Week 7. Chris Beem

VOA[M4] and Vafa–Witten.

State of the art consists of works by Feigin–Gukov; Gukov + collaborators | Physics

Dylan Butson | More mathematical.

— connections to lots of other things in Susy QFTs.

Recall:

$$Z^{G}_{VW}(M_4) = \sum_n VW(G, M_4, n)\, q^n \qquad q = e^{2\pi i \tau}, \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{ym}}.$$

G: compact Lie gp

M_4: 4-manifold

n: $c_2(G\text{-bundle} \to M_4)$

is expected to satisfy some modularity properties predicted from Physics.

— General state of modularity predictions, e.g. including 4-manifolds with boundary, is pretty vague.

Claim: These q-series should be characters of (modules for) some vertex algebras, associated to (G, M_4), that are nice enough to get modularity.

e.g. C_2 cofinite, or quasi-lisse, vertex algebras.

Why should we expect this?

Claim: $\exists$ a family (labelled by a simply-laced Lie algebra $\mathfrak{a} \in \{A, D, E\}$) of 6d QFTs ((2,0)-theories)

Large space time symmetry algebra: $OSp(8^*|4)$ superconformal algebra.

— Mysterious theories! Argued to exist using other mysterious things (like String / M theory).

Don't have a Lagrangian / path-integral description.

— Using String Theory (M-theory tricks), various "facts" known:

after compactifying to $d < 6$, can often identify resulting lower-dimensional theory as something more conventional. Gives geometric organization of some collection of lower-dim'l theories.

Early 2000's: industry of guessing lower-dim'l theories.

$$(2,0)_{\mathfrak{a}} \text{ on } X_5 \times S^1_R \xrightarrow{\text{choices here}} \text{5d } N=2 \text{ SYM theory}$$

mysterious

radius

$R \ll \mathrm{Vol}(X_5)$'s

with gauge group G, Lie $G = \mathfrak{a}$

Choices here.

$(g^2_{ym} \sim R)$ less mysterious.

coupling constant

elliptic curve parameter τ

$$(2,0)_{\text{on}} \; X_4 \times \mathbb{E}_\tau \xrightarrow[\text{vol}(\mathbb{E}_\tau)^{1/2} \ll \text{vol}(X_4)^{1/4}]{\text{choice here}} \text{4d } N=4 \text{ SYM with gauge group } G$$

$G \leftarrow$ choice here.

$$\frac{\theta}{2\pi} + \frac{4\pi i}{g_{ym}^2} = \tau$$

complexified coupling constant.

"Geometrizes S-duality": $\mathbb{E}_\tau$ depends on τ only up to $PSL(2,\mathbb{Z})$ action on $\mathbb{H}$;

so expect r.h.s. to depend on τ only up to $PSL(2,\mathbb{Z})$ too.

$$(2,0)_{\mathfrak{g}} \text{ on } X_4 \times C_{g,n} \xrightarrow{\text{vol}(C_{g,n})^{1/2} \ll \text{vol}(X_4)^{1/4}} \text{"Theory of class } S\text{"}.$$

Mysterious

Riemann surface genus g, n marked pts

Less mysterious.

Gukov, Gaiotto / Dimofte / Gukov, ... on guesses for many other complicated lower dimensional limits.

Compactifying on a 2-manifold 2-d things is easy and understood very well. Compactifying on 3d things is more complicated.

A simplification that is supposed to exist for these theories

Twisting. (2,0) theories has $Spin(5)_R$

R-symmetry: preferred global symmetry of theory $Spin(5) = Sp(4)$.

— There is no topological twist of the 6-d theory.

— But can restrict by hand to backgrounds of form $X_6 = X_4 \times X_2$

$$Spin(6) \supset Spin(4) \times Spin(2)$$

$$\begin{array}{ccc} & & Spin(3) \times Spin(2) \\ Spin(4) \cong SU(2)_+ \times SU(2)_- \to 1 & & \cap \\ \rho: Spin(4) \times Spin(2) & \longrightarrow & Spin(5) \end{array}$$

R-symmetry group

in this twist: which properties of theory are trivialized in a cohomological sense?

— In this twist, X_4 becomes topological. X_2 becomes holomorphic

Only features of X_2 you care about are holomorphic $\longrightarrow$

"Holomorphic topological twist".

$(2,0)_{\mathfrak{g}}$ SCFT $\xrightarrow{\text{twist (hol / top)}}$ Holomorphic$_2$ Topological$_4$ field theory, labelled by $\mathfrak{g}$.

Explanation for VOA(M_4) story:

$(2,0)_{\mathfrak{g}}$ HT on $\overset{\text{Kähler}}{X_4} \times \overset{\mathbb{E}_\tau}{X_2}$

$\mathrm{vol}(X_4)^{1/4} \ll \mathrm{vol}(X_2)^{1/2}$

$\equiv$

use those susy's to do twist

hol twist of $T(X_4, \mathfrak{g})$ evaluated on X_2 / $\mathbb{E}_\tau$

2-d theory

(0,2) SCFT if X_4 general

(0,4) SCFT if X_4 Kähler.

