Talk for week 1

References on V–W equations:

Vafa–Witten, hep-th/9408074.

B. A. Mares, PhD thesis, 2010.

Tanaka arXiv: 1312.2673

Tanaka–Thomas arXiv: 1702.08487,
1702.08488.

The Vafa–Witten invariants started (Vafa–Witten 1994) as a gauge theory problem: we are asked to "count" solutions of the Vafa–Witten equations.

Background data: (X, g)

compact, oriented, Riemannian 4-manifold.

G compact Lie group (e.g. $SU(2)$, $U(N)$).

$P \to X$ principal G-bundle.

The Vafa-Witten equations concern triples (A, B, Γ), where:

* A is a connection on P
* $B \in \Gamma^\infty(\Lambda^2_+ T^*X \otimes \mathrm{ad}(P))$

self-dual 2-forms on X

$P \times_G \mathfrak{g}$, vector bundle with fibre Lie algebra $\mathfrak{g}$ of G.

* $\Gamma \in \Gamma^\infty(\mathrm{ad}(P))$

satisfying the equations

$$(1) \quad d_A \Gamma + d_A^* B = 0 \quad \text{in } \Gamma^\infty(T^*X \otimes \mathrm{ad}(P))$$

uses $[\,,\,] : \mathrm{ad}(P) \times \mathrm{ad}(P) \to \mathrm{ad}(P)$.

$$(2) \quad F_A^+ + [B \cdot B] + [B \cdot \Gamma] = 0 \quad \text{in } \Gamma^\infty(\Lambda^2_+ T^*X \otimes \mathrm{ad}(P))$$

combines

$[\,,\,] : \Lambda^2_+ T^*X \otimes \Lambda^2_+ T^*X \to \Lambda^2_+ T^*X$

and $[\,,\,] : \mathrm{ad}(P) \times \mathrm{ad}(P) \to \mathrm{ad}(P)$.

Then we form the V–W moduli space

$$\mathcal{M}_P^{VW} = \frac{\{(A, B, \Gamma) : (1), (2) \text{ hold}\}}{\mathrm{Aut}(P)}.$$

— This is a nonlinear elliptic system with virtual dimension zero.

— In <u>really nice</u> cases, we can hope $\mathcal{M}_P^{VW}$ is a compact, oriented 0-manifold, <u>ie</u> a finite set with signs, and we can define $VW(P) = \# \mathcal{M}_P^{VW}$ to be the signed count: the <u>Vafa-Witten invariant</u>.

— For this to be interesting, we would

hope that $VW(P)$ is independent of the Riemannian metric g on X

(At least if $b^2_+(X) > 1$.)

This looks quite like the set-up for other gauge theory 4-manifold invariants: Donaldson invariants and Seiberg-Witten invariants. Both of these can be made to work (for $b^2_+(X) > 1$), and give numbers that are independent of the metric g, and tell you something interesting about the smooth structure of X.

First approximation to Vafa-Witten's String Theory Conjecture

* One should be able to define such invariants $VW(P) \in \mathbb{Z}$ for all X, G, P (at least if $b_+^2(X) > 1$), and they should be independent of g.

* If $G = SU(2)$, then P is classified by $c_2(P) \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$

Then $VW(X, q) := \sum_{q \in \mathbb{Z}} q^n \, VW(P : c_2(P) = n)$

should (after normalization) be a <u>modular form</u>: an interesting class of functions from Number Theory.

— Roughly, $VW(X, q)$ has an unexpected, hidden symmetry under $\frac{\log q}{2\pi i} \mapsto \frac{1}{\frac{\log q}{2\pi i}}$.

They call this hidden symmetry "S-duality".
It is geometrically mysterious.

Problems with the Vafa-Witten programme.

There are 3 problems with dealing with moduli spaces $\mathcal{M}$ of this set:

(a) obstructions: $\mathcal{M}$ is not smooth

(b) reducibles: (A, B, Γ) may have nontrivial symmetry group in $\mathrm{Aut}(P)$

(c) noncompactness: $\mathcal{M}$ may not be compact.

There are standard techniques to deal with (a), and with (b) if $b_+^2 > 1$.

The real problem is (c), and it looks insoluble: the V-W moduli spaces really should be noncompact, and this should kill the independence of the metric g.

So, Vafa-Witten invariants were a stupid idea in the first place.

However, things get better if we either to X being a <u>Kähler manifold</u>.

