Enumerative invariants in Geometry

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These slides available at
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Manifolds

Geometers study different kinds of space. Our favourite spaces are called *manifolds*. Fix $n \geqslant 0$. An *n-dimensional manifold* is a Hausdorff, second countable topological space X equipped with a cover $\{U_i : i \in I\}$ of open sets $U_i \subset X$ with homeomorphisms $\varphi_i : U_i \to V_i \subseteq \mathbb{R}^n$ with open subsets of $\mathbb{R}^n$. In effect the 'chart' (U_i, V_i, φ_i) is a choice of *local coordinates* $(x_1, \ldots, x_n)$ on $U_i \subseteq X$. Given two charts (U_i, V_i, φ_i), (U_j, V_j, φ_j) we have a *transition function* $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1}$, a homeomorphism between open subsets $\varphi_i^{-1}(U_j), \varphi_j^{-1}(U_i) \subseteq \mathbb{R}^n$. It writes the φ_j coordinates $(y_1, \ldots, y_n)$ as functions of the φ_i coordinates $(x_1, \ldots, x_n)$ on $U_i \cap U_j$.

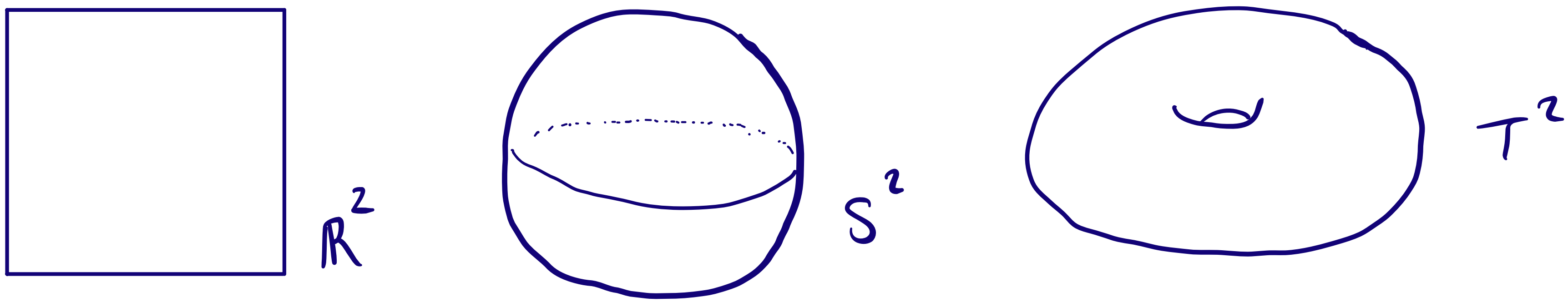
There are different kinds of manifolds:

- For a *topological manifold*, the φ_{ij} must be continuous (automatic).
- For a *smooth manifold*, the φ_{ij} must be smooth, that is, all partial derivatives $\frac{\partial^k}{\partial x_{a_1} \cdots \partial x_{a_k}} \varphi_{ij}$ must exist and be continuous.
- For a *complex manifold*, take V_i, V_j open in $\mathbb{C}^n$ not $\mathbb{R}^n$, and require the φ_{ij} to be holomorphic.

By 'manifold' we generally mean 'smooth manifold'.

Examples, geometric structures, applications

Some examples of n-manifolds are $\mathbb{R}^n$, $\mathcal{S}^n$, $T^n = (\mathcal{S}^1)^n$.



We can also define continuous maps between topological manifolds, smooth maps between (smooth) manifolds, holomorphic maps between complex manifolds, making them into categories.

Much of geometry studies a manifold *with a geometric structure*, such as a Riemannian metric. You can do calculus on manifolds: differentiation and integration, and study p.d.e.s on manifolds.

General Relativity models space-time as a 4-dimensional manifold $\mathbb{M}$ with a geometric structure (a Lorentzian metric, matter fields) satisfying a p.d.e. (Einstein's equations).

