# Hypercomplex and Quaternionic Manifolds

and

# Scalar Curvature on Connected Sums


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# Abstract


There are two parts to this thesis. In the first part, three different methods are given to construct hypercomplex and quaternionic manifolds. The first method, a quotient construction, is similar to the quotient constructions defined for hyperkähler manifolds by Hitchin et al. and for quaternionic Kähler manifolds by Galicki and Lawson. The second method involves constructing one hypercomplex or quaternionic manifold from another such manifold equipped with a group action and an 'instanton' connection, and the third method uses the structure theory of Lie groups to define hypercomplex and quaternionic structures on homogeneous manifolds.

As an application of these methods it is shown that there exist compact, simply-connected, irreducible, inhomogeneous hypercomplex and quaternionic manifolds in all dimensions greater than four, and that there are many homogeneous hypercomplex and quaternionic manifolds. In dimension four the methods produce families of self-dual conformal metrics upon $n\mathbb{CP}^2$, including LeBrun's metrics and some new self-dual metrics. Nontrivial families of many anticommuting complex structures upon open manifolds are also constructed.

The second part of the thesis presents explicit analytic constructions of metrics of constant scalar curvature, in the conformal class of a connected sum of constant scalar curvature manifolds. As well as verifying the solution of the Yamabe problem in some specific cases, the work shows how Yamabe metrics behave as the underlying conformal structure develops 'long necks', and enables the construction of many metrics of constant, positive scalar curvature that are not Yamabe metrics, as they are not absolute minima of the Hilbert action.

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# Introduction

This thesis is of two independent parts, with no cross-references, so that each part makes sense on its own as a complete piece of work. The first part, referred to as Part I, is on the subject of quaternionic geometry, and its main accomplishment is the construction of many examples of hypercomplex and quaternionic manifolds. The second part, referred to as Part II, is about finding metrics of constant scalar curvature in the conformal class of a Riemannian manifold constructed as a connected sum. We shall shortly introduce the two parts in more depth separately.

The connections between the two parts are the emphasis on conformal geometry, for 4-dimensional quaternionic geometry is self-dual conformal geometry, and in particular on connected sums and decays of conformal manifolds, which come into the first part as a main theme running through the work on connected sums of copies of $\mathbb{CP}^2$. Apart from this, the methods used are quite different, the first part coming under the heading of Geometry and the second under the heading of Analysis, and the conformal geometry of the second part is much more general, applying to conformal manifolds of dimensions other than four.

**Introduction to Part I**

The field of quaternionic geometry is the study of four different geometric structures on manifolds, the hypercomplex, hyperkähler, quaternionic and quaternionic Kähler structures. A hypercomplex manifold is a manifold $M$ with three complex structures $I_1, I_2, I_3$ satisfying the quaternion relations $I_1 I_2 = -I_2 I_1 = I_3$. Equivalently, it is a manifold with a $GL(n, \mathbb{H})$- structure preserved by a torsion-free connection. Similarly, a quaternionic manifold is a manifold with a $GL(n, \mathbb{H})GL(1, \mathbb{H})$- structure preserved by a torsion-free connection. In four dimensions, a quaternionic manifold is a self-dual conformal manifold. Hyperkähler and quaternionic Kähler manifolds are respectively hypercomplex and quaternionic manifolds that also possess a compatible Riemannian metric, whose Levi-Civita connection preserves the $G$- structure.

Hyperkähler and quaternionic Kähler manifolds are already well studied, and much work has been done on both their general theory and the construction of examples. In contrast, very little is known about examples of hypercomplex and quaternionic manifolds except in dimension four, and the subject has not received much attention. The purpose of the first part of this thesis is to remedy this by developing a battery of techniques for constructing and modifying hypercomplex and quaternionic manifolds, each technique being based upon something from self-dual, hyperkähler or quaternionic Kähler geometry.

By applying these we shall show that higher-dimensional, compact examples of the manifolds are numerous and varied. A difference between the world of hyperkähler and quaternionic Kähler geometry and the world of hypercomplex and quaternionic geometry will become apparent: in each case the absence of metrics gives the theory a much more flexible character, in that we are free to choose data for each construction that is predetermined in the metric case.

To construct hypercomplex and quaternionic manifolds, three main techniques will be used. The first of these techniques involves quotient constructions. A *quotient construction*, or *reduction*, is a process which, given a manifold with a structure such as a symplectic or a hyperkähler structure that is preserved by the action of a Lie group, produces a manifold of lower dimension with the same type of structure. The new manifold is a quotient of a submanifold of the original manifold by the Lie group.

The idea originally arose in classical mechanics: if a mechanical system moving under Hamilton's equations has a symmetry, then there is an associated momentum that is conserved. It can be shown that restricting to states of the system for which this momentum takes some fixed value, and dividing by the symmetry, gives a new system of a lower dimension that also satisfies Hamilton's equations. Now Hamilton's formulation of classical mechanics is really symplectic geometry in disguise, and this device is a method for constructing a lower-dimensional symplectic manifold from a symplectic manifold with symmetry. In this geometrical guise it is called the Marsden-Weinstein reduction [MW].

Similar reductions exist for geometric structures related to symplectic structures, such as Kähler, hyperkähler and quaternionic Kähler structures. The idea of momentum has been brought from mechanics into this setting, and so each construction works by defining a *moment map*, and dividing the zeros of this by the symmetry group. Reductions for hypercomplex and quaternionic manifolds will be described that are closely related to the known reductions for hyperkähler and quaternionic Kähler manifolds. The technical innovation that makes them possible is a new definition of moment map in the metric-free setting.

For the known reductions, the moment map is defined up to a constant by the group action and the geometric structures. In the new cases, the moment map is defined to be a solution of a differential equation involving the group action and the geometric structures, but this equation is not strong enough to determine the solution up to a constant. There may therefore be many solutions to the equation, or none, and to perform the quotient a moment map must be taken as part of the data, along with the manifold and the group action. The new definition of moment map is rather different from the original concept in mechanics, but we have chosen to retain the name anyway.

The second technique for constructing hypercomplex and quaternionic structures is that of *twisting constructions*. Suppose that $M$ is a manifold with a smooth action $\Psi$ of a Lie group $G$. Let $P$ be a principal $G$- bundle over $M$, with a given lifting of the action $\Psi$ to $P$. The Lie group $G$ thus acts on the total space of $P$ in two different ways: there is the lifted action $\Psi$, and also an action from the principal bundle structure, which we will call $\Phi$. Dividing the total space of $P$ by the action $\Phi$ recovers the manifold $M$. But one may also divide $P$ by the action $\Psi$, or the diagonal action of $G$ combining $\Phi$ and $\Psi$. If the action is free and $G$ is compact, then a new manifold $N$ will result that may be different from $M$. We say that $N$ is $M$ *twisted by* $P$, using the action $\Psi$.

It will be shown that if $M$ is a hypercomplex or quaternionic manifold, and $P$ carries a $\Psi$- invariant connection that is quaternionic (a generalization of instantons) then one can define a hypercomplex or quaternionic structure on $N$ that is nonsingular wherever the connection is transverse to the group action in a suitable sense. This idea is a generalization to the quaternionic case of a process described by Jones and Tod [JT] for constructing a self-dual conformal metric from a magnetic monopole on an Einstein-Weyl space.

Using this construction compact hypercomplex and quaternionic manifolds can be made in all dimensions greater than four, that are simply-connected, not homogeneous, and not (even locally) hyperkähler or quaternionic Kähler. We believe that these are the first such examples to be described in higher dimensions. For comparison, the only known examples of compact quaternionic Kähler manifolds with positive scalar curvature are Riemannian symmetric spaces. The quaternionic Kähler Riemannian symmetric spaces have been classified ([Bs], §14E), and there is a short list given in Table 14.52 of [Bs]. The classification depends upon that of compact Lie groups, for it turns out that exactly one quaternionic Kähler symmetric space can be made from each compact semisimple Lie group.

The third technique for constructing hypercomplex and quaternionic manifolds is the use of Lie algebra theory to find homogeneous hypercomplex and quaternionic structures on compact Lie groups and homogeneous spaces. We have just remarked that many Riemannian symmetric spaces are quaternionic Kähler. These turn out to be the most basic examples of large families of hypercomplex and quaternionic structures on compact groups and homogeneous spaces.

Also in this part we prove that nontrivial examples of manifolds with four or more anticommuting complex structures exist, using the last two techniques discussed above. Moreover, some of them can be made compact. I hope that these will be of use to physicists working in the field of supersymmetry, where attempts have already been made to construct examples of this sort, but without success.

Much of the material in the first part of the thesis is contained in the author's three papers [J1], [J2], [J3]; broadly speaking, [J1] has become Chapter 2, [J2] Chapters 3 and 5, and [J3] Appendix B.

**Introduction to Part II**

In the second part of the thesis we consider another topic, that of Riemannian manifolds with constant scalar curvature. The Yamabe problem, first proposed by Yamabe in 1960, is to find a metric of constant scalar curvature within the conformal class of any given compact conformal manifold of dimension at least three. There is a functional $Q$ upon the space of Riemannian metrics on a compact manifold $M$ called the *Hilbert action*, whose critical points on any conformal class are exactly the metrics of constant scalar curvature in that class. Yamabe's approach was to try to use analysis to construct an absolute minimum of the Hilbert action, which would then automatically be a critical point and so have constant scalar curvature. Such a minimal metric is called a *Yamabe metric*.

Although Yamabe failed to solve the problem, progress was made by many authors, and the solution was completed in 1984. In our second part we shall use analysis to construct constant scalar curvature metrics on connected sums of Riemannian manifolds of constant scalar curvature, without invoking the solution to the Yamabe problem. (Connected sums are defined in §4.1.) The existence proofs we give are of course somewhat simpler than the solution of the Yamabe problem, as to invoke the solution to prove existence would be using a powerful tool to do an easy job.

Our results do verify the solution of the Yamabe problem in some particular cases, but they do more than that. For we are able to describe explicitly what happens to Yamabe metrics as the underlying conformal manifold decays into a connected sum; we can for instance say that one obvious sort of behaviour, that of developing long 'tubes' resembling $\mathcal{S}^{n-1} \times \mathbb{R}$ as Riemannian manifolds, does not happen, but that small 'pinched necks' may develop instead, or else one part of the manifold may be crushed very small by the conformal factor.

Also, it will be shown that positive scalar curvature metrics exist that are not covered by the Yamabe problem, as they are higher stationary points of the Hilbert action, and not minima. And although Yamabe metrics may be expected to be generically unique, we shall produce general codimension one examples where there are two Yamabe metrics. In the final chapter we shall formulate a conjecture about the number of stationary points of the Hilbert action in any conformal class, counted with appropriate signs. These ideas are interesting because the non-uniqueness of constant scalar curvature metrics in the positive case is not well understood, and data like this may help to build up a picture.

Our method of proof has two stages. Firstly, a metric on the connected sum will be written down whose scalar curvature approximates a constant function; such metrics will often be referred to as *approximate metrics*. This can be done because it is assumed that the component manifolds already have constant scalar curvature, and so it is only the scalar curvature upon the small region of joining that we have to worry about. Secondly, it will be shown that any Riemannian manifold with scalar curvature sufficiently close to constant — where 'sufficiently close' depends on other geometrical invariants of $M$ — can be adjusted by a small conformal change to give a metric of constant scalar curvature.

The families of metrics defined in the first stage are of three sorts. The first sort consists of a small asymptotically flat manifold of zero scalar curvature glued into a small hole in a Riemannian manifold with a fixed metric of constant scalar curvature, and the second sort consists of two Riemannian manifolds with fixed metrics of equal constant scalar curvature, with a small hole cut out of each, joined by a small 'neck'. This will be more easily understood by looking at the series of diagrams in §11.1. Using these two families we can, in the positive scalar curvature case, define three different families of approximate metrics in the same conformal classes, and so get three distinct metrics of constant scalar curvature in each conformal class.

The third sort is an oddity, what you get when you glue a zero scalar curvature manifold into a negative scalar curvature manifold. The zero scalar curvature manifold is shrunk small by a homothety, and is glued into the negative scalar curvature manifold by a 'neck' that is still smaller.

The second stage applies to any compact Riemannian manifold with scalar curvature sufficiently close to constant – connected sums are just the application we are interested in. Thus the results could in principle be used in other situations, for instance to give an alternative proof of known results on the existence of a nearby metric of constant scalar curvature in every nearby conformal class, and perhaps also for more general surgeries on Riemannian manifolds than connected sums.

The main result of the second stage is proved by a sequence method. We extract from the geometry an equation that the small conformal change must satisfy, write the equation in terms of an invertible linear operator and an 'error term', and then inductively define a sequence of functions that converge to a suitable conformal change function when the error term is sufficiently small. As this error term depends on the difference between the scalar curvature and its average value, it is small when the scalar curvature is close to being constant.

What we do in this part is not dissimilar to the work of Floer in [F] and Donaldson and Friedman in [DF], except that the problem we consider is a lot easier. In these papers the authors construct self-dual metrics on the connected sum with small neck of two self-dual 4-manifolds. Our problem substitutes the simple condition of constant scalar curvature for the complex condition of self-dual curvature.

My motivation for undertaking the work of the second part was firstly to acquire some knowledge and experience of analytical methods, which I hope to be able to apply to geometrical questions, and secondly and more specifically, to try using constant scalar curvature metrics as a tool to understand better the decays possible at the edges of moduli spaces of self-dual metrics on 4-manifolds. It is difficult to get a handle upon conformal geometry, precisely because there is no easy way of assigning sizes to things, and by choosing a preferred metric in the conformal class, this problem at once disappears. The solution of the Yamabe problem makes constant scalar curvature metrics an obvious choice for these preferred metrics.

There may also be a way to relate the constant scalar curvature condition to self-duality; for instance, a zero scalar curvature metric upon a self-dual conformal manifold has self-dual Levi-Civita connection upon one of its spin bundles, and so zero scalar curvature metrics on a self-dual 4-manifold translate to some holomorphic object on the twistor space, by the Penrose transform. We note that King and Kotschick use constant scalar curvature metrics in the way suggested above, to study the moduli space of self-dual structures on a 4-manifold ([KK], §3).

