The Pandharipande-Thomas rationality conjecture for superpositive curve classes

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Based on arXiv:2111.04694 and arXiv:2604.05664,

joint work with Reginald Anderson.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`.

Introduction

Let X be a smooth projective 3-fold over $\mathbb{C}$. A *Pandharipande–Thomas* (PT) *stable pair* (F, s) is a pure 1-dimensional coherent sheaf F on X together with a section $s : \mathcal{O}_X \to F$ with 0-dimensional cokernel. We allow $F = s = 0$. We say that (F, s) has *class* $(\beta, n) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$ if $\mathrm{Pd}(c_2(F)) = \beta$ and $\chi(F) = n$. Then either $\beta = n = 0$, or β is an effective curve class. By Pandharipande–Thomas arXiv:0709.3823, there is a proper moduli scheme $P_n(X, \beta)$ of PT pairs in class (β, n), which has a perfect obstruction theory, and a virtual class $[P_n(X, \beta)]_{\mathrm{virt}}$ in $H_{2c_1(X)\cdot\beta}(P_n(X, \beta), \mathbb{Z})$. There is a *universal sheaf* $\mathfrak{F} \to X \times P_n(X, \beta)$ with $\mathfrak{F}|_{X\times\{[F,s]\}} \cong F$. For all $k \in \mathbb{N}$ and $\gamma \in H^l(X, \mathbb{Q})$, define $\tau_k(\gamma) \in H^{2k+l-2}(P_n(X, \beta), \mathbb{Q})$ by

$$\tau_k(\gamma) = (\Pi_{P_n(X,\beta)})_*\big(\Pi_X^*(\gamma) \cup \mathrm{ch}_{2+k}(\mathfrak{F})\big).$$

We have $P_n(X, \beta) = \emptyset$ if $n \ll 0$.

Let $k_i \in \mathbb{N}$, $\eta_i \in H^{l_i}(X,\mathbb{Q})$ for $i=1,\ldots,m$ with $\sum_{i=1}^m (2k_i + l_i - 2) = 2c_1(X)\cdot\beta$. Define the *Pandharipande–Thomas invariant*

$$PT_{\beta,n}\bigl(\textstyle\prod_{i=1}^m \tau_{k_i}(\eta_i)\bigr) = \bigl(\textstyle\prod_{i=1}^m \tau_{k_i}(\eta_i)\bigr)\cdot[P_n(X,\beta)]_{\rm virt} \quad \text{in } \mathbb{Q}.$$

Combine these into a generating function

$$PT_\beta\bigl(\textstyle\prod_{i=1}^m \tau_{k_i}(\eta_i),q\bigr) = \sum_{n\in\mathbb{Z}} PT_{\beta,n}\bigl(\textstyle\prod_{i=1}^m \tau_{k_i}(\eta_i)\bigr)q^n \quad \text{in } \mathbb{Q}[[q]][q^{-1}].$$

Conjecture 1 (Pandharipande–Thomas 2007, Pandharipande 2017.)

$PT_\beta\bigl(\prod_{i=1}^m \tau_{k_i}(\eta_i),q\bigr)$ *is the Laurent expansion of a rational function* $F(q)$, *which has poles only at* $q=0$ *and at roots of unity.*

This is known in several cases including X Calabi–Yau and X toric by work of Bridgeland, Toda, and Pandharipande–Pixton.

Pandharipande–Thomas invariants are curve-counting invariants. They are conjecturally equivalent to Gromov–Witten invariants, and rank 1 Donaldson–Thomas invariants of X, by explicit formulae.

Definition

An effective curve class $\beta \in H_2(X,\mathbb{Z})$ is called *positive* if $c_1(X)\cdot\beta > 0$, and *superpositive* if whenever $\beta = \beta' + \beta''$ with β', β'' effective then β', β'' are positive.
If X is a Fano 3-fold then all curve classes are superpositive.

The goal of this talk is to explain:

Theorem 1 (Anderson–Joyce arXiv:2604.05664.)

Conjecture 1 holds if β is a **superpositive** *curve class.*
If a linear algebraic $\mathbb{C}$-group G acts on X then the analogue holds for G-equivariant Pandharipande–Thomas invariants.

Karpov–Moreira independently proved the first part of Theorem 1 in arXiv:2604.06023, released on the arXiv on the same day by agreement. They also prove a $\mathbb{Z}_2$-symmetry of the PT generating function, which we do not prove for technical reasons.

The proof of Theorem 1 is an application of my theory of enumerative invariants of abelian categories in homology in arXiv:2111.04694. Define an abelian category $\mathcal{A}$ to have objects (F,V,ϕ) where $F\in\mathrm{coh}_{\leqslant 1}(X)$ is a coherent sheaf of dimension $\leqslant 1$, and V is a finite-dimensional vector space over $\mathbb{C}$, and $\phi:\mathcal{O}_X\otimes_{\mathbb{C}}V\rightarrow F$ is a morphism in $\mathrm{coh}(X)$. By wall-crossing between two different stability conditions on $\mathcal{A}$, I prove an identity relating PT virtual classes $[P_n(X,\beta)]_{\mathrm{virt}}$ and Donaldson–Thomas invariants $[\mathcal{M}^{\mathrm{ss}}_{(\beta,n)}(\tau)]_{\mathrm{inv}}$ counting 1-dimensional sheaves. We need β superpositive for $[\mathcal{M}^{\mathrm{ss}}_{(\beta,n)}(\tau)]_{\mathrm{inv}}$ to be well defined.

