Gamma(X \times Y)$, where Γ has $2k + 2l$ vertices. Firstly, we know that the holomorphic symplectic form, its dual, and the Riemann curvature tensor of $X \times Y$ are all given by sums of the corresponding objects pulled back from X and Y using p_1^* and p_2^* respectively. For example

$$\omega^{(X \times Y)} = p_1^* \omega^{(X)} + p_2^* \omega^{(Y)}.$$

If we take local coordinates on $X \times Y$ coming from local coordinates on X and Y, then this means that the matrix

$$\omega_{ij}^{(X\times Y)} = \begin{pmatrix} \omega_{ij}^{(X)} & 0 \\ 0 & \omega_{ij}^{(Y)} \end{pmatrix}$$

of $\omega^{(X\times Y)}$ splits into block form, with the matrices of $\omega^{(X)}$ and $\omega^{(Y)}$ along the diagonal. Similarly for the dual symplectic form, the Riemann curvature tensor, and hence also the section

$$\Phi^{(X\times Y)} = p_1^*\Phi^{(X)} + p_2^*\Phi^{(Y)}.$$

Let γ be a connected component of Γ, with $2m$ vertices. Note that we can define

$$\gamma(\Phi^{(X\times Y)}) \in \Omega^{0,2m}(X\times Y)$$

as before and it makes perfectly good sense. In fact, taking the wedge product of all these forms for all connected components of Γ gives us precisely

$$\Gamma(\Phi^{(X\times Y)}) \in \Omega^{0,2k+2l}(X\times Y).$$

Now when it comes to calculating $\gamma(\Phi^{(X\times Y)})$, the indices must belong either all to X-coordinates or all to Y-coordinates, for if they are mixed then at some stage we will need to contract an X index with a Y index using $\tilde{\omega}^{(X\times Y)}$, and this will give zero as $\tilde{\omega}^{(X\times Y)}$ splits into block form in local coordinates. Note that we have used the connectedness of γ here. Therefore

$$\gamma(\Phi^{(X\times Y)}) = p_1^*\gamma(\Phi^{(X)}) + p_2^*\gamma(\Phi^{(Y)}).$$

It follows that when calculating $\Gamma(\Phi^{(X\times Y)})$, we must decompose Γ into two disjoint subgraphs γ and γ', and then $\Gamma(\Phi^{(X\times Y)})$ is given by the sum of

$$p_1^*\gamma(\Phi^{(X)}) \wedge p_2^*\gamma'(\Phi^{(Y)})$$

over all such decompositions. Now

$$\begin{aligned}(\omega^{(X\times Y)})^{k+l} &= (p_1^*\omega^{(X)} + p_2^*\omega^{(Y)})^{k+l} \\ &= \binom{k+l}{k} p_1^*(\omega^{(X)})^k \wedge p_2^*(\omega^{(Y)})^l\end{aligned}$$

and therefore

$$\begin{aligned} b_\Gamma(X\times Y) &= \frac{1}{(8\pi^2)^{k+l}(k+l)!}\int_{X\times Y}\Gamma(\Phi^{(X\times Y)})(\omega^{(X\times Y)})^{k+l} \\ &= \frac{1}{(8\pi^2)^{k+l}k!l!}\int_{X\times Y}\sum_{\gamma\sqcup\gamma'=\Gamma} p_1^*\gamma(\Phi^{(X)}) \wedge p_2^*\gamma'(\Phi^{(Y)}) p_1^*(\omega^{(X)})^k \wedge p_2^*(\omega^{(Y)})^l \\ &= \sum_{\gamma\sqcup\gamma'=\Gamma}\left(\frac{1}{(8\pi^2)^k k!}\int_X \gamma(\Phi^{(X)})(\omega^{(X)})^k\right)\left(\frac{1}{(8\pi^2)^l l!}\int_Y \gamma'(\Phi^{(Y)})(\omega^{(Y)})^l\right) \\ &= \sum_{\gamma\sqcup\gamma'=\Gamma} b_\gamma(X) b_{\gamma'}(Y)\end{aligned}$$

If γ has $2m$ vertices where $m > k$ then $\gamma(\Phi^{(X)})$ must vanish as it would be given by projecting an element of $C^\infty(X, (\bar{T}^*)^{\otimes 2m})$ to the exterior product $\Omega^{0,2m}(X)$, and the latter vanishes for $\dim_{\mathbb{R}} X = 4k$. Likewise if γ' has more than $2l$ vertices then $\gamma'(\Phi^{(Y)})$ must vanish. Hence on the right hand side of the above formula, the terms in the sum vanish unless γ and γ' have precisely $2k$ and $2l$ vertices respectively. This formula can be rewritten in the following way.

Proposition 4 *There is a correspondence between products of hyperkähler manifolds and coproducts of graphs. More precisely, if X and Y are compact hyperkähler manifolds then*

$$b_\Gamma(X \times Y) = b_{\Delta(\Gamma)}(X, Y).$$

One important corollary of this proposition is that if Γ is a connected graph then its Rozansky-Witten invariant will vanish on reducible hyperkähler manifolds.

As another example, take Γ to be the graph Θ^{k+l} given by $k+l$ disjoint copies of theta. Since γ must have $2k$ vertices, the only possibility is Θ^k, but there are $\binom{k+l}{k}$ different ways to choose which k copies of theta make up γ. Therefore

$$\frac{1}{(k+l)!} b_{\Theta^{k+l}}(X \times Y) = \frac{1}{k!} b_{\Theta^k}(X) \frac{1}{l!} b_{\Theta^l}(Y)$$

or in other words $\frac{1}{k!} b_{\Theta^k}$ is a multiplicative invariant on hyperkähler manifolds.

1.7 Example calculation

Let S be a K3 surface, which is the unique compact irreducible hyperkähler manifold of real-dimension four. In this subsection we shall explicitly compute the Rozansky-Witten invariant $b_\Theta(S)$ using the different approaches described in Subsection 1.5. We shall assume S is the Kummer surface constructed as follows. Take a complex torus $T^2 = \mathbb{C}^2/\mathbb{Z}^4$, and act on it with the involution

$$(z_1, z_2) \mapsto (-z_1, -z_2).$$

There are sixteen fixed-points: $(0,0)$, $(1/2, 0)$, $(i/2, 0)$, etc. We blow-up the torus at these sixteen points to get $\widehat{T^2}$ and then quotient by the involution. The resulting (smooth) surface S is a K3 surface. Denote the sixteen exceptional curves of the blow-up in $\widehat{T^2}$ by $D_1, \ldots, D_{16}$; these are exceptional curves of the first kind. They are fixed by the involution, and hence we shall use the same notation $D_1, \ldots, D_{16}$ to denote the images of these curves in S, which are really exceptional curves of the second kind.

Near these exceptional curves S looks like the cotangent bundle of the projective line, $T^*\mathbb{P}^1$, with D_i the zero section. The space $T^*\mathbb{P}^1$ is a smooth resolution of the double cone $\mathbb{C}^2/\pm 1$. Away from the origin, the latter can be described by coordinates $\pm(z_1, z_2)$. Suppose $\mathbb{P}^1$ has coordinates ζ and $\tilde{\zeta}$ on the complements of the north and south poles respectively, with $\tilde{\zeta} = -\zeta^{-1}$. A differential form $\eta d\zeta$ becomes $\tilde{\eta} d\tilde{\zeta}$ in the other coordinate patch, where $\tilde{\eta} = \eta\zeta^2$. So $T^*\mathbb{P}^1$ can be covered by two coordinate patches U_1 and $\tilde{U}_1$ whose coordinates (ζ, η) and $(\tilde{\zeta}, \tilde{\eta})$ are related in the above way. The map onto $\mathbb{C}^2/\pm 1$ is given by

$$(\zeta, \eta), (\tilde{\zeta}, \tilde{\eta}) \mapsto \pm(z_1, z_2)$$

where

$$(\eta, \eta\zeta, \eta\zeta^2) = (\tilde{\eta}\tilde{\zeta}^2, -\tilde{\eta}\tilde{\zeta}, \tilde{\eta}) = (z_1^2, z_1 z_2, z_2^2).$$

An open cover for S is given by

$$\{U_0, U_1, \tilde{U}_1, \ldots, U_{16}, \tilde{U}_{16}\}$$

where each pair $\{U_i, \tilde{U}_i\}$ cover a small neighbourhood of D_i (so that $U_i \cup \tilde{U}_i$ does not intersect $U_j \cup \tilde{U}_j$ for $i \neq j$), and the open set U_0 is the complement of the exceptional curves.

To get a representation of the Atiyah class α_T of S in Čech cohomology we choose a holomorphic connection over each open set in our cover. In fact, we may as well choose flat connections. For example, on U_0 we choose the connection ∇_0 characterized by

$$\nabla_0(\frac{\partial}{\partial z_1}) = \nabla_0(\frac{\partial}{\partial z_2}) = 0.$$

In U_1 coordinates this becomes

$$\begin{aligned} \nabla_0(\frac{\partial}{\partial \zeta}) &= \frac{1}{2\eta}(d\eta \otimes \frac{\partial}{\partial \zeta}) \\ \nabla_0(\frac{\partial}{\partial \eta}) &= \frac{1}{2\eta}(d\zeta \otimes \frac{\partial}{\partial \zeta} - d\eta \otimes \frac{\partial}{\partial \eta}) \end{aligned}$$

So on $U_0 \cap U_1$ we find

$$(\alpha_T)_{01} = \nabla_0 - \nabla_1 = \frac{1}{2\eta}\begin{pmatrix} d\eta & d\zeta \\ 0 & -d\eta \end{pmatrix} \in \mathrm{H}^0(U_0 \cap U_1, T^* \otimes \mathrm{End}T)$$

in U_1 coordinates. Similarly for $(\alpha_T)_{0\tilde{1}}$, $(\alpha_T)_{02}$, etc. The flat connection $\tilde{\nabla}_1$ on $\tilde{U}_1$ is characterized by

$$\tilde{\nabla}_1(\frac{\partial}{\partial \tilde{\zeta}}) = \tilde{\nabla}_1(\frac{\partial}{\partial \tilde{\eta}}) = 0.$$

In U_1 coordinates this becomes

$$\begin{aligned} \tilde{\nabla}_1(\frac{\partial}{\partial \zeta}) &= \frac{2}{\zeta}(-d\zeta \otimes \frac{\partial}{\partial \zeta} + d\eta \otimes \frac{\partial}{\partial \eta}) + \frac{6\eta}{\zeta^2}(d\zeta \otimes \frac{\partial}{\partial \eta}) \\ \tilde{\nabla}_1(\frac{\partial}{\partial \eta}) &= \frac{2}{\zeta}(d\zeta \otimes \frac{\partial}{\partial \eta}) \end{aligned}$$

So on $U_1 \cap \tilde{U}_1$ we find

$$(\alpha_T)_{\tilde{1}1} = \tilde{\nabla}_1 - \nabla_1 = \frac{2}{\zeta^2}\begin{pmatrix} -\zeta d\zeta & 0 \\ \zeta d\eta + 3\eta d\zeta & \zeta d\zeta \end{pmatrix} \in \mathrm{H}^0(U_1 \cap \tilde{U}_1, T^* \otimes \mathrm{End}T)$$

in U_1 coordinates. Similarly for $(\alpha_T)_{2\tilde{2}}$, etc. Thus we have an element α_T in the Čech cohomology group $\mathrm{H}^1(S, T^* \otimes \mathrm{End}T)$ which represents the Atiyah class.

Next we rewrite $T^* \otimes \mathrm{End}T$ as $(T^*)^{\otimes 3}$ by using the holomorphic symplectic form ω on S to identify T and T^*. Since Rozansky-Witten invariants do not change under rescalings of the symplectic form, we can choose ω to look like $2d\zeta \wedge d\eta$ on U_1. Then the matrices ω_{ij} and ω^{ij} both equal

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and hence the identification of T and T^* is given by

$$\frac{\partial}{\partial\zeta} \leftrightarrow -d\eta$$

$$\frac{\partial}{\partial\eta} \leftrightarrow d\zeta.$$

Thus using ω to rewrite α_T, we find

$$\begin{aligned} (\alpha_T)_{01} &= \frac{-1}{2\eta}(d\eta \otimes d\zeta \otimes d\eta + d\zeta \otimes d\eta \otimes d\eta + d\eta \otimes d\eta \otimes d\zeta) \qquad (1) \\ (\alpha_T)_{\tilde{1}1} &= \frac{2}{\zeta}(d\zeta \otimes d\zeta \otimes d\eta + d\eta \otimes d\zeta \otimes d\zeta + d\zeta \otimes d\eta \otimes d\zeta) + \frac{6\eta}{\zeta^2}(d\zeta \otimes d\zeta \otimes d\zeta). \end{aligned}$$

We get similar formula for $(\alpha_T)_{0\tilde{1}}$, $(\alpha_T)_{02}$, etc. and in particular we see that α_T is in $\mathrm{H}^1(S, \mathrm{Sym}^3 T^*)$.

To represent the Atiyah class α_T of S in Dolbeault cohomology we begin with a smooth global connection ∇ of type $(1,0)$ on S. To obtain such a connection we could take the flat connections on each of the open sets

$$\{U_0, U_1, \tilde{U}_1, \ldots, U_{16}, \tilde{U}_{16}\}$$

and then patch them together using a partition of unity

$$\{\psi_0, \psi_1, \tilde{\psi}_1, \ldots, \psi_{16}, \tilde{\psi}_{16}\}.$$

For example, in U_1 coordinates this would look like

$$\begin{aligned} \nabla|_{U_1} &= \psi_0\nabla_0 + \psi_1\nabla_1 + \tilde{\psi}_1\tilde{\nabla}_1 \\ &= \psi_0\left(d + \frac{1}{2\eta}\begin{pmatrix} d\eta & d\zeta \\ 0 & -d\eta \end{pmatrix}\right) + \psi_1 d + \tilde{\psi}_1\left(d + \frac{2}{\zeta^2}\begin{pmatrix} -\zeta d\zeta & 0 \\ \zeta d\zeta + 3\eta d\zeta & \zeta d\zeta \end{pmatrix}\right) \\ &= d + \psi_0\frac{1}{2\eta}\begin{pmatrix} d\eta & d\zeta \\ 0 & -d\eta \end{pmatrix} + \tilde{\psi}_1\frac{2}{\zeta^2}\begin{pmatrix} -\zeta d\zeta & 0 \\ \zeta d\eta + 3\eta d\zeta & \zeta d\zeta \end{pmatrix} \end{aligned}$$

since $\psi_0+\psi_1+\tilde{\psi}_1$ is identically one on U_1. The $(1,1)$ part of the curvature of this connection gives a Dolbeault representative of the Atiyah class. If we write the above expression $\nabla=d+A$, then this is precisely $\bar{\partial}A$, and

$$\alpha_T=\left[\frac{1}{2\eta}\begin{pmatrix} d\eta & d\zeta \\ 0 & -d\eta\end{pmatrix}\bar{\partial}\psi_0+\frac{2}{\zeta^2}\begin{pmatrix} -\zeta d\zeta & 0 \\ \zeta d\eta+3\eta d\zeta & \zeta d\zeta\end{pmatrix}\bar{\partial}\tilde{\psi}_1\right].$$

Using ω to rewrite our of representative of α_T as an element of $\Omega^{0,1}(S,\mathrm{Sym}^3T^*)$, we get

$$\begin{aligned}\alpha_T \;=\; & \left[\frac{-1}{2\eta}(d\eta\otimes d\zeta\otimes d\eta+d\zeta\otimes d\eta\otimes d\eta+d\eta\otimes d\eta\otimes d\zeta)\bar{\partial}\psi_0\right. \qquad (2)\\ & \left.+\left(\frac{2}{\zeta}(d\zeta\otimes d\zeta\otimes d\eta+d\eta\otimes d\zeta\otimes d\zeta+d\zeta\otimes d\eta\otimes d\zeta)+\frac{6\eta}{\zeta^2}(d\zeta\otimes d\zeta\otimes d\zeta)\right)\bar{\partial}\tilde{\psi}_1\right]\end{aligned}$$

in $\mathrm{H}^{0,1}_{\bar{\partial}}(S,\mathrm{Sym}^3T^*)$.

Our final description of the Atiyah class is as the residue of a meromorphic connection. Let D be the divisor on S given by the sum of the sixteen exceptional curves

$$D_1+\ldots+D_{16}$$

which are given locally by $\eta_i=\tilde{\eta}_i=0$. In U_1 coordinates, the flat connection ∇_0 on U_0 contains a factor of η^{-1}, and similarly for $\tilde{U}_1$, U_2, etc. Thus ∇_0 can be extended to a meromorphic connection on S with a simple pole along D, which we continue to denote by ∇_0. Let $L=\mathcal{O}(D)$ be the line bundle associated to D, and s the canonical section, which is given locally by

$$s=\begin{cases} 1 & \text{in } U_0\\ \eta_i & \text{in } U_i\\ \tilde{\eta}_i & \text{in } \tilde{U}_i.\end{cases}$$

Then $s\nabla_0$ is non-singular on S, and its restriction to D is the image of the residue β_T in $\mathrm{H}^0(D,\mathrm{Sym}^3T^*\otimes L)$ as in Subsection 1.5, where as usual we have identified T and T^*. A local calculation shows that

$$\begin{aligned}(\beta_T)_1 &= \frac{-1}{2}(d\eta\otimes d\zeta\otimes d\eta+d\zeta\otimes d\eta\otimes d\eta+d\eta\otimes d\eta\otimes d\zeta) \qquad (3)\\ (\beta_T)_{\tilde{1}} &= \frac{-1}{2}(d\tilde{\eta}\otimes d\tilde{\zeta}\otimes d\tilde{\eta}+d\tilde{\zeta}\otimes d\tilde{\eta}\otimes d\tilde{\eta}+d\tilde{\eta}\otimes d\tilde{\eta}\otimes d\tilde{\zeta})\end{aligned}$$

where we should remember that we have restricted to D so that $\eta=\tilde{\eta}=0$, and also that this is a section of a bundle which has been twisted by L. We get similar formulae for $(\beta_T)_2$, $(\beta_T)_{\tilde{2}}$, etc. and the Atiyah class is given by $\alpha_T=\delta\beta_T$.

Now we calculate the Rozansky-Witten invariant $b_\Theta(S)$ using the three different descriptions of the Atiyah class just given. We begin with the Čech cohomology description.

We place a copy of the Atiyah class at each of the two vertices of Θ, and then contract along the edges using three copies of $\tilde{\omega}$. The product of two Čech cohomology classes $\{a_{ij}\}$ and $\{b_{ij}\}$ is given by

$$(ab)_{ijk} = \frac{1}{6}(a_{ij}b_{ik} - a_{ik}b_{ij} + a_{jk}b_{ji} - a_{ji}b_{jk} + a_{ki}b_{kj} - a_{kj}b_{ki}).$$

In the case of the Atiyah class (1) this simplifies to

$$(\alpha_T\alpha_T)_{0\tilde{1}1} = \frac{1}{2}((\alpha_T)_{01}(\alpha_T)_{\tilde{1}1} - (\alpha_T)_{\tilde{1}1}(\alpha_T)_{01})$$

and similarly for $(\alpha_T\alpha_T)_{0\tilde{2}2}$, etc. In U_1 coordinates $\tilde{\omega}$ looks like

$$2\frac{\partial}{\partial\zeta} \wedge \frac{\partial}{\partial\eta}$$

and a calculation shows that using this to contract along the edges we get

$$\Theta(\alpha_T)_{0\tilde{1}1} = \frac{-3}{\zeta\eta}.$$

Similar for $\Theta(\alpha_T)_{0\tilde{2}2}$, etc. and together these give us $\Theta(\alpha_T)$ in $\mathrm{H}^2(S, \mathcal{O}_S)$. The Serre duality pairing with ω in $\mathrm{H}^0(S, \Lambda^2 T^*)$ can be calculated by multiplying by this section and then integrating along contours. For example, in U_1 coordinates we integrate along the contour

$$\{|\zeta| = |\eta| = 1\} \subset U_0 \cap \tilde{U}_1 \cap U_1$$

to obtain

$$\int_{|\zeta|=1}\int_{|\eta|=1} \frac{-3}{\zeta\eta} 2d\zeta \wedge d\eta = -6(2\pi i)^2.$$

Summing over all sixteen exceptional curves and then dividing by the factor $8\pi^2$ gives us the Rozansky-Witten invariant

$$b_\Theta(S) = 48.$$

Next we do the same calculation with Dolbeault cohomology classes. We work in U_1 coordinates. Taking two copies of our representative of the Atiyah class from Equation (2), and contracting with $\tilde{\omega}$ gives us the section

$$\frac{-3}{\zeta\eta}(\bar{\partial}\psi_0 \otimes \bar{\partial}\tilde{\psi}_1 - \bar{\partial}\tilde{\psi}_1 \otimes \bar{\partial}\psi_0) \in C^\infty(S, (\bar{T}^*)^{\otimes 2})$$

and projecting to the exterior product gives us a Dolbeault representative of

$$\Theta(\alpha_T) = \left[\frac{-6}{\zeta\eta}\bar{\partial}\psi_0 \wedge \bar{\partial}\tilde{\psi}_1\right] \in \mathrm{H}^{0,2}_{\bar{\partial}}(S, \mathcal{O}_S).$$

Multiplying by the symplectic form and integrating gives us

$$\int_{U_1} \frac{-12}{\zeta\eta} \bar{\partial}\psi_0 \wedge \bar{\partial}\tilde{\psi}_1 \wedge d\zeta \wedge d\eta.$$

Since the integrand is supported inside U_1 anyway, we can think of this as being an integral over all of $T^*\mathbb{P}^1$, and because $d\zeta \wedge d\zeta$ and $d\eta \wedge d\eta$ vanish, we can write

$$\begin{aligned}
\bar{\partial}\psi_0 \wedge \bar{\partial}\tilde{\psi}_1 \wedge d\zeta \wedge d\eta &= \bar{\partial}\psi_0 \wedge \bar{\partial}\tilde{\psi}_1 \wedge d\zeta \wedge d\eta + \bar{\partial}\psi_0 \wedge \partial\tilde{\psi}_1 \wedge d\zeta \wedge d\eta \\
&= \bar{\partial}\psi_0 \wedge d\tilde{\psi}_1 \wedge d\zeta \wedge d\eta \\
&= \bar{\partial}\psi_0 \wedge d\tilde{\psi}_1 \wedge d\zeta \wedge d\eta + \partial\psi_0 \wedge d\tilde{\psi}_1 \wedge d\zeta \wedge d\eta \\
&= d\psi_0 \wedge d\tilde{\psi}_1 \wedge d\zeta \wedge d\eta.
\end{aligned}$$

Substituting this back into the above integral we obtain

$$\int_{T^*\mathbb{P}^1} \frac{-12}{\zeta\eta} d\psi_0 \wedge d\tilde{\psi}_1 \wedge d\zeta \wedge d\eta.$$

Now ψ_0 vanishes on $U_1 \backslash U_0$ and is identically one on $U_0 \backslash (U_1 \cap \tilde{U}_1)$ (at least away from the other exceptional curves), and $\tilde{\psi}_1$ vanishes on $U_1 \backslash \tilde{U}_1$ and is identically one on $\tilde{U}_1 \backslash (U_0 \cap U_1)$. In fact, we can choose these functions to be invariant under the torus action on $T^*\mathbb{P}^1$. In other words, converting to polar coordinates $\zeta = re^{i\theta}$ and $\eta = se^{i\phi}$, we can choose ψ_0 and $\tilde{\psi}_1$ to be functions of only r and s, with ψ_0 vanishing for $s = 0$ and identically one for s large, and $\tilde{\psi}_1$ vanishing for $r = 0$ or s large, and identically one for r large and $s = 0$. Hence the integral becomes

$$\int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \int_{s=0}^{\infty} \int_{\phi=0}^{2\pi} 12 d\psi_0 \wedge d\tilde{\psi}_1 \wedge d\theta \wedge d\phi = 12(2\pi)^2 \int_{r=0}^{\infty} \int_{s=0}^{\infty} d\psi_0 \wedge d\tilde{\psi}_1.$$

Writing the integrand $d(\psi_0 d\tilde{\psi}_1)$ and applying Stokes' theorem gives us an integral along the oriented boundary of the first quadrant in $\mathbb{R}^2$, namely

$$48\pi^2 \int_{\{r=0\}\cup\{s=\infty\}\cup\{\overline{r=\infty}\}\cup\{\overline{s=0}\}} \psi_0 d\tilde{\psi}_1$$

where $\{\overline{r=\infty}\}$ and $\{\overline{s=0}\}$ denote $\{r=\infty\}$ and $\{s=0\}$, respectively, but with orientations opposite to the standard ones. Observe that $\tilde{\psi}_1$ vanishes on $\{r=0\}$ and $\{s=\infty\}$, and ψ_0 vanishes on $\{\overline{s=0}\}$. On $\{\overline{r=\infty}\}$, ψ_1 vanishes and therefore $\tilde{\psi}_1 \equiv 1-\psi_0$ (as these functions form a partition of unity, and our integral is supported in an open set which is away from the other fifteen exceptional curves). Hence our integral becomes

$$\begin{aligned}
48\pi^2 \int_{\{\overline{r=\infty}\}} \psi_0 d(1-\psi_0) &= 48\pi^2 \int_{s=0}^{\infty} \psi_0 d\psi_0|_{r=\infty} \\
&= 48\pi^2 \left[\frac{1}{2}\psi_0^2\right]_{r=\infty,s=0}^{r=\infty,s=\infty} \\
&= 24\pi^2.
\end{aligned}$$

Again we need to sum over all sixteen exceptional curves and divide by the factor $8\pi^2$, obtaining

$$b_\Theta(S) = 48.$$

Finally we calculate $b_\Gamma(S)$ using the residue description of the Atiyah class (3). First we compute $\Theta(\alpha_T, \beta_T)$ in $\mathrm{H}^1(D, L)$ using a Čech cohomology class for α_T. In U_1 coordinates this is given by multiplying $(\alpha_T)_{1\tilde{1}}$ by $(\beta_T)_1$, and then contracting with three copies of $\tilde{\omega}$, and the result is

$$\Theta(\alpha_T, \beta_T)_{1\tilde{1}} = \frac{3}{\zeta}$$

where we have used the fact that η vanishes on D. Next we need to take the Serre duality pairing of this element with the section $\omega|_D$ of $\Lambda^2 T^*|_D$. Since D is given locally by $\eta = 0$, we write

$$\omega|_D = d\eta \wedge (-2d\zeta)$$

and hence we should multiply $\Theta(\alpha_T, \beta_T)$ by the section $-2\eta d\zeta$ of $\mathcal{K}_D \otimes L^*|_D$ and then perform an integral along the contour

$$\{|\zeta| = 1\} \subset U_1 \cap \tilde{U}_1.$$

Note that $\Theta(\alpha_T, \beta_T)_{1\tilde{1}}$ is a local section of L, or in other words a local meromorphic function with a simple pole along D. The latter description requires us to include a factor of η^{-1} which cancels with the η in $-2\eta d\zeta$, and hence we get

$$\int_{|\zeta|=1} \frac{3}{\zeta}(-2d\zeta) = -6(2\pi i)$$

Alternatively, using a Dolbeault cohomology class for α_T we obtain the expression

$$\Theta(\alpha_T, \beta_T) = \left[\frac{3}{\zeta}\bar{\partial}\tilde{\psi}_1|_D\right] \in \mathrm{H}^{0,1}_{\bar{\partial}}(D, L)$$

in U_1 coordinates. Here we regard $\tilde{\psi}_1|_D$ as simply a function of ζ and $\bar{\zeta}$. In fact, converting to polar coordinates $\zeta = re^{i\theta}$ as before, we may choose $\tilde{\psi}_1|_D$ to be a function of r only, vanishing for $r = 0$ and identically one for r large. Multiplying by $-2d\zeta$ as above and integrating over D_1 gives us

$$\begin{aligned}
\int_{D_1} \frac{3}{\zeta}\bar{\partial}\tilde{\psi}_1|_D \wedge (-2d\zeta) &= -6i \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} d\tilde{\psi}_1|_D \wedge d\theta \\
&= -6(2\pi i) \int_{r=0}^{\infty} d\tilde{\psi}_1|_D \\
&= -6(2\pi i) \left[\tilde{\psi}_1|_D\right]_{r=0}^{r=\infty} \\
&= -6(2\pi i)
\end{aligned}$$

as before. Including the factor $2\pi i$ (from the Cauchy residue formula), along with the usual factors 16 and $(8\pi^2)^{-1}$, gives us

$$b_\Theta(S) = 48.$$

2 Characteristic numbers

2.1 Chern numbers and Chern-Weil theory

By Chern-Weil theory we can write the Chern numbers of a manifold as integrals of copies of the curvature, multiplied in some way. We will make this precise below, but first let us note that Rozansky-Witten invariants were defined in a similar way, and hence it is reasonable to expect some relation between them. In fact all the Chern numbers can be expressed in terms of Rozansky-Witten invariants, as we shall show in this chapter. Whether or not the converse is true, ie. that all Rozansky-Witten invariants can be expressed in terms of the Chern numbers, is a fundamental question of this theory and one we shall investigate further in Chapter 5.

For a hyperkähler manifold we shall see that all the odd Chern classes vanish, and the remaining even Chern classes are really the Pontryagin classes, which are topological invariants. So although we find it easier to work with Chern classes, our results will be completely independent of the choice of compatible complex structure used to define them. So let X be a compact hyperkähler manifold of real-dimension $4k$ and fix a complex structure I. By Chern-Weil theory we can represent the Chern character of X by traces of powers of the Riemann curvature tensor

$$K^{i}{}_{jk\bar{l}} \in \Omega^{1,1}(X, \mathrm{End}T).$$

More precisely, we have

$$\begin{aligned} ch(T) &= 2k + ch_1(T) + ch_2(T) + \ldots + ch_{2k}(T) \\ &= \sum_{m=0}^{2k} \frac{(-1)^m}{m!(2\pi i)^m}[\mathrm{Tr}(K^m)] \end{aligned}$$

where we multiply K's by composing $\mathrm{End}T$ and taking the wedge product on forms, and then take the trace in $\mathrm{End}T$. Note that since we can identify T and T^* using ω, we have

$$ch_m(T) = ch_m(T^*) = (-1)^m ch_m(T)$$

and hence all odd components of the Chern character vanish for hyperkähler manifolds, as claimed above.

The Chern numbers of X are obtained by taking components of the Chern character and wedging them together to get something which can be integrated over X. This means taking an even partition $(\lambda_1, \ldots, \lambda_j)$ of $2k$ (ie. one for which every λ_i is even) and then calculating

$$ch_{\lambda_1} \cdots ch_{\lambda_j}(X) = \int_X ch_{\lambda_1}(T) \wedge \cdots \wedge ch_{\lambda_j}(T).$$

The integrand lies in $\Omega^{2k,2k}(X)$. Following Hirzebruch's notation [29], we shall work with a rescaling

$$s_\lambda(T) = \lambda! ch_\lambda(T) = \frac{1}{(2\pi i)^\lambda}[\mathrm{Tr}(K^\lambda)] \in \mathrm{H}^{2\lambda}(X, \mathbb{Z})$$

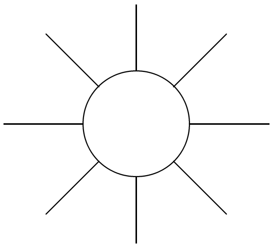
of the components of the Chern character (note that we have dropped the sign $(-1)^\lambda$ since λ is even). Then we get

$$\begin{aligned} s_{\lambda_1}\cdots s_{\lambda_j}(X) &= \int_X s_{\lambda_1}(T)\wedge\cdots\wedge s_{\lambda_j}(T) \\ &= \frac{1}{(2\pi i)^{2k}}\int_X \mathrm{Tr}(K^{\lambda_1})\wedge\cdots\wedge\mathrm{Tr}(K^{\lambda_j}) \end{aligned}$$

which we shall still call Chern numbers.

2.2 Relation to Rozansky-Witten invariants

We wish to express the Chern numbers of X in terms of the Rozansky-Witten invariants $b_\Gamma(X)$ for the appropriate choice of graph homology class Γ. We will see that *wheels* play an important role. Suppose that the graph Γ contains a λ-wheel w_λ, ie. a closed path with distinct vertices and λ edges. Each vertex has a third outgoing edge, and we shall call these the spokes of the wheel. For example, an 8-wheel looks like:



It is not too difficult to show using the AS and IHX relations that a graph containing an odd wheel must be homologous to zero, so we shall assume that λ is even. We must be careful with our orientation, which is induced from the planar embedding as before. To see this as a Rozansky-Witten orientation let us assume, without loss of generality, that the λ vertices of the wheel are ordered $1,\ldots,\lambda$ in an anticlockwise manner, and that the flags at the mth vertex are labelled $m, \lambda+m, 2\lambda+m$ in an anticlockwise manner, with m labelling the flag corresponding to the spoke. The corresponding orientation of $\det F(w_\lambda)$ is given by the element

$$(f_1\wedge f_{\lambda+1}\wedge f_{2\lambda+1})\wedge(f_2\wedge f_{\lambda+2}\wedge f_{2\lambda+2})\wedge\cdots\wedge(f_\lambda\wedge f_{2\lambda}\wedge f_{3\lambda})$$

where the f_i's are a basis for $\mathbb{R}^{F(w_\lambda)}$ (recall the example in the previous chapter). We can rewrite this element as

$$-(f_1\wedge f_2\wedge\cdots\wedge f_\lambda)\wedge(f_{\lambda+1}\wedge f_{2\lambda+2})\wedge(f_{\lambda+2}\wedge f_{2\lambda+3})\wedge\cdots\wedge(f_{2\lambda}\wedge f_{2\lambda+1})$$

and this shows that the edges of $\bar{w}_\lambda = -w_\lambda$ should be oriented in the anticlockwise direction, where we have ordered the flags $1,2,\ldots,\lambda$ corresponding to the spokes of

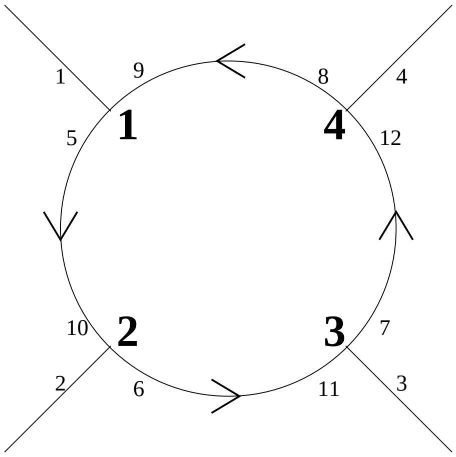


Figure 3: Orientation of a wheel

the wheel in the same order as the vertices themselves. This is best illustrated by an example, and Figure 3 shows the case $\lambda = 4$.