Let X have complex structure J and Kähler form ω. Then

$$\Lambda^2_+ T^*X \cong \Lambda^{2,0} T^*X \oplus \langle \omega \rangle_{\mathbb{R}},$$

and can write $B = \varphi \oplus (b \otimes \omega)$

$$\varphi \in \Gamma^\infty(\Lambda^{2,0} T^*X \otimes_{\mathbb{R}} \mathrm{ad}(P)),$$

$$b \in \Gamma^\infty(\mathrm{ad}(P))$$

Often in the Kähler case, one can show that $b = \Gamma = 0$ (e.g. if $G = SU(2)$ and A is irreducible, so $\operatorname{Ker} d_A = 0$)

and the the Vafa–Witten invariants

reduce to equations:

(A, φ) for A a connection on P

and $\varphi \in \Gamma^\infty(\Lambda^{2,0} T^*X \otimes_{\mathbb{R}} \mathrm{ad}(P))$,

satisfying

$$\bar\partial_A \varphi = 0, \qquad F_A^{0,2} = 0,$$

$$F_A^{1,1} \wedge \omega + [\varphi, \bar\varphi] = 0.$$

For $U(n)$ case, allow $c\,\mathbb{1} \cdot \omega^2$ here,

$c \in \mathbb{R}$, $\mathbb{1} \in \mathfrak{u}(n)$ identity.

coming from centre of $\mathfrak{u}(n)$.

Something important has happened here:

$F_A^{0,1} = 0$ means (P, A) comes from a holomorphic principal $G^{\mathbb{C}}$-bundle, and $\bar\partial_A \varphi = 0$ means that φ is a holomorphic section of the holomorphic vector bundle $\Lambda^{2,0} T^* X \otimes_{\mathbb{R}} \mathrm{ad}(P)$.

Tanaka arXiv: 1312.2673:

there is a "Hitchin – Kobayashi correspondence":

given a holomorphic $G^{\mathbb{C}}$ bundle $P^{\mathbb{C}} \to X$ and a holomorphic section φ in

$H^0(K_X \otimes_{\mathbb{C}} \mathrm{ad}(P^{\mathbb{C}}))$,

$(P^{\mathbb{C}}, \varphi)$ come from a

flat connection on P satisfying

the V-W equations iff

$(P^{\mathbb{C}}, \varphi)$ satisfy an algebraic

(poly) stability condition.

Conclusion so far:

If we take (X, g) to be a Kähler manifold, the Vafa-Witten moduli spaces

$\mathcal{M}_P$ can be constructed using <u>holomorphic geometry</u>.

<u>Plan</u>: suppose X is a <u>complex projective surface</u>. Take $G = SU(r)$ or $U(r)$. Rewrite the problem as a problem in <u>complex algebraic geometry</u>.

<u>Reason</u>: Algebraic Geometers are much better at compactifying moduli spaces than Differential Geometers are.

Moduli spaces of holomorphic vector

bundles, can often be compactified as moduli spaces of torsion-free coherent sheaves. In this context, may get compact moduli spaces, and get invariants independent of the choice of metric g.

Tanaka-Thomas arXiv: 1702.08487
arXiv: 1702.08488

explained how to do this.

They replace (P, ϕ) by (E, φ), where E is a (semistable) rank r torsion-free sheaf, and $\varphi \in \mathrm{Hom}(E, E \otimes K_X)$.

— This is for the $U(r)$ case. For the $SU(r)$ case, fix $\det E \cong \mathcal{O}_X$ and take $\mathrm{Tr}\,\varphi = 0$ in $H^0(K_X)$.

Such pairs (E, ϕ) must satisfy a semistability condition:

$$\frac{\int_X c_1(F)\, v(\omega)}{\operatorname{rank} F} \underset{(-)}{<} \frac{\int_X c_1(E)\, v(\omega)}{\operatorname{rank} E}$$

for all proper ϕ-invariant subsheaves $0 \neq F \subsetneq E$.

— Can also interpret (E, ϕ) as a torsion sheaf on the noncompact Calabi–Yau 3-fold K_X.

Using Algebraic Geometry, Tanaka–Thomas define a moduli space $M^{ss}_{VW}(r, d, k)$

of (semi) stable VW pairs (E, φ)

with $\operatorname{rk} E = r$, $c_1(E) = d$, $c_2(E) = k$.

