Geometry advanced class. Week 3. (P. Bousseau)

<u>Plan</u>: 1/ $\mathcal{N}=4$ SYM and S-duality

2/ Vafa-Witten equations from topologically twisting $\mathcal{N}=4$ SYM

1/ $\mathcal{N}=4$ SYM and S-duality

1.1. Classical and quantum field theories.

M : spacetime (Lorentzian or Riemannian manifold)

$\mathcal{F}$: space of fields (e.g. connections, sections of bundles, ...) "∞-dim manifold"

$S : \mathcal{F} \to \mathbb{C}$: "action", $S = \int_M \mathcal{L}\, \mathrm{vol}$ $\mathcal{L}$: "Lagrangian density" local functional of fields.

. Classical field theory:

Euler-Lagrange equation: $\delta S = 0$ System of PDE

. Quantum field theory (QFT) $\mathcal{O}_1, \dots, \mathcal{O}_n \in \underbrace{C^\infty_{loc}(\mathcal{F}, \mathbb{C})}_{\text{local functional of fields}}$:

Correlation functions:

$$\left\langle \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = \text{"} \int_{\phi \in \mathcal{F}} D\phi \, e^{iS} \prod_{i=1}^n \mathcal{O}_i(x_i) \text{"} \quad \text{(Lorentzian)}$$

$$\text{or "} \int_{\phi \in \mathcal{F}} D\phi \, e^{-S} \prod_{i=1}^n \mathcal{O}_i(x_i) \text{"} \quad \text{(Riemannian)}$$

. Belief: QFT exists if "$\underbrace{\text{effective couplings}}_{\text{Can be checked in some leading approximation}}$ do not grow at short distances"

$\cdot\ M = \underbrace{N}_{\text{space}} \times \underbrace{\mathbb{R}_t}_{\text{time}}$ $\rightarrow$ Usual quantum mechanical formalism can be reconstructed: . Hilbert space of states

. Hamiltonian operator

. Particle states ....

<u>Remark</u>: To incorporate fermions, M needs to be a $\underbrace{\text{super}}_{\mathbb{Z}/2\text{-graded}}$<u>manifold</u>.

Finite dimensional supermanifold:

$$\left.\begin{array}{lll} x_1, \ldots, x_n & \text{even coordinates} & x_i x_j = x_j x_i \\ \theta_1, \ldots, \theta_m & \text{odd coordinates} & \theta_i \theta_j = -\theta_j \theta_i \end{array}\right\} \mathbb{R}^{n|m}$$

$$C^\infty(\mathbb{R}^{n|m}) = C^\infty(\mathbb{R}^n) \otimes \Lambda^*(\mathbb{R}^m)^\vee$$

(Technically: sheaf of $\mathbb{Z}/2$-graded commutative algebras over a manifold)

Parity changing operation: $\Pi\, \mathbb{R}^{n|m} = \mathbb{R}^{m|n}$

<u>Example</u>: M manifold ΠTM : supermanifold s.t

$$C^\infty(\Pi TM) = C^\infty(M, \Omega^\bullet M)$$

Vector fields on supermanifolds:

$\hookrightarrow$ super Lie algebra

<u>Example</u>: exterior derivative

$$d : \Omega^i M \to \Omega^{i+1} M$$

defines an odd vector field Q

ΠTM s.t. $[Q, Q] = 0$.

[since $d^2 = 0$]

Integration on supermanifold:

$$\int_{\mathbb{R}^{0|m}} d\theta_1 \ldots d\theta_m\, f(\theta_1, \ldots, \theta_m)$$

$= \text{Coefficient of } \theta_1 \ldots \theta_m \text{ in } f(\theta_1, \ldots, \theta_m)$

$$= \sum_{k=0}^{m} \sum_{i_1 < \cdots < i_k} f_{i_1 \ldots i_k} \theta_{i_1} \cdots \theta_{i_k}$$

1.2. Classical $\mathcal{N}=4$ SYM (super Yang-Mills)

$M = \mathbb{R}^4$ $\quad \mathcal{F} = C^\infty(M, \Omega^1 M \otimes \mathrm{ad}(\mathfrak{g}))$

G compact Lie group $\quad A$ connection 1-form

$\mathfrak{g} = \mathrm{Lie}(G)$ $\quad \oplus\, C^\infty(M, \Pi(S^+ \oplus S^-) \otimes \mathbb{C}^4 \otimes \mathrm{ad}(\mathfrak{g}))$

