Chenjing Bu. Week 6

Dimensional reduction

$$\mathcal{X} \quad T^*(1)\,\mathcal{Y}, \qquad \text{oriented } (-1) \text{ shifted symplectic structure.}$$

$$\downarrow$$

$\mathcal{Y}$ quasi-smooth derived Artin stack over $\mathbb{C}$

(+ assumption)

Examples: (1) $\mathcal{Y} = \mathrm{Coh}_c(S)$ S surface

$\mathcal{X} = \mathrm{Coh}_c(K_S)$ with CY3 structure on K_S

(2) Q quiver.

Q_3

$$\begin{tikzcd} \bullet \arrow[loop left, "e_1"] \arrow[r, shift left, "a"] & \bullet \arrow[loop above, "e_2"] \arrow[l, shift left, "a^*"] \arrow[r, shift left, "b"] & \bullet \arrow[loop above, "e_3"] \arrow[l, shift left, "b^*"] \end{tikzcd}$$

CY2 | Π_Q : preprojective algebra

Q_2 relations: $\sum [a, a^*] = 0$

$(Q_2, \text{relations})$ (e.g. $\cancel{a^*a = 0}$ $a^*a - b^*b = 0$, $bb^* = 0$)

$$\mathcal{Y}\text{-Rep}(\Pi_Q) \simeq T^* \operatorname{Rep} Q$$

0-shifted symplectic
$\Rightarrow$ quasi-smooth.

$$\mathcal{X} = \operatorname{Rep}(Q_3, W)$$

$$W = \sum_{a \text{ edge of } Q} (a, a^*) \cdot \sum_i \ell_i$$

add ℓ_i | add a^*

$$\simeq T^*(-1)\, T^* \operatorname{Rep} Q$$

$$\simeq T^*(-1)\, T(-1) \operatorname{Rep} Q$$

add a^* | add ℓ_i

E.g. (local model for everything).

$V \circlearrowleft G$ vector bundle, G-equivariant.

$p \downarrow \big) s$ G-equivariant section.

smooth $\mathbb{C}$-scheme $\to \mathcal{U} \circlearrowleft G$ affine algebraic $\mathbb{C}$-group

U_y

$$Y = s^{-1}(0)$$

$$\mathcal{Y} = Y/G.$$

$$\mathcal{X} = T^*(-1)\mathcal{Y}:$$

? Good local model if there are enough points with reductive stabilizers, e.g. if there exists a good moduli space.

$$\begin{tikzcd} V^* \arrow[r] \arrow[d, "f"] & U \\ \mathbb{A}^1 = \mathbb{C} & \end{tikzcd} \qquad f(u, \alpha) = \alpha(s|_u)$$

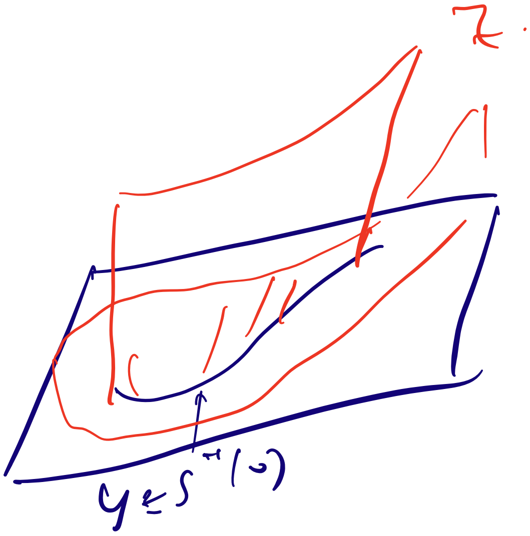
$$X = \mathrm{Crit}(f)$$

$$\mathcal{X} = X/G = T^*(-1)\mathcal{Y}.$$

$$Z = f^{-1}(0).$$

At $u \in U$, fibers of X, Y are

$$\begin{cases} \text{if } s(u) \neq 0, & Y_u = \phi,\ Z_u = s(u)^{\perp} = \mathbb{A}^{\mathrm{rk} V - 1},\ X_u = \phi. \\ \text{if } s(u) = 0, & Y_u = *,\ Z_u = s(u)^{\perp} = \mathbb{A}^{\mathrm{rk} V},\ X_u \neq \phi. \end{cases}$$



