Week 8.   Chenjing Bu

Construction of VOA(M4).

– Two construction with the same setup.

(conjecturally equivalent.)

(Algebraic) Butson   2312.03648

(Geometric) Butson–Rapcak.   2309.16582

↑ homology of moduli space.

Setup: X affine toric singular Calabi–Yau 3-fold $/\mathbb{C}$

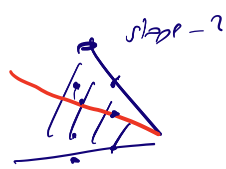
$\pi: Y \to X$ crepant resolution of toric 3-folds,

such that $\pi^{-1}(x) = *$ or $\mathbb{P}^1$ $\forall x \in X$.

Y Calabi–Yau

Example: $X = \{xy = z^2\} \times \mathbb{C}$

toric fan



$\times\ (0, \infty)$

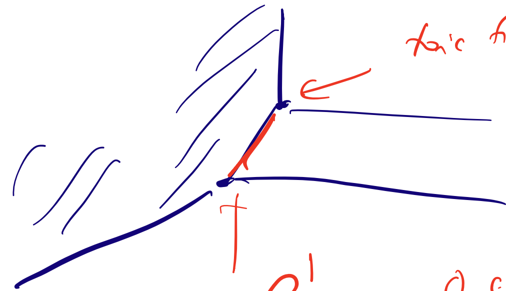
$$Y = \mathcal{O}(-2) \to \mathbb{CP}^1 \times \mathbb{C}$$

euler fn.

toric fixed pts

4d divisor for S_i: 4d faces.

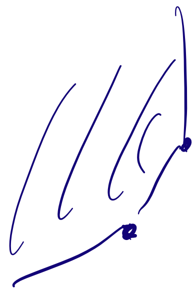
dual:



$\mathbb{P}^1$ or $0 \in \mathbb{C}$.

$\longrightarrow$

$\mathbb{C}$ direction



$= \quad \times \quad (0, \infty)$

S = toric divisor $\quad \sum_{i=1}^{n} r_i (S_i)$ divisor,

$S_i \subset Y$ toric hypersurfaces $\quad (r_i > 0)$.

making $S_i \neq S_j$, $i \neq j$.

— Then Butson constructs

$\mathbb{V}(Y, S)$ vertex algebra.

(torus equivariant construction.

depends on equivariant parameters.)

Expectation: When $X = \mathfrak{g}(S_1)$

$\Rightarrow$ VOA $(S_1, \mathfrak{gl}(r_1)]$.

I. Algebraic version.

Idea: Heisenberg on $S_i \xrightarrow{\text{glue}} V(\mathfrak{g}, S)$.

(From F-G, $\mathfrak{gl}(1) \simeq S \Rightarrow$ Heisenberg.)

Heisenberg + Kac-Moody vertex algebra.

$\mathfrak{g}$: reductive Lie algebra / $\mathbb{C}$ ($\mathfrak{g}$ abelian for today.)

abelian $\downarrow$ Heisenberg.

nonabelian $\rightarrow$ Kac-Moody

Basis $e^1, \dots, e^{\dim \mathfrak{g}} \in \mathfrak{g}$ $[e^a, e^b] = f_c^{ab} e^c$.

$\langle -, - \rangle$ invariant pairing on $\mathfrak{g}$ $\langle e^a, e^b \rangle = d^{ab}$.

Loop algebra $\hat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C} \cdot K$

$\mathfrak{g}[t, t^{-1}]$

$= \bigoplus_{1 \le a \le \dim \mathfrak{g},\ n \in \mathbb{Z}} \mathbb{C} e^a_n \oplus \mathbb{C} \cdot K$

$$[\underset{\in \mathfrak{g}}{a} \cdot \underset{\in \mathbb{C}[t,t^{-1}]}{f(t)},\ \underset{\in \mathfrak{g}}{b} \cdot \underset{\in \mathbb{C}[t,t^{-1}]}{g(t)}] = [a,b]\cdot f(t)g(t) - \mathrm{Res}_t(f\,dg)\langle a,b\rangle K$$

$\Longleftrightarrow$

$$[e_n^a, e_m^b] = f^{ab}_c\, e^c_{m+n} + n\,\delta_{n,-m}\, d^{ab} K.$$

E.g.

$$\hat{\mathfrak{g}}_{e_1} = \bigoplus \mathbb{C}\cdot b_n \oplus \mathbb{C}\cdot K$$

$$[b_n, b_m] = n\,\delta_{n,-m}\, K.$$

Kac-Moody vertex algebra associated to $\mathfrak{g}$, with level $k \in \mathbb{C}$ is (or vector space)

$$V_k(\mathfrak{g}) = \frac{U(\hat{\mathfrak{g}})\cdot 1}{\left\{\begin{array}{l} \mathfrak{g}[t]\cdot 1 = 0 \quad t \geq 0 \\ K\cdot 1 = k\cdot 1 \end{array}\right\}}$$

= algebra / ideal, module for $U(\hat{\mathfrak{g}})$.