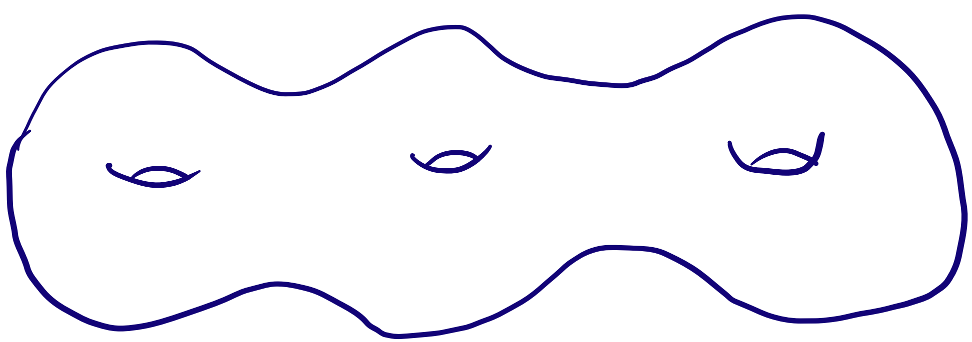
Mechanics can be formulated using manifolds: the phase space of positions and velocities of a mechanical system are a manifold with a symplectic structure.

Classification of manifolds

It is a fundamental problem in geometry to classify n-manifolds X (say, compact, connected, and perhaps simply-connected) up to isomorphism. Implicitly we expect the answer to depend on *invariants* of X from Algebraic Topology such as cohomology $H^*(X,\mathbb{Z})$, fundamental group $\pi_1(X)$, characteristic classes, ...

The only compact, connected 1-manifold is $\mathcal{S}^1$.

We can understand compact, connected 2-manifolds X in terms of the *Euler characteristic* $\chi(X) = \sum_{k=0}^{\dim X} (-1)^k \dim H^k(X,\mathbb{R})$ and its *orientability* $o(X)$, where $o(X) = 0$ if X is orientable and $o(X) = 1$ otherwise. Then X is classified by $(o(X), \chi(X))$, which takes values in $\{0\} \times \{2, 0, -2, -4, \ldots\} \amalg \{1\} \times \{1, 0, -1, -2, \ldots\}$



Σ_g, sphere with g holes (here $g = 3$). Orientable.

$\chi(\Sigma_g) = 2g - 2.$

Classification of manifolds

A compact, connected 3-manifold X can be understood by its fundamental group $\pi_1(X)$. The Poincaré Conjecture (Perelman 2003) says $\pi_1(X) = \{1\} \Leftrightarrow X \cong \mathcal{S}^3$. The more general Thurston Geometrization Programme breaks any X into nice pieces.

Compact, connected, simply-connected topological and smooth n-manifolds X for $n \geqslant 5$ were classified in the 1950s-1960s using algebro-topological invariants (cohomology $H^*(X, \mathbb{Z})$ + extra data).

Compact, connected, simply-connected *topological* 4-manifolds X were classified in the 1980s by Freedman. They are classified by $(H^2(X, \mathbb{Z})$, intersection form $\cup)$, and their *Kirby–Siebenmann invariant* $\kappa(X) \in \{0, 1\}$, where $\kappa(X) = 0 \Leftrightarrow X \times \mathbb{R}$ is smoothable.

This leaves a big open question: what is the classification of compact, connected, simply-connected *smooth* 4-manifolds?

For example, the 4-dimensional Poincaré Conjecture (unknown) says that any smooth 4-manifold homeomorphic to $\mathcal{S}^4$ must be diffeomorphic (isomorphic as a smooth manifold) to $\mathcal{S}^4$.

Donaldson theory, prototypical enumerative invariants

Let X be a compact oriented smooth 4-manifold. The *Donaldson invariant* is a linear map $D_X : \mathrm{Sym}^*(H_0(X,\mathbb{Z}) \oplus H_2(X,\mathbb{Z})) \to \mathbb{Z}$, due to Simon Donaldson in the 1980s. The definition is difficult and weird. You fix some geometric data: a metric g on X and principal $\mathrm{SU}(2)$-bundle $P \to X$. You form a *moduli space* $\mathcal{M}_P$ of 'instantons' (solutions of a nonlinear p.d.e.) on P, and compactify it to $\bar{\mathcal{M}}$ by adding singular solutions. Under good conditions, $\bar{\mathcal{M}}_P$ is a compact, oriented manifold with some natural cohomology classes; $D_X(\text{input data from } P) = \int_{\overline{\mathcal{M}}_P}(\text{cohomology class})$.

We call D_X an *enumerative invariant* as it 'counts' something (solutions of the instanton equation) living on X.

It turns out that D_X is independent of choices $g, P, \ldots$ and depends only on X as a smooth 4-manifold.

There are many examples of 4-manifolds X, Y which are isomorphic as topological 4-manifolds, but have $D_X \not\cong D_Y$.

So Donaldson invariants know something (we don't yet know what) about the classification of smooth 4-manifolds.

