# Week 3, Hulya

## What is the CoHA of a surface?

Plan:
* Review CoHAs.
* Borel-Moore homology of stacks; virtual pullbacks.

Kapranov-Vasserot arXiv: 1901.07641, §3-§4.
+ Adeel Khan arXiv: 19

* Calculation of CoHA for 0-dim'l sheaves

Mellit-Minets-Schiffmann-Vasserot arXiv: 2311.13415

1. $S$ a smooth, connected, quasi-projective surface.

$A = \mathrm{Coh}_c S$, abelian category of coherent sheaves on $S$ with proper support.

$\mathcal{M} = \mathcal{C}oh(S)$ : moduli stack of objects in $A$.

Singular; not a smooth Artin stack in general.

Let $\mathcal{E} \in \mathrm{Coh}(S) \Rightarrow$ def theory of $\mathcal{E}$ is governed by Ext groups, $\mathrm{Ext}^i(\mathcal{E},\mathcal{E})$

$\mathrm{Ext}^0(\mathcal{E},\mathcal{E}) = \mathrm{Hom}(\mathcal{E},\mathcal{E}) = T_1 \mathrm{Aut}(\mathcal{E})$.

$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) =$ space of first order deformations of $\mathcal{E}$.

If $S$ were a curve then $\mathrm{Coh}(S)$ would be a smooth Artin stack of dimension $\dim \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) - \dim \mathrm{Ext}^0(\mathcal{E},\mathcal{E})$
$= -\chi(\mathcal{E},\mathcal{E})$.

Generally, $\chi(\mathcal{E},\mathcal{E}) = \sum_{i=0}^{\dim S} (-1)^i \dim \mathrm{Ext}^i(\mathcal{E},\mathcal{E})$

$S$ surface: $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \neq 0$, $\dim \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ can vary.

__Goal:__ Define $H_*^{BM}(\mathrm{Coh}(S))$

Recall:

$$0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0$$

$$\begin{tikzcd} & \mathrm{Exact} \arrow[dl, "q"', "\pi_1\times\pi_2"] \arrow[dr, "p", "\pi_2"'] & \\ \mathrm{Coh}(S)\times\mathrm{Coh}(S) \arrow[rr, bend left] & & \mathrm{Coh}(S) \end{tikzcd}$$

$(\mathcal{E},\mathcal{F})$ $\qquad$ $\mathcal{G}$

CoHA multiplication

$p_* \circ q^* \leftarrow q^!$, virtual $q^!$

Subtlety: pull back via a "wrong way map", not functorial in homology

$$H_*^{BM}(\mathrm{Coh}(S)) = H^{-*}(f_* \omega_{\mathrm{Coh}(S)})$$

<u>Goal:</u> $f : \mathrm{Coh}(S) \longrightarrow *$ projection.

$$\begin{array}{c} \omega_{\mathrm{Coh}\,S} \in D_{con}(\mathrm{Coh}(S)) \\ \downarrow f_* \\ D_{con}(\mathrm{Spec}\,\mathbb{C}) \end{array}$$

Derived category of constructible complexes of sheaves

<u>First goal:</u> explain $D_{con}(\mathrm{Coh}(S))$

first for schemes, then for stacks

Let $X$ be a "reasonable" top space.

<u>Recall:</u> Constructible sheaf $\mathcal{F}$ on a space $X$ with a stratification is a sheaf which is locally constant on each stratum.

$X = \coprod_{i \in I} X_i$ $\quad X_i$ locally closed subset

Complex of sheaves of $\mathbb{Q}$-vector spaces

$$F^\bullet = \cdots \to F_1 \xrightarrow{d_1} F_2 \xrightarrow{d_2} F_3 \xrightarrow{d_3} \cdots$$

$$d_i \circ d_{i-1} = 0$$

$\mathcal{H}^i(F^\bullet)$ is a constructible sheaf of $\mathbb{Q}$ vector spaces

constructible complexes of sheaves } triangulated category

$\Uparrow$

complexes of constructible sheaves.

Enhanced to a dg-category or stable $\infty$-category (Adeel Khan).

$D_{con}(X)$ defined by Verdier

$X$ say: underlying complex analytic space of a scheme.