When calculating $\Gamma(\Phi)$ this part of the graph will contribute the section

$$w_\lambda(\Phi) \in C^\infty(X, (T^* \otimes \bar{T}^*)^{\otimes\lambda})$$

which with respect to local complex coordinates has components

$$\begin{aligned}
w_\lambda(\Phi)_{k_1\bar{l}_1\cdots k_\lambda\bar{l}_\lambda} &= -\Phi_{i_1 j_1 k_1 \bar{l}_1}\omega^{j_1 i_2}\Phi_{i_2 j_2 k_2 \bar{l}_2}\omega^{j_2 i_3}\cdots\Phi_{i_\lambda j_\lambda k_\lambda \bar{l}_\lambda}\omega^{j_\lambda i_1} \\
&= -\omega_{i_1 m_1}K^{m_1}{}_{j_1 k_1 \bar{l}_1}\omega^{j_1 i_2}\omega_{i_2 m_2}K^{m_2}{}_{j_2 k_2 \bar{l}_2}\omega^{j_2 i_3}\cdots\omega_{i_\lambda m_\lambda}K^{m_\lambda}{}_{j_\lambda k_\lambda \bar{l}_\lambda}\omega^{j_\lambda i_1} \\
&= -K^{m_1}{}_{j_1 k_1 \bar{l}_1}(-\delta^{j_1}_{m_2})K^{m_2}{}_{j_2 k_2 \bar{l}_2}(-\delta^{j_2}_{m_3})\cdots(-\delta^{j_{\lambda-1}}_{m_\lambda})K^{m_\lambda}{}_{j_\lambda k_\lambda \bar{l}_\lambda}(-\delta^{j_\lambda}_{m_1}) \\
&= -(-1)^\lambda K^{m_1}{}_{m_2 k_1 \bar{l}_1}K^{m_2}{}_{m_3 k_2 \bar{l}_2}\cdots K^{m_\lambda}{}_{m_1 k_\lambda \bar{l}_\lambda} \\
&= -\mathrm{Tr}(K^{\otimes\lambda})_{k_1\bar{l}_1\cdots k_\lambda\bar{l}_\lambda}
\end{aligned}$$

where $k_1, \ldots, k_\lambda$ denote the indices attached to the spokes, and we have used the fact that λ is even. Note that in $K^{\otimes\lambda}$ we take the tensor product of forms, whereas K^λ is obtained by taking the wedge product. The latter can be recovered from the former by projecting to the exterior product. Let us denote this projection by S and $\bar{S}$, for projection to the holomorphic and anti-holomorphic exterior products respectively; then

$$S\bar{S}(w_\lambda(\Phi)) = -\mathrm{Tr}(K^\lambda) \in \Omega^{0,\lambda}(X, \Lambda^\lambda T^*) = \Omega^{\lambda,\lambda}(X).$$

We will reorder the indices to separate the holomorphic and anti-holomorphic forms; note that $\mathrm{Tr}(K^\lambda) \in \Omega^{\lambda,\lambda}(X)$ involves taking the wedge product of λ copies of K, so with respect to local coordinates it will look like (some coefficient multiplied by)

$$dz_{k_1} \wedge d\bar{z}_{l_1} \wedge dz_{k_2} \wedge d\bar{z}_{l_2} \wedge \cdots \wedge dz_{k_\lambda} \wedge d\bar{z}_{l_\lambda}.$$

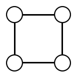
We can rewrite this as

$$(-1)^{\lambda/2} dz_{k_1} \wedge dz_{k_2} \wedge \cdots \wedge dz_{k_\lambda} \wedge d\bar{z}_{l_1} \wedge d\bar{z}_{l_2} \wedge \cdots \wedge d\bar{z}_{l_\lambda}$$

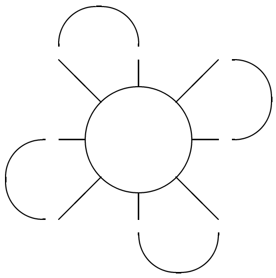
and hence

$$S\bar{S}(w_\lambda(\Phi))_{k_1 \cdots k_\lambda \bar{l}_1 \cdots \bar{l}_\lambda} = -(-1)^{\lambda/2} \mathrm{Tr}(K^\lambda)_{k_1 \bar{l}_1 \cdots k_\lambda \bar{l}_\lambda}.$$

Suppose that Γ is the necklace graph with k beads Θ_k. For example, Θ_4 looks like:



This clearly contains an 8-wheel. Indeed, the remainder of the graph is simply four connecting edges:



More generally, Θ_k is made up of a $2k$-wheel and k edges connecting adjacent spokes. To calculate the Rozansky-Witten invariant $b_{\Theta_k}(X)$ we construct $\Theta_k(\Phi) \in \Omega^{0,2k}(X)$. From our previous discussion, we know that this is obtained from

$$w_{2k}(\Phi) = -\mathrm{Tr}(K^{\otimes 2k}) \in C^\infty(X, (T^* \otimes \bar{T}^*)^{\otimes 2k})$$

by contracting with the k copies of $\tilde{\omega}$ corresponding to the k connecting edges

$$-\mathrm{Tr}(K^{\otimes 2k})_{k_1 \bar{l}_1 \cdots k_{2k} \bar{l}_{2k}} \omega^{k_1 k_2} \cdots \omega^{k_{2k-1} k_{2k}} d\bar{z}_{l_1} \otimes \cdots \otimes d\bar{z}_{l_{2k}}$$

and then taking the projection $\bar{S}$ to the exterior product $\Omega^{0,2k}(X)$. As always, we must be careful with the orientation, but recall that the orientation was determined by the element

$$-(f_1 \wedge f_2 \wedge \cdots \wedge f_{2k}) \wedge (f_{2k+1} \wedge f_{4k+2}) \wedge (f_{2k+2} \wedge f_{4k+3}) \wedge \cdots \wedge (f_{4k} \wedge f_{4k+1})$$

which implies that the connecting edges should be oriented from v_1 to v_2, $\ldots$, from v_{2k-1} to v_{2k} (the minus sign already appears in our formula.) In other words, our use of $\omega^{k_1 k_2}$, $\ldots$, $\omega^{k_{2k-1} k_{2k}}$ agrees with the orientation.

Now suppose that instead of the connecting edges joining v_1 to v_2, $\ldots$, v_{2k-1} to v_{2k}, they join $v_{\pi(1)}$ to $v_{\pi(2)}$, $\ldots$, $v_{\pi(2k-1)}$ to $v_{\pi(2k)}$ for some permutation π of $2k$ elements. Call the graph obtained in this way Γ_π (note that $\Gamma_{\mathrm{Id}} = \Theta_k$). Since

$$f_{\pi(1)} \wedge f_{\pi(2)} \wedge \cdots \wedge f_{\pi(2k)} = (\mathrm{sgn}\pi) f_1 \wedge f_2 \wedge \cdots \wedge f_{2k}$$

it follows that the orientation on Γ_π induced from the planar embedding is $-(\mathrm{sgn}\pi)$ times the orientation given by orienting the edges from $v_{\pi(1)}$ to $v_{\pi(2)}$, ..., from $v_{\pi(2k-1)}$ to $v_{\pi(2k)}$, where we assume that the vertices are labelled 1 to $2k$ in an anti-clockwise manner and the edges of the wheel are oriented in an anti-clockwise manner, as before. It follows that $\Gamma_\pi(\Phi) \in \Omega^{0,2k}(X)$ is given by taking the projection $\bar{S}$ of

$$-(\mathrm{sgn}\pi)\mathrm{Tr}(K^{\otimes 2k})_{k_1\bar{l}_1\cdots k_{2k}\bar{l}_{2k}}\omega^{k_{\pi(1)}k_{\pi(2)}}\cdots\omega^{k_{\pi(2k-1)}k_{\pi(2k)}}d\bar{z}_{l_1}\otimes\cdots\otimes d\bar{z}_{l_{2k}}$$

to the exterior product $\Omega^{0,2k}(X)$.

Let

$$\Gamma = \sum_{\pi\in\mathcal{S}_{2k}}\Gamma_\pi$$

where we have summed over all permutations of $2k$ objects. Each term in this sum actually occurs $2^k k!$ times, because we can permute the k connecting edges in $k!$ ways and each one can be used to join two spokes in two different ways (ie. the first spoke to the second, or the second spoke to the first). We denote the graph homology class given by summing over all graphs obtained by joining the spokes of w_{2k} in pairs by $\langle w_{2k}\rangle$, and call it the *closure* of the wheel w_{2k}. Then $\Gamma = 2^k k!\langle w_{2k}\rangle$.

From what we have shown above, $\Gamma(\Phi) \in \Omega^{0,2k}(X)$ is given by taking the projection $\bar{S}$ of

$$-\sum_{\pi\in\mathcal{S}_{2k}}(\mathrm{sgn}\pi)\mathrm{Tr}(K^{\otimes 2k})_{k_1\bar{l}_1\cdots k_{2k}\bar{l}_{2k}}\omega^{k_{\pi(1)}k_{\pi(2)}}\cdots\omega^{k_{\pi(2k-1)}k_{\pi(2k)}}d\bar{z}_{l_1}\otimes\cdots\otimes d\bar{z}_{l_{2k}}$$

to the exterior product. In constructing the Rozansky-Witten invariant $b_\Gamma(X)$, we next multiply $\Gamma(\Phi)$ by ω^k to obtain an element of $\Omega^{2k,2k}(X)$. So we should consider

$$-\sum_{\pi\in\mathcal{S}_{2k}}(\mathrm{sgn}\pi)\bar{S}(\mathrm{Tr}(K^{\otimes 2k}))_{k_1\bar{l}_1\cdots k_{2k}\bar{l}_{2k}}\omega^{k_{\pi(1)}k_{\pi(2)}}\cdots\omega^{k_{\pi(2k-1)}k_{\pi(2k)}}\omega^k d\bar{z}_{l_1}\wedge\cdots\wedge d\bar{z}_{l_{2k}}.$$

Since this is a skewed-sum over all permutations of $2k$ objects, we can insert the projection S to the (holomorphic) exterior product in front of $\bar{S}(\mathrm{Tr}(K^{\otimes 2k}))$ and this will not change the formula. It does mean, however, that $S\bar{S}(\mathrm{Tr}(K^{\otimes 2k}))$ can be replaced by $\mathrm{Tr}(K^{2k})$. We also reorder the indices, introducing the appropriate sign adjustment, to get

$$(-1)^{k+1}\sum_{\pi\in\mathcal{S}_{2k}}(\mathrm{sgn}\pi)\mathrm{Tr}(K^{2k})_{k_1\cdots k_{2k}\bar{l}_1\cdots\bar{l}_{2k}}\omega^{k_{\pi(1)}k_{\pi(2)}}\cdots\omega^{k_{\pi(2k-1)}k_{\pi(2k)}}\omega^k d\bar{z}_{l_1}\wedge\cdots\wedge d\bar{z}_{l_{2k}}.$$

If $a_1,\ldots,a_{2k}$ are (local) holomorphic vector fields on X then the algebra of exterior products tells us that

$$\omega^k(a_1,\ldots,a_{2k}) = \frac{1}{(2k)!}\sum_{\pi\in\mathcal{S}_{2k}}(\mathrm{sgn}\pi)\omega(a_{\pi(1)},a_{\pi(2)})\cdots\omega(a_{\pi(2k-1)},a_{\pi(2k)})$$

where the sum is over all permutations of $2k$ objects. Dually, if $\alpha_1, \ldots, \alpha_{2k}$ are (local) holomorphic sections of the cotangent bundle T^* of X, then

$$\alpha_1 \wedge \cdots \wedge \alpha_{2k} = \frac{1}{2^{2k}(k!)^2} \sum_{\pi \in \mathcal{S}_{2k}} (\mathrm{sgn}\pi)\tilde{\omega}(\alpha_{\pi(1)}, \alpha_{\pi(2)}) \cdots \tilde{\omega}(\alpha_{\pi(2k-1)}, \alpha_{\pi(2k)})\omega^k.$$

By writing $\mathrm{Tr}(K^{2k})$ in local coordinates, we can show that

$$\begin{aligned} &2^{2k}(k!)^2\mathrm{Tr}(K^{2k}) \\ &\quad = \sum_{\pi \in \mathcal{S}_{2k}} (\mathrm{sgn}\pi)\mathrm{Tr}(K^{2k})_{k_1 \cdots k_{2k}\bar{l}_1 \cdots \bar{l}_{2k}} \omega^{k_{\pi(1)}k_{\pi(2)}} \cdots \omega^{k_{\pi(2k-1)}k_{\pi(2k)}} \omega^k d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_{2k}}. \end{aligned}$$

Substituting this into the above formula we get

$$\Gamma(\Phi)\omega^k = (-1)^{k+1}2^{2k}(k!)^2\mathrm{Tr}(K^{2k}) \in \Omega^{2k,2k}(X)$$

and therefore

$$\begin{aligned} b_\Gamma(X) &= \frac{1}{(8\pi^2)^k k!} \int_X \Gamma(\Phi)\omega^k \\ &= \frac{-(-1)^k 2^{2k}(k!)^2}{(8\pi^2)^k k!} \int_X \mathrm{Tr}(K^{2k}) \\ &= -2^k k! \frac{1}{(2\pi i)^{2k}} \int_X \mathrm{Tr}(K^{2k}) \\ &= -2^k k! s_{2k}(X). \end{aligned}$$

In particular

$$\begin{aligned} s_{2k}(X) &= \frac{-1}{2^k k!} b_\Gamma(X) \\ &= -b_{\langle w_{2k} \rangle}(X) \end{aligned}$$

gives a way of expressing the characteristic number $s_{2k}(X)$ in terms of the Rozansky-Witten invariants.

Suppose we wish to write the characteristic number

$$s_{\lambda_1} \cdots s_{\lambda_j}(X) = \frac{1}{(2\pi i)^{2k}} \int_X \mathrm{Tr}(K^{\lambda_1}) \wedge \cdots \wedge \mathrm{Tr}(K^{\lambda_j})$$

in a similar way, where $(\lambda_1, \ldots, \lambda_j)$ is an even partition of $2k$. Following the calculations presented above, we arrive at the following formula.

Proposition 5 *The Chern numbers of a compact hyperkähler manifold can all be expressed in terms of the Rozansky-Witten invariants as*

$$s_{\lambda_1} \cdots s_{\lambda_j}(X) = (-1)^j b_{\langle w_{\lambda_1} \cdots w_{\lambda_j} \rangle}(X)$$

where $\langle w_{\lambda_1} \cdots w_{\lambda_j} \rangle$ *denotes the sum of all graphs obtained by taking the disjoint union of a* λ_1*-wheel, a* λ_2*-wheel, ..., and a* λ_j*-wheel and joining their spokes in pairs.*

Note that this sum of graphs includes examples where two spokes of the same wheel may be joined. We call this sum the *closure* of the disjoint union of the j wheels $w_{\lambda_1}, \ldots, w_{\lambda_j}$, though more often we shall call it a *polywheel*. The only possibly mysterious part in this formula is the sign $(-1)^j$. Recall that our wheels have a canonical orientation given by their planar embedding, but in constructing the Rozansky-Witten invariants it was more convenient to reverse this, and hence the minus sign in our formula for $s_{2k}(X)$. Similarly, with $\langle w_{\lambda_1} \cdots w_{\lambda_j} \rangle$ we need to reverse the orientation of all j wheels, and hence the sign $(-1)^j$.

Any characteristic number can be expressed as a linear combination of the Chern numbers $s_{\lambda_1} \cdots s_{\lambda_j}(X)$, and therefore as a linear combination of Rozansky-Witten invariants corresponding to polywheels $\langle w_{\lambda_1} \cdots w_{\lambda_j} \rangle$. In Appendix A.1 we expand out the polywheels with up to ten vertices to see which graph homology classes they give. For two, four, and six vertices it is possible to invert these relations, so that it is possible to write all graph homology classes as linear combinations of polywheels (see Appendix A.2). It follows that for $k = 1$, 2, and 3 all Rozansky-Witten invariants can be expressed in terms of Chern numbers.

For larger degree k it is not possible to invert the polywheel relations. To see why consider the dimension of $\mathcal{A}(\emptyset)^k_{\mathrm{conn}}$ (the connected graph homology classes with $2k$ vertices). For degree less than or equal to three this dimension is one, and so the dimension of $\mathcal{A}(\emptyset)^k$ is precisely $p(k)$, the number of partitions of k. This is the same as the number of polywheels. On the other hand, for degree greater than or equal to four the dimension of $\mathcal{A}(\emptyset)^k_{\mathrm{conn}}$ is greater than one, and so there are more than $p(k)$ distinct graph homology classes, but still only $p(k)$ polywheels. In general, the behaviour of $\dim \mathcal{A}(\emptyset)^k_{\mathrm{conn}}$ is not known, but for small k we have the following sequences of numbers.

k	1	2	3	4	5	6	7	8	9	10
$\dim \mathcal{A}(\emptyset)^k_{\mathrm{conn}}$	1	1	1	2	2	3	4	5	6	8
$\dim \mathcal{A}(\emptyset)^k$	1	2	3	6	9	16	25	42	65	105
$p(k)$	1	2	3	5	7	11	15	22	30	42

Thus the polywheels span a subspace of graph homology which is of codimension

$$\dim \mathcal{A}(\emptyset)^k - p(k)$$

in degree k. We shall call this the *polywheel subspace*. In degree four this codimension is one. It is somewhat surprising then that Θ^4 and $\Theta^2\Theta_2$ both lie in this subspace (see Appendix A.2). In degree five the polywheel subspace has codimension two, but again it contains both Θ^5 and $\Theta^3\Theta_2$. Later we will see that this is no accident, but is a consequence of a deep result known as the Wheeling Theorem which arose from the study of finite-type invariants of knots.

We have shown that the Rozansky-Witten invariants associated to polywheels give the Chern numbers, and hence any graph homology class in the polywheel subspace will give rise to a Rozansky-Witten invariant which is a characteristic number. At this stage we are unable to conclude anything about the graphs which lie outside the polywheel subspace, though later we will prove that some of them give rise to invariants which are not characteristic numbers. However, we will also show that some of these Rozansky-Witten invariants, which appear to be more general, are related in other ways to the characteristic numbers.

2.3 Products of virtual hyperkähler manifolds

In the previous chapter we mentioned briefly that elements of graph cohomology $\mathcal{A}(\emptyset)^*$ may be treated as "virtual" hyperkähler manifolds, since actual hyperkähler manifolds give rise to such elements. We now wish to show that this association is compatible with characteristic numbers under taking products. Firstly let us see how the characteristic numbers behave under taking products of actual hyperkähler manifolds.

Let X and Y be compact hyperkähler manifolds of real-dimensions $4k$ and $4l$ respectively. The tangent bundle of their product is given by

$$T(X \times Y) = p_1^* TX \oplus p_2^* TY$$

where p_1 and p_2 are the projections from $X \times Y$ onto X and Y respectively. Therefore

$$\begin{aligned} s(T(X \times Y)) &= s(p_1^* TX \oplus p_2^* TY) \\ &= p_1^* s(TX) + p_2^* s(TY) \end{aligned}$$

where

$$s(T(X \times Y)) = 2(k+l) + s_2(T(X \times Y)) + s_4(T(X \times Y)) + \dots$$

and similarly for $s(TX)$ and $s(TY)$. It follows that

$$\begin{aligned} & s_{\lambda_1} \cdots s_{\lambda_j}(X \times Y) \\ &\quad = \int_{X \times Y} s_{\lambda_1}(T(X \times Y)) \wedge \cdots \wedge s_{\lambda_j}(T(X \times Y)) \\ &\quad = \int_{X \times Y} (p_1^* s_{\lambda_1}(TX) + p_2^* s_{\lambda_1}(TY)) \wedge \cdots \wedge (p_1^* s_{\lambda_j}(TX) + p_2^* s_{\lambda_j}(TY)) \\ &\quad = \int_{X \times Y} \sum_{S \subset \{\lambda_1, \dots, \lambda_j\}} (\Lambda_{\lambda_i \in S} p_1^* s_{\lambda_i}(TX)) \wedge (\Lambda_{\lambda_i \notin S} p_2^* s_{\lambda_i}(TY)) \\ &\quad = \sum_{S \subset \{\lambda_1, \dots, \lambda_j\}} \int_{X \times Y} p_1^* \left(\Lambda_{\lambda_i \in S} s_{\lambda_i}(TX)\right) \wedge p_2^* \left(\Lambda_{\lambda_i \notin S} s_{\lambda_i}(TY)\right) \\ &\quad = \sum_{S \subset \{\lambda_1, \dots, \lambda_j\}} \left(\int_X \Lambda_{\lambda_i \in S} s_{\lambda_i}(TX) \right) \left(\int_Y \Lambda_{\lambda_i \notin S} s_{\lambda_i}(TY) \right) \\ &\quad = \sum_{S \subset \{\lambda_1, \dots, \lambda_j\}} \left(\prod_{\lambda_i \in S} s_{\lambda_i}(X) \right) \left(\prod_{\lambda_i \notin S} s_{\lambda_i}(Y) \right) \end{aligned}$$

where we regard $\prod_{\lambda_i\in S} s_{\lambda_i}(X)$ as being zero unless $\sum_{\lambda_i\in S}\lambda_i = 2k$ (and similarly $\sum_{\lambda_i\notin S}\lambda_i = 2l$).

Let $B_k \in (\mathcal{A}(\emptyset)^k)^*$ and $B_l \in (\mathcal{A}(\emptyset)^l)^*$ be (homogeneous) elements of graph cohomology. Recall that the product in graph cohomology is dual to the coproduct in graph homology. Thus if Γ is a trivalent graph with $2(k+l)$ vertices then

$$\begin{aligned}(B_kB_l)(\Gamma) &= (B_k, B_l)(\Delta\Gamma)\\ &= \sum_{\gamma\sqcup\gamma'=\Gamma} B_k(\gamma)B_l(\gamma')\end{aligned}$$

where B_k and B_l give zero unless evaluated on graphs with $2k$ and $2l$ vertices, respectively. Thus $B_kB_l \in (\mathcal{A}(\emptyset)^{k+l})^*$. If these elements are to be regarded as virtual hyperkähler manifolds of real-dimensions $4k$ and $4l$ respectively, then their Rozansky-Witten invariants should be given by

$$b_\Gamma(B_k) = B_k(\Gamma)$$

for Γ a graph with $2k$ vertices (and similarly for B_l). Their product $B_k \times B_l$ as virtual hyperkähler manifolds should correspond to their product B_kB_l as graph cohomology elements. Therefore if Γ is a trivalent graph with $2(k+l)$ vertices then

$$\begin{aligned}b_\Gamma(B_k\times B_l) &= (B_kB_l)(\Gamma)\\ &= \sum_{\gamma\sqcup\gamma'=\Gamma} B_k(\gamma)B_l(\gamma')\\ &= \sum_{\gamma\sqcup\gamma'=\Gamma} b_\gamma(B_k)b_{\gamma'}(B_l)\\ &= b_{\Delta(\Gamma)}(B_k, B_l).\end{aligned}$$

Finally, we wish to show that this corresponds to the product formula for characteristic numbers when we make the appropriate choice for Γ, namely, a polywheel $\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle$ where $(\lambda_1,\ldots,\lambda_j)$ is an even partition of $2(k+l)$. Recall that for an actual hyperkähler manifold X the Rozansky-Witten invariant corresponding to this polywheel is the Chern number $(-1)^j s_{\lambda_1}\cdots s_{\lambda_j}(X)$. Suppose

$$\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle = \gamma\sqcup\gamma'$$

is a decomposition of Γ into two disjoint graphs. If at least one vertex of w_{λ_i} lies in γ then the whole wheel must, and it would follow that no vertices of w_{λ_i} could lie in γ'. Therefore we can find a subset S of $\{\lambda_1,\ldots,\lambda_j\}$ such that if $\lambda_i\in S$ then the wheel w_{λ_i} lies in γ, and otherwise it lies in γ'. It follows that γ and γ' are graphs occurring in

$$\langle \cup_{\lambda_i\in S} w_{\lambda_i}\rangle \qquad \text{and} \qquad \langle \cup_{\lambda_i\notin S} w_{\lambda_i}\rangle$$

respectively. Furthermore, every such pair of graphs occurring in the above two polywheels give a decomposition of Γ, and this is true for all possible subsets S of

$\{\lambda_1, \dots, \lambda_j\}$. Therefore

$$\begin{aligned}\Delta\langle w_{\lambda_1} \cdots w_{\lambda_j}\rangle &= \sum_{\gamma \sqcup \gamma' = \Gamma} \gamma \otimes \gamma' \\ &= \sum_{S \subset \{\lambda_1, \dots, \lambda_j\}} \langle \cup_{\lambda_i \in S} w_{\lambda_i}\rangle \otimes \langle \cup_{\lambda_i \notin S} w_{\lambda_i}\rangle.\end{aligned}$$

Taking the corresponding Rozansky-Witten invariants and introducing the additional signs we get

$$\begin{aligned}(-1)^j b_{\langle w_{\lambda_1} \cdots w_{\lambda_j}\rangle}(B_k \times B_l) &= (-1)^j b_{\Delta\langle w_{\lambda_1} \cdots w_{\lambda_j}\rangle}(B_k, B_l) \\ &= \sum_{S \subset \{\lambda_1, \dots, \lambda_j\}} (-1)^{|S|} b_{\langle \cup_{\lambda_i \in S} w_{\lambda_i}\rangle}(B_k)(-1)^{j-|S|} b_{\langle \cup_{\lambda_i \notin S} w_{\lambda_i}\rangle}(B_l)\end{aligned}$$

where as usual we regard the terms on the right hand side as being zero unless $\sum_{\lambda_i \in S} \lambda_i = 2k$ and $\sum_{\lambda_i \notin S} \lambda_i = 2l$. For actual hyperkähler manifolds this is precisely the product formula for characteristic numbers, and therefore virtual hyperkähler manifolds are well behaved with respect to characteristic numbers under taking products.

2.4 The disconnected graph Θ^k

In Chapter 4 we will prove that Θ^k lies in the polywheel subspace for all k. Thus the corresponding Rozansky-Witten invariants b_{Θ^k} are characteristic numbers. This result relies on the Wheeling Theorem, which arose in the study of knot invariants. First though, let us look at some of the more basic properties of this invariant.

We have seen that the invariant $\frac{1}{k!} b_{\Theta^k}$ is multiplicative, so let us assume now that X is irreducible. The special property we shall need for irreducible X is the following (see Beauville [10], for example):

$$\mathrm{H}^{0,2m}_{\bar{\partial}}(X) = \langle [\bar{\omega}^m] \rangle$$

for $0 \leq m \leq k$. In other words, this cohomology group is one-dimensional and the Dolbeault class represented by the form $\bar{\omega}^m$ gives a generator.

If Γ has $2m$ vertices where $m < k$ then we can still define

$$[\Gamma(\Phi)] \in \mathrm{H}^{0,2m}_{\bar{\partial}}(X)$$

as before, and it will be some multiple β_Γ of $[\bar{\omega}^m]$. Note that the constant β_Γ still depends on X. Taking the disjoint union of several graphs will simply correspond to multiplying the β's, ie.

$$[\gamma\gamma'(\Phi)] = \beta_\gamma \beta_{\gamma'} [\bar{\omega}^m] \in \mathrm{H}^{0,2m}_{\bar{\partial}}(X)$$

where $2m$ is the total number of vertices in $\gamma\gamma'$. In particular, if $m = k$ we find

$$b_{\gamma\gamma'}(X) = \frac{1}{(8\pi^2)^k k!}\beta_\gamma\beta_{\gamma'}\int_X \bar{\omega}^k\omega^k.$$

For example

$$b_{\Theta^k}(X) = \frac{1}{(8\pi^2)^k k!}\beta_\Theta^k\int_X \bar{\omega}^k\omega^k.$$

Since ω_2 is a Kähler form for X, a volume form is given by

$$\begin{aligned}\frac{1}{(2k)!}\omega_2^{2k} &= \frac{1}{(2k)!}(\frac{\omega+\bar{\omega}}{2})^{2k}\\ &= \frac{1}{2^{2k}(k!)^2}\omega^k\bar{\omega}^k.\end{aligned}$$

Therefore

$$\int_X \bar{\omega}^k\omega^k = 2^{2k}(k!)^2\mathrm{vol}(X)$$

and hence

$$b_{\Theta^k}(X) = \frac{k!}{(2\pi^2)^k}\beta_\Theta^k\mathrm{vol}(X).$$

To determine β_Θ observe that

$$\beta_\Theta\int_X \bar{\omega}^k\omega^k = \int_X [\Theta(\Phi)][\bar{\omega}^{k-1}][\omega^k].$$

On the other hand, since X is Ricci-flat we have (see Besse [12])

$$\begin{aligned}\|K\|^2 &= \frac{8\pi^2}{(2k-2)!}\int_X c_2\omega_2^{2k-2}\\ &= -\frac{4\pi^2}{(2k-2)!}\int_X s_2(\frac{\omega+\bar{\omega}}{2})^{2k-2}\\ &= -\frac{4\pi^2}{2^{2k-2}((k-1)!)^2}\int_X s_2\omega^{k-1}\bar{\omega}^{k-1}\end{aligned}$$

where $\|K\|^2$ is the $\mathcal{L}^2$-norm of the curvature, and c_2 is the second Chern class of the tangent bundle, which is equal to $-s_2/2$. We wish to relate the two integrals

$$\int_X s_2\omega^{k-1}\bar{\omega}^{k-1} \qquad \text{and} \qquad \int_X [\Theta(\Phi)][\bar{\omega}^{k-1}][\omega^k].$$

Observe that Θ can be obtained by joining the spokes of a 2-wheel. So consider the section

$$w_2(\Phi) \in C^\infty(X, (T^*\otimes\bar{T}^*)^{\otimes 2})$$

given by

$$w_2(\Phi)_{k_1\bar{l}_1k_2\bar{l}_2} = -\mathrm{Tr}(K^{\otimes 2})_{k_1\bar{l}_1k_2\bar{l}_2}.$$

Contracting with $\tilde{\omega}$ then taking the projection $\bar{S}$ to the exterior product we get (paying careful attention to the orientation, as always)

$$\Theta(\Phi) = -\bar{S}\mathrm{Tr}(K^{\otimes 2})_{k_1\bar{l}_1k_2\bar{l}_2}\omega^{k_1k_2}d\bar{z}_{l_1}\wedge d\bar{z}_{l_2} \in \Omega^{0,2}(X).$$

Since $\omega^{k_1k_2}$ is skew-symmetric, we can rewrite the right hand side as

$$-S\bar{S}\mathrm{Tr}(K^{\otimes 2})_{k_1\bar{l}_1k_2\bar{l}_2}\omega^{k_1k_2}d\bar{z}_{l_1}\wedge d\bar{z}_{l_2} = -\mathrm{Tr}(K^2)_{k_1\bar{l}_1k_2\bar{l}_2}\omega^{k_1k_2}d\bar{z}_{l_1}\wedge d\bar{z}_{l_2}.$$

Taking the corresponding cohomology classes, we find

$$[\Theta(\Phi)] = -[\mathrm{Tr}(K^2)_{k_1\bar{l}_1k_2\bar{l}_2}\omega^{k_1k_2}d\bar{z}_{l_1}\wedge d\bar{z}_{l_2}] \in \mathrm{H}^{0,2}_{\bar{\partial}}(X).$$

In general, for a $(2,0)$-form $\alpha = \sum \alpha_{ij}dz_i\wedge dz_j$ we have

$$\alpha\wedge\omega^{k-1} = \frac{1}{2k}(\sum \omega^{ij}\alpha_{ij})\omega^k.$$

Applying this to

$$\begin{aligned}\mathrm{Tr}(K^2) &= \mathrm{Tr}(K^2)_{k_1\bar{l}_1k_2\bar{l}_2}dz_{k_1}\wedge d\bar{z}_{l_1}\wedge dz_{k_2}\wedge d\bar{z}_{l_2}\\ &= -\mathrm{Tr}(K^2)_{k_1\bar{l}_1k_2\bar{l}_2}dz_{k_1}\wedge dz_{k_2}\wedge d\bar{z}_{l_1}\wedge d\bar{z}_{l_2}\end{aligned}$$

we find

$$\mathrm{Tr}(K^2)\wedge\omega^{k-1} = \frac{-1}{2k}(\sum \omega^{k_1k_2}\mathrm{Tr}(K^2)_{k_1\bar{l}_1k_2\bar{l}_2}d\bar{z}_{l_1}\wedge d\bar{z}_{l_2})\omega^k$$

and taking cohomology classes

$$\begin{aligned}(2\pi i)^2 s_2[\omega^{k-1}] &= [\mathrm{Tr}(K^2)][\omega^{k-1}]\\ &= \frac{-1}{2k}[\sum \omega^{k_1k_2}\mathrm{Tr}(K^2)_{k_1\bar{l}_1k_2\bar{l}_2}d\bar{z}_{l_1}\wedge d\bar{z}_{l_2}][\omega^k]\\ &= \frac{1}{2k}[\Theta(\Phi)][\omega^k].\end{aligned}$$

Therefore

$$-8\pi^2 k s_2[\omega^{k-1}][\bar{\omega}^{k-1}] = [\Theta(\Phi)][\omega^k][\bar{\omega}^{k-1}]$$

and hence

$$-8\pi^2 k\int_X s_2\omega^{k-1}\bar{\omega}^{k-1} = \int_X[\Theta(\Phi)][\bar{\omega}^{k-1}][\omega^k].$$

We can now determine β_Θ, and we find

$$\beta_\Theta = \frac{1}{2k}\frac{\|K\|^2}{\mathrm{vol}(X)}$$

which gives the following proposition.

