The structure of invariants counting coherent sheaves on complex surfaces

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These slides available at
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1. Introduction

Let X be a complex projective surface, with geometric genus $p_g = \dim H^0(K_X)$. We usually restrict to $p_g > 0$, that is, $b^2_+(X) > 1$. Let $\kappa \in K^0_{\rm top}(X)$ be a topological K-theory class on X. We often write $\kappa = (r, \alpha, k)$ for $r = \operatorname{rank}\kappa$, $\alpha = c_1(\kappa) \in H^2(X, \mathbb{Z})$ and $k = \operatorname{ch}_2(\kappa) \in \frac{1}{2}\mathbb{Z}$ with $\int_X \alpha^2 + 2k \in 2\mathbb{Z}$, and usually restrict to $r > 0$. Choose a Kähler class ω on X. Then we can define *Gieseker (semi)stability* τ of coherent sheaves on X using ω, and can form moduli stacks $\mathcal{M}^{\rm st}_\kappa(\tau) \subseteq \mathcal{M}^{\rm ss}_\kappa(\tau)$ of τ-(semi)stable coherent sheaves on X with class κ. Here $\mathcal{M}^{\rm st}_\kappa(\tau)$ has a Behrend–Fantechi obstruction theory (which is *reduced* if $p_g > 0$) and $\mathcal{M}^{\rm ss}_\kappa(\tau)$ has a projective coarse moduli scheme. Thus, if $\mathcal{M}^{\rm st}_\kappa(\tau) = \mathcal{M}^{\rm ss}_\kappa(\tau)$ (if there are no strictly τ-semistable sheaves in class κ) then $\mathcal{M}^{\rm ss}_\kappa(\tau)$ is proper with a B–F obstruction theory, and so has a *virtual class* $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}$ in $H_*(\mathcal{M}^{\rm ss}_\kappa(\tau), \mathbb{Z})$. In nice cases (e.g. Hilbert schemes) $\mathcal{M}^{\rm ss}_\kappa(\tau)$ is smooth and $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt} = [\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm fund}$ is the fundamental class of $\mathcal{M}^{\rm ss}_\kappa(\tau)$ as a compact complex manifold.

We can construct many *universal cohomology classes* S_{jkl} on $\mathcal{M}^{\rm ss}_\kappa(\tau)$ — in the case when $\mathcal{M}^{\rm ss}_\kappa(\tau)$ is a fine moduli space, by $S_{jkl} = \mathrm{ch}_l(\mathcal{U})\backslash e_{jk}$ for $\mathcal{U} \to X \times \mathcal{M}^{\rm ss}_\kappa(\tau)$ the universal sheaf and e_{jk} a basis element for $H_k(X, \mathbb{Q})$. Then we can form *enumerative invariants* $I_P = \int_{[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}} P(S_{jkl})$ for any polynomial $P(S_{jkl})$ in these universal classes homogeneous of the correct dimension. There is a huge literature by many authors studying invariants of this kind for particular κ (e.g. rank $r = 2$) and $P(S_{jkl})$. They include *Donaldson invariants* of the underlying oriented 4-manifold X, *K-theoretic Donaldson invariants*, *Vafa–Witten invariants* (instanton branch), *Segre integrals*, *Verlinde integrals*, *virtual Euler characteristics* and χ_y*-genera* of $\mathcal{M}^{\rm ss}_\kappa(\tau)$, and so on. Often people show that these invariants I_P can be encoded in generating functions of a nice form. There are also many open conjectures like this by Göttsche, Kool and others. In fact, for rank $r > 1$ and $c_1(X) \neq 0$ there are lots of conjectures and few theorems.

I will report on a project which in some sense determines *all possible* invariants $I_P = \int_{[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}} P(S_{jkl})$, as it determines the virtual classes $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}$. We give an expression for $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}$ in terms of non-explicit universal functions in infinitely many variables $\dot{r}_0, r_1, r_2, \ldots, v_1, v_2, \ldots$, depending on the rank r of κ, with coefficients in a number field $\mathbb{F}_r \subset \mathbb{C}$. This proves at least the structural part of many conjectures in the literature (i.e. it gives the shape and symmetries of the invariants' generating function, but may not determine the particular power series appearing in it). This is an application of my Monster Wall Crossing Formula paper arXiv:2111.04694 (302 pages), which defined enumerative invariants in very general settings and proved they satisfy a WCF. My current document on invariants of surfaces is 355 pages, not yet finished, and horribly complex. Today I am going to try to explain just the statement of the main theorem in the case $p_g > 0$. I may not have time to talk about the proof.

