

Vertex algebra and Lie algebra structures on the homology of moduli spaces

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Plan of talk:

- 1 Introduction
- 2 Vertex algebras on homology of moduli stacks
- 3 Lie algebras on homology of moduli stacks
- 4 Examples

1.1. Introduction

Vertex algebras are a very complicated, very rich algebraic structure coming from Conformal Field Theory in Physics, which also occur in Moonshine and other areas of Representation Theory. All interesting vertex algebras are infinite-dimensional.

I will explain a new construction of (graded) vertex algebra structures on the homology $H_*(\mathfrak{M})$ of certain moduli stacks \mathfrak{M} . It is extraordinarily general, and produces a huge number of examples. There are versions in Algebraic Geometry, Differential Geometry (using topological stacks), and Representation Theory. There is a functor from (graded) vertex algebras V to (graded) Lie algebras V_{Lie} . Roughly, the Lie algebra $H_*(\mathfrak{M})_{\text{Lie}}$ is the homology $H_*(\mathfrak{M}^{\text{pl}})$ of the associated ‘projective linear’ moduli stack $\mathfrak{M}^{\text{pl}} = \mathfrak{M}/[*/\mathbb{G}_m]$. Thus, we have a parallel new construction of infinite-dimensional (graded) Lie algebras $H_*(\mathfrak{M}^{\text{pl}})$.

This is interesting for lots of reasons, for example:

- Vertex algebras are often rather rigid and have well behaved representation theory, e.g. their characters may have modular properties. Use this to understand homology of moduli spaces.
- Hope to explain work of Grojnowski, Nakajima, ... on Heisenberg algebra actions on homology of Hilbert schemes.
- Understand correct notion of ‘quantum vertex algebra’?
- Applications to enumerative invariant theories in Algebraic and Differential Geometry, e.g. counting semistable coherent sheaves on surfaces, D–T style invariants for Fano 3-folds and C–Y 4-folds, Casson invariants, Donaldson and S–W invariants of 4-manifolds. In the AG case, semistable moduli schemes $\mathcal{M}_\alpha^{\text{st}}(\tau)$ are open substacks of the projective linear moduli stack \mathfrak{M}^{pl} . So a virtual class for $\mathcal{M}_\alpha^{\text{st}}(\tau)$ (if stable=semistable in class α) gives an element $[\mathcal{M}_\alpha^{\text{st}}(\tau)]_{\text{virt}}$ in $H_*(\mathfrak{M}^{\text{pl}})$, which is now a Lie algebra. I believe there is a universal wall-crossing formulae for $[\mathcal{M}_\alpha^{\text{st}}(\tau)]_{\text{virt}}$ under change of stability condition, in terms of the Lie bracket on $H_*(\mathfrak{M}^{\text{pl}})$.

1.2. Vertex algebras (don't try to understand this slide.)

Let R be a commutative ring. A *vertex algebra* over R is an R -module V equipped with morphisms $D^{(n)} : V \rightarrow V$ for $n = 0, 1, 2, \dots$ with $D^{(0)} = \text{id}_V$ and $v_n : V \rightarrow V$ for all $v \in V$ and $n \in \mathbb{Z}$, with v_n R -linear in v , and a distinguished element $\mathbb{1} \in V$ called the *identity* or *vacuum vector*, satisfying:

- (i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.
- (ii) If $v \in V$ then $\mathbb{1}_{-1}(v) = v$ and $\mathbb{1}_n(v) = 0$ for $-1 \neq n \in \mathbb{Z}$.
- (iii) If $v \in V$ then $v_n(\mathbb{1}) = D^{(-n-1)}(v)$ for $n < 0$ and $v_n(\mathbb{1}) = 0$ for $n \geq 0$.
- (iv) $u_n(v) = \sum_{k \geq 0} (-1)^{k+n+1} D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.
- (v) $(u_l(v))_m(w) = \sum_{n \geq 0} (-1)^n \binom{l}{n} (u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w)))$

for all $u, v, w \in V$ and $l, m \in \mathbb{Z}$, where the sum makes sense by (i). We can also define *graded vertex algebras* and *vertex superalgebras*.

