

Week 3, Hulya

What is the CohA of a surface?

Plan:

- * Review CohAs.
- * Borel-Moore homology of stacks, virtual pullbacks.

↑
Kapranov-Vasserot arXiv: 1901.07641, §3-§4.
= Aded Khan arXiv: 19

- * Calculation of CohA for \mathcal{O} -divisor sheaves

↑
Mellit-Minets-Schiffmann-Vasserot arXiv: 2311.13417

1. S a smooth, connected, quasi-projective surface.

$A = \text{Coh}_c S$, abelian category of coherent sheaves on S with proper support.

$\mathcal{M} = \overline{\text{Coh}}(S)$: moduli stack of objects in A .

↑
Singular; not a smooth Artin stack in general.

Let $E \in \text{Coh}(S) \Rightarrow$ def theory of E

i) governed by Ext group, $\text{Ext}^i(E, E)$

$$\text{Ext}^0(E, E) = \text{Hom}(E, E) = T_1 \text{Aut}(E).$$

$\text{Ext}^1(E, E)$ = space of first order deformations of E .

If S were a curve then

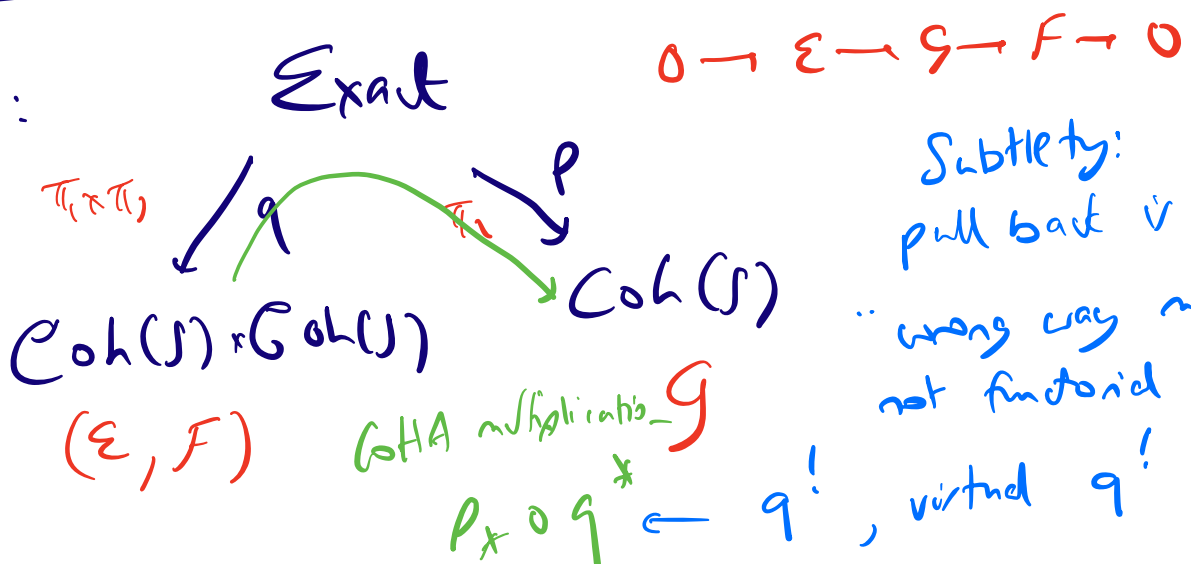
$\text{Coh}(S)$ would be a smooth Artin stack
 of dimension $\dim \text{Ext}^1(E, E) - \dim \text{Ext}^0(E, E)$
 $= -\chi(E, E).$

Generally, $\chi(E, E) = \sum_{i=0}^{\dim S} (-1)^i \dim \text{Ext}^i(E, E)$

S surface: $\text{Ext}^2(E, E) \neq 0$, $\dim \text{Ext}^1(E, E)$ can vary.