Proposition 6 *For an irreducible hyperkähler manifold X of real-dimension $4k$*

$$b_{\Theta^k}(X) = \frac{k!}{(4\pi^2 k)^k} \frac{\|K\|^{2k}}{\mathrm{vol}(X)^{k-1}}.$$

This formula is intricately related to the metric on the hyperkähler manifold; it involves both the curvature and the volume of X. However, in Chapter 4 we will see that Θ^k is in the polywheel subspace. Thus b_{Θ^k} is a characteristic number, which in the hyperkähler case means that it is (somewhat remarkably) a topological invariant of the manifold.

3 Relations in graph homology I

3.1 Perturbative Chern-Simons theory

The Rozansky-Witten invariants allow us to associate an element of graph cohomology $\mathcal{A}(\emptyset)^*$, the dual of graph homology, to a hyperkähler manifold. Other elements of $\mathcal{A}(\emptyset)^*$ can tell us something about relations in graph homology, ie. if trivalent graphs Γ_1 and Γ_2 are homologous and $B \in \mathcal{A}(\emptyset)^*$, then

$$B(\Gamma_1) = B(\Gamma_2).$$

This statement is completely tautologous, but we shall use it to deduce some interesting relations. In particular, the elements of $\mathcal{A}(\emptyset)^*$ we shall use come from the weights $c_\Gamma(\mathfrak{g})$ arising from a semisimple Lie algebra, as in perturbative Chern-Simons theory.

The partition function of Chern-Simons theory gives us an invariant of three-manifolds. A Feynman diagram calculation (as in Axelrod and Singer [2, 3]) shows that it is of finite type and looks like

$$\sum_\Gamma c_\Gamma(\mathfrak{g}) I_\Gamma^{\mathrm{CS}}(M)$$

where the weights $c_\Gamma(\mathfrak{g})$ depend on the Lie algebra $\mathfrak{g}$ of the gauge group and $I_\Gamma^{\mathrm{CS}}(M)$ depends on the three-manifold M. More precisely, it is believed that this sum occurs as the expansion of the contribution coming from the trivial connection. In any case, we shall only be concerned here with the weights $c_\Gamma(\mathfrak{g})$. For any quadratic Lie algebra $\mathfrak{g}$, ie. Lie algebra with an invariant inner product, these weights can be rigorously defined as follows.

Choose a basis $\{x_1, \ldots, x_n\}$ for $\mathfrak{g}$ and let the invariant inner product be σ^{ij} with respect to this basis. The structure constants ${c_{ij}}^k$ of $\mathfrak{g}$ are defined by

$$[x_i, x_j] = {c_{ij}}^k x_k.$$

Using $\sigma^{-1} = \sigma_{ij}$ to lower indices we obtain

$$c_{ijk} = \sum_m {c_{ij}}^m \sigma_{mk}.$$

This is totally skew-symmetric since the Lie bracket is skew and the inner product invariant.

Let Γ be an oriented trivalent graph with $2k$ vertices. We use the standard notion of orientation here, namely an equivalence class of cyclic orderings of the outgoing edges at each vertex, with two orderings being equivalent if they differ at even number of vertices. Recall that this is equivalent to a Rozansky-Witten orientation. Place a copy of c_{ijk} at each vertex and attach the indices ijk to the outgoing edges in accordance with the cyclic ordering given by the orientation, ie. if we first attach i to one of the edges, then which way we attach j and k to the remaining two edges is determined by the cyclic ordering. Note that this is well-defined as

$$c_{ijk} = c_{jki} = c_{kij}$$

so it does not depend on which edge we first attach i to. Now use σ^{ij} to contract the indices along edges. Thus if two vertices are connected by an edge, and the two ends of the edge are labelled i_t and i_s, then we contract with $\sigma^{i_t i_s}$. Since σ^{ij} is symmetric this does not require an orientation of the edge. The resulting number is the weight $c_\Gamma(\mathfrak{g})$, and it does not depend on the choice of basis for $\mathfrak{g}$.

For example, if $\Gamma = \Theta$ (with the canonical orientation coming from drawing in the plane) then

$$c_\Theta(\mathfrak{g}) = \sum_{i_1,j_1,k_1,i_2,j_2,k_2} c_{i_1 j_1 k_1}\sigma^{i_1 i_2}\sigma^{j_1 k_2}\sigma^{k_1 j_2}c_{i_2 j_2 k_2}.$$

Note that we have used sub-indices 1 and 2 but this construction does not depend on an ordering of the vertices.

Reversing the orientation of the graph by reversing the cyclic ordering at one vertex changes the indices used to label the outgoing edges at that vertex. This is equivalent to swapping two indices on a c_{ijk}. Since

$$c_{ikj} = -c_{ijk}$$

this reverses the sign of $c_\Gamma(\mathfrak{g})$, ie.

$$c_{\bar{\Gamma}}(\mathfrak{g}) = -c_\Gamma(\mathfrak{g}).$$

The IHX relation also holds as a consequence of the Jacobi identity, ie.

$$c_{\Gamma_I}(\mathfrak{g}) = c_{\Gamma_H}(\mathfrak{g}) - c_{\Gamma_X}(\mathfrak{g}).$$

Thus if we linearly extend $c_\Gamma(\mathfrak{g})$ to rational linear combinations of trivalent graphs, then the weights we get are well defined on graph homology and so define an element of the dual space $\mathcal{A}(\emptyset)^*$. In the next subsection we shall calculate some of these numbers explicitly when $\mathfrak{g} = \mathfrak{su}(2)$.

3.2 Gauge group SU(2)

In the case of Lie algebras coming from semisimple Lie groups an invariant inner product is given by $-1/2$ times the Killing form, which is given by taking the trace of the product in the adjoint representation. Therefore

$$\sigma_{ij} = -\frac{1}{2}\mathrm{Tr}(\mathrm{ad}x_i \mathrm{ad}x_j).$$

The structure constants are given by the adjoint representation, ie.

$$[x_i, x_j] = (\mathrm{ad}x_i)x_j = c_{ij}{}^k x_k$$

so that

$$c_{ij}{}^k = (\mathrm{ad}x_i)_{jk}$$

where the jk coefficients indicate that we regard $\mathrm{ad}x_i$ as an $n \times n$ matrix.

When $\mathfrak{g} = \mathfrak{su}(2)$ we can take as a basis the 3×3 matrices

$$x_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

written in the adjoint representation (we drop the 'ad' notation from here onwards). The Killing form is given by $\mathrm{Tr}(x_i x_j)$ and a calculation shows that this is -2 times the identity matrix, so that the inner product is simply δ. The structure constants are given by the unique totally skew-symmetric tensor in three-dimensions

$$c_{ijk} = \epsilon_{ijk}$$

(ie. unique up to scale, but the scale here is one). More often we shall use the 3×3 matrix x_i to represent $c_{ij}{}^k$.

Let us now calculate some of the weights arising from the $\mathfrak{su}(2)$ Lie algebra, beginning with $\Gamma = \Theta$. Then

$$\begin{aligned} c_\Theta(\mathfrak{su}(2)) &= \sum_{i_1,j_1,k_1,i_2,j_2,k_2} \epsilon_{i_1 j_1 k_1} \delta^{i_1 i_2} \delta^{j_1 k_2} \delta^{k_1 j_2} \epsilon_{i_2 j_2 k_2} \\ &= -\sum_{i,j,k} (\epsilon_{ijk})^2 \\ &= -6. \end{aligned}$$

For a disconnected graph in this theory, we simply multiply the corresponding weights for the connected components. For example

$$c_{\Theta^k}(\mathfrak{su}(2)) = c_\Theta(\mathfrak{su}(2))^k = (-6)^k.$$

Suppose that Γ contains a 2-wheel w_2. The contribution to $c_\Gamma(\mathfrak{su}(2))$ from the 2-wheel is

$$\begin{aligned} w_2(\mathfrak{su}(2)) &= \sum_{j_1,k_1,j_2,k_2} \epsilon_{i_1 j_1 k_1} \delta^{k_1 j_2} \epsilon_{i_2 j_2 k_2} \delta^{k_2 j_1} \\ &= -\sum_{j,k} \epsilon_{i_1 jk} \epsilon_{i_2 jk} \\ &= -2\delta_{i_1 i_2}. \end{aligned}$$

Therefore

$$c_\Gamma(\mathfrak{su}(2)) = -2c_{\Gamma'}(\mathfrak{su}(2))$$

where Γ' is the graph constructed from Γ by removing the 2-wheel and joining the two univalent vertices (ie. the two loose ends created).

More generally, suppose that Γ contains a λ-wheel w_λ. The contribution to $c_\Gamma(\mathfrak{su}(2))$ from this part of the graph would be

$$\begin{aligned}
w_\lambda(\mathfrak{su}(2)) &= \sum_{j_1,k_1,\ldots,j_\lambda,k_\lambda} \epsilon_{i_1 j_1 k_1}\delta^{k_1 j_2}\epsilon_{i_2 j_2 k_2}\delta^{k_2 j_3}\cdots\epsilon_{i_\lambda j_\lambda k_\lambda}\delta^{k_\lambda j_1} \\
&= \sum_{j_1,\ldots,j_\lambda} c_{i_1 j_1}{}^{j_2} c_{i_2 j_2}{}^{j_3}\cdots c_{i_\lambda j_\lambda}{}^{j_1} \\
&= \sum_{j_1,\ldots,j_\lambda} (x_{i_1})_{j_1 j_2}(x_{i_2})_{j_2 j_3}\cdots(x_{i_\lambda})_{j_\lambda j_1} \\
&= \mathrm{Tr}(x_{i_1}x_{i_2}\cdots x_{i_\lambda})
\end{aligned}$$

where the indices $i_1,\ldots,i_\lambda$ label the spokes of the wheel. Thus if Γ is the polywheel $\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle$ we find that

$$c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) = \sum \mathrm{Tr}(x_{i_{11}}x_{i_{12}}\cdots x_{i_{1\lambda_1}})\cdots\mathrm{Tr}(x_{i_{j1}}x_{i_{j2}}\cdots x_{i_{j\lambda_j}})$$

where the sum is over all ways of contracting together the spokes using the inner product, or since this is δ, simply pairing the indices and summing. We can assume all λ's are even as otherwise the polywheel vanishes. Let $\lambda_1+\ldots+\lambda_j=2k$, so that the polywheel $\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle$ has $2k$ vertices. By considering what happens when we join two spokes, we will express the above weight in terms of weights of polywheels with fewer vertices, thus arriving at a recursive formula. In order to proceed we need the following two identities.

Lemma 7 *Let A be a 3×3 matrix. Then*

$$\sum_i x_i A x_i = A^t - (\mathrm{Tr}A)I.$$

Lemma 8 *Let A and B be 3×3 matrices. Then*

$$\sum_i \mathrm{Tr}(Ax_i)\mathrm{Tr}(Bx_i) = -\frac{1}{2}\mathrm{Tr}((A-A^t)(B-B^t)).$$

Proof Simply let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} m & n & o \\ p & q & r \\ s & t & u \end{pmatrix}$$

then expand both sides of the relations. $\square$

The first lemma will describe the effect of joining two spokes on the same wheel, and the second will describe the effect of joining two spokes on different wheels.

So first consider a factor in $c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2))$ which looks like

$$\sum_k \mathrm{Tr}(x_{i_1}\cdots x_{i_{a-1}}x_k x_{i_{a+1}}\cdots x_{i_{a+b-1}}x_k x_{i_{a+b+1}}\cdots x_{i_\lambda})$$

where $1 \le b \le \lambda - 1$ and $1 \le a \le \lambda - b$. The first lemma allows us to rewrite this as two terms

$$\begin{aligned}\mathrm{Tr}(x_{i_1}\cdots x_{i_{a-1}}&(x_{i_{a+1}}\cdots x_{i_{a+b-1}})^t x_{i_{a+b+1}}\cdots x_{i_\lambda})\\ &= (-1)^{b-1}\mathrm{Tr}(x_{i_1}\cdots x_{i_{a-1}}x_{i_{a+b-1}}\cdots x_{i_{a+1}}x_{i_{a+b+1}}\cdots x_{i_\lambda})\end{aligned}$$

and

$$-\mathrm{Tr}(x_{i_1}\cdots x_{i_{a-1}}x_{i_{a+b+1}}\cdots x_{i_\lambda})\mathrm{Tr}(x_{i_{a+1}}\cdots x_{i_{a+b-1}}).$$

Substituting these two terms back into the expression for $c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2))$ we find that the first gives us

$$(-1)^{b-1}c_{\langle w_{\lambda_1}\cdots w_{\lambda_s-2}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2))$$

and the second gives

$$-c_{\langle w_{\lambda_1}\cdots w_{\lambda_{s-1}}w_{\lambda_s-b-1}w_{b-1}w_{\lambda_{s+1}}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2))$$

where we originally joined two spokes of the sth wheel, and the second expression vanishes unless b is odd. Summing from $a = 1$ to $\lambda_s - b$ and $b = 1$ to $\lambda_s - 1$, the first expression gives us

$$\frac{\lambda_s}{2}c_{\langle w_{\lambda_1}\cdots w_{\lambda_s-2}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2))$$

and summing from $a = 1$ to $\lambda_s - b$ and rewriting b as $l + 1$, the second expression gives us

$$-\sum_{l=0,\text{even}}^{\lambda_s-2}(l+1)c_{\langle w_{\lambda_1}\cdots w_{\lambda_{s-1}}w_{\lambda_s-l-2}w_l w_{\lambda_{s+1}}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)).$$

We also need to sum from $s = 1$ to j but we shall do this later.

Now consider what happens when we join spokes from different wheels. There are λ_1 choices for the spoke on the first wheel and λ_2 choices for the spoke on the second wheel, and each term looks like

$$\sum_k \mathrm{Tr}(x_{i_{11}}x_{i_{12}}\cdots x_{i_{1(\lambda_1-1)}}x_k)\mathrm{Tr}(x_{i_{21}}x_{i_{22}}\cdots x_{i_{2(\lambda_2-1)}}x_k).$$

By the second lemma we can rewrite this as the sum of four terms, all of which give the same expression (after some relabelling of indices)

$$-\frac{1}{2}\mathrm{Tr}(x_{i_{11}}x_{i_{12}}\cdots x_{i_{1(\lambda_1-1)}}x_{i_{21}}x_{i_{22}}\cdots x_{i_{2(\lambda_2-1)}}).$$

Substituting this back into the expression for $c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2))$ we get

$$-2c_{\langle w_{\lambda_1}\cdots w_{\lambda_{s-1}}w_{\lambda_{s+1}}\cdots w_{\lambda_{t-1}}w_{\lambda_{t+1}}\cdots w_{\lambda_j}w_{\lambda_s+\lambda_t-2}\rangle}(\mathfrak{su}(2))$$

where we originally joined spokes coming from the sth and tth wheels. Recall that there are $\lambda_s\lambda_t$ such terms, coming from the different ways of choosing the spokes on each wheel.

Combining the above results we can obtain a recursion relation for the weight $c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2))$. It remains to note that a given graph occurring in $\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle$ has k pairs of spokes joined together; on the other hand, we've considered what happens when we join just one pair of spokes, for which there are k choices. Thus we've counted everything k times, and hence the relation is

$$\begin{aligned}
kc_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) = {} & \sum_s \frac{\lambda_s}{2} c_{\langle w_{\lambda_1}\cdots w_{\lambda_s-2}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) \\
& -\sum_s \sum_{l=0,\text{even}}^{\lambda_s-2} (l+1) c_{\langle w_{\lambda_1}\cdots w_{\lambda_{s-1}}w_{\lambda_s-l-2}w_l w_{\lambda_{s+1}}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) \\
& -\sum_{\text{pairs } s,t} 2\lambda_s\lambda_t c_{\langle w_{\lambda_1}\cdots w_{\lambda_{s-1}}w_{\lambda_{s+1}}\cdots w_{\lambda_{t-1}}w_{\lambda_{t+1}}\cdots w_{\lambda_j}w_{\lambda_s+\lambda_t-2}\rangle}(\mathfrak{su}(2)).
\end{aligned}$$

The initial conditions are

1. $c_{\emptyset}(\mathfrak{su}(2)) = 1$,
2. $c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}w_0\rangle}(\mathfrak{su}(2)) = 3c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2))$.

The second of these follows from the fact that we regard the wheel with no spokes, w_0, as a closed loop; thus it contributes a factor of

$$\mathrm{Tr}(\mathrm{Id}_3) = 3.$$

Next we solve the recursion relation.

Proposition 9 *Let $\lambda_1, \ldots, \lambda_j$ be positive even integers whose sum is $2k$. Then*

1. $c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) = 2^{j-1}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2))$,
2. $c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) = \frac{(-1)^k(2k+1)!}{2^{k-1}k!}$.

Proof We will use induction on k, the first case ($k = j = 1$, $\lambda_1 = 2$) being trivial. Suppose the results are true up to level k, and consider the next level $k+1$.

Suppose that all λ's are greater than 2. Using the recursion relation and applying the inductive hypothesis to the right hand side we find

$$
\begin{aligned}
&(k+1)c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) \\
&\quad= \sum_s \frac{\lambda_s}{2}2^{j-1}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) - \sum_s \sum_{l=2,\mathrm{even}}^{\lambda_s-4}(l+1)2^j c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) \\
&\qquad -\sum_s c_{\langle w_{\lambda_1}\cdots w_{\lambda_{s-1}}w_{\lambda_s-2}w_0 w_{\lambda_{s+1}}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) \\
&\qquad -\sum_s(\lambda_s-1)c_{\langle w_{\lambda_1}\cdots w_{\lambda_{s-1}}w_0 w_{\lambda_s-2}w_{\lambda_{s+1}}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) \\
&\qquad -\sum_{\text{pairs } s,t} 2\lambda_s\lambda_t 2^{j-2}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) \\
&\quad= \sum_s \lambda_s 2^{j-2}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) - \sum_s\left(\left(\frac{\lambda_s-2}{2}\right)^2-1\right)2^j c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) \\
&\qquad -\sum_s 3\lambda_s 2^{j-1}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) - \sum_{s\neq t}\lambda_s\lambda_t 2^{j-2}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) \\
&\quad= \left(\sum_s(\lambda_s-\lambda_s^2+4\lambda_s-6\lambda_s) - \sum_{s\neq t}\lambda_s\lambda_t\right)2^{j-2}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) \\
&\quad= -\left(\sum_s\lambda_s + \left(\sum_s\lambda_s\right)^2\right)2^{j-2}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) \\
&\quad= -(k+1)(2k+3)2^{j-1}c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)).
\end{aligned}
$$

When $j=1$ this tells us

$$
\begin{aligned}
(k+1)c_{\langle w_{2(k+1)}\rangle}(\mathfrak{su}(2)) &= -(k+1)(2k+3)c_{\langle w_{2k}\rangle}(\mathfrak{su}(2)) \\
&= (k+1)\frac{(-1)^{k+1}(2k+3)!}{2^k(k+1)!}
\end{aligned}
$$

which proves the second part of the proposition. It also shows that

$$(k+1)c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) = (k+1)2^{j-1}c_{\langle w_{2(k+1)}\rangle}(\mathfrak{su}(2))$$

which proves the first part, though we assumed that all λ's were greater than 2. If this assumption is not satisfied there are some additional 0-wheels occurring in the formulae of the proof but the proposition is still satisfied. □

These calculations of the $\mathfrak{su}(2)$ weight system are of use in deriving relations in graph homology, for example those occurring in Appendix A. In the next subsection we shall describe how this process works.

3.3 Graph homology relations

The $\mathfrak{su}(2)$ weight system gives us an element of the dual space $\mathcal{A}(\emptyset)^*$. Equivalently, by breaking this up into separate degrees, we get homogeneous elements C_k in each of the dual spaces $(\mathcal{A}(\emptyset)^k)^*$. Recall that since graph homology has a Hopf algebra structure we can multiply these C_k's, the product in graph cohomology being the dual of the coproduct in graph homology. If Γ has $2(k+l)$ vertices, then $C_kC_l \in (\mathcal{A}(\emptyset)^{k+l})^*$ is given by

$$\begin{aligned}(C_kC_l)(\Gamma) &= (C_k, C_l)(\Delta\Gamma) \\ &= \sum_{\gamma\sqcup\gamma'=\Gamma} C_k(\gamma)C_l(\gamma')\end{aligned}$$

where C_k and C_l give zero unless evaluated on graphs with $2k$ and $2l$ vertices, respectively. More generally we can form

$$C_{m_1}C_{m_2}\cdots C_{m_j} \in (\mathcal{A}(\emptyset)^{m_1+\ldots+m_j})^*.$$

We know that the number of polywheels with $2k$ vertices is equal to $p(k)$, the number of partitions of k. Suppose a graph Γ with $2k$ vertices belongs to the polywheel subspace. Then we can write

$$\Gamma = \sum_{(m_1,\ldots,m_j)=\text{partition of}\,k} a_{(m_1,\ldots,m_2)}\langle w_{2m_1}\cdots w_{2m_j}\rangle.$$

To determine the coefficients $a_{(m_1,\ldots,m_j)}$ we simply need to act on both sides with $C_{n_1}\cdots C_{n_j}$ for all partitions $(n_1,\ldots,n_j)$ of k then solve the set of linear equations.

For example, take

$$\Gamma = \Theta^2 = a_{(1,1)}\langle w_2w_2\rangle + a_{(2)}\langle w_4\rangle.$$

Acting with C_2 and C_1C_1 gives

$$\begin{aligned}(-6)^2 &= 60a_{(1,1)} + 30a_{(2)} \\ 2(-6)^2 &= 2(-6)^2a_{(1,1)}\end{aligned}$$

respectively, and hence solving we find

$$\Theta^2 = \langle w_2w_2\rangle - \frac{4}{5}\langle w_4\rangle.$$

An equivalent formulation of the above procedure is given by regarding the elements C_k as virtual hyperkähler manifolds of real-dimension $4k$, as described in the previous chapters. The Rozansky-Witten invariants of C_k are given by

$$b_\Gamma(C_k) = C_k(\Gamma) = c_\Gamma(\mathfrak{su}(2))$$

for trivalent graphs with $2k$ vertices. The polywheels correspond to Chern numbers

$$\begin{aligned} s_{\lambda_1}\cdots s_{\lambda_j}(C_k) &= (-1)^j c_{\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle}(\mathfrak{su}(2)) \\ &= \frac{(-1)^{k+j}(2k+1)!}{2^{k-j}k!} \end{aligned}$$

and the product $C_kC_l \in (\mathcal{A}(\emptyset)^{k+l})^*$ corresponds to the product manifold $C_k \times C_l$ of dimension $4(k+l)$.

If Γ lies in the polywheel subspace we know that the corresponding Rozansky-Witten invariant will be a characteristic number, ie. a linear combination of Chern numbers. Let us suppose this is the case for $\Gamma = \Theta^k$, and hence $\frac{1}{48^k k!}b_{\Theta^k}$ is a characteristic number. We have included the extra factor of $k!$ because this makes the invariant multiplicative (see Subsection 1.6); the power of 48 is simply to make it more convenient to state our end result. Following Hirzebruch's work on multiplicative sequences (see [29]), this means that the generating sequence

$$g(x) = 1 + \frac{1}{48}b_\Theta x^2 + \frac{1}{48^2 2!}b_{\Theta^2}x^4 + \frac{1}{48^3 3!}b_{\Theta^3}x^6 + \ldots$$

is determined by a single power series $f(x)$ which enables us to write each b_{Θ^k} as a characteristic number. More specifically, in $4k$ dimensions let $\gamma_1, \ldots, \gamma_{2k}$ be Chern roots, so that $s_\lambda = \gamma_1^\lambda + \ldots + \gamma_{2k}^\lambda$. Then

$$g(x) = f(x\gamma_1)\cdots f(x\gamma_{2k})$$

and by expanding this out and writing the coefficients in terms of s_λ we get precise expressions for b_{Θ^k} in terms of Chern numbers. It is convenient to write

$$\ln f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

and hence

$$\ln g(x) = 2ka_0 + a_1 s_1 x + a_2 s_2 x^2 + a_3 s_3 x^3 + a_4 s_4 x^4 + \ldots$$

Since we are in the hyperkähler case, all odd coefficients vanish, and a_0 must also vanish so that the series is independent of k. Then

$$\begin{aligned} g(x) &= \exp(a_2 s_2 x^2 + a_4 s_4 x^4 + a_6 s_6 x^6 + \ldots) \\ &= 1 + a_2 s_2 x^2 + (a_4 s_4 + \frac{1}{2}a_2^2 s_2^2)x^4 + (a_6 s_6 + a_2 a_4 s_2 s_4 + \frac{1}{6}a_2^3 s_2^3)x^6 + \ldots \end{aligned}$$

The coefficients a_i can now be determined by evaluating on a *basic sequence* of manifolds, which generate the cobordism ring (see Hirzebruch [29]).

In our case

$$s_{2k}(C_k) = \frac{(-1)^{k+1}(2k+1)!}{2^{k-1}k!}$$

is non-zero, so the virtual manifolds C_k form a basic sequence. For this sequence of manifolds

$$\begin{aligned} g(x) &= 1+\frac{1}{48}b_\Theta(C_1)x^2+\frac{1}{48^2 2!}b_{\Theta^2}(C_2)x^4+\frac{1}{48^3 3!}b_{\Theta^3}(C_3)x^6+\ldots \\ &= 1-\frac{1}{48}6x^2+\frac{1}{48^2 2!}6^2x^4-\frac{1}{48^3 3!}6^3x^6+\ldots \\ &= 1-\frac{1}{8}x^2+\frac{1}{8^2 2!}x^4-\frac{1}{8^3 3!}x^6+\ldots \\ &= \exp(-\frac{1}{8}x^2) \end{aligned}$$

and we also have

$$\begin{aligned} g(x) &= 1+a_2s_2(c_1)x^2+(a_4s_4+\frac{1}{2}a_2^2s_2^2)(c_2)x^4+(a_6s_6+a_2a_4s_2s_4+\frac{1}{6}a_2^3s_2^3)(c_3)x^6+\ldots \\ &= 1+a_2s_2(c_1)x^2+(a_4-a_2^2)s_4(c_2)x^4+(a_6-2a_2a_4+\frac{4}{6}a_2^3)s_6(c_3)x^6+\ldots \end{aligned}$$

Comparing this to the sequence

$$\begin{aligned} f(x)^{-2} &= \exp(-2\ln f(x)) \\ &= \exp(-2a_2x^2-2a_4x^4-2a_6x^6-\ldots) \\ &= 1-2a_2x^2+(-2a_4+2a_2^2)x^4+(-2a_6+4a_2a_4-\frac{8}{6}a_2^3)x^6+\ldots \end{aligned}$$

we see that $f(x)^{-2}$ has coefficients equal to

$$-2\frac{(-1)^k}{8^k k! s_{2k}(C_k)}=\frac{1}{4^k(2k+1)!}$$

and therefore

$$\begin{aligned} f(x)^{-2} &= 1+\sum_{k=1}^{\infty}\frac{1}{4^k(2k+1)!}x^{2k} \\ &= \frac{e^{x/2}-e^{-x/2}}{x} \\ &= \frac{\sinh(x/2)}{(x/2)}. \end{aligned}$$

The generating sequence for the Todd genus Td is determined by the function $\frac{x}{1-e^x}$. In the hyperkähler case the Chern roots occur in plus/minus pairs, so we can take instead the generating function

$$(\frac{x}{1-e^x}\times\frac{-x}{1-e^{-x}})^{1/2}=\frac{x}{e^{x/2}-e^{-x/2}}=\frac{(x/2)}{\sinh(x/2)}.$$

Comparing this with the formula for $f(x)^{-2}$, we see that the generating sequence for $\frac{1}{48^k k!} b_{\Theta^k}$ is precisely $\mathrm{Td}^{1/2}$. Of course, as stated earlier all our results are independent of the choice of compatible complex structure. In particular, if we wish to write things in a form that is manifestly independent of this choice then we can rephrase our results in terms of the (topologically invariant) Pontryagin classes instead of the Chern classes. This would also mean replacing the Todd genus by the $\hat{A}$-genus, and so it is really $\hat{A}^{1/2}$ which generates $\frac{1}{48^k k!} b_{\Theta^k}$.

The above arguments were based upon the assumption that b_{Θ^k} should be a characteristic number, and so do not constitute a complete proof. However, in the next chapter we shall prove

Theorem 10 *A generating sequence for* $\frac{1}{48^k k!} b_{\Theta^k}$ *is*

$$\begin{aligned} \mathrm{Td}^{1/2} &= 1 - \frac{1}{48} s_2 + \frac{1}{48^2 2!}(s_2^2 + \frac{4}{5} s_4) - \frac{1}{48^3 3!}(s_2^3 + \frac{12}{5} s_2 s_4 + \frac{64}{35} s_6) + \\ &\quad + \frac{1}{48^4 4!}(s_2^4 + \frac{24}{5} s_2^2 s_4 + \frac{48}{25} s_4^2 + \frac{256}{35} s_2 s_6 + \frac{1152}{175} s_8) - \ldots \end{aligned}$$

In fact, we shall prove the corresponding relation in graph homology from which this result follows. All we need to show is that Θ^k lies in the polywheel subspace for all k, but actually our methods will reproduce the above formula.

3.4 The SU(2) virtual hyperkähler manifolds

Before ending this chapter let us say a few more words about the virtual hyperkähler manifolds C_k arising from the perturbative Chern-Simons theory with gauge group SU(2). One could ask whether there is a family of hyperkähler manifolds whose Rozansky-Witten invariants realize the $\mathfrak{su}(2)$ weight system. Appendix E contains tables of values of Rozansky-Witten invariants in low degrees for the compact hyperkähler manifolds $S^{[k]}$ and $T^{[[k]]}$ and their products. The former is the Hilbert scheme of k points on a K3 surface S and the latter is the generalized Kummer variety. We will say much more about these manifolds in Chapter 5, where we will also perform the calculations which lead to the values in Appendix E. The last part of that appendix contains values for the $\mathfrak{su}(2)$ weight system, or equivalently, values of the Rozansky-Witten invariants for the virtual hyperkähler manifolds C_k. Let us compare this weight system to the hyperkähler weight system. Firstly, using Hilbert schemes of points on a K3 surface S, we find

$$C_1 \sim -\frac{1}{8} S$$

$$C_2 \sim -\frac{1}{12} S^{[2]} + \frac{7}{96} S^2$$

$$C_3 \sim -\frac{3}{64} S^{[3]} + \frac{5}{48} S \times S^{[2]} - \frac{85}{1536} S^3$$

where $\sim$ means that both sides have the same Rozansky-Witten invariants, ie. give the same weight system. Secondly, using generalized Kummer varieties (note that $T^{[[1]]}$ is the Kummer model of the K3 surface, so it is actually the same as S), we get

$$C_1 \sim -\frac{1}{8}T^{[[1]]}$$

$$C_2 \sim -\frac{1}{36}T^{[[2]]} + \frac{1}{32}(T^{[[1]]})^2$$

$$C_3 \sim -\frac{3}{320}T^{[[3]]} + \frac{29}{1440}T^{[[1]]} \times T^{[[2]]} - \frac{17}{1536}(T^{[[1]]})^3$$

Of course in degrees one, two, and three we only require one, two, and three weights, respectively, to span graph cohomology, so it is no surprise that we can express the $\mathfrak{su}(2)$ weight system in terms of these hyperkähler weight systems. However, in degree four we find that this is not possible using only the Hilbert schemes of points on a K3 surface, or using only the generalized Kummer varieties. Instead we must combine both these hyperkähler weight systems, and then we can write

$$C_4 \sim \frac{1}{32}S^{[4]} - \frac{7}{800}T^{[[4]]} + \frac{5}{256}S \times S^{[3]} + \frac{1}{48}S^{[2]} \times S^{[2]} - \frac{73}{768}S^2 \times S^{[2]} + \frac{263}{6144}S^4.$$

There does not appear to be any nice pattern emerging in this behaviour.

Instead let us suppose that the virtual manifolds C_k are represented by *actual* hyperkähler manifolds, which are connected and irreducible. We know that

$$\begin{aligned} s_{\lambda_1} \ldots s_{\lambda_j}(C_k) &= (-1)^j c_{\langle w_{\lambda_1} \cdots w_{\lambda_j} \rangle}(\mathfrak{su}(2)) \\ &= (-1)^j 2^{j-1} c_{\langle w_{2k} \rangle}(\mathfrak{su}(2)) \\ &= (-2)^{j-1} s_{2k}(C_k) \end{aligned}$$

where $\lambda_1 + \ldots + \lambda_j = 2k$. Only the relative values are important here: an irreducible hyperkähler manifold of real-dimension $4k$ must have Todd genus $k+1$, so we need to rescale C_k to begin with in order for there to be any hope that it be represented by an actual hyperkähler manifold. Solving for the Chern classes, we find that

$$c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_j}(C_k)$$

is independent of the (even) partition $(\lambda_1, \ldots, \lambda_j)$ of $2k$. Furthermore, after rescaling so that $\mathrm{Td}(C_k) = k+1$ all of these Chern numbers remain integral. Hence there is no immediate reason why there shouldn't exist actual hyperkähler manifolds to represent (the rescaled) virtual hyperkähler manifolds C_k.

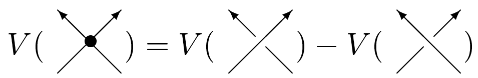
4 Relations in graph homology II

4.1 Knot theory, wheels, and wheeling

In this subsection we describe a result known as the Wheeling Theorem which arose from the study of finite type invariants of knots, and which has been recently proved by Bar-Natan, Le, and Thurston [8]. In the following subsection we shall use this result to show that the graphs Θ^k lie in the polywheel subspaces, and we shall obtain precise expressions for them in terms of polywheels. Of course, these are precisely the expressions described in the previous chapter.