$\mathrm{ad}(\mathfrak{g}) : \mathfrak{g}$ $\quad \Psi \quad S^+, S^-$: spinor bundles on M

$$\mathrm{Spin}(4) = \underset{\curvearrowright \mathbb{C}^2}{SU(2)} \times \underset{\curvearrowright \mathbb{C}^2}{SU(2)}$$

$\oplus\, C^\infty(M, \mathbb{R}^6 \otimes \mathrm{ad}(\mathfrak{g}))$

ϕ

$$\mathcal{L} = \frac{1}{g^2} \mathrm{Tr}\Big(\underbrace{F_A \wedge * F_A}_{\text{Yang-Mills}} + \underbrace{(\Psi, \not{D}_A \Psi)}_{\text{Dirac}} + \underbrace{(d_A\phi, d_A\phi)}_{\text{Klein-Gordon}} + \phi\Psi\Psi + [\phi,\phi]^2 \Big) - i\theta\, \mathrm{Tr}(F_A \wedge F_A)$$

Schematically

Coupling constant

$$\boxed{\tau := \frac{\theta}{2\pi} + \frac{i}{g^2}}$$

θ-term

$\int \mathrm{Tr}\, F\wedge F \sim c_2$

$G = U(N), SU(N)$

$\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$

Why this very particular form?

$\mathrm{isom}(\mathbb{R}^4) \curvearrowright \mathcal{F}$

$\underset{\text{Translations}}{\mathbb{R}^4} \to \mathrm{isom}(\mathbb{R}^4) \to \underset{\text{rotations}}{\mathfrak{so}(4)}$

Supersymmetry extension: $\mathcal{N}=4$

$\underset{\text{super Lie algebra}}{\mathrm{isom}(\mathbb{R}^4)_{\mathbb{C}} \oplus (S^+\oplus S^-)\otimes \mathbb{C}^4} \oplus \underbrace{\mathfrak{so}(6)_{\mathbb{C}}}_{\text{R-symmetry}} \curvearrowright \mathcal{F}$

preserving S

$\varepsilon_+ \in S^+$ $\quad S^+ \otimes S^- \simeq TM \otimes \mathbb{C}$

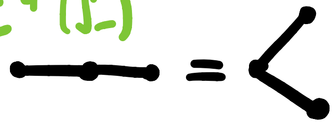
$\varepsilon_- \in S^-$ $\quad [\varepsilon_+, \varepsilon_-] = \text{Translation by } \varepsilon_+ \otimes \varepsilon_-$

R-symmetry $so(6) \curvearrowright \mathbb{R}^6$ (fundamental representation of $so(6)$)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

Exceptional isomorphism $\longrightarrow$ $su(4) \curvearrowright \mathbb{C}^4$ (S_+) (spinor representation of $so(6)$)

$\mathbb{C}^4$ (S_-)

$(A_3 = D_3$  $)$

Non-trivial Calculation: $\mathcal{N}=1$ 10d SYM + $SL(2,\mathbb{O}) = Spin(1,9)$

. Equation of motions: $\delta S = 0$:

Coupled system of PDEs.

$$\begin{cases} d_A^* F = \psi\psi + \phi\, d_A \phi \\ \not{D}_A \psi = \phi\psi \\ \Box\phi = \psi\psi + \phi[\phi,\phi] \end{cases}$$

1.3. Quantum $\mathcal{N}=4$ SYM and S-duality

$\mathcal{N}=4$ SYM $\Rightarrow \tau$ remains a constant at the quantum level.

Belief: $\forall \tau \in \mathbb{H}$, $\exists$ well-defined QFT. $\quad T(\tau, G)$
$\quad\quad\quad\quad G$

Weak coupling: $g \to 0$, $\mathrm{Im}\,\tau \to \infty$: perturbation theory / Feynman diagram

Strong coupling?