2. Set up of the problem

For reasons explained in a moment, we work with moduli stacks of objects in the *derived category* $D^b \operatorname{coh}(X)$, rather than objects in $\operatorname{coh}(X)$. Write $\mathcal{M}$ for the moduli stack of objects in $D^b \operatorname{coh}(X)$, a higher $\mathbb{C}$-stack. It has a splitting $\mathcal{M} = \coprod_{\kappa \in K^0_{\rm top}(X)} \mathcal{M}_\kappa$ with $\mathcal{M}_\kappa$ the substack of $E^\bullet$ with class $[\![E^\bullet]\!] = \kappa$. There is a morphism $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ acting by $([E^\bullet], [F^\bullet]) \to [E^\bullet \oplus F^\bullet]$ on $\mathbb{C}$-points. Now $\mathbb{G}_m$ acts on objects $E^\bullet$ in $D^b \operatorname{coh}(X)$ with $\lambda \in \mathbb{G}_m$ acting as $\lambda \operatorname{id}_{E^\bullet} : E^\bullet \to E^\bullet$. This induces an action $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ of the group stack $[*/\mathbb{G}_m]$ on $\mathcal{M}$. We write $\mathcal{M}^{\rm pl} = \mathcal{M}/[*/\mathbb{G}_m]$ for the quotient, called the 'projective linear' moduli stack. It has a splitting $\mathcal{M}^{\rm pl} = \coprod_{\kappa \in K^0_{\rm top}(X)} \mathcal{M}^{\rm pl}_\kappa$ with $\mathcal{M}^{\rm pl}_\kappa = \mathcal{M}_\kappa/[*/\mathbb{G}_m]$. There is a morphism $\mathcal{M} \to \mathcal{M}^{\rm pl}$ which is a $[*/\mathbb{G}_m]$-fibration on $\mathcal{M} \setminus \{[0]\}$. We consider τ-(semi)stable moduli stacks $\mathcal{M}^{\rm st}_\kappa(\tau) \subseteq \mathcal{M}^{\rm ss}_\kappa(\tau)$ to be open substacks of $\mathcal{M}^{\rm pl}$. This is because τ-stable sheaves E have $\operatorname{Aut}(E) = \mathbb{G}_m$, so quotienting by $\mathbb{G}_m$ gives $\mathcal{M}^{\rm st}_\kappa(\tau)$ trivial isotropy groups, that is, $\mathcal{M}^{\rm st}_\kappa(\tau)$ is actually a $\mathbb{C}$-*scheme*, not an Artin stack.

Theorem 2.1 (Simons PhD student Jacob Gross arXiv:1907.03269)

Let X be a connected complex projective surface. Write $\mathcal{M}$ for the moduli stack of objects in $D^b\operatorname{coh}(X)$ and $K^0_{\rm sst}(X)$ for the **semi-topological K-theory** *of X (equal to $\mathrm{Image}(K^0(\mathrm{coh}(X))\to K^0_{\rm top}(X))$ for X a surface). Then $\mathcal{M}=\coprod_{\kappa\in K^0_{\rm sst}(X)}\mathcal{M}_\kappa$ with $\mathcal{M}_\kappa$ connected, and*

$$H_*(\mathcal{M}_\kappa,\mathbb{Q})\cong \mathrm{Sym}^*\bigl(H^{\rm even}(X,\mathbb{Q})\otimes_{\mathbb{Q}} t^2\mathbb{Q}[t^2]\bigr)\otimes_{\mathbb{Q}} \textstyle\bigwedge^*\bigl(H^{\rm odd}(X,\mathbb{Q})\otimes_{\mathbb{Q}} t\mathbb{Q}[t^2]\bigr). \tag{2.1}$$

A similar equation holds for cohomology $H^*(\mathcal{M}_\kappa,\mathbb{Q})$.

This says we can describe $H_*(\mathcal{M})$ completely explicitly. It is why we take $\mathcal{M}$ to be the moduli stack of objects in $D^b\operatorname{coh}(X)$: we do not have an explicit description of the homology of the moduli stack of objects in $\operatorname{coh}(X)$.

Definition

Let $X,\mathcal{M},\mathcal{M}_\kappa$ be as in Theorem 2.1, and write $\mathcal{U}_\kappa^\bullet\rightarrow X\times\mathcal{M}_\kappa$ for the universal complex. Write $b^k=b^k(X)$ for $k=0,\ldots,4$, and choose bases $(e_{jk})_{j=1}^{b^k}$ for $H_k(X,\mathbb{Q})$ with $e_{10}=1$ and $e_{14}=[X]$. Write $(\epsilon_{jk})_{j=1}^{b^k}$ for the dual basis for $H^k(X,\mathbb{Q})$. For $l>k/2$ define $S_{jkl}\in H^{2l-k}(\mathcal{M}_\kappa)$ by $S_{jkl}=\mathrm{ch}_l(\mathcal{U}_\kappa^\bullet)\backslash e_{jk}$. Regard S_{jkl} as of degree $2l-k$, and as an even (odd) variable if k is even (odd). Then Theorem 2.1 shows $H^*(\mathcal{M}_\kappa)$ is the graded polynomial superalgebra

$$H^*(\mathcal{M}_\kappa)\cong\mathbb{Q}[S_{jkl}:0\leqslant k\leqslant 2m,\ 1\leqslant j\leqslant b^k,\ l>k/2]. \qquad (2.2)$$

We also give a dual description of homology $H_*(\mathcal{M}_\kappa)$ by

$$H_*(\mathcal{M}_\kappa)\cong e^\kappa\otimes\mathbb{Q}[s_{jkl}:0\leqslant k\leqslant 2m,\ 1\leqslant j\leqslant b^k,\ l>k/2], \qquad (2.3)$$

where e^κ is a formal symbol to remember κ, and

$$\Bigl(\prod_{j,k,l}S_{jkl}^{m_{jkl}}\Bigr)\cdot\Bigl(e^\kappa\prod_{j,k,l}s_{jkl}^{m'_{jkl}}\Bigr)=\begin{cases}\pm\prod_{j,k,l}m_{jkl}!, & m_{jkl}=m'_{jkl}\text{ all } j,k,l,\\ 0, & \text{otherwise.}\end{cases}$$