It is usual to encode the maps $u_n : V \rightarrow V$ for $n \in \mathbb{Z}$ in generating function form as R -linear maps for each $u \in V$

$$Y(u, z) : V \longrightarrow V[[z, z^{-1}]], \quad Y(u, z) : v \longmapsto \sum_{n \in \mathbb{Z}} u_n(v) z^{-n-1},$$

where z is a formal variable. The $Y(u, z)$ are called *fields*, and have a meaning in Physics. Parts (i)–(v) may be rewritten as properties of the $Y(u, z)$. One interesting property is this: for all $u, v, w \in V$ there exist $N \gg 0$ depending on u, v such that

$$(y - z)^N Y(u, y) Y(v, z) w = (y - z)^N Y(v, z) Y(u, y) w. \quad (1)$$

There may be a V -valued rational function $R(y, z)$ with poles when $y = 0$, $z = 0$ and $y = z$, such that the l.h.s. of (1) is a formal Laurent series convergent to $R(y, z)$ when $0 < |y| < |z|$, and the r.h.s. converges to $R(y, z)$ when $0 < |z| < |y|$.

Think of $u *_z v = Y(u, z)v$ as a multiplication on V depending on a complex variable z , with poles at $z = 0$. Very roughly, V is a commutative associative algebra under $*_z$, with identity $\mathbb{1}$, except the formal power series and poles make everything more complicated.

Any commutative algebra $(V, \mathbb{1}, \cdot)$ with derivation D is a vertex algebra, with $Y(u, z)v = (e^{zD}u) \cdot v$, so no poles, where $u_n(v) = (\frac{1}{(n+1)!} D^{n+1}u) \cdot v$ for $n \geq -1$, and $u_n(v) = 0$ for $n < -1$. We call such V a *commutative vertex algebra*. All non-commutative vertex algebras are infinite-dimensional, so even the simplest nontrivial examples are large, complicated objects, which are difficult to write down.

Let R be a field of characteristic zero. A *vertex operator algebra* (VOA) over R is a vertex algebra V over R , with a distinguished *conformal element* $\omega \in V$ and a *central charge* $c_V \in R$, such that writing $L_n = \omega_{n+1} : V_* \rightarrow V_*$, the L_n define an action of the *Virasoro algebra* on V_* , with central charge c_V , and $L_{-1} = D^{(1)}$. VOAs are important in Physics. We will give a geometric construction of vertex algebras, but often they will *not* be VOAs.

If V is a (graded/super) vertex algebra then $V/\langle D^{(k)}(V), k \geq 1 \rangle$ is a (graded/super) Lie algebra, with Lie bracket

$$[u + \langle D^{(k)}(V), k \geq 1 \rangle, v + \langle D^{(k)}(V), k \geq 1 \rangle] = u_0(v) + \langle D^{(k)}(V), k \geq 1 \rangle.$$

Vertex algebras were introduced in mathematics by Borchers, who noticed that certain infinite-dimensional Lie algebras important in Representation Theory were constructed as $V/\langle D^{(k)}(V), k \geq 1 \rangle$. For example, Kac–Moody Lie algebras are (Lie subalgebras of) the Lie algebras associated to lattice vertex algebras.

Vertex algebras are used in Representation Theory, both of infinite-dimensional Lie algebras, and in Moonshine – the Monster may be characterized as the symmetry group of a certain infinite-dimensional vertex algebra.

2. Vertex algebras on homology of moduli stacks

We explain the Algebraic Geometry version first. Let \mathbb{K} be a field, and \mathcal{A} a \mathbb{K} -linear abelian category satisfying some conditions, e.g. \mathcal{A} could be the category $\text{mod-}\mathbb{K}Q$ of representations of a quiver Q , or the category $\text{coh}(X)$ of coherent sheaves on a smooth projective \mathbb{K} -scheme X . Write \mathfrak{M} for the moduli stack of objects in \mathcal{A} , which should be an Artin \mathbb{K} -stack, locally of finite type. Suppose we have a homology theory $H_*(-)$ of Artin \mathbb{K} -stacks over a commutative ring R (e.g. $R = \mathbb{Q}$), satisfying some axioms. Given some extra data on \mathfrak{M} , we will define a vertex algebra structure on the homology $H_*(\mathfrak{M})$. We also define a graded Lie bracket $[,]$ on either a modification $H_*(\mathfrak{M})^{t=0}$ of $H_*(\mathfrak{M})$, or the homology $H_*(\mathfrak{M}^{\text{pl}})$ of a modification $\mathfrak{M}^{\text{pl}} = \mathfrak{M}/[*/\mathbb{G}_m]$ of \mathfrak{M} , making $H_*(\mathfrak{M}^{\text{pl}})$ into a *graded Lie (super)algebra* (with a nonstandard grading).