$\mathrm{vol}(X_2)^{1/2} \ll \mathrm{vol}(X_4)^{1/4}$

$\equiv$

use these susy's to do the twist.

4d $N=2$ SCFT if X_2 general.

4d $N=4$ SCFT if $X_2 = \mathbb{E}_\tau$.

top twist of $T(X_2, \mathfrak{g})$ evaluated on X_4.

Evaluate on $X_2 \times X_4$

VW twist

However, as theory is holomorphic/topological, answer should be independent of volumes, so all 3 things should be equivalent.

in Holomorphic twist of

$(0,2)$ or $(0,4)$ SQFT

Algebra of local operators becomes a vertex algebra.

determines $q = e^{2\pi i \tau}$

Evaluating on E_τ gives the vacuum character of vertex algebra.

— roughly

VOA(X_4)

Vafa-Witten partition function

"Compactifying on X"

= "taking indices of X very small"

Key thing: after taking twist, answer independent of the sizes, so need not take the limit.

Caveats / nuances in this picture.

If 2-d theory $T(X_4, \alpha)$ is nice enough local operator algebra (compact SCFT at low energies) then [holomorphic] twist of $T(X_4, \alpha)$ has a natural, "left-moving chiral algebra":

simple sub-VOA

— and this is what Feigin–Gukov call $VOA(X_4)_{FG}$

— And the full local operator algebra in particular is a module for this.

— The thing they say is a module, is also a VOA in its own right

— the full VOA may not be simple (can have proper ideal) | Nice enough case.

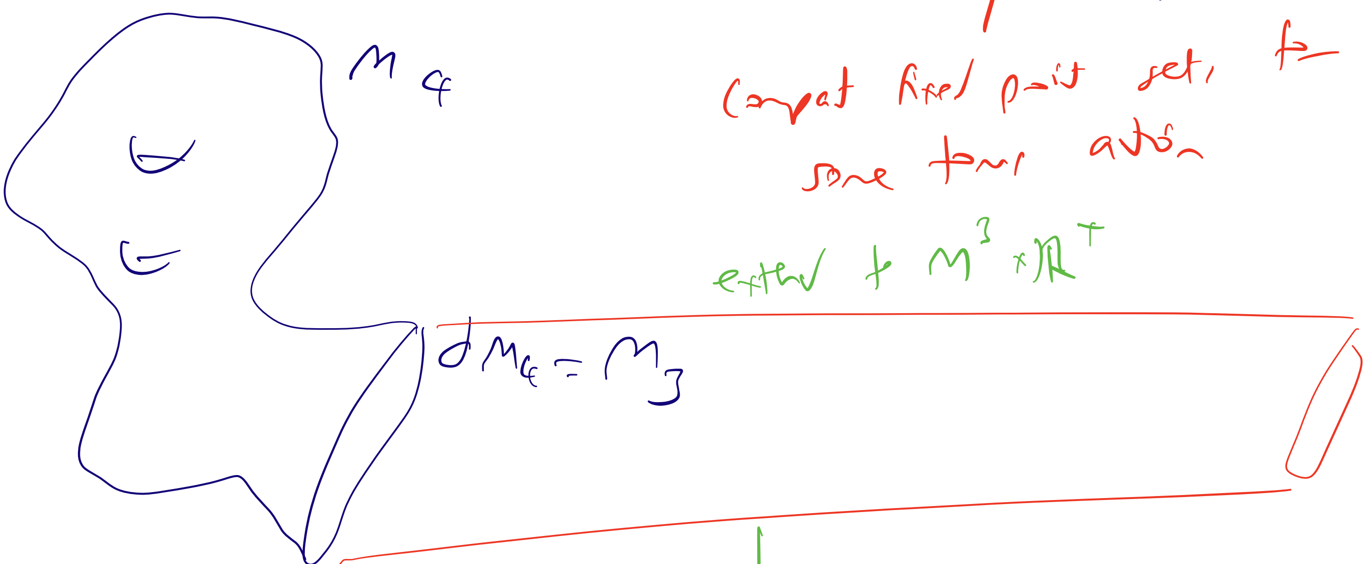
(Chris doesn't know: which X give nice enough theory; e.g. which moving at X.)

— Feigin–Gukov don't restrict to "nice enough" X_4.

Two special classes of interesting cases:

Suppose X_4 non-compact (not equivariantly compact)

compact fixed point set, for some torus action



extend to $M^3 \times \mathbb{R}^+$

(Question: can you do V-W invariants of 4-mflds with boundary?)

(2,0) HT theory reduced on this

$T(M_3, \sigma)$ evaluated on solid torus 

holomorphic

$\mathbb{R}^2 \times \mathbb{R}^+$ ← topological. (HT)

⇒ solid torus partition function

Hol twisted VA lives on ∂(solid torus)

insert line operator in centre of torus.

— So: get lots of some modules

expectation: solid tori that have modular symmetry.