This pairing has the property that if $\Phi : \mathcal{M}\times\mathcal{M}\rightarrow\mathcal{M}$ maps $([E^\bullet],[F^\bullet])\mapsto[E^\bullet\oplus F^\bullet]$ then

$$H_*(\Phi)\bigl(e^\kappa P(s_{jkl})\boxtimes e^\lambda Q(s_{jkl})\bigr)=e^{\kappa+\lambda}P(s_{jkl})Q(s_{jkl})$$

for polynomials P,Q. Also $-\cap S_{jkl}$ acts as $\frac{\partial}{\partial s_{jkl}}$.
It will be convenient to restrict to sheaves of *positive rank*. Write $\mathcal{M}_{\mathrm{rk}>0}=\coprod_{\kappa\in K^0_{\mathrm{sst}}(X):\mathrm{rk}\,\kappa>0}\mathcal{M}_\kappa$, and similarly for $\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0}$. Then $\Pi_{\mathrm{rk}>0}:\mathcal{M}_{\mathrm{rk}>0}\rightarrow\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0}$ induces a surjective morphism $H_*(\mathcal{M}_{\mathrm{rk}>0})\rightarrow H_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})$. It turns out this induces an isomorphism from $\mathrm{Ker}(-\cap S_{101})$ to $H_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})$, where $\mathrm{Ker}(-\cap S_{101})$ is functions independent of s_{101}. Thus we identify

$$H_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})\cong\bigoplus_{\kappa\in K^0_{\mathrm{sst}}(X):\mathrm{rk}\,\kappa>0}e^\kappa\otimes\mathbb{Q}\bigl[s_{jkl}:0\leqslant k\leqslant 2m,\ 1\leqslant j\leqslant b^k,\ l>k/2,\ (j,k,l)\neq(1,0,1)\bigr].\tag{2.4}$$

Thus, if κ satisfies $\operatorname{rank}\kappa > 0$ and $\mathcal{M}^{\rm st}_\kappa(\tau)=\mathcal{M}^{\rm ss}_\kappa(\tau)$ we have

$$[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}\in H_{2+2p_g-2\chi(\kappa,\kappa)}(\mathcal{M}^{\rm pl}_\kappa,\mathbb{Q})\cong e^\kappa\mathbb{Q}[s_{jkl},\ (j,k,l)\neq(1,0,1)],$$

where $\chi: K^0_{\rm top}(X)\times K^0_{\rm top}(X)\to\mathbb{Z}$ is the symmetrized Euler form. We write $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}=e^\kappa P_\kappa(s_{jkl})$, for $P_\kappa(s_{jkl})$ a $\mathbb{Q}$-polynomial in the infinitely many graded variables s_{jkl}, homogeneous of degree $2+2p_g-2\chi(\kappa,\kappa)$. Our mission, should we choose to accept it, is to compute the polynomials $P_\kappa(s_{jkl})$ (or better, generating functions encoding the $P_\kappa(s_{jkl})$) as explicitly as possible. Knowing $P_\kappa(s_{jkl})$ tells us $I_P=\int_{[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}}P(S_{jkl})$ for all $P(S_{jkl})$.
My Monster WCF paper defines invariants $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm inv}$ in *rational* homology $H_*(\mathcal{M}^{\rm pl}_\kappa,\mathbb{Q})$ for *all* classes κ, not just those with stable=semistable, with $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm inv}=[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}$ in $H_*(\mathcal{M}^{\rm pl}_\kappa,\mathbb{Z})$ when $\mathcal{M}^{\rm st}_\kappa(\tau)=\mathcal{M}^{\rm ss}_\kappa(\tau)$. These $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm inv}$ satisfy identities (Wall Crossing Formulae) which are powerful tools for computations. We aim to compute $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm inv}$ for all κ with $\operatorname{rank}\kappa>0$.

Example

Donaldson invariants are defined when $\mathrm{rk}\,\kappa = 2$ as integrals $\int_{[\mathcal{M}^{\mathrm{ss}}_\kappa(\tau)]_{\mathrm{inv}}} Q(S_{102}, S_{j22} : j = 1, \ldots, b^2)$ of polynomials Q in $S_{102} \in H^4(\mathcal{M}_\kappa)$ and $S_{j22} \in H^2(\mathcal{M}_\kappa)$. So they are determined by taking $P_\kappa(s_{jkl})$ and setting $s_{jkl} = 0$ if $(j, k, l) \neq (1, 0, 2)$ or $(j, 2, 2)$.

This illustrates the fact that Donaldson invariants, and other invariants in the literature, are just a small slice of the information in $[\mathcal{M}^{\mathrm{ss}}_\kappa(\tau)]_{\mathrm{inv}}$, which depends on infinitely many variables. To use my WCF, we usually have to compute with *the whole of* $[\mathcal{M}^{\mathrm{ss}}_\kappa(\tau)]_{\mathrm{inv}}$, not just small pieces like Donaldson invariants.
There is an important difference between $p_g = 0$ and $p_g > 0$. If $p_g = 0$ (i.e. $b^2_+ = 1$) then $[\mathcal{M}^{\mathrm{ss}}_\kappa(\tau)]_{\mathrm{inv}}$ depends on the Kähler form ω used to define τ, but if $p_g > 0$ (i.e. $b^2_+ > 1$) it is independent. For $p_g > 0$ we define $[\mathcal{M}^{\mathrm{ss}}_\kappa(\tau)]_{\mathrm{inv}}$ using *reduced* obstruction theories. The WCF for $p_g = 0$ and $p_g > 0$ are different (there are more terms when $p_g = 0$). Today I discuss only $p_g > 0$. The same techniques should work for $p_g = 0$, with a more complex answer.