The Wheeling Theorem is an isomorphism of algebras constructed from certain diagrams. We begin by briefly describing the knot theory motivation behind these algebras. Almost all of the results stated here require us to be working over a field of characteristic zero, and we shall assume this to be the case from the outset. Indeed, we are really only concerned with the base field being the rational numbers $\mathbb{Q}$. A good reference for this material is Bar-Natan's article [4].

Given an invariant of framed oriented knots V, we can extend it to an invariant of knots with singular points by the rule



where the three knots are identical except in a small ball where they are as indicated. We call V a *Vassiliev invariant of type* m if the extended invariant vanishes on knots with more than m singular points. Vassiliev invariants, also known as finite type invariants, encompass the Alexander-Conway, Jones, HOMFLY, and Kauffman polynomials, and the Reshetikhin-Turaev quantum invariants. It is still an open problem whether they separate all knots.

The set of all Vassiliev invariants of $\mathcal{V}$ is a linear space filtered by type

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots$$

The fundamental theorem of Vassiliev invariants (due to Kontsevich [36]) says that the graded space $\mathrm{gr}\mathcal{V}$ associated to $\mathcal{V}$ can be identified with the dual $\mathcal{A}(S^1)^*$ of a certain space $\mathcal{A}(S^1)$ of linear combinations of certain diagrams modulo certain relations. This is not quite a precise statement, as there is an additional subtlety concerning the parity of the framing number of the knot. If we temporarily forget the framing, we can say that the graded space associated to the set of all (unframed) Vassiliev invariants is isomorphic to the dual of $\mathcal{A}(S^1)/1\mathrm{T}$, where we have quotiented by an additional relation 1T known as the *framing independence relation*. However, let us stick to framed knots and ignore the parity problem, as it won't affect what we have to say here.

The diagrams of $\mathcal{A}(S^1)$ are unitrivalent graphs together with an oriented circle S^1, upon which all of the univalent (or *external*) vertices lie. We call this circle the *skeleton* of the diagram, and shall usually draw these diagrams with it broken at one point to produce a directed line (precisely where we break it is not important). These diagrams are oriented by an equivalence class of cyclic orderings of the trivalent (or *internal*) vertices, with two orderings being equivalent if they differ on an even number of vertices. We further require that every connected component of the unitrivalent graph has at least one external vertex (so that the overall diagram is connected). The relations are the IHX relations on internal vertices, and the STU relations near the skeleton, and the AS relation which simply says that reversing the orientation is the same as changing the sign. In fact, it can be shown that the IHX and AS relations follow from the STU relations. All of this is perhaps best illustrated in the pictures below.

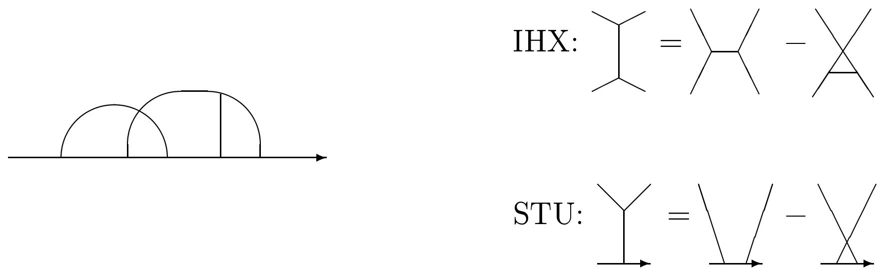


Figure 4: A diagram in $\mathcal{A}(S^1)$, and the IHX and STU relations

We call the elements of $\mathcal{A}(S^1)$ *chord diagrams*, and this space is graded by half the total number of vertices in the diagram. We allow elements given by infinite series so long as there are only a finite number of terms of any given degree.

The proof of $\mathrm{gr}\mathcal{V} \cong \mathcal{A}(S^1)^*$ relies on the construction of a universal Vassiliev invariant of knots, taking values in $\mathcal{A}(S^1)$. A *weight* of degree m is an element of the dual of $\mathcal{A}(S^1)^m$, and composing with the universal invariant gives us a Vassiliev knot invariant of finite type m. Conversely, all Vassiliev invariants of finite type m can be constructed from some system of weights of degrees 0 to m. Taking particular systems of weights (in all degrees) allows us to recover the Jones polynomial, Alexander-Conway polynomial, and all of the other invariants of knots mentioned above.

Up to normalization, all known examples of the universal Vassiliev invariant are believed to give the same answer as the original example, the Kontsevich integral Z [36]. Actually, this is really the framed Kontsevich integral, and composing with the natural projection from $\mathcal{A}(S^1)$ to $\mathcal{A}(S^1)/1\mathrm{T}$ will give us the original Kontsevich integral (of unframed knots). Now Z is extremely difficult to calculate, and it has only recently been discovered how to calculate it for the most trivial knot, the unknot U. In order to describe $Z(U)$ we first need to introduce another description of the space $\mathcal{A}(S^1)$.

Let $\mathcal{B}$ be the space of linear combinations of oriented unitrivalent graphs, modulo the IHX and AS relations. In other words, the diagrams in $\mathcal{B}$ are essentially the same as those in $\mathcal{A}(S^1)$, except that we remove the skeletons. Thus the STU relations are no longer relevant, but we retain the other relations. As before, each connected component must have at least one univalent vertex, though the overall diagram need not be connected. The space $\mathcal{B}$ is graded by half the total number of vertices. Some examples of elements of $\mathcal{B}$ are wheels w_{2m} and their disjoint unions.

Figure 5: A diagram in $\mathcal{B}$, and the IHX relations

There is a natural map $\mathcal{B} \to \mathcal{A}(S^1)$ given by averaging over all ways of joining the univalent vertices to a skeleton, and in fact this gives an isomorphism between $\mathcal{B}$ and $\mathcal{A}(S^1)$. The proof relies on using the equivalence relations to rewrite an arbitrary element of $\mathcal{A}(S^1)$ like something which is clearly in the image of the above map. We note here that $\mathcal{B}$ has a product given by taking the disjoint union of unitrivalent graphs. We denote this product by $\cup$. The space $\mathcal{A}(S^1)$ also has a product given by juxtaposition of skeletons. This is why we choose to write skeletons as directed lines rather than closed circles, but recall that it doesn't matter where we break the circle in order to get a directed line (in other words, this product is well-defined). Using the isomorphism $\mathcal{B} \cong \mathcal{A}(S^1)$ we can transfer this product to $\mathcal{B}$, but note that it is not the same as the disjoint union product. Instead we denote this second product by $\times$.

The following theorem is known as the Wheels Theorem and was recently proved by Bar-Natan, Le, and Thurston [8].

Theorem 11 *The framed Kontsevich integral $Z(U) \in \mathcal{A}(S^1) \cong \mathcal{B}$ of the unknot is given by*

$$\Omega = \exp_\cup \sum_{m=1}^{\infty} b_{2m} w_{2m}$$

where the 'modified Bernoulli numbers' b_{2m} are defined by

$$\sum_{m=0}^{\infty} b_{2m} x^{2m} = \frac{1}{2} \log \frac{\sinh x/2}{x/2}$$

and $\exp_\cup$ means that we exponentiate using the disjoint union product.

The first few terms of Ω look like

$$1 + \frac{1}{48}w_2 + \frac{1}{48^2 2!}(w_2^2 - \frac{4}{5}w_4) + \frac{1}{48^3 3!}(w_2^3 - \frac{12}{5}w_2 w_4 + \frac{64}{35}w_6) + \ldots$$

From the definition of the coefficients b_{2m}, it is clear that this formula can be obtained from $\mathrm{Td}^{-1/2}$ by replacing the characteristic classes s_{2m} by wheels w_{2m}.

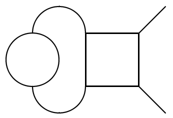
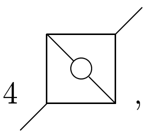
Although we will not make use of this theorem, we do need to know Ω which plays a part in the next theorem we will describe. We wish to extend the space $\mathcal{B}$ by including diagrams which have connected components with no univalent vertices, and we call this larger space $\mathcal{B}'$. Likewise, we extend $\mathcal{A}(S^1)$ by including diagrams with connected components consisting of trivalent graphs with no univalent vertices, and we call this larger space $\mathcal{A}(S^1)'$. The isomorphism $\mathcal{B} \cong \mathcal{A}(S^1)$ extends to $\mathcal{B}' \cong \mathcal{A}(S^1)'$.

We have seen that the space $\mathcal{B}$ admits two different products, $\cup$ and $\times$. The disjoint union product $\cup$ clearly extends to $\mathcal{B}'$. Also, the juxtaposition product on $\mathcal{A}(S^1)$ clearly extends to $\mathcal{A}(S^1)'$, and then can be transferred to $\mathcal{B}'$ as these spaces are isomorphic. We continue to denote these two products by $\cup$ and $\times$, and denote the two algebras given by these products by $\mathcal{B}'_\cup$ and $\mathcal{B}'_\times$ respectively.

Given a unitrivalent graph $C \in \mathcal{B}'$, we get an operator

$$\hat{C} : \mathcal{B}' \to \mathcal{B}'$$

defined in the following way. If C has no more univalent vertices than C', then $\hat{C}(C')$ is given by summing over all the ways of joining them to some (or all) of the univalent vertices of C'; otherwise $\hat{C}(C')$ is zero. We can then extend this definition linearly to any element $C \in \mathcal{B}'$, including infinite series provided that C contains only a finite number of terms of any given degree. For example, when C and C' are wheels we get

$\widehat{w_2}(w_4) = 8$  $+\,4$  , $\qquad \widehat{w_4}(w_2) = 0.$

Acting with the operator $\hat{\Omega}$ is known as *wheeling*, since Ω is made from wheels.

The following theorem is known as the Wheeling Theorem, and was recently proved by Bar-Natan, Le, and Thurston [8]. It is also allegedly a corollary of Kontsevich's results [38].

Theorem 12 *The operator associated to Ω intertwines the two different product structures on $\mathcal{B}'$, ie. $\hat{\Omega} : \mathcal{B}'_\cup \to \mathcal{B}'_\times$ is an isomorphism of algebras.*

The Wheeling Theorem was conjectured by Bar-Natan, Garoufalidis, Rozansky, and Thurston in [7], and also independently by Deligne [19]. In the former article the Wheels Theorem was also conjectured, and both were proved "at the level of Lie algebras", though knowing this is not sufficient to deduce the theorems in their full generality. It simply means verifying the results when evaluated in a Lie algebra weight system, as in the previous chapter where we evaluated trivalent graphs in the $\mathfrak{su}(2)$ weight system. In a Lie algebra weight system, the Wheeling Theorem reduces to the Duflo isomorphism [21]. If these weight systems generated graph cohomology (or more generally, the dual space $(\mathcal{B}')^*$) then this would suffice, but unfortunately it is believed that they do not. For example, it is known that there is an element of $\mathcal{B}$ of degree 16 which is killed by all simple Lie algebra weight systems (see Vogel [50]), and it is suspected that this element will in fact vanish in all Lie algebra weight systems. Of course, there should also be corresponding elements of graph homology of degree 16 which satisfy the same vanishing properties in Lie algebra weight systems. An interesting question, which we do not answer here, is whether these graph homology classes give us Rozansky-Witten invariants which are non-zero on some hyperkähler manifold. Such a manifold would need to have real-dimension 64, which is considerably larger than in the examples we will present in the next chapter.

In the following subsection we shall use the Wheeling Theorem to prove Theorem 10 from Chapter 3.

4.2 The proof of Theorem 10

The Wheeling Theorem says that wheeling with $\hat{\Omega}$ intertwines the two different product structures on $\mathcal{B}'$. Let ℓ be the unitrivalent graph given by a single line, ie. with two univalent vertices connected by a single edge. Then by the Wheeling Theorem

$$\hat{\Omega}(\ell^{\cup k}) = (\hat{\Omega}(\ell))^{\times k}$$

where the superscripts $\cup$ and $\times$ are to indicate that we are using the two different products to calculate the kth powers. We can consider the terms on each side of this equality which have no univalent vertices, and equate these parts. This makes sense because the number of univalent vertices is preserved under the IHX and AS relations.

The left hand side is given by

$$\hat{\Omega}(\underbrace{\ell \cup \cdots \cup \ell}_{k}).$$

To get a term with no univalent vertices, we need to take the kth term Ω_k in the series for Ω and join all of its univalent vertices to those of $\ell^{\cup k}$. Up to a factor, this is the same as taking the sum over all ways of joining the $2k$ univalent vertices of Ω_k to themselves, ie. the closure $\langle \Omega_k \rangle$ of Ω_k. The factor is precisely $2^k k!$ since there are $k!$ ways to order the copies of ℓ and two ways of joining each at a given location (ie. we can reverse the ends). We already remarked that Ω looks like $\mathrm{Td}^{-1/2}$ with the characteristic classes s_{2m} replaced by wheels w_{2m}. Therefore modulo terms with univalent vertices, the left hand side equals $2^k k!$ times $\mathrm{Td}_k^{-1/2}$ with the characteristic classes replaced by wheels, and summed over all the ways of joining the spokes of these wheels to produce a polywheel. For example, the first few terms are

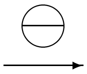
$$\begin{array}{rcl}
\hat{\Omega}(\ell) & = & \frac{1}{24}\langle w_2\rangle \\
\hat{\Omega}(\ell^{\cup 2}) & = & \frac{1}{24^2}(\langle w_2^2\rangle - \frac{4}{5}\langle w_4\rangle) \\
\hat{\Omega}(\ell^{\cup 3}) & = & \frac{1}{24^3}(\langle w_2^3\rangle - \frac{12}{5}\langle w_2 w_4\rangle + \frac{64}{35}\langle w_6\rangle) \\
\hat{\Omega}(\ell^{\cup 4}) & = & \frac{1}{24^4}(\langle w_2^4\rangle - \frac{24}{5}\langle w_2^2 w_4\rangle + \frac{48}{25}\langle w_4^2\rangle + \frac{256}{35}\langle w_2 w_6\rangle - \frac{1152}{175}\langle w_8\rangle)
\end{array}$$

modulo terms with univalent vertices.

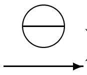
Now consider the right hand side. Firstly, if we include terms with univalent vertices we find

$$\begin{array}{rcl}
\hat{\Omega}(\ell) & = & \hat{1}(\ell) + \dfrac{1}{48}\widehat{w_2}(\ell) \\
 & = & \ell + \dfrac{1}{24}\langle w_2\rangle \\
 & = & \ell + \dfrac{1}{24}\Theta
\end{array}$$

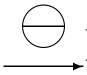
which looks like

 $+\ \dfrac{1}{24}$ 

when written as an element of $\mathcal{A}(S^1)'$. Therefore

$(\hat{\Omega}(\ell))^{\times k} = ($  $+\ \dfrac{1}{24}$  $)^{\times k}$

and modulo terms with univalent vertices this equals

$\dfrac{1}{24^k}($  $)^{\times k}.$

Written as an element of $\mathcal{B}'$ this is simply $\frac{1}{24^k}\Theta^k$. Comparing with the left hand side we see that

$$\begin{array}{rcl}
\Theta & = & \langle w_2\rangle \\
\Theta^2 & = & \langle w_2^2\rangle - \frac{4}{5}\langle w_4\rangle \\
\Theta^3 & = & \langle w_2^3\rangle - \frac{12}{5}\langle w_2 w_4\rangle + \frac{64}{35}\langle w_6\rangle \\
\Theta^4 & = & \langle w_2^4\rangle - \frac{24}{5}\langle w_2^2 w_4\rangle + \frac{48}{25}\langle w_4^2\rangle + \frac{256}{35}\langle w_2 w_6\rangle - \frac{1152}{175}\langle w_8\rangle
\end{array}$$

etc. Therefore we can conclude:

Proposition 13 *For all k, the graph Θ^k lies in the polywheel subspace. The formula expressing it as a linear combination of polywheels is given by taking $48^k k!$ times the kth term $\mathrm{Td}_k^{-1/2}$ of $\mathrm{Td}^{-1/2}$, turning the characteristic classes s_{2m} into wheels w_{2m}, and then summing over all the ways of joining the spokes of these wheels.*

To prove Theorem 10 it only remains to take the Rozansky-Witten invariants corresponding to the above graphs. We get the following sequence of relations:

$$\begin{aligned} b_{\Theta} &= -s_2 \\ b_{\Theta^2} &= s_2^2 + \tfrac{4}{5}s_4 \\ b_{\Theta^3} &= -s_2^3 - \tfrac{12}{5}s_2s_4 - \tfrac{64}{35}s_6 \\ b_{\Theta^4} &= s_2^4 + \tfrac{24}{5}s_2^2s_4 + \tfrac{48}{25}s_4^2 + \tfrac{256}{35}s_2s_6 + \tfrac{1152}{175}s_8 \end{aligned}$$

etc. By taking the Rozansky-Witten invariant corresponding to the polywheel $\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle$ we get $(-1)^j s_{\lambda_1}\cdots s_{\lambda_j}$, so we have effectively turned the wheels w_{2m} back into characteristic classes. The overall effect of the sign changes $(-1)^j$ is to change the generating sequence from $\mathrm{Td}^{-1/2}$ to $\mathrm{Td}^{1/2}$. Therefore the general case looks like

$$b_{\Theta^k} = 48^k k!\mathrm{Td}_k^{1/2}$$

or in other words a generating sequence for $\frac{1}{48^k k!}b_{\Theta^k}$ is $\mathrm{Td}^{1/2}$, which is precisely Theorem 10.

4.3 The bubbling effect

Let Γ be a connected trivalent graph with $2k$ vertices. Consider replacing an edge of Γ by a two-wheel w_2, which we shall call a *bubble*, to get a new graph Γ' with $2k+2$ vertices. We claim that the graph homology class of Γ' is independent of where we add the bubble. It suffices to show that the bubble commutes past vertices, and this follows from the IHX relation: a graph with the bubble near a vertex is equivalent to twice the graph with the bubble at the vertex (see Figure 6). Thus we can move the bubble to the vertex then out the other side, and this proves our claim. We call this the *bubbling effect*. Note that if Γ has several connected components then there is some choice as to where we add the bubble, and different choices may give non-homologous graphs. Nevertheless, if Γ decomposes as $\gamma_1\cup\cdots\cup\gamma_m$ where each component γ_i is connected with $2k_i$ vertices, then we can define

$$\Gamma' = \sum_{i=1}^{m}\frac{k_i}{k}\gamma_1\cup\cdots\cup\gamma_{i-1}\cup\gamma_i'\cup\gamma_{i+1}\cup\cdots\cup\gamma_m.$$

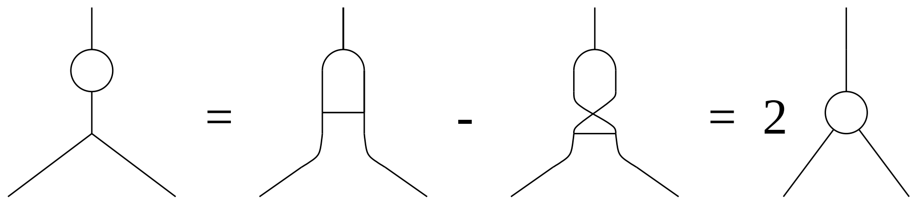


Figure 6: The bubbling effect

Suppose that Γ is in the polywheel subspace, and write

$$\Gamma = \sum a_{(\lambda_1,\ldots,\lambda_j)}\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle$$

where the sum is over all even partitions $(\lambda_1,\ldots,\lambda_j)$ of $2k$. Let us add another two-wheel w_2 to the polywheels. Then either the two spokes of w_2 join to each other and we get a copy of Θ or they join to the other wheels and this is equivalent to adding a bubble. There are k choices of where the bubble is added, corresponding to the k pairs of spokes which are joined, and two ways to connect the bubble at each location (as we can reverse the ends of w_2). If Γ decomposes as $\gamma_1\cup\cdots\cup\gamma_m$ where each component γ_i is connected with $2k_i$ vertices, then k_i of the pairs will be on the part giving rise to γ_i. Therefore

$$\begin{aligned}
&\sum a_{(\lambda_1,\ldots,\lambda_j)}\langle w_2 w_{\lambda_1}\cdots w_{\lambda_j}\rangle\\
&\quad = \sum a_{(\lambda_1,\ldots,\lambda_j)}\Theta\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle + \sum_{i=1}^{m} 2k_i\gamma_1\cup\cdots\cup\gamma_i'\cup\cdots\cup\gamma_m\\
&\quad = \Theta\Gamma + 2k\Gamma'
\end{aligned}$$

where we write the disjoint union $\Theta\cup\Gamma$ simply as $\Theta\Gamma$. We can conclude that $\Theta\Gamma+2k\Gamma'$ is also in the polywheel subspace.

For example, we have a generating function $\mathrm{Td}^{-1/2}$ which tells us how to express Θ^k as a linear combination of polywheels (for all k). Let us write

$$\Theta^k = \sum a^{(k)}_{(\lambda_1,\ldots,\lambda_j)}\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle.$$

Then the formula above says that

$$\sum a^{(k)}_{(\lambda_1,\ldots,\lambda_j)}\langle w_2 w_{\lambda_1}\cdots w_{\lambda_j}\rangle = \Theta^{k+1} + 2k(\Theta^k)'.$$

Now

$$\Theta^{k+1} = \sum a^{(k+1)}_{(\eta_1,\ldots,\eta_j)}\langle w_{\eta_1}\cdots w_{\eta_j}\rangle$$

summed over all even partitions $(\eta_1,\ldots,\eta_j)$ of $2k+2$, and $\Theta'=\Theta_2$, so that

$$\begin{aligned}
k(\Theta^k)' &= \sum_{i=1}^{k}\Theta\cup\cdots\cup\Theta'\cup\cdots\Theta\\
&= k\Theta^{k-1}\Theta_2.
\end{aligned}$$

Then the above formula tells us that $\Theta^{k-1}\Theta_2$ can also be expressed as a linear combination of polywheels, and we can determine this precisely since we know the coefficients $a^{(k)}$ and $a^{(k+1)}$. For example, if $k=3$ we have

$$\begin{aligned}
\Theta^2\Theta_2 &= \frac{1}{6}(\langle w_2^4\rangle - \frac{12}{5}\langle w_2^2 w_4\rangle + \frac{64}{35}\langle w_2 w_6\rangle)\\
&\quad -\frac{1}{6}(\langle w_2^4\rangle - \frac{24}{5}\langle w_2^2 w_4\rangle + \frac{48}{25}\langle w_4^2\rangle + \frac{256}{35}\langle w_2 w_6\rangle - \frac{1152}{175}\langle w_8\rangle)\\
&= \frac{2}{5}\langle w_2^2 w_4\rangle - \frac{8}{25}\langle w_4^2\rangle - \frac{32}{35}\langle w_2 w_6\rangle + \frac{192}{175}\langle w_8\rangle.
\end{aligned}$$

Now we take Γ to be $\Theta^{k-2}\Theta_2$, which can be written

$$\Theta^{k-2}\Theta_2 = \sum a^{(k)}_{(\lambda_1,\ldots,\lambda_j)}\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle$$

and hence

$$\sum a^{(k)}_{(\lambda_1,\ldots,\lambda_j)}\langle w_2 w_{\lambda_1}\cdots w_{\lambda_j}\rangle = \Theta^{k-1}\Theta_2 + 2k(\Theta^{k-2}\Theta_2)'.$$

As before, the first term on the right can be written

$$\Theta^{k-1}\Theta_2 = \sum a^{(k+1)}_{(\eta_1,\ldots,\eta_j)}\langle w_{\eta_1}\cdots w_{\eta_j}\rangle$$

and in the second term

$$k(\Theta^{k-2}\Theta_2)' = (k-2)\Theta^{k-3}\Theta_2^2 + 2\Theta^{k-2}\Theta_3$$

which we can therefore express as a linear combination of polywheels. In particular $\Theta^{k-2}\Theta_3$ can be expressed as a linear combination of polywheels and $\Theta^{k-3}\Theta_2^2$.

Next we take Γ to be

$$\begin{aligned}(k-1)(\Theta^{k-3}\Theta_2)' &= (k-3)\Theta^{k-4}\Theta_2^2 + 2\Theta^{k-3}\Theta_3\\ &= (k-1)\sum a^{(k)}_{(\lambda_1,\ldots,\lambda_j)}\langle w_{\lambda_1}\cdots w_{\lambda_j}\rangle\end{aligned}$$

and hence

$$\begin{aligned}&(k-1)\sum a^{(k)}_{(\lambda_1,\ldots,\lambda_j)}\langle w_2 w_{\lambda_1}\cdots w_{\lambda_j}\rangle\\ &\quad= (k-3)\Theta^{k-3}\Theta_2^2 + 2\Theta^{k-2}\Theta_3 + 2k((k-3)\Theta^{k-4}\Theta_2^2 + 2\Theta^{k-3}\Theta_3)'\\ &\quad= (k-3)\Theta^{k-3}\Theta_2^2 + 2\Theta^{k-2}\Theta_3 + 2(k-3)((k-4)\Theta^{k-5}\Theta_2^3 + 4\Theta^{k-4}\Theta_2\Theta_3)\\ &\qquad +4((k-2)\Theta^{k-4}\Theta_2\Theta_3 + 2\Theta^{k-3}\Theta_4)\\ &\quad= (k-3)\Theta^{k-3}\Theta_2^2 + 2\Theta^{k-2}\Theta_3 + 2(k-3)(k-4)\Theta^{k-5}\Theta_2^3\\ &\qquad +4(3k-8)\Theta^{k-4}\Theta_2\Theta_3 + 8\Theta^{k-3}\Theta_4.\end{aligned}$$

It is beginning to become rather difficult to keep track of all the terms here, but the important point we wish to observe is that we can express $\Theta^{k-3}\Theta_4$ as a linear combination of polywheels and graphs made from the disjoint union of copies of the necklace graphs Θ, Θ_2, and Θ_3. Furthermore, continuing with these kinds of calculations allows us to prove inductively the following result.

Proposition 14 *For $m = 1$ to k, the graph $\Theta^{k-m}\Theta_m$ with $2k$ vertices can be expressed as a linear combination of polywheels and graphs made from the disjoint union of copies of the necklace graphs Θ, Θ_2, $\ldots$, and Θ_{m-1}.*

For $m = 1$ this is just the statement that Θ^k is in the polywheel subspace. In terms of Rozansky-Witten invariants, this says that b_{Θ^k} is a characteristic number. The significance of this result to the Rozansky-Witten theory for $m > 1$ will be discussed in the next chapter.

5 Calculations

5.1 Consequences of our graph homology relations

In the last two chapters we have derived certain relations in graph homology. Our primary aim was to express certain graphs or combinations of graphs as linear combinations of polywheels, because this implies that the corresponding Rozansky-Witten invariants are characteristic numbers. For example, we proved in Subsection 4.2 that a generating sequence for $\frac{1}{48^k k!}b_{\Theta^k}$ is $\mathrm{Td}^{1/2}$. Combining this with our expression for b_{Θ^k} from Subsection 2.4 proves the following theorem.

Theorem 15 *The $\mathcal{L}^2$-norm of the curvature of an irreducible compact hyperkähler manifold can be expressed in terms of the volume and characteristic numbers. More specifically, we have*

$$\frac{1}{(192\pi^2 k)^k}\frac{\|K\|^{2k}}{\mathrm{vol}(X)^{k-1}} = \int_X \mathrm{Td}_k^{1/2}(X).$$

When $k=1$ we get

$$\|K\|^2 = -4\pi^2 s_2(X)$$

so a K3 surface S has $\|K\|^2 = 192\pi^2$. For $k=2$ we have

$$\|K\|^4 = 32\pi^4 \mathrm{vol}(X)(s_2^2(X) + \frac{4}{5}s_4(X))$$

from which we conclude that

$$s_2^2(X) + \frac{4}{5}s_4(X) > 0.$$

Writing this as

$$\frac{4}{5}(7c_2^2(X) - 4c_4(X))$$

and noting that the Todd genus of an irreducible eight-dimensional hyperkähler manifold must equal three, ie.

$$\frac{1}{720}(3c_2^2(X) - c_4(X)) = 3$$

tells us that the Euler characteristic $c_4(X)$ must be less than 3024. Note that a reformulation by Bogolomov of a result of Verbitsky [15] (see also Beauville [11]) shows that $c_4(X) \leq 324$, so our bound is quite crude. Actually, $c_4(S^{[2]}) = 324$ for the Hilbert scheme of two points on a K3 surface S, so the upper bound 324 is sharp. We will discuss these specific examples of compact hyperkähler manifolds in later subsections.

In general, since $\|K\|$ is positive we get the following bounds on the characteristic numbers.

Corollary 16 *Suppose X is an irreducible compact hyperkähler manifold of real-dimension $4k$. Then we have*

$$\int_X \mathrm{Td}_k^{1/2}(X) > 0.$$

Equality would imply that $\|K\| = 0$ and hence X is flat, ie. a complex torus T^k, but such an X is not irreducible.

We saw in the previous chapter that $\Theta^{k-2}\Theta_2$ also lies in the polywheel subspace, and therefore the corresponding Rozansky-Witten invariant $b_{\Theta^{k-2}\Theta_2}$ is also a characteristic number. Suppose that X is irreducible, and recall the discussion in Subsection 2.4. We showed there that the Rozansky-Witten invariant corresponding to the disjoint union of several graphs is given by

$$b_{\gamma\gamma'}(X) = (2\pi^2)^{-k} k! \beta_\gamma \beta_{\gamma'} \mathrm{vol}(X)$$

where β_γ and $\beta_{\gamma'}$ are some scalars depending on X and the graphs γ and γ' respectively. In particular

$$b_{\Theta^k}(X) = (2\pi^2)^{-k} k! \beta_\Theta^k \mathrm{vol}(X)$$

and

$$b_{\Theta^{k-2}\Theta_2}(X) = (2\pi^2)^{-k} k! \beta_\Theta^{k-2} \beta_{\Theta_2} \mathrm{vol}(X).$$

Now observe that

$$\begin{aligned} b_{\Theta^{k-4}\Theta_2^2}(X) &= (2\pi^2)^{-k} k! \beta_\Theta^{k-4} \beta_{\Theta_2}^2 \mathrm{vol}(X) \\ &= \frac{((2\pi^2)^{-k} k! \beta_\Theta^{k-2} \beta_{\Theta_2} \mathrm{vol}(X))^2}{(2\pi^2)^{-k} k! \beta_\Theta^k \mathrm{vol}(X)} \\ &= \frac{b_{\Theta^{k-2}\Theta_2}(X)^2}{b_{\Theta^k}(X)} \end{aligned}$$

and more generally

$$b_{\Theta^{k-2m}\Theta_2^m}(X) = \frac{b_{\Theta^{k-2}\Theta_2}(X)^m}{b_{\Theta^k}(X)^{m-1}}$$

for $m = 1$ to $\lfloor k/2 \rfloor$, where $\lfloor k/2 \rfloor$ is $k/2$ if k is even, and $(k-1)/2$ if it is odd. Since b_{Θ^k} and $b_{\Theta^{k-2}\Theta_2}$ are characteristic numbers, we conclude that for *irreducible* hyperkähler manifolds $b_{\Theta^{k-2m}\Theta_2^m}$ can be written as a rational function of characteristic numbers.

We also saw that $\Theta^{k-3}\Theta_3$ could be expressed as a linear combination of polywheels and $\Theta^{k-4}\Theta_2^2$, and thus $b_{\Theta^{k-3}\Theta_3}$ can also be written as a rational function of characteristic numbers. As before we can write

$$b_{\Theta^{k-5}\Theta_2\Theta_3}(X) = \frac{b_{\Theta^{k-2}\Theta_2}(X) b_{\Theta^{k-3}\Theta_3}(X)}{b_{\Theta^k}(X)}$$

and hence this Rozansky-Witten invariant is also a rational function of characteristic numbers. In fact, this is true for b_Γ where Γ is any graph made from a disjoint union of copies of Θ, Θ_2, and Θ_3. Proceeding by induction and using Proposition 14 proves the following theorem.

Theorem 17 *Let Γ be a trivalent graph with $2k$ vertices constructed by taking a disjoint union of copies of the necklace graphs Θ, Θ_2, $\ldots$, and Θ_k. Then for an irreducible hyperkähler manifold X, $b_\Gamma(X)$ can be expressed as a rational function of the characteristic numbers of X.*

Note that the only denominators in these expressions are powers of $b_\Theta(X)$, or in other words $\mathrm{Td}^{1/2}(X)$. We have already noted in Corollary 16 that this is strictly positive for irreducible X, by virtue of the fact that $\|K\|$ is strictly positive, hence none of these rational functions can be singular.

Recall that for reducible hyperkähler manifolds we have a product formula

$$b_\Gamma(X \times Y) = \sum_{\gamma \sqcup \gamma' = \Gamma} b_\gamma(X) b_{\gamma'}(Y).$$

More generally, if X decomposes into irreducible factors $X_1 \times \cdots \times X_m$ then

$$b_\Gamma(X_1 \times \cdots \times X_m) = \sum_{\gamma_1 \sqcup \cdots \sqcup \gamma_m = \Gamma} b_{\gamma_1}(X_1) \cdots b_{\gamma_m}(X_m).$$

If Γ is constructed from a disjoint union of necklace graphs Θ, Θ_2, $\ldots$, and Θ_k, then each of $\gamma_1, \ldots, \gamma_m$ must be too. Thus $b_{\gamma_i}(X_i)$ can be determined from the characteristic numbers of X_i. So to determine $b_\Gamma(X_1 \times \cdots \times X_m)$ we need to know the characteristic numbers of all the irreducible factors of X. These cannot necessarily be determined from the knowledge of the characteristic numbers of X.

For degree $k \leq 5$ the graphs constructed from the polywheels and the necklace graphs Θ, Θ_2, $\ldots$, and Θ_5 span graph homology (see Appendix A). Therefore if we know the characteristic numbers of X (or the characteristic numbers of the irreducible factors of X, in the case when X is reducible) we can determine all the Rozansky-Witten invariants of X. So for real-dimension less than or equal to 20 the Rozansky-Witten invariants are *essentially* nothing more than the characteristic numbers. We say 'essentially' because normally we would only take linear combinations of characteristic numbers, not rational functions. We will prove in Subsection 5.5 that some Rozansky-Witten invariants can certainly not be written as linear combinations of characteristic numbers.