**Overview of Part I**

The introductory chapter, Chapter 1, discusses different ways of defining the four geometric structures, and reviews the idea of a twistor space, and the quotient constructions that are known already. In Chapter 2, quotient constructions for hypercomplex and quaternionic manifolds are described. They are similar to the known quotient constructions for hyperkähler and quaternionic Kähler manifolds, and the new idea is the definition that will be given for moment maps in the metric-free setting.

Chapter 3 is on the subject of twisting constructions. Suppose that $M$ is a hypercomplex or quaternionic manifold with a $G$- bundle $P$ and a smooth action of $G$ for some Lie group $G$, and that $P$ carries a quaternionic connection, invariant under the group action. Let $N$ be $M$ twisted by $P$ under this group action. Then we shall define a hypercomplex or quaternionic structure on $N$, that is nonsingular wherever the connection is transverse to the group action in a suitable sense. Using this construction many compact, simply-connected hypercomplex and quaternionic manifolds will be made, in all dimensions greater than four.

Chapter 4 is a chapter of examples. We apply the work of Chapters 2 and 3 to construct explicit examples of self-dual conformal structures upon complex weighted projective spaces, and the compact 4-manifolds $n\mathbb{CP}^2$. Families of such metrics have already been found by Poon and LeBrun, and we show that the known metrics may all be constructed both by using the methods of Chapter 2 and also using the methods of Chapter 3, and moreover that these methods construct other families of self-dual metrics on $n\mathbb{CP}^2$, that I believe are new.

In Chapter 5 the theory of compact, homogeneous hypercomplex and quaternionic manifolds will be developed, using the structure theory of Lie groups. To do this we shall draw upon results about compact homogeneous complex manifolds, which are already well understood. To close the first part, Chapter 6 considers the most natural generalization of hypercomplex manifolds, that of manifolds with several anticommuting complex structures. Using material from Chapters 3 and 5 we show that there do indeed exist collections of arbitrarily many anticommuting complex structures on manifolds, that are not locally flat, and that a few of these examples can be made compact.

The appendices to Part I are Appendix A and Appendix B. Appendix A offers an alternative proof for a theorem in Chapter 3. Appendix B outlines a method for using the quotient constructions of Chapter 2 to make self-dual metrics on 4-manifolds and orbifolds, in particular upon connected sums of $\mathbb{CP}^2$'s. We show that all LeBrun's metrics on $n\mathbb{CP}^2$ [L2] arise as quaternionic quotients of $\mathbb{HP}^{n+1}$ by $U(1)^n$, and explain how to construct some new families of metrics on $n\mathbb{CP}^2$, which have also been described by a twisting method in Chapter 4.

**Overview of Part II**

The background to the problem is contained in Chapter 7, which introduces the Yamabe problem and the analytical tools such as Sobolev spaces and embedding theorems to be used later. We look at asymptotically flat manifolds in some detail, as they are important in the sequel. Chapter 8 then defines two families of approximate metrics upon connected sums, makes estimates of the scalar curvature of the metrics, and proves a uniform bound on a Sobolev constant for a particular embedding of Sobolev spaces, that is needed in the next chapters.

The main existence results for metrics of constant scalar curvature conformal to the approximate metrics above are proved in Chapter 9. We begin with a quite general existence proof and then apply it, first to the case of scalar curvature $-1$, and then to the case of scalar curvature 1. The former case is simple, but the latter is more difficult, and the proof of a result on the eigenvalues of the Laplacian $\Delta$ on approximate metrics in the positive case has been deferred until Appendix D, as it is long and not easy to follow.

In Chapter 10 we deal with the exceptional cases left over from Chapter 9, which are all the connected sums involving manifolds of zero scalar curvature. It will be seen in Chapter 9 that positive and negative scalar curvature manifolds behave rather differently, and the zero scalar curvature cases, being the junction between them, behave differently again. The first two cases, the connected sum of a Riemannian manifold of zero scalar curvature and one of positive scalar curvature, and the connected sum of two Riemannian manifolds of zero scalar curvature, turn out to give metrics with small, constant negative scalar curvature. The second case has the attractive property of balancing the volumes of the component manifolds in the connected sum. The third case, the connected sum of a zero scalar curvature manifold and a negative scalar curvature manifold, is also interesting.

The main text closes in Chapter 11 with diagrams of the different approximate metrics, as a visual aid, and some discussion of issues raised by the results of the second part. The appendices to Part II are Appendix C and Appendix D. The first sketches a proof for a Sobolev embedding theorem for asymptotically flat manifolds, that I have been unable to find a reference for. The second proves results on the spectrum of the Laplacian on connected sums, that were deferred from Chapters 9 and 10.

# Part I: Hypercomplex and Quaternionic Manifolds

## Chapter 1: Background Material for Part I

This chapter goes briefly over some topics in quaternionic geometry, setting down definitions and ideas for use in Chapters 2-6. In §1.1 we define four different geometric structures on manifolds based on the quaternions, the hypercomplex, hyperkähler, quaternionic and quaternionic Kähler structures, and review alternative ways of making the definitions. Section 1.2 explains how the concept of twistor space arising in 4-dimensional self-dual geometry, which encodes the conformal structure of a self-dual 4-manifold in the complex structure of a complex 3-manifold, may be extended to the four types of manifold in a natural way. Finally, §1.3 summarizes several different quotient constructions, which are a means of taking a manifold with a particular geometric structure and a symmetry group, and generating another manifold of smaller dimension with the same structure.

### 1.1. Definitions

To begin with we shall explain several equivalent ways of defining hypercomplex, hyperkähler, quaternionic and quaternionic Kähler manifolds. Here and throughout manifolds will be taken to be smooth and connected, and objects such as bundles and sections or subbundles of bundles will be taken to be smooth unless stated otherwise. A good reference for the material in this section is [S2], Chapters 8 and 9.

We will write the quaternions $\mathbb{H}$, invented by Hamilton, as

$$\mathbb{H} = \{a + bi_1 + ci_2 + di_3 : a, b, c, d \in \mathbb{R}\},$$

which are a real vector space in the obvious way, and become a unital, non-commutative algebra under the quaternion relations

$$i_1 i_2 = -i_2 i_1 = i_3, \qquad i_2 i_3 = -i_3 i_2 = i_1, \qquad i_3 i_1 = -i_1 i_3 = i_2.$$

The unit quaternions $\{a + bi_1 + ci_2 + di_3 \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\}$ are a subgroup of $\mathbb{H}$ under multiplication, isomorphic to $SU(2)$. The imaginary quaternions are those for which $a = 0$. There is a natural norm upon $\mathbb{H}$, satisfying $\|q_1 q_2\| = \|q_1\| \cdot \|q_2\|$, given by $\|a + bi_1 + ci_2 + di_3\|^2 = a^2 + b^2 + c^2 + d^2$.

In what follows, modules over $\mathbb{H}$ will always be left modules, so the quaternions will act by multiplication on the left.

Now real manifolds can be thought of as spaces that are modelled locally upon vector spaces over $\mathbb{R}$, and Riemannian manifolds as spaces modelled locally upon normed vector spaces over $\mathbb{R}$. Similarly, complex manifolds are spaces modelled locally upon vector spaces over $\mathbb{C}$, and Kähler manifolds spaces modelled locally upon normed vector spaces over $\mathbb{C}$. These four structures on manifolds have proved to be of great interest. As the quaternions are in some sense the next step up from the real and the complex numbers, it is worthwhile to look for suitable definitions for spaces modelled locally upon $\mathbb{H}$- modules and normed $\mathbb{H}$- modules (i.e. $\mathbb{H}^n$).

### 1.1.1. Hypercomplex and hyperkähler manifolds

The most natural quaternionic analogues of complex and Kähler manifolds are called hypercomplex and hyperkähler manifolds. A *hypercomplex manifold* $M$ ([S2], p. 137; [S3], §6) is defined to be a $4n$-dimensional manifold $M$ with 3 integrable complex structures $I_1, I_2, I_3$ that satisfy the quaternion relations $I_1I_2 = I_3, I_3I_1 = I_2, I_2I_3 = I_1$. A *hyperkähler manifold* ([HKLR] p. 538; [S2], p. 114) is defined to be a $4n$- dimensional manifold $M$ with 3 integrable complex structures $I_1, I_2, I_3$ satisfying the quaternion relations, and a Riemannian metric $g$ preserved by and Kähler w.r.t. each $I_i$. Thus a hyperkähler manifold is a hypercomplex manifold with an extra structure, a metric that satisfies a condition. The simplest hyperkähler manifolds are the $\mathbb{H}$- modules $\mathbb{H}^n$. Pseudo-hyperkähler manifolds are defined in the obvious way using a pseudo-Riemannian metric.

An important feature of real and complex manifolds that does not hold for hypercomplex manifolds is that real and complex manifolds are locally trivial — a sufficiently small neighbourhood of any point in a real or complex manifold is isomorphic to a standard model — whereas hypercomplex manifolds are not, and have local structure and curvature. A calculation using the Bianchi identity shows that hyperkähler manifolds are Ricci-flat. Four-dimensional hyperkähler manifolds can be defined in another way: a self-dual, Ricci-flat manifold is (locally) hyperkähler. The extra structure comes from the fact that these two curvature conditions imply that the Levi-Civita connection upon one of the spin bundles is flat, and hence (locally) has a natural trivialization, and on a conformal 4-manifold a trivialization of one of the spin bundles defines an almost hypercomplex structure.

At this point our conventions on orientation should be explained. On the one hand, a complex structure (and hence also a hypercomplex structure) on a manifold naturally defines an orientation. With respect to this natural complex orientation a hyperkähler 4-manifold is anti-self-dual. On the other hand, it is usual to define four-dimensional quaternionic manifolds to be self-dual, and self-dual conformal structures are usually studied in preference to anti-self-dual ones. To reconcile these two points of view we will define the orientation of a hypercomplex or a hyperkähler manifold to be the opposite of the natural complex orientation, and then we may continue to talk of self-duality rather than anti-self-duality.

Each hypercomplex manifold has a unique torsion-free connection $\nabla^M$ called the *Obata connection* ([S3], §6) satisfying $\nabla^M I_i = 0$. Conversely, a manifold $M$ that has three almost complex structures $I_1, I_2, I_3$ satisfying $I_1 I_2 = I_3$ and a torsion-free connection $\nabla^M$ with $\nabla^M I_i = 0$ is hypercomplex, for $\nabla^M I_i = 0$ implies that $I_i$ is integrable. This gives a second definition of hypercomplex manifold. For a hyperkähler manifold this unique connection is the Levi-Civita connection $\nabla$, and a second definition of hyperkähler manifold is a Riemannian manifold $M$ with metric $g$ and three almost structures $I_1, I_2, I_3$ satisfying $I_1 I_2 = I_3$ and $\nabla I_i = 0$.

### 1.1.2. Quaternionic and quaternionic Kähler manifolds

We now come to quaternionic and quaternionic Kähler manifolds, whose definitions are more difficult to motivate. The main reason for introducing them will be explained more fully in the next section, and is this: quaternionic and quaternionic Kähler structures possess in higher dimensions the main distinctive features of the geometry of self-dual conformal and self-dual Einstein structures respectively in four dimensions.

For $n \geq 2$, a *quaternionic structure* ([S2], p. 135; [S3], Definition 1.1) on a $4n$- dimensional manifold $M$ is defined to be a vector subbundle $\mathcal{G}$ of $\mathrm{End}(TM)$ allowing at each point a basis $I_1, I_2, I_3$ of almost complex structures satisfying $I_1 I_2 = I_3$, that admits a torsion-free connection $\nabla^M$ preserving $\mathcal{G}$. This connection is not in fact unique, which is why we do not include it in the structure but only require that it should exist.

The condition involving the connection $\nabla^M$ is a sort of integrability condition, as it implies that the bundle of almost complex structures in $\mathcal{G}$ should possess a local section through each point that is an integrable complex structure.

For $n \geq 2$, a *quaternionic Kähler structure* ([S1], p. 143; [S2], p. 124) on a $4n$- dimensional manifold $M$ is defined to be a Riemannian metric $g$ together with a vector subbundle $\mathcal{G}$ of $\mathrm{End}(TM)$ allowing at each point a basis $I_1, I_2, I_3$ of almost complex structures preserving $g$ and satisfying $I_1 I_2 = I_3$, such that the Levi-Civita connection $\nabla$ preserves $\mathcal{G}$.

Thus a quaternionic Kähler manifold is a quaternionic manifold $M$ together with a metric $g$ preserved by the complex structures in $\mathcal{G}$, such that $\nabla^M$ is the Levi-Civita connection of $g$. Again, the Bianchi identity implies that quaternionic Kähler manifolds are Einstein. The model examples of quaternionic Kähler manifolds are the quaternionic projective spaces $\mathbb{HP}^n$, which are defined by $\mathbb{HP}^n = (\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^*$, where the non-zero quaternions $\mathbb{H}^*$ act by left multiplication upon $\mathbb{H}^{n+1}$.

In four dimensions we make the special definitions that a quaternionic manifold is a self-dual conformal manifold and a quaternionic Kähler manifold is a Riemannian manifold that is self-dual and Einstein. (Self-duality is treated in the next section.)

### 1.1.3. Connections and $G$- structures

The four geometric structures defined above appear fairly dissimilar – the only obvious unifying factor is the existence at each point of three complex structures $I_1, I_2, I_3$ satisfying $I_1 I_2 = I_3$. These in fact make the tangent space into a left $\mathbb{H}$- module: if $a + bi_1 + ci_2 + di_3 \in \mathbb{H}$ and $v \in T_x M$ one can define $(a + bi_1 + ci_2 + di_3) \cdot v = av + bI_1 v + cI_2 v + dI_3 v$, which is a left action of $\mathbb{H}$.

However, a striking similarity in the definitions may be brought out by reformulating them in the language of $G$- structures and special holonomy. If $G$ is a Lie group, a *$G$- structure* $Q$ on a manifold $M$ is a principal bundle $Q$ over $M$ with group $G$, that is a subbundle of the frame bundle of $M$. Let $\nabla^M$ be a torsion-free connection upon the tangent bundle of $M$. (The torsion-free condition is there to tell the geometry that it is dealing with the tangent bundle. If it were not imposed we would be free to choose locally an isomorphism of the tangent bundle with a 4-dimensional vector bundle with connection of our choice, and induce the connection from that.) Then $\nabla^M$ induces a connection upon the frame bundle of $M$, and we say $\nabla^M$ preserves the subbundle $Q$ of the frame bundle if $Q$ is closed under parallel transport by this induced connection.