3. The main results. 3.1. Normalizing $c_1(\kappa)$

Let $L \to X$ be a line bundle with $c_1(L) = \lambda \in H^2(X, \mathbb{Z})$. Then $- \otimes L : D^b \operatorname{coh}(X) \to D^b \operatorname{coh}(X)$ is an equivalence inducing an isomorphism $\mathcal{M}_\kappa \to \mathcal{M}_{\kappa \otimes [\![L]\!]}$. Under the isomorphism $H_*(\mathcal{M}_\kappa, \mathbb{Q}) \cong \mathbb{Q}[s_{jkl}]$, this is identified with an algebra isomorphism $\Omega_\lambda : \mathbb{Q}[s_{jkl}] \to \mathbb{Q}[s_{jkl}]$ acting on generators by

$$\Omega_\lambda : s_{jkl} \longmapsto \sum_{j',k',l' : 2l-k=2l'-k'} A_{jk}^{j'k'} s_{j'k'l'},$$

where $(A_{jk}^{j'k'})$ is the matrix of $- \otimes L$ on $K^0_{\mathrm{top}}(X)$, and is polynomial in λ. Thus Ω_λ makes sense for $\lambda \in H^2(X, \mathbb{Q})$, as well as $\lambda \in H^2(X, \mathbb{Z})$. We have $\Omega_\lambda([\mathcal{M}_\kappa^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}) = [\mathcal{M}_{\kappa \otimes [\![L]\!]}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}$. So for $\kappa = (r, \alpha, k)$ with $r > 0$, we find it helpful to consider $\Omega_{-\alpha/r}([\mathcal{M}_{(r,\alpha,k)}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}})$. Effectively, we are tensoring by a 'fractional line bundle' L with $c_1(L) = -\alpha/r$, to modify $\kappa = (r, \alpha, k)$ so that it has $c_1(\kappa) = 0$. The advantage is that formulae for $\Omega_{-\alpha/r}([\mathcal{M}_{(r,\alpha,k)}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}})$ are nearly independent of α (they depend on $\int_X \alpha \cup \beta \mod r$ for $\beta \in \mathrm{SW}(X)$).

3.2. The universal variables $\dot r_0, r_l, v_l$. The number field $\mathbb{F}_r$

We want to give an expression for $\Omega_\lambda([\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm inv})$ involving universal functions independent of X, and of the bases (e_{jk}) for $H_k(X,\mathbb{Q})$ and (ϵ_{jk}) for $H^k(X,\mathbb{Q})$ which determine the (co)homology variables s_{jkl}, S_{jkl}. To do this we will use 'universal variables' $\dot r_0, r_l, v_l$ where $\dot r_0, r_l \in H^*(X,\mathbb{Q})\otimes\mathbb{Q}[s_{jkl}]$ are given by

$$\dot r_0 = \sum_{j,k,j',k':k>0} \lambda_{jk}^{j'k'}\epsilon_{jk}\boxtimes s_{j'k'2}, \tag{3.1}$$

$$r_l = \frac{1}{l!}\sum_{j,k,j',k'} \lambda_{jk}^{j'k'}\epsilon_{jk}\boxtimes s_{j'k'(l+2)}, \quad l=1,2,\ldots, \tag{3.2}$$

with $(\lambda_{jk}^{j'k'})$ the inverse matrix of $(\alpha,\beta)\mapsto\int_X \alpha\cup\beta$ on $H^*(X)$, and $v_l = \frac{1}{l!}s_{14(l+2)}$. We write $\boldsymbol{r}=(r_1,r_2,\ldots)$ and $\boldsymbol{v}=(v_1,v_2,\ldots)$. For $r\geqslant 1$ (the rank of κ) define a number field $\mathbb{F}_r\subset\mathbb{C}$ by

$$\mathbb{F}_r = \begin{cases} \mathbb{Q}, & r=1 \text{ or } 2,\\ \mathbb{Q}[e^{\frac{\pi i}{2r}}], & r\geqslant 3 \text{ is odd},\\ \mathbb{Q}[e^{\frac{\pi i}{r}}], & r\geqslant 3 \text{ is even}.\end{cases}$$

3.3. The main theorem

Theorem 3.1

When $p_g > 0$, for $r \geqslant 1$ and $(r,\alpha,k) \in K^0_{\rm sst}(X)$ there is a formula

$$\Omega_{-\alpha/r}\bigl([\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm fd}\bigr) = \bigl[q^{\mathrm{vdim}\,\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)_{\rm fd}}\bigr] \tag{3.3}$$
$$\left(\sum_{\substack{\beta_1,\ldots,\beta_{r-1}\\ \in H^2(X,\mathbb{Z})^{1,1}:\\ \mathfrak{s}_{\beta_a}\in \mathrm{SW}(X),\\ a=1,\ldots,r-1}} \begin{array}{l} r^2\cdot \rho_r^{\int_X \mathrm{td}_2(X)}\cdot \eta_r^{\int_X c_1(X)^2}\cdot \prod_{1\leqslant a\leqslant b\leqslant r-1} \zeta_{r,ab}^{\int_X \beta_a\cup\beta_b}\cdot \\ \phi_r^{\int_X \alpha\cup c_1(X)}\cdot \prod_{a=1}^{r-1}\bigl(\mathrm{SW}([\mathfrak{s}_{\beta_a}])\theta_{r,a}^{\int_X\alpha\cup\beta_a}\bigr)\cdot \\ \exp\Bigl[\int_X A_r(\beta_1,\ldots,\beta_{r-1},c_1(X),\mathrm{td}_2(X),q,\dot{r}_0,\boldsymbol{r},\boldsymbol{v})\Bigr]\end{array}\right).$$

Here $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm fd}$ is the 'fixed determinant' invariant, equal to $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ when $b^1(X)=0$, and $\rho_r,\eta_r,\zeta_{r,ab},\phi_r,\theta_{r,a}\in\mathbb{F}_r\setminus\{0\}$, and A_r is a universal function independent of X, and $\mathrm{SW}(\mathfrak{s}_{\beta_a})\in\mathbb{Z}$ are Seiberg–Witten invariants of X. Furthermore:

Theorem 3.1 (Continued)

(i) $\rho_r = \pm\frac{1}{r}$.