Rem: $\mathcal{N}=4$ SCFT (superconformal field theory)

$T(\tau, G)$
$\|$
$T(\tau+1, G)$
since $\theta \sim \theta + 2\pi$

S-duality conjecture ($\sim$ Montonen-Olive, 1975)

$$T(\tau, G) = T\left(-\frac{1}{\tau}, G^\vee\right)$$

$\tau \mapsto -\frac{1}{\tau}$

$\tau = i\infty \longleftrightarrow \tau = 0$

Weak Coupling $\quad$ Strong Coupling $(g \to \infty)$

$G^\vee$: Langlands dual group of G

$(G = SU(2),\ G^\vee = SO(3))$

$(G^\vee)^\vee = G \quad\quad (E_8^\vee = E_8)$

A $\quad$ B $\leftrightarrow$ C $\quad$ D $\quad$ E $\quad$ F $\quad$ G

Weight lattice $\longleftrightarrow$ dual weight lattice

G simple:

Center $Z \longleftrightarrow \pi_1$

Montonen-Olive: in gauge theory, electric charge $\in$ Weight lattice of G
G

$\updownarrow$ S?

Semiclassically: magnetic monopoles, magnetic charge $\in$ dual of weight lattice of G = weight lattice of $G^\vee$

$\left\{ \phi \xrightarrow[\infty \text{ of } \mathbb{R}^3]{} \phi_0 \right\}$ "Coulomb branch"

Conjecture not considered very seriously until

1994. Sen: S-duality $\Rightarrow$ Existence of dyons

($SL(2,\mathbb{Z}) \curvearrowright (1,0)$, $(0,1)$, $(2,1)$)

$\Downarrow$

Semiclassically: L^2-cohomology Classes on moduli spaces of monopoles. / Explicit check for $(2,1)$

(Witten convinced by this result)

Dualities in string theory

- IIB string, 10d, coupling constant $\tau \in \mathbb{H}$
  S-duality conjecture $SL(2,\mathbb{Z}) \curvearrowright \tau$

N D3-branes $\subset$ IIB

$U(N)$ 4d $\mathcal{N}=4$ SYM $\circlearrowleft S$

$\circlearrowleft$
S

Rem: No "proof" of S-duality Conj even at the physics level (eg by formal manipulations of path-integrals)

- $IIB / \mathbb{C}^2/\Gamma_g \times \mathbb{R}^6 \rightsquigarrow \mathcal{N}=(2,0)$ 6d SCFT
  ADE

$T^2_\tau \times \underbrace{\mathbb{R}^4}_{\mathcal{N}=4 \text{ 4d SYM}}$

$SL(2,\mathbb{Z}) \curvearrowright T^2_\tau$

6d / $\mathbb{R}^2 \times M_4$

$\uparrow$

2d CFT / VOA s.t. Z_{M_4} = character of VOA ...

2/ Vafa-Witten equations from topologically twisting 4d $\mathcal{N}=4$ SYM

Vafa-Witten 1994 motivation: provide further evidence for the S-duality conjecture.

<u>Basic idea</u>: Replace $\mathbb{R}^4$ by a compact Riemannian spin 4-manifold $M = (M,g)$

and consider the partition function $Z_{G,(M,g)}(\tau,\bar\tau) = \int_{\mathcal{F}} e^{-S}$ ~ Includes ~ $\sum$ over all principal G-bundles $P \to M$

Should be <u>modular</u>

A G-connection on P

$\psi \in C^\infty((S_+ \oplus S_-) \otimes \mathbb{C}^4 \otimes \mathrm{ad}(P))$

$\varphi \in C^\infty(\mathbb{R}^6 \otimes \mathrm{ad}(P))$

. Complicated function of (M,g) (e.g. for "free theory" ~ $\det \Delta_{(M,g)}$) and of $(\tau,\bar\tau)$ (not holomorphic)

. No supersymmetry

No constant spinor on general (M,g)

R-symmetry : $\mathrm{Spin}(6) \curvearrowright \mathbb{C}^4, \mathbb{R}^6$

"Couple to a background $\mathrm{Spin}(6)$-gauge field"

Fix $R \to M$ a $\mathrm{Spin}(6)$-bundle with a connection.

Twist the fields: ψ $\mathbb{C}^4$ $\quad (S_+ \oplus S_-) \otimes \mathbb{C}^4 \otimes \mathrm{ad}(P) \times_M R$

ϕ $\mathbb{R}^6$ $\quad \mathbb{R}^6 \otimes \mathrm{ad}(P) \times_M R$

$$Z_{G,(M,g),R}(\tau,\bar\tau) = \int_{\mathcal{F}} e^{-S}$$

Twisted partition function.

Even more complicated in general.

