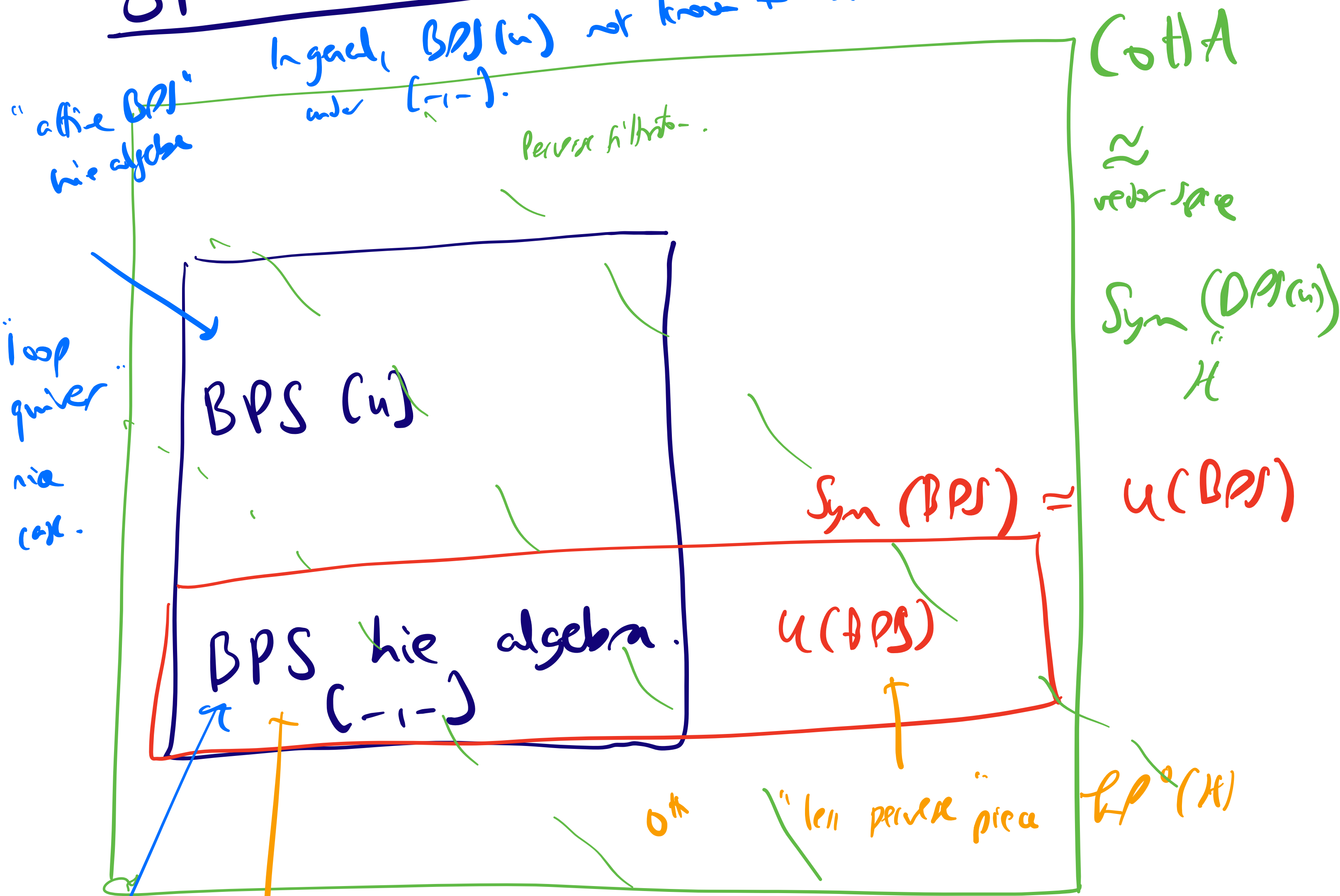
Week 7. Chenjing.

Structure of CoHA

In general, BPS(u) not known to be closed under [-,-].



BPS:
* closed under [-,-] in CoHA.
* Lie algebra = BPS is commutator

⇒ U(BPS) → CoHA
algebra morphism.

loop quiver: only case in which everything is known + explicit: preprojective algebra of loop quiver. $\bullet \circlearrowright$ $\rightsquigarrow$ coh sheaves on $\mathbb{A}^2$.

[Quivers with no relations: CoHA is commutative, not interesting.]

Set up: $\mathcal{Y}$: quasi-smooth stack. (take to be $\mathrm{Rep}(\Pi_Q)$.) $\rightarrow$ $\mathrm{coh}_c(\mathbb{A}^2)$ loop quiver

$$\mathcal{X} = T^*(-1)\mathcal{Y} \quad (\mathrm{Rep}\, Q, W) \rightarrow \mathrm{coh}_c(\mathbb{A}^3) \text{ loop quiver}$$

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\pi_{\mathcal{Y}}"] \arrow[d, dash] & Y \arrow[d, dash, "p"] \\ \mathcal{X} \arrow[r, "\pi_{\mathcal{X}}"] & X \end{tikzcd}$$

good moduli spaces

$\mathcal{Y}$: 2d CoHA

$\mathcal{X}$: 3d CoHA

BPS: lives in 2d $\mathcal{Y}$ CoHA, but defined using $\mathcal{X}$ 3d CoHA

$$\mathrm{BPS}_X = {}^p\mathcal{H}^1\big((\pi_{\mathcal{X}})_*(\varphi_X)\big)$$

Support Lemma ✓ Needs CY 2 $\quad BPS_X \simeq BPS_{\mathcal{Y}} \otimes \mathbb{Q}_{\mathbb{A}^1}$!