The four structures defined above are each equivalent to a $G$- structure $Q$ preserved by a torsion-free connection, for four different groups $G$. To show this, for each of the four types of manifold let $G$ be the group of automorphisms of the tangent plane at a point that preserve the structures defined on it. Thus for the hypercomplex manifolds $G$ must preserve three complex structures, so $G = GL(n, \mathbb{H})$, the set of invertible quaternion matrices acting upon $\mathbb{H}^n$ on the right, as this is the group commuting with the left $\mathbb{H}$- module structure of $\mathbb{H}^n$. For the hyperkähler manifolds $G$ preserves three complex structures and a metric, so $G = Sp(n)$ acting on the right, where the group $Sp(n)$ is the intersection of $GL(n, \mathbb{H})$ and $SO(4n)$.

In the quaternionic and quaternionic Kähler cases elements of $G$ also are allowed to act on the family of complex structures in $\mathcal{G}$, so $G$ is $GL(n,\mathbb{H})GL(1,\mathbb{H})$ and $Sp(n)Sp(1)$ respectively. Here $GL(n,\mathbb{H})$ acts upon $\mathbb{H}^n$ (thought of as row matrices of quaternions) by matrix multiplication on the right, and $GL(1,\mathbb{H}) = \mathbb{H}^*$ acts upon $\mathbb{H}^n$ by matrix multiplication (i.e. quaternion scalar multiplication) on the left. The product of these groups inside $GL(4n,\mathbb{R})$ is the group $GL(n,\mathbb{H})GL(1,\mathbb{H})$. Similarly, $Sp(n)Sp(1)$ is the intersection of $GL(n,\mathbb{H})GL(1,\mathbb{H})$ and $SO(4n)$.

The structures – triples of almost complex structures and metrics – on the four families of manifolds are encapsulated by a $G$- structure $Q$ on $M$ for the four families of groups. Apart from the 4-dimensional cases of quaternionic and quaternionic Kähler structures, the additional integrability conditions can now be summed up by saying that there should exist a torsion-free connection $\nabla^M$ on $M$ that preserves $Q$. In the hypercomplex case this connection is the Obata connection — recall the second definition of hypercomplex manifold. Similarly, in the hyperkähler and the quaternionic Kähler cases the connection is the Levi-Civita connection. In the quaternionic case the definition explicitly requires some such connection $\nabla^M$ to exist. So this gives a unified way of looking at the four definitions.

In four dimensions $GL(n,\mathbb{H})GL(1,\mathbb{H})$ is just the conformal group, and $Sp(n)Sp(1)$ is just $SO(4)$. So extending the $G$- structure definition of quaternionic and quaternionic Kähler manifolds to four dimensions just gives a conformal and a Riemannian 4-manifold respectively. But the actual definitions also require self-duality and the Einstein condition, so the $G$- structure definition is not sufficiently strong in four dimensions. Motivation for the stronger definitions comes in the next section.

### 1.1.4. Holonomy groups

Closely linked with $G$- structures preserved by a connection, is the idea of holonomy. Let $M$ be a manifold with a connection $\nabla^M$ on its tangent bundle, usually supposed to be torsion-free. Fix a point $x$ in $M$. Let $\gamma$ be some (piecewise smooth) directed loop based at $x$. Parallel transport around $\gamma$ using $\nabla^M$ gives an automorphism of $T_xM$. Let $\mathrm{Hol}_x(M)$ be the set of all such automorphisms coming from loops $\gamma$ based at $x$.

By the usual arguments of composing loops and reversing their directions, it is easy to see that $\mathrm{Hol}_x(M)$ is in fact a subgroup of $\mathrm{Aut}(T_xM)$. Moreover, if $y$ is another point of $M$ and $\alpha$ is any (piecewise smooth) path between $x$ and $y$, then the parallel transport of $\mathrm{Hol}_x(M)$ along $\alpha$ under $\nabla^M$ is $\mathrm{Hol}_y(M)$. So the group $\mathrm{Hol}_x(M)$ does not depend upon the point $x$, and this group is called the *holonomy group*

$\mathrm{Hol}(M)$ of the connection $\nabla^M$. (Implicitly we mean not just the group, but also its representation upon a tangent space of $M$.)

The connection with $G$- structures is this. Suppose $G$ is some subgroup of $\mathrm{Aut}(T_x(M))$ and $\mathrm{Hol}_x(M) \subseteq G$. Choose a frame for $T_xM$ and let $Q_x$ be the orbit of this frame under $G$. Then the parallel transport of $Q_x$ over $M$ defines a $G$- structure $Q$ on $M$ that is preserved by $\nabla^M$. Thus given $M$ and $\nabla^M$, $M$ admits a $G$- structure preserved by $\nabla^M$ if and only if $\mathrm{Hol}(M) \subseteq G$ as an inclusion of groups and representations.

So hypercomplex, hyperkähler and quaternionic Kähler manifolds can be defined as manifolds $M$ with a torsion-free connection $\nabla^M$ together with a given inclusion of $\mathrm{Hol}(M)$ in $GL(n, \mathbb{H}), Sp(n)$ and $Sp(n)Sp(1)$ respectively, as an inclusion of groups and representations. Quaternionic manifolds can also be defined this way, but with the technical reservation that the connection is not unique and not part of the quaternionic structure.

The theory of holonomy groups says that not all representations of connected Lie groups can actually be realized as the holonomy group of a torsion-free connection on a manifold, but many can be excluded by algebraic calculations. The most important case is that of holonomy groups of Riemannian manifolds, for which the list of possible local holonomy groups for Riemannian manifolds that are neither locally a product nor symmetric is $SO(n), U(n/2), SU(n/2), Sp(n/4), Sp(n/4)Sp(1), G_2, Spin(7)$ or $Spin(9)$. This result was proved by Berger [Br] in 1955; for a modern, more readable and shorter presentation we recommend [S2], Chapter 10, or the survey of Riemannian holonomy groups in Chapter 10 of [Bs].

The more general case of classification of possible holonomy groups of torsion-free connections is also handled by Berger [Br], who gives a rather longer list of possible holonomy groups. The methods of proof involve a careful study of the curvature of the connection. If $\nabla$ preserves a $G$- structure, then the curvature $R^i{}_{jkl}$ of $\nabla$ lies in $\mathfrak{g} \otimes \Lambda^2 T^*$ at each point, where $\mathfrak{g}$ is the Lie algebra of $G$. If $\nabla$ is also torsion-free, then $R$ satisfies the Bianchi identities. It is the algebraic interplay between these two facts that is exploited in his proof.

Berger shows that groups and representations fall into three classes: a first class, for which one can show that any torsion-free connection with holonomy in this class has vanishing curvature ($R = 0$), a second class, for which one can show any torsion-free connection with holonomy in this class has constant curvature ($\nabla R = 0$), and a comparatively small third class, the ones that are left. In the first case, the connection must be flat, and the holonomy group is trivial; therefore the first class do not arise as holonomy groups. In the second case, the curvature being constant implies that the manifold and connection are locally symmetric, and are thus locally homogeneous. So groups of the second class arise locally as holonomy groups only of (unique) homogeneous symmetric spaces.

We are then left with the reduced list of the third class, that represent possibly interesting geometric structures. Much work has since gone into the production of examples realizing the holonomy groups of the third list, to show that these possible holonomy groups do actually exist in nature. The first part of the thesis is a small step in this programme.

### 1.2. Twistor spaces and related structures

We begin this section with a brief summary of the some of the geometry of self-duality in four dimensions. The foundation of this branch of geometry is the splitting of the special orthogonal group $SO(4) = SU(2) \times_{\{\pm 1\}} SU(2)$, a splitting which happens in no other dimension. The two $SU(2)$ factors are interchanged by an orientation-reversing automorphism of $\mathbb{R}^4$.

One consequence of this is that representation spaces of $SO(4)$ become representation spaces of the two $SU(2)$ factors, and thus frequently split up into quite small irreducible chunks. In particular, the space containing the Weyl conformal curvature now splits into two irreducible subspaces, which are interchanged by an orientation-reversing automorphism. Thus the Weyl curvature $W$ of a conformal metric in four dimensions splits into two components, called the self-dual and anti-self-dual components $W^+$ and $W^-$.

They are called this because the easiest way to define the two components is using the Hodge star operator. On an even-dimensional oriented Riemannian manifold $M^{2l}$ the Hodge star $* : \Lambda^p \to \Lambda^{2l-p}$ is defined by $\alpha \wedge (*\beta) = (\alpha \cdot \beta) dV_g$ for all $\alpha, \beta \in \Lambda^p$, where $\Lambda^p$ is the bundle of exterior $p$- forms, the inner product comes from the Riemannian metric, and the volume form $dV_g$ is defined using the metric and the orientation.

An important fact about the Hodge star is that $*^2 = (-1)^l$. So in dimension 4, $*$ gives an involution of $\Lambda^2$, which in fact only depends on the conformal structure of $M$. This involution splits $\Lambda^2$ into a sum of subbundles $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, which are the $+1$- and $-1$- eigenspaces of $*$. Now the Weyl conformal curvature $W = W^i_{\ jkl}$ can be thought of as an endomorphism-valued 2-form, and so in four dimensions the Hodge star acts upon the 2-form part to give $*W$. We define $W^+ = (W + *W)/2$ and $W^- = (W - *W)/2$; then $W = W^+ + W^-$ and $*W^+ = W^+$, $*W^- = -W^-$. So $W^+$ is self-dual and $W^-$ is anti-self-dual under Hodge duality.

The splitting of the conformal curvature into two comparable components means that one can impose a condition upon a 4-dimensional conformal structure that is half-way to conformal flatness, by requiring one of $W^+, W^-$ to vanish. A conformal 4-manifold is called *self-dual* if $W = W^+$ and *anti-self-dual* if $W = W^-$. These two change round under change of orientation, as the Hodge star changes sign.

Note: some authors use self-dual in the stronger sense that the Riemannian curvature must be self-dual, which means the manifold should be both conformally self-dual and Ricci-flat. For us, self-dual will mean conformally self-dual.

The interest in self-duality comes from the remarkable connection between the self-dual condition and complex geometry found by Penrose, and written up in the context of four-dimensional Riemannian geometry by Atiyah et al. in their paper [AHS], from which all this material comes.

Let $M$ be an oriented conformal 4-manifold. Let $Z \to M$ be the bundle of almost complex structures on $M$ that have natural complex orientation opposite to the chosen orientation on $M$. Equivalently, $Z$ is the negative projective spin bundle, $P(V_-)$. Then $Z$ is a bundle over $M$ with fibre $SO(4)/U(2) = \mathbb{CP}^1$. Now there is a natural almost complex structure on the total space of $Z$. Broadly, this comes from taking the usual complex structure on $\mathbb{CP}^1$ in the fibre directions, and the tautological almost complex structure in the manifold directions. This is not quite enough to define a complex structure because there are no preferred horizontal subspaces in the fibration. The remaining information comes from the Cartan conformal connection, or local twistor transport in twistor language.

The connection between the self-dual condition and complex geometry is this: the self-duality of the conformal structure on $M$ is a necessary and sufficient condition for the integrability of the almost complex structure on $Z$. Thus if $M$ is self-dual, $Z$ is a complex manifold fibring over $M$, called the *twistor space* of $M$.

Now $Z$ has a natural involution $\sigma$ given by the antipodal map on the fibres, which changes the sign of each almost complex structure. This involution changes the sign of the almost complex structure on $Z$. So if $M$ is self-dual then $Z$ is a complex manifold with an antiholomorphic involution $\sigma$, called a real structure. The fibres of the map $Z \to M$ are complex lines, and as they are preserved by the real structure they are called real lines. It can be shown that the real lines have normal bundle $\mathcal{O}(1) + \mathcal{O}(1)$.

There is a converse to this construction, which is this: let $Z$ be a 3-dimensional complex manifold with an antiholomorphic involution $\sigma$ and containing a real line $C$ with normal bundle $\mathcal{O}(1) + \mathcal{O}(1)$. Then a neighbourhood of $C$ is a fibration over some real four dimensional manifold with real lines as fibres (this is proved using complex deformation theory), and a self-dual conformal structure on the 4-manifold can be reconstructed from the complex structure of $Z$. (In fact the set of all complex lines close to $C$ is a complex 4-manifold, and intersection of neighbouring lines defines the cones of a complex conformal structure on this 4-manifold. This conformal structure may be restricted to the real lines, where it becomes a real conformal structure.)

So there is a correspondence between real self-dual 4-manifolds $M$, and complex 3-manifolds $Z$ with an antiholomorphic involution $\sigma$ containing complex lines with normal bundle $\mathcal{O}(1)+\mathcal{O}(1)$. This is called the *(Penrose) twistor correspondence*. The idea of the twistor programme is to translate differential geometric structures on $M$ to complex analytic structures on $Z$, and then to learn about the differential geometry of $M$ using the powerful methods of complex analysis. Note that this involves using *global* complex geometry to study *local* differential geometry, as the structure of $Z$ is locally trivial, being just a complex manifold, and the local structure of $M$ is encoded in the global structure of $Z$.

Two examples of the twistor correspondence that we will return to are firstly, the translation of self-dual Yang-Mills instantons on $M$ to holomorphic bundles on $Z$ that are trivial on real lines, and secondly, the translation of an Einstein metric in the conformal class of $M$ to a real Kähler metric on $Z$.

A large part of the theory of quaternionic and quaternionic Kähler manifolds extends and generalizes this twistor correspondence. We will explain below how to define the twistor space of a quaternionic manifold, and that the twistor space of a quaternionic Kähler manifold has a (pseudo-) Kähler structure upon it. Thus quaternionic manifolds are suitable generalizations of self-dual conformal 4-manifolds because they have complex twistor spaces, and quaternionic Kähler manifolds are good generalizations of self-dual Einstein 4-manifolds because they have complex Kähler twistor spaces. This is the promised motivation for the special definitions of quaternionic and quaternionic Kähler manifolds in the four-dimensional case.

