Geometry Advanced Class TT 26

Cohomological Hall Algebras

Week 1. Introduction, by Dominic.

First idea

Let $\mathcal{A}$ be an abelian category over $\mathbb{C}$, and $\mathcal{M}$ the moduli stack of objects in $\mathcal{A}$, an Artin $\mathbb{C}$-stack.

Then we can form the cohomology $H^*(\mathcal{M})$ (or K theory, Borel-Moore homology, ...). We aim to make $H^*(\mathcal{M})$ into an associative algebra, a Cohomological Hall Algebra (CoHA).

Writing Exact for the stack of exact sequences $0 \to E_1 \to E_2 \to E_3 \to 0$, the multiplication $*$ in $H^*(M)$

is $(\pi_2)_* \circ (\pi_1 \otimes \pi_3)^*$, $\pi_i : \mathcal{E}xact \to M$

map sequence to E_i.

This only works for nice categories

A satisfying conditions (needed to define $(\pi_2)_*$).

There may be other moduli stacks

N (e.g. moduli stacks of framed objects in A) such that

$H^*(N)$ is a module over $H^*(M)$.

If we are lucky, then $H^*(M)$

is an infinite-dimensional algebra

already understood in Representation Theory,

and then we can use this to understand

$H^*(M)$ and $H^*(N)$.

Overview of the field

1990s, 2000s. E.g. Green, 1995, Hall Algebras and quantum groups $\rightarrow$

⓪ Hall Algebra over finite fields

$\leftarrow$ e.g. Schiffmann "Lectures on Hall Algebras", 2006.

$\downarrow$

Kontsevich–Soibelman 2010 §2–§3. Defined on $H^*(M)$.

① CoHAs of 1-dimensional abelian categories over $\mathbb{C}$: $\operatorname{coh}(\Sigma_g)$, Σ_g curve, $\operatorname{mod}\mathbb{C}Q$, Q quiver.

Latyntsev 2021: compatible with vertex coalgebra structure. $\rightarrow$

$\swarrow \qquad \searrow$

② CoHAs of $\operatorname{coh}_{cs}(S)$, S complex quasi-projective surface. Defined on Borel-Moore homology $H_*^{BM}(M)$

$\uparrow$ Kapranov–Vasserot 2019.

③ CoHAs of a 3-Calabi-Yau category $\operatorname{coh}(X)$, X Calabi-Yau 3-fold, or $\operatorname{mod}\mathbb{C}Q/W$, quiver with superpotential W. Defined on critical cohomology $\mathbb{H}^*(\mathcal{P}_M)$.

$\uparrow$ Kontsevich–Soibelman 2010, §4.

④ Representations of associative algebras $\mathbb{A}$ on $H^*(\mathcal{N})$, $\mathcal{N}$ moduli stack associated to $\mathbb{A}$, which look like they <u>ought</u> to be representations of a CoHA $H^*(\mathcal{M})$.

— Generally defined using <u>correspondences</u> in $\mathcal{N} \times \mathcal{N}$. $\mathbb{A}$ is <u>not geometrically defined</u>, i.e. not presented as $\mathbb{A} = H^*(\mathcal{M})$, but defined by generators and relations.

— Example: Grojnowski, Nakajima 1994: action of Heisenberg algebra on $H^*(\mathrm{Hilb}^*(S))$, S complex surface. CoHA interpretation by Mellit - Minets - Schiffmann - Vasserot 2023.

Lots of work on H^*(quiver moduli spaces) by Nakajima, ...

0. Hall Algebras over finite fields

Reference: Schiffmann, "Lectures on Hall algebras", 2006.

An abelian category $\mathcal{A}$ is finitary if for all $E, F \in \mathcal{A}$, $|\mathrm{Hom}(E,F)|, |\mathrm{Ext}^1(E,F)| < \infty$.

An important class of examples is if p is a prime, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the finite field, and $\mathcal{A}$ is $\mathbb{F}_p$-linear with

$$\dim_{\mathbb{F}_p} \mathrm{Hom}(E,F), \quad \dim_{\mathbb{F}_p} \mathrm{Ext}^1(E,F) < \infty.$$

Write $\mathbb{H} = \mathbb{Q}\{\text{Iso. classes } [E] \text{ of objects } E \in \mathcal{A}\}$.