In work with Reginald Anderson, the Pandharipande–Thomas rationality conjecture is then deduced from a periodicity property of the $[\mathcal{M}^{\mathrm{ss}}_{(\beta,n)}(\tau)]_{\mathrm{inv}}$ in $n\in\mathbb{Z}$, coming from the fact that if $L\rightarrow X$ is the ample line bundle used to define semistability then $-\otimes L$ gives an isomorphism $\mathcal{M}^{\mathrm{ss}}_{(\beta,n)}(\tau)\rightarrow\mathcal{M}^{\mathrm{ss}}_{(\beta,n+c_1(L)\cdot\beta)}(\tau)$.

The method is similar to Bridgeland/Toda in the Calabi–Yau 3-fold case, but with a different invariants and wall-crossing set-up.

Let X be a smooth complex projective m-fold. Write $\mathcal{M}$ for the moduli stack of objects in $D^b\operatorname{coh}(X)$, a higher $\mathbb{C}$-stack. Now $[*/\mathbb{G}_m]$ is a group stack with an action $\Psi : [*/\mathbb{G}_m]\times\mathcal{M}\to\mathcal{M}$ which acts on isotropy groups $\mathrm{Iso}_{[*/\mathbb{G}_m]}(*)=\mathbb{G}_m$, $\mathrm{Iso}_{\mathcal{M}}([E^\bullet])=\mathrm{Aut}(E^\bullet)$ by $(\lambda,\alpha)\mapsto\lambda\,\mathrm{id}_{E^\bullet}\cdot\alpha$. The quotient $\mathcal{M}/[*/\mathbb{G}_m]$ is the 'projective linear' moduli stack $\mathcal{M}^{\mathrm{pl}}$ of objects in $D^b\operatorname{coh}(X)$. Points of $\mathcal{M}^{\mathrm{pl}}$ are isomorphism classes $[E^\bullet]$ of $E^\bullet$ in $D^b\operatorname{coh}(X)$, but isotropy groups are $\mathrm{Iso}_{\mathcal{M}}([E^\bullet])=\mathrm{Aut}(E^\bullet)/\mathbb{G}_m\cdot\mathrm{id}_{E^\bullet}$. Here $\mathcal{M}^{\mathrm{pl}}$ is often called the *rigidification* of $\mathcal{M}$. There is a projection $\Pi:\mathcal{M}\to\mathcal{M}^{\mathrm{pl}}$ which is a $[*/\mathbb{G}_m]$-fibration except over 0.

Work by me in 2017 makes $H_*(\mathcal{M},\mathbb{Q})$ into a *graded vertex algebra* over $\mathbb{Q}$. Then Borcherds makes $H_*(\mathcal{M},\mathbb{Q})/D(H_*(\mathcal{M},\mathbb{Q}))$ into a *graded Lie algebra*. I show that $H_*(\mathcal{M}^{\mathrm{pl}},\mathbb{Q})$ has the structure of a graded Lie algebra over $\mathbb{Q}$, and $\Pi_*:H_*(\mathcal{M},\mathbb{Q})\to H_*(\mathcal{M}^{\mathrm{pl}},\mathbb{Q})$ factorizes via a Lie algebra morphism $H_*(\mathcal{M},\mathbb{Q})/D(H_*(\mathcal{M},\mathbb{Q}))\to H_*(\mathcal{M}^{\mathrm{pl}},\mathbb{Q})$, which is an isomorphism except over complexes with Chern character zero.

Vertex algebras (don't try to understand this slide.)

Let R be a commutative ring. A *vertex algebra* over R is an R-module V equipped with morphisms $D^{(n)}: V \to V$ for $n = 0, 1, 2, \ldots$ with $D^{(0)} = \mathrm{id}_V$ and $v_n : V \to V$ for all $v \in V$ and $n \in \mathbb{Z}$, with v_n R-linear in v, and a distinguished element $\mathbb{1} \in V$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $v \in V$ then $\mathbb{1}_{-1}(v) = v$ and $\mathbb{1}_n(v) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(iii) If $v \in V$ then $v_n(\mathbb{1}) = D^{(-n-1)}(v)$ for $n < 0$ and $v_n(\mathbb{1}) = 0$ for $n \geqslant 0$.

(iv) $u_n(v) = \sum_{k \geqslant 0} (-1)^{k+n+1} D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.

(v) $$(u_l(v))_m(w) = \sum_{n \geqslant 0} (-1)^n \binom{l}{n} \bigl(u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w))\bigr)$$

for all $u, v, w \in V$ and $l, m \in \mathbb{Z}$, where the sum makes sense by (i).

We can also define *graded vertex algebras* and *vertex superalgebras*.

It is usual to encode the maps $u_n : V \to V$ for $n \in \mathbb{Z}$ in generating function form as R-linear maps for each $u \in V$

$$Y(u,z) : V \longrightarrow V[[z,z^{-1}]], \quad Y(u,z) : v \longmapsto \sum_{n\in\mathbb{Z}} u_n(v) z^{-n-1},$$

where z is a formal variable. The $Y(u,z)$ are called *fields*, and have a meaning in Physics. Parts (i)–(v) may be rewritten as properties of the $Y(u,z)$. One interesting property is this: for all $u,v,w \in V$ there exist $N \gg 0$ depending on u,v such that

$$(y-z)^N Y(u,y) Y(v,z) w = (y-z)^N Y(v,z) Y(u,y) w. \qquad (1)$$

There may be a V-valued rational function $R(y,z)$ with poles when $y=0$, $z=0$ and $y=z$, such that the l.h.s. of (1) is a formal Laurent series convergent to $R(y,z)$ when $0<|y|<|z|$, and the r.h.s. converges to $R(y,z)$ when $0<|z|<|y|$.