No known conventional algebro-topological invariants of 4-manifolds capture the same data as Donaldson invariants.

Triumphs of Donaldson theory: the rise . . .
. . . and the fall: Seiberg–Witten theory

In the 1980s-1990s, Donaldson theory proved amazing things about smooth 4-manifolds. Some highlights:

- If X has $b^2_-(X) = 0$ then $(H^2(X,\mathbb{Z}),\cup)$ is diagonalizable. So many topological 4-manifolds admit no smooth structures.
- There are uncountably many distinct smooth structures on $\mathbb{R}^4$.
- The smooth h-cobordism theorem fails in 4 dimensions.

In 1994, the String Theorists Seiberg and Witten proposed a new enumerative invariant of 4-manifolds: *Seiberg–Witten invariants* SW:$\{\mathrm{Spin}^c\text{-structures on } X\} \to \mathbb{Z}$. They 'count' Seiberg–Witten monopoles on X, not instantons. They are much simpler, in both theory (moduli spaces $\mathcal{M}_{\mathrm{SW}}$ are compact, so no issues of compactification) and structure. Conjecturally, Donaldson invariants may be written in terms of Seiberg–Witten invariants. In a few months, many major problems in Donaldson theory were solved using Seiberg–Witten invariants. There was then nothing left for Donaldson theorists to do. They were all fired, and they and their tiny children died in poverty.

Mirror Symmetry and Gromov–Witten invariants

A *Calabi–Yau m-fold* X is a compact complex m-dimensional manifold with a 'holomorphic volume form'. Calabi–Yau 3-folds are important in String Theory. String Theorists say that the universe is really 10 dimensional, and looks like $X \times \mathbb{R}^{3,1}$, where X is a Calabi–Yau 3-fold of diameter $\approx 10^{-33}$cm and $\mathbb{R}^{3,1}$ is Minkowski space. Quantum Theory should associate a Super Conformal Field Theory to X, on which our observable low-energy physics depends.
Circa 1990, String Theorists Candelas et al. proposed that Calabi–Yau 3-folds come in 'mirror pairs' $X, \check{X}$ with SCFTs isomorphic under an involution. Geometric invariants of X could be related via the SCFT to completely different invariants of $\check{X}$.
They proposed that invariants counting rational curves $\mathbb{CP}^1$ in X (later called 'Gromov–Witten invariants') are coefficients in the power series expansion of a function measuring deformation of complex structure of $\check{X}$. Using this, they computed the GW invariants of X the quintic in $\mathbb{CP}^4$ (it was later proved they got them right). Gromov–Witten invariants had not yet been defined, so mathematicians had to do this to catch up.
Mirror Symmetry grew into a huge subject over 30 years.

4-manifolds and complex surfaces

A 4-manifold X is locally modelled on $\mathbb{R}^4$. A complex surface (complex 2-manifold) Y is locally modelled on $\mathbb{C}^2$. As $\mathbb{C}^2 \cong \mathbb{R}^4$, Y has an underlying smooth 4-manifold Y^{sm}. We can regard Y as a pair (Y^{sm}, J), where J is a *complex structure* on Y^{sm}. It tells us when a smooth function $f : Y^{\mathrm{sm}} \to \mathbb{C}$ is holomorphic.

Complex surfaces Y are particularly nice examples of smooth 4-manifolds. Donaldson and Seiberg–Witten invariants can be defined using the complex geometry of Y (instantons are roughly holomorphic vector bundles), and become simpler.

If Y is complex algebraic (e.g. if Y is *projective*, a complex submanifold of $\mathbb{CP}^N$ for $N \gg 0$), we can use *complex algebraic geometry*. This gives much more powerful tools: algebraic geometry is much better at dealing with singular objects than differential geometry. It is often easy to produce *compact* moduli spaces $\overline{\mathcal{M}}$ in algebraic geometry (needed so that we can integrate $\int_{\overline{\mathcal{M}}}$(cohomology class)), but in differential geometry, finding good compactifications is usually hard. Many enumerative invariant theories exist only in algebraic geometry.