— Gives general set up to discuss Poincaré duality.

6 functor formalism $\qquad f: X \to Y$

1) R $f_*$ : $D_{c}(X) \to D_c(Y)$
2) L $f^*$ : $D_c(Y) \to D_c(X)$
3) R $f_!$ : $D_c(X) \to D_c(Y)$
4) L $f^!$ : $D_c(Y) \to D_c(X)$
5) $\mathcal{H}om$
6) $\otimes$

Let $f: X \to *$, $\quad \mathbb{Q}_X$ constant sheaf on $X$.

$$H^i(X, \mathbb{Q}_X) = H^i(f_* \mathbb{Q}_X)$$

* Compatibility between $f^* \xleftarrow[\text{adjoint}]{} f_*$

$$\mathrm{Hom}(f^* F, G) \simeq \mathrm{Hom}(F, f_* G)$$

$\forall \; F \in D_c(Y), \; G \in D_c(X)$.

$f_!$ : pushforward with compact support.

$\mathcal{F}$ : sheaf on $X$, $\quad f : X \to Y$.

Classical versions.

Our $f_*$, $f_!$ are right derived versions of these.

$(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ ✓

$(f_! \mathcal{F})(U) =$ all sections, where support in $f^{-1}(U)$ is proper over $U$.

For complexes of sheaves, there are derived versions of these operations.

Q: $f : X \to pt$.

$$H^i_{cs}(X, \mathbb{Q}_X) = H^i(f_!(\mathbb{Q}_X)).$$

with some conditions.

Let $f : X \to Y$

$\exists !$ $\quad f^! : D_{c}(Y) \to D_{c}(X)$

s.t. $(f_!, f^!)$ are adjoint functors.

Ex. Let $f : X \longrightarrow pt$

"Dualizing complex" $\omega_X = f^!(\mathbb{Q}_{pt}) \in D_{cs}(X)$

Let $\mathcal{F}^\cdot$ complex of sheaves on $X$.

$$R\mathrm{Hom}(f_! \mathcal{F}, \mathbb{Q}_{pt}) = R\mathrm{Hom}(\mathcal{F}, \underbrace{f^! \mathbb{Q}_{pt}}_{\omega_X})$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$(H_{cs}(X, \mathcal{F}))^\vee \qquad\qquad R\Gamma(\underbrace{R\mathcal{H}om(\mathcal{F}, f^! \mathbb{Q})}_{})$$

$D\mathcal{F} =$ Verdier dual to $\mathcal{F}$

$$\Rightarrow \quad H_c^i(X, \mathcal{F})^\vee = H^{-i}(X, D(\mathcal{F}))$$

Ex: Let $X$ oriented and of dim $n$.

↑ not nec compact

$\omega_X = \mathbb{Q}_X(n)$. $\qquad \mathcal{F} = \mathbb{Q}_X \Rightarrow D(\mathcal{F})$

$= R\mathcal{H}om(\mathbb{Q}_X, \mathbb{Q}_X(n))$

$= \mathbb{Q}_X(n)$.

$$\Rightarrow \quad H_c^i(X,\mathbb{Q})^\vee = H^{n-i}(X,\mathbb{Q}).$$

Poincaré duality.

For any $X$,

$$H_c^{-i}(X,\mathbb{Q})^\vee = H^{i}(X, \omega_X)$$

$D(\mathbb{Q}_X)$

$$\| $$

$$H_i^{BM}(X,\mathbb{Q})$$

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Now let $X$ be an Artin stack.

$$D_{cons}(X) := \lim_{\longleftarrow} D_{cons}(T)$$

projective limit

$T \longrightarrow X$ arbitrary morphism

$T$ scheme of finite type over $\mathbb{C}$

Doesn't make sense for triangulated categories.