Think of $u *_z v = Y(u,z)v$ as a multiplication on V depending on a complex variable z, with poles at $z=0$. Very roughly, V is a commutative associative algebra under $*_z$, with identity $\mathbb{1}$, except the formal power series and poles make everything more complicated.

Lie algebras from vertex algebras

If V is a (graded/super) vertex algebra then $V/D(V)$ is a (graded/super) Lie algebra, with Lie bracket

$$\big[u + D(V), v + D(V)\big] = u_0(v) + D(V).$$

Vertex algebras were introduced in mathematics by Borcherds, who noticed that certain infinite-dimensional Lie algebras important in Representation Theory were constructed as $V/D(V)$. For example, Kac–Moody Lie algebras are (Lie subalgebras of) the Lie algebras associated to lattice vertex algebras.
Vertex algebras are used in Representation Theory, both of infinite-dimensional Lie algebras, and in Moonshine – the Monster may be characterized as the symmetry group of a certain infinite-dimensional vertex algebra.

Write $K_i^{\text{s-t}}(X)$, $i \geqslant 0$ for the *semi-topological K-theory* of X (see papers by Friedlander–Walker). There are natural decompositions $\mathcal{M} = \coprod_{\alpha \in K_0^{\text{s-t}}(X)} \mathcal{M}_\alpha$, $\mathcal{M}^{\text{pl}} = \coprod_{\alpha \in K_0^{\text{s-t}}(X)} \mathcal{M}_\alpha^{\text{pl}}$, where $\mathcal{M}_\alpha, \mathcal{M}_\alpha^{\text{pl}}$ are the moduli stacks of $E^\bullet \in D^b \operatorname{coh}(X)$ with $[\![E^\bullet]\!] = \alpha$ in $K_0^{\text{s-t}}(X)$.

Theorem 2 (Jacob Gross arXiv:1907.03269.)

For all $\alpha \in K_0^{\text{s-t}}(X)$, $\mathcal{M}_\alpha$ is nonempty and connected, with

$$H_*(\mathcal{M}_\alpha, \mathbb{Q}) \cong \operatorname{SSym}^*\Bigl(\bigoplus_{i \geqslant 1} K_i^{\text{s-t}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}\Bigr). \tag{2}$$

Here $\operatorname{SSym}^*(-) = \bigoplus_{l \geqslant 0} \operatorname{SSym}^l(-)$ is the supersymmetric algebra of a $\mathbb{Z}$-graded $\mathbb{Q}$-vector space, and $K_i^{\text{s-t}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ has degree i.

In many examples (e.g. if X is a curve, surface, or toric) we have

$$K_i^{\text{s-t}}(X) \cong \begin{cases} K^0_{\text{top}}(X^{\text{an}}), & i \geqslant 1 \text{ is even,} \\ K^1_{\text{top}}(X^{\text{an}}), & i \geqslant 1 \text{ is odd.} \end{cases} \tag{3}$$

So $H_*(\mathcal{M}_\alpha, \mathbb{Q}), H_*(\mathcal{M}, \mathbb{Q})$ can be written down explicitly. The vertex algebra on $H_*(\mathcal{M}, \mathbb{Q})$ is also explicit, a super-lattice vertex algebra.

The graded vertex algebra structure on $H_*(\mathcal{M},\mathbb{Q})$

For $\alpha,\beta\in K_0^{\text{s-t}}(X)$ and $u\in H_*(\mathcal{M}_\alpha,\mathbb{Q})$, $v\in H_*(\mathcal{M}_\beta,\mathbb{Q})$, define

$$Y(u,z)v=Y(z)(u\otimes v)=(-1)^{\chi(\alpha,\beta)}\sum\nolimits_{i,j\geqslant 0} z^{\chi(\alpha,\beta)+\chi(\beta,\alpha)-i+j}\cdot$$
$$\bigl(\Phi_{\alpha,\beta}\circ(\Psi_\alpha\times\mathrm{id}_{\mathcal{M}_\beta})\bigr)_*\bigl(t^j\boxtimes((u\boxtimes v)\cap c_i((\mathcal{E}xt^\bullet_{\alpha,\beta})^\vee\oplus\sigma^*_{\alpha,\beta}(\mathcal{E}xt^\bullet_{\beta,\alpha})))\bigr).$$