(ii) $\theta_{r,a} \in \left\{e^{\frac{2\pi ib}{r}} : 1 \leqslant b < r\right\}$ *is a nontrivial* r^{th} *root of unity.*

(iii) $\phi_r \in \left\{e^{\frac{2\pi ib}{r}} : 1 \leqslant b \leqslant r\right\}$ *is an* r^{th} *root of unity.*

(iv) η_r *and* $\zeta_{r,ab}$ *for* $1 \leqslant a \leqslant b < r$ *lie in* $\mathbb{F}_r \setminus \{0\}$.

(v) A_r *lies in the quotient of* $\mathbb{F}_r[\beta_1,\ldots,\beta_{r-1},c_1(X),\mathrm{td}_2(X),\dot{r}_0,r_1,r_2,\ldots,v_1,v_2,\ldots][[q]]_{q^{>0}}$ *by an ideal generated by things like* $c_1(X)^3, c_1(X)\cup\mathrm{td}_2(X),\ldots$. *Here to regard* A_r *as independent of* X, *we just consider* $\beta_a, c_1(X),\ldots$ *to be formal variables. But when we fix a surface* X, *then we regard* $A_r(\beta_1,\ldots,\boldsymbol{v})$ *as lying in* $H^*(X,\mathbb{Q})\otimes\mathbb{Q}[s_{jkl}][[q]]$, *where* $\beta_a, c_1(X),\mathrm{td}_2(X)\in H^*(X,\mathbb{Q})$ *are the given values, and* $\dot{r}_0, r_l\in H^*(X,\mathbb{Q})\otimes\mathbb{Q}[s_{jkl}]$ *are as in* (3.1)–(3.2), *and* $v_l = \frac{1}{l!}s_{14(l+2)}$. *Then* $\int_X A_r(\cdots)$ *applies* $\int_X : H^*(X,\mathbb{Q})\to\mathbb{Q}$ *so that* $\int_X A_r(\cdots)\in\mathbb{Q}[s_{jkl}][[q]]_{q^{>0}}$.

Note that α appears in (3.3) only through $[q^{\mathrm{vdim}\,\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)_{\mathrm{fd}}}]$ and $\phi_r^{\int_X \alpha\cup c_1(X)}, \theta_{r,a}^{\int_X \alpha\cup\beta_a}$, and so via $\int_X \alpha\cup c_1(X), \int_X \alpha\cup\beta_a \mod r$.

3.4. Example: Hilbert schemes

For rank $r = 1$, fixed determinant moduli spaces $\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)_{\mathrm{fd}}$ are basically Hilbert schemes $\mathrm{Hilb}^n(X)$. Also there are no Seiberg–Witten terms in (3.3). In this case we can rewrite and strengthen Theorem 3.1 to give:

Theorem 3.2

Writing $\boldsymbol{u} = (u_1, u_2, \ldots)$, *there exist formal functions* $A(q, \boldsymbol{u})$, $B(q, \boldsymbol{u}), C(q, \boldsymbol{u}), D(q, \boldsymbol{u})$, *defined uniquely as the solutions to p.d.e.s, such that for any complex projective surface* X *we have*

$$\sum_{n\geqslant 0} q^n[\mathrm{Hilb}^n(X)]_{\mathrm{fund}} = \exp\Big[\int_X \Big(A(qe^{-\dot{r}_0}, \boldsymbol{r}) + c_1(X)\cup B(qe^{-\dot{r}_0}, \boldsymbol{r}) + c_1(X)^2\cup C(qe^{-\dot{r}_0}, \boldsymbol{r}) + \mathrm{td}_2(X)\cup D(qe^{-\dot{r}_0}, \boldsymbol{r})\Big)\Big]. \tag{3.4}$$

We can compute $A(q, \boldsymbol{u}), \ldots, D(q, \boldsymbol{u})$ *up to some order in* q *using Mathematica.*

3.5. Example: Donaldson invariants in arbitrary rank

Let $L \in H^2(X,\mathbb{Q})$, and write $L = \sum_{j=1}^{b^2} L_j \epsilon_{j2}$. The *rank r Donaldson invariants* of X are

$$D^X_{(r,\alpha,k)}(L + \mathrm{upt}) = \int_{[\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)]_{\mathrm{fd}}} \exp\Bigl(\sum_{j=1}^{b^2} L_j S_{j22} + S_{102}u\Bigr).$$

Theorem 3.3

$$D^X_{(r,\alpha,k)}(L + upt) = \bigl[q^{\mathrm{vdim}\,\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)_{\mathrm{fd}}}\bigr] \tag{3.5}$$
$$\left(\sum_{\substack{\beta_1,\ldots,\beta_{r-1}\\ \in H^2(X,\mathbb{Z})^{1,1}:\\ \mathfrak{s}_{\beta_a}\in \mathrm{SW}(X),\\ a=1,\ldots,r-1}} r^2 \rho_r^{\int_X \mathrm{td}_2(X)} \eta_r^{\int_X c_1(X)^2} \phi_r^{\int_X \alpha\cup c_1(X)} \prod_{1\leqslant a\leqslant b\leqslant r-1} \zeta_{r,ab}^{\int_X \beta_a\cup\beta_b} \prod_{a=1}^{r-1}\bigl(\mathrm{SW}([\mathfrak{s}_{\beta_a}])\theta_{r,a}^{\int_X \alpha\cup\beta_a}\bigr)\cdot \exp\Bigl[q^2\Bigl(\tfrac{1}{2}\int_X L^2 + ru\Bigr) + q\Bigl(\int_X L\cup\Bigl(C_r c_1(x) + \sum_{a=1}^{r-1} C_{r,a}\beta_a\Bigr)\Bigr)\Bigr]\right).$$

