# Cohomological Hall algebras. Week 2

## Recap.

$\mathcal{A}$ nice abelian category over $\mathbb{C}$,

e.g. $\mathcal{A} = \text{mod-}\mathbb{C}Q$, $Q$ quiver,

$\mathcal{A} = \text{coh } X$, $X$ smooth projective $\mathbb{C}$-scheme of low dimension.

$\mathcal{M}$ = moduli stack of objects in $\mathcal{A}$, an Artin $\mathbb{C}$-stack.

$\mathbb{H} = H^*(\mathcal{M}, \mathbb{Q})$, or $H^{BM}_*(\mathcal{M}, \mathbb{Q})$, or some other generalized cohomology of $\mathcal{M}$.

Define a product $*: \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ to make $\mathbb{H}$ into an associative $\mathbb{Q}$-algebra: a <u>Cohomological Hall Algebra</u> (<u>CoHA</u>).

$\mathcal{E}xact$: moduli stack of exact sequences

$0 \to E_1 \to E_2 \to E_3 \longrightarrow 0$.

$\pi_i : \mathcal{E}xact \longrightarrow \mathcal{M}$ for $i = 1, 2, 3$ map to $E_i$.

Then $\alpha * \beta = (\pi_2)_* \circ (\pi_1 \times \pi_3)^* (\alpha \boxtimes \beta)$

$\pi_2 : \mathcal{E}xact \longrightarrow \mathcal{M}$

$\pi_1 \times \pi_3 : \mathcal{E}xact \longrightarrow \mathcal{M} \times \mathcal{M}$.

For $H^*(-)$: need $\pi_2$ <u>proper and quasi-smooth</u>
to define $(\pi_2)_*$. $(\pi_1 \times \pi_3)^*$ always defined.

For $H^{BM}_*(-)$: need $\pi_2$ proper to define $(\pi_2)_*$
and $\pi_1 \times \pi_3$ quasi-smooth to define $(\pi_1 \times \pi_3)^*$.

## Plan for today:

* Crash course on Artin stacks.
* Cohomology of Artin stacks.
* Quasi-smoothness, virtual classes, wrong-way maps.
* Why is $*$ associative?
* Identities for CoHAs.
* The simplest example: $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$.
* Latyntsev: compatibility with braided vertex coalgebra structure.

# 1. Crash course on Artin stacks

$\mathbb{C}$-schemes are geometric spaces $X$ locally modelled on $\operatorname{Spec} A$ for $A$ a commutative $\mathbb{C}$-algebra. Roughly speaking, Artin $\mathbb{C}$-stacks $S$ are a class of geometric spaces locally modelled on quotients $[X/G]$ for $X$ a $\mathbb{C}$-scheme and $G$ an algebraic $\mathbb{C}$-group.

Then $\mathbb{C}$-points of $[X/G]$ are points $x$ $G$-orbits in $X$. But $X(\mathbb{C})/G(\mathbb{C})$, i.e. the stack $[X/G]$ also remembers the stabilizer group $\mathrm{Stab}_G(x)$, in the isotropy group $\mathrm{Iso}_{[X/G]}(xG)$, which is part of the structure. So, for example, the quotient $[*/G]$ of a point $*$ by a trivial action of a group $G$ is not just the point $*$, it remembers the group $G$.

Formally, $\mathbb{C}$-schemes $X$ can be identified with special functors $\mathrm{Alg}_{\mathbb{C}} \longrightarrow \mathrm{Sets}$ by $A \mapsto \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} A, X)$, and $\mathbb{C}$-stacks are defined as special functors $\mathrm{Alg}_{\mathbb{C}} \longrightarrow \mathrm{Groupoids}$. But this gives little geometric intuition.

Stacks are useful in moduli problems. In essentially all interesting moduli problems in geometry, a <u>moduli stack</u> exists. For example, if $X$ is a projective $\mathbb{C}$-scheme, there is a moduli stack $\mathcal{M}$ of coherent sheaves on $X$. $\mathbb{C}$ points $[E]$ of $\mathcal{M}$ are isomorphism classes of $E$ in $\mathrm{coh}(X)$, and the isotropy groups are $\mathrm{Iso}_{\mathcal{M}}([E]) = \mathrm{Aut}(E)$.