Goal: Define $H_*^{\text{BM}}(\text{Coh}(S))$

Recall:



Subtlety:
 pull back is a
 "wrong way map,"
 not functorial in
 homology

$$H_*^{B^m}(\text{Coh}(S)) = H_*^{-*}(f_* \mathcal{W}_{\text{Coh}(S)})$$

Goal: $f: \text{Coh}(S) \rightarrow *$ projection.

$$\mathcal{W}_{\text{Coh}(S)} \in \mathcal{D}_{\text{con}}(\text{Coh}(S)) \xleftarrow{\text{Derived category of constructible complexes (if-else)}}$$

$$\downarrow F_x$$

$$\mathcal{D}_{\text{con}}(\text{Spec } \mathbb{C})$$

First goal: explain $\mathcal{D}_{\text{con}}(\text{Coh}(S))$

first for sheaves, then for stacks.

Let X be a "reasonable" top space.

Recall: Constructible sheaf \mathcal{F} on a space X with a stratification \mathcal{S} is a sheaf which is locally constant on each stratum.

$$X = \bigsqcup_{i \in I} X_i \quad X_i \text{ locally const. sheaf}$$

Complex of sheaves, of \mathbb{Q} -vector spaces
 $F^\bullet = \dots \rightarrow F_1 \xrightarrow{d_1} F_2 \xrightarrow{d_2} F_3 \rightarrow \dots$
 $d_i \circ d_{i-1} = 0$

$H^i(F^\bullet)$ is a constructible sheaf.
of \mathbb{Q} -vector spaces

Constructible complexes of sheaves } triangulated category

Complexes of constructible sheaves.

Enhanced to a
dg-category or stable ∞ -category
(Adeel Khan).

$D_{\text{con}}(\mathbb{N})$ defined by Verdier

say: underlying complex analytic space of a sphere.

— Gives general set up to divisors Poincaré Duality.

6 functor families

$$f: X \rightarrow Y$$

- 1) $R f_*$: $D_{(2)}(X) \rightarrow D_{(2)}(Y)$
- 2) $L f^*$: $D_{(2)}(Y) \rightarrow D_{(2)}(X)$
- 3) $R f_!$: $D_{(2)}(X) \rightarrow D_{(2)}(Y)$
- 4) $L f^!$: $D_{(2)}(Y) \rightarrow D_{(2)}(X)$
- 5) Hom
- 6) \otimes

Let $f: X \rightarrow *$, \mathbb{Q}_X unit Sect $\cong X$.

$$H^i(X, \mathbb{Q}_X) = H^i(f_* \mathbb{Q}_X)$$

* Compatibility between $f^* \dashv f_*$ adjoint

$$\text{Hom}(f^* F, G) \cong \text{Hom}(F, f_* G)$$

$$\forall F \in D_{(2)}(Y), G \in D_{(2)}(X).$$

$f_!$: pull forward with compact support.

$$F: \text{sheaf} \rightarrow X, \quad F: X \rightarrow Y$$

(Classical
version)

Our f_*

$f_!$ are

right
derived

versions of f
then.

$$(f_* F)(U) = F(F^{-1}(U))$$

✓

$$(f_! F)(U) = \text{all sections over } U \text{ whose support is proper over } U.$$

for complexes of sheaves, there are
derived versions of these operations.

$$Q: f: X \rightarrow \text{pt.}$$

$$H_{cs}^i(X, \mathcal{Q}_X) = H^i(f_!(\mathcal{Q}_X)).$$

Let $f: X \rightarrow Y$ with some conditions.

$$\exists! f^!: \mathcal{D}_{cs}(Y) \rightarrow \mathcal{D}_{cs}(X)$$

st. $(f_!, f^!)$ are adjoint functors.