Need to work with dg-categories or stable ∞-categories.

$$\begin{tikzcd} T' \arrow[dr, "\pi"] \arrow[d, "f'"'] & \\ X & T \arrow[l, "f"] \end{tikzcd} \qquad \begin{tikzcd} D_{con}(T') & \\ D_{con}(X) \arrow[u, "f'^*"] \arrow[r, "f^*"'] & D_{con}(T) \arrow[ul, "\pi^*"'] \arrow[l, phantom, "\Rightarrow"] \end{tikzcd}$$

Can also define $(f_*, f^*)$

adjoint functors $(f_!, f^!)$

$$H^i_c(X, \mathbb{Q}) := H^i(f_! \mathbb{Q}_X)$$

$$\parallel$$

$$H_i^{BM}(X, \mathbb{Q}) = H^{-i}(X, \omega_X)$$

$i \in \mathbb{Z}$ not $i \in \mathbb{N}$

for instance: $\omega_X$: can only be nonzero in negative degrees

# CoHA product

$$\begin{tikzcd}
 & \mathcal{E}xact \arrow[dl, "q"'] \arrow[dr, "p"] & \\
Coh(S) \times Coh(S) & & Coh(S)
\end{tikzcd}$$

$$m : H_*^{BM}(Coh(S)) \times H_*^{BM}(Coh(S)) \longrightarrow H_*^{BM}(Coh(S))$$

$$m = p_* q^!$$

Subtleties

$p$ : proper map.

## Working with derived stacks

$\exists\ p_*$ for proper maps

$q^!$ : not smooth map between stacks,

but quasi-smooth.

"Relative tangent complex $\mathbb{T}_q^{\times n}$"

Perfect of amplitude $[-1, 1]$

$$H^{-1}(\mathbb{T}_q)_{\varepsilon} = \mathrm{Ext}^0(E_1, E_2)$$

$$H^0(\mathbb{T}_q) = \mathrm{Ext}^1(E_1, E_2)$$

$$H^1(\mathbb{T}_q) = \mathrm{Ext}^2(E_1, E_2)$$

for $q^{-1}(E_1, E_2)$

Adeel Khan: $\exists\, (q^!)^{vir}$:

$$\begin{tikzcd}
H_i^{BM}(\mathrm{Coh}(1) \times \mathrm{Coh}(1)) \arrow[d, "(q^!)^{vir}"] \\
H^{BM}_{i+2\chi(E_1,E_2)}(\mathrm{Exact}) \arrow[d, "p_*"] \\
H^{BM}_{i+2\chi(E_1,E_2)}(\mathrm{Coh}(1))
\end{tikzcd}$$

$p_* \, q^{!vir}$

associative

CoHA

product.

Mellit – Minets – Schiffmann – Vasserot's

Definition of $H_0(S) := H_*^{BM}(\mathrm{Coh}_0(S), \mathbb{Q})$

stack of dimension – 0 sheaves

subalgebra

$H(S)$

$H_*^{BM}(\mathrm{Coh}(S), \mathbb{Q})$

$\mathrm{Coh}_0(S) = \coprod_{n \geq 0} \mathrm{Coh}_{0,n}(S)$.

__MMSV:__ generators + relations

__Theorem:__ $H_0(S)$ is generated as an algebra by $H_*^{BM}(\mathrm{Coh}_{0,1}(S), \mathbb{Q})$

$$\| $$

$$H_*^{BM}(\underbrace{S \times B\mathbb{C}^*}_{\text{smooth stack of dimension 1}}, \mathbb{Q})$$

- $H^*(B\mathbb{C}^*, \mathbb{Q}) = \mathbb{Q}(u)$ $\deg u = 2$.

$(S \times B\mathbb{C}^*) \in H_*^{BM}(S \times B\mathbb{C}^*, \mathbb{Q})$

$$H^i(S \times B\mathbb{C}^*, \mathbb{Q}) \cong H^{BM}_{2-i}(S \times B\mathbb{C}^*, \mathbb{Q})$$

$$\to \; - \cap [S \times B\mathbb{C}^*]$$

lives in degree $\leq 2$ in $\mathbb{Z}$.

$$H^*(S, \mathbb{Q})[u]$$

basis: $\lambda_i u^n$

$\lambda_i$: basis elt for $H^*(S, \mathbb{Q})$

$$T_n(\lambda) := \lambda u^n \cap [S \times B\mathbb{C}^*].$$

Basis for $H^{BM}_*(\mathrm{Coh}_{0,1}(S))$.

$H^0(S)$: Deformed $W_{1+\infty}$ algebra

null-def.

$W^+$