Vafa–Witten invariants and modularity

In 1994, the String Theorists Vafa and Witten proposed that for a compact oriented 4-manifold X, a Lie group G and $n \in \mathbb{Z}$ one could define *Vafa–Witten invariants* VW_n^G by 'counting' moduli spaces $\mathcal{M}_{G,n}^{\mathrm{VW}}$ of 'Vafa–Witten monopoles' on X (solutions of another nonlinear p.d.e.). Oversimplifying a bit, they proposed that the generating function $VW^G(q) = \sum_{n\in\mathbb{Z}} VW_n^G q^n$ should be a *modular form*, a class of functions from Number Theory with deep hidden symmetries. This is surprising, and mathematically mysterious! (More precisely, there should be a modular transformation relating $VW^G(q)$ and $VW^{G^\vee}(q)$, where $G^\vee$ is the Langlands dual group of G.) For general X, the moduli spaces $\mathcal{M}_{G,n}^{\mathrm{VW}}$ are noncompact, so VW_n^G does not really make sense. But if X is a (Kähler) complex surface and $G = \mathrm{SU}(r)$ or $\mathrm{U}(r)$, the moduli spaces are compact; in nice cases, VW_n^G are the Euler characteristics of instanton moduli spaces. There is now a lot of evidence from calculations in examples that Vafa–Witten generating functions are (mock) modular, at least when $G = \mathrm{SU}(r)$ or $\mathrm{U}(r)$ for $r = 2, 3, 4$.

Coherent sheaves on algebraic complex manifolds

Let X be a compact algebraic complex manifold. We can define *coherent sheaves* E on X. These are (sheaves of) modules over (the sheaf of) algebraic holomorphic functions on X. They include *vector bundles* on X (a family of $\mathbb{C}$-vector spaces E_x for $x \in X$, depending holomorphically on x). A general coherent sheaf is roughly a (singular) vector bundle on a (singular) complex submanifold of X. They form an abelian category $\mathrm{coh}(X)$. Algebraic geometers care a lot about $\mathrm{coh}(X)$, and its derived category $D^b\,\mathrm{coh}(X)$. One version of Mirror Symmetry for Calabi–Yau 3-folds $X, \check{X}$ is as an equivalence $D^b\,\mathrm{coh}(X) \simeq D^b\mathcal{F}(\check{X})$. We can define *stability conditions* τ on $\mathrm{coh}(X)$: simple algebraic criteria for a coherent sheaf E to be τ-stable, τ-semistable, or τ-unstable. A stability condition τ on $\mathrm{coh}(X)$ assigns a (generalized) number $\tau(\alpha)$ to each nonzero topological type α, and $0 \neq F \in \mathrm{coh}(X)$ is τ-stable (or τ-semistable) if for all exact sequences $0 \to E \to F \to G \to 0$ with $E, G \neq 0$ we have $\tau(\llbracket E \rrbracket) < \tau(\llbracket G \rrbracket)$ (or $\tau(\llbracket E \rrbracket) \leqslant \tau(\llbracket G \rrbracket)$). Sometimes τ-semistability corresponds to existence of a solution of a p.d.e. On a complex surface X, τ-semistable vector bundles correspond to instantons.

Moduli spaces of (semistable) coherent sheaves

Let X be a compact algebraic complex manifold. Fix a stability condition τ on $\mathrm{coh}(X)$, and a topological type (Chern character) α of coherent sheaves on X. Algebraic geometers like to study the moduli spaces $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) \subseteq \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ of τ-(semi)stable coherent sheaves E in class α. These are geometric spaces (schemes or stacks) whose points correspond to isomorphism classes $[E]$ of such E.

We need to restrict to τ-semistables to make $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ well-behaved (actually, for it even to exist, in the world of schemes). It is compact and Hausdorff, and in nice cases (e.g. X a curve or Fano surface, α primitive) is a compact complex manifold.

There are different choices of stability conditions: τ depends on the Kähler class $[\omega]$ of a Kähler metric g on X. This $[\omega]$ lives in the Kähler cone $\mathcal{K}$, an open convex cone in $H^{1,1}_{\mathbb{R}}(X)$. As we vary $[\omega]$ continuously in $\mathcal{K}$, $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ can change discontinuously as we cross real hyperplanes ('walls') in $\mathcal{K}$. There is a *wall-and-chamber structure* on $\mathcal{K}$, such that $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ is constant on the interior of each chamber, and changes by deletion/addition as we cross walls. This is the beginning of the idea of 'wall-crossing'.