PBW basis:

$$V_k(\mathfrak{g}) \underset{\substack{\text{vector}\\ \text{space}}}{\cong} \mathbb{C}\left[e^a_n \; : \; \begin{array}{l} 1 \le a \le \dim \mathfrak{g} \\ n = -1, -2, \dots \end{array} \right]$$

Vertex algebra structure:

$$e^a_{-1-n}(z) = \frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n e^a(z)$$

where $e^a(z) = \sum_{m \in \mathbb{Z}} \boxed{e^a_m} \, z^{-m-1}$

This acts on $V_k(\mathfrak{g})$ by the $\hat{\mathfrak{g}}$ action on $V_k(\mathfrak{g})$.

— Given a set of generating fields. Other elements, standard recipe using normally ordered products.

<u>Heisenberg</u>: $\pi_k = V_k(\mathfrak{gl}_1)$ with $d^{11} = 1$.

$= V_1(\mathfrak{gl}_1)$ with $d'' = k$.

($\hat{\mathfrak{g}}$ only depends on $d \cdot k$, not d, k.)

Go back to $\pi : Y \to X$ resolution.

$T = (\mathbb{C}^*)^2 \subset (\mathbb{C}^*)^3$: subtorus preserving CY structure.

$H_T^*(pt) = \mathbb{Q}(\varepsilon_1, \varepsilon_2)$.

$\mathbb{K} = \mathrm{Frac}\left(H_T^*(pt)\right) = \mathbb{Q}(\varepsilon_1, \varepsilon_2)$.

for localized T equivariant cohomology.

We will construct a VOA over $\mathbb{K}$

$\Leftrightarrow$ VOA depends on parameters $\varepsilon_1, \varepsilon_2$

Notations:

$F \subset Y$: (discrete) set of $(\mathbb{C}^*)^3$ fixed points.

$F_i := F \cap S_i$.

Heisenberg $\pi_i = \bigotimes_{i \in F_i} V_{k_y}(\mathfrak{gl}_1)$

level $k_y = \dfrac{-1}{e_T(T_y S_i)} \in \mathbb{K}$

equivariant Euler class.

(Heisenberg of dim $= |F_i|$)

$\cong V_1(\mathfrak{h}_i)$ $\qquad$ $\mathfrak{h}_i = \bigoplus_{y \in F_i} \mathbb{K} \cdot 1_y$ abelian Lie algebra

? Could think of as $H^*(S_i^{(\mathbb{C}^*)^3})$.

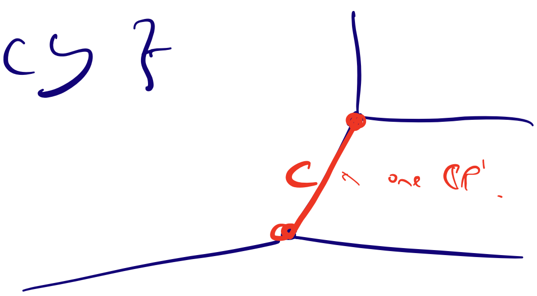
$$d^{y,y'} = k_y \cdot \delta_{y,y'}.$$

Lattice VOA:

construct a bigger VOA from $V_1(\mathfrak{h}_i)$.

$C = \{\text{compact toric curves } \mathbb{P}^1 \subset Y\}$

$C_i = \{c \in C : c \subset S_i\}$



For $c \in C_i$, define $y \in F_i$

$$\lambda_y^c = \begin{cases} e_T(N_c S_i|_y), & y \in c \\ 0, & y \notin c \end{cases}$$

associated to fixed pt y, curve c, surface S_i.

Define

$$\Pi_i = \bigoplus_{\ell \in \underbrace{\mathbb{Z}^{C_i}}} \Pi_{i,\ell}$$

lattice VOA

pairing on this lattice: intersections of curves c inside surface S_i.

$$\Pi_{i,\ell} = \frac{U(\hat{h}_i)\cdot 1}{\left\{ \begin{array}{l} \underbrace{t\, h_i(t)}_{t^{>0}=0} \cdot 1 = 0 \\ 1_y \cdot 1 = \sum_{c \in C_i} \ell_c \cdot \lambda_y^c \\ K \cdot 1 = 1 \end{array} \right\}}$$

Has a lattice VOA structure.

don't need k, level is encoded as pairing on h.

$$\Pi_{i,0} = \pi_i$$

$$\boxed{\Pi_i = \mathbb{V}(y, s_i), \quad \text{lattice VOA} = \text{VOA}(S_i; \mathfrak{g}^{e_1}).}$$

General case: Glue these together, in a nontrivial way.

Take $\Pi(y,s) = \bigotimes_i \Pi_i^{\otimes r_i}$

Not the final answer.

$$s = \sum_i r_i s_i$$

$$\pi(y,s) \subset \Pi(y,s)$$

$$\pi(y,J) = \bigotimes_i \pi_i^{\otimes r_i} \subset \bigotimes_i \Pi_i^{\otimes r_i}.$$

find conditions

$$W(y,J) = \ker\Big(Q : \Pi(y,J) \to \bigoplus_{k=1}^{N-1} \bigoplus_{\gamma \subset S_{i_k} \subset S_{i_{k+1}}} \Pi_{\partial\gamma}\Big)$$

γ: non compact toric curve.

