Week 2. On definition of Vafa-Witten invariants

Recap. X complex projective surface.

Tanaka-Thomas define Vafa-Witten invariants as follows:

Fix a Chern character (r, α, k)

rank ≥ 1 first Chern class second Chern class

Define a moduli scheme $\mathcal{M}^{VW}_{r,\alpha,k}$ parametrizing pairs (E, ϕ), where E is a torsion-free coherent sheaf in class (r, α, k) and $\varphi \in \mathrm{Hom}(E, E \otimes K_X)$.

(This is for the $U(r)$ case. For $SU(r)$, fix $\det E \cong \mathcal{O}_X$ and trace $\varphi = 0$.)

Such pairs (E, φ) must satisfy a <u>semistability condition</u>:

Better: should use Gieseker stability. This formula gives the right answer for vector bundles (Tanaka, H-K correspondence.)

$$\frac{\int_X c_1(F)\, v(\omega)}{\operatorname{rk} F} \underset{(-)}{<} \frac{\int_X c_1(E)\, v(\omega)}{\operatorname{rk} E}$$

for all proper φ-invariant subsheaves $0 \neq F \subsetneq E$.

Then $M^{VW}_{r,\alpha,k}$ should carry a Behrend-Fantechi obstruction theory.

There is a $\mathbb{G}_m$-action on $M^{VW}_{r,\alpha,k}$ mapping $\lambda : (E, \varphi) \longmapsto (E, \lambda\varphi)$.

The $\mathbb{G}_m$-fixed locus $\left(M^{VW}_{r,\alpha,k}\right)^{\mathbb{G}_m}$ is <u>compact</u> (proper)

Algebraic Geometry now gives a prescription to define an invariant by "virtual localization":

we define (Provisional definition, will improve later)

$$VW(r,\alpha,k) = \int_{\underbrace{[M''_{VW}(r,\alpha,k)^{\mathbb{C}^*_m}]^{virt}}_{\text{"virtual class" of fixed point moduli space in homology}}} \frac{1}{e(N_{(r,\alpha,k)})} \in \mathbb{Q}$$

$N_{(r,\alpha,k)}$: Normal bundle (complex) of $M''_{VW}(r,\alpha,k)^{\mathbb{C}^*_m}$ in $M''_{VW}(r,\alpha,k)$.

Remarks * This uses the Algebraic Geometry technology of moduli schemes, Behrend-Fantechi obstruction theories and virtual classes. This is a means of dealing with moduli spaces not being smooth.

* This definition only works if there are no strictly semistables. If there are semistable points, the obstruction theory does not work. Tanaka-Thomas [T] give a partly-conjectural extension to including strictly semistables, the conjectures proved by Henry Liu in arXiv: 2309.03673.

– Actually, there are a bunch of different issues to discuss at this point.

An issue: defining invariants via virtual classes, compared to defining invariants via weighted Euler characteristics.

One easy thing to do would be to treat Vafa–Witten invariants as examples of Donaldson–Thomas invariants, counting dimension 2 torsion sheaves on the non-compact Calabi–Yau 3-fold X_X, and define the invariants as motivic invariants other than using homology / cohomology classes, so the invariants would become weighted Euler characteristics of the moduli spaces $\mathcal{M}^{ss}_{vw}(r,\alpha,k)$.

— There is already a very well-developed theory of Donaldson–Thomas invariants (Joyce–Song, Kontsevich–Soibelman, ...)

— The Donaldson–Thomas approach has two disadvantages:

(a) It does not give a deformation-invariant answer.

(b) When Tanaka–Thomas compute it in examples, the answers do not have the modularity properties conjectured in Physics (except when $K_X \leq 0$).

So Tanaka–Thomas conclude that the

Donaldson – Thomas' approach is the wrong one, and the virtual localization approach the right one.

$$\boxed{\text{An issue: reduced invariants.}}$$

– The provisional definition for $U(r)$

$$VW(r, \alpha, k) = \int_{[\mathcal{M}''_{VW}(r,\alpha,k)^{\mathbb{C}^*}]^{vir}} \frac{1}{e(N_{(r,\alpha,k)})}$$

above automatically gives zero if

$h^{1,0}(X) > 0$ or $h^{2,0}(X) > 0$,

which excludes most interesting surfaces.

Tanaka–Thomas' solution is to work with $SU(r)$ invariants, i.e. fix determinants $\det E = \mathcal{O}_X$, which do not automatically vanish.

There is an alternative approach using reduced invariants for $U(r)$ which give non-zero answers, at least if $h^{1,0}(X)=0$.

$$\boxed{\text{Instanton branch and monopole branch}}$$

We are considering

$\mathcal{M}^{ss}_{VW}(r,\alpha,k)^{\mathbb{C}^*}$, where the

$\mathbb{C}^*$ action is $\lambda : [E,\phi] \longmapsto [E,\lambda\phi]$.