It is worth considering what happens as k gets larger. We want to roughly count how many Rozansky-Witten invariants there are and how many can be expressed as rational functions of characteristic numbers according to Theorem 17. In particular, we would like to know how many graphs *cannot* be expressed as linear combinations of polywheels and disjoint unions of necklace graphs Θ_i. We will think of the latter as being part of a 'conventional' basis for graph homology, and the polywheels as being certain linear combinations of the completed basis.

Firstly, in degree k our conventional basis contains $\dim \mathcal{A}(\emptyset)^k$ graphs, of which $\dim \mathcal{A}(\emptyset)^k_{\mathrm{conn}}$ are connected. The disjoint unions of necklace graphs Θ_i account for $p(k)$ of these graphs, where $p(k)$ is the number of partitions of k. This leaves

$$\dim \mathcal{A}(\emptyset)^k - p(k)$$

graphs. The polywheels span a $p(k)$-dimensional subspace, and so there are $p(k)$ equations given by writing the polywheels in terms of our conventional basis. However, we can write Θ^k and $\Theta^{k-2}\Theta_2$ in terms of polywheels. We can also write $\Theta^{k-3}\Theta_3$ in terms of polywheels and $\Theta^{k-4}\Theta_2^2$. Indeed, for $m = 1$ to k we can write $\Theta^{k-m}\Theta_m$ in terms of polywheels and disjoint unions of necklace graphs Θ_i with $i < m$ (this is precisely Proposition 14). Thus there is a subspace of the span of the $p(k)$ polywheel equations which has dimension at least k and involves only disjoint unions of necklace graphs Θ_i. So overall we are left with at most $p(k) - k$ equations involving polywheels, disjoint unions of necklace graphs Θ_i, and the remaining

$$\dim\mathcal{A}(\emptyset)^k - p(k)$$

graphs. It follows that if

$$p(k) - k < \dim\mathcal{A}(\emptyset)^k - p(k)$$

then there are certainly some graphs which cannot be expressed as linear combinations of polywheels and disjoint unions of necklace graphs Θ_i. We summarize these values for $k \leq 10$ in the table below.

k	1	2	3	4	5	6	7	8	9	10
$\dim\mathcal{A}(\emptyset)^k_{\mathrm{conn}}$	1	1	1	2	2	3	4	5	6	8
$\dim\mathcal{A}(\emptyset)^k$	1	2	3	6	9	16	25	42	65	105
$p(k)$	1	2	3	5	7	11	15	22	30	42
$\dim\mathcal{A}(\emptyset)^k - p(k)$	0	0	0	1	2	5	10	20	35	63
$p(k) - k$	0	0	0	1	2	5	8	14	21	32

It can be seen that for $k \geq 7$ there are clearly graphs which cannot be expressed as linear combinations of polywheels and disjoint unions of necklace graphs Θ_i. We can conclude that these graphs give rise to Rozansky-Witten invariants which cannot be determined from characteristic numbers *using the methods described above*. There may, however, be other ways to determine them using simply the knowledge of the characteristic numbers.

5.2 Deriving Chern numbers from the χ_y-genus

From the discussion of the previous subsection we know that when k is small the Rozansky-Witten invariants can all be calculated from knowledge of the Chern numbers . In particular, this works for $k \leq 5$ and possibly for $k = 6$. On the other hand, these methods break down for $k \geq 7$, meaning that it isn't possible to determine *all* Rozansky-Witten invariants in this way.

In the next subsections we will look at some specific examples of compact hyperkähler manifolds, namely the Hilbert schemes of points on a K3 surface and the generalized Kummer varieties. At this stage, it suffices to say that for these examples the Chern numbers are not well known, meaning that there are no explicit formulas for calculating them in all dimensions. We discuss some partial answers to this question in the next subsections, but first we wish to describe a way of obtaining some preliminary information about the Chern numbers from the Hirzebruch χ_y-genus, which is known explicitly (in terms of generating functions) in all dimensions for the examples of compact hyperkähler manifolds we wish to study.

Our method is to use the Riemann-Roch formula to express the χ_y-genus in terms of Chern numbers. In real-dimensions 4, 8, and 12 we can invert these relations. In dimension 16 we are left with an additional unknown variable. Although we will not proceed beyond dimension 16, let us note the following: in real-dimension $4k$ there are $p(k)$ Chern numbers. On the other hand, the Hirzebruch χ_y-genus

$$\chi_y = \chi^0 + \chi^1 y + \ldots + \chi^{2k-1}y^{2k-1} + \chi^{2k}y^{2k}$$

contains $2k+1$ terms, though it is symmetric ($\chi^0 = \chi^{2k}$, $\chi^1 = \chi^{2k-1}$, $\ldots$). Since we are in the hyperkähler case, it also satisfies

$$\sum_{m=0}^{2k}(-1)^m(6m^2 - k(6k+1))\chi^m = 0$$

due to a result of Salamon [46]. So writing the χ_y-genus in terms of Chern numbers gives us no more than k equations for the Chern numbers. Note that the relations $\chi^0 = \chi^{2k}$, etc. and Salamon's result don't give us any additional relations on the Chern numbers; ie. these relations are tautologous when we write the χ^m in terms of Chern numbers. So in general we would expect there to be at least $p(k) - k$ additional unknown variables when we try to write the Chern numbers in terms of the χ^m. In other words, for large k the Hirzebruch χ_y-genus will contain far less information than is required to determine the Chern numbers.

Let X have Chern roots $\{\gamma_1, \gamma_2, \ldots, \gamma_{2k-1}, \gamma_{2k}\}$. For a hyperkähler manifold they occur in plus/minus pairs, but we shall explain the theory for a general manifold. The characteristic classes are given by

$$s_\lambda = \gamma_1^\lambda + \gamma_2^\lambda + \ldots + \gamma_{2k-1}^\lambda + \gamma_{2k}^\lambda.$$

Let $h^{p,q}$ be the Hodge numbers, defined by $\dim\mathrm{H}^q(X, \Lambda^p T^*)$. Then the Hirzebruch χ_y-genus is defined to be

$$\chi_y(X) = \sum_{p,q=0}^{2k}(-1)^q h^{p,q} y^p$$

so that each individual coefficient is given by the Euler characteristic of a bundle of forms

$$\begin{aligned}\chi^m(X) &= \chi(\Lambda^m T^*) \\ &= \sum_{q=0}^{2k}(-1)^q h^{m,q}.\end{aligned}$$

The Riemann-Roch theorem says

$$\chi^m(X) = \int_X [\mathrm{ch}(\Lambda^m T^*)\mathrm{Td}(X)]_{4k}$$

where $\mathrm{ch}(\Lambda^m T^*)$ is the Chern character of $\Lambda^m T^*$, $\mathrm{Td}(X)$ is the Todd genus of (the tangent bundle of) X, and $[\ldots]_{4k}$ means that we pick out the component of degree $4k$. In terms of the Chern roots we can write these terms as

$$\mathrm{ch}(\Lambda^m T^*) = \sum_{i_1<i_2<\ldots<i_m} e^{-\gamma_{i_1}-\gamma_{i_2}-\ldots-\gamma_{i_m}}$$

and

$$\mathrm{Td}(X) = \prod_{i=1}^{2k} \frac{\gamma_i}{1-e^{\gamma_i}}.$$

Substituting these into the Riemann-Roch formula allows us to express each $\chi^m(X)$ as an integral of some product of Chern roots. In fact the integrand is symmetric in permutations of the roots and hence can be written in terms of the characteristic classes, and picking the degree $4k$ component means taking terms $s_{\lambda_1}\cdots s_{\lambda_j}$ where $(\lambda_1,\ldots,\lambda_j)$ is a partition of $2k$. Overall we obtain an expression for the χ_y-genus in terms of the Chern numbers. Note that these calculations are simpler in the hyperkähler case once we pair the Chern roots into plus/minus pairs.

These expressions are shown in the Appendix B for $k\leq 4$. In fact a different basis for the Chern numbers has been used there, namely $c_{\lambda_1}\cdots c_{\lambda_j}$. It is of course possible to rewrite these in terms of the 's' basis, but we do this later, after attempting to invert the relations. We only show χ^m for $m\leq k$, as $\chi^{k+1}=\chi^{k-1}$, $\chi^{k+2}=\chi^{k-2}$, etc. Indeed by Salamon's result χ^k can also be expressed as χ^m for $m<k$. Thus in the subsequent table, where we attempt to invert the relations, χ^k does not appear. We observe that for $k=1$, 2, and 3 it is possible to fully invert these relations, whereas for $k=4$ there is an additional unknown variable s. It is these expressions which we rewrite in the 's' basis.

In the next two subsections we will look at some particular examples of compact hyperkähler manifolds, for which the χ_y-genus is known for all k. Substituting into the equations derived above gives us all the Chern numbers in real-dimensions 4, 8, and 12. From these we can calculate all of the Rozansky-Witten invariants using the results of the first subsection in this chapter. In dimension 16 there is still the unknown s. However, by directly calculating one of the Rozansky-Witten invariants we can determine s for these examples. This will then enable us to calculate all of the Chern numbers and hence also the remaining Rozansky-Witten invariants in this dimension.

In particular, in dimension 16 we can write b_{Θ^4} in terms of the Chern numbers. This leads to

$$\begin{aligned} b_{\Theta^4} &= s_2^4+\frac{24}{5}s_2^2s_4+\frac{48}{25}s_4^2+\frac{256}{35}s_2s_6+\frac{1152}{175}s_8 \\ &= \frac{1}{175}(-21936s+4396904448\chi^0-7259904\chi^1+2472960\chi^2+278784\chi^3). \end{aligned}$$

We will calculate b_{Θ^4}, and hence determine s from the above formula. This method leads to the complete tables of Chern numbers and Rozansky-Witten invariants which appear in Appendices D and E.1.

5.3 The Hilbert scheme of points on a K3 surface

For a long time the only known compact hyperkähler manifold was the K3 surface in real-dimension four. Fujiki [24] was the first to discover higher dimensional examples, of dimension eight, and shortly afterwards Beauville [10] generalized these to produce two families of examples in all dimensions. They are the Hilbert scheme of points on a K3 surface and the generalized Kummer varieties. The latter are also constructed via Hilbert schemes, which we will discuss shortly. Other examples of compact hyperkähler manifolds were discovered by Mukai [41] around the same time by exhibiting a holomorphic symplectic structure on the moduli space of sheaves on a K3 surface or complex torus. These later proved to be deformations of Beauville's examples. Indeed apart from these two main families there is just one other known compact hyperkähler manifold, in real-dimension 20. It was constructed by O'Grady [42] as a desingularization of a certain singular moduli space of sheaves on a K3 surface.

Let M be an algebraic variety, or more generally a scheme (although the following discussion holds in a more general setting, we will only be interested in base field $\mathbb{C}$). The *Hilbert scheme of k points on M* is the moduli space of zero-dimensional subschemes of length k, and we denote it by $M^{[k]}$. The generalization to non-algebraic M is known as the Douady space, which we also denote by $M^{[k]}$. An example of a length k subscheme is given by a collection of k distinct unordered points $\{x_1, \ldots, x_k\}$. More generally, we could allow some of these points to collide, in which case there should be some additional information at those points. For two points colliding, this amounts to an element of the projectivization of the tangent space at the double point, which corresponds to the direction the two points collided along. In fact this gives a complete description of the Hilbert scheme of two points as

$$M^{[2]} = \mathrm{Blow}_\Delta(M \times M)/\mathcal{S}_2$$

where Blow_Δ denotes the blow-up along the diagonal and $\mathcal{S}_2$ the symmetric group on two elements, acting by interchanging the two factors of $M \times M$. Unfortunately for $k > 2$ it is not possible to describe $M^{[k]}$ in such a simple way.

There is a surjective morphism

$$\pi : M^{[k]} \to \mathrm{Sym}^k M = M^k/\mathcal{S}_k$$

to the kth symmetric product known as the Hilbert-Chow morphism, given by

$$\xi \mapsto \sum_{x \in M} \mathrm{length}(\xi)_x [x].$$

This map is a bijection on the open subset where elements are given by k distinct points. When the complex-dimension of M is greater than one, the symmetric product is singular. However, a theorem of Fogarty [23] says that the Hilbert scheme gives a smooth resolution when M is a complex surface, ie. has complex-dimension two. In particular, $M^{[k]}$ is of complex-dimension $2k$ and non-singular in this case.

Suppose that the complex surface M has a holomorphic symplectic form ω. Then on M^k there is a natural holomorphic symplectic form given by

$$p_1^*\omega + \ldots + p_k^*\omega$$

where p_i is projection onto the ith factor. This is clearly $\mathcal{S}_k$-invariant, and hence we get a holomorphic two-form on $\mathrm{Sym}^k M$. We claim that the pull-back of this two-form to $M^{[k]}$ is non-degenerate. Consider $M^{[2]}$ for example: the corresponding holomorphic two-form on

$$\mathrm{Blow}_\Delta(M \times M)$$

is degenerate on the blown-up diagonal, but a direct calculation shows that when we take the quotient by $\mathcal{S}_2$ we get a non-degenerate two-form on the Hilbert scheme. For general k, the same direct calculation shows that the holomorphic two-form on $M^{[k]}$ is non-degenerate on the open subset where at most two points collide. The complement of this open subset has codimension two, hence by Hartog's theorem the two-form extends to a holomorphic two-form on all of $M^{[k]}$. Furthermore, if this two-form were to degenerate anywhere, it would have to degenerate on an entire divisor, ie. codimension one submanifold, which is clearly impossible. This is Beauville's construction of a holomorphic symplectic form on $M^{[k]}$ (see [10]). More generally, Mukai [41] constructed a holomorphic symplectic form on the moduli space of stable sheaves (of fixed rank and Chern classes) on a K3 surface or abelian surface, ie. on compact surfaces with holomorphic symplectic forms. This is a generalization as the Hilbert scheme $M^{[k]}$ can be thought of as the moduli space of rank-one torsion-free sheaves on M.

In the compact Kähler case, the existence of a holomorphic symplectic form is equivalent to the existence of a hyperkähler metric by Yau's theorem [52] and the results of Bochner and Yano [13]. This means that we should choose M to be a K3 surfaces S or an abelian surface T, ie. a torus. We shall discuss the latter case in the next subsection; first we wish to look at the Hilbert scheme of k points on S.

To begin with, the Hodge numbers of the Hilbert scheme of points on an arbitrary smooth projective surface M were calculated by Cheah [18]. Writing

$$h(M^{[k]}) = \sum_{p,q=1}^{2k} h^{p,q}(M^{[k]})x^q y^p$$

then in terms of the Hodge numbers $h^{p,q}(M)$ of M we have

$$\sum_{k=0}^{\infty} h(M^{[k]})t^k = \prod_{n=1}^{\infty}\left(\frac{\prod_{p+q\,\mathrm{odd}}(1+x^{p+n-1}y^{q+n-1}t^n)^{h^{p,q}(M)}}{\prod_{p+q\,\mathrm{even}}(1+x^{p+n-1}y^{q+n-1}t^n)^{h^{p,q}(M)}}\right).$$

In particular, putting $x = -1$ gives us the Hirzebruch χ_y-genus. The Hodge diamond of a K3 surface S looks like

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

Using this to calculate the χ_y-genus of $S^{[k]}$ we get the results in Appendix C for $k \leq 4$. Substituting these values into our Riemann-Roch formula gives us the results in Appendix D. In particular, we know all the Chern numbers and hence all the Rozansky-Witten invariants for $k = 1$, 2, and 3. Our next aim is to calculate $b_{\Theta^4}(S^{[4]})$ and hence determine s, in order to complete the tables in the appendices up to $k = 4$. In fact we will derive a formula for $b_{\Theta^k}(S^{[k]})$ for all k.

Recall that the Rozansky-Witten invariants are invariant under rescaling of the holomorphic symplectic form. So let us normalize ω on S so that

$$\int_S \omega\bar{\omega} = 1.$$

Let us also denote the induced holomorphic symplectic form on $S^{[k]}$ by ω; if there is likely to be some confusion we shall specify which space ω is on with a subscript, eg. $\omega_{S^{[k]}}$. It will have the normalization

$$\begin{aligned} \int_{S^{[k]}} \omega^k\bar{\omega}^k &= \int_{\mathrm{Sym}^k S} (p_1^*\omega + \ldots + p_k^*\omega)^k (p_1^*\bar{\omega} + \ldots + p_k^*\bar{\omega})^k \\ &= \int_{S^k/\mathcal{S}_k} (k!)^2 p_1^*\omega \cdots p_k^*\omega p_1^*\bar{\omega} \cdots p_k^*\bar{\omega} \\ &= \int_{S^k} k! p_1^*(\omega\bar{\omega}) \cdots p_k^*(\omega\bar{\omega}) \\ &= k! \end{aligned}$$

Recall from Subsection 2.4 that for irreducible X we can write

$$b_{\Theta^k}(X) = \frac{1}{(8\pi^2)^k k!} \beta_\Theta^k \int_X \omega^k\bar{\omega}^k$$

where β_Θ is a scalar whose value is given by

$$[\Theta(\Phi)] = \beta_\Theta[\bar{\omega}] \in \mathrm{H}^{0,2}_{\bar{\partial}}(X).$$

Actually β_Θ is not a canonically defined number, for if we rescale the holomorphic symplectic form ω by λ then β_Θ rescales by λ^{-2}. We needn't be concerned however, as we shall be looking at the Hilbert scheme $S^{[k]}$ for which we have fixed a normalization of the holomorphic symplectic form, and we shall assume this throughout. Indeed, this normalization leads to the formula

$$b_{\Theta^k}(S^{[k]}) = \frac{1}{(8\pi^2)^k} \beta_\Theta^k$$

with

$$\Theta(\alpha_T) = \beta_\Theta[\bar{\omega}] \in \mathrm{H}^{0,2}_{\bar{\partial}}(S^{[k]}).$$

Note that we have replaced $[\Theta(\Phi)]$ by $\Theta(\alpha_T)$, where the Atiyah class α_T is represented by Φ in Dolbeault cohomology. This is because we wish to use descriptions of the Atiyah class other than the Dolbeault description. More generally, if Γ decomposes into connected components as $\gamma_1 \cdots \gamma_j$ then

$$b_\Gamma(S^{[k]}) = \frac{1}{(8\pi^2)^k}\beta_{\gamma_1}\cdots\beta_{\gamma_j}$$

with

$$\gamma_i(\alpha_T) = \beta_{\gamma_i}[\bar{\omega}^{m_i}] \in \mathrm{H}^{0,2m_i}_{\bar{\partial}}(S^{[k]})$$

where $2m_i$ is the number of vertices in γ_i. In general β_{γ_i} will rescale by λ^{-2m_i} under a rescaling of the holomorphic symplectic form, though once again we assume the normalization of ω has been fixed, and hence β_{γ_i} is a scalar which depends only on $S^{[k]}$ (in other words on k), and which can be evaluated according to the formula

$$\begin{aligned} k!\beta_{\gamma_i} &= \int_{S^{[k]}} \beta_{\gamma_i}[\omega^k][\bar{\omega}^k] \\ &= \int_{S^{[k]}} \gamma_i(\alpha_T)[\omega^k][\bar{\omega}^{k-m_i}]. \end{aligned}$$

In fact this approach is not the simplest. Let us drop the subscript 'i' in what follows. Instead of multiplying $\gamma(\alpha_T)[\omega^m]$ by $[\omega^{k-m}\bar{\omega}^{k-m}]$ and then integrating over the entire space $S^{[k]}$, it is easier to integrate the former over a complex submanifold X of real-dimension $4m$. We can restrict

$$\gamma(\alpha_T)[\omega^m] \in \mathrm{H}^{2m,2m}_{\bar{\partial}}(S^{[k]})$$

to X, and there are projections $T^*_{S^{[k]}} \to T^*_X$ and $\bar{T}^*_{S^{[k]}} \to \bar{T}^*_X$, which induce a projection

$$\mathrm{H}^0(X, \Lambda^{2m}T^*_{S^{[k]}} \wedge \Lambda^{2m}\bar{T}^*_{S^{[k]}}|_X) \to \mathrm{H}^0(X, \Lambda^{2m}T^*_X \wedge \Lambda^{2m}\bar{T}^*_X).$$

Applying this to $\gamma(\alpha_T)[\omega^m]|_X$ gives us something we can integrate over X. Actually, we shall assume when we integrate that forms with some normal component to X give zero, so that the projection is implicitly understood. The integral equals

$$\int_X \gamma(\alpha_T)[\omega^m]|_X = \beta_\gamma \int_X \omega^m\bar{\omega}^m|_X$$

from which we can determine β_γ.

From a Čech cohomology point of view, we begin with

$$\gamma(\alpha_T) \in \mathrm{H}^{2m}(S^{[k]}, \mathcal{O}_{S^{[k]}})$$

and restrict this to a cohomology class in $\mathrm{H}^{2m}(X, \mathcal{O}_X)$. Note that there is no reason to expect this cohomology group to be one-dimensional. We also take the section

$$\omega^m \in \mathrm{H}^0(S^{[k]}, \Lambda^{2m} T^*_{S^{[k]}})$$

restrict it to X, and project to a holomorphic $2m$-form on X. This gives us

$$\omega_X^m \in \mathrm{H}^0(X, \Lambda^{2m} T^*_X)$$

where ω_X is the restriction and projection of ω to X, and it is a holomorphic two-form which will most likely be degenerate. The Serre duality pairing of $\gamma(\alpha_T)$ with ω_X^m then gives us a canonical number, which is equal to β_γ multiplied by

$$\int_X \omega^m \bar{\omega}^m|_X.$$

As before, this integral factor must be included as β_γ on its own is not a canonically defined scalar.

We can also calculate β_γ via the residue approach. Recall that this involves a meromorphic connection on $S^{[k]}$ with simple poles along a smooth divisor D. The element β_T is the residue of this connection, and we can use it along with the Atiyah class α_T to construct an element

$$\gamma(\alpha_T, \beta_T) \in \mathrm{H}^{2m-1}(D, L|_D)$$

which we restrict to $D \cap X$. Now we require that D intersects X normally, or in other words $D \cap X$ is of codimension one in X, and hence a divisor in X. In particular, this means that if D is given in local holomorphic coordinates $z_1, \ldots, z_{2k}$ by $f(z) = 0$, then the normal form df in $T^*_{S^{[k]}}$ will project to a non-zero element in T^*_X. Now consider

$$\omega^m \in \mathrm{H}^0(S^{[k]}, \Lambda^{2m} T^*_{S^{[k]}})$$

and restrict and project to X as before to get

$$\omega_X^m \in \mathrm{H}^0(X, \Lambda^{2m} T^*_X).$$

Then we can construct

$$\gamma(\alpha_T, \beta_T)\omega_X^m|_{D\cap X} \in \mathrm{H}^{2m-1}(D \cap X, \mathcal{K}_X \otimes L|_{D\cap X})$$

and by the adjunction formula the canonical line bundle of $D \cap X$ is

$$\mathcal{K}_{D\cap X} = \mathcal{K}_X \otimes L|_{D\cap X}.$$

Therefore the above element is really in $\mathrm{H}^{2m-1}(D\cap X, \mathcal{K}_{D\cap X})$. In the case that $D\cap X$ is connected this cohomology group is canonically isomorphic to $\mathbb{C}$, with the map to the complex numbers given by contour integration in the case of Čech cohomology and integration over all of $D \cap X$ in the case of Dolbeault cohomology. More generally we need to sum the contributions coming from the different connected components of $D \cap X$. This gives us the integral of the element

$$\gamma(\alpha_T, \beta_T)\omega_X^m|_{D\cap X}$$

which by the residue formula equals β_γ, up to the usual normalization factor

$$\int_X \omega^m \bar{\omega}^m|_X$$

and the addition $2\pi i$ factor. As before, note that the Poincaré residue formula implies that if we write

$$\frac{\omega_X^m}{f}|_{D\cap X} = \frac{df}{f} \wedge \nu'_{D\cap X}$$

locally, then the section

$$\omega_X^m|_{D\cap X} \in \mathrm{H}^0(D \cap X, \mathcal{K}_X|_{D\cap X})$$

should be replaced by the section

$$f\nu'_{D\cap X} \in \mathrm{H}^0(D \cap X, \mathcal{K}_{D\cap X} \otimes L^*|_{D\cap X})$$

in the above integral.

This approach can be extended to the case when D is a normal crossing divisor in $S^{[k]}$, although describing the residue β_T in that case is slightly more complicated. On the other hand, we only need to know $\beta_T|_{D\cap X}$ so if we assume that $D \cap X$ in X is still a smooth divisor then the calculation is no different to before. The reason this last approach is so useful is that it enables us to restrict the calculations to integrals over submanifolds of small dimension, namely $D \cap X$. We then only need a description of the Atiyah class in a neighbourhood of these submanifolds, which potentially allows us to avoid regions in $S^{[k]}$ where the Atiyah class becomes more difficult to describe. This reasoning should become clearer in the calculation of β_Θ, which we now turn to.

In the case $\gamma = \Theta$ the submanifold X that we should choose is

$$\widehat{S} = \{\xi \in S^{[k]} | \{x_2, x_3, \ldots, x_k\} \subset \mathrm{supp}(\xi)\}$$

where $\{x_2, \ldots, x_k\}$ is an unordered set of $k-1$ distinct fixed points chosen to lie in generic position in S. Let us write

$$\pi(\xi) = [x_1] + [x_2] + [x_3] + \ldots + [x_k] \in \mathrm{Sym}^k S$$

where π is the Hilbert-Chow morphism. Here x_1 is an arbitrary point in S, which may coincide with one of $x_2, \ldots, x_k$, and therefore

$$\pi(\widehat{S}) \cong S.$$

In $S^{[k]}$ however, a neighbourhood where x_1 collides with one of the fixed points, say x_2, looks locally like the product of a neighbourhood of the blown-up double point $2[x_2]$ in the Hilbert scheme of two points

$$S^{[2]} = \mathrm{Blow}_\Delta(S \times S)/\mathcal{S}_2$$

and neighbourhoods of the remaining $k-2$ fixed points $x_3, \dots, x_k$. The intersection of $\widehat{S}$ with this neighbourhood is a neighbourhood of the blow-up of x_2 in S. This explains our notation, as $\widehat{S}$ is isomorphic to S with the $k-1$ fixed points blown-up.

From the construction of ω on $S^{[k]}$ it is clear that restricting to $\widehat{S}$ gives us the holomorphic symplectic form on S extended over the exceptional curves $C_2, \dots, C_k$ of the blow-ups in some way. Since these curves have codimension one we can ignore them in calculations of volume, and hence

$$\begin{aligned}\int_{\widehat{S}} \omega\bar{\omega}|_{\widehat{S}} &= \int_{S\backslash\{x_2,\dots,x_k\}} \omega_S \bar{\omega}_S \\ &= 1.\end{aligned}$$

Therefore

$$\beta_\Theta = \int_{\widehat{S}} \Theta(\alpha_T)[\omega]|_{\widehat{S}}.$$

We claim that the right hand side is the sum

$$8\pi^2 b_\Theta(S) + (k-1)\delta$$

of

$$\int_S \Theta(\alpha_S)[\omega_S] = 8\pi^2 b_\Theta(S)$$

where α_S is the Atiyah class of the tangent bundle of S, and $k-1$ additional (identical) contributions δ to the integral coming from the neighbourhoods $V_2, \dots, V_k$ of $C_2, \dots, C_k$ in $\widehat{S}$. The point to note is that this expression is linear in k.

Proposition 18 *We know that for some scalar* β_Θ

$$\Theta(\alpha_T) = \beta_\Theta[\bar{\omega}] \in \mathrm{H}^{0,2}_{\bar{\partial}}(S^{[k]}).$$

The dependence of β_Θ *on* $S^{[k]}$ *is that it is a linear expression in* k.

Proof As with our calculation of $b_\Theta(S)$ in Subsection 1.7 we prove this result in three different ways, based on the Čech, Dolbeault, and residue descriptions of the Atiyah class. First let us make some general comments. Recall that we had an open cover $\{U_0, U_1, \dots, \tilde{U}_{16}\}$ of the Kummer surface S. Let Ξ be the support of $\xi \in S^{[k]}$, which we think of as an unordered set of k points of S with multiplicities. Define

$$\begin{aligned}\Delta &= \{\xi \in S^{[k]} | \text{at least two points of } \Xi \text{ coincide}\} \\ \Delta_2 &= \{\xi \in S^{[k]} | \text{exactly two points of } \Xi \text{ coincide}\}.\end{aligned}$$

Then Δ is a divisor in $S^{[k]}$ which we call the *large diagonal*, and Δ_2 is a dense open subset of Δ which we call the *small diagonal*. Let $\bar{\Delta}^{(2)}$ be a small neighbourhood of the large diagonal in $S^{[2]}$ (which is the same as the small diagonal in $S^{[2]}$). Define open sets in $S^{[k]}$ by

$$\begin{aligned}
U_{\underbrace{0\cdots000}_{k}} &= \{\xi \in S^{[k]} | \Xi \cap \Delta = \emptyset, \Xi \subset U_0\} \\
U_{\underbrace{0\cdots00}_{k-1}i} &= \{\xi \in S^{[k]} | \Xi \cap \Delta = \emptyset, \exists x \in \Xi \cap U_i \text{ such that } \Xi\backslash\{x\} \subset U_0\} \\
U_{\underbrace{0\cdots00}_{k-1}\tilde{i}} &= \{\xi \in S^{[k]} | \Xi \cap \Delta = \emptyset, \exists x \in \Xi \cap \tilde{U}_i \text{ such that } \Xi\backslash\{x\} \subset U_0\} \\
U_{\underbrace{0\cdots0}_{k-2}\bar{\Delta}} &= \{\xi \in S^{[k]} | \exists \xi_2 \in \bar{\Delta}^{(2)} \text{ such that } \Xi_2 \subset \Xi, \Xi \cap \Delta \subset \Xi_2, \Xi\backslash\Xi_2 \subset U_0\}
\end{aligned}$$

where Ξ_2 obviously means the support of ξ_2. In other words, the first of these is the set where the support of ξ consists of k distinct points in U_0, the second and third are the sets where the support of ξ consists of k distinct points, $k-1$ of which are in U_0 and the remaining one in U_i or $\tilde{U}_i$ (respectively), and the last is the set where the support of ξ contains two points which either collide or are very close together, with the remaining $k-2$ being distinct points in U_0.

The submanifold $\widehat{S}$ of $S^{[k]}$ is contained in the union of the above sets. Indeed

$$\begin{aligned}
U_{0\cdots0} \cap \widehat{S} &\cong U_0\backslash\{x_2, \ldots, x_k\} \\
U_{0\cdots0i} \cap \widehat{S} &\cong U_i \\
U_{0\cdots0\tilde{i}} \cap \widehat{S} &\cong \tilde{U}_i \\
U_{0\cdots0\bar{\Delta}} \cap \widehat{S} &\cong V_2 \cup \cdots \cup V_k
\end{aligned}$$

where the open sets $V_2, \ldots, V_k$ are the small neighbourhoods of the exceptional curves $C_2, \ldots, C_k$ in $\widehat{S}$, and are disjoint. Note that by choosing $x_2, \ldots, x_k$ to lie in *generic* position in S we mean that they lie in

$$U_0\backslash(U_1 \cup \cdots \cup \tilde{U}_{16}).$$

It is clear that the collection

$$\{U_{0\cdots0}, U_{0\cdots01}, \ldots, U_{0\cdots0\tilde{16}}, U_{0\cdots0\bar{\Delta}}\}$$

can be completed to an open cover of $S^{[k]}$ in such a way that the additional sets will not intersect with $\widehat{S}$. Since we will only perform calculations on $\widehat{S}$ (or on divisors in $\widehat{S}$) it follows that we only need a description of the Atiyah class over the above sets.

Let us begin with a Čech cohomology description. Recall that this is given by taking local holomorphic connections and looking at their differences on intersections of open sets. Take the product of k copies of the flat connection ∇_0, and quotient by the action of $\mathcal{S}_k$. This gives the flat connection $\nabla_{0\cdots0}$ on $U_{0\cdots0}$ (as this open set avoids the large diagonal). Similarly, the flat connections $\nabla_{0\cdots 0i}$ on $U_{0\cdots 0i}$ and $\nabla_{0\cdots 0\tilde{i}}$ on $U_{0\cdots 0\tilde{i}}$ can be constructed from $k-1$ copies of ∇_0 and a copy of the flat connections ∇_i and $\tilde{\nabla}_i$ on U_i and $\tilde{U}_i$ respectively. With $U_{0\cdots 0\bar{\Delta}}$ we encounter a slight problem: there does not exist a flat (or even holomorphic) connection on $\bar{\Delta}^{(2)}$ and hence there will not exist such a connection on $U_{0\cdots 0\bar{\Delta}}$. However, we can cover $\bar{\Delta}^{(2)}$ with open sets which do admit holomorphic (even flat) connections. Taking the products of these with $k-2$ copies of ∇_0, and quotienting by $\mathcal{S}_k$ as before, will give us holomorphic (flat) connections on the corresponding open cover of $U_{0\cdots 0\bar{\Delta}}$. The important feature here is that the $k-2$ extra factors ∇_0 are all *flat* connections and are common to all of the connections over the open sets in the cover of $U_{0\cdots 0\bar{\Delta}}$.