In contrast, <u>moduli schemes</u> $\mathscr{M}$ may not exist. If one does exist, there is a projection

$$\underset{\uparrow \atop \text{moduli stack}}{\mathcal{M}} \longrightarrow \underset{\uparrow \atop \text{moduli scheme}}{\mathscr{M}}$$

which forgets information, in particular, it forgets automorphism groups $\mathrm{Aut}(E)$ of objects $E$. Nontrivial automorphism groups may cause moduli schemes not to exist.

Artin stacks form a 2-category.

There is a notion of (2-)Cartesian square

$$\begin{tikzcd} W \arrow[r, "f"'] \arrow[d, "e"] \arrow[dr, phantom, "\Rightarrow"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

so that $W = X \times_Z Y$ is the fibre product.

An Artin $\mathbb{C}$-stack $S$ has a <u>classifying space</u> $S^{cla}$, a topological space defined up to homotopy equivalence.

If $S = [X/G]$ is a quotient stack then $S^{cla} \simeq \dfrac{X^{an} \times EG^{an}}{G^{an}}$, where $X^{an}$ is the complex analytic topological space of the $\mathbb{C}$-scheme $X$, and

$$\begin{tikzcd} EG^{an} \circlearrowleft G^{an} \arrow[d] \\ BG^{an} \end{tikzcd}$$

is a classifying space for the complex Lie group $BG^{an}$.

If $X = *$ is a point then $[*/G]^{cla} \simeq BG^{an}$. So $[*/\mathbb{C}^*]^{cla} \simeq \mathbb{CP}^\infty$.

## 2. (Co)homology of Artin $\mathbb{C}$-stacks

If $S$ is an Artin $\mathbb{C}$-stack, we define the (co)homology to be

$$H_*(S,\mathbb{Q}) = H_*(S^{cla},\mathbb{Q}),$$

$$H^*(S,\mathbb{Q}) = H^*(S^{cla},\mathbb{Q})$$

usual (co) homology in Algebraic Topology.

In particular, if $S = [X/G]$ is a quotient stack, then

$$H^*([X/G]) \cong H^*_{G^{an}}(X^{an})$$

is the $G^{an}$-<u>equivariant cohomology</u> of $X^{an}$.

<u>Warning:</u> Borel-Moore homology of (nice) Artin stacks $S$ <u>cannot</u> be defined using $S^{cla}$. Borel-Moore homology is not preserved by homotopy equivalences.

__Example__. $[*/GL(n,\mathbb{C})]$ is the classifying stack for principal $GL(n,\mathbb{C})$-bundles, or equivalently, rank $n$ vector bundles. If $S$ is a $\mathbb{C}$-scheme then rank $n$ vector bundles $E \to S$ are in 1-1 correspondence with stack morphisms $S \xrightarrow{\phi_E} [*/GL(n,\mathbb{C})]$. A rank $n$ vector bundle has Chern classes $c_i(E)$ for $i=1,\dots,n$. These correspond to elements $c_i \in H^{2i}([*/GL_n(\mathbb{C})],\mathbb{Z})$ for $i=1,\dots,n$ with $c_i(E) = \phi_E^*(c_i)$.

Then $H^*([*/GL_n(\mathbb{C})], \mathbb{Q}) = \mathbb{Q}[c_1,c_2,\dots c_n]$ is the polynomial algebra freely generated by $c_i$ in degree $2i$ for $i=1,\dots,n$.

Note that $H^{2k}([*/GL_n(\mathbb{C})],\mathbb{Q}) \neq 0$ for all $k \geq 0$.

# 3. Quasi-smoothness, virtual classes, wrong-way maps

Suppose $S$ is a proper, smooth $\mathbb{C}$-scheme of $\dim_{\mathbb{C}} S = n$. Then $S$ is a compact, complex manifold, and has a fundamental class $[S]_{fund} \in H_{2n}(S, \mathbb{Z})$.

Next suppose $S$ is a proper, quasi-smooth derived $\mathbb{C}$-scheme (in the world of derived algebraic geometry), or a proper $\mathbb{C}$-scheme with a Behrend-Fantechi perfect obstruction theory (in the world of classical algebraic geometry). Then $S$ has a virtual class $[S]_{virt} \in H_{2 \operatorname{vdim}_{\mathbb{C}} S}(S, \mathbb{Z})$.