Here $\Phi_{\alpha,\beta}:\mathcal{M}_\alpha\times\mathcal{M}_\beta\to\mathcal{M}_{\alpha+\beta}$ maps $([E],[F])\mapsto[E\oplus F]$,
$\Psi_\alpha:[*/\mathbb{G}_m]\times\mathcal{M}_\alpha\to\mathcal{M}_\alpha$ is the group stack action,
$\sigma_{\alpha,\beta}:\mathcal{M}_\alpha\times\mathcal{M}_\beta\to\mathcal{M}_\beta\times\mathcal{M}_\alpha$ exchanges the factors,
$t^j\in H_{2j}([*/\mathbb{G}_m],\mathbb{Q})\cong\mathbb{Q}$ is the generator, $\mathcal{E}xt^\bullet_{\alpha,\beta}\to\mathcal{M}_\alpha\times\mathcal{M}_\beta$ is the Ext complex, and $\chi:K_0^{\text{s-t}}(X)\times K_0^{\text{s-t}}(X)\to\mathbb{Z}$ is the Euler form.
The identity $\mathbb{1}\in H_0(\mathcal{M}_0,\mathbb{Q})\subset H_*(\mathcal{M},\mathbb{Q})$ is the image of $1\in H_0(*,\mathbb{Q})$ under $[0]:*\to\mathcal{M}_0$. The translation operator $D:H_*(\mathcal{M}_\alpha)\to H_{*+2}(\mathcal{M}_\alpha)$ is $D(u)=(\Psi_\alpha)_*(t\boxtimes u)$.
We shift the grading on $H_*(\mathcal{M},\mathbb{Q})$ by

$$\hat{H}_n(\mathcal{M}_\alpha)=H_{n-2\chi(\alpha,\alpha)}(\mathcal{M}_\alpha),\quad \hat{H}_n(\mathcal{M})=\bigoplus_{\alpha\in K_0^{\text{s-t}}(X)}\hat{H}_n(\mathcal{M}_\alpha).$$

When we can define enumerative invariants 'counting' moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-semistable coherent sheaves or complexes on X with Chern character $\alpha\in H^{\rm even}(X,\mathbb{Q})$, for example, if X is a curve, surface, Fano 3-fold, or Calabi–Yau 3-fold or 4-fold, in my theory from arXiv:2111.04694, the invariants should be defined as homology classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}$ in $H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$. If $\tau,\tilde\tau$ are two stability conditions, we prove a universal wall-crossing formula expressing $[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm inv}$ as a $\mathbb{Q}$-linear combination of iterated Lie brackets of $[\mathcal{M}_{\alpha_i}^{\rm ss}(\tilde\tau)]_{\rm inv}$ for $i=1,\ldots,n$, with $\alpha=\alpha_1+\cdots+\alpha_n$, using the graded Lie algebra structure on $H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$.
As both $H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$ and the Lie bracket on it are difficult to write down, for explicit calculations, it is usually better to lift to the vertex algebra $H_*(\mathcal{M},\mathbb{Q})$. Pandharipande–Thomas virtual classes $[P_n(X,\beta)]_{\rm virt}$ have a natural lift to $H_*(\mathcal{M},\mathbb{Q})$, as there is a universal complex $\mathcal{O}_{X\times P_n(X,\beta)}\xrightarrow{\mathfrak{s}}\mathfrak{F}$ on $X\times P_n(X,\beta)$.

One-dimensional Donaldson–Thomas invariants

Now fix $m = \dim_{\mathbb{C}} X = 3$. Then we can form moduli stacks $\mathcal{M}_\alpha^{\rm ss}(\tau)$ of Gieseker semistable coherent sheaves F on X with Chern character $\alpha \in H^{\rm even}(X, \mathbb{Q})$. The natural obstruction theory $\mathcal{E}^\bullet$ on $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is perfect in $[-2, 1]$, but to define a Behrend–Fantechi virtual class for $\mathcal{M}_\alpha^{\rm ss}(\tau)$ we need it to be perfect in $[-1, 1]$. We have $H^{-2}(\mathcal{E}^\bullet|_{[F]}) \cong \mathrm{Ext}^3(F, F)^*$, so the problems are caused by the groups $\mathrm{Ext}^3(F, F)$ for $[F] \in \mathcal{M}_\alpha^{\rm ss}(\tau)$. There are *two different* cases in which we can define virtual classes:

(a) If X is a Calabi–Yau 3-fold and $\mathcal{M}_\alpha^{\rm st}(\tau) = \mathcal{M}_\alpha^{\rm ss}(\tau)$ then $\mathrm{Ext}^3(F, F) \cong \mathrm{Hom}(F, F)^* \cong \mathbb{C}$ by Serre duality.

(b) Under a Fano type condition on X, α we ensure that $\mathrm{Ext}^3(F, F) = 0$ for all $[F] \in \mathcal{M}_\alpha^{\rm ss}(\tau)$.

For counting 1-*dimensional* sheaves F with $\mathrm{Pd}(c_2(F)) = \beta$ in $H_2(X, \mathbb{Z})$ and $\chi(F) = n$, if β is *superpositive* then (b) holds, so my theory defines Donaldson–Thomas invariants $[\mathcal{M}_{(\beta,n)}^{\rm ss}(\tau)]_{\rm inv}$ in $H_2(\mathcal{M}_{(\beta,n)}^{\rm pl}, \mathbb{Q})$, which have dimension 2 in homology.

Define an abelian category $\mathcal{A}$ to have objects (F, V, ϕ) where $F \in \mathrm{coh}_{\leqslant 1}(X)$ is a coherent sheaf of dimension $\leqslant 1$, and V is a finite-dimensional vector space over $\mathbb{C}$, and $\phi : \mathcal{O}_X \otimes_{\mathbb{C}} V \to F$ is a morphism in $\mathrm{coh}(X)$. Define $K(\mathcal{A}) = \mathbb{Z} \oplus H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$. Define the *class* of (F, V, ϕ) to be $[\![F, V, \phi]\!] = (\dim V, \mathrm{Pd}(c_2(F)), \chi(F))$. Write $\mathcal{N}^{\mathrm{pl}}$ for the projective linear moduli stack of objects in $\mathcal{A}$. It has a decomposition $\mathcal{N}^{\mathrm{pl}} = \bigoplus_{(d,\beta,n)\in K(\mathcal{A})} \mathcal{N}^{\mathrm{pl}}_{(d,\beta,n)}$, where $\mathcal{N}^{\mathrm{pl}}_{(d,\beta,n)}$ is the substack of objects in class (d, β, n).