Invariants counting coherent sheaves on projective surfaces

Let X be a projective complex surface. Choose a stability condition τ on $\mathrm{coh}(X)$ and topological type α such that $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)=\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$, that is, all τ-semistable sheaves in class α are τ-stable. Then $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ may not be a complex manifold, but (Behrend–Fantechi 1997) it behaves like one, in the sense that it has a *virtual fundamental class* $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ in $H_*(\mathcal{M}_\alpha^{\mathrm{ss}}(\tau),\mathbb{Z})$. There is a large supply of natural cohomology classes $\Psi\in H^*(\mathcal{M}_\alpha^{\mathrm{ss}}(\tau),\mathbb{Q})$, from Chern characters of the universal sheaf. So we can make numbers $I_\alpha(\Psi)=\int_{[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}}\Psi\in\mathbb{Q}$. These are unchanged under deformations of the complex structure of X, so they are called *invariants*. Donaldson invariants, and 'instanton branch' Vafa–Witten invariants of X, are of the form $I_\alpha(\Psi)$ for particular Ψ. There is a literature on computing $I_\alpha(\Psi)$ for other Ψ (Segre invariants, Verlinde invariants, virtual χ_{-y}-genus, ...). Important contributions by Mochizuki 2009, Göttsche–Kool, ...

Donaldson–Thomas invariants of Calabi–Yau 3-folds

Let X be a Calabi–Yau 3-fold, and τ a stability condition on $\mathrm{coh}(X)$. *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ in $\mathbb{Z}$ or $\mathbb{Q}$ 'count' the moduli space $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$. They were defined by Thomas in 2000 when $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)=\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ using virtual classes, and by Joyce–Song in 2008 for general α using 'motivic' methods. They are unchanged under deformations of the complex structure of X. In String Theory they are 'numbers of BPS states', or 'numbers of black holes'. Kontsevich–Soibelman 2008 gave a 'refined' version $DT^\alpha_{\mathrm{ref}}(\tau)$. If $\tau,\tilde\tau$ are two stability conditions, Joyce–Song prove a *Wall-Crossing Formula (WCF)* giving $DT_\alpha(\tilde\tau)$ in terms of the $DT_\beta(\tau)$:

$$DT_\alpha(\tilde\tau)=\sum_{\substack{\text{directed trees }\Gamma,\\ \text{vertices } v \text{ labelled by}\\ \text{topological types } \alpha_v,\\ \sum_{\text{vertices } v}\alpha_v=\alpha}} V(\Gamma,(\alpha_v),\tau,\tilde\tau)\prod_{\substack{\text{vertices}\\ v\text{ in }\Gamma}} DT_{\alpha_v}(\tau)\prod_{\substack{\text{edges}\\ v\to w\text{ in }\Gamma}}\chi(\alpha_v,\alpha_w), \tag{1}$$

where $V(\Gamma,(\alpha_v),\tau,\tilde\tau)\in\mathbb{Q}$ is a combinatorial coefficient. This built on a WCF by Joyce 2004 for other motivic invariants. Kontsevich–Soibelman gave a WCF equivalent to this for $DT^\alpha_{\mathrm{ref}}(\tau)$.

Wall-Crossing Formulae: crossing a 'simple wall'

Suppose we have stability conditions $\tau, \tilde\tau$ on X, and a topological type α such that the only change between $\mathcal{M}^{\rm ss}_\alpha(\tau)$ and $\mathcal{M}^{\rm ss}_\alpha(\tilde\tau)$ comes from a single splitting $\alpha = \alpha_1 + \alpha_2$ with $\tau(\alpha_1) > \tau(\alpha_2)$ and $\tilde\tau(\alpha_1) < \tilde\tau(\alpha_2)$. Then $\mathcal{M}^{\rm ss}_\alpha(\tau)$ and $\mathcal{M}^{\rm ss}_\alpha(\tilde\tau)$ differ by:

- If we have a non-split exact sequence $0 \to E_1 \to F \to E_2 \to 0$ with $[E_i] \in \mathcal{M}^{\rm ss}_{\alpha_i}(\tau) = \mathcal{M}^{\rm ss}_{\alpha_i}(\tilde\tau)$ for $i = 1, 2$ then $[F]$ lies in $\mathcal{M}^{\rm ss}_\alpha(\tilde\tau)$ but not in $\mathcal{M}^{\rm ss}_\alpha(\tau)$. The family of such $[F]$ is $\mathbb{P}(\mathrm{Ext}^1(E_2, E_1))$.
- If instead we have $0 \to E_2 \to F' \to E_1 \to 0$ then $[F']$ lies in $\mathcal{M}^{\rm ss}_\alpha(\tau)$ but not in $\mathcal{M}^{\rm ss}_\alpha(\tilde\tau)$. The family of such $[F']$ is $\mathbb{P}(\mathrm{Ext}^1(E_1, E_2))$.