The extra data we need

We need some extra data, a perfect complex Θ^\bullet on $\mathfrak{M} \times \mathfrak{M}$ satisfying some assumptions; the formulae for the vertex and Lie algebra structures involve $\text{rank } \Theta^\bullet$ and Chern classes $c_i(\Theta^\bullet)$. We also need signs $\epsilon_{\alpha,\beta}$ related to ‘orientation data’ for \mathcal{A} . For graded antisymmetry of $[,]$ we need $\sigma^*(\Theta^\bullet) \cong (\Theta^\bullet)^\vee[2n]$ for some $n \in \mathbb{Z}$, where $\sigma : \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M} \times \mathfrak{M}$ exchanges the factors, as then $c_i(\sigma^*(\Theta^\bullet)) = (-1)^i c_i(\Theta^\bullet)$. In our examples there is a natural perfect complex $\mathcal{E}xt^\bullet$ on $\mathfrak{M} \times \mathfrak{M}$ with $H^i(\mathcal{E}xt^\bullet|_{([E],[F])}) \cong \text{Ext}_{\mathcal{A}}^i(E, F)$ for $E, F \in \mathcal{A}$ and $i \in \mathbb{Z}$. If \mathcal{A} is a $2n$ -Calabi–Yau category then $\sigma^*((\mathcal{E}xt^\bullet)^\vee) \cong \mathcal{E}xt^\bullet[2n]$, and we put $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$. Otherwise we put $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee + \sigma^*(\mathcal{E}xt^\bullet)[2n]$. Thus examples split into ‘even Calabi–Yau’ and ‘general’ vertex algebras.

More detail on the basic set-up

Let $K(\mathcal{A})$ be a quotient group of the Grothendieck group $K_0(\mathcal{A})$ of \mathcal{A} such that $\mathfrak{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathfrak{M}_\alpha$, with \mathfrak{M}_α the moduli stack of objects $E \in \mathcal{A}$ in class α in $K(\mathcal{A})$, an open and closed substack in \mathfrak{M} . We suppose we are given a biadditive map $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \rightarrow \mathbb{Z}$ called the *Euler form*, with $\chi(\alpha, \beta) = \chi(\beta, \alpha)$. The restriction $\Theta_{\alpha, \beta}^\bullet = \Theta^\bullet|_{\mathfrak{M}_\alpha \times \mathfrak{M}_\beta}$ should have rank $\Theta_{\alpha, \beta}^\bullet = \chi(\alpha, \beta)$.

There should be an Artin stack morphism $\Phi : \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M}$ mapping $\Phi(\mathbb{K}) : ([E], [F]) \mapsto [E \oplus F]$ on \mathbb{K} -points, from direct sum in \mathcal{A} . It is associative and commutative. In perfect complexes on $\mathfrak{M}_\alpha \times \mathfrak{M}_\beta \times \mathfrak{M}_\gamma$ for $\alpha, \beta, \gamma \in K(\mathcal{A})$ we should have

$$(\Phi_{\alpha, \beta} \times \text{id}_{\mathfrak{M}_\gamma})^*(\Theta_{\alpha+\beta, \gamma}^\bullet) \cong \Pi_{\mathfrak{M}_\alpha \times \mathfrak{M}_\gamma}^*(\Theta_{\alpha, \gamma}^\bullet) \oplus \Pi_{\mathfrak{M}_\beta \times \mathfrak{M}_\gamma}^*(\Theta_{\beta, \gamma}^\bullet),$$

needed for the graded Jacobi identity for $[\ , \]$, and corresponding to

$$\text{Ext}_{\mathcal{A}}^i(E \oplus F, G)^* \cong \text{Ext}_{\mathcal{A}}^i(E, G)^* \oplus \text{Ext}_{\mathcal{A}}^i(F, G)^*.$$

The stack $[*/\mathbb{G}_m]$ and morphism Ψ

Write $\mathbb{G}_m = \mathbb{K} \setminus \{0\}$ as an algebraic \mathbb{K} -group under multiplication, and $[*/\mathbb{G}_m]$ for the quotient stack, where $*$ = $\text{Spec } \mathbb{K}$ is the point.