— There is an open and closed subvariety of $\mathcal{M}^{ss}_{VW}(r,\alpha,k)^{\mathbb{C}^*}$ of (E,ϕ) with $\phi=0$, and this is just the moduli space $\mathcal{M}^{ss}_{tf}(r,\alpha,k)$ of semistable torsion-free sheaves.

The contribution this makes to the Vafa–Witten invariant is the virtual Euler characteristic $\chi^{vir}(M^{\eta}_{sheaf}(r,\alpha,k))$.

This is called the instanton branch.

The parts of $M^{ss}_{VW}(r,\alpha,k)^{\mathbb{T}_m}$ of $[E,\phi]$ with $\phi \neq 0$ are called the monopole branch.

Vafa–Witten invariants are the sum of two contributions,

$$VW(r,\alpha,k) = VW(r,\alpha,k)^{inst} + VW(r,\alpha,k)^{mono},$$

both deformation invariant.

Tanaka–Thomas show that if $\deg K_X \leq 0$ then any stable (E, ϕ) has $\phi = 0$, & in the stable = semistable case, the monopole branch is empty, and

$VW(r,d,k) = VW(r,d,k)^{inst}$.

However, in the general case, they show you only get modular form for generating function which sum up $VW(r,d,k)^{inst}$ and $VW(r,d,k)^{mono}$:

individually, instanton branch and monopole branch do not give modular forms.

Refined Vafa-Witten invariants

Thomas arXiv: 1810.00078

defines a refined version of Vafa-Witten invariants. The unrefined version is:

$$VW(r,d,k) = \int_{\left(M''_{VW}(r,d,k)^{\mathbb{C}_m}\right)_{virt}} \frac{1}{e(N_{(r,d,k)})}$$

Here $N_{(r,d,k)}$ is the virtual normal bundle of $M''_{VW}(r,d,k)^{\mathbb{C}_m}$ in $M''_{VW}(r,d,k)$.

Now $N_{(r,d,k)}$ has a natural $\mathbb{C}_m$-action, & $N_{(r,d,k)} = \bigoplus_{m \in \mathbb{Z} \setminus 0} N_{(r,d,k)}^{m}$, where $N_{(r,d,k)}^{m}$ is the subcomplex with $\mathbb{C}^*$ weight $m \in \mathbb{Z}$.

Thomas defines a complicated refined version

$$VW^{ref}(r,d,k) \in \mathbb{Q}(t^{1/2}),$$ which

uses the decomposition $N_{(r,d,k)} = \bigoplus_{m \in \mathbb{Z}\setminus 0} N_{(r,d,k)}^{m}$,

and is natural in K theory.

For the instanton branch, the contribution to $VW^{ref}(r,d,k)$ is the virtual χ_{-t} genus

$$\chi_{-t}^{vir}\left(M^{ss}_{inst}(r,d,k)\right).$$

At $t=1$ it specializes to ordinary Vafa-Witten invariants.

The generating functions are <u>Jacobi</u> modular forms.

More things to talk about:

- Basic orientation on coherent sheaves.
- Definition, examples.
- Generalization of vector bundles.

Example: Hilbert schemes of points.

Example: a family of rank 2 vector bundles on $\mathbb{C}^2$ whose limit is a torsion-free coherent sheaf:

Consider (x, y) coord. on $\mathbb{C}^2$.

$t \in \mathbb{C}$ fixed.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{C}^2} \arrow[rr, "\begin{pmatrix} x \\ y \\ t \end{pmatrix}"] & & \mathcal{O}_{\mathbb{C}^2}^{\oplus 3} \arrow[r] & E_t \arrow[r] & 0 \end{tikzcd}$$

exact sequence of coherent sheaves

If $t \neq 0$, $\begin{pmatrix} x \\ y \\ t \end{pmatrix}$ is injective as a map of vector bundles, and E_t is a vector bundle.

— If $t = 0$, $\begin{pmatrix} x \\ y \\ t \end{pmatrix}$ is injective as a morphism of sheaves, but not injective on the fibres at $(x,y) = (0,0)$.

$$E_0 = \{ f \in \mathcal{O}_X : f|_{(0,0)} = 0 \}$$

like in Hilb$^1(\mathbb{C}^2)$ $\uparrow$ $\oplus$

$\mathcal{O}_X$] from 3rd coord $\hookrightarrow \mathcal{O}_{\mathbb{C}^2}^{\oplus 2}$.

This shows why moduli spaces of vector bundles are generally non compact, but moduli spaces of (semi stable) torsion-free sheaves on projective schemes are often compact (proper). We need compactness to define virtual classes and invariants.

<u>Also:</u> dim will how pairs (E, Q) on X correspond to a dimension 2 torsion sheaf on K_X.

As K_X is a non compact Calabi-Yau 3-fold, this brings ideas of Donaldson-Thomas theory to bear.