$T(M_3, \sigma)$

$T(X_4, \sigma)$ ← boundary condition for $T(M_3, \sigma)$

Hence: VW invariants for 4 mfds with boundary

⟹ creates modularity expectations.

Feigin – Gukov: V–W invariants on

4 mfd with boundary; need to choose a

flat connection on the boundary. (⟹ fiber nearby to 3 mfd?)

Case when things are most concrete, with

actual constructions:

Suppose X is toric, so $U(1) \times U(1)$ ~ $\mathbb{C}^* \times \mathbb{C}^*$ acts

(fixed set is points)

equivariant localization?

expect to have a refinable equivariant version

of the theory. ⟹ Expect vertex algebras

with parameters (equivariant parameters) that you

can set to values.

"Physics in the Ω-background."

Expectation, because in 6-d.

$$(2,0)^{HT}_{\mathfrak{g}=\mathfrak{sl}_N} \qquad \mathbb{C}\times\mathbb{R}^4_{\varepsilon_1\,\varepsilon_2}$$

equivariant under $(\mathbb{C}^\times)^2$.

AGT correspondence

and Beem-Rastelli-Van Rees

Get vertex algebra on lines on $\mathbb{C}$

vertex algebra $\rightarrow$ W_N algebra with central charge

$$c=(N-1)+(N^3-N)\left(b+\frac{1}{b}\right)$$

$$b^2=\varepsilon_1/\varepsilon_2$$

This is equivariant $VOA(M_4)$ of $(\mathbb{R}^4 \circlearrowleft \mathbb{C}^{\times 2})$.
$\mathbb{R}^4 \cong \mathbb{C}^2$

— For more generality toric X_4,

let $\{p_1,\dots,p_k\}$ be the $U(1)\times U(1)$ fixed points expect as a subalgebra of Equivariant VOA($X_{\mathbb{C}}$, $\varepsilon_1, \varepsilon_2$, S^1_N)

extension $\cup$ perturbed central charge.

cf (Feigin Gukov)

$$\bigotimes_{i=1}^{N} W_N(C(\varepsilon_{1,i}, \varepsilon_{2,i}))$$

fact: each W_N includes

$W_n(\ldots)$

$\wedge$

Heis $\otimes N$

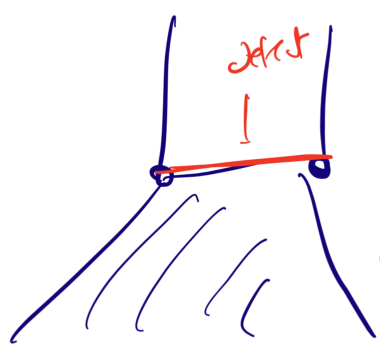
Butson:

— Expect $\mathcal{W}$ VOA($X_{\mathbb{C}}$, equiv) is a

extension of $\bigotimes_{c=1}^{N} W_N(\ldots)$

(Butson thinks of $X_{\mathbb{C}}$ as divisor in CY_3).

$\mathcal{O}(-1)\oplus\mathcal{O}(0) \longrightarrow \mathbb{P}^1$



Butson.

Get the thing which [illegible] thing [illegible] OP!

divisor: $\mathcal{O}(-2)\longrightarrow\mathbb{P}^1$.

Physics expectation: which modules of

$\sim \bigotimes_{i \leq 1} W_n(c(\dots))$

appear in VOA ($\chi_{\epsilon}, \mathcal{E}_1, \mathcal{E}_2$).

$\bigoplus$ (infinitely many terms of $\bigotimes W_N$)

— Get predictions for the vector space of VOA ($\chi_{\epsilon}\dots$) as a module for

$\bigotimes W_N(\dots)$

— The way to get VOA structure on VOA($\chi_{\epsilon}\dots$)

$$\mathrm{VOA}(\chi_{\epsilon}) \supset \bigotimes W_N$$

$\cap$ ← characterize

$\cap$ ← character by screening operators

e.g. (lattice VOA) $\supset (H^{\otimes l})^{\otimes N}$

$=$ VOA(min tor, of $\mathfrak{sl}_N$) ?

$\mathfrak{g} = \mathfrak{gl}_1$: nice theory: Nakajima - Grojnowski?

One thing to advertize:

— Case which is not toric, but should have nice structure if work equivariantly w.r.t. a single $U(1)$.

(Tanaka - Thomas: equivariance to make things work at all. BJ asks: what if the 4-manifold X has a group action too?)

equivariantly compact.

Case. $X_4 = T^* C_g$ ⟲ $\mathbb{C}^*$ action on fibre.

then VOA (X_4) should be $\cong$ Chiral algebra of class S
conjectured

[constructed by T. Arakawa;

very explicit character formula.

(2nd q-deformed Yang-Mills partition fn.)

U(1)-equivariant theories: don't get extra parameters.

What difference does X Kähler make?

Feigin Gukov VOA(M₄) ⟳ (modules).

↑

the thinks is the VOA.

↑

X Kähler gives some extra information here

Identify vector space of VOA(M₄)? In terms of moduli space? Borcherds-Kapustin: say something.