If S is an Artin \mathbb{K} -stack and $s \in S(\mathbb{K})$ a \mathbb{K} -point there is an *isotropy group* $\text{Iso}_S(s)$, an algebraic \mathbb{K} -group. We have

$\text{Iso}_{\mathfrak{M}}([E]) \cong \text{Aut}(E)$ for $E \in \mathcal{A}$. There is a natural morphism

$\mathbb{G}_m \rightarrow \text{Aut}(E)$ mapping $\lambda \mapsto \lambda \cdot \text{id}_E \in \text{Aut}(E) \subset \text{Hom}_{\mathcal{A}}(E, E)$.

There should be an Artin stack morphism $\Psi : [*/\mathbb{G}_m] \times \mathfrak{M} \rightarrow \mathfrak{M}$

mapping $(*, [E]) \mapsto [E]$ on \mathbb{K} -points, and acting on isotropy groups by

$$\Psi_* : \text{Iso}_{[*/\mathbb{G}_m] \times \mathfrak{M}}(*, [E]) \cong \mathbb{G}_m \times \text{Aut}(E) \longrightarrow \text{Iso}_{\mathfrak{M}}([E]) \cong \text{Aut}(E),$$

$$\Psi_* : (\lambda, \mu) \longmapsto (\lambda \cdot \text{id}_E) \circ \mu.$$

Here $[*/\mathbb{G}_m]$ is a *group stack*, and Ψ is an *action of $[*/\mathbb{G}_m]$ on \mathfrak{M}* , which is free except over $[0] \in \mathfrak{M}$. This Ψ encodes the natural

morphisms $\mathbb{G}_m \rightarrow \text{Iso}_{\mathfrak{M}}([E])$ for all $[E] \in \mathfrak{M}(\mathbb{K})$.

We require a compatibility between Ψ and Θ^\bullet , roughly that

$$(\Psi \times \text{id}_{\mathfrak{M}})^*(\Theta^\bullet) \cong \Pi_{[*/\mathbb{G}_m]}^*(L) \otimes \Pi_{\mathfrak{M} \times \mathfrak{M}}^*(\Theta^\bullet)$$

where L is the line bundle on $[*/\mathbb{G}_m]$ associated to the obvious representation of \mathbb{G}_m on \mathbb{K} . This corresponds to $\lambda \text{id}_E \in \text{Aut}(E)$ acting by multiplication by $\lambda \in \mathbb{G}_m$ on $\text{Ext}^i(E, F)^*$.

We should be given $\epsilon_{\alpha, \beta} = \pm 1$ for $\alpha, \beta \in K(\mathcal{A})$ satisfying

$$\epsilon_{\alpha, \beta} \cdot \epsilon_{\beta, \alpha} = (-1)^{\chi(\alpha, \beta) + \chi(\alpha, \alpha)\chi(\beta, \beta)},$$

$$\epsilon_{\alpha, \beta} \cdot \epsilon_{\alpha + \beta, \gamma} = \epsilon_{\alpha, \beta + \gamma} \cdot \epsilon_{\beta, \gamma}.$$

They are needed to correct signs in defining $[\ , \]$. Such $\epsilon_{\alpha, \beta}$ always exist. They are related to ‘orientation data’ as follows: if we have chosen ‘orientations’ for $\mathfrak{M}_\alpha, \mathfrak{M}_\beta, \mathfrak{M}_{\alpha + \beta}$, then $\epsilon_{\alpha, \beta}$ should be the natural sign comparing the orientations at $[E] \in \mathfrak{M}_\alpha(\mathbb{K})$, $[F] \in \mathfrak{M}_\beta(\mathbb{K})$ and $[E \oplus F] = \Phi([E], [F]) \in \mathfrak{M}_{\alpha + \beta}(\mathbb{K})$. (See work by Cao–Gross–Joyce–Tanaka–Upmeyer on orienting \mathfrak{M}_α , in arXiv:1811.01096, arXiv:1811.02405, arXiv:1811.09658.)

The homology of $[*/\mathbb{G}_m]$, and its action on $H_*(\mathfrak{M})$

Let $H_*(-)$ be a homology theory of Artin \mathbb{K} -stacks over a commutative ring R , satisfying some natural axioms. Then

$$H_i([*/\mathbb{G}_m]) \cong \begin{cases} R, & i = 0, 2, 4, 6, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

(This holds as the ‘classifying space’ of $[*/\mathbb{G}_m]$ is $\mathbb{K}\mathbb{P}^\infty$.) So we may write $H_*([*/\mathbb{G}_m]) \cong R[t]$, for t a formal variable of degree 2, such that t^n is a basis element for $H_{2n}([*/\mathbb{G}_m])$.

