

Geometry Advanced Class TT26

Organizers: Dominic Joyce, Pierrick Bousseau, Hulya Arguz, Chenjing Bu

Enquiries: e-mail dominic.joyce@maths.ox.ac.uk.

Subject: Cohomological Hall Algebras

Time and place: Tuesdays 9.30-11.0 weeks 1,2,3,5,6,7,8, in L4 week 1, and in C5 for the following weeks. No meeting fourth week.

Introduction

Let \mathcal{A} be a geometric \mathbb{C} -linear abelian category, such as the category $\text{coh}(X)$ of coherent sheaves on a smooth projective complex manifold X , or the category $\text{mod-}\mathbb{C}Q$ of representations of a quiver Q . Let \mathcal{M} be the moduli stack of objects in \mathcal{A} . A *Cohomological Hall Algebra (CoHA)* is the structure of an associative algebra on the cohomology $H^*(\mathcal{M}, \mathbb{Q})$. These are a part of Geometric Representation Theory.

In good cases, people can identify the CoHA as some interesting infinite-dimensional algebra studied in Representation Theory, such as a Heisenberg algebra, a W -algebra, ... CoHAs may also have interesting representations, for example, the CoHA of 0-dimensional sheaves on a complex projective surface X has a representation on the cohomology of the moduli scheme of torsion-free rank one sheaves on X (and so on the cohomology of Hilbert schemes of points on X).

This is a large subject. We are open to suggestions from the audience, but provisionally we plan to focus on the following areas:

- Classical CoHAs associated to representations of quivers, coherent sheaves on curves, and coherent sheaves on surfaces, following Schiffmann and other authors.
- Grojnowski-Nakajima's work on Heisenberg algebras acting on cohomology of Hilbert schemes of points on complex surfaces X . Extension by Mellit-Minets-Schiffmann-Vasserot framing this as a representation of a CoHA of 0-dimensional sheaves on X , generalization to $W_{1+\infty}$ -algebras and to higher rank sheaves.
- Kontsevich-Soibelman 2010 definition of CoHAs for moduli stacks of objects in 3-Calabi-Yau categories using 'critical cohomology', which is essentially the hypercohomology $\mathbb{H}^*(\mathcal{M}, \mathcal{P}_{\mathcal{M}})$ of a certain special perverse sheaf $\mathcal{P}_{\mathcal{M}}$ on \mathcal{M} .
- Subsequent developments of Kontsevich-Soibelman style 3CY CoHAs, including Kinjo-Park-Safronov 2024.

Plan of classes:

Week 1. Dominic: Classical CoHAs in the style of Schiffmann.

Week 2. Hulya: More about classical CoHAs, coherent sheaves on curves and surfaces.

Week 3. Pierrick: Kontsevich-Soibelman critical CoHAs.

Week 4. No meeting.

Week 5. TBA

Week 6. TBA

Week 7. TBA

Week 8. TBA

Main references:

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T. Kinjo, H. Park, and P. Safronov, *Cohomological Hall algebras for 3-Calabi-Yau categories*, [arXiv:2406.12838](#)

M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants*, Commun. Number Theory Phys. 5 (2011), 231–352. [arXiv:1006.2706](#).

A. Mellit, A. Minets, O. Schiffmann, and E. Vasserot, *Coherent sheaves on surfaces, COHAs and deformed $W_{1+\infty}$ -algebras*, [arXiv: 2311.13415](#), 2023.

H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. 145 (1997), 379–388.

H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, vol. 18, University Lecture Series, American Mathematical Society, Providence, RI, 1999.

O. Schiffmann, *Lectures on Hall algebras*, in: *Geometric methods in representation theory. II*, vol. 24-II, Sémin. Congr., Soc. Math. France, Paris, 2012, 1–141. [math/0611617](#).

Other useful sources:

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D. Butson, *Vertex algebras from divisors on Calabi-Yau threefolds*, [arXiv:2312.03648](#)

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- D.-E. Diaconescu, M. Porta, F. Sala, O. Schiffmann, and E. Vasserot, *Cohomological Hall algebras of one-dimensional sheaves on surfaces and Yangians*, [arXiv:2603.03386](https://arxiv.org/abs/2603.03386)
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- O. Schiffmann and E. Vasserot, *Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on \mathbb{A}^2* , *Publ. math. IHES* 118 (2013), 213–342. [arXiv:1202.2756](https://arxiv.org/abs/1202.2756)
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