Universal structures in enumerative invariant theories

Dominic Joyce, Oxford University

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These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`.

1. Outline of the conjectural picture

An *enumerative invariant theory* in Algebraic or Differential Geometry is the study of invariants $I_\alpha(\tau)$ which 'count' τ-semistable objects E with fixed topological invariants $[\![E]\!]=\alpha$ in some geometric problem, usually by means of a virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ for the moduli space $\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-semistable objects in some homology theory, with $I_\alpha(\tau)=\int_{[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}}\mu_\alpha$ for some natural cohomology class μ_α. We call the theory $\mathbb{C}$-*linear* if the objects E live in a $\mathbb{C}$-linear additive category $\mathcal{A}$. For example:

- Mochizuki-style invariants counting coherent sheaves on surfaces. (Think of as algebraic Donaldson invariants.)
- Donaldson–Thomas invariants of Calabi–Yau or Fano 3-folds.
- Donaldson–Thomas type invariants of Calabi–Yau 4-folds.
- $\mathrm{U}(m)$ Donaldson invariants of 4-manifolds.
- Invariants counting representations of quivers Q.

I conjecture that many such theories share a common universal structure. Here is an outline of this structure:

(a) We form two moduli stacks $\mathcal{M},\mathcal{M}^{\rm pl}$ of all objects E in $\mathcal{A}$, where $\mathcal{M}$ is the usual moduli stack, and $\mathcal{M}^{\rm pl}$ the 'projective linear' moduli stack of objects E modulo 'projective isomorphisms', i.e. quotient by $\lambda\,{\rm id}_E$ for $\lambda\in\mathbb{G}_m$ or $\mathrm{U}(1)$.

(b) We are given a quotient $K_0(\mathcal{A})\twoheadrightarrow K(\mathcal{A})$, where $K(\mathcal{A})$ is the lattice of topological invariants $[\![E]\!]$ of E (e.g. fixed Chern classes). We split $\mathcal{M}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha$, $\mathcal{M}^{\rm pl}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha^{\rm pl}$.

(c) There is a symmetric biadditive *Euler form* $\chi:K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$.

(d) We can form the homology $H_*(\mathcal{M}),H_*(\mathcal{M}^{\rm pl})$ over $\mathbb{Q}$, with $H_*(\mathcal{M})=\bigoplus_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}_\alpha)$, $H_*(\mathcal{M}^{\rm pl})=\bigoplus_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}_\alpha^{\rm pl})$. Define shifted versions $\hat H_*(\mathcal{M}),\check H_*(\mathcal{M}^{\rm pl})$ by $\hat H_n(\mathcal{M}_\alpha)=H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$, $\check H_n(\mathcal{M}_\alpha^{\rm pl})=H_{n+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl})$. Then previous work by me (later) makes $\hat H_*(\mathcal{M})$ into a *graded vertex algebra*, and $\check H_*(\mathcal{M}^{\rm pl})$ into a *graded Lie algebra*.

(e) There is a notion of *stability condition* τ on $\mathcal{A}$. When $\mathcal{A}=\mathrm{coh}(X)$, this can be Gieseker stability for a polarization on X. For Donaldson theory for a compact oriented 4-manifold X with $b^2_+(X)=1$, the stability condition is the splitting $H^2_{\mathrm{dR}}(X,\mathbb{R})=H^2_+(X)\oplus H^2_-(X)$ induced by a metric g. For each $\alpha\in K(\mathcal{A})$ we can form moduli spaces $\mathcal{M}^{\mathrm{st}}_\alpha(\tau)\subseteq\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$ of τ-(semi)stable objects in class α. Here $\mathcal{M}^{\mathrm{st}}_\alpha(\tau)$ is a substack of $\mathcal{M}^{\mathrm{pl}}_\alpha$, and has the structure of a 'virtual oriented manifold' (in Algebraic Geometry, it may be a $\mathbb{C}$-scheme with perfect obstruction theory; in Differential Geometry, under genericness it may be an oriented manifold). Also $\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$ is compact (proper). Thus, if $\mathcal{M}^{\mathrm{st}}_\alpha(\tau)=\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$ we have a virtual class $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$, which we regard as an element of $H_*(\mathcal{M}^{\mathrm{pl}}_\alpha)$. The virtual dimension is $\mathrm{vdim}_{\mathbb{R}}[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}=2-\chi(\alpha,\alpha)$, so $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ lies in $\check{H}_0(\mathcal{M}^{\mathrm{pl}}_\alpha)\subset\check{H}_0(\mathcal{M}^{\mathrm{pl}})$, which is a Lie algebra by (b).

We can prove all of (a)–(e) already in the cases we care about.