Here $C_r, C_{r,a} \in \mathbb{F}_r$. The $\exp[\cdots]$ term comes from the terms in $q^2\dot{r}_0^2, q^2\dot{r}_0, qc_1(x)\cup\dot{r}_0, q\beta_a\cup\dot{r}_0$ in A_r, just $r+2$ coefficients.

3.6. Symmetries of the generating function

Here is (3.3) again:

$$\Omega_{-\alpha/r}\bigl([\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm fd}\bigr)=\bigl[q^{\mathrm{vdim}\,\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)_{\rm fd}}\bigr]$$

$$\left(\sum_{\substack{\beta_1,\ldots,\beta_{r-1}\\ \in H^2(X,\mathbb{Z})^{1,1}:\\ \mathfrak{s}_{\beta_a}\in\mathrm{SW}(X),\\ a=1,\ldots,r-1}}\begin{array}{l} r^2\cdot\rho_r^{\int_X\mathrm{td}_2(X)}\cdot\eta_r^{\int_X c_1(X)^2}\cdot\prod\limits_{1\leqslant a\leqslant b\leqslant r-1}\zeta_{r,ab}^{\int_X\beta_a\cup\beta_b}\cdot\\ \phi_r^{\int_X\alpha\cup c_1(X)}\cdot\prod\limits_{a=1}^{r-1}\bigl(\mathrm{SW}([\mathfrak{s}_{\beta_a}])\theta_{r,a}^{\int_X\alpha\cup\beta_a}\bigr)\cdot\\ \exp\Bigl[\int_X A_r(\beta_1,\ldots,\beta_{r-1},c_1(X),\mathrm{td}_2(X),q,\dot{r}_0,\boldsymbol{r},\boldsymbol{v})\Bigr]\end{array}\right).$$

This has an obvious symmetry group S_{r-1} by permutation of $\beta_1,\ldots,\beta_{r-1}$. Less obvious, if β is a Seiberg–Witten class then so is $-c_1(X)-\beta$, with $\mathrm{SW}([\mathfrak{s}_{-c_1(X)-\beta}])=(-1)^{\int_X\mathrm{td}_2(X)}\,\mathrm{SW}([\mathfrak{s}_\beta])$. So replacing β_a by $-c_1(X)-\beta_a$, and ρ_r by $-\rho_r$, gives a $\mathbb{Z}_2$-symmetry for $a=1,\ldots,r-1$. This gives a symmetry group $\Gamma_r=S_{r-1}\ltimes\mathbb{Z}_2^{r-1}$ acting on choices of $\rho_r,\eta_r,\phi_r,\theta_{r,a},\zeta_{r,ab},A_r$.

Symmetries of the generating function

(a) It turns out that the data $\rho_r, \eta_r, \phi_r, \theta_{r,a}, \zeta_{r,ab}, A_r$ is unique up to this action of $\Gamma_r = S_{r-1} \ltimes \mathbb{Z}_2^{r-1}$. We can conjugate everything by an element of the Galois group $\mathrm{Gal}(\mathbb{F}_r)$; this is equivalent to the action of an element of Γ_r, giving a morphism $\mathrm{Gal}(\mathbb{F}_r) \to \Gamma_r$.
(b) We can use the Γ_r-action to standardize the constants $\rho_r, \eta_r, \phi_r, \theta_{r,a}, \zeta_{r,ab}$: after applying an element of Γ_r we can take

$$\rho_r = \tfrac{1}{r}, \quad \phi_r = 1, \quad \theta_{r,a} = e^{\frac{2\pi i a}{r}}, \quad a = 1, \ldots, r-1.$$

There are also conjectural values for $\eta_r, \zeta_{r,ab}$ due to Göttsche 2021, but I haven't proved these yet, except for small r.
(c) If r is odd then $\mathrm{vdim}\,\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)_{\mathrm{fd}}$ is always even. Then all q^{odd} terms in the whole sum (3.3) are zero, even though individual terms in the sum can have nonzero q^{odd} terms.
(d) $\mathrm{vdim}\,\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\mu^\omega)_{\mathrm{fd}} \equiv \int_X \alpha \cup c_1(X) + \int_X \mathrm{td}_2(X) \mod 2$ if r is even. If $n \not\equiv \int_X \alpha \cup c_1(X) + \int_X \mathrm{td}_2(X) \mod 2$ then q^n terms in the whole sum (3.3) are zero.
(e) Parts (c),(d) give an extra $\mathbb{Z}_2$ symmetry of (3.3) under $q \mapsto -q$.