Given a principal bundle $P$ over $M$ for the group $G$, to each representation $V$ of $G$ one may associate a bundle $\mathcal{V}$ over $M$ defined by $\mathcal{V} = P \times_G V$. The bundle $\mathcal{V}$ has fibre $V$. Let us return to the situation of the previous section with a manifold $M$ with $G$- structure $Q$, where $G$ is one of the holonomy groups $GL(n,\mathbb{H})GL(1,\mathbb{H})$ and $Sp(n)Sp(1)$. These groups both have double covers, $SL(n,\mathbb{H}) \times GL(1,\mathbb{H})$ and $Sp(n)\times Sp(1)$. Thus for each quaternionic or quaternionic Kähler manifold $M$ with $G$- structure $Q$, locally there is a double cover $\tilde{Q}$ which is a principal bundle with group $SL(n,\mathbb{H}) \times GL(1,\mathbb{H})$ or $Sp(n) \times Sp(1)$. (Note $\tilde{Q}$ need not exist globally.)

Let us take $V$ to be the natural representation of $GL(1,\mathbb{H})$ or $Sp(1)$ on the right on the quaternions $\mathbb{H}$, with $SL(n,\mathbb{H})$ or $Sp(n)$ acting trivially. Actually, in the quaternionic case it is necessary to let the scalars $\mathbb{R}^* \subset GL(n,\mathbb{H})GL(1,\mathbb{H})$ act to some prescribed power on $V$; this corresponds to regarding $GL(n,\mathbb{H})GL(1,\mathbb{H})$ not as $GL(n,\mathbb{H})Sp(1)$ nor as $SL(n,\mathbb{H})GL(1,\mathbb{H})$ but as a mixture of the two.

Forming the bundle associated to the local principal bundle $\tilde{Q}$ gives a local fibre bundle $\tilde{\mathcal{V}}$ over $M$ with fibre $\mathbb{H}$, called the *natural quaternionic line bundle*. To make a bundle that exists globally it is necessary to divide $\tilde{\mathcal{V}}$ by $\pm 1$, as a double cover of $Q$ may not exist, and this gives a global bundle $\mathcal{U}(M)$ over $M$ with fibre $\mathbb{H}/\{\pm 1\}$, called the *associated bundle*. This is defined just before Theorem 3.5 in [Sw].

By projectivizing the fibre $\mathbb{H}/\{\pm 1\}$ with respect to any of the left actions of $\mathbb{C}^*$, we get a fibre bundle $Z$ over $M$ with fibre $\mathbb{CP}^1$ called the *twistor space*. In [S3], Corollary 7.4, Salamon proves that the twistor space of a quaternionic manifold is complex by first proving that the total space of the associated bundle, which he calls $Y$, is hypercomplex. This then implies that the twistor space $Z$ is complex. Quaternionic manifolds can therefore be regarded as the most general sort of manifolds that admit a complex twistor space in analogy with the four-dimensional case (see, e.g. [S2], p. 135).

As a quaternionic Kähler manifold is quaternionic, its associated bundle will certainly be hypercomplex. However, Swann goes on to prove ([Sw], Theorem 3.5) that the Riemannian metric on $M$ induces a pseudo-Riemannian metric on $\mathcal{U}(M)$ which together with the hypercomplex structure makes it pseudo-hyperkähler. Then the projectivization of $\mathcal{U}(M)$ to give $Z$ may be viewed as a pseudo-Kähler quotient by $U(1)$, which gives a pseudo-Kähler structure upon the twistor space $Z$, as promised.

The model examples of a quaternionic Kähler manifold, its associated bundle and its twistor space are quaternionic projective space $\mathbb{HP}^n$, the fibration $\big(\mathbb{H}^{n+1}\setminus\{0\}\big)/\{\pm 1\} \to \mathbb{HP}^n$, and the projectivization $\mathbb{CP}^{2n+1}$ of $\big(\mathbb{H}^{n+1}\setminus\{0\}\big)/\{\pm 1\}$ respectively.

### 1.3. Existing quotient constructions

There are several different families of manifolds – symplectic, Kähler, hyperkähler and quaternionic Kähler manifolds – for which it is known that when a connected group of automorphisms acts upon such a manifold preserving the structure, there is a process called a reduction or a quotient that produces a new manifold of the same family but of smaller dimension.

These constructions are all based upon the concept of a *moment map*, which is a map from the original manifold into the dual of the Lie algebra of the group satisfying a certain differential equation, and work by showing that such a moment map exists and that the zero set of the moment map divided by the action of the group has the structure of the original manifold defined on it in a natural way. We now summarize briefly the existing quotient constructions.

### 1.3.1. The Marsden-Weinstein symplectic quotient

Let $M$ be a symplectic manifold with symplectic form $\omega$, and let $H$ be a connected group of symplectomorphisms acting freely on $M$. Let $\mathfrak{h}$ be the Lie algebra of $H$, and suppose $h \in \mathfrak{h}$. Then the action of $H$ on $M$ induces a vector field $X$ on $M$ corresponding to $h$. Because $H$ preserves $\omega$, $X$ must satisfy $\mathcal{L}_X\omega = 0$, where $\mathcal{L}_X$ is the Lie derivative. Therefore by the classical formula for Lie derivatives

$$\mathcal{L}_X\omega = d(i_X\omega) + i_X(d\omega) = d(i_X\omega) = 0,$$

as $\omega$ is closed, where $i_X$ is contraction with $X$. So $d(i_X\omega) = 0$ and locally $i_X\omega = d(f_h)$, where $f_h$ is a real function on $M$, defined up to a constant. If $M$ is simply connected then $f_h$ is defined globally on $M$ up to a constant.

So for each $h \in \mathfrak{h}$ a function $f_h$ can be defined such that $df_h = i_X\omega$. Under mild conditions on the group $H$ these functions may be put together to form a 'moment map' $\mu : M \longrightarrow \mathfrak{h}^*$ such that for each $h \in \mathfrak{h}$, $h \cdot \mu$ is equal to $f_h$ as a function on $M$, and $\mu$ is $H$- equivariant with respect to the coadjoint action on $\mathfrak{h}^*$. Then $\mu$ is uniquely defined up to addition of an element of the centre of $\mathfrak{h}^*$. Define $N = \{m \in M : \mu(m) = 0\}/H$. Then $N$ is a smooth manifold (as $H$ acts freely on $M$) and $\omega$ descends to give a 2-form $\pi(\omega)$ on $N$. This is not obvious, and may be proved as follows.

If $\pi(m)$ is a point in $N$ and $Y, Z$ are two vectors in $T_{\pi(m)}N$, then $Y, Z$ lift to two vectors $Y', Z'$ in $T_mM$ which are defined up to the addition of any $X(m)$, where $X$ is the vector field associated to $h \in \mathfrak{h}$. But then $\omega(Y', Z')$ does not depend on the choice of $Y'$ or $Z'$. This is because $Y', Z'$ both lie in the plane $d\mu = 0$ in $T_mM$ and so $(i_X\omega)(Y') = \omega(X, Y') = 0$, $(i_X\omega)(Z') = \omega(X, Z') = 0$ for any such $X$. Therefore defining $\pi(\omega)(X, Y) = \omega(X', Y')$ gives a 2-form $\pi(\omega)$ on $N$, that is well-defined because of the $H$- invariance of $\omega$. This 2-form is closed, as its lift to $\{m \in M : \mu(m) = 0\}$ is the restriction of a closed form $\omega$, and it is easily shown to be nondegenerate. So $N$ is a symplectic manifold. This is called the Marsden-Weinstein symplectic quotient [MW].

### 1.3.2. The Kähler quotient

Let $M$ be a Kähler manifold, with metric $g$, complex structure $I$, and Kähler form $\omega$ defined by $\omega(X, Y) = g(IX, Y)$. Suppose that a connected group $H$ acts freely on $M$, preserving the complex structure and the metric. As in the previous section, we can define the symplectic quotient $N$ of $M$ by $H$. However, we also have a quotient metric $g$ on $N$. The two structures are in fact compatible, and there is a complex structure $I$ on $N$ such that $\omega(X, Y) = g(IX, Y)$ on $N$.

So the symplectic quotient defines a new Kähler manifold $N$ of lower dimension. There is a different way to think about this. Because $M$ is complex and $H$ acts preserving the complex structure, we can try to complexify the action of $H$ to give an action of $H^c$ on $M$. This can always be done locally — for each point of $M$ there is a neighbourhood of 1 in $H^c$ for which a holomorphic action can be defined close to the point — but the global action exists only if the vector fields induced by $i\mathfrak{h}$ are *complete* (can be exponentiated arbitrarily far from each point).

Suppose then that the action of $H$ on $M$ can be complexified to give an action of $H^c$. We would like to be able to form $M/H^c$ as a complex manifold and show that it is canonically isomorphic as a complex manifold to $N$. The problem with doing this is that because $H^c$ is noncompact, $M/H^c$ may not be Hausdorff. This problem is solved by restricting to an open set of points in $M$. For the purposes of this section, a point of $M$ is defined to be *stable* if its orbit under $H^c$ meets the zero set of the moment map. Let $M^\circ$ be the set of stable points of $M$. Then $M^\circ/H^c$ is a Hausdorff complex manifold, and under reasonable conditions is isomorphic to $N$. In many cases in algebraic geometry, this notion of stability coincides with a pre-existing algebraic definition called *Mumford stability*.

### 1.3.3. The hyperkähler quotient

The metric $g$ of a hyperkähler manifold is Kähler with respect to each of the three complex structures $I_1, I_2, I_3$. There therefore exist three linearly independent symplectic forms $\omega_1, \omega_2, \omega_3$, the Kähler forms of the complex structures. Suppose that a connected group of diffeomorphisms $H$ acts freely on $M$ preserving the structures. Then there are three moment maps $\mu_1, \mu_2, \mu_3 : M \longrightarrow \mathfrak{h}^*$, one for each of the Kähler forms $\omega_j$. We form the manifold $N = \{m \in M : \mu_1(m) = \mu_2(m) = \mu_3(m) = 0\}/H$, of dimension $4(n - \dim H)$. As $N$ is a submanifold of the symplectic quotients of $M$ by $H$ with respect to the symplectic forms $\omega_j$, the closed 2-forms $\omega_j$ on the symplectic quotients restrict to $N$. So $N$ has three 2-forms $\omega_1, \omega_2, \omega_3$, and also a natural quotient metric induced from $M$.

These four structures turn out to be compatible, so that there are three complex structures $I_1, I_2, I_3$ on $N$ which satisfy $I_1 I_2 = I_3$, with respect to which the quotient metric $g$ is Kähler, and which give $\omega_1, \omega_2, \omega_3$ as the Kähler forms. Thus the quotient $N$ has a canonical hyperkähler structure defined upon it. This is the hyperkähler quotient of Hitchin, Karlhede, Lindström and Roček [HKLR].

We may again think about this from the point of view of the complex structures, regarding the metric as secondary. Here the important point is that $\mu_2 + i\mu_3$ is a holomorphic function with respect to $I_1$, $\mu_3 + i\mu_1$ is holomorphic w.r.t. $I_2$ and $\mu_1 + i\mu_2$ holomorphic w.r.t. $I_3$. Therefore $N$ can be identified with the quotient of $\{m \in M : \mu_2 + i\mu_3 = 0\}$ by the complexification of $H$ with respect to $I_1$, and because $\mu_2 + i\mu_3$ is holomorphic w.r.t. $I_1$ this is complex w.r.t. $I_1$, being the quotient of a complex manifold by a complex group. Similarly $N$ has complex structures $I_2, I_3$.

In this way we are led very simply to the guiding idea of Chapter 2: in our description of the complex structures on the quotient $N$ we have only used the fact that $\mu_2 + i\mu_3$ is holomorphic with respect to $I_1$, $\mu_3 + i\mu_1$ w.r.t. $I_2$ and $\mu_1 + i\mu_2$ w.r.t. $I_3$. Thus quotients with hypercomplex structures can be formed in the absence of a metric provided there exist functions $\mu_1, \mu_2, \mu_3$ satisfying these conditions.

### 1.3.4. The quaternionic Kähler quotient

Suppose $M$ is quaternionic Kähler with non-zero scalar curvature, and that a connected group of diffeomorphisms $H$ acts freely on $M$ preserving the quaternionic Kähler structure. We shall explain what the natural definition of a moment map is in the quaternionic Kähler case. Let $h \in \mathfrak{h}$ and $X$ be the vector field on $M$ induced by $h$. Let $I_1, I_2, I_3$ be a local basis for $\mathcal{G}$ as before, and $\omega_1, \omega_2, \omega_3$ the corresponding 2-forms. Let $i_X$ denote contraction with the vector field $X$.

Define $\Theta_h = \Sigma_{j=1}^{3}(i_X\omega_j) \otimes I_j$. This definition is independent of the basis $I_1, I_2, I_3$, and so $\Theta_h$ is globally defined. Galicki and Lawson prove ([GaL], Theorem 2.4) that under these conditions there is a unique section $f_h$ of $\mathcal{G}$ such that $\nabla f_h = \Theta_h$. Their proof uses the fact that when the scalar curvature is non-zero, then the map $d^\nabla d^\nabla : \mathcal{G} \longrightarrow \Lambda^2 T^*M \otimes \mathcal{G}$ is linear and injective, and thus $f_h$ can be explicitly defined in terms of $\nabla\Theta_h$. When the scalar curvature is zero we have the hyperkähler case.

This section $f_h$ is analogous to the functions $f_h$ constructed in the symplectic case. However, in the quaternionic Kähler case $f_h$ is uniquely defined, rather than being defined up to a constant. There is therefore no obstruction to putting the functions together to make a moment map $\mu$, which is an equivariant section of the bundle $\mathfrak{h}^* \otimes \mathcal{G}$. As $\mathcal{G}$ is not trivial there is only one meaningful level set of the moment map, the zeros, and Galicki and Lawson prove ([GaL], §3) that defining $N = \{m \in M : \mu(m) = 0\}/H$ as the quaternionic Kähler quotient of $M$ by $H$, the quaternionic Kähler structure of $M$ descends to $N$ to make a new quaternionic Kähler manifold.