Next we look at the differences

$$(\alpha_T)_{(0\cdots 0)(0\cdots 0i)} = \nabla_{0\cdots 0} - \nabla_{0\cdots 0i} \in \mathrm{H}^0(U_{0\cdots 0} \cap U_{0\cdots 0i}, T^* \otimes \mathrm{End}T)$$

etc. These give a Čech representative of the Atiyah class α_T of $S^{[k]}$. Of course we wish to restrict to $\widehat{S}$ so we need only consider those open sets which intersect this submanifold. For example, consider the above section over $U_{0\cdots 0} \cap U_{0\cdots 0i}$, which we can write in local coordinates on either $U_{0\cdots 0}$ or $U_{0\cdots 0i}$. The connections $\nabla_{0\cdots 0}$ and $\nabla_{0\cdots 0i}$ both contain the same $k-1$ flat factors ∇_0 which contribute nothing to the connection one-forms in either coordinate patch. Therefore

$$(\alpha_T)_{(0\cdots 0)(0\cdots 0i)} = p_k^*(\nabla_0 - \nabla_i)$$

where p_k is the projection to the kth factor (this makes sense at a local level, but not globally as the $\mathcal{S}_k$ action permutes the factors). Now

$$\nabla_0 - \nabla_i = (\alpha_S)_{0i} \in \mathrm{H}^0(U_0 \cap U_i, T_S^* \otimes \mathrm{End}T_S)$$

gives a Čech representative of the Atiyah class on S. In particular, restricting the Atiyah class α_T to

$$\widehat{S} \backslash (V_2 \cup \cdots \cup V_k)$$

gives us the restriction of the Atiyah class α_S to S with neighbourhoods of the fixed points $x_2, \ldots, x_k$ removed. As these points lie in

$$U_0 \backslash (U_1 \cup \cdots \cup \tilde{U}_{16})$$

their small neighbourhoods will not intersect with any two-fold intersections of sets on the open cover

$$\{U_0, U_1, \ldots, \tilde{U}_{16}\}$$

of S. In other words removing these neighbourhood will not change the Čech representative of α_S. It follows that restricting

$$\Theta(\alpha_T) \in \mathrm{H}^2(S^{[k]}, \mathcal{O}_{S^{[k]}})$$

to a cohomology class in $\mathrm{H}^2(\widehat{S}, \mathcal{O}_{\widehat{S}})$ gives us

$$\Theta(\alpha_S) \in \mathrm{H}^2(S, \mathcal{O}_S)$$

away from the small neighbourhoods $V_2, \ldots, V_k$ of the exceptional curves $C_2, \ldots, C_k$. Since the restricted holomorphic symplectic form $\omega_{\widehat{S}}$ equals ω_S on this region, we see that the Serre duality pairing of $\Theta(\alpha_T)$ with $\omega_{\widehat{S}}$ is equal to the Serre duality pairing of $\Theta(\alpha_S)$ with ω_S, plus $k-1$ additional terms coming from the neighbourhoods $V_2, \ldots, V_k$. In other words, β_Θ is equal to $8\pi^2 b_\Theta(S)$ plus these additional terms.

It is clear by symmetry that these $k-1$ additional terms must be identical, but we also need them to be independent of k. Recall that we covered $U_{0\cdots 0\bar{\Delta}}$ with open sets such the (flat) holomorphic connections on these sets all looked locally like a product of a flat connection on some open set in $\bar{\Delta}^{(2)}$ and $k-2$ copies of the flat connection ∇_0. The flat connection $\nabla_{0\cdots 0}$ on $U_{0\cdots 0}$ also factorizes locally into a product of k copies of ∇_0. Therefore the differences of the connections on the two-fold intersections of these open sets with $U_{0\cdots 0}$ (or with themselves) will still contain these $k-2$ flat factors which contribute nothing to the connection one-forms in either coordinate patch. Note that $U_{0\cdots 0\bar{\Delta}}$ does not intersect $U_{0\cdots 0i}$ or $U_{0\cdots 0\tilde{i}}$ (at least not near $\widehat{S}$). Thus the Atiyah class on these two-fold intersections of sets does not depend on how many additional flat factors we add to the local connections. In other words it does not vary as we change k. Now if we restrict to $\widehat{S}$, we can observe that in the Serre duality pairing of $\Theta(\alpha_T)$ with $\omega_{\widehat{S}}$ the contributions which come from the neighbourhoods $V_2, \ldots, V_k$ will be identical and independent of k. If we call these terms δ, it follows that

$$\beta_\Theta = 8\pi^2 b_\Theta(S) + (k-1)\delta$$

is linear in k.

We will now repeat the proof using the Dolbeault representative of the Atiyah class, which is given by the $(1,1)$ part of the curvature of a smooth global connection of type $(1,0)$ on $S^{[k]}$. Such a connection may be obtained by patching together local connections of type $(1,0)$ using a partition of unity. In fact, we can use the local flat connections

$$\nabla_{0\cdots 0}, \nabla_{0\cdots 01}, \ldots, \nabla_{0\cdots 0\tilde{16}}$$

from the Čech description, plus a connection of type $(1,0)$ on $U_{0\cdots 0\bar{\Delta}}$. The latter can be constructed by taking the product of some arbitrary type $(1,0)$ connection on $\bar{\Delta}^{(2)}$ with $k-2$ copies of the flat connection ∇_0, and quotienting by the action of $\mathcal{S}_k$. As before, we need only consider the collection of open sets

$$\{U_{0\cdots 0}, U_{0\cdots 01}, \ldots, U_{0\cdots 0\tilde{16}}, U_{0\cdots 0\bar{\Delta}}\}$$

which intersect $\widehat{S}$.

Let $\{\psi_0, \psi_1, \ldots, \tilde{\psi}_{16}\}$ be a partition of unity subordinate to the open cover $\{U_0, U_1, \ldots, \tilde{U}_{16}\}$ of S. We can use this to construct a partition of unity for the open cover of $S^{[k]}$. For example, the product of k copies of ψ_0, quotiented by the action of $\mathcal{S}_k$, will give a partition function $\psi_{0\cdots 0}$ for $U_{0\cdots 0}$. Similarly, on $U_{0\cdots 0i}$ and $U_{0\cdots 0\tilde{i}}$ we can take the product of $k-1$ copies of ψ_0 with a copy of ψ_i or $\tilde{\psi}_i$, quotient by the action of $\mathcal{S}_k$, to get $\psi_{0\cdots 0i}$ and $\psi_{0\cdots 0\tilde{i}}$ respectively. On $U_{0\cdots 0\bar{\Delta}}$ we take the product of $k-2$ copies of ψ_0 with some arbitrary partition function $\psi_{\bar{\Delta}}$ supported in $\bar{\Delta}^{(2)}$ and identically one on the diagonal in $S^{[2]}$. This gives $\psi_{0\cdots 0\bar{\Delta}}$. For these to really be partition functions on $S^{[k]}$ we may need to modify them slightly in a neighbourhood of the large diagonal, but nowhere else.

Using this partition of unity we can patch together the local connections in order to get a global smooth connection of type $(1,0)$ on $S^{[k]}$. In a neighbourhood of $\widehat{S}$ this will look like

$$\psi_{0\cdots 0}\nabla_{0\cdots 0}+\psi_{0\cdots 01}\nabla_{0\cdots 01}+\ldots+\psi_{0\cdots 0\tilde{16}}\nabla_{0\cdots 0\tilde{16}}+\psi_{0\cdots 0\bar{\Delta}}\nabla_{0\cdots 0\bar{\Delta}}$$

from which we see that the Atiyah class will look like

$$\alpha_T=A_{0\cdots 0}\bar{\partial}\psi_{0\cdots 0}+A_{0\cdots 01}\bar{\partial}\psi_{0\cdots 01}+\ldots+A_{0\cdots 0\tilde{16}}\bar{\partial}\psi_{0\cdots 0\tilde{16}}+\bar{\partial}(A_{0\cdots 0\bar{\Delta}}\psi_{0\cdots 0\bar{\Delta}})$$

where $A_{0\cdots 0}$ is the connection one-form of the local connection $\nabla_{0\cdots 0}$, etc. Restrict to $\widehat{S}$ and ignore the last term which is only supported in the neighbourhoods $V_2,\ldots,V_k$. Let ξ in $S^{[k]}$ have support $\{x_1,x_2,\ldots,x_k\}$ and lie on

$$\widehat{S}\backslash(V_2\cup\ldots\cup V_k).$$

In other words x_1 lies in S but not in a neighbourhood of any of the points $x_2,\ldots,x_k$. Then

$$\begin{aligned}\psi_{0\cdots 0}(\xi)&=\psi_0(x_1)\\ \psi_{0\cdots 0i}(\xi)&=\psi_i(x_1)\\ \psi_{0\cdots 0\tilde{i}}(\xi)&=\tilde{\psi}_i(x_1)\end{aligned}$$

as ψ_0 is identically one on the neighbourhoods of the points $x_2,\ldots,x_k$. Similarly, the connection one forms at ξ are given by

$$\begin{aligned}A_{0\cdots 0}(\xi)&=p_1^*A_0(x_1)\\ A_{0\cdots 0i}(\xi)&=p_1^*A_i(x_1)\\ A_{0\cdots 0\tilde{i}}(\xi)&=p_1^*\tilde{A}_i(x_1)\end{aligned}$$

where p_1 is the projection to the first factor (which makes sense locally, though not globally due to the action of the symmetric group permuting the factors). Therefore

$$\begin{aligned}\alpha_T|_{\widehat{S}\backslash(V_2\cup\cdots\cup V_k)}&=(p_1^*A_0\bar{\partial}\psi_0+p_1^*A_1\bar{\partial}\psi_1+\ldots+p_1^*A_{\tilde{16}}\bar{\partial}\psi_{\tilde{16}})(x_1)\\ &=p_1^*\alpha_S(x_1)\end{aligned}$$

where α_S is the Dolbeault representative of the Atiyah class on S. Since this is supported in a neighbourhood of the sixteen exceptional curves $D_1,\ldots,D_{16}$ in S, removing the neighbourhoods of the points $x_2, \ldots, x_k$ (where α_S vanishes anyway) will have no effect. Therefore

$$\begin{aligned}\int_{\widehat{S}\backslash(V_2\cup\cdots\cup V_k)} \Theta(\alpha_T)[\omega]|_{\widehat{S}} &= \int_S \Theta(\alpha_S)[\omega_S] \\ &= 8\pi^2 b_\Theta(S)\end{aligned}$$

as we have already seen that ω restricted to $\widehat{S}$ is just ω_S away from the blow-ups of $x_2, \ldots, x_k$. As before, this just leaves the terms

$$\int_{V_2} \Theta(\alpha_T)[\omega]|_{\widehat{S}} + \ldots + \int_{V_k} \Theta(\alpha_T)[\omega]|_{\widehat{S}}$$

which we can assume are identical by symmetry. It only remains to prove that each of these integrals is independent of k. As in the Čech cohomology case, this follows from the observation that $\nabla_{0\cdots 0\bar{\Delta}}$ contains $k-2$ copies of the *flat* connection ∇_0 as factors, which contribute nothing to the connection one-form $A_{0\cdots 0\bar{\Delta}}$. Also the partition function $\psi_{0\cdots 0\bar{\Delta}}$ contains $k-2$ factor equal to ψ_0 which are identically one on $V_2, \ldots, V_k$ (since ψ_0 is identically one on the neighbourhoods of the fixed points $x_2, \ldots, x_k$). In other words, over $V_2, \ldots, V_k$ the Atiyah class α_T does not vary as we change k, as this merely involves adding 'redundant' factors. In particular

$$\int_{V_j} \Theta(\alpha_T)[\omega]|_{\widehat{S}} = \delta$$

is independent of k (for $j = 2, \ldots, k$), and therefore

$$\begin{aligned}\beta_\Theta &= \int_{\widehat{S}} \Theta(\alpha_T)[\omega]|_{\widehat{S}} \\ &= \int_{\widehat{S}\backslash(V_2\cup\cdots\cup V_k)} \Theta(\alpha_T)[\omega]|_{\widehat{S}} + \int_{V_2} \Theta(\alpha_T)[\omega]|_{\widehat{S}} + \ldots + \int_{V_k} \Theta(\alpha_T)[\omega]|_{\widehat{S}} \\ &= 8\pi^2 b_\Theta(S) + (k-1)\delta\end{aligned}$$

is linear in k.

Finally, we show how this result can be proved using the residue approach. Recall that on the Kummer surface S we took the flat connection ∇_0 on U_0 and observed that it could be extended to a global meromorphic connection with a simple pole along the smooth divisor

$$D = D_1 + \ldots + D_{16}$$

where $D_1, \ldots, D_{16}$ are the sixteen exceptional curves in S. In the case of $S^{[k]}$ we take the flat connection $\nabla_{0\ldots 0}$ on $U_{0\ldots 0}$. Note that this open set is dense in $S^{[k]}$ and its complement is the union of the large diagonal Δ and the divisors

$$E_i = \{\xi \in S^{[k]} | \exists x \in \Xi \cap D_i\}.$$

In fact, $\nabla_{0\cdots0}$ can be extended to a global meromorphic connection on $S^{[k]}$ with a simple pole along the divisor

$$E = E_1 + \ldots + E_{16} + \Delta.$$

Although this is a normal crossing divisor, its intersection $E \cap \widehat{S}$ with $\widehat{S}$ is a smooth divisor in this submanifold. Indeed

$$\begin{aligned} E \cap \widehat{S} &= E_1 \cap \widehat{S} + \ldots + E_{16} \cap \widehat{S} + \Delta \cap \widehat{S} \\ &= D_1 + \ldots + D_{16} + C_2 + \ldots + C_k \end{aligned}$$

and the terms in the last line are all smooth curves in $\widehat{S}$. Consider how this connection looks in a neighbourhood of

$$\xi \in D_i \subset \widehat{S} \subset S^{[k]}.$$

Let the support of ξ be $\Xi = \{x_1, x_2, \ldots, x_k\}$. The points $x_2, \ldots, x_k$ lie in U_0 over which ∇_0 is holomorphic, whereas x_1 lies on D_i along which ∇_0 has a simple pole. Therefore the residue of $\nabla_{0\cdots0}$ at ξ is simply the residue of ∇_0 at x_1, as the $k-1$ additional holomorphic factors do not contribute to the residue. Thus

$$\beta_T|_{D_i} = p_1^*\beta_S|_{D_i}$$

where p_1 is the (locally well-defined) projection to the first factor and β_S is the residue of the meromorphic connection ∇_0 on S. Consider also how the connection $\nabla_{0\cdots0}$ looks in a neighbourhood of

$$\xi \in C_2 \subset \widehat{S} \subset S^{[k]}.$$

Writing the support of ξ as above, we see that we can still say that ∇_0 is holomorphic in neighbourhood of the $k-2$ distinct points $x_3, \ldots, x_k$. Hence these $k-2$ factors do not contribute to the residue, which therefore does not vary with k. Similarly for neighbourhoods of ξ in $C_3, \ldots, C_k$.

Now the residue calculation of β_Θ involves an integral of

$$\Theta(\alpha_T, \beta_T)\omega_{\widehat{S}}|_{E\cap\widehat{S}}$$

which is either a contour integral if we use Čech cohomology or an integral over $E \cap \widehat{S}$ if we use Dolbeault cohomology. Firstly, from what we know about the forms of the Atiyah class α_T (in either Čech or Dolbeault descriptions), the holomorphic symplectic form ω, and the residue β_T when they are restricted to $\widehat{S}$, we can observe that the integrand restricted to

$$D = D_1 + \ldots + D_{16}$$

is just

$$\Theta(\alpha_S, \beta_S)\omega_S|_D$$

and restricted to

$$\Delta \cap \widehat{S} = C_2 + \ldots + C_k$$

gives $k-1$ terms which are independent of k, and identical by symmetry. Therefore, in Dolbeault cohomology for example

$$\begin{aligned}
\beta_\Theta &= 2\pi i \int_{E\cap\widehat{S}} \Theta(\alpha_T, \beta_T)\omega_{\widehat{S}}|_{E\cap\widehat{S}} \\
&= 2\pi i \int_D \Theta(\alpha_S, \beta_S)\omega_S|_D + (k-1)\delta \\
&= 8\pi^2 b_\Theta(S) + (k-1)\delta
\end{aligned}$$

and similarly if we use Čech cohomology. So again we see that β_Θ is linear in k. □

Since we know $b_\Theta(S) = 48$ and $b_{\Theta^2}(S^{[2]}) = 3600$, this is enough to determine

$$\beta_\Theta = (8\pi^2)12(k+3)$$

and hence

Proposition 19 *For the Hilbert scheme of k points on a K3 surface S we have*

$$b_{\Theta^k}(S^{[k]}) = 12^k(k+3)^k$$

and therefore

$$\int_{S^{[k]}} \mathrm{Td}_k^{1/2}(S^{[k]}) = \frac{(k+3)^k}{4^k k!}.$$

We can verify that this gives $b_{\Theta^3}(S^{[3]}) = 373248$, which is in agreement with the value we have already obtained from the Chern numbers. Also, knowing $b_{\Theta^4}(S^{[4]})$ allows us to determine s and hence complete our tables of Chern numbers and Rozansky-Witten invariants for $k \leq 4$ (see Appendices D and E.1).

We stated earlier that there are no explicit formulae for the Chern numbers of these Hilbert schemes in all dimensions. However, there are some implicit formulae for the Chern numbers of Hilbert schemes of points on an arbitrary complex surface due to Ellingsrud, Göttsche, and Lehn [22], and these methods have enabled Göttsche [25] to calculate the Chern numbers of $S^{[k]}$ for $k \leq 6$ using the Bott residue formula. It is reassuring that these values agree with those obtained above for $k \leq 4$. Furthermore, when k is 5 and 6 we can use Göttsche's calculations to compare our values for $b_{\Theta^5}(S^{[5]})$ and $b_{\Theta^6}(S^{[6]})$ as calculated above with the value calculated by using the formulae for b_{Θ^5} and b_{Θ^6} in terms of Chern numbers. Once again we find the values agree.

In order to extend our calculations to higher dimensions we need to know something about the form of β_γ in the case that γ is a trivalent graph with $2m > 2$ vertices, ie. not simply just Θ. In fact there is strong evidence to suggest that β_γ is linear in k for these cases also. Indeed the calculations we have already done verify this for β_{Θ_2} with $k \leq 4$. Furthermore, we can calculate a few more values of Rozansky-Witten invariants by using Göttsche's values for the Chern numbers for $k \leq 6$, and this allows us to verify the linearity of β_γ in several other cases.

If we attempt to prove the linearity of β_γ using either the Čech or Dolbeault descriptions of the Atiyah class things soon become complicated: the submanifold X in $S^{[k]}$ which we need to perform calculations on will have real-dimension $4m$ and be at least as complicated as $S^{[m]}$. In the β_Θ case performing calculations on $\widehat{S}$ was just manageable. For $m > 1$, the approach most likely to succeed would probably involve meromorphic connections. However, there does not appear to be a way to reduce the calculation to one over a smooth divisor, so we ultimately need a good description of the residue of a meromorphic connection with a simple pole along a normal crossing divisor. Note that in the case of connections with *logarithmic* singularities such a theory is well-developed, and Kapranov has used this approach to derive quite explicit formulae for the Rozansky-Witten invariants of a hyperkähler manifold which admits such a connection (see [32]). These formulae generalize those of Ohtsuki [43] for Chern numbers. Unfortunately, it seems that such a connection does not exist on the Hilbert scheme of points on a K3 surface (for example, the poles of ∇_0 on S are certainly not logarithmic).

5.4 Generalized Kummer varieties

The Hilbert scheme of k points on a torus T has a hyperkähler metric, but these spaces are not irreducible. There is a map p to the torus itself given by composing the Hilbert-Chow morphism with addition (since the torus is an abelian surface, we can add k points together and the result will lie in T). The fibres of this map are all isomorphic hyperkähler manifolds of dimension $4(k-1)$, and are also irreducible. We denote them by $T^{[[k-1]]}$, so that $T^{[[k]]}$ would be the $4k$-dimensional space obtained by taking the fibres in $T^{[k+1]}$. This is the second main family of examples of compact hyperkähler manifolds, and they are known as generalized Kummer varieties (as $T^{[[1]]}$ is simply the Kummer model of a K3 surface).

The Hodge numbers of these spaces were calculated by Göttsche and Soergel [26]. In particular, the χ_y-genus is given by

$$\chi_y(T^{[[k]]}) = (k+1)\sum_{d|(k+1)} d^3(1 - y + y^2 - \ldots + (-y)^{\frac{k+1}{d}-1})^2(-y)^{k+1-\frac{k+1}{d}}.$$

For $k \leq 4$ these polynomials are written out in Appendix C. Once again we can substitute these values into the Riemann-Roch formula to give the results in Appendix D, from which we can determine all the Rozansky-Witten invariants for $k = 1$, 2, and 3. The next step is to compute $b_{\Theta^k}(T^{[[k]]})$, and in particular $b_{\Theta^4}(T^{[[4]]})$ in order to determine s, and hence complete the tables up to $k = 4$.

To begin with let us normalize the holomorphic symplectic form on T so that

$$\int_T \omega\bar{\omega} = 1.$$

The same calculation as before shows that the induced holomorphic symplectic form on $T^{[k+1]}$, also denoted ω, will have normalization

$$\int_{T^{[k+1]}} \omega^{k+1}\bar{\omega}^{k+1} = (k+1)!$$

The generalized Kummer variety $T^{[[k]]}$ sits inside this Hilbert scheme as a fibre of the fibration over T. Denote by i the inclusion of the fibre; thus we have

$$\begin{array}{ccc} T^{[[k]]} & \stackrel{i}{\hookrightarrow} & T^{[k+1]} \\ & & \downarrow p \\ & & T. \end{array}$$

We wish to calculate the volume of the generalized Kummer variety, or rather we wish to know

$$\int_{T^{[[k]]}} \omega^k \bar{\omega}^k.$$

Recall that the holomorphic symplectic form ω on the Hilbert scheme $T^{[k+1]}$ is given by

$$\pi^*(p_0^*\omega + \ldots + p_k^*\omega)$$

where p_j is the projection onto the $(j-1)$th factor in T^{k+1}, the sum descends to the symmetric product $\mathrm{Sym}^{k+1}T$, and then we pull-back to the Hilbert scheme using the Hilbert-Chow morphism π. The holomorphic symplectic form on the generalized Kummer variety $T^{[[k]]}$, also denoted ω, is defined by the restriction of $\omega_{T^{[k+1]}}$. According to Subsection 2.4 the volume form on $T^{[k+1]}$ is given by

$$\begin{aligned} V_{T^{[k+1]}} &= \frac{1}{2^{2(k+1)}((k+1)!)^2}\omega^{k+1}\bar{\omega}^{k+1} \\ &= \frac{1}{2^{2(k+1)}((k+1)!)^2}\pi^*(p_0^*\omega + \ldots + p_k^*\omega)^{k+1}\pi^*(p_0^*\bar{\omega} + \ldots + p_k^*\bar{\omega})^{k+1} \\ &= \frac{1}{2^{2(k+1)}}\pi^*(p_0^*\omega \cdots p_k^*\omega p_0^*\bar{\omega} \cdots p_k^*\bar{\omega}) \end{aligned}$$

on $T^{[[k]]}$ by

$$\begin{aligned} V_{T^{[[k]]}} &= \frac{1}{2^{2k}(k!)^2}\omega^k\bar{\omega}^k \\ &= \frac{1}{2^{2k}(k!)^2}\pi^*(p_0^*\omega + \ldots + p_k^*\omega)^k\pi^*(p_0^*\bar{\omega} + \ldots + p_k^*\bar{\omega})^k \\ &= \frac{1}{2^{2k}}\pi^*\left(\sum_{j=0}^k p_0^*\omega \cdots \widehat{p_j^*\omega} \cdots p_k^*\omega\right)\pi^*\left(\sum_{j=0}^k p_0^*\bar{\omega} \cdots \widehat{p_j^*\bar{\omega}} \cdots p_k^*\bar{\omega}\right) \end{aligned}$$

where $\widehat{p_j^*\omega}$ and $\widehat{p_j^*\bar{\omega}}$ mean that we omit these terms, and on T by

$$V_T = \frac{1}{2^2}\omega\bar{\omega}.$$

We wish to compare the volume form on $T^{[k+1]}$ with the one induced by the fibration from the volume forms on the fibre $T^{[[k]]}$ and the base T, ie. we wish to compare

$$V_{T^{[k+1]}} \qquad \text{and} \qquad i_* V_{T^{[[k]]}} \wedge p^* V_T.$$

Consider the map $p_0 + \ldots + p_k$ from T^{k+1} to T. Clearly it is symmetric and so descends to a map from $\mathrm{Sym}^{k+1}T$ to T. In fact p is given by composing the Hilbert-Chow morphism π with $p_0 + \ldots + p_k$. Now

$$p^* V_T = \frac{1}{2^2} p^*\omega p^*\bar{\omega}$$

and $p^*\omega$ is 'almost' $\pi^*(p_0^*\omega + \ldots + p_k^*\omega) = \omega_{T^{[k+1]}}$ (similarly for $p^*\bar{\omega}$). Indeed given two vector fields v and u on $T^{[k+1]}$, we find

$$\begin{aligned}
p^*\omega(v,u) &= \omega(p_* v, p_* u) \\
&= \omega((p_{0*} + \ldots + p_{k*})\pi_* v, (p_{0*} + \ldots + p_{k*})\pi_* u) \\
&= \sum_{i,j=0}^{k} \omega(p_{i*}\pi_* v, p_{j*}\pi_* u) \\
&= \sum_{j=0}^{k} \omega(p_{j*}\pi_* v, p_{j*}\pi_* u) + \sum_{i\neq j} \omega(p_{i*}\pi_* v, p_{j*}\pi_* u) \\
&= \sum_{j=0}^{k} \pi^* p_j^* \omega(v,u) + \sum_{i\neq j} \omega(p_{i*}\pi_* v, p_{j*}\pi_* u) \\
&= \omega_{T^{[k+1]}}(v,u) + \sum_{i\neq j} \omega(p_{i*}\pi_* v, p_{j*}\pi_* u)
\end{aligned}$$

However, when we calculate $i_* V_{T^{[[k]]}} \wedge p^* V_T$ the second term above gives zero contribution (this can easily be seen in local coordinates) and therefore

$$\begin{aligned}
i_* V_{T^{[[k]]}} \wedge p^* V_T &= \frac{1}{2^{2k}(k!)^2}\omega^k\bar{\omega}^k \frac{1}{2^2}\omega\bar{\omega} \\
&= \frac{1}{2^{2(k+1)}(k!)^2}\omega^{k+1}\bar{\omega}^{k+1} \\
&= (k+1)^2 V_{T^{[k+1]}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{2^2}\frac{1}{2^{2k}(k!)^2}\int_{T^{[[k]]}} \omega^k\bar{\omega}^k &= \mathrm{vol}(T)\mathrm{vol}(T^{[[k]]}) \\
&= (k+1)^2 \mathrm{vol}(T^{[k+1]}) \\
&= (k+1)^2 \frac{1}{2^{2(k+1)}((k+1)!)^2}\int_{T^{[k+1]}} \omega^{k+1}\bar{\omega}^{k+1} \\
&= \frac{1}{2^2}\frac{1}{2^{2k}(k!)^2}(k+1)!
\end{aligned}$$

and hence
$$\int_{T^{[[k]]}} \omega^k \bar{\omega}^k = (k+1)!$$

As in the last subsection, if Γ decomposes into connected components as $\gamma_1 \cdots \gamma_j$ then
$$\begin{aligned} b_\Gamma(T^{[[k]]}) &= \frac{1}{(8\pi^2)^k k!}\beta_{\gamma_1}\cdots\beta_{\gamma_j}\int_{T^{[[k]]}} \omega^k\bar{\omega}^k \\ &= \frac{(k+1)}{(8\pi^2)^k}\beta_{\gamma_1}\cdots\beta_{\gamma_j} \end{aligned}$$
with
$$\gamma_i(\alpha_T) = \beta_{\gamma_i}[\bar{\omega}^{m_i}] \in \mathrm{H}^{0,2m_i}_{\bar{\partial}}(T^{[[k]]})$$
where γ_i has $2m_i$ vertices, and the scalars β_{γ_i} depend on $T^{[[k]]}$, in other words on k. When γ is the graph Θ we can show that this dependence is linear.

Proposition 20 *We know that for some scalar β_Θ*
$$\Theta(\alpha_T) = \beta_\Theta[\bar{\omega}] \in \mathrm{H}^{0,2}_{\bar{\partial}}(T^{[[k]]}).$$
The dependence of β_Θ on $T^{[[k]]}$ is that it is a linear expression in k.

Proof The proof is similar to the proof of Proposition 18 and will be omitted. □

From $b_\Theta(T^{[[1]]}) = 48$ and $b_{\Theta^2}(T^{[[2]]}) = 3888$ we can determine
$$\beta_\Theta = (8\pi^2)12(k+1)$$
and hence

Proposition 21 *For the generalized Kummer variety $T^{[[k]]}$ we have*
$$b_{\Theta^k}(T^{[[k]]}) = 12^k(k+1)^{k+1}$$
and therefore
$$\int_{T^{[[k]]}} \mathrm{Td}^{1/2}_k(T^{[[k]]}) = \frac{(k+1)^{k+1}}{4^k k!}.$$

We can then verify $b_{\Theta^3}(T^{[[3]]}) = 442368$, and use $b_{\Theta^4}(T^{[[4]]})$ to determine s and complete our tables of Chern numbers and Rozansky-Witten invariants for $k \leq 4$ (see Appendices D and E.1).

Unlike for the Hilbert schemes of points on a K3 surface, these values for the Chern numbers of the generalized Kummer varieties appear to be completely new. In other words, this method of using the Rozansky-Witten invariants to determine the Chern numbers for $k = 4$ appears to give results not previously known.

For trivalent graphs γ with $2m > 2$ vertices we can probably also expect β_γ to be linear in k for generalized Kummer varieties. As with the Hilbert schemes of points on a K3 surface there is some evidence to support this, and a proof will most likely follow from a generalization of the residue approach to meromorphic connections with simple poles along normal crossing divisors.

5.5 Cobordism classes

Up to now in this chapter we have been dealing only with irreducible hyperkähler manifolds. For reducible manifolds we can use the product formula. Since we know the values of all the Rozansky-Witten invariants up to $k = 4$ for the Hilbert schemes of points on a K3 surface and for the generalized Kummer varieties, we can therefore calculate the Rozansky-Witten invariants for all hyperkähler manifolds constructed by taking products of these (ie. this includes reducible manifolds of arbitrarily large dimension). Some examples are given in Appendix E.2. It follows that we can, in principle, calculate the characteristic numbers for these products as well. Indeed up to real-dimension 20 we have explicit formulae for the Chern numbers in terms of Rozansky-Witten invariants by using the expansions of polywheels as given in Appendix A.1.

We know that in dimensions 4, 8, and 12 that the Rozansky-Witten invariants are all characteristic numbers. However, in dimension 16 we only know that b_{Θ^4} and $b_{\Theta^2\Theta_2}$ are characteristic numbers. At this stage we can only write the remaining invariants as rational functions of the characteristic numbers (for irreducible hyperkähler manifolds). In this subsection we will prove that they are not characteristic numbers, ie. they cannot be written as a *linear combinations* of Chern numbers. It suffices to show this for $b_{\Theta_2^2}$, as the other invariants can be written as linear combinations of $b_{\Theta_2^2}$ and Chern numbers (see Appendix A.2).

All we shall actually do is exhibit an example of two hyperkähler manifolds which have the same Chern numbers but different values of $b_{\Theta_2^2}$. Recall that in the hyperkähler case, the Chern numbers are equivalent to the Pontryagin numbers. By a theorem of Thom [47] (see also Hirzebruch's book [29]) two manifolds have the same Pontryagin numbers if and only if they represent the same class in the oriented rational cobordism ring "modulo torsion". We will ignore subtleties arising from torsion elements. Two oriented manifolds X and Y of real-dimension $4k$ are cobordant if there exists an oriented manifold M of real-dimension $4k + 1$ with boundary $\partial M = X + (-Y)$, ie. the disjoint sum of X and the manifold Y with its orientation reversed. The oriented rational cobordism ring is the ring of cobordism classes over the rational numbers, with multiplication given by taking products of manifolds and addition given by disjoint sums. Stated in these terms, we wish to find two cobordant hyperkähler manifolds which are nonetheless distinguished by the Rozansky-Witten invariants.

The construction is simple enough. We consider all hyperkähler manifolds in dimension 16 obtained by taking products of Hilbert schemes of points on a K3 surface. They are:

$$S^{[4]} \qquad S \times S^{[3]} \qquad S^{[2]} \times S^{[2]} \qquad S^2 \times S^{[2]} \qquad S^4$$

Now we take a linear combinations of these (ie. a disjoint sum) and choose the rational coefficients in such a way that the resulting (disconnected, reducible) hyperkähler manifold has the same Chern numbers as the generalized Kummer variety

$T^{[[4]]}$. We find that

$$X = 7S^{[4]} - \frac{49}{8}S \times S^{[3]} - 3S^{[2]} \times S^{[2]} + \frac{67}{12}S^2 \times S^{[2]} - \frac{21}{16}S^4$$

is the required manifold. However, calculating the value of $b_{\Theta_2^2}$ we find that

$$\begin{aligned} b_{\Theta_2^2}(X) &= 278784 \\ &\neq 288000 \\ &= b_{\Theta_2^2}(T^{[[4]]}) \end{aligned}$$

so this Rozansky-Witten invariant distinguishes the two manifolds and therefore cannot be a linear combination of Chern numbers.

Theorem 22 *The Rozansky-Witten invariant $b_{\Theta_2^2}$ is not a characteristic number on hyperkähler manifolds of real-dimension 16. However, it can be expressed as a rational function of characteristic numbers on irreducible hyperkähler manifolds.*

Note that the linear combination of manifolds in X involves both rational and negative coefficients. Instead we can rearrange things so that we have

$$336S^{[4]} + 268S^2 \times S^{[2]} \sim 48T^{[[4]]} + 294S^{[3]} + 144S^{[2]} \times S^{[2]} + 63S^4$$

where $\sim$ means the two manifolds are cobordant (in this case, we have two hyperkähler manifolds in the same integral cobordism class). The value of $b_{\Theta_2^2}$ on each manifold is 19795968 and 19353600 respectively, and hence the Rozansky-Witten invariants distinguish these two manifolds.