$\mathcal{U}$

Dimensional reduction

(Davison, local model

Kinjo, global version as constructible sheaf

Kinjo–Koseki) as mixed Hodge modules.

$$\mathcal{X} = T^*(-1)\mathcal{Y} \quad \mathbb{Q}_{\mathcal{X}} = \text{DT perverse sheaf or Mixed Hodge module.}$$

$$\begin{array}{c}\mathcal{X}\\ \downarrow \pi\\ \mathcal{Y}\end{array}$$

$$\pi_*(\mathbb{Q}_{\mathcal{X}}) \cong \underbrace{\mathbb{D}\mathbb{Q}_{\mathcal{Y}}}_{\text{dualizing complex } \omega_{\mathcal{Y}}} \otimes \mathbb{L}^{\operatorname{vdim} \mathcal{Y}/2}$$

Verdier dual

Also.

$$\pi_!(\mathbb{Q}_{\mathcal{X}}) \cong \mathbb{Q}_{\mathcal{Y}} \otimes \mathbb{L}^{-\operatorname{vdim}\mathcal{Y}/2}$$

$\mathbb{L} = H_c^*(\mathbb{A}^1)$. Has degree 2

$\mathbb{L}^{1/2}$ has degree 1. (i.e. in H^1)

At level of constructible sheaves, $\mathbb{L} = \mathbb{Q}(-1)$

$\mathbb{L}^{1/2} = \mathbb{Q}(-1)$

— If $\mathcal{Y}$ is smooth, $\mathbb{P}\mathbb{Q}_{\mathcal{Y}} =$

$$\mathbb{Q}_{\mathcal{Y}}[2\dim_{\mathbb{C}} \mathcal{Y}]$$

$$\mathbb{P}\mathbb{Q}_{\mathcal{Y}} \otimes \mathbb{L}^{\mathrm{vdim}\,\mathcal{Y}/2}$$

$$= \mathbb{Q}_{\mathcal{Y}}(\dim_{\mathbb{C}} \mathcal{Y})$$

In particular (by $*$ pushing forward to a point)

$$\underbrace{H^*_{\mathrm{crit}}(\mathfrak{X})}_{\text{3-d CoHA.}} = \mathbb{H}^*(\varphi_{\mathfrak{X}})$$

$\circledast$

$$\underbrace{H^{BM}_{-*}(\mathcal{Y};\mathbb{Q})}_{\text{2d CoHA.}}[-\mathrm{vdim}\,\mathcal{Y}]$$

dual vector space statement

$$H^*_{\mathrm{crit},c}(\mathfrak{X}) \cong H^*_c(\mathcal{Y})(\mathrm{vdim}\,\mathcal{Y})$$

Aside: Remark: Let $\mathcal{X}$ be moduli stack of objects in a 3CY abelian category + generic stability (or zero Euler form). Then:

$$\boxed{H^*_{\mathrm{crit}}(\mathcal{X}) = \mathrm{Sym}\left(H^*_{\mathrm{BPS}}(\mathcal{X}) \otimes \mathbb{Q}[t]\right)}$$

Proof path: 1d $\longrightarrow$ 3d $\longrightarrow$ 2d

smooth $\qquad \mathcal{X} \qquad \mathcal{Y}$

Proof idea for $(*)$:

Reduce to local model (a $\mathcal{Y}$) with V: trivial vector bundle (non-trivial G action)

$$\begin{array}{c} U \times \mathbb{A}^n \\ \downarrow \uparrow s \\ U \end{array} \qquad \mathcal{Y} = s^{-1}(0), \quad X = \mathrm{Crit}\, f$$

$$f(u, v) = \langle s(u), v \rangle$$

$u \in U$, $v \in \mathbb{A}^n$.

$$H^*_c(\phi_f \mathbb{Q}_V) = H^*_c(\underbrace{\{f=0\}}_{Z}, \cdot) \Big/ H^*_c\{f=1\}$$

for proof to work

(lose in derived category)

Z has two parts:

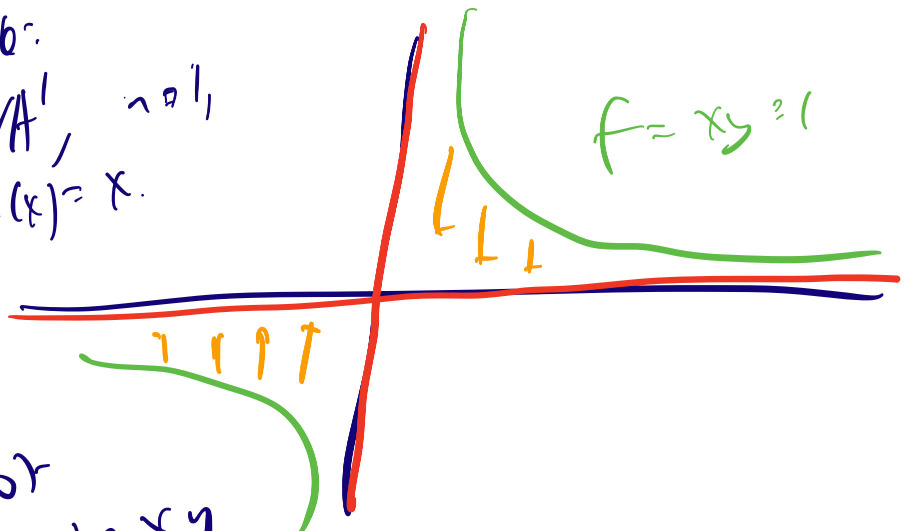
$\{f=0\} \cong \{s=0\} \times \mathbb{A}^n \quad \cup (s \neq 0) \times \mathbb{A}^{n-1}$

$(f=1) = ((s \neq 0) \times \mathbb{A}^{n-1})$

Example:

$U = \mathbb{A}^1$, $n=1$,

$s(x) = x$.



$Z = xy = 0$

$Y = \mathbb{A}^2$

$f(x,y) = xy$

$$\simeq H_c(S \times \mathbb{A}^n)$$

$$\simeq H_c^*(Y)(-2n)$$

$\pi : V^\vee \to U$

$\underline{\text{Sei}}$ / pose $\quad \pi_! \varphi_f \mathbb{Q}_{U \times \mathbb{A}^n} = \mathbb{Q}_Y(-2n)$

$\pi_! = \mathbb{D} \pi_* \mathbb{D}$

$\mathbb{D} \varphi_f = \varphi_f$

Apply Verdier dual $\mathbb{D}$

$\Rightarrow \quad \pi_* \varphi_f = \mathbb{D} \mathbb{Q}_Y(n - \dim U)$ $\square$

Hence: we can use the 3-d CoHA to define the 2-d CoHA. Claim by Kinjo, proof not available yet:

(Davison, Kinjo-Park-Safronov) /

– differ by a sign from Kapranov-Vasserot 2d CoHA.