Let $\Omega : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \rightarrow [*/\mathbb{G}_m]$ be the stack morphism induced by the group morphism $\omega : \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m$ mapping $\omega : (\lambda, \mu) \mapsto \lambda\mu$. Define $\star : H_*([*/\mathbb{G}_m]) \times H_*([*/\mathbb{G}_m]) \rightarrow H_*([*/\mathbb{G}_m])$ by $\zeta \star \eta = H_*(\Omega)(\zeta \boxtimes \eta)$. Then \star makes $H_*([*/\mathbb{G}_m]) \cong R[t]$ into a commutative R -algebra, with $t^m \star t^n = \binom{m+n}{m} t^{m+n}$.

Define $\diamond : H_*([*/\mathbb{G}_m]) \times H_*(\mathfrak{M}) \rightarrow H_*(\mathfrak{M})$ by $\zeta \diamond \theta = H_*(\Psi)(\zeta \boxtimes \theta)$. Then \diamond makes $H_*(\mathfrak{M})$ into a module over $H_*([*/\mathbb{G}_m]) \cong R[t]$.

Bilinear operations $u_n(v)$ on $H_*(\mathfrak{M})$ and vertex algebras

Let $\alpha, \beta \in K(\mathcal{A})$ and $a, b \geq 0$, $n \in \mathbb{Z}$. Define an R -bilinear operation

$$H_a(\mathfrak{M}_\alpha) \times H_b(\mathfrak{M}_\beta) \longrightarrow H_{a+b-2n-2\chi(\alpha,\beta)-2}(\mathfrak{M}_{\alpha+\beta})$$

by, for all $u \in H_a(\mathfrak{M}_\alpha)$ and $v \in H_b(\mathfrak{M}_\beta)$,

$$u_n(v) = \sum_{\substack{i \geq 0: 2i \leq a+b, \\ i \geq n+\chi(\alpha,\beta)+1}} \epsilon_{\alpha,\beta} (-1)^{a\chi(\beta,\beta)} \cdot H_{a+b-2n-2\chi(\alpha,\beta)-2}(\Phi_{\alpha,\beta} \circ (\Psi_\alpha \times \text{id}_{\mathfrak{M}_\beta})) (t^{i-n-\chi(\alpha,\beta)-1} \boxtimes [(u \boxtimes v) \cap c_i([\Theta_{\alpha,\beta}^\bullet])]), \quad (2)$$

where $t^k \in H_{2k}([*/\mathbb{G}_m])$ as above. Define $D^{(k)}(u) = t^k \diamond u$, and let the vacuum vector $\mathbb{1}$ be $1 \in H_0(\mathfrak{M}_0)$.

Theorem

All this makes $H_*(\mathfrak{M}) = \bigoplus_{\alpha \in K(\mathcal{A})} H_*(\mathfrak{M}_\alpha)$ into a **graded vertex superalgebra** over R , with the shifted grading

$$\tilde{H}_i(\mathfrak{M}_\alpha) = H_{i-\chi(\alpha,\alpha)}(\mathfrak{M}_\alpha).$$

The proof uses properties of Chern classes, and combinatorial identities.

When R is a \mathbb{Q} -algebra, for $u \in H_a(\mathfrak{M}_\alpha)$ and $v \in H_b(\mathfrak{M}_\beta)$, we can rewrite (2) using Chern characters $\text{ch}_j(-)$ in the suggestive form

$$\begin{aligned} Y(u, z)v &= \sum_{n \in \mathbb{Z}} u_n(v) z^{-n-1} \\ &= \epsilon_{\alpha,\beta} (-1)^{a\chi(\beta,\beta)} z^{\chi(\alpha,\beta)} \cdot H_*(\Phi_{\alpha,\beta} \circ (\Psi_\alpha \times \text{id}_{\mathfrak{M}_\beta})) \\ &\quad \left\{ \left(\sum_{i \geq 0} z^i t^i \right) \boxtimes [(u \boxtimes v) \cap \exp\left(\sum_{j \geq 1} (-1)^{j-1} (j-1)! z^{-j} \text{ch}_j([\Theta_{\alpha,\beta}^\bullet])\right)] \right\}. \end{aligned}$$

Question

What is the interpretation of these vertex algebras in Physics?