A connection between the quaternionic Kähler quotient and the hyperkähler quotient has been found by Swann. Recall that to each quaternionic Kähler manifold $M$ there is associated a bundle $\mathcal{U}(M)$ with (pseudo-) hyperkähler total space. If a Lie group $F$ acts on $M$ preserving the structure, then $F$ also acts on $\mathcal{U}(M)$, and so one can do a pseudo-hyperkähler quotient of $\mathcal{U}(M)$ by $F$. Swann shows ([Sw], Theorem 4.6) that by constraining the moment map to vanish on the zero section of the fibration, the moment map is uniquely defined and the resulting pseudo-hyperkähler quotient is the associated bundle of the quaternionic Kähler quotient of $M$ by $F$. Thus, applying the hyperkähler quotient in the associated bundle gives a reduction for quaternionic Kähler manifolds.

# Chapter 2: Quotient Constructions

In §1.3 the idea of a quotient construction was discussed, some known quotient constructions were briefly described, and a connection was given between the hyperkähler quotient and quaternionic Kähler quotient, found by Swann. This chapter presents quotients for hypercomplex and quaternionic manifolds that are analogous to those for hyperkähler and quaternionic Kähler manifolds. A quotient for hypercomplex manifolds will be defined in §2.1, and then Swann's idea will be used to extend this to a reduction for quaternionic manifolds in §2.3.

The Marsden-Weinstein reduction and the others are two-stage processes. Firstly, a *moment map* is defined, which is a map from the manifold $M$ into a vector space or vector bundle, satisfying a certain differential equation. Under reasonable conditions it is shown that the moment map exists and is unique, up to at most the addition of a constant vector. Secondly, it is shown that the quotient of the zero set of the moment map by the group $F$ inherits the structure of the original manifold.

There is an essential difference between the new constructions and the known ones. In the processes we shall describe, there is a concept of moment map, but in a given situation moment maps need not exist, or if they do they may not be unique. However, once a moment map for a particular group action is chosen, the second stage of defining structure on the quotient of the zero set presents no problems. Thus in some cases the reduction of a hypercomplex or quaternionic manifold by a respectable group action cannot be defined because no moment map exists, but in others there may be families of distinct reductions of a manifold by a fixed group action, that much exceed the freedom to add a constant vector to the moment map in the hyperkähler quotient.

One special class of quaternionic manifolds are Kähler surfaces with zero scalar curvature; they are quaternionic because they are conformally anti-self-dual. (See for instance [Pt].) In §2.4 it will be shown that the zero-scalar-curvature Kähler condition fits in well with the quotient picture for quaternionic manifolds, and the Kähler metrics in the conformal class of a quotient can be easily described. We shall define a family of higher-dimensional analogues of zero-scalar-curvature Kähler manifolds, to be called *quaternionic complex* manifolds, which are manifolds with holonomy $SL(n,\mathbb{H})U(1)$. The quaternionic quotient then extends to give a quotient for these manifolds.

## 2.1. The hypercomplex quotient

Let $M$ be a hypercomplex manifold with complex structures $I_1, I_2$ and $I_3$ and $F$ be a compact Lie group acting smoothly and freely on $M$ preserving $I_i$. Let the Lie algebra of $F$ be $\mathfrak{F}$. Then $F$ acts on $\mathfrak{F}^*$ by the coadjoint action.

We define a hypercomplex moment map to be a triple $\mu = (\mu_1, \mu_2, \mu_3)$ of $F$- equivariant maps $\mu_i : M \longrightarrow \mathfrak{F}^*$ satisfying the following two conditions:

$(i)$ $\mu$ satisfies the 'Cauchy-Riemann equations'

$$I_1 d\mu_1 = I_2 d\mu_2 = I_3 d\mu_3, \tag{1}$$

where $I_i$ acts on the cotangent bundle of $\mathfrak{F}^*$-valued 1-forms, $T^*M \otimes \mathfrak{F}^*$.

$(ii)$ Let $X : \mathfrak{F} \longrightarrow \Gamma(TM)$ be the map assigning to each $f \in \mathfrak{F}$ the vector field it induces on $M$. Then for every non-zero $f$ in $\mathfrak{F}$, $\mu$ must satisfy the 'transversality condition'

$$\big(I_1 d\mu_1(f)\big)\big(X(f)\big) \text{ does not vanish on } M. \tag{2}$$

We make three remarks about these conditions. Firstly, an equivalent formulation of condition $(i)$ is

$$I_1(d\mu_2 + id\mu_3) = -i(d\mu_2 + id\mu_3)$$

$$I_2(d\mu_3 + id\mu_1) = -i(d\mu_3 + id\mu_1) \tag{3}$$

$$I_3(d\mu_1 + id\mu_2) = -i(d\mu_1 + id\mu_2),$$

and these three are the Cauchy-Riemann conditions for $\mu_2 + i\mu_3$ to be a holomorphic function with respect to the complex structure $I_1$, $\mu_3 + i\mu_1$ to be holomorphic w.r.t. $I_2$ and $\mu_1 + i\mu_2$ to be holomorphic w.r.t. $I_3$. This is why they were called Cauchy-Riemann conditions.

Secondly, it will be sufficient in $(ii)$ for $(I_1 d\mu_1(f))(X(f))$ not to vanish upon the level set of $\mu$ that will be used for the quotient, rather than on the whole of $M$. Thirdly, $(ii)$ can actually be replaced with the weaker but more complicated condition that for every non-zero $f \in \mathfrak{F}$ and $m \in M$ there should exist some $f'$ in $\mathfrak{F}$ such that

$$\big(I_1 d\mu_1(f')\big)_m \big(X(f)\big)_m \neq 0.$$

We shall prove the following proposition.

**Proposition 2.1.1.** *Let $M$, $F$ and $\mu$ be as above, and let $\zeta_1, \zeta_2, \zeta_3$ be elements of the centre of $\mathfrak{F}^*$. Define $P = \big\{m \in M : \mu_1(m) = \zeta_1, \mu_2(m) = \zeta_2, \mu_3(m) = \zeta_3\big\}$ and $N = P/F$. Then $N$ has a natural hypercomplex structure.*

Two proofs will be given, the first one being the informal proof that led us to the result, and the second being more technically satisfactory as it does not rely on complexifying group actions. The second method of proof extends to the quaternionic quotient, where it has the advantage of being direct, and not an application of the hypercomplex quotient in the associated bundle.

*First Proof.* To show that the quotient has three integrable complex structures, observe that restricting to the solutions of $\mu_j = \zeta_j$ $(j = 1, 2, 3)$ and dividing by $F$ is locally equivalent to restricting to the solutions of $\mu_2 + i\mu_3 = \zeta_2 + i\zeta_3$ and dividing by the complexification of $F$ by $I_1$. This is because condition $(ii)$ ensures that locally each orbit of the complexification of $F$ meets the solutions of $\mu_1 = \zeta_1$ in only one orbit of $F$.

But by $(i)$, $\mu_2 + i\mu_3 = \zeta_2 + i\zeta_3$ is a holomorphic condition w.r.t. $I_1$, and so the quotient $N$ is equivalent to the quotient of a complex manifold by a complex group, and is complex with complex structure $I_1$. Thus $N$ has three complex structures upon it.

To show that these satisfy the relation $I_1 I_2 = I_3$, for each $p$ in $P$ let $V_p$ be defined by

$$\begin{aligned} V_p &= \{v \in T_pM : d\mu_1(v) = d\mu_2(v) = d\mu_3(v) = (I_1 d\mu_1)(v) = 0\} \\ &= \{v \in T_pP : (I_1 d\mu_1)(v) = 0\}. \end{aligned} \tag{4}$$

This defines a vector bundle $V$ over $P$ that is a subbundle of $TP \subset TM|_P$. Now by $(ii)$ the condition $(I_1 d\mu_1)(v) = 0$ is transverse to the infinitesimal action of $F$, and thus there is a natural isomorphism between $V_p$ and $T_{\pi(p)}N$.

So there is an isomorphism between $\pi^*(TN)$ and the subbundle $V$ of $TP$. But condition $(i)$ implies that $V$, considered as a subbundle of $TM|_P$, is closed under $I_1, I_2, I_3$. As $I_1, I_2, I_3$ are $F$- invariant, this defines actions of $I_1, I_2, I_3$ on $TN$ which are clearly the same as the ones above defining integrable complex structures on $N$. Because $I_1, I_2, I_3$ satisfy $I_1 I_2 = I_3$ on $V$, this relation also holds on $N$. □

*Second Proof.* Using $\nabla^M, I_1, I_2$ and $I_3$, a connection $\nabla^N$ and three almost complex structures $I_1, I_2, I_3$ will be defined on $N$. It will then be shown that $\nabla^N$ is torsion-free and satisfies $\nabla^N I_i = 0$. From §1.1, this will imply that $N$ is hypercomplex.

As $F$ acts freely the map $X_m : \mathfrak{F} \longrightarrow T_m M$ is an injection for each $m$ in $M$. By abuse of notation $\mathfrak{F}$ will be identified with its image $X_m(\mathfrak{F})$ in each $T_m M$. Now by $(ii)$ and the definition of $\mu$,

$$TM|_P = TP \oplus I_1\mathfrak{F} \oplus I_2\mathfrak{F} \oplus I_3\mathfrak{F},$$

where both sides are vector bundles over $P$.

Because of the transversality condition on the moment map, $\mathfrak{F}$ is transverse to the annihilator of $I_1 d\mu_1$ in $TP$. So there is a direct sum decomposition $TP = V \oplus \mathfrak{F}$, where $V$ is the bundle defined above.

Thus $P$ satisfies $TM|_P = TP \oplus \mathrm{Im}\,\mathbb{H} \cdot \mathfrak{F}$ and $TP = V \oplus \mathfrak{F}$, where $V$ is some $\mathbb{H}$- invariant vector subbundle of $TM|_P$. It will be shown that under these conditions, $N = P/F$ is a hypercomplex manifold.

**Lemma 2.1.2.** *Let $M$ be hypercomplex and acted on smoothly and freely by a compact Lie group $F$ preserving the structure. Suppose $P$ is an $F$- invariant submanifold of $M$ satisfying $TM|_P = TP \oplus \mathrm{Im}\,\mathbb{H} \cdot \mathfrak{F}$ and $TP = V \oplus \mathfrak{F}$, where $V$ is an $\mathbb{H}$- invariant vector subbundle of $TM|_P$. Then there is a natural hypercomplex structure on $N = P/F$.*

*Proof.* A torsion-free connection $\nabla^N$ on $N$ and three almost complex structures $I_1, I_2, I_3$ will be defined on $N$ and it will be shown that $\nabla^N I_i = 0$. Let $\pi$ be the projection from $P$ to $N$. Observe that $V_p$ is identified with $T_{\pi(p)}N$ by $\pi$. Now the complex structures $I_1, I_2, I_3$ on $M$ act on $V_p$, and therefore also on $T_{\pi(p)}N$. The actions are $F$- invariant, and so descend to give three almost complex structures $I_1, I_2, I_3$ on $N$.

To define the connection on $N$, let $u, v$ be vector fields on $N$. They lift uniquely to give $F$- invariant sections $\tilde{u}, \tilde{v}$ of $V$ over $P$. We shall think of $\tilde{u}$ as a section of $TP$ and $\tilde{v}$ as a section of $TM|_P$.

Now if $a, b$ are vector fields on $M$, the vector field $\nabla^M_a b$ may be formed. This action of $\nabla^M$ can be restricted to $P$: if $a$ is a section of $TP$ and $b$ is a section of $TM|_P$, then $\nabla^{M|P}_a b$ is a section of $TM|_P$.

Thus $\nabla^{M|P}_{\tilde{u}} \tilde{v}$ is defined as a section of $TM|_P$. As $\nabla^M$ is unique, it is $F$- invariant, and so is this section. To get a vector field on $N$, project to $V$ and then push down. So $\nabla^N$ is defined by the equation

$$\nabla^N_u v = \pi \circ \rho \left( \nabla^{M|P}_{\tilde{u}} \tilde{v} \right), \tag{5}$$

where $\rho$ is projection to the first factor in the vector bundle decomposition $TM|_P = V \oplus \mathbb{H} \cdot \mathfrak{F}$. By $F$-invariance, $\nabla^F$ is well-defined.

This definition gives a connection $\nabla^N$ on $N$. Note that in the hyperkähler case where there are metrics, this definition is the same as the usual one involving orthogonal projection. It will now be shown that the connection is torsion-free.

It is sufficient to show that whenever $u, v$ are vector fields on $N$, then $\nabla^N_u v - \nabla^N_v u = [u, v]$. The left hand side is

$$\nabla^N_u v - \nabla^N_v u = \pi \circ \rho \left( \nabla^{M|P}_{\tilde{u}} \tilde{v} - \nabla^{M|P}_{\tilde{v}} \tilde{u} \right) = \pi \circ \rho([\tilde{u}, \tilde{v}]).$$

Now $[\tilde{u}, \tilde{v}]$ is a vector field on $P$, and therefore is a section of $TP \subset TM|_P$. On this subbundle $TP$ we have $(\pi \circ \rho)|_{TP} = \pi$, as restricted to $TP$ the kernel of $\rho$ is equal to the kernel of $\pi$. Thus $\nabla^N_u v - \nabla^N_v u = \pi([\tilde{u}, \tilde{v}])$. But in general, if $\pi : P \to N$ is a submersion and $\tilde{u}, \tilde{v}$ are vector fields on $P$ that are lifts of vector fields $u, v$ on $N$, then $\pi([\tilde{u}, \tilde{v}]) = [\pi(\tilde{u}), \pi(\tilde{v})] = [u, v]$. So $\nabla^N_u v - \nabla^N_v u = [u, v]$, and the connection $\nabla^N$ is torsion-free.