The CY3 property implies that Euler characteristics satisfy $\chi(\mathbb{P}(\mathrm{Ext}^1(E_2, E_1))) - \chi(\mathbb{P}(\mathrm{Ext}^1(E_1, E_2))) = \chi(\alpha_1, \alpha_2)$. We deduce

$$\chi(\mathcal{M}^{\rm ss}_\alpha(\tilde\tau)) = \chi(\mathcal{M}^{\rm ss}_\alpha(\tau)) + \chi(\alpha_1, \alpha_2) \cdot \chi(\mathcal{M}^{\rm ss}_{\alpha_1}(\tau)) \cdot \chi(\mathcal{M}^{\rm ss}_{\alpha_2}(\tau)).$$

As DT invariants $DT_\alpha(\tau)$ are weighted Euler characteristics of $\mathcal{M}^{\rm ss}_\alpha(\tau)$, by a proof including weights we can show

$$DT_\alpha(\tilde\tau) = DT_\alpha(\tau) + (-1)^{\chi(\alpha_1,\alpha_2)} \chi(\alpha_1, \alpha_2) \cdot DT_{\alpha_1}(\tau) \cdot DT_{\alpha_2}(\tau). \quad (2)$$

Equation (1) can be deduced from this by crossing many simple walls.

Motivic and homological invariants

There are two fundamentally different kinds of invariants $I_\alpha(\tau)$ 'counting' moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau)$:

- *Motivic invariants*, taking values in a ring R (such as $R=\mathbb{Z}$ or $\mathbb{Q}$) and defined using an invariant $\Upsilon(S)\in R$ of schemes S, such as the Euler characteristic $\chi(S)$, which is additive: $\Upsilon(S)=\Upsilon(S\setminus T)+\Upsilon(T)$ for $T\subset S$ closed.
- *Homological invariants*, which are basically virtual fundamental classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}\in H_*(\mathcal{M}_\alpha^{\rm ss}(\tau),\mathbb{Q})$, at least when $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$. We can get numbers by integrating canonical cohomology classes over these homology classes, but the homology class should be considered to be the underlying invariant.

Donaldson–Thomas invariants $DT_\alpha(\tau)$ are both: they are numbers in $\mathbb{Z}$ or $\mathbb{Q}$ which are also 0-dimensional homology classes. Although the two kinds of invariants have similarities (for example, wall-crossing formulae) they use rather different techniques. Motivic invariants are generally easier.

Universal structure of enumerative invariants

I have shown that many different enumerative invariant theories share a common universal structure. Begin with the ingredients:

• Let $\mathcal{A}$ be a $\mathbb{C}$-linear additive category coming from geometry or algebra. Examples include $\mathrm{coh}(X)$, $D^b\,\mathrm{coh}(X)$ for X a projective complex manifold (usually of small dimension $\dim X = 1, 2, 3$ or 4, and maybe with additional assumptions like Calabi–Yau or Fano), and categories of representations from algebra, e.g. $\text{mod-}\mathbb{C}Q$. 'Additive' means we have direct sums $E \oplus F$ of objects E, F in $\mathcal{A}$.

• Let $K(\mathcal{A})$ be an abelian group, with a notion of topological type $[\![E]\!] \in K(\mathcal{A})$ of objects E in $\mathcal{A}$.

• Let τ be a (nice) stability condition on $\mathcal{A}$ which factors via $K(\mathcal{A})$.

Then we can form moduli spaces $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) \subseteq \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ of τ-(semi)stable objects E in $\mathcal{A}$ with $[\![E]\!] = \alpha$ in $K(\mathcal{A})$.

We consider enumerative invariants $I_\alpha(\tau)$ which 'count' $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$.

For two stability conditions $\tau, \tilde{\tau}$, we expect there to to be a wall-crossing formula which writes $I_\alpha(\tilde{\tau})$ in terms of the $I_\beta(\tau)$.

Universal structure: Lie algebras

All the invariant theories of additive categories I have worked with have this in common: you can write them in terms of a Lie algebra $\mathcal{L} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{L}_\alpha$ graded over $K(\mathcal{A})$, such that $I_\alpha(\tau)$ lies in $\mathcal{L}_\alpha$, and the wall-crossing formula has the form

$$I_\alpha(\tilde\tau) = \sum_{\substack{\alpha=\alpha_1+\cdots+\alpha_n:\\ \alpha_i \in K(\mathcal{A})}} \tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot [\cdots[[I_{\alpha_1}(\tau), I_{\alpha_2}(\tau)],\ldots], I_{\alpha_n}(\tau)], \tag{3}$$

where $\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\in\mathbb{Q}$ are combinatorial coefficients defined in my 2004 papers, and independent of the invariant theory. But different invariant theories have different Lie algebras $\mathcal{L}$.