3. Lie algebras on homology of moduli stacks

There are many different versions of our Lie algebra construction. Here is one of the simplest, which is well known in the theory of vertex algebras. Write $I_t = \langle t, t^2, t^3, \dots \rangle_R$ for the ideal in $H_*([*/\mathbb{G}_m]) = R[t]$ spanned over R by all positive powers of t . For each $\alpha \in K(\mathcal{A})$, define

$$H_*(\mathfrak{M}_\alpha)^{t=0} = H_*(\mathfrak{M}_\alpha)/(I_t \diamond H_*(\mathfrak{M}_\alpha)),$$

using the representation \diamond of $(R[t], \star)$ on $H_*(\mathfrak{M}_\alpha)$. Now define

$$[\ ,]^{t=0} : H_a(\mathfrak{M}_\alpha)^{t=0} \times H_b(\mathfrak{M}_\beta)^{t=0} \longrightarrow H_{a+b-2\chi(\alpha,\beta)-2}(\mathfrak{M}_{\alpha+\beta})^{t=0}$$

by $[u + (I_t \diamond H_*(\mathfrak{M}_\alpha)), v + (I_t \diamond H_*(\mathfrak{M}_\beta))]^{t=0} = u_0(v) + (I_t \diamond H_*(\mathfrak{M}_{\alpha+\beta}))$.

Define an alternative grading on $H_*(\mathfrak{M}_\alpha)^{t=0}$ by

$$\check{H}_i(\mathfrak{M}_\alpha)^{t=0} = H_{i+2-\chi(\alpha,\alpha)}(\mathfrak{M}_\alpha)^{t=0}.$$

Then using $\chi(\alpha, \beta) = \chi(\beta, \alpha)$ we find that $[\ ,]^{t=0}$ maps

$$[\ ,]^{t=0} : \check{H}_{\check{a}}(\mathfrak{M}_\alpha)^{t=0} \times \check{H}_{\check{b}}(\mathfrak{M}_\beta)^{t=0} \longrightarrow \check{H}_{\check{a}+\check{b}}(\mathfrak{M}_{\alpha+\beta})^{t=0}.$$

Using identities on the $u_n(v)$, we find that if $u \in \check{H}_{\check{a}}(\mathfrak{M}_\alpha)^{t=0}$, $v \in \check{H}_{\check{b}}(\mathfrak{M}_\beta)^{t=0}$ and $w \in \check{H}_{\check{c}}(\mathfrak{M}_\gamma)^{t=0}$ then

$$\begin{aligned} [v, u]^{t=0} &= (-1)^{\check{a}\check{b}+1}[u, v]^{t=0}, \\ (-1)^{\check{c}\check{a}}[[u, v]^{t=0}, w]^{t=0} &+ (-1)^{\check{a}\check{b}}[[v, w]^{t=0}, u]^{t=0} \\ &+ (-1)^{\check{b}\check{c}}[[w, u]^{t=0}, v]^{t=0} = 0. \end{aligned}$$

That is, $[\ ,]^{t=0}$ is a *graded Lie bracket* on $\check{H}_*(\mathfrak{M})^{t=0} = \bigoplus_{\alpha \in K(\mathcal{A})} \check{H}_*(\mathfrak{M}_\alpha)^{t=0}$, as we want.

The 'projective linear' Lie algebra

A disadvantage of the ' $t = 0$ ' version is that $H_*(\mathfrak{M})^{t=0}$ is not presented as the homology of a nice space. The 'projective linear' version corrects this. Recall that $[*/\mathbb{G}_m]$ is a group stack, and $\Psi : [*/\mathbb{G}_m] \times \mathfrak{M} \rightarrow \mathfrak{M}$ is an action of $[*/\mathbb{G}_m]$ on \mathfrak{M} , which is free on $\mathfrak{M}' = \mathfrak{M} \setminus \{[0]\}$. We can form a quotient $\mathfrak{M}^{\text{pl}} = \mathfrak{M}'/[*/\mathbb{G}_m]$ called the 'projective linear moduli stack', with a morphism $\Pi^{\text{pl}} : \mathfrak{M}' \rightarrow \mathfrak{M}^{\text{pl}}$ which is a principal $[*/\mathbb{G}_m]$ -bundle.

Then \mathbb{K} -points of \mathfrak{M}^{pl} are isomorphism classes $[E]$ of nonzero $E \in \mathcal{A}$, and isotropy groups are

$$\text{ISO}_{\mathfrak{M}^{\text{pl}}}([E]) \cong \text{Aut}(E)/(\mathbb{G}_m \cdot \text{id}_E).$$

That is, we make \mathfrak{M}^{pl} from \mathfrak{M}' by quotienting out \mathbb{G}_m from each isotropy group, a process called 'rigidification'. For moduli of stable coherent sheaves, the stable moduli scheme is the rigidification of the stable moduli stack.