Define a functor $\Upsilon : \mathcal{A} \to D^b \mathrm{coh}(X)$ to map (F, V, ϕ) to the complex $\mathcal{E}^\bullet = \big[\underset{-1}{\mathcal{O}_X \otimes_{\mathbb{C}}} V \xrightarrow{\phi} \underset{0}{F}\big]$, with F in degree 0. Then Υ induces a morphism of stacks $\Upsilon^{\mathrm{pl}} : \mathcal{N}^{\mathrm{pl}} \to \mathcal{M}^{\mathrm{pl}}$, where $\mathcal{M}^{\mathrm{pl}}$ is the projective linear moduli stack of objects in $D^b \mathrm{coh}(X)$. It will be important that Υ^{pl} is *not an open inclusion*. At points $(F, V, 0)$ where $\dim V > 0$, $\phi = 0$, and $\mathrm{Ext}^2(\mathcal{O}_X, F) \cong H^1(F \otimes K_X)^* \neq 0$, the complex $\big[\underset{-1}{\mathcal{O}_X \otimes_{\mathbb{C}}} V \xrightarrow{0} \underset{0}{F}\big]$ has more deformations in $D^b \mathrm{coh}(X)$ than $(F, V, 0)$ does in $\mathcal{A}$.

We define a 1-parameter family of stability conditions τ_t on $\mathcal{A}$ for $t \in [c_-, c_+]$, and apply my wall-crossing theory to invariants counting τ_t-semistable objects in $\mathcal{A}$. To define virtual classes of τ_t-semistable moduli spaces $\mathcal{N}^{\rm ss}_{(d,\beta,n)}(\tau_t)$, we take the obstruction theory on $\mathcal{N}^{\rm pl}$ to be the pullback under $\Upsilon_* : \mathcal{N}^{\rm pl} \to \mathcal{M}^{\rm pl}$ of the natural obstruction theory on $\mathcal{M}^{\rm pl}$, because this is the obstruction theory used to define Pandharipande–Thomas invariants. This can be badly-behaved at points of $\mathcal{N}^{\rm pl}$ for *two different* reasons. Firstly, if $\mathcal{E}^\bullet = \Upsilon(F, V, \phi)$ has $\mathrm{Ext}^3(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \neq 0$ then the obstruction theory has cohomology in degree -2, and is not perfect in $[-1, 1]$. We restrict to *superpositive* curve classes β to avoid such points. Secondly, if $\Upsilon^{\rm pl}$ is not an open inclusion at $[F, V, \phi]$ then the obstruction theory on $\mathcal{M}^{\rm pl}$ does not pull back to an obstruction theory on $\mathcal{N}^{\rm pl}$ at $[F, V, \phi]$. This will force us to restrict to PT moduli spaces $P_n(X, \beta)$ with $n \gg 0$, to ensure that $H^1(F \otimes K_X) = 0$ for all $[F, s]$ in $P_n(X, \beta)$.

Fix a superpositive curve class $\beta \in H_2(X,\mathbb{Z})$ and $n \gg 0$, and an ample line bundle $L \to X$. We define a 1-parameter family τ_t of stability conditions on $\mathcal{A}$ for $t \in [c_-, c_+]$, such that:

- τ_t-semistable objects $(0,F,0)$ in class $(0,\beta',n')$ for β' an effective curve class and all t are μ-semistable sheaves F in class (β',n'), where μ is slope stability defined using $c_1(L)$. These are counted by one-dimensional Donaldson–Thomas invariants $[\mathcal{M}^{\rm ss}_{(\beta',n')}(\mu)]_{\rm inv}$.
- All objects $(F,\mathbb{C},\phi)$ in class $(1,\beta,n)$ are τ_{c_-}-unstable, as they are destabilized by $0 \to (F,0,0) \to (F,\mathbb{C},\phi) \to (0,\mathbb{C},0) \to 0$.
- The moduli spaces $\mathcal{N}^{\rm ss}_{(1,\beta',n')}(\tau_{c_+})$ for β' a factor of β and n' satisfying an inequality are PT moduli spaces $P_{n'}(X,\beta')$. Here we regard a PT pair $s:\mathcal{O}_X \to F$ as an object $(F,\mathbb{C},s)$ in $\mathcal{A}$.
- The obstruction theory on $\mathcal{N}^{\rm pl}$ is well-behaved on all τ_t-semistable moduli spaces for $t \in [c_-,c_+]$ involved in the wall-crossing formula for class $(1,\beta,n)$. This uses both β superpositive, and $n \gg 0$.

My Monster WCF paper now proves:

Theorem 3 (Joyce arXiv:2111.04694, version 2, to appear.)