Here is the conjectural part (mostly proved now) of the picture:

(f) For many theories, there is a problem defining the invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$, i.e. when the moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau)$ contain *strictly* τ*-semistable points* (in gauge theory, these are *reducible connections*).
We conjecture there is a systematic way to define $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in homology over $\mathbb{Q}$ (not $\mathbb{Z}$) in these cases. (In gauge theory, this requires a condition analogous to $b^2_+\geqslant 1$.)

(g) If $\tau,\tilde\tau$ are stability conditions and $\alpha\in K(\mathcal{A})$, we expect that

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm virt}=\sum_{\alpha_1+\cdots+\alpha_n=\alpha}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm virt},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm virt}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm virt}\bigr],\qquad(1)$$

where $\tilde U(-)$ are combinatorial coefficients defined in my previous work on wall-crossing formulae for motivic invariants, and $[\,,\,]$ is the Lie bracket on $\check H_0(\mathcal{M}^{\rm pl})$ from (b).

(h) We can often give an explicit, inductive definition of the $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ using (1) and the method of *pair invariants*.

In the Algebraic Geometry case, the theory above is appropriate in cases when the natural obstruction theories on moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau)=\mathcal{M}_\alpha^{\rm st}(\tau)$ are perfect in $[-1,0]$. There are two situations when this is not true/not what we want, so we modify the picture:

(i) When $\mathcal{A}=\mathrm{coh}(X)$ or $D^b\,\mathrm{coh}(X)$ for X a Calabi–Yau 3-fold, the natural obstruction theory on $\mathcal{M}_\alpha^{\rm ss}(\tau)$ has terms in degree -2 from $\mathrm{Ext}^3(E,E)$. We can remove these by taking trace-free Ext to define Donaldson–Thomas invariants, changing the real virtual dimension by 2.
To include these in the theory, for $\mathcal{A}$ odd Calabi–Yau we can modify (d) above to make $\hat H_*(\mathcal{M})$ into a *graded vertex Lie algebra* (with grading changed by 2) and $\check H_*(\mathcal{M}^{\rm pl})$ into a *graded Lie algebra* (with grading changed by 2), as before.
So we can include Donaldson–Thomas theory in our picture.

(j) Let X be a projective surface with $h^{0,2}(X) > 0$, i.e. $b_+^2 > 1$, and consider moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau) = \mathcal{M}_\alpha^{\rm st}(\tau)$ of Gieseker stable sheaves in $\mathrm{coh}(X)$ for $\mathrm{rank}\,\alpha > 0$. Then the natural obstruction theory on $\mathcal{M}_\alpha^{\rm ss}(\tau)$ has a factor $H^{0,2}(X)^*$ in degree -1 which forces $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt} = 0$. So our theory works, but is boring, as the invariants are zero.
By deleting $H^{0,2}(X)^*$ from the obstruction theory, we can define *reduced invariants* $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$, which may be nonzero. I have a general version of (f)–(h) above for reduced invariants, in which a trivial bundle of rank o_α is deleted from obstruction theories on $\mathcal{M}_\alpha^{\rm ss}(\tau)$, so that $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red} \in \check{H}_{2o_\alpha}(\mathcal{M}_\alpha^{\rm pl})$. The wall-crossing formula (1) for $[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm red}$ is modified by only summing over $\alpha_1, \ldots, \alpha_n$ with $\alpha = \alpha_1 + \cdots + \alpha_n$ and $o_\alpha = o_{\alpha_1} + \cdots + o_{\alpha_n}$. This theory can handle algebraic Donaldson invariants when $b_+^2 > 1$, and categories combining Donaldson and Seiberg–Witten invariants (L-Bradlow pairs).

In Gross–Joyce–Tanaka 2005.05637 we prove our conjectures when $\mathcal{A} = \text{mod-}\mathbb{C}Q$ is the category of representations of a quiver Q without oriented cycles, and stability conditions τ are slope stability.

In work in progress, I have nearly finished proving the conjectures for a wide range of situations in Algebraic Geometry in which virtual classes are defined using Behrend–Fantechi perfect obstruction theories. This includes invariants counting coherent sheaves on curves, surfaces and Fano 3-folds.

I hope that my proof will also extend to Calabi–Yau 4-fold virtual classes (Borisov–Joyce/Oh–Thomas) without a huge amount of work. This would give a theory of Donaldson–Thomas type invariants for Calabi–Yau 4-folds.

It is a complete mystery to me why vertex algebras appear in this problem. I reinvented the Borcherds definition of vertex algebra when I was trying to write down the Lie bracket $[\,,\,]$ on $H_*(\mathcal{M}^{\rm pl})$ in the wall-crossing formula (1) – at the time I didn't know what a vertex algebra was. Maybe there is a Physics explanation?

Remarks on counting strictly τ-semistables

When $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$, the virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ are defined using a geometric structure on $\mathcal{M}_\alpha^{\rm ss}(\tau)$ (e.g. smooth $\mathbb{C}$-schemes, or $\mathbb{C}$-schemes with perfect obstruction theories, or -2-shifted symplectic derived schemes) by a known construction.