Fix a filtration of O_S with quotients $O_{S_{i_1}}(\dots), \dots, O_{S_{i_N}}(\dots)$, $N = \sum r_i$

$$\partial_\gamma : h \longrightarrow \mathbb{K}$$

$$\bigotimes_i h_i^{\otimes r_i} = \bigotimes_i \bigotimes_{x \in F_i} \bigotimes_{j=1}^{r_i} \mathbb{K}\, 1_y^{(i,j)}$$

$$1_y^{c_{i,j}} \longmapsto \begin{cases} e_T(N_C S_{i_k} \mid y) & \text{if } y \in \gamma \cap S_{i_k} \\ -e_T(N_\gamma S_{i_{k+1}} \mid y) & \text{if } y \in \gamma \cap S_{i_{k+1}} \\ & c = c_{i_{k+1}},\ j = j_{k+1} \\ 0 & \text{otherwise.} \end{cases}$$

Picks out information of unexpected term coming from is adjacent term in the filtration.

j_k = # times c_{i_k} appears in $(c_{i_1}, \dots, c_{i_k})$

$$Q = {:}\exp\left(\sum_{y \in F} \gamma_y(1_y)\, \phi_y(T)\right){:}$$

normally ordered product.

Heisenberg field.

[Take-away: $\mathbb{V}(\mathcal{Y},\mathcal{S})$ is a vertex operator subalgebra of a big VOA $\Pi(\mathcal{Y},\mathcal{S})$.]

II. Geometric construction in BDGN – Rapčák is conjecturally the same.

– Easier to explain.

<u>Perverse coherent sheaves</u> on $\pi : \mathcal{Y} \to X$.

– An alternative heart of a t-structure.

$$P\,\mathrm{Coh}(\mathcal{Y} \to X) \subset D^b_{\mathrm{coh}}\,\mathcal{Y}.$$

<u>Contains:</u> $\mathcal{O}_Y(1)$,

$\mathcal{O}_{\text{surface}}(1)$, $\mathcal{O}_{\text{point}}$,

$\mathcal{O}_{\mathbb{P}^1}$, $\mathcal{O}_{\mathbb{P}^1}(-1)[1]$, ...

(dim ≥ 2: shift by 1.
dim $= 1$: shift the negative one.)

— As a heart is closed under extensions, arbitrary extensions of these objects still live in the heart.

<u>E.g.</u> For surface $i: S \hookrightarrow Y$

$\mathbb{I}(1) = i_*$ $\quad I_{pt}(1) \in \mathcal{P}((0, \, y/x))$

because $\mathcal{O}_{pt} \to I(1) \to \mathcal{O}_S(1)$ sequence in heart

Inside $\mathbb{P}\mathrm{Coh}(\mathcal{Y}/X)$, have smaller category: perverse coherent extensions.

$\mathcal{O}_{pt}$ fixed, $\mathcal{O}_{\mathbb{P}^1}$, $\mathcal{O}_{\mathbb{P}^1}(-1)(1)$

$$PC\mathcal{E} = \langle \mathcal{O}_{S_i}(1), \text{ compactly supported simple objects} \rangle$$

$$\subset \mathbb{P}\mathrm{Coh}(\mathcal{Y}/X)^{(\mathbb{C}^*)^3}.$$

$$PC\mathcal{E} \simeq \mathrm{Rep}(Q, W) \qquad (Q, W)$$

quiver with potential.

– explicit examples in paper.

Fix f: iterated extension of $\mathcal{O}_{S_i}(1)$

(with some subtleties.)

$\iff$ framing for quiver

Two choices in paper: $\underline{f}_S$ gives $\boxed{\mathcal{O}_S(1)}$

Memo: thicken S_i to order r_i.

$\underline{\mathcal{O}}$: trivial or $\bigoplus \mathcal{O}_{S_i}(1)^{\oplus r_i}$

framings.

Define: $\mathbb{V}_S^{f} = \bigoplus H^{T}_{*}\left(\mathcal{M}(Q^{f})_{\underline{d}}^{\text{?}}, \varphi_W^{f}\right)$

$\underline{d}$: dim vector

f, same

? Equivariant cohomology, the framing dual vector space.

moduli of representations

Potential depending $\underline{f}$.

Vanishing cycle sheaf

Conjecture: $\mathbb{V}(Y,S) \curvearrowright \mathbb{V}_S^{f}$

$\exists$ action

in Butson paper

algebraic version

such that $\mathrm{COHA} \curvearrowright \mathbb{V}_S^{f_S} \cong \mathbb{V}(Y,S)$

small part of $\mathbb{V}(Y,S)$: positive part.

VA rep on itself: gives VA structure

$$\mathbb{V}_S^{\,0} \cong \Pi(\mathfrak{g}, S)$$

Case of no $\mathbb{P}^1$'s (eg $S = \mathbb{A}^2$),

one special case: Nakajima's construction

$\mathrm{Hei} \circlearrowright H_*(\mathrm{Hilb}^*).$