If $\beta \in H_2(X,\mathbb{Z})$ is superpositive and $n \gg 0$ then the following identity holds in the graded Lie algebra $H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$:

$$0 = \sum_{\substack{1\leqslant j\leqslant k,\\ \beta=\beta_1+\cdots+\beta_k,\\ n=n_1+\cdots+n_k,\\ \beta_i \text{ effective and}\\ n_i\in\mathbb{Z},\ i\neq j,\\ \text{either } \beta_j \text{ effective}\\ \text{and } n_j\in\mathbb{Z},\\ \text{or } (\beta_j,n_j)=(0,0)}} \begin{aligned}[t] &\tilde{U}\bigl((0,\beta_1,n_1),\ldots,(0,\beta_{j-1},n_{j-1}),(1,\beta_j,n_j),\\ &(0,\beta_{j+1},n_{j+1}),\ldots,(0,\beta_k,n_k);\tau_{c_+},\tau_{c_-}\bigr)\cdot\\ &\bigl[\bigl[\cdots\bigl[[\mathcal{M}^{\rm ss}_{(\beta_1,n_1)}(\mu)]_{\rm inv},[\mathcal{M}^{\rm ss}_{(\beta_2,n_2)}(\mu)]_{\rm inv}\bigr],\ldots,\\ &[\mathcal{M}^{\rm ss}_{(\beta_{j-1},n_{j-1})}(\mu)]_{\rm inv}\bigr],[P_{n_j}(X,\beta_j)]_{\rm virt}\bigr],\\ &[\mathcal{M}^{\rm ss}_{(\beta_{j+1},n_{j+1})}(\mu)]_{\rm inv}\bigr],\ldots,[\mathcal{M}^{\rm ss}_{(\beta_k,n_k)}(\mu)]_{\rm inv}\bigr]. \end{aligned} \tag{4}$$

Here $\tilde{U}(\cdots;\tau_{c_+},\tau_{c_-})\in\mathbb{Q}$ are universal combinatorial coefficients defined in Joyce 2004, which occur in many different wall crossing formulae. There are only finitely many nonzero terms in the sum.

This is an identity relating Pandharipande–Thomas invariants and one-dimensional Donaldson–Thomas invariants.

When $\beta = 1$ we have $\tilde{U}((1,\beta,n);\tau_{c_+},\tau_{c_-}) = 1$. Thus we can rewrite (4) as

$$
\begin{aligned}
&\bigl[P_n(X,\beta)\bigr]_{\mathrm{virt}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5)\\
&= -\sum_{\substack{2\leqslant j\leqslant k,\\ \beta=\beta_1+\cdots+\beta_k,\\ n=n_1+\cdots+n_k,\\ \beta_i \text{ effective and}\\ n_i\in\mathbb{Z},\ i\neq j,\\ \text{either } \beta_j \text{ effective}\\ \text{and } n_j\in\mathbb{Z},\\ \text{or } (\beta_j,n_j)=(0,0)}} \begin{array}{l}\tilde{U}\bigl((0,\beta_1,n_1),\ldots,(0,\beta_{j-1},n_{j-1}),(1,\beta_j,n_j),\\ (0,\beta_{j+1},n_{j+1}),\ldots,(0,\beta_k,n_k);\tau_{c_+},\tau_{c_-}\bigr)\cdot\\ \bigl[\bigl[\cdots\bigl[[\mathcal{M}^{\mathrm{ss}}_{(\beta_1,n_1)}(\mu)]_{\mathrm{inv}},[\mathcal{M}^{\mathrm{ss}}_{(\beta_2,n_2)}(\mu)]_{\mathrm{inv}}\bigr],\ldots,\\ [\mathcal{M}^{\mathrm{ss}}_{(\beta_{j-1},n_{j-1})}(\mu)]_{\mathrm{inv}}\bigr],[P_{n_j}(X,\beta_j)]_{\mathrm{virt}}\bigr],\\ [\mathcal{M}^{\mathrm{ss}}_{(\beta_{j+1},n_{j+1})}(\mu)]_{\mathrm{inv}}\bigr],\ldots,[\mathcal{M}^{\mathrm{ss}}_{(\beta_k,n_k)}(\mu)]_{\mathrm{inv}}\bigr].\end{array}
\end{aligned}
$$

For example, if β is an *irreducible* curve class then (5) reduces to

$$[P_n(X,\beta)]_{\mathrm{virt}} = \bigl[[\mathcal{M}^{\mathrm{ss}}_{(\beta,n)}(\mu)]_{\mathrm{inv}},[P_0(X,0)]_{\mathrm{virt}}\bigr], \qquad (6)$$

where $[P_0(X,0)]_{\mathrm{virt}} \in H_0(\mathcal{M}^{\mathrm{pl}}_{(1,0,0)},\mathbb{Q}) \cong \mathbb{Q}$ is the class of the point $[0,\mathbb{C},0]$.

You might think that (5) allows us to construct PT classes $[P_n(X,\beta)]_{\rm virt}$ for $n\gg 0$ solely from Donaldson–Thomas invariants $[\mathcal{M}^{\rm ss}_{(\beta',n')}(\mu)]_{\rm inv}$, by induction on the number of irreducible factors of β. But this is not true. Equation (5) gives $[P_n(X,\beta)]_{\rm virt}$ only when $n\gg 0$, say when $n\geqslant N_\beta$, but it involves $[P_{n_j}(X,\beta_j)]_{\rm virt}$ for n_j which may not have $n_j\geqslant N_{\beta_j}$, so that the $[P_{n_j}(X,\beta_j)]_{\rm virt}$ are not determined solely by the $[\mathcal{M}^{\rm ss}_{(\beta',n')}(\mu)]_{\rm inv}$.