3.7. Sketch of the proof: rank 1 case

First I prove the rank 1 case, Theorem 3.2 on Hilbert schemes. Define $\mathrm{Hilb}(X,q)=\sum_{n\geqslant 0}q^n[\mathrm{Hilb}^n(X)]_{\mathrm{fund}}\in\mathbb{Q}[s_{jkl}][[q]]$. Using Ellingsrud–Göttsche–Lehn 2001 I show that

$$\mathrm{Hilb}(X,q)=1+q(\cdots), \tag{3.6}$$

$$\begin{aligned}\frac{\partial}{\partial q}\mathrm{Hilb}(X,q)&=\\ \int_X\mathrm{Res}_z\Bigl\{z^{-1}\exp\Bigl[-&\sum_{\substack{j,k,j',k',\\ l'>k'/2:\ l'\geqslant(k+k')/2}}\frac{z^{(k+k')/2-l'}}{(l'-(k+k')/2)!}\mu_{jk}^{j'k'}\epsilon_{jk}\boxtimes s_{j'k'l'}\Bigr]\\ \circ\exp\Bigl[-z^2\,\epsilon_{14}\boxtimes q\frac{\partial}{\partial q}+&\sum_{j,k,\ l>k/2}(l-1)!z^l\epsilon_{jk}\boxtimes\frac{\partial}{\partial s_{jkl}}\Bigr]\cdot\mathrm{Hilb}(X,q)\Bigr\},\end{aligned} \tag{3.7}$$

where $(\mu_{jk}^{j'k'})$ is the inverse Mukai pairing. Then I show that (3.4) is the unique solution to (3.6)–(3.7), where $A(q,\boldsymbol{u}),\ldots,D(q,\boldsymbol{u})$ are solutions to p.d.e.s derived from (3.6)–(3.7).

3.8. Constructing invariants by induction on rank

There is a method to compute invariants $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ by induction on the rank $r=1,2,\ldots$ starting from rank 1 data. This is due to Mochizuki 2009 in the algebraic case, and is the analogue of the construction of Donaldson invariants from Seiberg–Witten invariants. Fix a line bundle $L\rightarrow X$, and define an auxiliary abelian category $\mathcal{A}$ with objects (V,E,ϕ), where V is a finite-dimensional $\mathbb{C}$-vector space, $E\in\mathrm{coh}(X)$, and $\phi:V\otimes_{\mathbb{C}}L\rightarrow E$ is a morphism. Write the class of (E,V,ϕ) as $[\![E,V,\phi]\!]=((r,\alpha,k),d)$ where $[\![E]\!]=(r,\alpha,k)$ and $\dim_{\mathbb{C}}V=d$. Starting from τ on $\mathrm{coh}(X)$ we define a 1-parameter family of stability conditions $\acute\tau_t$ on $\mathcal{A}$ for $t\in[0,\infty)$. Thus we get semistable moduli stacks $\mathcal{M}^{\rm ss}_{((r,\alpha,k),d)}(\acute\tau_t)$ of objects in $\mathcal{A}$. My theory defines 'pair invariants' $[\mathcal{M}^{\rm ss}_{((r,\alpha,k),d)}(\acute\tau_t)]_{\rm inv}$ (at least when $r>0$ and $d=0,1$) satisfying a wall-crossing formula under change of stability condition $\acute\tau_t$.

It turns out that:

- When $d=0$, $\mathcal{M}^{\rm ss}_{((r,\alpha,k),0)}(\acute\tau_t)=\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)$. Thus the sheaf invariants $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ are pair invariants with $d=0$.
- If $r=1$, $\mathcal{M}^{\rm ss}_{((1,\alpha,k),1)}(\acute\tau_t)$ is independent of t and may be written using Seiberg–Witten invariants and Hilbert schemes.
- If $r>1$, $d=1$ and $t\gg 0$ then $\mathcal{M}^{\rm ss}_{((r,\alpha,k),1)}(\acute\tau_t)=\emptyset$, so $[\mathcal{M}^{\rm ss}_{((r,\alpha,k),d)}(\acute\tau_t)]_{\rm inv}=0$. Thus wall-crossing from $t\gg 0$ to $t=0$ gives a WCF of the general form
- $[\mathcal{M}^{\rm ss}_{((r,\alpha,k),1)}(\acute\tau_0)]_{\rm inv}=$ sum of repeated Lie brackets of $[\mathcal{M}^{\rm ss}_{((1,\alpha',k'),1)}(\acute\tau_0)]_{\rm inv}$ and $[\mathcal{M}^{\rm ss}_{(r'',\alpha'',k'')}(\tau)]_{\rm inv}$ for $r''<r$,

  using a Lie bracket on $H_*(\mathcal{M}^{\rm pl}_{\mathcal{A}})$ from my vertex algebra theory.
- If $L=\mathcal{O}_X(-N)$ for $N\gg 0$ we can recover $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ from $[\mathcal{M}^{\rm ss}_{((r,\alpha,k),1)}(\acute\tau_0)]_{\rm inv}$.
- By induction we may now compute $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}\Rightarrow [\mathcal{M}^{\rm ss}_{((r+1,\alpha,k),1)}(\acute\tau_0)]_{\rm inv}\Rightarrow[\mathcal{M}^{\rm ss}_{(r+1,\alpha,k)}(\tau)]_{\rm inv}\Rightarrow\cdots$.
- Thus, we can compute $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ for $r>1$ in terms of classes of $\mathrm{Hilb}^n(X)$, $\mathrm{Pic}^0(X)$ and Seiberg–Witten invariants.