Under some assumptions (including R a \mathbb{Q} -algebra) we can show that $H_*(\Pi^{\text{pl}}) : H_*(\mathfrak{M}') \rightarrow H_*(\mathfrak{M}^{\text{pl}})$ induces an isomorphism $H_*(\mathfrak{M}')^{t=0} \cong H_*(\mathfrak{M}^{\text{pl}})$. Thus, the Lie bracket $[\cdot, \cdot]^{t=0}$ on $H_*(\mathfrak{M}')^{t=0}$ induces a Lie bracket $[\cdot, \cdot]^{\text{pl}}$ on $H_*(\mathfrak{M}^{\text{pl}})$. Actually, even without an isomorphism $H_*(\mathfrak{M}')^{t=0} \cong H_*(\mathfrak{M}^{\text{pl}})$ we can define a graded Lie bracket $[\cdot, \cdot]^{\text{pl}}$ on $H_*(\mathfrak{M}^{\text{pl}})$ in a different way.

Here $[\cdot, \cdot]^{\text{pl}}$ is graded for the alternative grading

$$\check{H}_i(\mathfrak{M}_\alpha^{\text{pl}}) = H_{i+2-\chi(\alpha,\alpha)}(\mathfrak{M}_\alpha^{\text{pl}}).$$

We should interpret $2 - \chi(\alpha, \alpha)$ as the (homological) *virtual dimension* of $\mathfrak{M}_\alpha^{\text{pl}}$, where the 2 is the (real) dimension of \mathbb{G}_m , which we quotiented from the isotropy groups to make \mathfrak{M}^{pl} .

There is also a triangulated category version of the construction, using higher stacks, which we can apply to moduli of objects in categories such as $D^b \text{coh}(X)$ for X a smooth projective \mathbb{K} -scheme.

4. Examples

Let $Q = (Q_0, Q_1, h, t)$ be a quiver and $\mathbb{K} = \mathbb{C}$, and apply our constructions to the abelian category $\mathcal{A} = \text{mod-}\mathbb{C}Q$ and the triangulated category $\mathcal{T} = D^b \text{mod-}\mathbb{C}Q$. Write

$K(\mathcal{A}) = K(\mathcal{T}) = \mathbb{Z}^{Q_0}$ for the lattice of dimension vectors. Define $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \rightarrow \mathbb{Z}$ by $\chi(\mathbf{d}, \mathbf{e}) = \sum_{v,w \in Q_0} a_{vw} \mathbf{d}(v) \mathbf{e}(w)$,

where $a_{vw} = 2\delta_{vw} - n_{vw} - n_{wv}$, for n_{vw} the number of edges $\bullet^v \rightarrow \bullet^w$ in Q , so that $A = (a_{vw})_{v,w \in Q_0}$ is the generalized Cartan matrix of Q . Write \mathfrak{M} and $\overline{\mathfrak{M}}$ for the (higher) moduli stacks of objects in \mathcal{A} and \mathcal{T} . Then we can work everything out very explicitly. We find:

- The vertex algebra $H_*(\overline{\mathfrak{M}})$ is the lattice vertex algebra of (\mathbb{Z}^{Q_0}, χ) .
- The full Lie algebra $\check{H}_*(\overline{\mathfrak{M}}^{\text{pl}})$ is rather large, but (for Q with no vertex loops) $\check{H}_0(\overline{\mathfrak{M}}^{\text{pl}})$ contains the derived Kac–Moody algebra $\mathfrak{g}'(A)$ with Cartan matrix A , with $\check{H}_0(\overline{\mathfrak{M}}^{\text{pl}}) = \mathfrak{g}'(A)$ if A is positive definite. Similarly, $\check{H}_0(\overline{\mathfrak{M}}^{\text{pl}})$ contains/equals the positive part \mathfrak{n}_+ of $\mathfrak{g}'(A)$.
- If $Q = \bullet$ has one vertex and no edges then $\check{H}_0(\overline{\mathfrak{M}}^{\text{pl}}) \cong \mathfrak{sl}(2, \mathbb{C})$.