Since $P_n(X,\beta)=\emptyset$ for $n\ll 0$, we have $[P_n(X,\beta)]_{\rm virt}=0$ for $n\leqslant M_\beta$, say. So this and (5) determine $[P_n(X,\beta)]_{\rm virt}$ for all $n\in\mathbb{Z}$ except for the *finitely many* cases $M_\beta<n<N_\beta$.

In general, (5) determines $[P_n(X,\beta)]_{\rm virt}$ in terms of D–T invariants $[\mathcal{M}^{\rm ss}_{(\beta',n')}(\mu)]_{\rm inv}$ and P–T $[P_{n'}(X,\beta')]_{\rm virt}$ for $M_{\beta'}<n'<N_{\beta'}$, both for β' one of the finitely many effective factors of β.

This will be enough for us to prove the Pandharipande–Thomas rationality conjecture for superpositive curve classes.

A periodicity property of the $[\mathcal{M}^{\rm ss}_{(\beta,n)}(\mu)]_{\rm inv}$ in n

We define slope stability μ on one-dimensional sheaves F on X using an ample line bundle $L\to X$. Then F is μ-semistable if and only if $F\otimes L$ is μ-semistable. Now $-\otimes L$ gives an autoequivalence of $D^b\operatorname{coh}(X)$, and isomorphisms $\Xi_L:\mathcal{M}\to\mathcal{M}$, $\Xi_L^{\rm pl}:\mathcal{M}^{\rm pl}\to\mathcal{M}^{\rm pl}$, which restrict to $\Xi_L^{\rm pl}:\mathcal{M}^{\rm ss}_{(\beta,n)}(\mu)\xrightarrow{\cong}\mathcal{M}^{\rm ss}_{(\beta,n+c_1(L)\cdot\beta)}(\mu)$. Therefore

$$(\Xi_L^{\rm pl})_*:\bigl[\mathcal{M}^{\rm ss}_{(\beta,n)}(\mu)\bigr]_{\rm inv}=\bigl[\mathcal{M}^{\rm ss}_{(\beta,n+c_1(L)\cdot\beta)}(\mu)\bigr]_{\rm inv}. \qquad (7)$$

Writing $d_\beta=c_1(L)\cdot\beta>0$, this shows $[\mathcal{M}^{\rm ss}_{(\beta,n)}(\mu)]_{\rm inv}$ has a periodicity in n with period d_β. Lift $[\mathcal{M}^{\rm ss}_{(\beta,n)}(\mu)]_{\rm inv}$ from $H_2(\mathcal{M}^{\rm pl}_{(\beta,n)},\mathbb{Q})$ to $H_2(\mathcal{M}_{(\beta,n)},\mathbb{Q})$ uniquely such that $\tau_1(c_1(L))\cdot[\mathcal{M}^{\rm ss}_{(\beta,n)}(\mu)]_{\rm inv}=0$. Identify $H_2(\mathcal{M}_{(\beta,n)},\mathbb{Q})$ with $H_2(\mathcal{M}_{(0,0)},\mathbb{Q})$ by direct sum with a fixed complex of class (β,n). Thus we regard $[\mathcal{M}^{\rm ss}_{(\beta,n)}(\mu)]_{\rm inv}$ for $n\in\mathbb{Z}$ as lying in $H_2(\mathcal{M}_{(0,0)},\mathbb{Q})$. For $a=1,\ldots,d_\beta$, using (7) and Theorem 2 we prove that the map $\mathbb{Z}\to H_2(\mathcal{M}_{(0,0)},\mathbb{Q})$, $k\mapsto[\mathcal{M}^{\rm ss}_{(\beta,a+kd_\beta)}(\mu)]_{\rm inv}$ is *polynomial* in k, of degree $\leqslant 6$, as (7) is a difference equation with polynomial solutions.

Quasipolynomials and piecewise quasipolynomials

Let V be a $\mathbb{Q}$-vector space. A function $f:\mathbb{Z}\to V$ is a *quasi-polynomial* if there exist $d\geqslant 1$ and polynomials $P_1,\ldots,P_d\in V[x]$ such that $f(n)=P_a(n)$ whenever $n\equiv a \mod d$. More generally, a function $f:\mathbb{Z}^k\to V$ is a *quasi-polynomial* if there exist $d\geqslant 1$ and polynomials $P_{\boldsymbol a}(x_1,\ldots,x_k)\in V[x_1,\ldots,x_k]$ for $\boldsymbol a\in\{1,2,\ldots,d\}^k$, such that

$$f(n_1,\ldots,n_k)=P_{\boldsymbol a}(n_1,\ldots,n_k)\quad\text{whenever } (n_1,\ldots,n_k)\equiv \boldsymbol a \mod d.$$

A *chamber decomposition* of $\mathbb{Z}^k$ is a finite partition $\mathbb{Z}^k=A_1\amalg\cdots\amalg A_N$ in which each chamber A_i is cut out by finitely many rational affine equalities and inequalities like $C_1n_1+\cdots+C_kn_k$ ($<$ or $\leqslant$ or $=$) D, for $C_j,D\in\mathbb{Q}$.
A function $f:\mathbb{Z}^k\to V$ is called *piecewise quasi-polynomial* if there is a chamber decomposition $\mathbb{Z}^k=A_1\amalg\cdots\amalg A_N$ and quasi-polynomials $g_1,\ldots,g_N:\mathbb{Z}^k\to V$ with $f|_{A_i}=g_i|_{A_i}$ for all i.
Ehrhart theory deals with sums of piecewise quasi-polynomials.