When $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$, we currently have *no definition* of $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in terms of a geometric structure on $\mathcal{M}_\alpha^{\rm ss}(\tau)$.

For quivers, our proof works by showing that there are unique $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$, extending the given ones when $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$, which also satisfy the wall-crossing formula (1). So the definition involves *all stability conditions*, not just one.

For motivic invariants, e.g. Joyce–Song Donaldson–Thomas invariants of Calabi–Yau 3-folds, there is a theory on how to count strictly τ-semistables (Joyce 2003-2008). It is a complicated mess, and uses rational weights. It is not directly applicable here, but motivates (1).

2. Vertex and Lie algebras on homology of moduli stacks

2.1. Vertex algebras (don't try to understand this slide.)

Let R be a commutative ring. A *vertex algebra* over R is an R-module V equipped with morphisms $D^{(n)} : V \to V$ for $n = 0, 1, 2, \ldots$ with $D^{(0)} = \mathrm{id}_V$ and $v_n : V \to V$ for all $v \in V$ and $n \in \mathbb{Z}$, with v_n R-linear in v, and a distinguished element $\mathbb{1} \in V$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $v \in V$ then $\mathbb{1}_{-1}(v) = v$ and $\mathbb{1}_n(v) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(iii) If $v \in V$ then $v_n(\mathbb{1}) = D^{(-n-1)}(v)$ for $n < 0$ and $v_n(\mathbb{1}) = 0$ for $n \geqslant 0$.

(iv) $u_n(v) = \sum_{k \geqslant 0} (-1)^{k+n+1} D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.

(v) $$(u_l(v))_m(w) = \sum_{n \geqslant 0} (-1)^n \binom{l}{n} \bigl(u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w))\bigr)$$

for all $u, v, w \in V$ and $l, m \in \mathbb{Z}$, where the sum makes sense by (i).

We can also define *graded vertex algebras* and *vertex superalgebras*.

It is usual to encode the maps $u_n : V \to V$ for $n \in \mathbb{Z}$ in generating function form as R-linear maps for each $u \in V$

$$Y(u,z) : V \longrightarrow V[[z,z^{-1}]], \quad Y(u,z) : v \longmapsto \sum_{n\in\mathbb{Z}} u_n(v) z^{-n-1},$$

where z is a formal variable. The $Y(u,z)$ are called *fields*, and have a meaning in Physics. Parts (i)–(v) may be rewritten as properties of the $Y(u,z)$. One interesting property is this: for all $u,v,w \in V$ there exist $N \gg 0$ depending on u,v such that

$$(y-z)^N Y(u,y) Y(v,z) w = (y-z)^N Y(v,z) Y(u,y) w. \qquad (2)$$

There may be a V-valued rational function $R(y,z)$ with poles when $y=0$, $z=0$ and $y=z$, such that the l.h.s. of (2) is a formal Laurent series convergent to $R(y,z)$ when $0<|y|<|z|$, and the r.h.s. converges to $R(y,z)$ when $0<|z|<|y|$.

Think of $u *_z v = Y(u,z)v$ as a multiplication on V depending on a complex variable z, with poles at $z=0$. Very roughly, V is a commutative associative algebra under $*_z$, with identity $\mathbb{1}$, except the formal power series and poles make everything more complicated.

Any commutative algebra $(V, \mathbb{1}, \cdot)$ with derivation D is a vertex algebra, with $Y(u,z)v = (e^{zD}u)\cdot v$, so no poles, where $u_n(v) = \bigl(\frac{1}{(n+1)!}D^{n+1}u\bigr)\cdot v$ for $n \geqslant -1$, and $u_n(v) = 0$ for $n < -1$. We call such V a *commutative vertex algebra*. All non-commutative vertex algebras are infinite-dimensional, so even the simplest nontrivial examples are large, complicated objects, which are difficult to write down.

Let R be a field of characteristic zero. A *vertex operator algebra* (*VOA*) over R is a vertex algebra V over R, with a distinguished *conformal element* $\omega \in V$ and a *central charge* $c_V \in R$, such that writing $L_n = \omega_{n+1} : V_* \to V_*$, the L_n define an action of the *Virasoro algebra* on V_*, with central charge c_V, and $L_{-1} = D^{(1)}$. VOAs are important in Physics. We will give a geometric construction of vertex algebras, but often they will *not* be VOAs.