Define a $\mathbb{Q}$-bilinear product $*$ on $\mathbb{H}$ by

$$[E_3] * [E_1] = \sum \frac{1}{|\mathrm{Aut}(E_1)|} \cdot \frac{1}{|\mathrm{Aut}(E_3)|} \cdot [E_2].$$

iso. classes $[E_2]$ in $\mathcal{A}$ $\quad \# \{$exact sequences

$$0 \to E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \to 0$$

in $\mathcal{A}$,

i.e. $\#$ pairs $(\alpha, \beta)\}$.

This is associative. To prove that

$$([E_3] * [E_2]) * [E_1] = [E_3] * ([E_2] * [E_1]),$$

we consider filtrations

$$0 = F_0 \subset F_1 \subset F_2 \subset F_3 = F \quad \text{in } \mathcal{A}$$

with $F_i / F_{i-1} \cong E_i$; essentially,

$$[E_3] * [E_2] * [E_1] = \sum_{(F)} \frac{1}{\prod_{i=1}^{3} |\mathrm{Aut}(E_i)|} \# \{\text{filtrations like this}\}.$$

Actually, Schiffmann works over $\mathbb{C}$ and includes an extra factor $\prod_{i>0} |\mathrm{Ext}^i(E_3, E_1)|^{(-1)^i}$. Doesn't change associativity but needed for compatibility with coproducts.

Green 1995: If $\mathrm{Ext}^i(E,F)=0$ for $i>1$ (i.e. $\dim A = 1$) then can define a coproduct $\Delta : \mathbb{H} \to \mathbb{H} \hat{\otimes} \mathbb{H}$ making A into a bialgebra, roughly by

in fact, a Hopf algebra

$$\Delta([E_2]) = \sum \frac{1}{|\mathrm{Aut}(E_2)|} [E_3] \otimes [E_1]$$

· extra twisting factor

iso. classes $[E_1], [E_3]$ in A

$\#\{$exact sequences $0 \to E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \to 0$ in A, i.e. # pairs $(\alpha, \beta)\}$.

There are interesting examples involving quivers Q over $\mathbb{F}_p$ in which one can make the set of isomorphism classes $[E]$ in $A = \text{mod-}\mathbb{F}_p Q$ independent of p. In fact, the indecomposable objects (E) correspond to positive roots of a Kac-Moody Lie algebra associated to Q. Then the structure constants $c^{(E_2)}_{[E_1][E_3]}$ in $[E_3] * (E_1) = \sum_{(E_2)} c^{(E_2)}_{(E_1)(E_3)} (E_2)$

become functions of p, $c^{[E_2]}_{(E_1)[E_3]}(p)$.

They often turn out to be polynomials in p.

Then as a Hopf algebra with the Green coproduct, $\mathbb{H}$ turns out to be isomorphic to the positive part of the quantum group of the Kac-Moody algebra associated to the quiver Q, at quantum parameter $v = p^{1/2}$. (Oversimplified a bit.)

Moral: Hall (bi) algebras can be computed explicitly, and even simple examples can give very interesting and deep algebras from Representation Theory.

Aside: it is an interesting problem to construct the full quantum group from a "Hall algebra" of $D^b \text{mod-}\mathbb{F}_p Q$. See Bridgeland arXiv: 1111.0745 for best known result.

(1) CoHAs of 1-dimensional abelian categories over $\mathbb{C}$.

Let $\mathcal{A}$ be an abelian category over $\mathbb{C}$, such that for all $E, F \in \mathcal{A}$, $\mathrm{Hom}(E,F)$, $\mathrm{Ext}^1(E,F)$ are finite-dimensional over $\mathbb{C}$, and $\mathrm{Ext}^i(E,F) = 0$ for $i > 1$. Then we call $\mathcal{A}$ <u>1-dimensional</u>.

The main examples are $\mathcal{A} = \text{mod } \mathbb{C}Q$ for a finite quiver Q, and $\mathcal{A} = \mathrm{coh}(C)$ for $C = \Sigma_g$ a smooth projective complex curve.

Let $\mathcal{M}$ be the moduli stack of objects in $\mathcal{A}$, an Artin stack over $\mathbb{C}$.

Then we can form the cohomology $H^*(\mathcal{M}, \mathbb{Q})$. Kontsevich - Soibelman 2010 §2-§3 explain how to make $H^*(\mathcal{M}, \mathbb{Q})$ into an associative algebra, a <u>Cohomological Hall Algebra</u> (<u>CoHA</u>).