Ex. Let $f: X \rightarrow pt$

Dualizing complex: $\omega_X = f^!(\mathbb{Q}_{pt}) \in \mathcal{D}_{con}(X)$

Let F complex of sheaves on X .

$$R\text{Hom}(f_! F, \mathbb{Q}_{pt}) = R\text{Hom}(F, \underbrace{f^!(\mathbb{Q}_{pt})}_{\omega_X})$$

$$\cong (H_{con}(X, F))^{\vee}$$

$$R\Gamma(R\text{Hom}(F, \omega_X))$$

$D F =$
vector dual to F .

$$\Rightarrow H_{con}^i(X, F)^{\vee} = H^i(X, D(F))$$

Ex: Let X orient and of dim n .

or see over

$$\omega_X \cong \mathbb{Q}_X(n) \quad F = \mathbb{Q}_X \Rightarrow D(F) = R\text{Hom}(\mathbb{Q}_X, \mathbb{Q}_X(n)) = \mathbb{Q}_X(n)$$

$$\Rightarrow H_c^i(X, \mathbb{Q})^v = H^{n-i}(X, \mathbb{Q}).$$

Poincaré duality.

For any X ,

$$H_c^i(X, \mathbb{Q})^v = H^{n-i}(X, \mathbb{Q}_X)$$

\parallel
 $H_i^{B\mathbb{M}}(X, \mathbb{Q})$

Now let X be an Artin stack.

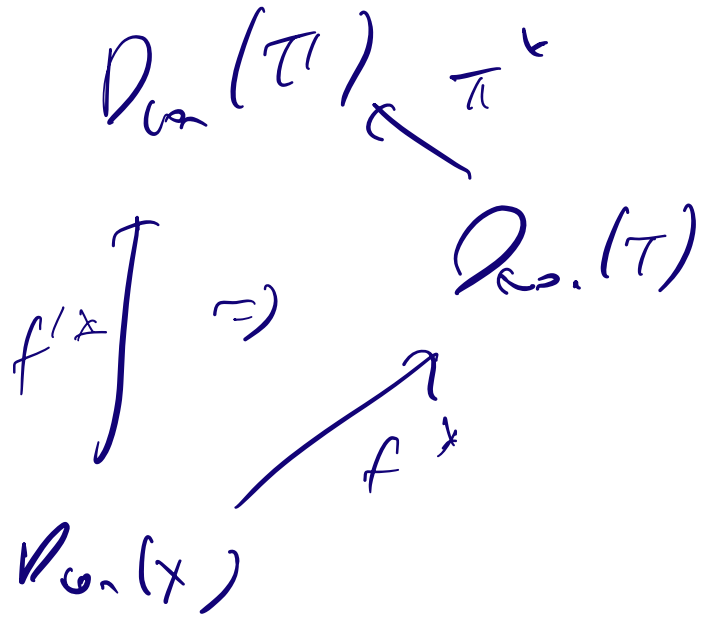
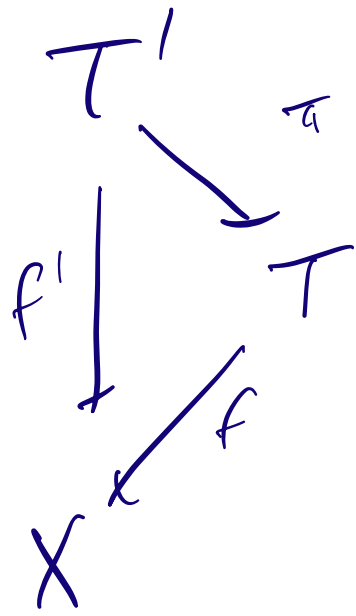
$$D_{\text{ét}}(X) := \varprojlim^{\text{pro}} D_{\text{ét}}(T)$$

Doesn't make sense for triangulated categories.

$$T \rightarrow X \quad \text{arbitrary map}$$

T scheme of finite type over \mathbb{C}

Need to work with dg-categories or stable ∞ -categories.