If V is a (graded/super) vertex algebra then $V/\langle D^{(k)}(V),\ k \geqslant 1\rangle$ is a (graded/super) Lie algebra, with Lie bracket

$$\big[u+\langle D^{(k)}(V),\ k\geqslant 1\rangle, v+\langle D^{(k)}(V),\ k\geqslant 1\rangle\big]=u_0(v)+\langle D^{(k)}(V),\ k\geqslant 1\rangle.$$

Vertex algebras were introduced in mathematics by Borcherds, who noticed that certain infinite-dimensional Lie algebras important in Representation Theory were constructed as $V/\langle D^{(k)}(V),\ k \geqslant 1\rangle$. For example, Kac–Moody Lie algebras are (Lie subalgebras of) the Lie algebras associated to lattice vertex algebras.
Vertex algebras are used in Representation Theory, both of infinite-dimensional Lie algebras, and in Moonshine – the Monster may be characterized as the symmetry group of a certain infinite-dimensional vertex algebra.

2.2. Vertex and Lie algebras on homology of moduli stacks

We will explain the Algebraic Geometry version of our theory. Let $\mathcal{A}$ be a $\mathbb{C}$-linear abelian or triangulated category from Algebraic Geometry or Representation Theory, e.g. $\mathcal{A} = \mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme, or $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q$ or $D^b \mathrm{mod}\text{-}\mathbb{C}Q$. Write $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A}$, which is an Artin $\mathbb{C}$-stack in the abelian case, and a higher $\mathbb{C}$-stack in the triangulated case. There is a morphism $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ acting by $([E],[F]) \to [E \oplus F]$ on $\mathbb{C}$-points.

Now $\mathbb{G}_m$ acts on objects E in $\mathcal{A}$ with $\lambda \in \mathbb{G}_m$ acting as $\lambda\,\mathrm{id}_E : E \to E$. This induces an action $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ of the group stack $[*/\mathbb{G}_m]$ on $\mathcal{M}$. We write $\mathcal{M}^{\mathrm{pl}} = \mathcal{M}/[*/\mathbb{G}_m]$ for the quotient, called the 'projective linear' moduli stack. There is a morphism $\mathcal{M} \to \mathcal{M}^{\mathrm{pl}}$ which is a $[*/\mathbb{G}_m]$-fibration on $\mathcal{M} \setminus \{[0]\}$.

We need some extra data:

- A quotient $K_0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$ giving splittings $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\rm pl} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha^{\rm pl}$.
- A symmetric biadditive *Euler form* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$.
- A perfect complex $\Theta^\bullet$ on $\mathcal{M} \times \mathcal{M}$ satisfying some assumptions, including $\mathrm{rank}\, \Theta^\bullet|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta} = \chi(\alpha, \beta)$.
  If $\mathcal{A}$ is a 4-Calabi–Yau category, and we will use Borisov–Joyce virtual classes, we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, where $\mathcal{E}xt^\bullet \to \mathcal{M} \times \mathcal{M}$ is the *Ext complex*. Otherwise we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)$, where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ swaps the factors.
- Signs $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ for $\alpha, \beta \in K(\mathcal{A})$ with $\epsilon_{\alpha,\beta} \cdot \epsilon_{\alpha+\beta,\gamma} = \epsilon_{\alpha,\beta+\gamma} \cdot \epsilon_{\beta,\gamma}$ and $\epsilon_{\alpha,\beta} \cdot \epsilon_{\beta,\alpha} = (-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}$.
  (These compare orientations on $\mathcal{M}_\alpha, \mathcal{M}_\beta, \mathcal{M}_{\alpha+\beta}$.)

Then we can make the homology $H_*(\mathcal{M})$, with grading shifted to $\hat{H}_*(\mathcal{M})$ as above, into a *graded vertex algebra*.

Writing $H_*([*/\mathbb{G}_m])=\mathbb{Q}[t]$ with $\deg t=2$, the state-field correspondence $Y(z)$ is given by, for $u\in H_a(\mathcal{M}_\alpha)$, $v\in H_b(\mathcal{M}_\beta)$

$$Y(u,z)v=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}z^{\chi(\alpha,\beta)}\cdot H_*\bigl(\Phi\circ(\Psi\times\mathrm{id})\bigr) \tag{2}$$
$$\Bigl\{\bigl(\sum_{i\geqslant 0}z^it^i\bigr)\boxtimes\bigl[(u\boxtimes v)\cap\exp\bigl(\sum_{j\geqslant 1}(-1)^{j-1}(j-1)!z^{-j}\,\mathrm{ch}_j([\Theta^\bullet])\bigr)\bigr]\Bigr\}.$$

The identity $\mathbb{1}$ is $1\in H_0(\mathcal{M}_0)$. Define $e^{zD}:\check H_*(\mathcal{M})\rightarrow\check H_*(\mathcal{M})[[z]]$ by $Y(v,z)\mathbb{1}=e^{zD}v$. Then $(\check H_*(\mathcal{M}),\mathbb{1},e^{zD},Y)$ is a graded vertex algebra, so $\check H_{*+2}(\mathcal{M})/D(\check H_*(\mathcal{M}))$ is a graded Lie algebra. In the abelian category case at least, there is a canonical isomorphism $\check H_*(\mathcal{M}^{\mathrm{pl}})\cong\check H_{*+2}(\mathcal{M})/D(\check H_*(\mathcal{M}))$. This makes $\check H_*(\mathcal{M}^{\mathrm{pl}})$ into a graded Lie algebra, and $\check H_0(\mathcal{M}^{\mathrm{pl}})$ into a Lie algebra.