For crossing a 'simple wall' this reduces to

$$I_\alpha(\tilde\tau) = I_\alpha(\tau) + [I_{\alpha_1}(\tau), I_{\alpha_2}(\tau)]. \tag{4}$$

For Donaldson–Thomas invariants of Calabi–Yau 3-folds we write $\mathcal{L} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathbb{Q}\cdot e^\alpha$ with Lie bracket $[e^\alpha, e^\beta] = (-1)^{\chi(\alpha,\beta)}\chi(\alpha,\beta)e^{\alpha+\beta}$, and set $I_\alpha(\tau) = DT_\alpha(\tau)e^\alpha$. Then (3)–(4) reduce to (1)–(2).

Universal structure: counting strictly semistables

When invariants can be defined, $I_\alpha(\tau)$ usually has a nice geometric definition when $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)=\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$: then $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ is a proper $\mathbb{C}$-scheme, and $I_\alpha(\tau)$ is a motivic invariant, or virtual class, of $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$. However, if $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)\neq\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ then defining well-behaved invariants $I_\alpha(\tau)$ is surprisingly difficult. We must take into account the automorphism groups $\mathrm{Aut}(E)$ of $[E]\in\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$. If E is τ-stable then $\mathrm{Aut}(E)=\mathbb{C}^*\cdot\mathrm{id}_E$, so dividing all automorphism groups out by $\mathbb{C}^*$ we can treat them as trivial, and model $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ as a scheme. But strictly τ-semistable E can have automorphism groups like $\mathrm{GL}(r,\mathbb{C})$. Then we must take $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ to be an *Artin stack*, a more complicated space remembering automorphism groups, modelled on quotients S/G for S a scheme and G a complex Lie group.
Defining motivic invariants of stacks is a problem. For Euler characteristics we want $\chi(S/G)=\chi(S)/\chi(G)$. But $\chi(G)=0$ for any G with $\mathrm{rank}\,G>0$, so we have to 'divide by zero'.
Defining virtual classes of stacks is also a problem.
It turns out that there is a complicated, essentially unique way to define $I_\alpha(\tau)$ when $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)\neq\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$, so that the WCF (3) holds.

Universal structure: homological invariants

There are enumerative invariants counting τ-semistable objects in additive categories $\mathcal{A}$ in which the primary invariant is a homology class in the homology $H_*(\mathcal{M},\mathbb{Q})$ of the moduli stack $\mathcal{M}$ of objects in $\mathcal{A}$. Theories of this kind include invariants counting coherent sheaves on complex projective curves, surfaces, Calabi–Yau 3-folds, Fano 3-folds, and Calabi–Yau 4-folds, and invariants counting representations of a quiver, or quiver with relations.

In arXiv:2111.04694 I develop a general theory of homological invariants and WCF in an additive category $\mathcal{A}$ satisfying a list of assumptions. This currently includes the cases of curves, surfaces, Fano 3-folds, and quivers (with relations) with no oriented cycles, and extensions to G-equivariant homology. I expect in future that the theory will extend to Calabi–Yau 4-folds, and other directions including various generalized homology theories.

Lie algebras for the homological invariant theory

To fit homological enumerative invariants into my universal structure, we need a Lie algebra $\mathcal{L}=\bigoplus_{\alpha\in K(\mathcal{A})}\mathcal{L}_\alpha$ such that the invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ take values in $\mathcal{L}_\alpha$, and the change of stability condition WCF (3) is written using the Lie bracket in $\mathcal{L}$. We can work out what the Lie algebra has to be by studying wall-crossing for simple walls. The answer is surprising. There are two different stacks $\mathcal{M},\mathcal{M}^{\rm pl}$ which are moduli spaces of all objects in $\mathcal{A}$, where for $E\in\mathcal{A}$, the usual moduli stack $\mathcal{M}$ remembers the symmetry group $\mathrm{Aut}(E)$, and the 'projective linear' moduli stack $\mathcal{M}^{\rm pl}$ remembers $\mathrm{Aut}(E)/\mathbb{C}^*\cdot\mathrm{id}_E$. There is a $B\mathbb{C}^*$-fibration $\Pi:\mathcal{M}\rightarrow\mathcal{M}^{\rm pl}$. It turns out that the homology $H_*(\mathcal{M},\mathbb{Q})$ has the structure of a *graded vertex algebra*, a very complicated object from Conformal Field Theory. (There should be a physical explanation for this, but I do not know it.) The vertex algebra induces a Lie algebra structure on $H_*(\mathcal{M},\mathbb{Q})/D(H_*(\mathcal{M},\mathbb{Q}))$ (Borcherds). The projection $\Pi:\mathcal{M}\rightarrow\mathcal{M}^{\rm pl}$ gives an isomorphism $H_*(\mathcal{M},\mathbb{Q})/D(H_*(\mathcal{M},\mathbb{Q}))\xrightarrow{\cong}H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$, so $H_*(\mathcal{M}^{\rm pl},\mathbb{Q})$ is a Lie algebra. We take $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}\in H_*(\mathcal{M}_\alpha^{\rm pl},\mathbb{Q})=\mathcal{L}_\alpha\subset H_*(\mathcal{M}^{\rm pl},\mathbb{Q})=\mathcal{L}$.