Let X be a smooth projective \mathbb{C} -scheme, and apply our theory to the abelian category $\mathcal{A} = \text{coh}(X)$, with moduli stack \mathfrak{M} , and the triangulated category $\mathcal{T} = D^b \text{coh}(X)$, with moduli stack $\overline{\mathfrak{M}}$. We either take X to be $2n$ -Calabi–Yau and set $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, or we set $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee + \sigma^*(\mathcal{E}xt^\bullet)$ for any X . Note that if X is $(2n+1)$ -Calabi–Yau this gives $c_i(\Theta^\bullet) = 0$, so our vertex algebras and Lie algebras are abelian, and boring.

I haven't worked out the details yet, but here are some highlights:

- For some nice classes of X (e.g. curves, some surfaces) we can compute $H_*(\overline{\mathfrak{M}})$ fairly explicitly as a vertex algebra. It is the tensor product of a lattice-type vertex algebra defined using $K^0(X)$ or $H^{\text{even}}(X)$, and a fermion vertex algebra defined using $K^1(X)$ or $H^{\text{odd}}(X)$. For general X we can produce vertex algebra morphisms from $H_*(\mathfrak{M})$, $H_*(\overline{\mathfrak{M}})$ to an explicit vertex algebra of this type.
- The Heisenberg algebra acting on homology of Hilbert schemes in Grojnowski–Nakajima should appear as a Lie subalgebra of $\check{H}_*(\overline{\mathfrak{M}}_{\dim 0}^{\text{pl}})$ for dimension 0 sheaves and complexes on X .

- When X is a Calabi–Yau 4-fold, one can define Donaldson–Thomas type invariants ‘counting’ moduli spaces $\mathfrak{M}^{\text{ss}}(\alpha)$ of (semi)stable coherent sheaves on X (Borisov–Joyce, Cao–Leung). We can think of these (i.e. the virtual classes of the moduli spaces) as taking values in $H_*(\mathfrak{M}^{\text{pl}})$.

I have a conjecture that the wall-crossing formula for these DT4 invariants under change of stability condition may be written using the Lie bracket $[\cdot, \cdot]^{\text{pl}}$ on $H_*(\mathfrak{M}^{\text{pl}})$, using the same universal wall-crossing formula in a Lie algebra that appears in my previous work on motivic and DT3 invariants.

The way I discovered the vertex algebra structure on $H_*(\mathfrak{M})$ was while trying to write down this Lie bracket $[\cdot, \cdot]^{\text{pl}}$, I accidentally reinvented the Borchers definition of vertex algebra, without knowing what these were at the time.

I expect the same wall-crossing formula will also work for Mochizuki invariants counting semistable coherent sheaves on surfaces, and other classes of enumerative invariants with wall-crossing.

The Differential Geometric version

Let X be a compact manifold with a real elliptic complex E^\bullet , e.g. for Donaldson theory on an oriented Riemannian 4-manifold (X, g) take

$$E^\bullet = (0 \rightarrow \Gamma(\Lambda^0 T^* X) \xrightarrow{d} \Gamma(\Lambda^1 T^* X) \xrightarrow{d_+} \Gamma(\Lambda^2_+ T^* X) \rightarrow 0).$$

Define a topological stack \mathfrak{M} to be either:

- The moduli stack of all connections ∇_P on all $U(n)$ -bundles $P \rightarrow X$, all $n \geq 0$, satisfying a curvature condition $F_+^\nabla = 0$.
- Moduli stack of $U(n)$ -connections, with no curvature condition.
- The topological space $\text{Map}_{C^0}(X, BU \times \mathbb{Z})$.

After choosing an ‘orientation’ on \mathfrak{M} (what you need to orient instanton moduli spaces, corresponding to $\epsilon_{\alpha, \beta}$ before) we can define a vertex algebra structure on $H_*(\mathfrak{M})$, and a Lie algebra structure on \mathfrak{M}^{pl} . Here if \mathfrak{M}_P is a quotient stack $[\mathcal{A}_P / \mathcal{G}_P]$ then $\mathfrak{M}_P^{\text{pl}} = [\mathcal{A}_P / \mathcal{G}_P^{\text{pl}}]$, where $U(1) \subset Z(\mathcal{G}_P)$ acts trivially on \mathcal{A}_P with $\mathcal{G}_P^{\text{pl}} = \mathcal{G}_P / U(1)$, and $H_*(\mathfrak{M}_P) \cong H_*^{\mathcal{G}_P}(\mathcal{A}_P)$, $H_*(\mathfrak{M}_P^{\text{pl}}) \cong H_*^{\mathcal{G}_P^{\text{pl}}}(\mathcal{A}_P)$ using equivariant homology.