Remarks

- One can often write down $\check{H}_*(\mathcal{M})$ and $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ with their algebraic structures explicitly. The answer is usually simpler in the derived category case. For example, my student Jacob Gross showed that if a smooth projective $\mathbb{C}$-scheme X is a curve, surface, or toric variety, and $\mathcal{M}$ is the moduli stack of $D^b \operatorname{coh}(X)$, then

$$\hat{H}_*(\mathcal{M},\mathbb{Q}) \cong \mathbb{Q}[K^0_{\mathrm{sst}}(X)] \otimes_{\mathbb{Q}} \mathrm{Sym}^*\bigl(K^0(X^{\mathrm{an}}) \otimes_{\mathbb{Z}} t^2\mathbb{Q}[t^2]\bigr) \otimes_{\mathbb{Q}} \textstyle\bigwedge^*\bigl(K^1(X^{\mathrm{an}}) \otimes_{\mathbb{Z}} t\mathbb{Q}[t^2]\bigr), \qquad (3)$$

with a super-lattice vertex algebra structure. Thus we can use this for explicit computations in examples, as well as for abstract theory.
- It helps to study $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ in $\operatorname{coh}(X)$ using $H_*(\mathcal{M})$, $H_*(\mathcal{M}^{\mathrm{pl}})$ for $D^b \operatorname{coh}(X)$, so we can use the presentation (3).
- Although Lie algebras are much simpler than vertex algebras, it is difficult to write down the Lie bracket on $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ explicitly: the best way seems to be via the vertex algebra structure on $\hat{H}_*(\mathcal{M})$.

3. Enumerative invariants

3.1. Virtual classes of moduli spaces

The vertex and Lie algebras $\hat{H}_*(\mathcal{M})$, $\check{H}_*(\mathcal{M}^{\rm pl})$ above work for $\mathcal{M}$ the moduli stack of objects in $\mathrm{coh}(X)$ or $D^b\,\mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme of any dimension. However, defining virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ is much more restrictive:

- If $\dim\mathcal{A}=1$, say if $\mathcal{A}=\text{mod-}\mathbb{C}Q$ or $\mathcal{A}=\mathrm{coh}(X)$ for X a curve, then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is a smooth projective $\mathbb{C}$-scheme, and has a fundamental class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm fund}$.
- If $\dim\mathcal{A}=2$, say if $\mathcal{A}=\text{mod-}\mathbb{C}Q/I$ or $\mathcal{A}=\mathrm{coh}(X)$ for X a surface, then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is a projective $\mathbb{C}$-scheme with obstruction theory, and has a Behrend–Fantechi virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$.
- If $\mathcal{A}=\mathrm{coh}(X)$ for X a Calabi–Yau or Fano 3-fold, one can also define Behrend–Fantechi virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$.
- If $\mathcal{A}=\mathrm{coh}(X)$ for X a Calabi–Yau 4-fold, Borisov–Joyce define a different kind of virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$, with *half the expected dimension* of the Behrend–Fantechi class.

On moduli stacks and moduli schemes

There are two main ways of forming moduli spaces in Algebraic Geometry: as *schemes* or *stacks*. An important difference is that if $\mathcal{M}$ is a moduli stack of objects E, then automorphism groups are remembered in the isotropy groups of $\mathcal{M}$ by $\mathrm{Iso}_{\mathcal{M}}([E]) = \mathrm{Aut}(E)$, but moduli schemes forget automorphism groups.

Our moduli stacks $\mathcal{M}, \mathcal{M}^{\mathrm{pl}}$ differ in that their isotropy groups are $\mathrm{Iso}_{\mathcal{M}}([E]) = \mathrm{Aut}(E)$, but $\mathrm{Iso}_{\mathcal{M}^{\mathrm{pl}}}([E]) = \mathrm{Aut}(E)/(\mathbb{G}_m \cdot \mathrm{id}_E)$.

If E is τ-stable then $\mathrm{Aut}(E) = \mathbb{G}_m \cdot \mathrm{id}_E$, so $\mathrm{Iso}_{\mathcal{M}^{\mathrm{pl}}}([E]) = \{1\}$. Because of this, the τ-stable moduli scheme $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ is actually an *open substack* in $\mathcal{M}^{\mathrm{pl}}$ (but not $\mathcal{M}$). This makes $\mathcal{M}^{\mathrm{pl}}$ useful for us.