Roughly, the quasi-smooth / obstruction theory condition means that $S$ can locally be written as $s^{-1}(0)$ for

$$\begin{array}{ll} E & \text{vector bundle} \\ \downarrow \uparrow s & \text{section,} \\ V & \text{smooth scheme} \end{array}$$

and $[S]_{virt}$ is locally $[V]_{fund} \cap c_{top}(E)$.

More generally, suppose

$f: S \to T$ is a proper, quasi-smooth, (representable) morphism of derived schemes / stacks, or a morphism with a relative perfect obstruction theory for classical schemes / stacks.

Then each fibre $f^{-1}(t)$ is quasi-smooth and has a virtual class $(f^{-1}(t))_{virt} \in H_{2vdim\, f}(f^{-1}(t), \mathbb{Q})$.

There is a <u>virtual pushforward</u>

$$H^k(S) \xrightarrow{f_*} H^{k-2vdim_{\mathbb{C}} f}(T)$$

and a <u>virtual pull back</u>

<u>See</u>: Adeel Khan, arXiv: 1909.01332.

$$H_k(T) \xrightarrow{f^*} H_{k+2vdim_{\mathbb{C}} f}(S).$$

There are 'wrong-way maps', that is, they go the opposite way to the usual functoriality of (co)homology. Think of them as integration over the virtual classes $(f^{-1}(t))_{virt}$, $t \in T$.

Here is an important property of pushforwards and pullbacks of morphisms of stacks: suppose we have a 2-Cartesian square

fibre product

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

with $f, g$ proper and quasi-smooth, representable

then

$$\begin{tikzcd} H^*(W) \arrow[r, "f_*"] & H^*(Y) \\ H^*(X) \arrow[u, "e^*"] \arrow[r, "g_*"] & H^*(Z) \arrow[u, "h^*"] \end{tikzcd}$$

commutes.

# 4. Associativity of CoHA product $*$

Let $\mathcal{A}$ be a nice $\mathbb{C}$-linear abelian category, and $\mathfrak{M}$ the moduli stack of objects in $\mathcal{A}$.

Write the stack of exact sequences, the stack $\mathfrak{Exact}$ as $\mathfrak{Filt}_2$, the stack of 2-step filtrations, $0 = E_0 \subset E_1 \subset E_2 = E$ in $\mathcal{A}$, $E_i$ subobjects (e.g. vector subspaces, vector subbundles)

$$
\begin{tikzcd}
 & 0 = E_0 \subset E_1 \subset E_2 = E \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & \\
E_1/E_0 & E_2/E_0 & E_2/E_1
\end{tikzcd}
$$

Also define $\mathfrak{Filt}_3$ to be the stack of 3-step filtrations $0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$.

Now consider the diagram of stacks

$$
\begin{tikzcd}
 & & \begin{matrix}\mathrm{Filt}_3 \\ 0=E_0\subset E_1\subset E_2\subset E_3\end{matrix} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\text{Cartesian}"] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
 & \begin{matrix}\mathrm{Filt}_2\times M \\ 0=E_0\subset E_1\subset E_2,\ E_3/E_2\end{matrix} \arrow[dl] \arrow[dr] & & \begin{matrix}\mathrm{Filt}_2 \\ 0=E_0\subset E_2\subset E_3\end{matrix} \arrow[dl] \arrow[dr] & \\
\begin{matrix}M\times M\times M \\ E_1/E_0,\ E_2/E_1,\ \frac{E_3}{E_1}\end{matrix} & & \begin{matrix}M\times M \\ E_2/E_0,\ E_3/E_2\end{matrix} & & \begin{matrix}M \\ E_3/E_0\end{matrix}
\end{tikzcd}
$$

$(*1)$

$$
\begin{tikzcd}
 & & H^*(\mathrm{Filt}_3) \arrow[dr, "(\ )_*"] \arrow[ddrr, bend left, "(\ )_*"] & & \\
 & H^*(\mathrm{Filt}_2\times M) \arrow[ur, "(\ )^*"] \arrow[dr, "(\ )_*"] \arrow[rr, phantom, "\text{Commutes}"] & & H^*(\mathrm{Filt}_2) \arrow[dr, "(\ )_*"] & \\
H^*(M\times M\times M) \arrow[ur, "(\ )^*"] \arrow[uurr, bend left, "(\ )^*"] & & H^*(M\times M) \arrow[ur, "(\ )^*"] & & H^*(M)
\end{tikzcd}
$$