In the representation (2.4), with $(\mathrm{N}_{jk}^{j'k'})$ the matrix of the symmetrized Mukai pairing, we may write the Lie bracket on $H_*(\mathcal{M}_{\mathrm{rk}>0}^{\mathrm{pl}})$ as

$$
\begin{aligned}
&\bigl[e^\alpha u(s_{jkl}), e^\beta v(s'_{j'k'l'})\bigr]_{\mathrm{rk}>0} = \mathrm{Res}_z\Bigl[(-1)^{\chi(\alpha,\beta)} z^{\chi(\alpha,\beta)+\chi(\beta,\alpha)} e^{\alpha+\beta}\cdot \\
&\Bigl\{\exp\Bigl(z\frac{\mathrm{rk}\,\beta}{\mathrm{rk}(\alpha+\beta)}\Bigl(\sum_{j,k}\alpha_{jk}s_{jk(1+k/2)}+\sum_{j,k,l}s_{jk(l+1)}\frac{\partial}{\partial s_{jkl}}\Bigr)\Bigr)\circ \\
&\exp\Bigl(-z\frac{\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\Bigl(\sum_{j',k'}\beta_{j'k'}s'_{j'k'(1+k'/2)}+\sum_{j',k',l'}s'_{j'k'(l'+1)}\frac{\partial}{\partial s'_{j'k'l'}}\Bigr)\Bigr)\circ \\
&\exp\Bigl(-\sum_{j,k,j',k',\,l>k/2}(-1)^l(l-k/2-1)!\,z^{k/2-l}\mathrm{N}_{jk}^{j'k'}\beta_{j'k'}\frac{\partial}{\partial s_{jkl}} \\
&-\sum_{j,k,j',k',\,l'>k'/2}(-1)^{k/2}(l'-k'/2-1)!\,z^{k'/2-l'}\mathrm{N}_{jk}^{j'k'}\alpha_{jk}\frac{\partial}{\partial s'_{j'k'l'}} \\
&-\sum_{\substack{j,k,j',k',\\ l>k/2,\ l'>k'/2}}(-1)^l(l+l'-(k+k')/2-1)!\,z^{(k+k')/2-l-l'}\cdot\mathrm{N}_{jk}^{j'k'}\frac{\partial^2}{\partial s_{jkl}\partial s'_{j'k'l'}}\Bigr) \\
&\bigl(u(s_{jkl})\cdot v(s'_{j'k'l'})\bigr)\Bigr\}\Bigr|_{s'_{jkl}=s_{jkl}}\Bigr]. \qquad (3.8)
\end{aligned}
$$

3.9. Changing the generating function to the right form

Equation (3.8) is a horribly complicated mess. What this means in practice: if you suppose (3.3) holds in rank r, and you use this to compute the generating function of invariants in rank $r+1$ using the inductive method, computing the Lie brackets using (3.8), and you get to the end without dying, the result does not look like (3.3) in rank $r+1$. Instead, it gives you a really complicated residue in an extra formal variable z, which depends on the line bundle $L \to X$, even though the answer $[\mathcal{M}^{\rm ss}_{(r+1,\alpha,k)}(\tau)]_{\rm fd}$ is independent of L. Worse, you can't use one L for the whole generating function, L must be more and more negative as the power of q increases. The most difficult part of the proof is to show this residue can actually be written in the form (3.3) for rank $r+1$.

To do this we change variables in the residue from z to another formal variable y. Then it turns out that there exists a smooth projective curve Σ, meromorphic functions $x_1, \ldots, x_r, y : \Sigma \to \mathbb{C} \cup \{\infty\}$, and points $\sigma_0, \sigma_\infty \in \Sigma$ with $y(\sigma_i) = i$, such that:

- The group Γ_{r+1} acts on Σ, and y is Γ_{r+1}-invariant and gives an isomorphism $\Sigma/\Gamma_{r+1} \cong \mathbb{C} \cup \{\infty\}$. Thus, any Γ_{r+1}-invariant meromorphic function on Σ is actually a rational function of $y \in \mathbb{C} \cup \{\infty\}$.
- Every part of the residue $\mathrm{Res}_y(y^{-1}W)$ which will define the generating function (3.3) in rank $r+1$ lifts to the curve Σ, as the Laurent expansion at $\sigma_\infty \in \Sigma$ of a $\mathbb{Q}$-rational function in $x_1, \ldots, x_r, y$, in the local coordinate y.
- The entire sum $y^{-1}W$ inside $\mathrm{Res}_y(y^{-1}W)$ is Γ_{r+1}-invariant, although the components are not. Thus, the entire sum is a rational function of $y \in \mathbb{C} \cup \{\infty\}$. It turns out to have a simple pole at $y = 0$, and no other poles in $\mathbb{C}$. Thus $\mathrm{Res}_y(yW) = W|_{y=0}$, or equivalently, $W|_{\sigma_0}$.

- Thus, we are dealing with meromorphic functions on Σ, which are presented initially as formal Laurent series in y near $\sigma_\infty \in \Sigma$. We want instead to evaluate these meromorphic functions at $\sigma_0 \in \Sigma$, and this evaluation gives (3.3) and the data $\rho_{r+1}, \eta_{r+1}, \phi_{r+1}, \theta_{r+1,a}, \zeta_{r+1,ab}, A_{r+1}$.
- $y^{-1}(0)$ is a free Γ_{r+1}-orbit in Σ, and $\sigma_0 \in y^{-1}(0)$ is chosen arbitrarily. Different choices give different data $\rho_{r+1}, \ldots, A_{r+1}$, differing by the action of Γ_{r+1}.
- All terms in (3.3) come from $\mathbb{Q}$-rational functions in $x_1, \ldots, x_r, y$ in Σ. But when we evaluate these at $\sigma_0 \in \Sigma$, which is not a $\mathbb{Q}$-point for $r+1>2$, we get coefficients in $\mathbb{F}_{r+1}$.
- The curve Σ can be written completely explicitly, though in a complicated way. This enables me to compute $\mathbb{F}_{r+1}, \rho_{r+1}, \phi_{r+1}, \theta_{r+1,a}$ explicitly.

Happy birthday, Alastair!