Advanced class, week 6.

Seiberg–Witten invariants and Vafa–Witten invariants

1. History of enumerative invariants of smooth 4-manifolds

1980's: Donaldson invariants.

— Count moduli spaces M of "instantons" on compact oriented Riemannian 4-manifold (X, g).

$b^2_+(X) = 0$: invariants cannot be defined.

$b^2_+(X) = 1$: invariants depend on g only via splitting

$$H^2_{dR}(X, \mathbb{R}) \cong \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$$

self-dual harmonic 2-forms — anti-self dual harmonic forms

[Compare: stability condition.]

Invariants change by Wall-Crossing Formula under change of splitting.

$b^2_+(X) > 1$: invariants independent of g.

Donaldson invariants can distinguish different smooth structures on the same topological 4-manifold — no other known invariants did this.

Moduli spaces M naturally noncompact, need to compactify them in a nice way to define invariants. This accounts for much of the difficulty of the theory.

Closely related to "instanton based" V-W invariants (Roughly: different cohomology class integrated on same moduli spaces.)

1994: Seiberg-Witten invariants.

— like Donaldson invariants, but technically much easier! S-W moduli spaces automatically compact, so do not need to compactify.

Structure of S-W invariants also much simpler.

Conjectural formula (now proved?) writes Donaldson invariants in terms of S-W invariants.

<u>Expectation</u>: S-W invariants are the most basic. Other 4-manifold invariants, including V-W invariants, can probably be written in terms of S-W invariants.

S-W invariants often computable, especially for complex surfaces.

2. Seiberg-Witten invariants

$n \geq 2$ $\quad \mathrm{Spin}(n) \xrightarrow{2:1} SO(n)$

double cover (universal cover if $n \geq 3$).

$$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times U(1) \big/ \pm(1,1)$$

exact sequence

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(n) \longrightarrow 1$$

Let (X,g) be an oriented Riemannian n-manifold. A $\underline{\mathrm{Spin}^c\text{-structure}}$ on X is

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\ & X & \end{tikzcd}$$

principal $\mathrm{Spin}^c(n)$-bundle on X

The oriented frame bundle of (X,g): principal $SO(n)$ bundle of oriented orthonormal frames for T_xX, $x \in X$.

morphism of principal bundles modelled on $\mathrm{Spin}^c(n) \longrightarrow SO(n)$.

So, P is a $U(1)$ bundle over Q.

Every oriented Riemannian 4-manifold admits a Spin^c structure. A Spin structure on X induces a Spin^c structure with trivial $U(1)$-bundle.

$Spin^c(X)$ = set of isomorphism classes of $Spin^c$ structures on X. It is a torsor for $H^2(X, \mathbb{Z})$, since two $Spin^c$ structures differ by a principal $U(1)$ bundle on X.

$Spin^c(4)$ has representations $\Delta^{\pm} \cong \mathbb{C}^2$ twisting the usual Clifford representations.

So a 4-manifold (X, g) with $Spin^c$ structure has spin bundles $\$_+, \$_- \longrightarrow X$ with fibre $\mathbb{C}^2$.

The <u>Seiberg - Witten equations</u> are for a pair (∇_P, ψ), where ∇_P is a connection on the principal $Spin^c(4)$-bundle $P \to X$

which lifts the Levi-Civita connection on the $SO(4)$ bundle $Q \to X$,

and $\psi \in \Gamma^\infty(\$_+)$,

satisfying $\not{D}_+^{\nabla_p} \psi = 0$ in $\Gamma^\infty(\$_-)$,

Dirac operator formed using ∇_p.

$$F_+^{\nabla_p} = q(\psi) \quad \text{in} \quad \Gamma^\infty(\Lambda^2_+ T^* X)$$

$U(1)$ part of curvature of ∇_p, piece in $\Lambda^2_+ T^* X$

q: quadratic map $\$_+ \to \Lambda^2 T^* X$.