$\alpha_1 \boxtimes \alpha_2 \boxtimes \alpha_3$ $\qquad$ $(\alpha_1 * \alpha_2) \boxtimes \alpha_3$ $\qquad$ $(\alpha_1 * \alpha_2) * \alpha_3$

and similarly consider

$$
\begin{tikzcd}[column sep=small]
& & \begin{array}{c}\mathrm{Filt}_3\\ 0=E_0\subset E_1\subset E_2\subset E_3\end{array} \arrow[dl] \arrow[dr] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
& \begin{array}{c}M\times\mathrm{Filt}_2\\ E_1/E_0,\ 0=\frac{E_1}{E_1}\subset\frac{E_2}{E_1}\subset\frac{E_3}{E_1}\end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c}\mathrm{Filt}_2\\ 0=E_0\subset E_1\subset E_3\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}M\times M\times M\\ E_1/E_0,\ E_2/E_1,\ \frac{E_3}{E_1}\end{array} & & \begin{array}{c}M\times M\\ E_1/E_0,\ E_3/E_1\end{array} & & \begin{array}{c}M\\ E_3/E_0\end{array}
\end{tikzcd}
$$

(*2)

$$
\begin{tikzcd}[column sep=small]
& & H^*(\mathrm{Filt}_3) \arrow[dr, "(\ )_*"] \arrow[ddrr, bend left, "(\ )_*"] & & \\
& H^*(M\times\mathrm{Filt}_2) \arrow[ur, "(\ )^*"] \arrow[dr, "(\ )_*"] \arrow[rr, phantom, "\text{(commute)}"] & & H^*(\mathrm{Filt}_2) \arrow[dr, "(\ )_*"] & \\
H^*(M\times M\times M) \arrow[ur, "(\ )^*"] \arrow[uurr, bend left, "(\ )^*"] & & H^*(M\times M) \arrow[ur, "(\ )^*"] & & H^*(M)
\end{tikzcd}
$$

$\alpha_1\boxtimes\alpha_2\boxtimes\alpha_3$ $\qquad$ $\alpha_1\boxtimes(\alpha_2*\alpha_3)$ $\qquad$ $\alpha_1*(\alpha_2*\alpha_3)$

From (*1)-(*2) we see that $(\alpha_1*\alpha_2)*\alpha_3=\alpha_1*(\alpha_2*\alpha_3)$.

# 5. Identities in CoHAs.

In nice cases we have a natural quotient

$$K^0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A}) \leftarrow$$ can be $K^{num}(\mathcal{A})$, numerical Grothendieck group.

$\top$ discrete lattice containing topological invariants.

E.g. $\mathcal{A} = \text{mod-}\mathbb{C}Q$, $K(\mathcal{A}) = \mathbb{Z}^{Q_0}$
= dimension vectors for $Q$.

$\mathcal{A} = \mathrm{Coh}(X)$, $ch : K^0(\mathcal{A}) \to H^*(X, \mathbb{Q})$
Chern character,
$K(\mathcal{A}) = \mathrm{Im}(ch) \subset H^*(X, \mathbb{Q})$

and a splitting $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$,

$\mathcal{M}_\alpha$ = substack of objects $E$ in class $\alpha \in K(\mathcal{A})$. $[\![E]\!] =$

Also expect that the only object in class $0 \in K(\mathcal{A})$ is the zero object $0 \in \mathcal{A}$,
and $\mathcal{M}_0 = \mathrm{Spec}\,\mathbb{C}$ is the point $[0]$.

Then $H^k(\mathcal{M}_0) = \begin{cases} \mathbb{Q} & k = 0 \\ 0 & k > 0, \end{cases}$ and
the identity is $\mathbb{1} = 1 \in H^0(\mathcal{M}_0) \subset H^*(\mathcal{M})$.
Easy to see $\mathbb{1} * \alpha = \alpha * \mathbb{1} = \alpha$.

## 6. The simplest example of a CoHA.

Take $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$, the category of finite-dimensional vector spaces over $\mathbb{C}$.
This is both mod-$\mathbb{C}Q$ for $Q = \bullet$,
and $\mathrm{coh}(X)$ for $X = \mathrm{Spec}\,\mathbb{C}$ the point.