The τ-semistable moduli scheme $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ has the *good property* that it is usually compact (proper). But it has the *bad properties* that it does not map to $\mathcal{M}^{\mathrm{pl}}$ or $\mathcal{M}$, and the obstruction theory (or other nice structure) on $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ does not extend to $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$, so we cannot define a virtual class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ unless $\mathcal{M}_\alpha^{\mathrm{st}}(\tau) = \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$.

3.2. The case of quivers

Let $Q=(Q_0,Q_1,h,t)$ be a quiver, with finite sets Q_0 of vertices and Q_1 of edges, and head and tail maps $h,t:Q_1\rightarrow Q_0$. Then we have a $\mathbb{C}$-linear abelian category $\text{mod-}\mathbb{C}Q$ of *representations* (V_v,ρ_e) of Q, comprising a finite-dimensional $\mathbb{C}$-vector space V_v for each $v\in Q_0$ and a linear map $\rho_e:V_{t(e)}\rightarrow V_{h(e)}$ for each $e\in Q_1$. The *dimension vector* of (V_v,ρ_e) is $\boldsymbol{d}\in\mathbb{N}^{Q_0}$, where $\boldsymbol{d}(v)=\dim V_v$.

We can work out our theory very explicitly for $\mathcal{A}=\text{mod-}\mathbb{C}Q$. We take $K(\mathcal{A})=\mathbb{Z}^{Q_0}$. Then $\mathcal{M}=\coprod_{\boldsymbol{d}\in\mathbb{N}^{Q_0}}\mathcal{M}_{\boldsymbol{d}}$, $\mathcal{M}^{\rm pl}=\coprod_{\boldsymbol{d}\in\mathbb{N}^{Q_0}}\mathcal{M}_{\boldsymbol{d}}^{\rm pl}$, where $\mathcal{M}_{\boldsymbol{d}}=[R_{\boldsymbol{d}}/\operatorname{GL}_{\boldsymbol{d}}]$, $\mathcal{M}_{\boldsymbol{d}}^{\rm pl}=[R_{\boldsymbol{d}}/\operatorname{PGL}_{\boldsymbol{d}}]$ with

$$R_{\boldsymbol{d}}=\textstyle\prod_{e\in Q_1}\operatorname{Hom}(\mathbb{C}^{t(\boldsymbol{d}(e))},\mathbb{C}^{h(\boldsymbol{d}(e))}),\;\; \operatorname{GL}_{\boldsymbol{d}}=\textstyle\prod_{v\in Q_0}\operatorname{GL}(\boldsymbol{d}(v)),$$

and $\operatorname{PGL}_{\boldsymbol{d}}=\operatorname{GL}_{\boldsymbol{d}}/\mathbb{G}_m$. Hence $H_*(\mathcal{M}_{\boldsymbol{d}})\cong H_*(B\operatorname{GL}_{\boldsymbol{d}})$ and $H_*(\mathcal{M}_{\boldsymbol{d}}^{\rm pl})\cong H_*(B\operatorname{PGL}_{\boldsymbol{d}})$, which we can write explicitly.

Slope stability conditions

Fix $\mu_v \in \mathbb{R}$ for all $v \in Q_0$. Define $\mu : \mathbb{N}^{Q_0} \setminus \{0\} \to \mathbb{R}$ by

$$\mu(\boldsymbol{d}) = \left(\sum_{v \in Q_0} \mu_v \boldsymbol{d}(v)\right) \Big/ \left(\sum_{v \in Q_0} \boldsymbol{d}(v)\right).$$

We call μ a *slope function*. An object $0 \neq E \in \text{mod-}\mathbb{C}Q$ is called μ*-semistable* (or μ*-stable*) if whenever $0 \neq E' \subsetneq E$ is a subobject we have $\mu(\mathbf{dim} E') \geqslant \mu(\mathbf{dim} E)$ (or $\mu(\mathbf{dim} E') > \mu(\mathbf{dim} E)$).

Recall that $\mathcal{M}_{\boldsymbol{d}}^{\rm pl} = [R_{\boldsymbol{d}} / \mathrm{PGL}_{\boldsymbol{d}}]$ as a quotient stack. King (1994) showed that there is a linearization θ of the action of $\mathrm{PGL}_{\boldsymbol{d}}$ on $R_{\boldsymbol{d}}$, such that a $\mathbb{C}$-point $[E] \in [R_{\boldsymbol{d}} / \mathrm{PGL}_{\boldsymbol{d}}]$ is μ-(semi)stable in $\text{mod-}\mathbb{C}Q$ iff the corresponding point in $R_{\boldsymbol{d}}$ is GIT (semi)stable.

Hence there are moduli schemes $\mathcal{M}_{\boldsymbol{d}}^{\rm st}(\mu) \subseteq \mathcal{M}_{\boldsymbol{d}}^{\rm ss}(\mu)$ which are the GIT (semi)stable quotients $R_{\boldsymbol{d}} /\!/_{\theta}^{\rm st} \mathrm{PGL}_{\boldsymbol{d}} \subseteq R_{\boldsymbol{d}} /\!/_{\theta}^{\rm ss} \mathrm{PGL}_{\boldsymbol{d}}$.