Key simplification for special choice of R

Odd vector field on $\mathcal{F}$

$\forall$ covariant section of $(S_+ \otimes S_-)\otimes \mathbb{C}^4 \times_M R$

Choose R s.t. this bundle contains a trivial sub-bundle

<u>Idea:</u> Pick $\rho : \mathrm{Spin}(4) \longrightarrow \mathrm{Spin}(6)$ and set $R = \underset{\text{bundle of } M}{\mathrm{Spin}} \times_\rho \mathrm{Spin}(6)$

$\mathrm{Spin}(4) = SU(2)_+ \times SU(2)_-$, $\mathrm{Spin}(6) = SU(4)$

VW <u>Twist:</u>

$$\left(\begin{array}{c|c} SU(2)_+ & 0 \\ \hline 0 & SU(2)_+ \end{array}\right)$$

$$4 \to (2,1)\oplus(2,1)$$
$$\bar 4 \to (2,1)\oplus(2,1)$$

- $S^+ \otimes \mathbb{C}^4$: $(2,1)\otimes((2,1)\oplus(2,1))$

$2\otimes 2 = 3 \oplus 1$

$$= (3,1)^{\oplus 2} \oplus (1,1)^{\oplus 2}$$

2 self-dual 2-forms $\Lambda^{2+}M$ — 2 scalars — $\psi^+_{\mu\nu}, \chi^+_{\mu\nu}$ — η, ζ

- $S^- \otimes \mathbb{C}^4$: $(1,2)\otimes((2,1)\oplus(2,1)) = (2,2)^{\oplus 2}$

2 vectors χ_μ, ψ_μ TM

- $\mathbb{R}^6$ $\quad 6 = \Lambda^2 4 = \Lambda^2((2,1)\oplus(2,1))$

$$= (1,1)\oplus(3,1)\oplus(1,1)\oplus(1,1) = (3,1)\oplus(1,1)^{\oplus 3}$$

B — 1 self-dual 2-form — $C, \phi, \bar\phi$ — 3 scalars

VW twist: 2 odd vector fields on $\mathfrak{F}$
(rotated by residual $U(1)_R$ symmetry)
↳ Pick 1: Q odd vector on $\mathfrak{F}$

s.t. $Q^2=0$

<u>Rem</u>: All bundles make sense $\forall M$ 4-manifold, no longer need to assume spin

$Q\mathcal{L}=0$? True on flat space
Curved space: need to add a "non-minimal coupling"

$$\frac{1}{4}B_{ij}\left(\frac{1}{6}(g_{ik}g_{jl}-g_{il}g_{jk})+W^+_{ijkl}\right)B_{kl}$$

Then:

$$\mathcal{L}=Q\Lambda-2i\pi\tau\,\mathrm{Tr}(F\wedge F)\quad\text{for some }\Lambda$$

Independent of g, holomorphic in τ

Formal conclusion: $Q\mathcal{L}=0$

$$\frac{\delta Z}{\delta g}=\int\frac{\delta S}{\delta g}e^{-S}=\int\left(Q\left(\frac{\delta\Lambda}{\delta g}\right)\right)e^{-S}$$

$$=\int Q\left(\frac{\delta\Lambda}{\delta g}e^{-S}\right)=0$$

Stokes (modulo ∂ term!)

$$\frac{\delta Z}{\delta\bar\tau}=\int Q\left(\frac{\delta\Lambda}{\delta\bar\tau}e^{-S}\right)=0$$

<u>Formally</u>: Z independent of g, holomorphic function of τ
$Z_{M,G}(\tau)$

"Calculation $Q\Lambda \sim (F^+ + [B,B] + [C,B])^2$
$+ (d_A C + d_A^* B)^2 \quad + \text{odd-part}$

<u>Rem:</u>

$F = F^+ + F^-$

$|F|^2 = F \wedge * F = (F^+ + F^-) \wedge (F^+ - F^-) = |F^+|^2 + |F^-|^2$

$F \wedge F = |F^+|^2 - |F^-|^2$

$$\frac{1}{2g^2}|F|^2 = \frac{1}{g^2}\Big(|F^+|^2 - \frac{1}{2}\big(\underbrace{|F^+|^2 - |F^-|^2}_{\operatorname{Tr} F\wedge F}\big)\Big)$$

$$Z = \int e^{-S} = \sum_n q^n \int e^{-\lambda Q \Lambda} \qquad q = e^{2i\pi z}$$

$$\frac{\delta}{\delta\lambda}\left(\int e^{-\lambda Q\Lambda}\right) = -\int Q\Lambda \, e^{-\lambda Q \Lambda} = -\int Q\left(\Lambda e^{-\lambda Q\Lambda}\right)$$

$= 0$
$\uparrow$ Stokes (modulo ∂ term!)