Applications of Wall-Crossing Formulae

Wall-Crossing Formulae are powerful tools. Some applications:

- Gromov–Witten invariants, rank 1 Donaldson–Thomas invariants, and Pandharipande–Thomas invariants are all used to count curves in projective complex 3-manifolds. There are conjectures relating GW/DT/PT invariants. WCF are used to prove the DT/PT correspondence and symmetries of PT generating functions (Bridgeland, Toda, Anderson–Joyce, Karpov–Moreira).
- We can use WCF to compute invariants counting rank $r \geqslant 1$ coherent sheaves (and 'pairs') on projective surfaces by induction on $r = 1, 2, \ldots,$ starting with rank 1 invariants and algebraic eiberg–Witten invariants. A 'by-hand' version of this for $r = 2, 3$ goes back to Mochizuki 2009, but my Lie algebras and WCF theory is more general and more structured.
- Feyzbakhsh–Thomas use WCF to show that rank r DT invariants of Calabi–Yau 3-folds can be determined by induction on r from rank 1. Extended to Fano 3-folds by Karpov–Moreira 2026 using (an extension of) my theory.

A return to Donaldson theory?

I expect that there should be a version of my universal invariant theory that works in gauge theory of 4-manifolds instead of algebraic geometry, and subsumes Donaldson theory, Seiberg–Witten theory, and maybe a kind of Vafa–Witten theory. Let X be a compact, oriented 4-manifold, and G a compact Lie group. Then fundamental classes of moduli spaces of connections on principal G-bundles can be defined in $H_*(\mathrm{Map}_{C^0}(X, BG), \mathbb{Q}))$. First restrict to $G = \mathrm{U}(r)$. (A version for $\mathrm{SU}(r)$ follows by fixing determinants.) Moduli spaces of $\mathrm{U}(r)$-instantons should be compactified by adding 'ideal instantons'. I expect this corresponds to replacing $B\mathrm{U}(r)$ by its completion $B\mathrm{U}$. I can define a vertex algebra structure on $H_*(\mathrm{Map}_{C^0}(X, B\mathrm{U} \times \mathbb{Z}), \mathbb{Q}))$ (the $\mathbb{Z}$ factor keeps track of the rank r). This induces a Lie algebra structure on $H_*(\mathrm{Map}_{C^0}(X, B\mathrm{U} \times \mathbb{Z})/B\mathrm{U}(1), \mathbb{Q}))$. We should show that fundamental classes of $\mathrm{U}(r)$ instanton moduli spaces in Donaldson theory can be defined in $H_*(\mathrm{Map}_{C^0}(X, B\mathrm{U} \times \mathbb{Z})/B\mathrm{U}(1), \mathbb{Q}))$, and satisfy identities such as the WCF (3) in this Lie algebra.

Witten's formula writing $\mathrm{SU}(2)$ Donaldson invariants in terms of Seiberg–Witten invariants can now be understood as the analogue of the induction by rank construction for invariants counting rank r coherent sheaves on a projective surface (which also involves algebraic Seiberg–Witten invariants), in the first case $r = 2$.
Assuming a conjecture, my methods provide a way to extend this to $\mathrm{SU}(r)$ and $\mathrm{U}(r)$ Donaldson invariants for all $r \geqslant 2$.
My methods should also provide a systematic way to define invariants counting instanton moduli spaces containing reducible connections – the analogue of strictly stable coherent sheaves in Donaldson theory. Reducibles are usually regarded as a problem to be avoided as much as possible, which is why much work in Donaldson theory focused on $G = \mathrm{SU}(2)$ only.