Sign of q is important for compactness.

$\mathcal{M}_p^{SW}$ = moduli space of such pairs (∇_p, ψ), modulo automorphisms ($U(1)$ gauge transformations) of P.

<u>Note:</u> as $1 \to U(1) \to Spin^c(4) \to SO(4) \to 1$ and the connection on the $SO(4)$ part is the fixed Levi-Civita connection, the connection ∇_ρ is in effect a <u>$U(1)$ connection</u>, modulo <u>$U(1)$ gauge transformations</u>. Since $U(1)$ is <u>abelian</u>, the gauge theory behaves nicely — in particular, no noncompactness phenomena coming from Lie group connections from nonabelian

<u>Surprising:</u> An essentially abelian problem. But it can see very nontrivial phenomena. But it can.

As the S-W eqns are nonlinear elliptic and form compact moduli spaces, by standard deformation invariant techniques can form a <u>virtual class</u> $[M^{sw}_\rho]_{vir}$.

When $b^2_+(X) > 1$, call X of <u>simple type</u> if $(M^{sw}_\rho)_{vir} = 0$ unless $vdim(M^{sw}_\rho)_{vir} = 0$.

It is conjectured that all 4-manifolds with $b^2_+ > 1$

are of simple type. Then the Seiberg–Witten invariant is just the number

$$SW(P) = \# [M_P^{SW}]_{vir} \in \mathbb{Z}.$$

Iso classes of Spin^c structures on X are a torsor for $H^2(X, \mathbb{Z})$. It is known that $SW(P) \neq 0$ for only finitely many Spin^c structures, called Seiberg–Witten classes, when $b^2_+(X) > 1$.

There is a conjectural formula for Donaldson invariants (at least for gp $U(r)$, $SU(r)$) of the form

$$\left(\text{rank } r \text{ Donaldson invariants}\right) = \sum_{\beta_1, \dots, \beta_{r-1} \in SW(X)} \prod_{i=1}^{r-1} SW(\beta_i).$$

(Universal formula involving algebraic numbers and exponentials).

— Expect similar gauge theory invariants (e.g. Vafa–Witten invariants) to have the same general form, but with different universal formulae.

When $b^2_+(X) = 1$, things are a lot more complicated — I won't discuss it today.

3. Seiberg–Witten invariants for Kähler surfaces

Recall that the Vafa–Witten equations simplify when the 4-manifold X is a Kähler surface. The same thing happens for SW equations.

Since we have a lift

$$\begin{tikzcd} & & U(2) \arrow[dl, dotted] \arrow[d] \\ 1 \arrow[r] & U(1) \arrow[r] & Spin^c(4) \arrow[r] & SO(4) \end{tikzcd}$$

every Kähler surface has a natural Spin^c structure. More generally, if

X is a Kähler surface then
Spin^c structures on X can be identified with complex line bundles, $L \to X$, with $U(1)$ structure, such that $\$_+ \to X$ is

$$L \oplus \left(L \otimes \overline{\Lambda^{0,2} T^* X}\right)$$
$$\cong L \oplus \left(L^{-1} \otimes K_X\right)$$

The SW eqn then become a $U(1)$ connection ∇_L on L and sections α in $\Gamma^\infty(L)$ and β in $\Gamma^\infty(L^{-1} \otimes K_X)$ such that $\Psi = \alpha \oplus \bar{\beta}$.

It turns out that for a solution of the SW equations, $\alpha\bar{\beta} = 0$, so either $\alpha = 0$ or $\beta = 0$, and $F^{0,2}_{\nabla_L} = 0$, so L is a holomorphic line bundle, and $\bar{\partial}\alpha = \bar{\partial}\beta = 0$, so α, β are holomorphic. Also

$$\deg L = \int_X c_1(L) \vee (\omega) = \frac{1}{i}\int_X (|\alpha|^2 - |\beta|^2)\, dV,$$

so if $\deg L > 0$ then $\alpha \neq 0$, $\beta = 0$
if $\deg L < 0$ then $\alpha = 0$, $\beta \neq 0$,
if $\deg L = 0$ then $\alpha = \beta = 0$.