6 An invariant of knots

6.1 Vector bundles on hyperkähler manifolds

Up to now we have studied exclusively the weight system on graph homology which gives rise to invariants of three-manifolds. Invariants of knots and links also arise naturally in the Rozansky-Witten theory. In this final chapter we wish to construct explicitly a weight system on chord diagrams from a hyperkähler manifold with a holomorphic vector bundle over it. This weight system is a natural extension of the Rozansky-Witten weight system on graph homology, and leads to potentially new finite-type invariants of knots (and links). We begin in this subsection by reviewing the subject of holomorphic vector bundles.

Let E be a smooth complex vector bundle of complex-rank r on a compact hyperkähler manifold X of real-dimension $4k$. Once again, we wish to use the techniques of complex geometry so we will choose a specific complex structure I from the space of compatible complex structures on X, and regard X as a complex manifold with respect to this choice. A connection ∇ on E splits into the sum of two differential operators

$$\partial_A : \Omega^{0,0}(E) \to \Omega^{1,0}(E)$$

and

$$\bar{\partial}_A : \Omega^{0,0}(E) \to \Omega^{0,1}(E).$$

Suppose the connection looks like $d+A$ locally, where A is an $\mathrm{End}E$-valued one-form. We can write

$$A = A^{1,0} + A^{0,1}$$

with $A^{p,q} \in \Omega^{p,q}(\mathrm{End}E)$, and then

$$\partial_A = \partial + A^{1,0}$$

and

$$\bar{\partial}_A = \bar{\partial} + A^{0,1}$$

locally. The curvature $R = \nabla \circ \nabla$ of the connection is an $\mathrm{End}E$-valued two-form which can also be decomposed into

$$R = R^{2,0} + R^{1,1} + R^{0,2}$$

with $R^{p,q} \in \Omega^{p,q}(\mathrm{End}E)$. Indeed

$$\begin{aligned} R^{2,0} &= \partial_A \circ \partial_A \\ R^{1,1} &= \partial_A \circ \bar{\partial}_A + \bar{\partial}_A \circ \partial_A \\ R^{0,2} &= \bar{\partial}_A \circ \bar{\partial}_A. \end{aligned}$$

If E is a holomorphic vector bundle we can choose a connection ∇ which is compatible with, or preserves, the complex structure. This means that relative to a local holomorphic trivialization

$$\bar{\partial}_A = \bar{\partial}$$

or $A^{0,1}=0$. For such a choice, the $(0,2)$ part of the curvature vanishes, ie.

$$R^{0,2}=\bar{\partial}_A\circ\bar{\partial}_A=0.$$

Conversely, if a complex vector bundle admits such a connection then it is holomorphic. Note that $\bar{\partial}$ gives the holomorphic structure on E and the connection ∇ is the sum of the differential operator ∂_A of type $(1,0)$ and this holomorphic structure.

An *Hermitian structure* h in a smooth complex vector bundle E is a smooth field of Hermitian inner products in the fibres of E, and we call (E,h) an *Hermitian vector bundle*. A connection ∇ is a *h-connection* if it preserves the Hermitian structure h, or in other words

$$d(h(s,t))=h(\nabla s,t)+h(s,\nabla t)$$

where s and t are (local) sections of E. For a holomorphic vector bundle E with a Hermitian structure h there is a unique h-connection ∇ which is compatible with the complex structure, and we call this the *Hermitian connection*. For example, recall that the Levi-Civita connection is the unique connection on the (holomorphic) tangent bundle T which preserves the metric and is torsion-free. In the Kähler case, the Levi-Civita connection is precisely the Hermitian connection for the Kähler metric.

For the Hermitian connection, the curvature is purely of type $(1,1)$

$$R\in\Omega^{1,1}(\mathrm{End}E)$$

and locally it can be expressed as $\bar{\partial}A^{1,0}$. In local complex coordinates it has components

$$R^I{}_{Jk\bar{l}}dz_k\wedge d\bar{z}_l$$

where I and J refer to a local basis of sections of E, and summation over repeated indices is assumed.

Recall from Subsection 1.5 that the Atiyah class

$$\alpha_E\in\mathrm{H}^1(X,T^*\otimes\mathrm{End}E)$$

of a holomorphic vector bundle E is the obstruction to the existence of a global holomorphic connection on E. In the Dolbeault model for cohomology it can be represented by the $(1,1)$ part of the curvature of a smooth global connection ∇ of type $(1,0)$ on E (ie. a complex structure preserving connection). Recall that for such a connection the $(0,2)$ part of the curvature vanishes, and

$$R=R^{2,0}+R^{1,1}.$$

The Bianchi identity tells us that $\nabla R=0$. Now

$$\begin{aligned}\nabla R &= (\partial_A+\bar{\partial})(R^{2,0}+R^{1,1})\\ &= \partial_A R^{2,0}+(\partial_A R^{1,1}+\bar{\partial}R^{2,0})+\bar{\partial}R^{1,1}\end{aligned}$$

where the three terms in the final line are of type $(3,0)$, $(2,1)$, and $(1,2)$ respectively, and hence each must vanish. In particular, $R^{1,1}$ is $\bar{\partial}$-closed, as we require for it to represent a Dolbeault cohomology class

$$[R^{1,1}] \in \mathrm{H}^{1,1}_{\bar{\partial}}(X, \mathrm{End}E) = \mathrm{H}^{0,1}_{\bar{\partial}}(X, T^* \otimes \mathrm{End}E).$$

In the more specialized case that we have an Hermitian structure h on E we know that the curvature R of the (unique) Hermitian connection is purely of type $(1,1)$, and hence is itself a representative of the Atiyah class α_E.

At the beginning of Chapter 2 we discussed how to represent (in Dolbeault cohomology) the Chern character $ch(T)$ of the tangent bundle to X by taking the trace of powers of the Riemann curvature tensor. More generally, if E is a smooth complex bundle with a connection ∇, then its Chern character can be expressed as

$$\begin{aligned} ch(E) &= r + ch_1(E) + ch_2(E) + \ldots + ch_{2k}(E) \\ &= \sum_{m=0}^{2k} \frac{(-1)^m}{m!(2\pi i)^m}[\mathrm{Tr}(R^m)] \end{aligned}$$

where

$$R \in \Omega^2(\mathrm{End}E)$$

is the curvature of ∇, its powers are obtained by composing in $\mathrm{End}E$ and taking the wedge product of forms, and the trace is in $\mathrm{End}E$. Thus the cohomology classes which $\mathrm{Tr}(R^m)$ represent are independent of the connection chosen. If E is a holomorphic vector bundle with an Hermitian structure h, then

$$R \in \Omega^{1,1}(\mathrm{End}E)$$

for the Hermitian connection on E. Therefore the component $ch_m(E)$ of the Chern character is of pure Hodge type (m,m). Furthermore, since R represents the Atiyah class, we can replace $[R]$ by α_E in the above formula. More specifically, $\mathrm{Tr}(\alpha_E^m)$ is given by composing in $\mathrm{End}E$ and taking tensor products, and $ch_m(E)$ is given by projecting

$$\frac{(-1)^m}{m!(2\pi i)^m}\mathrm{Tr}(\alpha_E^m) \in \mathrm{H}^m(X, (T^*)^{\otimes m})$$

to

$$\mathrm{H}^m(X, \Lambda^m) = \mathrm{H}^{m,m}_{\bar{\partial}}(X).$$

Note that because we are in the Kähler case the topological Chern character coincides with the one constructed from the Atiyah class. Indeed we can use any description of the Atiyah class to construct the Chern character. For example, forget about the Hermitian structure h and simply choose a connection on E which is compatible with the complex structure. Then the $(1,1)$ part of the curvature will represent α_E and hence

$$ch_m(E) = \frac{(-1)^m}{m!(2\pi i)^m}[\mathrm{Tr}((R^{1,1})^m)] \in \mathrm{H}^{m,m}_{\bar{\partial}}(X).$$

Finally, let us note that if the bundle carries a symplectic structure then the odd components of the Chern character vanish (as for the tangent bundle), though this is not always the case.

6.2 Wilson lines in Chern-Simons theory

In this subsection we give a brief overview of the process of obtaining invariants of knots and links in Chern-Simons theory due to Witten [51]. These comments are intended to serve as motivation for our construction in the following subsection of a "Rozansky-Witten" weight system on chord diagrams, which will be an analogue of the weight system obtained in perturbative Chern-Simons theory.

Let $\mathcal{L}$ be a framed oriented link in a three-manifold M. Each component C_a of the link is an embedding of an oriented circle in M, called a *Wilson line*. Taking the holonomy of the Chern-Simons connection A_i around C_a gives us an observable of the theory

$$W_{V_a}(C_a) = \mathrm{Tr}_{V_a}\mathrm{Pexp}\int_{C_a} A_i dx_i$$

where the Feynman path integral takes values in the gauge group G, we choose a representation V_a of G for each link component, and we take the trace in the corresponding representation. Including these in the partition function of the theory gives us an invariant of the link $\mathcal{L}$ in M

$$Z(M;\mathcal{L}) = \int D\mathcal{A}\exp(iL)\prod\nolimits_a W_{V_a}(C_a)$$

where L is the Lagrangian, or Chern-Simons action, and the integral is over the moduli space of gauge equivalence classes of connections. This partition function of $\mathcal{L}$ in M is also known as the *expectation value* of $\mathcal{L}$, or as the *Wilson correlation function*.

In the case that M is the three-sphere S^3, $G = \mathrm{SU}(2)$, and all the V_a are the standard representation on $\mathbb{C}^2$, then we recover the Jones polynomial of the link $\mathcal{L}$ in S^3. Indeed, the motivation behind Witten's work was to find an intrinsically three-dimensional interpretation of the Jones polynomial, which would then allow generalizations to links in arbitrary three-manifolds. More generally, taking G to be $\mathrm{SU}(N)$ and all the V_a to be the standard representation on $\mathbb{C}^N$ allows us to obtain the HOMFLY polynomial of a link in S^3. On the other hand, if we choose a representation to be the trivial one, the corresponding Wilson line vanishes. Choosing all the representation to be trivial makes the link vanish and we are left simply with the Chern-Simons three-manifold invariant.

Taking a Feynman diagram expansion of the partition function of $\mathcal{L}$ in M gives us

$$Z(M;\mathcal{L}) = \sum_D c_D(\mathfrak{g};V_a)Z_D(M;\mathcal{L})$$

where the weights $c_D(\mathfrak{g}; V_a)$ depend only on the Lie algebra $\mathfrak{g}$ of the gauge group and the representations V_a, and $Z_D(M; \mathcal{L})$ depends on the link $\mathcal{L}$ embedded in M. The sum is over all Feynman diagrams with external vertices on Wilson lines. More precisely, D is an oriented unitrivalent graph with univalent vertices attached to m oriented circles S^1 (the skeleton), where m is the number of components of $\mathcal{L}$. The orientation is an equivalence class of cyclic orderings at the trivalent, or internal, vertices of D, with two such being equivalent if they differ by an even number of changes. We can regard the skeleton as part of the graph, so as to obtain a purely trivalent graph. Then because the skeleton is made up of *oriented* circles, there is automatically a cyclic ordering at the external vertices (those lying on the skeleton) given by taking the order: incoming part of skeleton, outgoing part of skeleton, remaining edge. Returning to the unitrivalent graph, each connected component should have at least one univalent vertex, and hence be attached to the skeleton. We denote the space of linear combinations of such diagrams modulo the IHX, AS, and STU relations (as in Chapter 4) by $\mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)$, where we take the disjoint union of m copies of S^1. We have already seen $\mathcal{A}(S^1)$, whose elements we called chord diagrams. We will continue to use this terminology when $m > 1$. Recall that $\mathcal{A}(S^1)'$ is the larger space obtained by allowing diagrams with connected components which are purely trivalent. Let $\mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)'$ be the corresponding space of diagrams with a skeleton consisting of m circles S^1, ie. allowing diagrams to have connected components consisting of purely trivalent graphs.

It is believed that the terms $Z_D(M; \mathcal{L})$ should be the same as the coefficients $Z_D^{\mathrm{Kont}}(M; \mathcal{L})$ of the framed Kontsevich integral

$$Z^{\mathrm{Kont}}(M; \mathcal{L}) = \sum_D Z_D^{\mathrm{Kont}}(M; \mathcal{L}) D \in \mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)$$

of the link $\mathcal{L}$ in M. In any case, we are more interested in the weight system $c_D(\mathfrak{g}; V_a)$.

As with the weight system $c_\Gamma(\mathfrak{g})$ on trivalent graphs described in Chapter 3, this weight system can be constructed for any Lie algebra $\mathfrak{g}$ with an invariant inner product and representations V_a. Recall that the structure constants of $\mathfrak{g}$ (with respect to some basis $\{x_1, \ldots, x_n\}$) give rise to a skew-symmetric tensor c_{ijk} and the inner product gives us a symmetric tensor σ^{ij}. In each representation V_a, the element x_i is mapped to some endomorphism of V_a. Thus we get tensors ${(B_a)_{iJ_a}}^{K_a}$ where J_a and K_a refer to some basis of V_a.

Suppose we have some chord diagram in $\mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)$. More generally, these weight systems are well defined on diagrams D in $\mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)'$. As with $c_\Gamma(\mathfrak{g})$, we place a copy of c_{ijk} at each trivalent vertex of D and attach the indices ijk to the outgoing edges in a way compatible with the cyclic ordering given by the orientation. If a univalent vertex is connected to the ath circle S^1 of the skeleton, then we place a copy of ${(B_a)_{iJ_a}}^{K_a}$ there. In this way the representations V_a are attached to the different Wilson lines. If some circle S^1 of the skeleton has no univalent vertices lying on it, then we simply introduce a factor of $\dim V_b$, where V_b is the representation associated to that circle. Returning to the univalent vertex, the index J_a is attached to the incoming part of the skeleton, the index K_a to the outgoing part, and the index i to the remaining edge. Next we use σ^{ij} to contract along all edges of D. On the skeleton we set adjacent indices to be equal, ie. if a part of the skeleton has been labelled with J_a at one end and with K_a at the other, with no univalent vertices lying between, then we set $J_a = K_a$. Equivalently, we contract along this part of the skeleton with a Kronecker delta $\delta_{J_a}^{K_a}$. The number that results from this process is the weight $c_D(\mathfrak{g}; V_a)$.

For example, let D be the chord diagram

where the fact that the skeleton has been broken is, of course, only a notational convenience. Then

$$c_D(\mathfrak{g}; R) = \sum_{i,j,K,L} B_{iK}{}^{L} B_{jL}{}^{K} \sigma^{ij}.$$

We chose bases for $\mathfrak{g}$ and its representations V_a in order to define these weights but they are in fact independent of these choices. Furthermore, they satisfy the IHX and AS relations for the same reasons that the weights $c_\Gamma(\mathfrak{g})$ do. They also satisfy the STU relations. This follows from the fact that V_a is a representation and so

$$\mathfrak{g} \to \mathrm{End}(V_a)$$

is a Lie algebra homomorphism; in particular, the bracket is preserved. Thus we have a well defined element of the dual space $(\mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)')^*$.

This is the weight system on chord diagrams which arises naturally in Chern-Simons theory. It has been extensively studied and generalized by knot theorists, to arbitrary quadratic Lie algebras and Lie super-algebras with representations. In the next subsection we will show how an analogous construction can be made with hyperkähler manifolds and vector bundles in Rozansky-Witten theory.

6.3 A new weight system on chord diagrams

Let X be a compact hyperkähler manifold of real-dimension $4k$ and choose a complex structure I from the family of complex structures on X compatible with the hyperkähler metric. Recall that the main ingredients in the construction of $b_\Gamma(X)$ were

$$\Phi \in \Omega^{0,1}(X, \mathrm{Sym}^3 T^*)$$

which was essentially the Riemannian curvature, and the dual of the holomorphic symplectic form

$$\tilde{\omega} \in \mathrm{H}^0(X, \Lambda^2 T).$$

In local complex coordinates these have components $\Phi_{ijk\bar{l}}$ and ω^{ij} respectively.

Let $E_1, \ldots, E_m$ be holomorphic vector bundles over X, of ranks $r_1, \ldots, r_m$ respectively. Suppose we have Hermitian structures h_a on each of the vector bundles E_a, and take the corresponding (unique) Hermitian connections on the bundles. Let the curvatures of these connections be

$$R_a \in \Omega^{1,1}(\mathrm{End}E_a).$$

In local complex coordinates they have components

$$(R_a)^{I_a}{}_{J_a k\bar{l}} dz_k \wedge d\bar{z}_l$$

where I_a and J_a refer to a local basis of sections of E_a.

Let $D \in \mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)'$ be a chord diagram with skeleton consisting of m copies of S^1. We wish to construct a weight system $b_D(X; E_a)$ which will be the natural analogue of $c_D(\mathfrak{g}; V_a)$. The weights $b_\Gamma(X)$ were analogous to $c_\Gamma(\mathfrak{g})$ in the sense that $\Phi_{ijk\bar{l}}$ replaced c_{ijk} and ω^{ij} replaced σ^{ij} in the construction. For $b_D(X; E_a)$ we also wish to replace $(B_a)_{iJ_a}{}^{K_a}$ in the construction of $c_D(\mathfrak{g}; V_a)$ by $(R_a)^{I_a}{}_{J_a k\bar{l}}$.

The construction should be obvious enough. We take a chord diagram D whose total number of vertices, trivalent and univalent, is $2k$. We place a copy of Φ at each trivalent vertex and attach the holomorphic indices i, j, and k to the outgoing edges in the usual way. We place copies of R_a at the univalent edges, where a is determined by which oriented circle S^1 of the skeleton the univalent edge is attached to. The index I_a is attached to the incoming part of the skeleton, J_a to the outgoing part, and k to the remaining outgoing edge. Then we contract along edges of D using the dual of the holomorphic symplectic form $\tilde{\omega}$, and along segments of the skeleton we contract using a Kronecker delta $\delta_{I_a}^{J_a}$ as in the Lie algebra case. This gives us a section of $(\bar{T}^*)^{\otimes 2k}$, which we project to the exterior product to get

$$D(\Phi; R_a) \in \Omega^{0,2k}(X).$$

Multiplying by the kth power of the holomorphic symplectic form gives us a $4k$-form which we can integrate to get a number. As with the Lie algebra case, we need some convention to deal with the situation where some circle S^1 of the skeleton has no univalent vertices lying on it. When this happens, we introduce a factor of $-\mathrm{rank}E_b$ where E_b is the holomorphic vector bundle corresponding to that circle. The minus sign is to ensure compatibility with a formula we shall state in Subsection 6.5.

Definition *We define the weight $b_D(X; E_a)$ corresponding to the chord diagram D to be*

$$\frac{1}{(8\pi^2)^k k!} \int_X D(\Phi; R_a)\omega^k.$$

In the case that $m = 0$, and we have no vector bundles, the chord diagrams are just trivalent graphs and the weights reduce to the Rozansky-Witten invariants of the manifold X. This explains our choice of factor in the above definition. We can also say something about the case of a trivial vector bundle E_b, for which a flat connection can be chosen, and hence the curvature R_b will vanish. This means that any chord diagram D with a univalent vertex on the bth circle S_b^1 of the skeleton will give rise to a vanishing weight $b_D(X; E_a)$. The remaining chord diagrams D will give weights which are $-\text{rank}E_b$ times

$$b_{D \backslash S_b^1}(X; E_a \backslash E_b)$$

ie. the weight corresponding to D with the bth circle S_b^1 of the skeleton removed, where we also remove E_b from the collection of vector bundles. In other words, except for the $-\text{rank}E_b$ factor, trivial vector bundles effectively vanish from the weight system. In some sense this is analogous to the vanishing of Wilson lines in Chern-Simons theory when the trivial representation has been associated to them.

So far we have neglected the role of the orientation in the above construction. Recall that an orientation of the chord diagram D is an equivalence class of cyclic orderings of the outgoing edges at each trivalent vertex, with two such being equivalent if the differ at an even number of vertices. We can regard the skeleton as being part of the graph, in which case what we have is a purely trivalent graph. In fact, it is oriented because we already have an equivalence class of cyclic orderings of the internal vertices and the vertices which lie on the skeleton (ie. the univalent vertices of D) have a canonical cyclic ordering given by: incoming part of skeleton, outgoing part of skeleton, remaining outgoing edge. We take the corresponding Rozansky-Witten orientation, ie. an equivalence class consisting of an ordering of the vertices and orientations of the edges. Now the m circles S^1 which make up the skeleton already have an orientation, so the last thing to do is to make sure that the orientations of our edges agree with this. Since we are dealing with an equivalence class, we can achieve this by making an even number of changes.

For example, if $D \in \mathcal{A}(S^1)'$ is the chord diagram

then the orientations of D in Figure 7 are equivalent. The first of these diagrams has the canonical orientation given by the anti-clockwise ordering of the outgoing edges at each vertex. In the second diagram we have converted this to the corresponding Rozansky-Witten orientation. In the last diagram we have reversed the orientations of the top and middle edges so that we have agreement with the original orientation of the skeleton.

Thus an orientation of the chord diagram D is the same as an equivalence class consisting of an ordering of the vertices (trivalent and univalent) and orientations of the edges (not including the skeleton, which has a canonical orientation); if the orderings differ by a permutation π and n edges are oriented in the reverse manner, then we regard these as equivalent if $\text{sgn}\pi = (-1)^n$. This is precisely what we need for the weight $b_D(X; E_a)$ to be well-defined. In particular, the ordering of the

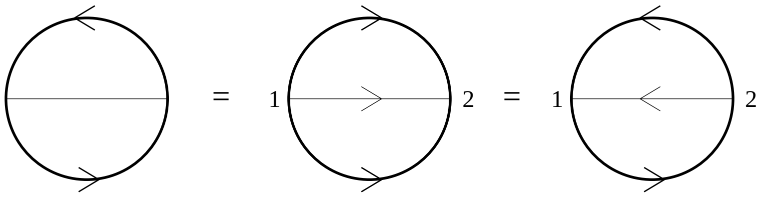


Figure 7: Three equivalent orientations

vertices tells us in which order to multiply the copies of Φ and R_a associated to the trivalent and univalent vertices respectively. The orientations of the edges tell us whether to contract with ω^{ij} or ω^{ji} along the edge.

Let us write out $b_D(X;E)$ explicitly when D is

and X is a K3 surface S with a vector bundle E over it. Using the third orientation in Figure 7 we see that

b  $(S;E) = \frac{1}{8\pi^2}\int_S R^I{}_{Jk_1\bar{l}_1} R^J{}_{Ik_2\bar{l}_2}\omega^{k_2k_1} d\bar{l}_1 \wedge d\bar{l}_2 \wedge \omega.$

We can imitate many of the ideas that we used for $b_\Gamma(X)$ in studying the weights $b_D(X;E_a)$. In particular, we will show the following:

- the construction of the weights can be formulated in cohomological terms, as in Kapranov's approach to the Rozansky-Witten invariants, and this enable us to prove they are independent of the choice of Hermitian structures h_a on the bundles E_a,
- the cohomological interpretation leads to a proof that the weights satisfy the STU relations, and hence only depend on the equivalence class of the chord diagram D in $\mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)'$,
- the Chern classes of the bundles E_a arise naturally for certain choices of chord diagrams,
- the Wheeling Theorem can be used to prove a result which generalizes Theorem 10 from Chapter 3.

6.4 The cohomological construction

Recall that an alternative definition of $b_\Gamma(X)$ due to Kapranov [33] replaced differential forms with the Dolbeault cohomology classes they represent. In particular, Φ

represent the Atiyah class of the tangent bundle

$$\alpha_T = [\Phi] \in \mathrm{H}^{0,1}_{\bar{\partial}}(X, \mathrm{Sym}^3 T^*).$$

We can do the same with $b_D(X; E_a)$, as the curvature R_a of the Hermitian connection on E_a is $\bar{\partial}$-closed and represents the Atiyah class of E_a

$$\alpha_{E_a} = [R_a] \in \mathrm{H}^{0,1}_{\bar{\partial}}(X, T^* \otimes \mathrm{End} E_a).$$

We can then construct

$$[D(\Phi; R_a)] \in \mathrm{H}^{0,2k}_{\bar{\partial}}(X)$$

as before, and it only depends on the cohomology classes of Φ and R_a. Multiplying by $[\omega^k]$ gives an element of $\mathrm{H}^{2k,2k}_{\bar{\partial}}(X)$ which we integrate to get

$$b_D(X; E_a) = \frac{1}{(8\pi^2)^k k!} \int_X [D(\Phi; R_a)][\omega^k].$$

More generally, we can take arbitrary representatives of the Atiyah classes α_T and α_{E_a}, and construct

$$D(\alpha_T; \alpha_{E_a}) \in \mathrm{H}^{2k}(X, \mathcal{O}_X).$$

This element can be paired with

$$\omega^k \in \mathrm{H}^0(X, \Lambda^{2k} T^*)$$

using Serre duality to produce the number $b_D(X; E_a)$, up to the factor $\frac{1}{(8\pi^2)^k k!}$.

Since the Atiyah class α_E of a holomorphic vector bundle E is well-defined without any reference to a Hermitian structure h on E, the next result follows immediately from the description above.

Proposition 23 *The weights $b_D(X; E_a)$ are independent of the choices of Hermitian structures h_a on the vector bundles E_a.*

It is clear that the weights $b_D(X; E_a)$ satisfy the AS relations, ie. they change sign under a change of orientation of D

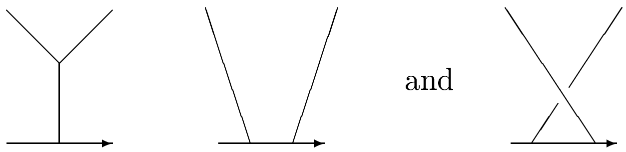
$$b_{\bar{D}}(X; E_a) = -b_D(X; E_a).$$

The argument is identical to the one for $b_\Gamma(X)$. They also satisfy the IHX relations: if three chord diagrams D_I, D_H, and D_X are identical except inside some small ball (away from the skeleton) where they look like I, H, and X respectively, then

$$b_{D_I}(X; E_a) = b_{D_H}(X; E_a) - b_{D_X}(X; E_a).$$

Again this follows from the same argument as for $b_\Gamma(X)$, using the cohomological approach described above. Indeed, the IHX relations only involve internal vertices, so the fact that the chord diagram also has a skeleton where we have attached the curvatures of vector bundles does not interfere with the argument. The STU relations, on the other hand, do depend on properties of these curvatures.

Suppose we are given three chord diagrams D_S, D_T, and D_X which are identical except inside some small ball near the skeleton where they look like



respectively. The orientations are induced from the planar embedding (for internal vertices) and the canonical cyclic orderings at external vertices induced from the orientation of the skeleton. Converting this to a Rozansky-Witten orientation, and remembering that we need to choose an equivalence class for which the orientation on the skeleton agrees with the given one, we find that we get the following: in D_S let the internal vertex be labelled t and the external vertex s, with the connecting edge oriented from t to s. Then in D_T the left vertex should be labelled s and the right one t, while in D_U the left vertex should be labelled t and the right one s. The orientations of the edges in D_T and D_U are induced from the orientation of the skeleton. We wish to show that

$$b_{D_S}(X;E_a) = b_{D_T}(X;E_a) - b_{D_U}(X;E_a).$$

Equivalently, we can take D_T with the reverse orientation (ie. switch the labels t and s so that they agree with those on D_U), in which case the equivalent relation is

$$b_{D_S}(X;E_a) + b_{\bar{D}_T}(X;E_a) + b_{D_U}(X;E_a) = 0.$$

As with the proof of the IHX relation, we will show that

$$D_S(\Phi;R_a) + \bar{D}_T(\Phi;R_a) + D_U(\Phi;R_a) \in \Omega^{0,2k}(X)$$

is $\bar{\partial}$-exact, and thus cohomologous to zero. Since the three chord diagrams differ only near one of the circles making up the skeleton, we will temporarily drop the subscript a for the holomorphic vector bundle E associated to that circle, and for its curvature R. Consider $d_A^2 R$, where d_A is the exterior derivative corresponding to the Hermitian connection on E (corresponding to some choice of Hermitian structure). By the Bianchi identity R is $\bar{\partial}$-closed, and hence

$$d_A^2 R = \bar{\partial}\partial_A R.$$

On the other hand

$$R \in \Omega^{1,1}(X, \mathrm{End}E) = \Omega^{0,1}(X, T^* \otimes \mathrm{End}E)$$

so d_A^2 gives the curvature of the associated connection on $T^* \otimes \mathrm{End}E$. This is the curvature of the Levi-Civita connection on T^* added to the curvature of the connection on $\mathrm{End}E$ induced from the Hermitian connection on E. Overall the curvature acts on $T^* \otimes \mathrm{End}E$ as $K \otimes 1 + 1 \otimes R$, where the second term act on $\mathrm{End}E$

via the Lie bracket. It is probably easiest to write this out in local coordinates, in which case we find

$$d_A^2 R \in \Omega^{1,2}(X, T^* \otimes \mathrm{End}E) = \Omega^{0,2}(X, T^* \otimes T^* \otimes \mathrm{End}E)$$

is given by

$$K^i{}_{jk\bar{l}} R^A{}_{Bi\bar{m}} + R^A{}_{Ck\bar{l}} R^C{}_{Bj\bar{m}} - R^A{}_{Cj\bar{m}} R^C{}_{Bk\bar{l}}.$$

Just to make things clear, A and B refer to $\mathrm{End}E$, j and k refer to $T^* \otimes T^*$, and $\bar{l}$ and $\bar{m}$ refer to $\Omega^{0,2}$ (we take a wedge product of forms in order to arrive in the exterior product). Since we are in this exterior product, we can interchange the anti-holomorphic indices in the last term, introducing a sign change

$$K^i{}_{jk\bar{l}} R^A{}_{Bi\bar{m}} + R^A{}_{Ck\bar{l}} R^C{}_{Bj\bar{m}} + R^A{}_{Cj\bar{l}} R^C{}_{Bk\bar{m}}.$$

Note that this is symmetric in jk, and hence we have an element which we write schematically as

$$KR + RR + RR \quad \in \quad \Omega^{0,2}(X, \mathrm{Sym}^2 T^* \otimes \mathrm{End}E). \tag{4}$$

It follows that

$$\partial_A R \in \Omega^{1,1}(X, T^* \otimes \mathrm{End}E) = \Omega^{0,1}(X, T^* \otimes T^* \otimes \mathrm{End}E)$$

must have the same symmetry, ie. must lie in

$$\Omega^{0,1}(X, \mathrm{Sym}^2 T^* \otimes \mathrm{End}E)$$

as

$$\bar{\partial}\partial_A R = KR + RR + RR.$$

Consider the S part of D_S, by which we mean the part of D_S which differs from D_T and D_U. Let the internal vertex be labelled with t and the external vertex with s, and assume $t < s$. We have seen that the edge connecting them must be oriented from t to s. When calculating $D_S(\Phi; R_a)$ this part contributes a copy of Φ and a copy of R "joined" by a copy of $\tilde{\omega}$, namely the section

$$S(\Phi; R_a) \in C^\infty(X, (T^*)^{\otimes 2} \otimes \mathrm{End}E \otimes (\bar{T}^*)^{\otimes 2})$$

with components

$$\begin{aligned} S(\Phi; R_a)_{j_t k_t}{}^{A_s}{}_{B_s \bar{l}_t \bar{l}_s} &= \Phi_{i_t j_t k_t \bar{l}_t} \omega^{i_t k_s} R^{A_s}{}_{B_s k_s \bar{l}_s} \\ &= \omega_{i_t m} K^m{}_{j_t k_t \bar{l}_t} \omega^{i_t k_s} R^{A_s}{}_{B_s k_s \bar{l}_s} \\ &= K^m{}_{j_t k_t \bar{l}_t} \delta^{k_s}_m R^{A_s}{}_{B_s k_s \bar{l}_s} \\ &= K^m{}_{j_t k_t \bar{l}_t} R^{A_s}{}_{B_s m \bar{l}_s}. \end{aligned}$$

The indices $j_t k_t$ refer to $(T^*)^{\otimes 2}$ and note that this term is symmetric in $j_t k_t$. Similarly, the $\bar{T}$ part of $\bar{D}_T$ contributes

$$\bar{T}(\Phi; R_a) \in C^\infty(X, (T^*)^{\otimes 2} \otimes \mathrm{End} E \otimes (\bar{T}^*)^{\otimes 2})$$

with components

$$\bar{T}(\Phi; R_a)_{j_t k_t}{}^{A_s}{}_{B_s \bar{l}_t \bar{l}_s} = R^{A_s}{}_{C j_t \bar{l}_t} R^C{}_{B_s k_t \bar{l}_s}$$

to $\bar{D}_T(\Phi; R_a)$, and the U part of D_U contributes

$$U(\Phi; R_a) \in C^\infty(X, (T^*)^{\otimes 2} \otimes \mathrm{End} E \otimes (\bar{T}^*)^{\otimes 2})$$

with components

$$U(\Phi; R_a)_{j_t k_t}{}^{A_s}{}_{B_s \bar{l}_t \bar{l}_s} = R^{A_s}{}_{C k_t \bar{l}_t} R^C{}_{B_s j_t \bar{l}_s}$$

to $U(\Phi; R_a)$. The sum of these three terms will be symmetric in $j_t k_t$, and will give us an element

$$S(\Phi; R_a) + \bar{T}(\Phi; R_a) + U(\Phi; R_a) \in C^\infty(X, \mathrm{Sym}^2 T^* \otimes \mathrm{End} E \otimes (\bar{T}^*)^{\otimes 2}).$$

Projecting to the exterior product $\Omega^{0,2}(X, \mathrm{Sym}^2 T^* \otimes \mathrm{End} E)$ gives us the term (4) which we described earlier in a schematic way as $KR + RR + RR$ (actually we need to interchange the second and third terms to get an exact correspondence).

It follows that

$$D_S(\Phi; R_a) + \bar{D}_T(\Phi; R_a) + D_U(\Phi; R_a) = D_*(\Phi; R_a, KR + RR + RR)$$

in $\Omega^{0,2k}(X)$ where D_* is the diagram which is identical to D_S away from the S part (and hence identical to $\bar{D}_T$ and D_U away from the $\bar{T}$ and U parts respectively), but contains one bivalent vertex lying on the skeleton instead of the S part. The construct of the right hand side is the same as for a chord diagram except that at the bivalent vertex we place $KR+RR+RR$. An orientation of D_* can be defined in such a way as to agree with those on D_S, $\bar{D}_T$, and D_U. However, such a choice only affects the overall sign so is not actually important here (since we are only interested in showing $\bar{\partial}$-exactness).

