Geometry Advanced Class MT25

Organizers: Dominic Joyce, Pierrick Bousseau, Hulya Arguz, Chenjing Bu

Enquiries: e-mail mailto:dominic.joyce@maths.ox.ac.uk.

Subject: Vafa-Witten invariants of projective surfaces.

Time and place: Wednesdays 9.30-11.0 room C3

Introduction

Vafa-Witten invariants are enumerative invariants of compact oriented 4-manifolds conjectured by the String Theorists Vafa and Witten in 1994. Notionally they should have a gauge theory definition - something like Euler characteristics of moduli space of instantons - but as the moduli spaces are potentially noncompact, this does not lead to a rigorous, deformation-invariant definition. For compact Kähler surfaces (better, projective surfaces), a rigorous definition is possible, due to Tanaka-Thomas and Thomas, roughly as invariants counting torsion-free coherent sheaves with Higgs fields.

Vafa and Witten proposed that generating functions of Vafa-Witten invariants should have modular properties: they should be modular forms (or possibly mock modular for surfaces with geometric genus $p_g = 0$, when there is a 'modular anomaly'). Modular forms are special functions coming from Number Theory. Their appearance in Vafa-Witten generating functions is surprising, and currently very mysterious geometrically.

For surfaces *X* with $p_g = 0$, the situation is different. Vafa-Witten invariants depend on the Kähler class of the surface, which defines a stability condition. String Theorists say the generating function of Vafa-Witten invariants should be modified to an 'almost holomorphic completion', which is non-holomorphic but exactly modular.

When Vafa-Witten invariants of projective surfaces have been calculated (often modulo some conjectures), the modular properties have been confirmed.

Topics we hope to cover:

- Geometric definition of Vafa-Witten invariants. Refinements of the definition.
- What String Theorists believe about Vafa-Witten invariants, and how best to explain this to mathematicians.
- Modular forms, and variants like quasimodularity and mock modularity. Overview of how modular forms (etc) appear in different corners of mathematics.
- What is known, and conjectured, about Vafa-Witten generating functions in the cases with $p_g > 0$ and $p_g = 0$.
- Feigin-Gukov's VOA[M4]: a programme that aims to associate a Vertex Operator Algebra to each 4-manifold *X* (plus extra data), with a representation whose characters are Vafa-Witten invariants.

Prerequisites: background in Algebraic Geometry up to the level of Hartshorne and beyond would definitely be helpful: we shall want to talk about things like torsion-free coherent sheaves (essentially, singular vector bundles), moduli schemes and moduli stacks of them, Behrend-Fantechi obstruction theories, and so on. I don't think we will have time to explain this background. If you are a String Theorist, just think 'branes' and ignore all Algebraic Geometry technicalities.

Practical details

Meetings will be weekly in MT25. Classes will generally last 90 minutes, with a speaker preparing a 70-80 minute talk in advance. Questions and audience participation are encouraged.

Pierrick not there 5th week, Chenjing not there 6th week. Chris Beem away weeks 2 and 4.

Provisional programme:

Week 1: Sergey Alexandrov: short discussion of Vafa-Witten invariants in String Theory.

Weeks 1-2: Dominic: What are Vafa-Witten invariants? Sketch of gauge theory 'definition', Tanaka-Thomas 1,2 and Thomas. Try to get to modularity statement, at least for SU(2).

Weeks 2-3: Pierrick: Overview of the Vafa-Witten paper, quantum field theory origin of the Vafa-Witten equations, physics motivation: S-duality of maximally supersymmetric Yang-Mills theory, Vafa-Witten equations from topological twist. If time: relation with exceptional holonomy.

Week 4: Andrew Graham (guest speaker from Number Theory): introduction to (quasi) modular forms, and their relation to special values of L-functions.

Week 5: Hulya: Explicit calculations and conjectures for generating series of Vafa-Witten invariants of surfaces with $p_g > 0$, in rank 2 and rank 3. Contributions of instanton and monopole branches to Vafa-Witten invariants, and discussion on modularity (in physics terms S-duality) exchanging these two contributions.

Rest of term:

Dominic: Invariants of projective surfaces (mostly with $p_g > 0$). Seiberg-Witten invariants, definitions for 4-manifolds and projective surfaces. The induction by rank construction which builds higher rank invariants out of Seiberg-Witten invariants and Hilbert scheme invariants. Structure theory for generating functions of invariants of surfaces with $p_g >$

Pierrick (?): What is known about invariants of surfaces with $p_g = 0$?

Chris Beem and Chenjing Bu: Geometric Representation Theory of 4-manifolds, based on Feigin-Gukov VOA[M4]. See also Butson, Butson-Rapcak, Dedushenko-Gukov-Putrov. Vafa-Witten invariants are supposed to be the characters of representations of VOAs associated to a 4-manifold. I think this might help to explain V-W modularity.

Papers we could cover during the class, and useful sources

S. Alexandrov, Vafa–Witten invariants from modular anomaly, Comm. Num. Theory Phys. 15 (2021), 149–219. https://arxiv.org/abs/2005.03680.

D. Butson, Vertex algebras from divisors on Calabi-Yau threefolds, https://arxiv.org/abs/2312.03648.

D. Butson and M. Rapcak, Perverse coherent extensions on Calabi-Yau threefolds and representations of cohomological Hall algebras, https://arxiv.org/abs/2309.16582

A. Dabholkar and P. Putrov, Three Avatars of Mock Modularity, https://arxiv.org/abs/2110.09790.

A. Dabholkar, P. Putrov, and E. Witten, Duality and Mock Modularity, https://arxiv.org/abs/2004.14387.

M. Dedushenko, S. Gukov, and P. Putrov, Vertex algebras and 4-manifold invariants, https://arxiv.org/abs/1705.01645

B. Feigin and S. Gukov, VOA[M4], https://arxiv.org/abs/1806.02470.

L. Göttsche and M. Kool, Refined SU(3) Vafa–Witten invariants and modularity, Pure Appl. Math. Q. 14 (2018), 467–513. https://arxiv.org/abs/1808.03245

L. Göttsche and M. Kool, Virtual refinements of the Vafa–Witten formula, Comm. Math. Phys. 376 (2020), 1–49. https://arxiv.org/abs/1703.07196

L. Göttsche, M. Kool, T. Laarakker, SU(r) Vafa-Witten invariants, Ramanujan's continued fractions, and cosmic strings, https://arxiv.org/abs/2108.13413

T. Laarakker, Monopole contributions to refined Vafa-Witten invariants, https://arxiv.org/abs/1810.00385.

Y. Tanaka and R. P. Thomas, Vafa-Witten invariants for projective surfaces I: stable case, J. Algebraic Geom. 29 (2020), 603–668. https://arxiv.org/abs/1702.08487.

Y. Tanaka and R. P. Thomas, Vafa-Witten invariants for projective surfaces II: semistable case, Pure Appl. Math. Q. 13 (2017), 517–562. https://arxiv.org/abs/1702.08488.

R. P. Thomas, Equivariant K-theory and refined Vafa–Witten invariants, Comm. Math. Phys. 378 (2020), 1451–1500. https://arxiv.org/abs/1810.00078

C. Vafa and E. Witten, A strong coupling test of S-duality, Nuclear Phys. B 431 (1994), 3–77. https://arxiv.org/abs/hep-th/9408074.