BPS sheaf is supported on image of $\mathbb{A}^1 \times \mathcal{Y}$ in $\mathcal{X}$

using $\mathbb{A}^1 \curvearrowright \mathcal{Y} \to X$ induced by $\mathbb{A}^1 \times \mathcal{Y} \to \mathcal{X}$.

fibre of $\mathcal{X} \to \mathcal{Y}$ like $\mathfrak{g}/G \to */G$

$G = \mathrm{Aut}(* \text{ in } \mathcal{Y})$

$X \to Y$ like $\mathfrak{g}/\!/G \to *$

$\cup$

$Z(\mathfrak{g})/\!/Z(G)$

$\shortparallel$

$Z(\mathfrak{g})$

$\cup$

$\mathbb{A}^1.$

at $E \in \mathcal{Y}$

$\mathrm{Hom}(E,E)/\mathrm{Aut}(E)$

$\boxed{\text{Require } \mathcal{Y} \text{ Calabi-Yau } 2}$

Aut^*

$\mathcal{Y}:$ Ext^2, Ext^1, Aut

$\mathcal{X}:$ Ext^2, $\mathrm{Ext}^{1\,*}$, Ext^1, $\boxed{\mathrm{Ext}^{2\,*}} = \mathrm{Hom}$, Aut

$\uparrow$

$\mathbb{A}^1$.

Another perverse sheaf on Y:

$$\mathbb{D}Q_y := {}^p\mathcal{H}^0((\pi_y)_* \mathbb{D}Q_y)$$

$\uparrow$

less perverse. filtration of CoHA.

Claim: ${}^p\mathcal{H}^0 \Rightarrow$ subspace of CoHA.

Davison — Hennecart — Schlegel Mejia

2 papers.

$$* \quad \mathcal{L}P^0 = H^*({}^p\mathcal{H}^0((\pi_y)_* \mathbb{D}Q_y))$$

$$\cap$$

$$\text{CoHA}.$$

Thm: $\mathcal{L}P^0 = U(\text{BPS})$ in 2CY setting.

(Thm for general 2CY categories)

— Generic subalgebra of the CoHA is a universal enveloping algebra.

* BPS Lie algebra explicitly computed: generalized Kac–Moody algebra

Special to type A.

$(K^{num}(2\text{-cycles}), \chi$ Euler pairing, T of 2-cycles $)$

symmetric

+ generators: $(\mathcal{JC}_{y,d,n})$ of $\pi_0(y)$.

K^{num} / A

n: hyperbolic index, d: divisibility

<u>GKM:</u> $\dfrac{\text{free Lie algebra}}{\text{Serre relations}}$ $ad(\cdots)^{(\cdots)} = 0$

— on the vector space of generators

<u>Memo:</u> they are contributions of d,

sum of $\mathbb{C}^{\oplus n}$, $\mathbb{C}$ is degree d/n.

Affine BPS Lie algebra

(only known for loop quivers Q)

$BPS(Q) \subset CoHA$ Lie subalgebra.

$CoHA = U(BPS(Q))$ (when it works)

[— Alg form equivariant CoHA, e.g. for $\mathbb{A}^2$.

— known that form equivariant CoHA is not the universal enveloping algebra of any Lie algebra).

coproduct Δ on CoHA.

Constructed using a coproduct Δ on CoHA.

= Joyce vertex algebra $\to$ $H_* = ((\mathrm{CoHA})^*$

↑

Or: proof that BPS(n) is closed under coproduct.

For loop quiver: Joyce VA is singular:

$\leftarrow$ commutative algebra with derivation.

↑

commutative in this case.

⇕

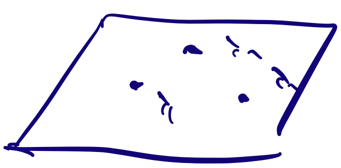
E_2 algebra given by eigenvalue decomposition along the loop.

$$\text{eig}: \mathcal{Y} \longrightarrow \mathrm{Sym}^*(\mathbb{A}^1)$$

$$\downarrow$$

$$\mathrm{Ran}(\mathbb{A}^1)$$

factorization structure.



over $\lambda_1, \lambda_2, \lambda_3$:

$\mathcal{Y}$ splits into pieces $\mathcal{Y}_i$.

— When you have this coproduct, can apply Milnor–Moore

$\Rightarrow$ CoHA $\mathcal{H} = U($primitive part of $\Delta)$.

=: affine BPS Lie algebra.

— deduce primitive part is closed under commutator of CoHA.