Replacing $KR + RR + RR$ by $\bar{\partial}\partial_A R$ and using the fact that Φ, R_a, and $\tilde{\omega}$ are all $\bar{\partial}$-closed, we get

$$D_*(\Phi; R_a, \bar{\partial}\partial_A R) = \bar{\partial} D_*(\Phi; R_a, \partial_A R)$$

where

$$D_*(\Phi; R_a, \partial_A R) \in \Omega^{0,2k-1}(X).$$

Thus we have shown that in Dolbeault cohomology

$$[D_S(\Phi; R_a)] + [\bar{D}_T(\Phi; R_a)] + [D_U(\Phi; R_a)] = 0 \in \mathrm{H}^{0,2k}_{\bar{\partial}}(X),$$

and therefore by the cohomological construction we see that

$$b_{D_S}(X;E_a) + b_{\bar{D}_T}(X;E_a) + b_{D_U}(X;E_a) = 0$$

as we wanted.

The fact that the weights $b_D(X;E_a)$ satisfy the STU relations immediately implies that they also satisfy the IHX and AS relations, although we have already seen that this is true. Combining these gives us the following result.

Proposition 24 *The dependence of the weight $b_D(X;E_a)$ on the chord diagram D is only through its equivalence class in $\mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)'$.*

6.5 Chern classes

To obtain Chern numbers from the Rozansky-Witten invariants $b_\Gamma(X)$ we needed to take a graph Γ somehow composed of wheels w_λ. More precisely, we chose Γ to be a polywheel, ie. the closure (sum over all the different ways of joining the spokes) of the disjoint union of a collection of wheels. Wheels arise naturally in chord diagrams as the skeleton is made up of a collection of oriented circles S^1, each of which can be regarded as a wheel. Thus we can expect that certain chord diagrams D will give us weights $b_D(X;E_a)$ which are related to the Chern classes of the holomorphic vector bundles E_a. Indeed this is the case, and the argument is virtually identical to that given for the Chern numbers of the tangent bundle in Chapter 2. Consequently we will state the results here without proof.

We will denote wheels which consist of a circle S^1 which is part of the skeleton of a chord diagram by a bold $\mathbf{w}_\lambda$. This looks like

where there are λ vertical spokes, and we have continued with our convention of breaking the circle S^1 at some point and drawing it as a directed line.

Suppose that we have a single holomorphic vector bundle E over a compact hyperkähler manifold X of real-dimension $4k$. We know that the closure $\langle w_{2k} \rangle$ of a $2k$-wheel will give rise to a Rozansky-Witten invariant which is (up to a factor) the Chern number given by the top component of the Chern character, ie.

$$b_{\langle w_{2k} \rangle}(X) = -s_{2k}(X).$$

The same argument shows that

$$b_{\langle \mathbf{w}_{2k} \rangle}(X;E) = -\int_X s_{2k}(E)$$

where

$$s_{2k}(E) = (2k)!ch_{2k}(E) \in \Omega^{2k,2k}(X)$$

is the usual rescaling of a component of the Chern character. More generally, we can show the following.

Proposition 25 *Suppose we have m holomorphic vector bundles E_a over a compact hyperkähler manifold X of real-dimension $2k$. Let D be the chord diagram given by the closure of a collection of wheels*

$$D = \langle w_{\lambda_1} \cdots w_{\lambda_j} \mathbf{w}_{\lambda_{j+1}} \cdots \mathbf{w}_{\lambda_{j+m}} \rangle \in \mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)'$$

where $\lambda_1 + \ldots + \lambda_{j+m} = 2k$ and there are m circles S^1 making up the skeleton. The weight corresponding to this chord diagram is

$$b_D(X; E_a) = (-1)^{j+m} \int_X s_{\lambda_1}(T) \wedge \cdots \wedge s_{\lambda_j}(T) \wedge s_{\lambda_{j+1}}(E_1) \wedge \cdots \wedge s_{\lambda_{j+m}}(E_m)$$

where T is the holomorphic tangent bundle of X.

Observe that both sides of this formula are zero unless $\lambda_1, \ldots, \lambda_j$ are all even. However, when some of $\lambda_{j+1}, \ldots, \lambda_{j+m}$ are odd we can still get a non-trivial result. We can also allow some of $\lambda_{j+1}, \ldots, \lambda_{j+m}$ to be zero, in which case $\mathbf{w}_0$ will give us a circle S^1 of the skeleton which has no univalent vertices lying on it. Recall that this introduces a factor of $-\mathrm{rank} E_b$ into $b_D(X; E_a)$, where E_b is the vector bundle associated to the circle. This agrees with the formula above, since $s_0(E_b) = \mathrm{rank} E_b$ and we have an additional minus sign in front of the integral coming from this bundle.

In Chapter 4 we used the Wheeling Theorem to show that Θ^k lies in the subspace generated by polywheels inside graph homology $\mathcal{A}(\emptyset)$, thereby proving Theorem 10. Recall that the Wheeling Theorem says that the map $\hat{\Omega}$ between $\mathcal{B}' \cong \mathcal{A}(S^1)'$ and itself is an algebra isomorphism, where the algebra structure comes from the two different products $\cup$ and $\times$. Our proof of Theorem 10 used the consequence

$$\hat{\Omega}(\ell^{\cup k}) = (\hat{\Omega}(\ell))^{\times k}$$

though actually we only used the terms on each side of this equation which have no univalent vertices, ie. consist of an element of graph homology $\mathcal{A}(\emptyset)$ with a disjoint skeleton. Now we shall consider the full equation, and the corresponding relations on weights $b_D(X; E_a)$. In fact, since the skeleton of an element of $\mathcal{A}(S^1)'$ consists of a single circle S^1, we need only one holomorphic vector bundle E over X.

We start with a low degree example, namely $k = 2$. Recall that the first few terms of Ω are

$$1 + \frac{1}{48} w_2 + \frac{1}{48^2 2!}(w_2^2 - \frac{4}{5} w_4) + \frac{1}{48^3 3!}(w_2^3 - \frac{12}{5} w_2 w_4 + \frac{64}{35} w_6) + \ldots$$

and therefore the left hand side looks like

$$\hat{\Omega}(\ell^{\cup 2}) = \hat{1}(\ell^{\cup 2}) + \frac{1}{48} \widehat{w_2}(\ell^{\cup 2}) + \frac{1}{48^2 2!}(\widehat{w_2^2} - \frac{4}{5} \widehat{w_4})(\ell^{\cup 2}).$$

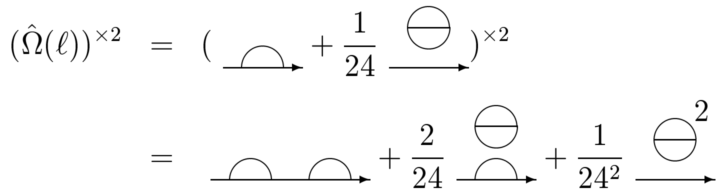
We wish to rewrite this element of $\mathcal{B}'$ as an element of $\mathcal{A}(S^1)'$. Recall that the required map is given by averaging over all the different ways of joining the univalent vertices to a skeleton S^1. Equivalently, on terms with λ univalent vertices, we can act with $\widehat{\mathbf{w}_\lambda}$, where this operator acts in an analogous way to $\widehat{w_\lambda}$. In fact, we also need to divide by $\lambda!$ in order to get an average and not a sum. Thus as an element of $\mathcal{A}(S^1)'$, we can write $\hat{\Omega}(\ell^{\cup 2})$ as

$$\frac{1}{4!}\widehat{\mathbf{w}_4}(\ell^{\cup 2}) + \frac{1}{2!48}\widehat{w_2\mathbf{w}_2}(\ell^{\cup 2}) + \frac{1}{48^2 2!}(\widehat{w_2^2\mathbf{w}_0} - \frac{4}{5}\widehat{w_4\mathbf{w}_0})(\ell^{\cup 2}).$$

Now acting on $\ell^{\cup 2}$ is the same as taking the closure of the above products of wheels, up to a factor of $8 = 2^2 2!$ which comes from the fact that each line ℓ can join two given spokes in two different ways, and the two lines can also be interchanged. Thus we get

$$8(\frac{1}{4!}\langle\mathbf{w}_4\rangle + \frac{1}{2!48}\langle w_2\mathbf{w}_2\rangle + \frac{1}{48^2 2!}(\langle w_2^2\mathbf{w}_0\rangle - \frac{4}{5}\langle w_4\mathbf{w}_0\rangle)).$$

In Chapter 4 we already saw that the right hand side looks like

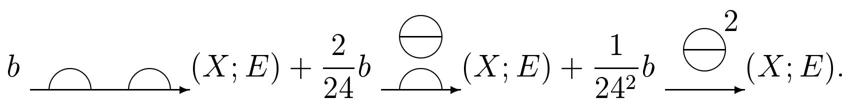


as an element of $\mathcal{A}(S^1)'$.

Now suppose that we have a holomorphic vector bundle E over a compact hyperkähler manifold X of real-dimension eight, and consider the weights $b_D(X;E)$ corresponding to the above chord diagrams $D \in \mathcal{A}(S^1)'$. Firstly, we have written the left hand side as a sum of closures of wheels, and we know that such diagrams give rise to integrals of Chern classes of E and the tangent bundle T. Indeed, the left hand side gives

$$\begin{aligned} -8\int_X \frac{s_4(E)}{4!} - \frac{s_2(T)}{48}\wedge\frac{s_2(E)}{2!} + \frac{(s_2^2(T)+\frac{4}{5}s_4(T))}{48^2 2!}\wedge s_0(E) \\ = -8\int_X \mathrm{Td}_0^{1/2}(T)\wedge ch_4(E) + \mathrm{Td}_2^{1/2}(T)\wedge ch_2(E) + \mathrm{Td}_4^{1/2}(T)\wedge ch_0(E) \\ = -8\int_X (\mathrm{Td}^{1/2}(T)\wedge ch(E))_4 \end{aligned}$$

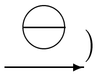
where the subscripts indicate which terms of $\mathrm{Td}^{1/2}(T)$ to take, and in the final line we integrate the fourth term of $\mathrm{Td}^{1/2}(T)\wedge ch(E)$. The right hand side then gives us a representation of this integral of characteristic numbers in terms of weights corresponding to simple chord diagrams, namely



We have just seen the $k = 2$ case. The general case of a real-dimension $4k$ hyperkähler manifold X is similar, and produces a formula expressing

$$-2^k k! \int_X (\mathrm{Td}^{1/2}(T) \wedge ch(E))_{2k}$$

as the weight $b_D(X; E)$ corresponding to the chord diagram

$D = ($  $+ \frac{1}{24}$  $)^{\times k}.$

The characteristic class

$$v(E) = \mathrm{Td}^{1/2}(T) \wedge ch(E) \in \bigoplus_{m=0}^{2k} \mathrm{H}^{2m}(X)$$

is known as the Mukai vector of E, and warrants a few comments.

The Mukai vector can be defined for any coherent sheaf $\mathcal{E}$ on a smooth manifold. It gives a map from K-theory to the cohomology ring which satisfies various functorial properties. In the more specific hyperkähler situation, it occurs in connection with moduli spaces of sheaves on a K3 surface S (see the book by Huybrechts and Lehn [31]). Let $\mathcal{M}(v)$ and $\mathcal{M}^s(v)$ be the moduli spaces of semi-stable and stable sheaves $\mathcal{E}$ over S, respectively, with Mukai vectors $v(\mathcal{E}) = v$. Provided the sheaves have rank greater than one, the moduli space $\mathcal{M}^s(v)$ is smooth. Its complex-dimension can be defined in terms of the Mukai vector v, as $(v, v) + 2$, where the inner product on cohomology is defined by

$$(v, w) = \int_S -v_0 \wedge w_4 + v_2 \wedge w_2 - v_4 \wedge w_0.$$

In this context, the Mukai vector v has also been of interest to physicists (see Dijkgraaf [20]), where it occurs as the *charge* of a *D-brane* state, which can be described in terms of a coherent sheaf $\mathcal{E}$ with Mukai vector $v \in \mathrm{H}^*(S, \mathbb{Z})$.

Returning to our result above, what we have is a formula for the integral of the top component of the Mukai vector in terms of weights of reasonably simple chord diagrams. For example, up to a factor the last of these chord diagrams is just Θ^k with a disjoint skeleton S^1.

Remarks

1. We know how to interpret the corresponding weight in terms of classical invariants of X and E, namely in terms of the $\mathcal{L}^2$-norm of the curvature of the Levi-Civita connection on X, the volume of X, and the rank of E. Unfortunately, we do not know how to interpret the weights of the remaining chord diagrams making up D in such a simple way, ie. in terms of classical invariants of X and E.

2. Another goal would be to find a representation of the entire Mukai vector in terms of weights of simple chord diagrams.

3. More generally, it would be interesting to know how these weights depend on the holomorphic structures on the vector bundles E_a.

6.6 Topological quantum field theory

In this chapter we have described how to construct a weight system $b_D(X; E_a)$ on chord diagrams D in $\mathcal{A}(S^1 \sqcup \ldots \sqcup S^1)'$ from a compact hyperkähler manifold X with a collection of m holomorphic vector bundles E_a over it, one for each circle S^1 in the skeleton of D. Composing this weight system with the framed Kontsevich integral

$$Z^{\mathrm{Kont}}(M; \mathcal{L}) = \sum_D Z_D^{\mathrm{Kont}}(M; \mathcal{L}) D$$

gives us a finite-type invariant

$$\sum_D b_D(X; E_a) Z_D^{\mathrm{Kont}}(M; \mathcal{L})$$

of the m-component framed oriented link $\mathcal{L}$ in the three-manifold M, which is similar in form to the invariant of links which arises from adding Wilson lines in Chern-Simons theory. Indeed we can ask: is there some natural extension of the Rozansky-Witten theory which leads to an invariant of links which, furthermore, gives rise to the weights $b_D(X; E_a)$ when expanded "perturbatively"? For an embedding of a knot in a three-manifold M, Rozansky and Witten [44] constructed an observable for their theory which depended on a tensor or spinor bundle, but this construction does not appear to extend to arbitrary holomorphic vector bundles. Let us consider instead the three-dimensional topological field theory.

In general, such a theory associates to a Riemann surface Σ a Hilbert space $\mathcal{H}_\Sigma$ and to a three-manifold M with boundary $\partial M = \Sigma$ a vector

$$v_M \in \mathcal{H}_\Sigma.$$

Suppose we have a knot $\mathcal{K}$ embedded in a closed three-manifold M. Removing a toroidal neighbourhood $\bar{\mathcal{K}}$ of the knot gives us a three-manifold $M \backslash \bar{\mathcal{K}}$ with boundary a torus T, and therefore a vector

$$v_{\mathcal{K} \subset M} \in \mathcal{H}_T$$

in the corresponding Hilbert space. If we choose a basis for $\mathcal{H}_T$, the components of $v_{\mathcal{K} \subset M}$ will give us a scalar-valued invariant of the knot $\mathcal{K}$ in M. Equivalently, we can choose some vector $w \in \mathcal{H}_T$ and take the inner product

$$\langle w, v_{\mathcal{K} \subset M} \rangle$$

with $v_{\mathcal{K}\subset M}$, and this will also give us a scalar-valued invariant.

In [44] Rozansky and Witten proposed the following topological quantum field theory. The Hilbert space associated to a genus g Riemann surface is the direct sum of cohomology groups

$$\bigoplus_{p,q} \mathrm{H}^q(X, (\Lambda^p T^*)^{\otimes g})$$

of the compact hyperkähler manifold X. Thus for a torus (ie. genus $g = 1$) we get the cohomology ring

$$\begin{aligned} \mathcal{H}_T &= \bigoplus_{p,q} \mathrm{H}^q(X, \Lambda^p T^*) \\ &= \bigoplus_{p,q} \mathrm{H}^{p,q}_{\bar{\partial}}(X) \end{aligned}$$

of X. According to the above argument, a knot $\mathcal{K}$ embedded in M will give rise to a cohomology class

$$v_{\mathcal{K}\subset M} \in \bigoplus_{p,q} \mathrm{H}^{p,q}_{\bar{\partial}}(X).$$

Now we must find some way to introduce the holomorphic vector bundle E into this construction. Of course, E gives rise naturally to characteristic classes in $\mathcal{H}_T$. Indeed we can take the Chern class, Chern character, or perhaps something like the Mukai vector, and these will all give us some element $w \in \mathcal{H}_T$. Pairing this with $v_{\mathcal{K}\subset M}$ then gives us a scalar-valued invariant

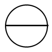
$$\int_X w \wedge v_{\mathcal{K}\subset M}$$

of the knot $\mathcal{K}$ in M. Now we may hope that a "perturbative" expansion of this knot invariant will give rise to the weights $b_D(X; E)$ where $D \in \mathcal{A}(S^1)$. However, the problem here is that our knot invariant depends on E only through its characteristic classes, and furthermore only in a linear way. In comparison, we would expect the weights $b_D(X; E)$ to depend on E in a more subtle way. Indeed, we saw in Chapter 5 that some of the Rozansky-Witten invariants $b_\Gamma(X)$ cannot be expressed as linear combinations of the characteristic numbers of X, so presumably a similar statement should hold for $b_D(X; E)$.

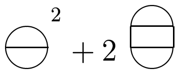
A Graph relations

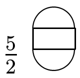
A.1 Expansions of polywheels

k = 1

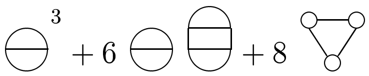
$\langle w_2 \rangle$ =


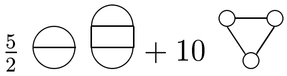
k = 2

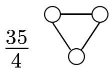
$\langle w_2^2 \rangle$ =


$\langle w_4 \rangle$ =


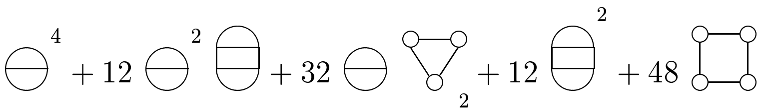
k = 3

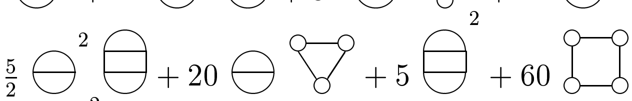
$\langle w_2^3 \rangle$ =


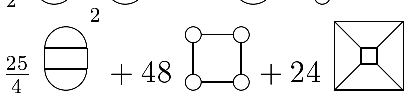
$\langle w_2 w_4 \rangle$ =


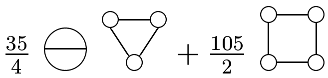
$\langle w_6 \rangle$ =


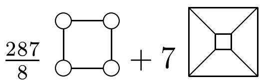
k = 4

$\langle w_2^4 \rangle$ =


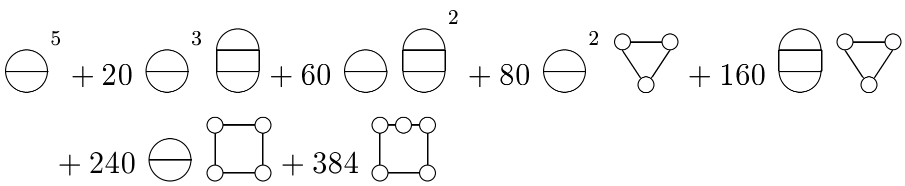
$\langle w_2^2 w_4 \rangle$ =


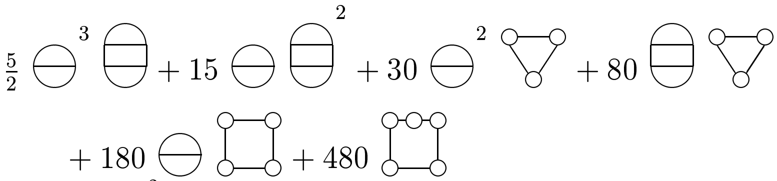
$\langle w_4^2 \rangle$ =


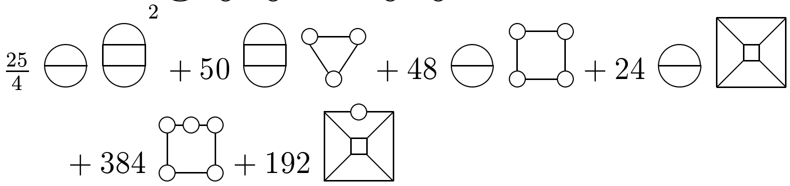
$\langle w_2 w_6 \rangle$ =


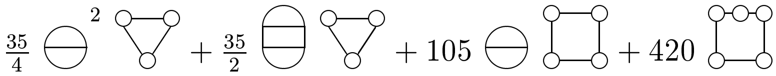
$\langle w_8 \rangle$ =


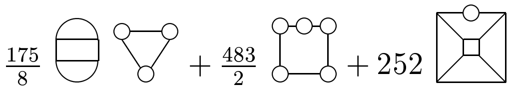
k = 5

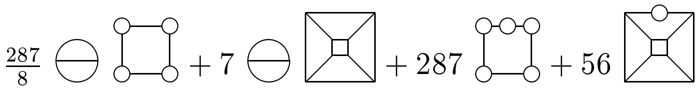
$\langle w_2^5 \rangle$ =


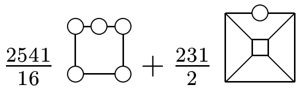
$\langle w_2^3 w_4 \rangle =$ 

$\langle w_2 w_4^2 \rangle =$ 

$\langle w_2^2 w_6 \rangle =$ 

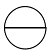
$\langle w_4 w_6 \rangle =$ 

$\langle w_2 w_8 \rangle =$ 

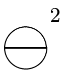
$\langle w_{10} \rangle =$ 

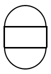
A.2 Writing graphs in terms of polywheels

k = 1

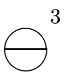
 $= \langle w_2 \rangle$

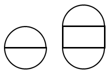
k = 2

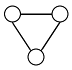
 $= \langle w_2^2 \rangle - \frac{4}{5}\langle w_4 \rangle$

 $= \frac{2}{5}\langle w_4 \rangle$

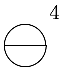
k = 3

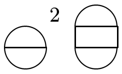
 $= \langle w_2^3 \rangle - \frac{12}{5}\langle w_2 w_4 \rangle + \frac{64}{35}\langle w_6 \rangle$

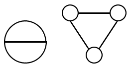
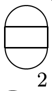
 $= \frac{2}{5}\langle w_2 w_4 \rangle - \frac{16}{35}\langle w_6 \rangle$

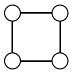
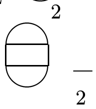
 $= \frac{4}{35}\langle w_6 \rangle$

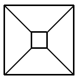
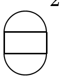
k = 4

 $= \langle w_2^4\rangle - \frac{24}{5}\langle w_2^2 w_4\rangle + \frac{48}{25}\langle w_4^2\rangle + \frac{256}{35}\langle w_2 w_6\rangle - \frac{1152}{175}\langle w_8\rangle$

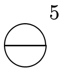
 $= \frac{2}{5}\langle w_2^2 w_4\rangle - \frac{8}{25}\langle w_4^2\rangle - \frac{32}{35}\langle w_2 w_6\rangle + \frac{192}{175}\langle w_8\rangle$

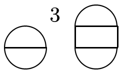
 $= -\frac{1}{2}$  $+ \frac{2}{25}\langle w_4^2\rangle + \frac{4}{35}\langle w_2 w_6\rangle - \frac{48}{175}\langle w_8\rangle$

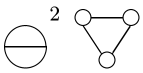
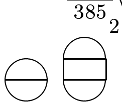
 $= \frac{1}{12}$  $- \frac{1}{75}\langle w_4^2\rangle + \frac{8}{175}\langle w_8\rangle$

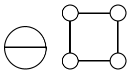
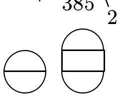
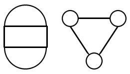
 $= -\frac{41}{96}$  $+ \frac{41}{600}\langle w_4^2\rangle - \frac{16}{175}\langle w_8\rangle$

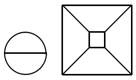
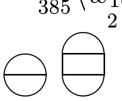
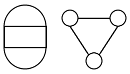
k = 5

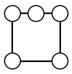
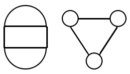
 $= \langle w_2^5\rangle - 8\langle w_2^3 w_4\rangle + \frac{48}{5}\langle w_2 w_4^2\rangle + \frac{128}{7}\langle w_2^2 w_6\rangle - \frac{512}{35}\langle w_4 w_6\rangle - \frac{1152}{35}\langle w_2 w_8\rangle + \frac{12288}{385}\langle w_{10}\rangle$

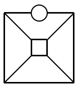
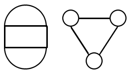
 $= \frac{2}{5}\langle w_2^3 w_4\rangle - \frac{24}{25}\langle w_2 w_4^2\rangle - \frac{48}{35}\langle w_2^2 w_6\rangle + \frac{64}{35}\langle w_4 w_6\rangle + \frac{576}{175}\langle w_2 w_8\rangle - \frac{1536}{385}\langle w_{10}\rangle$

 $= -$  $+ \frac{4}{25}\langle w_2 w_4^2\rangle + \frac{4}{35}\langle w_2^2 w_6\rangle - \frac{16}{35}\langle w_4 w_6\rangle - \frac{96}{175}\langle w_2 w_8\rangle + \frac{384}{385}\langle w_{10}\rangle$

 $= \frac{1}{12}$  $-$  $- \frac{1}{75}\langle w_2 w_4^2\rangle + \frac{8}{105}\langle w_4 w_6\rangle + \frac{8}{175}\langle w_2 w_8\rangle - \frac{64}{385}\langle w_{10}\rangle$

 $= -\frac{41}{96}$  $- \frac{9}{8}$  $+ \frac{41}{600}\langle w_2 w_4^2\rangle - \frac{11}{105}\langle w_4 w_6\rangle - \frac{16}{175}\langle w_2 w_8\rangle + \frac{184}{1155}\langle w_{10}\rangle$

 $= \frac{5}{24}$  $- \frac{1}{105}\langle w_4 w_6\rangle + \frac{8}{385}\langle w_{10}\rangle$

 $= -\frac{55}{192}$  $+ \frac{11}{840}\langle w_4 w_6\rangle - \frac{23}{1155}\langle w_{10}\rangle$

B Riemann-Roch formula

B.1 χ_y-genus in terms of Chern numbers

k	χ^m	Chern number expression
1	χ^0	$c_2/12$
	χ^1	$-10c_2/12$
2	χ^0	$(3c_2^2 - c_4)/720$
	χ^1	$(12c_2^2 - 124c_4)/720$
	χ^2	$(18c_2^2 + 474c_4)/720$
3	χ^0	$(10c_2^3 - 9c_2c_4 + 2c_6)/60480$
	χ^1	$(60c_2^3 - 306c_2c_4 - 492c_6)/60480$
	χ^2	$(150c_2^3 - 1143c_2c_4 + 13134c_6)/60480$
	χ^3	$(200c_2^3 - 1692c_2c_4 - 33224c_6)/60480$
4	χ^0	$(21c_2^4 - 34c_2^2c_4 + 5c_4^2 + 13c_2c_6 - 3c_8)/3628800$
	χ^1	$(168c_2^4 - 872c_2^2c_4 + 1240c_4^2 - 1816c_2c_6 - 744c_8)/3628800$
	χ^2	$(588c_2^4 - 4552c_2^2c_4 + 7340c_4^2 + 3964c_2c_6 + 86316c_8)/3628800$
	χ^3	$(1176c_2^4 - 10904c_2^2c_4 + 18280c_4^2 + 32408c_2c_6 - 857688c_8)/3628800$
	χ^4	$(1470c_2^4 - 14380c_2^2c_4 + 24350c_4^2 + 53230c_2c_6 + 1739310c_8)/3628800$

B.2 Chern numbers in terms of χ_y-genus

k	Chern no.	χ_y-genus
1	c_2	$12\chi^0$
2	c_2^2	$248\chi^0 - 2\chi^1$
	c_4	$24\chi^0 - 6\chi^1$
3	c_2^3	$7272\chi^0 - 184\chi^1 - 8\chi^2$
	c_2c_4	$1368\chi^0 - 208\chi^1 - 8\chi^2$
	c_6	$36\chi^0 - 16\chi^1 + 4\chi^2$
4	c_2^4	$3s$
	$c_2^2c_4$	$2s - 116032\chi^0 - 372\chi^1 + 112\chi^2 + 12\chi^3$
	c_4^2	$s - 74960\chi^0 + 777\chi^1 + 332\chi^2 + 33\chi^3$
	c_2c_6	$4512\chi^0 - 1278\chi^1 + 168\chi^2 + 18\chi^3$
	c_8	$48\chi^0 - 27\chi^1 + 12\chi^2 - 3\chi^3$

k	Chern no.	χ_y-genus
1	s_2	$-24\chi^0$
2	s_2^2	$992\chi^0 - 8\chi^1$
	s_4	$400\chi^0 + 20\chi^1$
3	s_2^3	$-58176\chi^0 + 1472\chi^1 + 64\chi^2$
	$s_2 s_4$	$-18144\chi^0 - 928\chi^1 - 32\chi^2$
	s_6	$-6552\chi^0 - 784\chi^1 - 56\chi^2$
4	s_2^4	$48s$
	$s_2^2 s_4$	$-8s + 1856512\chi^0 + 5952\chi^1 - 1792\chi^2 - 192\chi^3$
	s_4^2	$-4s + 657152\chi^0 + 18384\chi^1 + 3520\chi^2 + 336\chi^3$
	$s_2 s_6$	$-12s + 1446528\chi^0 - 10872\chi^1 + 672\chi^2 + 72\chi^3$
	s_8	$-6s + 664128\chi^0 - 3924\chi^1 + 1680\chi^2 + 204\chi^3$

C Hirzebruch χ_y-genus of $S^{[k]}$ and $T^{[[k]]}$

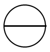
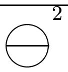
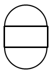
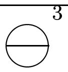
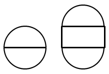
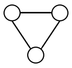
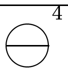
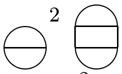
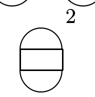
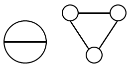
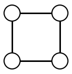
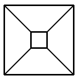
k	Space	χ_y-genus
1	S	$2 - 20y + 2y^2$
2	$S^{[2]}$	$3 - 42y + 234y^2 - 42y^3 + 3y^4$
3	$S^{[3]}$	$4 - 64y + 508y^2 - 2048y^3 + 508y^4 - 64y^5 + 4y^6$
4	$S^{[4]}$	$5 - 86y + 785y^2 - 4556y^3 + 14786y^4 - 4556y^5 + 785y^6 - 86y^7 + 5y^8$
2	$T^{[[2]]}$	$3 - 6y + 90y^2 - 6y^3 + 3y^4$
3	$T^{[[3]]}$	$4 - 8y + 44y^2 - 336y^3 + 44y^4 - 8y^5 + 4y^6$
4	$T^{[[4]]}$	$5 - 10y + 15y^2 - 20y^3 + 650y^4 - 20y^5 + 15y^6 - 10y^7 + 5y^8$

D Chern numbers in low degree

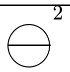
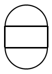
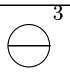
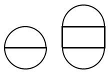
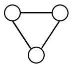
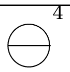
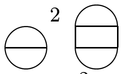
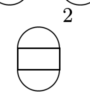
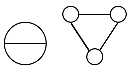
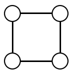
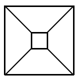
k	Chern number	$S^{[k]}$	$T^{[[k]]}$
1	s_2	-48	-48
2	s_2^2 s_4	3312 360	3024 1080
3	s_2^3 s_2s_4 s_6	-294400 -29440 -4480	-241664 -66560 -22400
4	s_2^4 $s_2^2s_4$ s_4^2 s_2s_6 s_8	$48s$ $-8s+8238720$ $-4s+2937120$ $-12s+8367120$ $-6s+4047480$	$48s$ $-8s+9200000$ $-4s+3148000$ $-12s+7350000$ $-6s+3381000$
4	s s_2^4 $s_2^2s_4$ s_4^2 s_2s_6 s_8	664080 31875840 2926080 280800 398160 63000	490000 23520000 5280000 1188000 1470000 441000

E Rozansky-Witten invariants in low degree

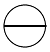
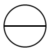
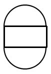
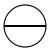
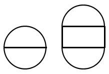
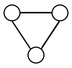
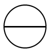
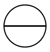
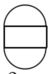
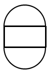
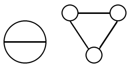
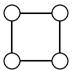
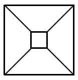
E.1 Irreducible manifolds

k	Γ	$b_\Gamma(S^{[k]})$	$b_\Gamma(T^{[[k]]})$
1		48	48
2		3600	3888
		-144	-432
3		373248	442368
		-13824	-36864
		512	2560
4		49787136	64800000
		-1693440	-4320000
		57600	288000
		56448	240000
		-1824	-12000
		348	-1500

E.2 Reducible manifolds

k	Γ				
2		$b_\Gamma(S^2)$			
		4608			
		0			
3		$b_\Gamma(S \times S^{[2]})$	$b_\Gamma(S \times T^{[[2]]})$	$b_\Gamma(S^3)$	
		518400	559872	663552	
		-6912	-20736	0	
		0	0	0	
4		$b_\Gamma(S \times S^{[3]})$	$b_\Gamma(S^{[2]} \times S^{[2]})$	$b_\Gamma(S^2 \times S^{[2]})$	$b_\Gamma(S^4)$
		71663616	77760000	99532800	127401984
		-1327104	-1036800	-663552	0
		0	41472	0	0
		24576	0	0	0
		0	0	0	0
		0	0	0	0

E.3 Virtual manifolds

k	Γ	$b_\Gamma(C_k)$
1		-6
2	^{2}	36
		12
3	^{3}	-216
		-72
		-24
4	^{4}	1296
	^{2} 	432
	^{2}	144
		144
		48
		24

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