Write $\mathcal{E}xact$ for the moduli stack of short exact sequences

$$0 \to E_1 \to E_2 \to E_3 \to 0,$$

and $\pi_i : \mathcal{E}xact \to \mathcal{M}$ for $i = 1, 2, 3$ for the morphism mapping this stack to E_i.

Then we have a diagram

$$\mathcal{M} \times \mathcal{M} \xleftarrow{\pi_1 \times \pi_3} \mathcal{E}xact \xrightarrow{\pi_2} \mathcal{M}.$$

Kontsevich–Soibelman define a bilinear product $*: H^*(M) \times H^*(M) \to H^*(M)$ by the composition

$$\begin{tikzcd} H^*(M) \times H^*(M) \arrow[r, "\boxtimes"] \arrow[d, "*"] & H^*(M \times M) \arrow[d, "(\pi_1 \times \pi_3)^*"] \\ H^*(M) & H^*(\mathrm{Exact}) \arrow[l, "(\pi_2)_*"'] \end{tikzcd}$$

Here $\boxtimes$ is the external tensor product and $(\pi_1 \times \pi_3)^*$ the pull back, both natural in cohomology.

The <u>subtle part</u> is the "pushforward" $(\pi_2)_*$. This is <u>not</u> natural in cohomology. To define it, we have to use the assumption that A has dimension 1.

Since $\mathcal{A}$ has dimension 1, $\mathcal{M}$ and $\mathcal{E}xact$ are <u>smooth</u> Artin stacks, their cotangent complexes $\mathbb{L}_{\mathcal{M}}$, $\mathbb{L}_{\mathcal{E}xact}$ are perfect in $(0,1)$. Thus

$\pi_2 : \mathcal{E}xact \to \mathcal{M}$ is quasi-smooth

– its cotangent complex is perfect in $(-1,1)$, it has a relative perfect obstruction theory.

Ignoring properness ~~for~~ the moment, this implies as in A. Khan arXiv:1902.01332 that there is a pushforward map

$$(\pi_2)_* : H^*(\mathcal{E}xact) \longrightarrow H^{*-2\,\mathrm{vdim}_{\mathbb{C}}\pi_2}(\mathcal{M}),$$

essentially a relative virtual class.

Think of $(\pi_2)_*$ as integration over virtual classes of the fibers of $\pi_2 : \mathcal{E}xact \to \mathcal{M}$.

There is another problem: $(\pi_2)_*$ should be defined if π_2 is quasi-smooth and ~~proper~~. For $\mathrm{Coh}(\Sigma_g)$, if we fix the Chern character, $\alpha_i = \mathrm{ch}\, E_i$, then $\pi_2^{\alpha_1,\alpha_2,\alpha_3} : \mathcal{E}xact_{\alpha_1,\alpha_2,\alpha_3} \longrightarrow \mathcal{M}_{\alpha_2}$ is proper, but π_2 is not in general.

Splitting
$$\mathcal{M} = \coprod_{\alpha \in \mathrm{Ch}(\mathrm{Coh}(\Sigma_g))} \mathcal{M}_\alpha,$$

we should define $*$ not on

allows infinitely nonzero terms

$$H^*(\mathcal{M}) = \prod_{\alpha \in \mathrm{Ch}(\mathrm{Coh}(\Sigma_g))} H^*(\mathcal{M}_\alpha),$$

but on

allows only finitely many nonzero terms

$$\tilde{H}^*(\mathcal{M}) = \bigoplus_{\alpha \in \mathrm{Ch}(\mathrm{Coh}(\Sigma_g))} H^*(\mathcal{M}_\alpha),$$

so properness of $\pi_2^{\alpha_1,\alpha_2,\alpha_3}$ is enough.

— Then one can prove that $*$ is associative. The argument uses the fact that in a Cartesian square of Artin stacks,

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z \end{tikzcd},$$

with f, g proper and quasi-smooth, the following commutes:

$$\begin{tikzcd} H^*(W) \arrow[r, "f_*"] & H^*(Y) \\ H^*(X) \arrow[u, "e^*"] \arrow[r, "g_*"] & H^*(Z) \arrow[u, "h^*"] \end{tikzcd}.$$

Latyntsev arXiv: 2110.14356: vertex coalgebra structure on $H^*(M)$, compatible with the CoHA multiplication.