Recall equation (5):

$$
\begin{aligned}
&\big[P_n(X,\beta)\big]_{\rm virt} \qquad (8)\\
&= - \sum_{\substack{2\leqslant j\leqslant k,\\ \beta=\beta_1+\cdots+\beta_k,\\ n=n_1+\cdots+n_k,\\ \beta_i \text{ effective and}\\ n_i\in\mathbb{Z},\ i\neq j,\\ \text{either } \beta_j \text{ effective}\\ \text{and } n_j\in\mathbb{Z},\\ \text{or } (\beta_j,n_j)=(0,0)}} \begin{aligned}[t] &\tilde U\big((0,\beta_1,n_1),\ldots,(0,\beta_{j-1},n_{j-1}),(1,\beta_j,n_j),\\ &(0,\beta_{j+1},n_{j+1}),\ldots,(0,\beta_k,n_k);\tau_{c_+},\tau_{c_-}\big)\cdot\\ &\big[\big[\cdots\big[[\mathcal{M}^{\rm ss}_{(\beta_1,n_1)}(\mu)]_{\rm inv},[\mathcal{M}^{\rm ss}_{(\beta_2,n_2)}(\mu)]_{\rm inv}\big],\ldots,\\ &[\mathcal{M}^{\rm ss}_{(\beta_{j-1},n_{j-1})}(\mu)]_{\rm inv}\big],[P_{n_j}(X,\beta_j)]_{\rm virt}\big],\\ &[\mathcal{M}^{\rm ss}_{(\beta_{j+1},n_{j+1})}(\mu)]_{\rm inv}\big],\ldots,[\mathcal{M}^{\rm ss}_{(\beta_k,n_k)}(\mu)]_{\rm inv}\big].\end{aligned}
\end{aligned}
$$

We lift this equation from $H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$ to $H_*(\mathcal{M},\mathbb{Q})$, so that we can write the Lie brackets using vertex algebra operations. We identify $H_*(\mathcal{M}_{(d,\beta,n)},\mathbb{Q})$ with $H_*(\mathcal{M}_{(0,0,0)},\mathbb{Q})$ by direct sum with a fixed complex, so we consider every term in the equation as lying in $H_*(\mathcal{M}_{(0,0,0)},\mathbb{Q})$. Then the $[\mathcal{M}^{\rm ss}_{(\beta_i,n_i)}(\mu)]_{\rm inv}$ are quasi-polynomial in n_i. We use (8) and induction on the number of effective factors of β to prove that $\big[P_n(X,\beta)\big]_{\rm virt}$ is a *piecewise quasi-polynomial* function of n.

To do this, we note that there are finitely many possibilities for k, $(\beta_1,\ldots,\beta_k)$ in the sum. For fixed $\beta_1,\ldots,\beta_k$, the function $(n_1,\ldots,n_k)\mapsto \tilde U\bigl((0,\beta_1,n_1),\ldots,(0,\beta_k,n_k);\tau_{c_+},\tau_{c_-}\bigr)$ is constant on the chambers of a chamber decomposition of $\mathbb{Z}^k$. Each term in the iterated Lie bracket in (8) is (piecewise) quasi-polynomial in n_i, by the inductive hypothesis. Using the definition of the vertex algebra, we show that taking the Lie brackets preserves the piecewise quasi-polynomial property, though it can increase the degree of the polynomials. So the entire term in the sum in (8) is piecewise quasi-polynomial in $(n_1,\ldots,n_k)\in\mathbb{Z}^k$. A basic result in Ehrhart theory then shows that the sum over $(n_1,\ldots,n_k)$ with $n=n_1+\cdots+n_k$ is piecewise quasi-polynomial in n.

This proves that the map $\mathbb{Z}\rightarrow H_{2c_1(X)\cdot\beta}(\mathcal{M}_{(0,0,0)},\mathbb{Q})$, $n\mapsto\bigl[P_n(X,\beta)\bigr]_{\rm virt}$, is piecwise quasi-polynomial in n. Since also $\bigl[P_n(X,\beta)\bigr]_{\rm virt}=0$ for $n\ll 0$, we see that $\sum_{n\in\mathbb{Z}}q^n\bigl[P_n(X,\beta)\bigr]_{\rm virt}$ is the Laurent expansion of a rational function, with poles only at $q=0$ and roots of unity. The Pandharipande–Thomas rationality conjecture follows.

With a minor increase in complexity, we can also prove the P–T rationality conjecture in the G-equivariant case.

One should be able to use a similar method to prove an additional $\mathbb{Z}_2$-symmetry of the P–T generating function. For Calabi–Yau 3-folds, Toda 2012 does this by wall-crossing in $D^b\,\mathrm{coh}(X)$ to a stability condition which is invariant under Verdier duality.

We did not do this because of a technical limitation in my enumerative invariant theory: to define the invariants and prove the wall-crossing formula, we need to define 'auxiliary pair invariants' or 'framing functors', and it is currently not clear how to do this in $D^b\,\mathrm{coh}(X)$ rather than $\mathrm{coh}(X)$.

Karpov–Moreira arXiv:2512.22360 give a K-theoretic version of my invariant theory which does not need framing functors, and using this in arXiv:2604.06023, under extra assumptions on X, they prove both the P–T rationality conjecture and the $\mathbb{Z}_2$-symmetry.