Take $K(\mathcal{A}) = \mathbb{Z}$, with
$K^0(\mathcal{A}) \to K(\mathcal{A})$ mapping $[E] \to \dim_{\mathbb{C}} E$.
Then $\mathcal{M} = \coprod_{n \geqslant 0} \mathcal{M}_n$, $\mathcal{M}_n = [*/GL_n(\mathbb{C})]$

$$H^*(\mathcal{M}_n) \cong H^*_{GL(n,\mathbb{C})}(*, \mathbb{Q})$$
$$\cong \mathbb{Q}[c_1, \dots, c_n], \quad \text{with } \deg c_i = 2i.$$

So $H^*(\mathcal{M}) = \prod_{n \geqslant 0} \mathbb{Q}[c_1, \dots, c_n]$

Sometimes $\oplus$ is preferable, need to modify notion of cohomology a bit to get $\oplus$.

An alternative presentation is

$$H^*_{GL(n,\mathbb{C})}(*, \mathbb{Q}) \cong \mathbb{Q}(x_1, \dots, x_n)^{S_n},$$

the algebra of symmetric polynomials in $x_1, \dots, x_n$, with $\deg x_i = 2$, where $x_1, \dots, x_n$ correspond to the Chern roots, that is,

$$(q - x_1)(q - x_2) \cdots (q - x_n) = q^n + q^{n-1} c_1 + \dots + c_n.$$

Kontsevich–Soibelman show the multiplication is

$$* : H^*([*/GL_m]) \times H^*([*/GL_n]) \longrightarrow H^*([*/GL_{m+n}])$$

is

$$f(x_1 \dots x_m) * g(x_1 \dots x_n)$$

$$= \sum_{\substack{\{1, \dots, m+n\} = \{i_1 < \dots < i_m\} \\ \amalg \{j_1 < \dots < j_n\}}} f(x_{i_1}, \dots, x_{i_m}) \cdot g(x_{j_1} \dots x_{j_n}) \cdot \prod_{\substack{a = 1 \dots m \\ b = 1 \dots n}} \frac{1}{(x_{i_a} - x_{j_b})}$$

The proof is by equivariant localization, with the factor $\prod \frac{1}{x_{i_a} - x_{j_b}}$ being $\frac{1}{e(N_{\dots})}$.

Globally, $\mathbb{H}$ is the free exterior algebra on infinitely many generators, $x_1^k \in H^*(*/GL_1)$,

$k = 0, 1, 2, \dots$

# 7. Latyntsev's vertex bialgebra structure.

Recall that for Hall algebras of 1-dimensional categories over finite fields, Green enhanced the Hall algebra to a bialgebra (Hopf algebra), which gave quantum groups in examples.

Let $\mathcal{A}$ be a 1-dimensional abelian category over $\mathbb{C}$, e.g. mod $\mathbb{C}Q$ or Coh $C$ for $C$ a smooth projective curve, and $\mathcal{M}$ the moduli stack of objects in $\mathcal{A}$.

Then: Kontsevich-Soibelman make $H^*(\mathcal{M})$ into an associative algebra (CoHA).

Joyce makes $H_*(\mathcal{M})$ into a quantum vertex algebra (braided vertex algebra)

so $H^*(\mathcal{M})$ is a braided vertex coalgebra.

Latyntsev arXiv 2110.14356 shows these are compatible:

$$
\begin{tikzcd}
H^*(M\times M) \arrow[r, "\text{CoHA } *"] \arrow[d, "\substack{\text{braided}\\ \text{vertex coalgebra}\\ Y^\vee \otimes Y^\vee}"'] & H^*(M) \arrow[dd, "\substack{\text{braided}\\ \text{vertex coalgebra}\\ Y^\vee}"] \\
H^*(M\times M\times M\times M)((z)) \arrow[d, "\substack{\text{braiding}\\ S(z) \text{ on 2nd + 3rd factors}}"'] \arrow[r, phantom, "\text{Commutes}"] & \\
H^*(M\times M\times M\times M)((z)) \arrow[r, "\text{CoHA } * \otimes *"] & H^*(M\times M)((z))
\end{tikzcd}
$$

Jindal–Kaubrys–Latyntsev
arXiv: 2603.21707 :
generalize this to Kontsevich–Soibelman critical CoHAs of quivers with superpotential. So, it may be a general phenomenon for many classes of CoHAs?