Finally, it will be shown that if $\nabla^M I_i = 0$, then $\nabla^N I_i = 0$. This is equivalent to the statement that whenever $u, v$ are vector fields on $N$, then $\nabla^N_u (I_i v) = I_i \nabla^N_u v$. Lifting to $TM|_P$, this equation is

$$\rho(\nabla^{M|P}_{\tilde{u}} (I_i \tilde{v})) = I_i \rho(\nabla^{M|P}_{\tilde{u}} \tilde{v}).$$

But since $\rho$ commutes with $I_i$, this is equivalent to showing that

$$\rho(\nabla^{M|P}_{\tilde{u}} (I_i \tilde{v})) = \rho(I_i \nabla^{M|P}_{\tilde{u}} \tilde{v}),$$

which is an immediate consequence of the fact that $\nabla^M I_i = 0$. Therefore $\nabla^N I_i = 0$. □

The lemma completes the second proof of Proposition 2.1.1. □

It is necessary to assume that $F$ is compact to ensure that the quotient $N$ is Hausdorff. One can remove this assumption by instead assuming that $N$ or $M/G$ is a manifold. The quotient may only be defined for a given group action on a hypercomplex manifold, if a moment map exists satisfying the conditions. It is therefore important to know whether such moment maps exist all the time, or only sometimes, or never. In fact moment maps sometimes exist, and sometimes do not. Moreover, when they do exist, there may be many possibilities.

An example of a situation in which no moment map exists is the dilation action of $U(1)$ on the Hopf surface, and an example of a situation in which many moment maps can exist is the quotient of $\mathbb{H}^n$ by a compact group. I believe that amongst all hypercomplex manifolds, moment maps are rather rare, so that they would not exist in a generic situation.

### 2.2. The quaternionic moment map

To define moment maps upon quaternionic manifolds, the most obvious approach is to use the torsion-free connection on the quaternionic manifold $M$ to define a connection on the bundle $\mathcal{G}$ defined in §1.1, and that as in the quaternionic Kähler case of [GaL], a moment map on a quaternionic manifold should be defined as a section of $\mathcal{G} \otimes \mathfrak{F}^*$, where $F$ is the quotient group.

For technical reasons this does not quite work. As explained in [S3], §5, to define invariant differential operators on vector bundles over quaternionic manifolds it is usually necessary to tensor through by some power of the real line bundle of volume forms on the manifold, because otherwise the operators defined will not be independent of the choice of connection.

Thus to define 'moment maps' which can be differentiated in a meaningful fashion we work not with the bundle $\mathcal{G}$, but with the bundle $\tilde{\mathcal{G}} = \mathcal{G} \otimes e$, which is $\mathcal{G}$ tensored with a non-zero power $e$ of the real line bundle of volume forms. (In fact, in the notation of [S3], the bundle $\mathcal{G}$ is $S^2H$, and the condition imposed on the moment maps is that their image under the operator $D : S^2H \to E \otimes S^3H$ should be zero. By Corollary 5.4 of [S3], this operator can only be defined from $S^2H'$ to $E' \otimes S^3H'$, where $E' = d^mE$, $H' = d^{-m}H$ with $m$ non-zero, and $d$ is a real line bundle whose $4n^{\text{th}}$ power is the bundle of volume forms.)

On $\tilde{\mathcal{G}}$ there is a connection $\nabla^M$ induced from a torsion-free connection on $M$. The condition we write down in terms of $\nabla^M$ will be independent of the choice of connection on $M$.

Suppose $F$ is a connected Lie group acting smoothly and freely on $M$, and let $\mathfrak{F}$ be its Lie algebra. Define a quaternionic moment map to be an $F$- equivariant section $\mu$ of $\tilde{\mathcal{G}} \otimes \mathfrak{F}^*$ which satisfies the following two conditions:

($i$) For $v$ some section of $T^*M \otimes e \otimes \mathfrak{F}^*$,

$$\nabla^M \mu = I_1 \otimes (I_1 v) + I_2 \otimes (I_2 v) + I_3 \otimes (I_3 v). \tag{6}$$

($ii$) Define $X : \mathfrak{F} \longrightarrow \Gamma(TM)$ to be the map assigning to each $f \in \mathfrak{F}$ the vector field induced by $f$. Then for each non-zero $f \in \mathfrak{F}$ the section $v$ must satisfy

$$v(f)X(f) \text{ does not vanish on the zero set of } \mu, \tag{7}$$

where $v(f)X(f)$ is a section of $e$.

These conditions are direct translations of conditions $(i)$ and $(ii)$ of §2.1 into the quaternionic context. Condition $(i)$ is independent of the local basis $I_1, I_2, I_3$ of $\mathcal{G}$ because it says that $\nabla^M \mu$ should be the contraction of $\Omega$ and $v$, where $\Omega$ is $I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$, which is independent of the choice of $I_1, I_2, I_3$.

Another way of writing $(i)$ is this: certainly $\nabla^M \mu = I_1 \otimes v_1 + I_2 \otimes v_2 + I_3 \otimes v_3$, where $v_1, v_2, v_3$ are sections of $T^*M \otimes e \otimes \mathfrak{F}^*$. Then $v_1, v_2, v_3$ must satisfy the condition $I_1 v_1 = I_2 v_2 = I_3 v_3$. As $v_1, v_2, v_3$ are the analogues of $d\mu_1, d\mu_2, d\mu_3$ in the hypercomplex case, this is a translation of $I_1 d\mu_1 = I_2 d\mu_2 = I_3 d\mu_3$. As in the hypercomplex case, $(ii)$ may be replaced by a weaker condition.

Because the class of hypercomplex manifolds is included in the class of quaternionic manifolds, this new definition also defines quaternionic moment maps on hypercomplex manifolds. However, in the original definition $\mu$ is a section of a trivial vector bundle, and in the new it is a section of a trivial vector bundle tensored with $e$. Therefore the two definitions are only consistent if the connection on $e$ is flat, that is, if the holonomy of the hypercomplex manifold reduces to $SL(n, \mathbb{H})$.

The explanation is this. There is a more general definition of moment map for hypercomplex and quaternionic manifolds that includes the definitions given as a special case; but the hypercomplex moment map and the quaternionic moment map for a hypercomplex manifold are different special cases of the general definition in the hypercomplex case. The general definition uses *quaternionic connections*, which will be considered in Chapter 3 and defined at the beginning of §3.1. Briefly, quaternionic connections on bundles over quaternionic manifolds are the appropriate generalization of self-dual connections on bundles over self-dual 4-manifolds.

Let $L$ be a real line bundle over a hypercomplex or quaternionic manifold $M$, and $\nabla^L$ a quaternionic connection on $L$. (With structure group the positive reals under multiplication.) The generalized definition of hypercomplex moment map is a triple $(\mu_1, \mu_2, \mu_3)$ of $F$- equivariant sections of $L \otimes \mathfrak{F}^*$ satisfying $I_1 \nabla^L \mu_1 = I_2 \nabla^L \mu_2 = I_3 \nabla^L \mu_3$ (the generalized condition $(i)$) and an obvious generalization of condition $(ii)$. The generalized definition of quaternionic moment map is an $F$- equivariant section of $\tilde{\mathcal{G}} \otimes L \otimes \mathfrak{F}^*$ satisfying the obvious generalizations of conditions $(i), (ii)$.

From [GaL], the condition on the quaternionic Kähler moment map is that for each $f \in \mathfrak{F}$, $\nabla^M \mu(f) = i_{X(f)}\Omega$, where $i_{X(f)}$ is contraction with $X(f)$ using the metric. This can be put in a neater form by observing that $X$ defines a section $v'$ of $T^*M \otimes \mathfrak{F}^*$ assigning to each $f$ the covector field associated by the metric to the vector field $X(f)$. Then the condition on $\mu$ is that $\nabla^M \mu$ should be the contraction of $v'$ and $\Omega$, as in the quaternionic case above.

Thus the difference between the quaternionic and the quaternionic Kähler case is that in the quaternionic case $\nabla^M \mu$ may be the contraction of $\Omega$ with any section $v$ of $T^*M \otimes e \otimes \mathfrak{F}^*$, whereas in the quaternionic Kähler case it must be the contraction with a particular section $v'$ given by the group action and the metric. This also holds for the hypercomplex and hyperkähler quotients, where for the former $I_1 d\mu_1$ can be any suitable section of a bundle and for the latter it must be a particular section given by the group action and the metric.

The definition of quaternionic moment map given above may be related to our previous definition of hypercomplex moment map on the associated bundle $\mathcal{U}(M)$, described in §1.2. A moment map $\mu$ on the associated bundle consists of three $\mathfrak{F}^*$- valued functions on $\mathcal{U}(M)$ which are quadratic on each fibre. By restricting (1) to the fibre, one finds that on each fibre the solutions $\mu$ form $\mathfrak{F}^*$ tensored with a three-dimensional vector space, and this vector space is simply $S^2H'$, the fibre of $\tilde{\mathcal{G}}$. So a quadratic moment map on the associated bundle $\mathcal{U}(M)$ gives a section of $\tilde{\mathcal{G}} \otimes \mathfrak{F}^*$, which is a quaternionic moment map. Conversely, a quaternionic moment map lifts to give a hypercomplex moment map on the associated bundle.

### 2.3. The quaternionic quotient

In the previous section the quaternionic moment map has been defined. It will now be shown that if such a moment map exists, then the quotient of the zero set by the group has a natural quaternionic structure. The first proof we give is by applying the hypercomplex quotient to the associated bundle $\mathcal{U}(M)$ (described in §1.2) of a quaternionic manifold, following the example of Swann in the quaternionic Kähler case, and was what led us to the result.

However, to understand the process on the level of quaternionic manifolds, a second proof will be given that does not use associated bundles. Again, the first proof is informal and the second more technical.

**Proposition 2.3.1.** *Let $M$ be a quaternionic manifold acted on freely and smoothly by a compact Lie group $F$ preserving the structure. Suppose that there exists a moment map $\mu$ for the action of the group. Let $P$ be the zero set of $\mu$ in $M$ and let $N = P/F$. Then $N$ has a natural quaternionic structure.*

*First Proof.* From [S2], p. 135, we know that the torsion-free connection $\nabla^M$ is not unique, but can be made unique by choosing a volume form for it to preserve. Choose an $F$- invariant volume form on $M$. Then there is a torsion-free connection $\nabla^M$ preserving the quaternionic structure and the volume form, and as it is unique it is $F$- invariant.

The choice of connection gives a hypercomplex structure on $\mathcal{U}(M)$, as described in §1.2, and since $\nabla^M$ is $F$- invariant, the induced action of $F$ on $\mathcal{U}(M)$ must preserve the hypercomplex structure. As we remarked at the end of the previous section, $\mu$ is simply a hypercomplex moment map for the induced action of $F$ on the associated bundle. By the results of §2.1, one can perform a hypercomplex quotient.

The new hypercomplex manifold has an $\mathbb{H}^*$- action induced from that on $\mathcal{U}(M)$ and is easily seen to fibre over $N$ with fibre $\mathbb{H}/\{\pm 1\}$. Projectivizing w.r.t. some $\mathbb{C}^* \subset \mathbb{H}^*$ gives a complex manifold, $Z$ say. Any $J \in \mathbb{H}^*$ that anticommutes with $I \in \mathbb{C}^*$ induces a real structure $\sigma$ on $Z$, and the orbits of the $\mathbb{H}^*$- action projectivize to give a fibration of $Z$ over $N$ by complex lines, upon which $\sigma$ acts as the antipodal map.

To show that $Z$ is a twistor space for a quaternionic structure on $N$ it only remains to prove that the normal bundle of the real lines is $2a\mathcal{O}(1)$. Now the normal bundle of a real line is the projectivization of the normal bundle of the corresponding $\mathbb{H}^*$- orbit in the hypercomplex manifold. But this normal bundle can be trivialized as a hypercomplex manifold. In fact the $\mathbb{H}^*$- action identifies all the fibres of the bundle, but this is not a triholomorphic identification as it permutes the complex structures.

However, the fibres are of the form $\mathbb{H}^a$, and composing the $\mathbb{H}^*$- action by premultiplication with the inverse element of $\mathbb{H}^*$, the complex structures are preserved and we have a triholomorphic trivialization of the normal bundle. The normal bundle of the real line in $Z$ is therefore isomorphic to that coming from a standard example, the normal bundle of $\mathbb{CP}^1$ in $\mathbb{CP}^{2a+1}$, which is $2a\mathcal{O}(1)$. So $Z$ is a twistor space and defines a quaternionic structure on $N$. $\square$

The above proof would be much shorter if we could conclude that because the quotient hypercomplex manifold has a suitable $\mathbb{H}^*$- action it is the associated bundle of some quaternionic structure. However, this is not in general true: as in the previous section, where the definition of moment map could be changed by tensoring by a real line bundle equipped with a *quaternionic connection* (see §3.1), here it is the case that a hypercomplex manifold with a suitable $\mathbb{H}^*$- action need not be an associated bundle but may instead be an associated bundle *twisted by* a real line bundle with quaternionic connection, as will be explained in Chapter 3. So we must projectivize to get rid of the real line bundle.

*Second Proof.* As above, choose $\nabla^M$ to be $F$- invariant. Let $I_1, I_2, I_3$ be a local $F$- invariant basis for the bundle of almost complex structures that satisfies $I_1 I_2 = I_3$, and $s$ be a local, smooth, non-vanishing $F$-invariant section of $e$. Then $\mu = \mu_1 s \otimes I_1 + \mu_2 s \otimes I_2 + \mu_3 s \otimes I_3$ where $\mu_1, \mu_2, \mu_3$ are scalar functions. At the points where $\mu = 0$, i.e. on $P$, we have

$$\nabla^M \mu = (d\mu_1) \otimes s \otimes I_1 + (d\mu_2) \otimes s \otimes I_2 + (d\mu_3) \otimes s \otimes I_3, \tag{8}$$

which is not generally true away from $P$ because $\nabla^M(s \otimes I_i)$ need not vanish. Then condition $(i)$ becomes $I_1 d\mu_1 = I_2 d\mu_2 = I_3 d\mu_3$, as in the hypercomplex case. As $P$ is defined by the vanishing of the scalar functions $\mu_1, \mu_2, \mu_3$, the vector bundle $V$ may be defined as in the hypercomplex case. Therefore $TM|_P = TP \oplus \mathrm{Im}\,\mathbb{H} \cdot \mathfrak{F}$ and $TP = V \oplus \mathfrak{F}$, where $V$ is a subbundle of $TP$ that is invariant under $I_1, I_2, I_3$ as a subbundle of $TM|_P$.