If Q has *no oriented cycles* then a $\mathbb{G}_m$ subgroup of $\mathrm{PGL}_{\boldsymbol{d}}$ acts on $R_{\boldsymbol{d}}$ with positive weights, so $\mathcal{M}_{\boldsymbol{d}}^{\rm ss}(\mu) = R_{\boldsymbol{d}} /\!/_{\theta}^{\rm ss} \mathrm{PGL}_{\boldsymbol{d}}$ is a projective $\mathbb{C}$-scheme. Also $\mathcal{M}_{\boldsymbol{d}}^{\rm st}(\mu) = R_{\boldsymbol{d}} /\!/_{\theta}^{\rm st} \mathrm{PGL}_{\boldsymbol{d}}$ is a smooth quasi-projective $\mathbb{C}$-scheme, an open substack of $\mathcal{M}_{\boldsymbol{d}}^{\rm pl} = [R_{\boldsymbol{d}} / \mathrm{PGL}_{\boldsymbol{d}}]$.

Thus, if Q has no oriented cycles, and μ is a slope function on $\text{mod-}\mathbb{C}Q$, and $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ with $\mathcal{M}^{\text{st}}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)$, then $\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)$ is a smooth projective $\mathbb{C}$-scheme and an open substack of $\mathcal{M}^{\text{pl}}_{\boldsymbol{d}}$, and has a fundamental class $[\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)]_{\text{fund}}$ in $H_*(\mathcal{M}^{\text{pl}}_{\boldsymbol{d}})$. It has dimension $2 - \chi(\boldsymbol{d}, \boldsymbol{d})$, where $\chi : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is

$$\chi(\boldsymbol{d}, \boldsymbol{e}) = 2 \sum_{v \in Q_0} \boldsymbol{d}(v)\boldsymbol{e}(v) - \sum_{e \in Q_1} \big(\boldsymbol{d}(h(e))\boldsymbol{e}(t(e)) + \boldsymbol{d}(t(e))\boldsymbol{e}(h(e))\big).$$

Theorem 1 (Gross–Joyce–Tanaka 2005.05637.)

Let Q be a quiver with no oriented cycles. Then for all slope functions μ on $\text{mod-}\mathbb{C}Q$ and $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$, there exist unique classes $[\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)]_{\text{virt}} \in H_{2-\chi(\boldsymbol{d},\boldsymbol{d})}(\mathcal{M}^{\text{pl}}_{\boldsymbol{d}}) = \check{H}_0(\mathcal{M}^{\text{pl}}_{\boldsymbol{d}})$ such that:

(i) *If $\mathcal{M}^{\text{st}}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)$ then $[\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)]_{\text{virt}} = [\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)]_{\text{fund}}$.*

(ii) *The $[\mathcal{M}^{\text{ss}}_{\boldsymbol{d}}(\mu)]_{\text{virt}}$ transform according to the wall-crossing formula (1) above in the Lie algebra $\check{H}_0(\mathcal{M}^{\text{pl}})$ under change of stability condition.*

We also prove:

Theorem 2 (Gross–Joyce–Tanaka 2005.05637.)

There is a notion of **morphism of quivers** $\lambda : Q \to Q'$, *which induces a functor* $\lambda_* : \text{mod-}\mathbb{C}Q \to \text{mod-}\mathbb{C}Q'$, *and morphisms of vertex algebras* $\Omega : \hat{H}_*(\mathcal{M}) \to \hat{H}_*(\mathcal{M}')$ *and of Lie algebras* $\Omega^{\text{pl}} : \check{H}_*(\mathcal{M}^{\text{pl}}) \to \check{H}_*(\mathcal{M}'^{\text{pl}})$. *If* μ' *is a slope function on* $\text{mod-}\mathbb{C}Q'$ *then* $\mu = \mu \circ \lambda_*$ *is a slope function on* $\text{mod-}\mathbb{C}Q$. *Then for each* $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ *with* $\lambda_*(\boldsymbol{d}) = \boldsymbol{d}' \in \mathbb{N}^{Q_0'} \setminus \{0\}$, *the virtual classes* $[\mathcal{M}_{\boldsymbol{d}}^{\text{ss}}(\mu)]_{\text{virt}}$ *of Theorem 1 satisfy*

$$\prod_{v\in Q_0} \boldsymbol{d}(v)! \cdot \Omega^{\text{pl}}\bigl([\mathcal{M}_{\boldsymbol{d}}^{\text{ss}}(\mu)]_{\text{virt}}\bigr) = \prod_{v'\in Q_0'} \boldsymbol{d}'(v')! \cdot [\mathcal{M}'^{\text{ss}}_{\boldsymbol{d}'}(\mu')]_{\text{virt}}.$$

3.3. Sketch proof of Theorems 1 and 2

We call a slope function μ *decreasing* if for all edges $\overset{v}{\bullet} \overset{e}{\longrightarrow} \overset{w}{\bullet}$ in Q we have $\mu_v > \mu_w$. Such μ exist if and only if Q has no oriented cycles. If μ is decreasing, for each $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$, either:

(a) $\boldsymbol{d} = \delta_v$ for some $v \in Q_0$, that is, $\boldsymbol{d}(v) = 1$ and $\boldsymbol{d}(w) = 0$ for $w \neq v$. Then $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)$ is a single point $*$.