Unfortunately $\mathcal{M}^{ss}_{VW}(r, d, k)$ is <u>still</u>

<u>non compact</u>. Observe that if (E, φ)

lies in the moduli space then so does

$(E, \lambda\varphi)$ for $\lambda \in \mathbb{C}^*$,

and as φ may have non zero eigenvalues

in K_X, letting $\lambda \to \infty$ has

no limit in the moduli space in general.

However, the <u>$\mathbb{G}_m$-fixed locus</u>

$$\mathcal{M}^{ss}_{VW}(r, d, k)^{\mathbb{G}_m} \text{ is compact.}$$

$\mathbb{G}_m$-fixed points can happen if $\varphi = 0$,

or if $E = E_1 \oplus \cdots \oplus E_k$ and $\varphi = \begin{pmatrix} 0 & * & * & * \\ & 0 & & * \\ 0 & & \ddots & 0 \end{pmatrix}$

is upper triangular.

Algebraic Geometry now gives a prescription to define an invariant by virtual localization:

we define (Provisional definition, will improve later.)

$$VW(r,\alpha,k) = \int_{\left(\mathcal{M}''_{VW}(r,\alpha,k)^{\mathbb{C}^*}\right)_{virt}} \frac{1}{e(N_{(r,\alpha,k)})} \in \mathbb{Q}$$

"virtual class" of fixed point moduli space in homology

Normal bundle (complex) of $\mathcal{M}''_{VW}(r,\alpha,k)^{\mathbb{C}^*}$ in $\mathcal{M}''_{VW}(r,\alpha,k)$.

— If $\mathcal{M}^{ss}_{VW}(r,\alpha,k)$ were compact, this would give the virtual Euler characteristic of $\mathcal{M}^{ss}_{VW}(r,\alpha,k)$.

Sergei Alexandrov: physical origin of VW invariants

$\mathcal{N}=4$ SYM in 4d, G.
- should possess "S-duality".

To simplify, take a "topological twist".

- A way of testing S-duality, which is not usually testable: S-duality swaps weak + strong coupling, and can only do perturbation theory around weak coupling.

- After topological twist, theory becomes topological.
- Do it so theory preserves some supersymmetry. Take Q-cohomology.
- Fermionic operator Q, $Q^2 = 0$.

Lagrangian $\mathcal{L} = \{Q, \Phi\}$ — Φ depends on metric, ...

but in cohomology, g-dependence disappears.

- Any data living only in Φ drops out in the topological twist.

$$\int D\varphi \{Q, \partial_s \bar{\Phi}\}$$

$$\langle \varphi' | Q\,\partial\varphi - \partial\varphi\, Q | \psi \rangle$$

complexified ops $\quad \tau = \theta + \dfrac{i}{g^2_{sn}}$

All dependence on $\bar{\tau}$ sits in φ,

so partition function of theory should be holomorphic, indep of $\bar{\tau}$.

$$\mathcal{L} = \lambda \{ Q, \bar{\Phi}_s \}.$$

as λ in $\bar{\Phi}_s$, answer indep of λ

$$= \int D\varphi \, \{Q, \delta\varphi\}\, e^{-\lambda \{Q, \varphi_g\} -}$$

$\lambda \to \infty$, saddle point approx.

$\Rightarrow$ can loc theory localizes $\to$ solutions in critical locus of $\{Q, \varphi_g\}$.

— Particularly in Kähler case, VW eqns come from supersymmetric localization.

— This is the naive picture. But sometimes there are problems.

— If something is noncompact, ...

e.g. for holomorphic anomaly of topological string theory,

Naive argument that partition fn is holomorphic fails because of noncompactness of moduli spaces.

How they appear here:

get

$$\int_{\mathcal{M}} dN \, \partial_N (\ldots)$$

moduli space

tot. derivative.

Naively this is zero by Stokes theorem.

— But if $b^2_+ = 1$, get nonvanishing contribution from $\partial\mathcal{M}$.

— So, argument that theory is independent of $\bar{\tau}$ fails, get a "holomorphic anomaly".

in particular, $b^2_+(X) = 1$.

Case $b^2_+(X) = 1$, $X = \mathbb{P}^2$, $G = SU(2)$

Dabholkar, Putrov, Vafa paper.

Check: Compute V-W invariants in some cases, see if they actually give a modular form (using computations by mathematicians?).

Case of $\mathbb{P}^2$: got mock modularity;

because partition function should be the modular completion: $h(\tau) \to \hat{h}(\tau) = h + \underbrace{\text{completing term}}$

non holomorphic. From integration by parts.