— Analogue of Green's coproduct in the finite field case.

(2) CoHAs of complex surfaces

The construction in (1) does not work when $A = \mathrm{coh}(S)$ for S a complex projective surface, so that $\dim A = 2$, since now M, $\mathcal{E}xact$ are not smooth (only quasi-smooth), so $\pi_2 : \mathcal{E}xact \to M$ is not quasi-smooth.

However, there is an alternative construction due to Kapranov–Vasserot 2019, which works on the Borel-Moore homology $H_*^{BM}(M)$.

(Co)homology theories in topology:

cohomology $H^*(X)$ compactly-supported cohomology $H^*_c(X)$

dual   dual

homology $H_*(X)$ Borel-Moore homology $H^{BM}_*(X)$

Isomorphic if X is an oriented manifold

$H^*_c(X)$ and $H^{BM}_*(X)$ are not invariant under homotopy equivalence, so more difficult to work with.

$H^{BM}_*(X)$ has pushforwards under <u>proper</u> $f : X \to Y$.

Borel-Moore homology can be defined for (nice) Artin stacks M:

$$H_k^{BM}(M, \mathbb{Q}) = \mathbb{H}^{-k}(M, \omega_M),$$

where ω_M is the dualizing complex in constructible sheaves $D^b_c(M, \mathbb{Q})$.

Note that it is defined for $k \in \mathbb{Z}$, not $k \in \mathbb{N}$.

If M is a smooth Artin $\mathbb{C}$-stack of $\mathbb{C}$-virtual dimension n then

$$\omega_M \cong \mathbb{Q}(n), \text{ so}$$

$$H_k^{BM}(M, \mathbb{Q}) \cong H^{2n-k}(M, \mathbb{Q}),$$

and $H_k^{BM}(M, \mathbb{Q}) = 0$ if $k \geq 2n$.

Let S be a projective complex surface and $\mathcal{M}$ the moduli stack of objects in $\mathrm{Coh}(S)$. Then we can consider

$$\mathcal{M} \times \mathcal{M} \xleftarrow{\pi_1 \times \pi_3} \mathcal{E}xact \xrightarrow{\pi_2} \mathcal{M}.$$

and define $*: H^{BM}_*(\mathcal{M}) \times H^{BM}_*(\mathcal{M}) \to H^{BM}_*(\mathcal{M})$ by the composition

$$\begin{tikzcd} H^{BM}_*(\mathcal{M}) \times H^{BM}_*(\mathcal{M}) \arrow[r, "\boxtimes"] \arrow[d, "*"] & H^{BM}_*(\mathcal{M} \times \mathcal{M}) \arrow[d, "(\pi_1 \times \pi_3)^*"] \\ H^{BM}_*(\mathcal{M}) & H^{BM}_*(\mathcal{E}xact). \arrow[l, "(\pi_2)_*"] \end{tikzcd}$$

In this case the pushforward $(\pi_2)_*$ is functorial in homology, (though we have a problem if π_2 is not proper).

The pullback $(\pi_1 \times \pi_3)^*$ is now the subtle thing, as pullbacks are not natural on (Borel-Moore) homology.

But for sheaves on surfaces,

$$\pi_1 \times \pi_2 : \mathcal{E}xact \longrightarrow M \times M$$

is quasi-smooth (has a perfect relative obstruction theory), so there is a virtual pullback map

$$(\pi_1 \times \pi_3)^* : H_k^{BM}(M \times M) \longrightarrow H^{BM}_{k+2\operatorname{vdim}_{\mathbb{C}} \pi_1 \times \pi_1}(\mathcal{E}xact).$$

— defined by the same technology used to define virtual classes (see Khan paper).

The proof that $*$ is associative uses the same properties of pullbacks and pushforwards on $H_*^{BM}(-)$.

If M is the moduli stack of objects in a 1-dimensional category, as in ①, then

$$H^k(M_\alpha) \cong H^{BM}_{2\operatorname{vdim}_{\mathbb{C}} M_\alpha - k}(M).$$

So we could write the CoHAs in ① in terms of either cohomology $H^*(M)$ or Borel-Moore homology $H^{BM}_*(M)$. The BM version is the one which extends to coherent sheaves on surfaces.