The proof of Proposition 2.3.1 will therefore be completed by the

**Lemma 2.3.2.** *Suppose $M$, $F$ and $\nabla^M$ are as above, and that $P$ is an $F$- invariant submanifold of $M$ satisfying $TM|_P = TP \oplus \mathrm{Im}\,\mathbb{H} \cdot \mathfrak{F}$ and $TP = V \oplus \mathfrak{F}$, where $V$ is a subbundle of $TP$ that is invariant under $I_1, I_2, I_3$ as a subbundle of $TM|_P$. Then $N = P/F$ has a natural quaternionic structure.*

*Proof.* A connection $\nabla^N$ and three almost complex structures $I_1, I_2$ and $I_3$ can be defined on $N$ exactly as in the proof of Lemma 2.1.2, with the proviso that $I_i$ are only local, and the proof there that $\nabla^N$ is torsion-free also transfers unchanged to this situation. It will now be shown that $\nabla^N$ preserves the family of almost complex structures on $N$. This then implies that $N$ is quaternionic, except in four dimensions, as the definition of quaternionic manifold is stronger in this case.

On $M$ the $I_i$ satisfy $\nabla^M I_i = \alpha_{ij} \otimes I_j$, using the summation convention, where $(\alpha_{ij})$ is an anti-symmetric $3 \times 3$ matrix of 1-forms on $M$.

Choosing the local almost complex structures $I_1, I_2, I_3$ to be $F$- invariant makes the 1-forms $\alpha_{ij}$ $F$-invariant. The horizontal parts of the $\alpha_{ij}$ project down to give an anti-symmetric matrix of 1-forms $(\alpha_{ij})$ on $N$. It will be shown that $\nabla^N I_i = \alpha_{ij} \otimes I_j$.

It is sufficient to show that $\nabla^N_u(I_i v) = I_i \nabla^N_u v + \alpha_{ij}(u) I_j v$, where $u, v$ are any vector fields on $N$ and $\alpha(v)$ is the contraction of the 1-form $\alpha$ with the vector field $v$. We lift this equation to $TM|_P$. The fields $u, v$ lift uniquely to give $F$- invariant sections $\tilde{u}, \tilde{v}$ of $V$ over $P$. We shall again think of $\tilde{u}$ as a section of $TP$ and $\tilde{v}$ as a section of $TM|_P$.

By definition of $\nabla^N$, the first two terms lift to $\rho(\nabla^{M|P}_{\tilde u}(I_i\tilde v))$ and $I_i\rho(\nabla^{M|P}_{\tilde u}\tilde v)$. The scalar field $\alpha_{ij}(u)$ lifts to $\alpha_{ij}(\tilde u)$, because although the vertical part of $\alpha_{ij}$ is lost on projection to $N$, $\tilde u$ is horizontal and so this does not matter. Also $I_j v$ lifts to $I_j\tilde v$.

Thus we must demonstrate that

$$\rho(\nabla^{M|P}_{\tilde u}(I_i\tilde v)) = I_i\rho(\nabla^{M|P}_{\tilde u}\tilde v) + \alpha_{ij}(\tilde u)I_j\tilde v.$$

But this is just the application of $\rho$ to the equation

$$\nabla^{M|P}_{\tilde u}(I_i\tilde v) = I_i\nabla^{M|P}_{\tilde u}\tilde v + \alpha_{ij}(\tilde u)I_j\tilde v,$$

which follows from $\nabla^M I_i = \alpha_{ij}\otimes I_j$. □

## 2.4. Quaternionic complex manifolds

Let $M$ be a quaternionic 4-manifold. The complex structures at a point $x$ in $M$ compatible with the quaternionic structure are parameterized by the points of the fibre of the twistor space $Z$ of $M$ over the point $x$. Thus an almost complex structure $I$ on $M$ compatible with the quaternionic structure is a section of the fibration $Z\to M$; $I$ is integrable whenever the section is a complex hypersurface in $Z$.

Recall the orientation convention of §1.1, that the orientations chosen upon hypercomplex and hyperkähler manifolds are opposite to the natural complex ones. In this section we will reverse this convention and work with the natural complex orientation, in order to fit in with other published papers. Thus we shall deal with Hermitian metrics that are anti-self-dual instead of self-dual, and this really does mean anti-self-dual with respect to the complex orientation.

We begin by quoting a result of Pontecorvo ([Pt], Theorem 2.1), that if $M$ is Hermitian with metric $g$ and complex structure $I$ and anti-self-dual, then $g$ is conformal to a Kähler metric if and only if the line bundle defined by the divisor $[X]$ is isomorphic to $K_Z^{-\frac12}$. Here $K_Z$ is the canonical line bundle over $Z$ and $[X]$ is the sum of the hypersurface $\Sigma$ in $Z$ defined by the complex structure $I$ and the hypersurface $\overline{\Sigma}$ defined by $-I$.

Thus the Kähler metrics in the conformal class of $M$ are exactly given by real holomorphic sections of $K_Z^{-\frac12}$. Using a calculation in [S1], Theorem 4.3, we find that a real holomorphic section of $K_Z^{-\frac12}$ is a complex function $\psi$ on the associated bundle that is quadratic, holomorphic with respect to $I_1$, and satisfies the reality condition $\psi(h) = \overline{\psi(I_2 h)}$ for $h$ in the associated bundle.

As $\psi$ is quadratic it has two zeros in each fibre of the twistor space $Z$, which are interchanged by the real structure $\sigma$ on $Z$ because $\psi$ is real. Therefore $\psi$ defines a complex structure $I$ and its conjugate $-I$ on $M$. Also, the 'norm' of $\psi$ on each fibre lies in a non-zero power of the volume forms, which gives a volume form on $M$. So $M$ has a complex structure, a conformal structure and a volume form, which together make $M$ Hermitian. Pontecorvo's result is that $M$ is in fact Kähler with zero scalar curvature.

We shall make an observation that will enable us to put this information in a form that does not single out the complex structure $I_1$ on the associated bundle. The quaternionic moment maps defined in §2.2 were interpreted on the associated bundle as triples of $\mathfrak{F}^*$- valued quadratic functions $\mu_1, \mu_2, \mu_3$ with $\mu_2 + i\mu_3$ holomorphic w.r.t. $I_1$. The remaining conditions imply the reality condition $(\mu_2+i\mu_3)(h) = (\mu_2-i\mu_3)(I_2 h)$, and these two conditions on $\mu_2, \mu_3$ are sufficient in the sense that there then exists a unique $\mu_1$ forming a quaternionic moment map. So a real holomorphic section $\psi$ of $K_Z^{-\frac{1}{2}}$ is equivalent to a triple $\mu$ of quadratic scalar functions $\mu_1, \mu_2, \mu_3$ on the associated bundle satisfying (1). On $M$ this is a section $\mu$ of $\tilde{\mathcal{G}}$ satisfying (6).

Define a *twistor function* $\mu$ on a quaternionic manifold $M$ to be a section $\mu$ of $\tilde{\mathcal{G}}$ satisfying (6). They are called twistor functions because they are in the kernel of a differential operator called the *quadratic twistor operator*. (See [S3], §5 for the theory of invariant differential operators on quaternionic manifolds and [S1], Lemma 6.4, for the definition of the quadratic twistor operator $D$ in the quaternionic Kähler case.) It is easily shown that the only twistor functions on a connected open set in $\mathbb{H}^n$, and hence $\mathbb{HP}^n$, are polynomials of degree at most two.

Let us digress for a moment to consider the relation between twistor functions and twistors. In Penrose's original formulation of twistor theory on flat space, twistors were the kernel of a related differential operator called the *(linear) twistor operator*, and thus were global objects defined on the whole manifold. However, on curved space it is very rare for the operator to have any nonzero kernel. Our twistor functions look like elements of the second symmetric product of the vector space of twistors, and can be thought of as quadratic twistors rather than linear twistors. Again, generically we expect a curved space to admit no nonzero twistor functions.

Pontecorvo's result can now be rewritten, to state that the Kähler metrics in the conformal class of a quaternionic 4-manifold $M$ are given by the non-vanishing twistor functions $\mu$ on $M$. Looked at this way, zero scalar curvature Kähler surfaces have an obvious generalization to higher dimensions. The manifolds are quaternionic with a preferred complex structure, so we shall adopt the name *quaternionic complex* for them. We define a quaternionic complex manifold to be a quaternionic manifold $M$ together with a twistor function $\mu$ that vanishes nowhere on $M$. It is sometimes convenient to allow $\mu$ to vanish at points on $M$, and these will be called *singular* points of the quaternionic complex manifold, so in general a quaternionic complex manifold will be an open set of a singular quaternionic complex manifold.

The quaternionic quotient generalizes very simply to the quaternionic complex case: if one does a quaternionic quotient of a quaternionic complex manifold by a group preserving the twistor function $\mu$ then it is easy to see that $\mu$ descends to a twistor function on the quotient, which will again be non-vanishing. (However, when dealing with singular quaternionic complex manifolds it is important to ensure that $\mu$ does not lie in the span of the moment maps chosen.)

The main result that we will prove about quaternionic complex manifolds is that they can alternatively be described as manifolds with an $SL(n,\mathbb{H})U(1)$- structure preserved by a torsion-free connection, and that as in the hypercomplex case this connection is unique. (The structure group is $SL(n,\mathbb{H})U(1)$ because, as in four dimensions, the twistor function gives a complex structure and a volume form, and the group preserving a quaternionic structure, a complex structure and a volume form is $SL(n,\mathbb{H})U(1)$.)

We note that in the classification by Berger [Br] of holonomy groups of manifolds with torsion-free connections, $SL(n,\mathbb{H})U(1)$ is given as a possible holonomy group in Theorem 4, p. 320; in Berger's notation, $SL(n,\mathbb{H})U(1)$ is $\mathbf{T}^1 \times \mathbf{SU}^*(2n)$.

**Theorem 2.4.1.** *Let $M$ be a quaternionic complex manifold. Then $M$ has a natural $SL(n,\mathbb{H})U(1)$-structure $Q$, and there is a unique torsion-free connection $\nabla^M$ preserving $Q$.*

*Proof.* As $M$ is quaternionic, it has a $GL(n,\mathbb{H})GL(1,\mathbb{H})$- structure $Q'$. A point $q'$ in the fibre of $Q'$ over $m \in M$ is an isomorphism of vector spaces $q' : \mathbb{H}^n \to T_mM$ inducing isomorphisms on the families of complex structures on $\mathbb{H}^n$ and $T_mM$.

But the twistor function on $M$ gives a non-zero volume form $\theta$ on $T_mM$ and selects one of the complex structures, denoted $I$. Define the subset $Q$ of $Q'$ as those $q' \in Q'$ taking $I_1$ to $I$ and the standard volume form on $\mathbb{H}^n$ to $\theta$. Clearly $Q$ fibres over $M$ with fibre $SL(n,\mathbb{H})U(1)$, so $Q$ is an $SL(n,\mathbb{H})U(1)$- structure on $M$.

To show that there exists a unique connection $\nabla^M$ on $M$ preserving $Q$, it is sufficient to find a unique $\nabla^M$ preserving $Q'$, $I$ and $\theta$. Recall that from [S2], p. 135, the torsion-free connection on a quaternionic manifold may be uniquely defined by giving a volume form for it to preserve. Let $\nabla^M$ be the torsion-free connection on $M$ preserving $Q'$ and the volume form $\theta$. We will show that $\nabla^M I = 0$.

Set $I_1 = I$ and choose $I_2, I_3$ locally in $\mathcal{G}$ such that $I_1 I_2 = I_3$. Then $\mu = s \otimes I_1$, where $s$ is a non-vanishing section of $e$. As $\nabla^M \theta = 0$ and $\theta$ is some non-zero power of $s$, we have $\nabla^M s = 0$. Also as $\nabla^M$ preserves $Q'$ we have $\nabla^M I_i = \alpha_{ij} I_j$, where $(\alpha_{ij})$ is an antisymmetric matrix of 1-forms. Thus $\nabla^M \mu = s \otimes \alpha_{1j} \otimes I_j$.

However, $\nabla^M \mu$ satisfies the quadratic twistor equation, and writing $\nabla^M \mu = s \otimes v_j \otimes I_j$ gives $I_1 v_1 = I_2 v_2 = I_3 v_3$. But $v_1 = \alpha_{11} = 0$, as $(\alpha_{ij})$ is anti-symmetric. So $v_2 = v_3 = 0$ and $\nabla^M \mu = 0$.

Therefore $\nabla^M I = 0$ and there is a unique torsion-free connection $\nabla^M$ preserving the quaternionic structure $Q'$ of $M$, the complex structure $I$ and the volume form $\theta$. □

As a corollary we reprove Pontecorvo's result quoted above.

**Corollary 2.4.2 ([Pt], Theorem 2.1).** *A four-dimensional quaternionic complex manifold is exactly a Kähler surface of zero scalar curvature.*

*Proof.* Let $M$ be a four-dimensional quaternionic complex manifold. Then $M$ has the structure of a Hermitian manifold, with Riemannian metric $g$ and compatible complex structure $I$. Since $\nabla^M$ is torsion-free and preserves $g$ it must be the Levi-Civita connection, and as $I$ satisfies $\nabla^M I = 0$, $M$ is by definition Kähler. But $M$ is conformally anti-self-dual, so it must have zero scalar curvature.

Conversely, if $M$ is a zero-scalar-curvature Kähler surface, it is quaternionic, and the volume form and complex structure together make up a section $\mu$ of $\tilde{\mathcal{G}}$ satisfying $\nabla^M \mu = 0$, and a fortiori the quadratic twistor equation. Thus $M$ is quaternionic complex. □

We also have an alternative definition for quaternionic complex manifolds:

**Corollary 2.4.3.** *In $4n$ dimensions with $n > 1$, a quaternionic complex manifold is a manifold with an $SL(n, \mathbb{H})U(1)$- structure preserved by a torsion-free connection. In four dimensions a quaternionic complex manifold is a Kähler surface with zero scalar curvature.*

A curious aspect of this work is that although we have a quotient for a type of Kähler manifold, it is not a Kähler quotient. This is because the higher dimensional manifolds do not have metrics. I have also found a pseudo-Kähler quotient for the zero-scalar-curvature Kähler surfaces given as examples of the quaternionic complex quotient in §4.2.1. But the two quotients seem almost unrelated and I do not know if there is a systematic way of producing zero-scalar-curvature Kähler surfaces as Kähler quotients.