Independent of λ.

$\lambda \to +\infty$: Localization on $Q\Lambda = 0$

$$\begin{cases} F^+ + [B,B] + [C,B] = 0 \\ d_A C + d_A^* B = 0 \end{cases}$$

Solutions of Vafa-Witten equations.
$\mathcal{M}_n$

Actually $\int e^{-\lambda Q \Lambda}$ calculates the Euler characteristic of the moduli space $\mathcal{M}_n$

$\uparrow$

| See Mathai-Quillen formalism

Atiyah-Jeffrey paper $\Pi T(\Pi E)$ $\downarrow$ ΠTM

or Vafa-Witten §2.

χ $Q\chi = H - A_i\psi^i\chi$

$QH = A_i\psi^i H - \frac{1}{2}\psi^i\psi^j F_{ij}\chi$

u^i $Qu^i = \psi^i$

ψ^i $Q\psi^i = 0$

$Q^2 = 0$

E – oriented real vector bundle $\mathrm{Rk}\ r$ | Metric $(-,-)$ connection A_i

$E \underset{s}{\rightleftarrows} M$

M ← compact oriented manifold

$$\Lambda = \frac{1}{2}(\chi, H + 2s)$$

$$\int e^{-\lambda Q\Lambda}$$

$$Q\Lambda = (H, H+2s) + 2\left(\chi, \frac{\partial s}{\partial u^i}\psi^i\right) - \frac{1}{2}\psi^i\psi^j F_{ij}\chi\chi$$

$$\int dH \quad \rightsquigarrow \frac{1}{2}(s,s) + \left(\chi, \frac{\partial s}{\partial u^i}\psi^i\right) - \frac{1}{2}\psi^i\psi^j F_{ij}\chi\chi$$

$s=0$:

$$\int e^{-\lambda Q\Lambda} = \int d\chi d\psi\, e^{-\frac{1}{2}\psi\psi F\chi\chi} = \mathrm{Pf}(F)$$

$s \mapsto ts$ $t \to \infty$

$$\int e^{-\lambda Q\Lambda} = [s^{-1}(0)] = \text{Euler class of } E = \mathrm{Pf}(F)$$

Incorporate G-action & G-quotient | add extra term (G-equivariant cohomology)

__CCL:__ S-duality conj for 4d $\mathcal{N}=4$ SYM $\Rightarrow$ $Z_G(\tau)$ modular function

↳ ∂ term issues: holomorphic anomaly

↳ Ambiguity of coupling to gravity: modular __form__

__Final remark:__ Other topological twists of 4d $\mathcal{N}=4$ SYM.

$$\rho: Spin(4) = SU(2)_+ \times SU(2)_- \longrightarrow Spin(6) = SU(4)$$

$$\left(\begin{array}{c|c} SU(2)_+ & 0 \\ \hline 0 & SU(2)_- \end{array}\right)$$

$TM \simeq T^*M$

$6 = \overbrace{(2,2)} \oplus (1,1)^{\oplus 2}$

Kapustin-Witten twist, related to geometric Langlands.

$$\rho: Spin(4) = SU(2)_+ \times SU(2)_- \rightarrow Spin(6) = SU(4)$$

$$\left(\begin{array}{c|c} SU(2)_+ & 0 \\ \hline 0 & I \end{array}\right)$$

$6 = (2,1)^{\oplus 2} \oplus (1,1)^{\oplus 2}$ 3rd

$$\rho: Spin(4) = SU(2)_+ \times SU(2)_- \rightarrow Spin(6) = SU(4) \quad \text{VW}$$

$$\left(\begin{array}{c|c} SU(2)_+ & 0 \\ \hline 0 & SU(2)_+ \end{array}\right)$$

$6 = \underbrace{(3,1)}_{\Lambda^{2,+}M} \oplus (1,1)^{\oplus 3}$

VW	KW	3rd
$IIB / \underbrace{\Lambda^{2+}M}_{G_2} \times \mathbb{R}^3$	$IIB / \underbrace{T^*M}_{SU(4)} \times \mathbb{R}^2$	$IIB / \underbrace{S^+_M}_{Spin(7)} \times \mathbb{R}^2$

$D3/M$ = Twisted $\mathcal{N}=4$ 4d SYM $U(N)$
N

CY4

Exceptional holonomy