(b) $\boldsymbol{d} = n\delta_v$ for some $v \in Q_0$ and $n > 1$. Then $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu) = \emptyset$ and $\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu) \cong [*/\mathrm{PGL}(n,\mathbb{C})]$. Note that $2-\chi(\boldsymbol{d},\boldsymbol{d}) = 2-2n^2 < 0$.

(c) $\boldsymbol{d} \neq n\delta_v$ for any $v \in Q_0$, $n \geqslant 1$. Then $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu) = \emptyset$.

Hence the classes $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}$ in Theorem 1 must be

$$[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt} = \begin{cases} 1 \in H_0(\mathcal{M}^{\rm pl}_{\boldsymbol{d}}) \cong R, & \boldsymbol{d} = \delta_v,\ v \in Q_0, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

as in case (b) $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt} \in H_{<0}(\mathcal{M}^{\rm pl}_{\boldsymbol{d}}) = 0$.

Equation (4) for some fixed decreasing μ, and the wall-crossing formula in Theorem 1(ii) from μ to $\dot\mu$, then determine unique classes $[\mathcal{M}^{\rm ss}_{\boldsymbol d}(\dot\mu)]_{\rm virt}$ for all slope functions $\dot\mu$ on $\text{mod-}\mathbb{C}Q$. We prove these satisfy Theorem 1(ii) for wall-crossing from $\dot\mu$ to $\ddot\mu$, for any two slope functions $\dot\mu,\ddot\mu$, by an associativity property of the wall-crossing formula proved in my 2003 work on motivic invariants. So far we have constructed classes $[\mathcal{M}^{\rm ss}_{\boldsymbol d}(\mu)]_{\rm virt}$ as in Theorem 1, satisfying Theorem 1(ii), but we do not yet know they satisfy (i). Next we prove these classes $[\mathcal{M}^{\rm ss}_{\boldsymbol d}(\mu)]_{\rm virt}$ satisfy Theorem 2, using the fact that since $\Omega^{\rm pl}:\check H_*(\mathcal{M}^{\rm pl})\to\check H_*(\mathcal{M}'^{\rm pl})$ is a Lie algebra morphism, it takes the wall-crossing formula (1) in $\check H_*(\mathcal{M}^{\rm pl})$ used to define $[\mathcal{M}^{\rm ss}_{\boldsymbol d}(\mu)]_{\rm virt}$ to an identity in $\check H_*(\mathcal{M}'^{\rm pl})$. The factors $\prod_v \boldsymbol d(v)!$, $\prod_{v'} \boldsymbol d'(v')!$ arise because of a combinatorial identity relating the number of different ways of splitting $\boldsymbol d=\boldsymbol d_1+\cdots+\boldsymbol d_n$ in $\mathbb{N}^{Q_0}\setminus\{0\}$, and $\boldsymbol d'=\boldsymbol d'_1+\cdots+\boldsymbol d'_n$ in $\mathbb{N}^{Q'_0}\setminus\{0\}$.

Finally we show the $[\mathcal{M}_{\boldsymbol{d}}^{\mathrm{ss}}(\mu)]_{\mathrm{virt}}$ satisfy Theorem 1(i). This is the most difficult part. If $\boldsymbol{d}(v) \in \{0,1\}$ and Q is a tree, we deduce the result using results of Joyce–Song on Donaldson–Thomas type invariants for quivers. Then we build up to progressively more general $Q, \boldsymbol{d}$ using Theorem 2 in different ways.

I am currently writing up a proof for the general case of Behrend–Fantechi obstruction theories in Algebraic Geometry, which include quivers as a special case. The methods are different, and a lot more complicated. In brief, I can prove the WCF (1) in some special cases ('simple walls') using $\mathbb{G}_m$-localization and 'master spaces'. Starting with invariants in an exact category $\mathcal{A}$, I consider invariants in auxiliary categories $\mathcal{B}$ in exact sequences

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \text{mod-}\mathbb{C}Q \longrightarrow 0.$$

I can find enough 'simple' wall-crossings in such categories $\mathcal{B}$ to force the invariants in $\mathcal{A}$ to be well defined and satisfy (1).