Where sign difference comes from:

$$\begin{tikzcd}
H^*_{crit}(\mathcal{X})^{\otimes 2} \arrow[r, "KPS", "\simeq"', "q"'] \arrow[d, "\wr"] \arrow[rd, bend left, "\text{Davison, Kinjo, ...}"] & H^*(\mathcal{X}^{exact}, p^! \phi_{\mathcal{X}}) \arrow[d, "p_!"] \arrow[ddl, leftrightarrow, "\text{not directly related.}"] \\
H^{BM}_{-*}(\mathcal{Y})^{\otimes 2}(\cdot \sim) \arrow[d, "q^*"] \arrow[rd, bend right, "\text{Kapranov–Vasserot}"'] & H_{crit}(\mathcal{X}) \arrow[d, "\wr"] \\
H^{BM}(\mathcal{Y}^{exact}) \arrow[r, "p^!"] & H^{BM}(\mathcal{Y})
\end{tikzcd}$$

"Joyce conjecture"

Not clear diagram commutes.

BPS Cohomology

Assumptions: $\mathfrak{X}$ −1 shifted symplectic

Artin stack over $\mathbb{C}$, * good moduli space

* symmetric (generic stability) else from 0.

cohomology of $B\mathbb{G}_m$.

$\in$

$\otimes \mathbb{Q}(t)$

$$H^*_{BPS}(\mathfrak{X}) \subset H^*_{cit}(\mathfrak{X})$$

$$\cong \qquad\qquad\qquad \cong$$

$$H^*_{BPS}(\mathcal{Y}) \subset H^{BM}_{-*}(\mathcal{Y})(\ldots)$$

if $\mathfrak{X} = T^*(-1)\mathcal{Y}$.

$$\mathfrak{X} \xrightarrow{\pi} X$$

here $\approx (*/\mathbb{G}_m)$ factor.

["Symmetry" in 2-d case: if $\chi(\alpha,\beta) \neq \chi(\beta,\alpha)$ the $\mu(\alpha) \notin N(\beta)$]

Define $\mathcal{Z} : {}^p\mathcal{H}^1(\pi_* \varphi_{\mathfrak{X}})$

$$\mathfrak{X} \xrightarrow{\pi} X$$

good moduli space

$$H_{BPS}(\mathfrak{X}) = H^*\left({}^p\mathcal{H}^1(\pi_*(\varphi_{\mathfrak{X}}))\right)$$

$$\Rightarrow \quad \begin{tikzcd}
\mathrm{Sym}\left(\bigoplus_{\gamma \in \pi_0(\mathfrak{X}) \setminus 0} H^*_{BPS}(\mathfrak{X}_\gamma) \otimes \mathbb{Q}(t)\right) \arrow[d, "\cong", "\text{CoHA multiplication}"'] \arrow[dl, "\text{iso of algebras}"'] & \\
\mathrm{gr}_P H_{ct}(\mathfrak{X}) \arrow[r] & H^*_{ct}(\mathfrak{X})
\end{tikzcd}$$

vector space iso.

or better:

$\mathrm{gr}_P H_{ct}(\mathfrak{X})$ graded object of perverse filtration

– To construct $H^*_{crit}\big(\mathcal{M}_{\text{everything}}\big)$ $:= A$

by choosing a generic stability condition:

$$H^*_{BPS}\left(\mathfrak{X}_\gamma^{\tau_0\text{-ss}}\right)$$

$$\cong \bigoplus_{\gamma=\gamma_1+\dots+\gamma_n} H_{BPS}\left(\mathfrak{X}_{\gamma_1}^{\tau_+\text{-ss}} \times \dots \times \mathfrak{X}_{\gamma_n}^{\tau_+\text{-ss}}\right)$$

if $\tau_\pm, \tau_0$ all generic wrt

More general: $H^*_{crit}(\mathfrak{X})$ ← Don't need generic stability..

T CoHA

$$= \bigotimes_{\text{slope } N} \mathrm{Sym}\left(\bigoplus_{\substack{\gamma \\ \text{slope}(\gamma)=N}} H^*_{BPS}\left(\mathfrak{X}_\gamma^{\tau\text{-ss}}\right) \otimes \mathbb{Q}[t]\right)$$

Davison — Hennecart — Schlegel Mejia

2 papers — BPS cohomology of 2CY cat.