Can do de Rham
adjoint functors

$$(f_*, f^*) \\
 (f_!, f^!)$$

$$H_c^i(X, \mathbb{Q}) := H^i(f_! \mathbb{Q}_X)$$

$$H_i^{\text{Borel}}(X, \mathbb{Q}) = \pi^{-i}(X \cup X)$$

$i \in \mathbb{Z}$ or $i \in \mathbb{N}$

for itaker: U_X : can now
be interpreted in
this sense

Cotta product

$$\begin{array}{ccc}
 & & \text{Exact} \\
 & \searrow q & \searrow p \\
 & & \text{GL}(S)
 \end{array}$$

$$m: H_x^{Dn}(\text{GL}(N)) \times H_x^{Dn}(\text{GL}(N))$$

$$\begin{array}{c}
 \parallel \\
 P_x \circlearrowleft q! \\
 \uparrow \\
 \text{Subtleher}
 \end{array}$$

$$\begin{array}{c}
 \downarrow \\
 H_x^{Dn}(\text{GL}(N))
 \end{array}$$

p: paper map.

Working with derived stacks

$\exists P_x$ for paper map,

$q!$: not smooth up below stack,
 so quasi-smooth.

"Relative target complex T_g^* "

Perfect of amplitude $(-1, 1)$

$$H^i(T_g^*)_{\mathbb{Z}} = \text{Ext}^0(\mathbb{Z}_i, \mathbb{Z}_i)$$

$$H^0(T_g) = \text{Ext}^1(\mathbb{Z}_i, \mathbb{Z}_i)$$

$$H^1(T_g) = \text{Ext}^2(\mathbb{Z}_i, \mathbb{Z}_i)$$

for $g^i(\mathbb{Z}_i, \mathbb{Z}_i)$

Ased Kha:

$$\mathcal{D}(g^i)^{u_i}: H_{i+2}^{pm}(\text{Coh}(1) \times \text{Coh}(1))$$

$\downarrow (g^i)^{u_i}$

$$H_{i+2}^{pm}(\text{Ext}^2)$$

$\downarrow \rho_a$

$$H_{i+2}^{pm}(\mathbb{Z}(\mathbb{Z}_i))$$

ρ_a g^i u_i
 ac-cubus
 C-HA
 fadant.

Mellit - Mietz - Schiffman - Venkates

Definition of $H_0(S) := H_x^{B_m}(\text{Coh}_0(S), \mathbb{Q})$

subalgebra



stack of dimension 0 sheaves

$$\text{Coh}_0(S) \cong \coprod \text{Coh}_{0, \alpha}(S)$$

$H(S)$

$$H_x^{B_m}(\text{Coh}(S), \mathbb{Q})$$

MMIV: generators + relations

Theorem: $H_0(S)$ is generated as an algebra by $H_x^{B_m}(\text{Coh}_{0,1}(S), \mathbb{Q})$

$$H_x^{B_m}(S \times B\mathbb{C}^*, \mathbb{Q})$$

smooth stack of dimension 1

• $H^*(B\mathbb{C}^*, \mathbb{Q}) = \mathbb{Q}(u)$ deg $u = 2$.

$$(S \times B\mathbb{C}^*) \in H_x^{B_m}(S \times B\mathbb{C}^*, \mathbb{Q})$$

$$H^i(S \times \mathbb{P}^k, \mathbb{Q}) \cong \bigoplus_{2-i}^{2m} H_{2-i}^{2m}(\mathbb{P}^k, \mathbb{Q})$$

$$H^i(S, \mathbb{Q}) \cong \bigoplus_{2-i}^{2m} H_{2-i}^{2m}(\mathbb{P}^k, \mathbb{Q})$$

(live in degree $\leq 2i - 2$.)

↑
basis: $\lambda_i u^n$

λ_i basis elt for $H^i(S, \mathbb{Q})$

$$T_n(\lambda) := \lambda u^n \cap (S \times \mathbb{P}^k).$$

Basis for $H_x^{2m}(\text{Coh}_{0,1}(U))$.

$H^0(U)$: Defomed Witt algebra
w/ ω^*