Hence we can identify the SW moduli space with either pairs (L, α) of a holomorphic line bundle L and a nonzero holomorphic section $0 \neq \alpha \in H^0(L)$ (if $\deg L > 0$)

or pairs (L, β) of a holomorphic line bundle L and a nonzero section $0 \neq \beta \in H^0(L^{-1} \otimes K_X)$ (if $\deg L < 0$),

or just L if $\deg L = 0$.

These are easy things to study using Algebraic Geometry.

If $b^2_+(X) > 1$ (equivalently, $h^{2,0}(X) > 0$, as $b^2_+(X) = 1 + h^{2,0}(X)$), it is a theorem that every Kähler surface is of simple type.

Although you might think that there are infinitely many holomorphic line bundles, L with nonzero holomorphic section, $0 \neq \alpha \in H^0(L)$ on any Kähler surface, if $b^2_+(X) > 1$ and $\deg L > \deg K_X$, it turns out that the obstruction bundle for the Seiberg-Witten moduli space has a constant factor $0 \neq H^{2,0}(X)$, so that

$$SW(L) = 0 \text{ if } \deg L > \deg K_X,$$

or if $\deg L < 0$. Because of this, there are only finitely many nonzero Seiberg-Witten invariants.

If X is a minimal projective surface of general type, there are only two nonzero Seiberg-Witten invariants for $L = \mathcal{O}_X$ and $L = K_X$, with $SW(\mathcal{O}_X) = 1$ and $SW(K_X) = (-1)^{\chi(\mathcal{O}_X)}$.

4. Vafa-Witten invariants of projective surfaces X when $b^2_+(X) > 1$.

There are conjectural formulae by Göttsche and Kool for generating functions of Vafa-Witten invariants of projective surfaces with $b^2_+(X) > 1$ for groups $SU(r)$ and $U(r)$ for $r = 2, 3$, written in terms of Seiberg-Witten invariants.

Here is a sample conjecture:

<u>Conjecture</u> (Göttsche–Kool 2020)

Let X be a projective surface with $b^2_+(X) > 1$ and $b^1(X) = 0$, and $(2, \alpha, k)$ be a Chern character of sheaves in rank 2, and $M^{Gi}_{2,\alpha,k}$ be the Gieseker moduli space.

Suppose $M^{Gi}_{2,\alpha,k}$ has stable = semistable.

Then $\chi^{vir}\left(M^{Gi}_{2,\alpha,k}\right)$ ← This is the "instanton branch" part of the V-W invariant

is the coefficient of $x^{\mathrm{vdim}\, M^{Gi}_{2,\alpha,k}}$ in

$$4\left(\frac{1}{2\bar{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_X)}\left(\frac{2\bar{\eta}(x^4)^2}{\theta_3(x)}\right)^{\int_X c_1(K_X)^2}\sum_{\beta\in H^2(X,\mathbb{Z})} SW(\beta)(-1)^{\int_X \alpha\cup\beta}\left(\frac{\theta_3(x)}{\theta_3(-x)}\right)^{\int_X \beta\cup c_1(K_X)},$$

where $\bar{\eta}(x)$ is a modular function and $\theta_3(x)$ a theta function.

— From conjectures like this, one gets complete expressions for V-W generating functions when $b_+^2(X) > 0$, which turn out to satisfy the V-W S-duality conjecture.

— Göttsche-Kool's conjecture is based on some knowledge of the overall shape of the generating function in terms of some unknown power series, coming mostly from the work of Mochizuki, plus extensive computer calculations of the first ≈ 50 coefficients of the power series, and matching these with power series of vector-valued modular forms as conjectured by Vafa-Witten.