# Chapter 3: Twisting Constructions

The idea that $U(1)$- invariant self-dual metrics can be constructed from solutions to a monopole equation on a 3-manifold with a special structure called an Einstein-Weyl structure has been known for some time. It first appeared, I believe, in 1977 in two papers [GH], [Ha] by Gibbons and Hawking. They used monopoles on the Einstein-Weyl space $\mathbb{R}^3$ to write down two families of complete, Ricci-flat, self-dual metrics called ALE spaces and ALF spaces.

Then in 1985, the construction was generalized to any Einstein-Weyl space by Jones and Tod ([JT], §6). Later, the construction was used by LeBrun [L2] on hyperbolic 3-space to make self-dual metrics on the connected sum of $n$ copies of $\mathbb{CP}^2$. In this chapter the construction will be generalized even further, replacing $U(1)$ by an arbitrary Lie group and moving from self-dual 4-manifolds to quaternionic manifolds in any dimension.

Suppose that we have chosen a hypercomplex or quaternionic manifold $M$, a Lie group $G$, an action $\Psi$ of $G$ on $M$ that preserves the structure, and a principal $G$- bundle $P$ over $M$ carrying a $\Psi$- invariant $G$-connection satisfying a certain curvature condition, that generalizes the instanton equations in the four-dimensional case. We shall show that one can, subject to a certain condition, define a new hypercomplex or quaternionic manifold $N$ that is $M$ 'twisted by' the $G$- bundle $P$. This is called a twisting construction because we imagine $P$ as having a twist in its topology, like a loop of paper with several half-turns in, that is applied to the manifold $M$ to give a new manifold $N$.

As an application of these ideas, in §3.2 many compact, nonsingular, simply-connected hypercomplex and quaternionic manifolds are constructed in dimensions greater than four, that are not products or joins of other manifolds, and even locally are not hyperkähler or quaternionic Kähler. (That is, the structure group cannot be reduced to $Sp(n)$ or $Sp(n)Sp(1)$.) These are interesting because they are (I believe) the first such examples, and show that there are many compact higher-dimensional hypercomplex and quaternionic manifolds, which is a contrast with the apparent scarcity of compact quaternionic Kähler manifolds.

As another application, in the next chapter twisting methods will be used to construct some new self-dual metrics on $n\mathbb{CP}^2$, in a similar way to LeBrun's description of his metrics. These new metrics are also briefly described in Appendix B at the end of §B.7. For its theoretical interest a generalization of the twisting construction is given in §3.3 that allows, in a certain sense, twisting by a homogeneous space instead of a group. I have not yet found any interesting examples of its use.

## 3.1. A twisting construction for hypercomplex and quaternionic manifolds

Suppose that $M$ is a quaternionic manifold, and let $Z$ be the twistor space of $M$. Then $Z$ is a complex manifold that is a fibre bundle over $M$ with fibres $\mathbb{CP}^1$ of normal bundle $n\mathcal{O}(1)$, and has an antiholomorphic involution $\sigma$ that fixes the fibres. Let $G$ be a compact Lie group, $P$ a principal $G$- bundle over $M$, and $A$ a connection on $P$. The curvature $\Omega$ of $A$ is a 2-form on $M$ with values in ad$P$, where ad$P$ is the bundle associated to the adjoint representation of $G$ on the Lie algebra $\mathfrak{g}$ of $G$.

At each point of $M$ there is a family of complex structures from the quaternionic structure, and for each such complex structure $I$, the 2-forms on $M$ can be decomposed into the $+1$- and $-1$- eigenspaces of $I$; the $+1$ eigenspace corresponds to the real $(1,1)$- forms and the $-1$ eigenspace to real combinations of $(2,0)$- and $(0,2)$- forms.

We define a *quaternionic connection* $A$ on $P$ to be a connection whose curvature $\Omega$ is in the $+1$-eigenspace for each $I$ in the family at every point. This definition coincides with the definition of a quaternionic connection given in [S3], Definition 7.1. Moreover, when $N$ is 4-dimensional, a quaternionic connection is just a (self-dual) instanton. So quaternionic connections are the natural generalization to quaternionic manifolds in all dimensions of the notion of an instanton in four dimensions.

Quaternionic connections are interesting because the following generalization of the Ward correspondence applies to them:

> *The Ward correspondence:* Let $M$ be a quaternionic manifold, $Z$ the twistor space of $M$, $G$ a Lie group, and $P$ a principal $G$- bundle over $M$. Let $\tilde{P}$ be the lift of $P$ to $Z$ and $\tilde{P}^c$ be the complexification of $\tilde{P}$, with fibre $G^c$, the complexification of $G$. Then quaternionic connections $A$ on $P$ are in one-to-one correspondence with real holomorphic structures on the $G^c$- bundle $\tilde{P}^c$ that are trivial on the fibres of $Z$.

By a real holomorphic structure we mean a holomorphic structure that changes sign under the composition of the real structure on $Z$ and complex conjugation on the fibres of $\tilde{P}^c$.

Because $P$ is a principal bundle there is an action of $G$ on $P$, which shall be called $\Phi$, that acts transitively on the fibres. Let $\Psi : G \to \mathrm{Aut}(M)$ be an action of $G$ upon $M$, that lifts to an action of $G$ on $P$ (some group actions may not lift to the bundle). We choose a particular lifting of $\Psi$ to $P$, that will also be called $\Psi$, preserving the principal bundle structure (i.e., commuting with $\Phi$). This lifting is not necessarily unique up to homotopy.

We shall now prove two theorems, which have very similar statements and proofs.

**Theorem 3.1.1.** *Let $M$, $P$, $\Phi$ and $\Psi$ be as above, and let $A$ be a $\Psi$- invariant quaternionic connection on $P$. Suppose $\Psi(G)$ acts freely on $P$. Then the manifold $N = P/\Psi(G)$ has a natural (possibly singular) quaternionic structure, which is nonsingular wherever the Lie algebra of $\Psi(G)$ is transverse to the horizontal subspaces of $A$ in $P$.*

**Theorem 3.1.2.** *Let $M$, $P$, $\Phi$ and $\Psi$ be as above, and let $A$ be a $\Psi$- invariant quaternionic connection on $P$. Let $\Delta : G \to \mathrm{Aut}(P)$ be the diagonal action of $G$ on $P$, given by $\Delta(g) = \Phi(g)\Psi(g)$. (This is a group homomorphism because $\Phi$ and $\Psi$ commute.) Suppose that $\Delta(G)$ acts freely on $P$. Then the manifold $N = P/\Delta(G)$ has a natural (possibly singular) quaternionic structure, which is nonsingular wherever the Lie algebra of $\Delta(G)$ is transverse to the horizontal subspaces of $A$ in $P$.*

If $M$ is hypercomplex rather than just quaternionic, and $\Psi$ preserves the hypercomplex structure as well as the quaternionic structure, then the manifolds $N$ constructed in Theorems 3.1.1 and 3.1.2 will also be hypercomplex. This is because if $M$ is hypercomplex then its twistor space $Z$ fibres over $\mathbb{CP}^1$, and this induces a fibration over $\mathbb{CP}^1$ of the twistor space $W$ of $N$ constructed in the proof below. The proofs of Theorems 3.1.1 and 3.1.2 are almost identical, so only the first will be given; to get the second, replace $\Psi$ by $\Delta$ throughout.

*Proof of Theorem 3.1.1.* By the Ward correspondence, the quaternionic connection $A$ on $P$ defines a holomorphic structure on the bundle $\tilde{P}^c$ over $Z$. The action $\Psi$ on $P$ lifts to $\tilde{P}$ and then to $\tilde{\Psi}$ on $\tilde{P}^c$, and as $A$ is $\Psi$- invariant, this action preserves the holomorphic structure. We define the antiholomorphic involution $\tilde{\sigma}$ of $\tilde{P}^c$ to be the composition of the antiholomorphic involution $\sigma$ on $Z$ and complex conjugation on the fibres $G^c$. Then $\tilde{\Psi}$ commutes with $\tilde{\sigma}$.

The action $\tilde{\Psi}$ of $G$ can be complexified to an action $\tilde{\Psi}^c$ of $G^c$ on $\tilde{P}^c$. Ideally we would like to say that $\tilde{P}^c/\tilde{\Psi}^c(G^c) \cong \tilde{P}/\tilde{\Psi}(G)$, because each $\tilde{\Psi}^c(G^c)$- orbit in $\tilde{P}^c$ contains exactly one $\tilde{\Psi}(G)$- orbit in $\tilde{P}$; thus $\tilde{P}/\tilde{\Psi}(G)$ would also be the quotient of a complex manifold by a complex group, and so would have a complex structure. However, this involves us in two sorts of problems: firstly, some $\tilde{\Psi}^c(G^c)$- orbits might contain no $G$- orbits in $\tilde{P}$, or more than one, and secondly, as $G^c$ is a non-compact group, topological restrictions on its action are necessary for the quotient even to be Hausdorff.

We shall overcome these problems as follows. Let $U$ be a small open neighbourhood of $G$ in $G^c$. We require that $U$ should be invariant under complex conjugation and the action of $G$ on the right, and that the closure of $U$ in $G^c$ should be compact. We also require that $U$ should be sufficiently small that if $x_1, x_2 \in \tilde{P}$ and $u_1, u_2 \in U$ and $\tilde{\Psi}^c(u_1)x_1 = \tilde{\Psi}^c(u_2)x_2$ then $x_1, x_2$ are in the same $\tilde{\Psi}(G)$- orbit in $\tilde{P}$. (This is possible at least for compact subsets of $\tilde{P}$. Here the transversality condition is needed to ensure that the action $\tilde{\Psi}^c(i\mathfrak{g})$ is transverse to $\tilde{P}$ in $\tilde{P}^c$, without which the result might fail.)

Let $S \subset \tilde{P}^c$ be the set $\tilde{\Psi}^c(U)[\tilde{P}]$. Then $S$ is an open neighbourhood of $\tilde{P}$ in $\tilde{P}^c$, and fibres over $\tilde{P}/\tilde{\Psi}(G)$ with fibre $U$. (Here we use the property of $U$ given in the previous paragraph, and also the right $G$- invariance of $U$.) As the fibres are locally $\tilde{\Psi}^c(G^c)$- orbits, they are complex submanifolds and the fibration is holomorphic. Since $\overline{U}$ is compact, the fibration is topologically well behaved. So $\tilde{P}/\tilde{\Psi}(G)$ is the base space of a holomorphic fibre bundle, and is thus a complex manifold. Also, $\tilde{\sigma}$ restricts to $S$, where it preserves the fibres, so it descends to an antiholomorphic involution $\sigma'$ of $\tilde{P}/\tilde{\Psi}(G)$.

Define $W = \tilde{P}/\tilde{\Psi}(G)$. Then from above, wherever $\Psi$ is transverse to the horizontal subspaces of $A$, $W$ is the base space of a holomorphic fibre bundle $S$, and so has a complex structure and an antiholomorphic involution. Moreover, $W$ fibres over $N = P/\Psi(G)$ with fibre $\mathbb{CP}^1$, since dividing by $\Psi(G)$ commutes with passage from $M$ to the twistor space $Z$. To prove that $W$ is a twistor space, it remains only to show that the normal bundle of the fibres is $n\mathcal{O}(1)$.

To do this requires a little algebraic geometry. The bundle $\tilde{P}^c$ is trivial over real lines as a holomorphic bundle, and so the normal bundle of a real line in $\tilde{P}^c$ is $n\mathcal{O}(1) + \mathfrak{g}^c \otimes \mathcal{O}$. But to get the normal bundle of the corresponding fibre of $W$ we have to divide by the part tangent to the orbit of $G^c$, which is isomorphic to $\mathfrak{g}^c \otimes \mathcal{O}$. However, one cannot simply cancel off the copies of $\mathcal{O}$; for example $\mathcal{O}$ may be embedded as a subbundle of $2\mathcal{O}(1) + \mathcal{O}$ so as to give the quotient bundle $\mathcal{O} + \mathcal{O}(2)$ rather than $2\mathcal{O}(1)$.

Now $n\mathcal{O}(1) + \mathfrak{g}^c \otimes \mathcal{O}$ has a canonical projection to $\mathfrak{g}^c \otimes \mathcal{O}$, and the condition that the normal bundle of the fibre should be $n\mathcal{O}(1)$ is that the map from $\mathfrak{g}^c \otimes \mathcal{O}$ to itself got by applying this projection to the subbundle tangent to the orbit of $G^c$, should be nondegenerate. We shall show that this is implied by the transversality condition.

Above, we used the fact that the natural twistor interpretation of the transversality condition is that $\tilde{\Psi}^c(i\mathfrak{g})$ is transverse to $\tilde{P}$ in $\tilde{P}^c$; that is, that at each point a certain natural linear map from $\mathfrak{g}$ to itself is nondegenerate. But on each real line in $\tilde{P}$, it can be seen that the map we want from $\mathfrak{g}^c \otimes \mathcal{O}$ to itself is the complexification of this natural map from $\mathfrak{g}$ to itself, and so the nondegeneracy of the complex map does follow from the transversality condition. Therefore the normal bundle of the fibres of $W$ is $n\mathcal{O}(1)$ wherever the transversality condition holds, and $W$ is the twistor space for a quaternionic structure on $N$, which is nonsingular wherever $\Psi(G)$ is transverse to the horizontal subspaces of $A$. $\square$

In Appendix A a rigorous proof of this Theorem will be given without invoking the Ward correspondence, by showing that the Nijenhuis tensor of each of the three almost complex structures on the associated bundle of $N$ vanishes, and thus that they are integrable. It is a long but elementary calculation that starts from the fact that the curvature is of type (1,1) with respect to each complex structure.

## 3.2. Compact hypercomplex and quaternionic manifolds

In this section we will apply Theorem 3.1.2 to construct compact, nonsingular, simply-connected examples of hypercomplex and quaternionic manifolds. Theorem 3.1.2 is actually more useful than Theorem 3.1.1, because there are many situations in which the image of the Lie algebra action $\psi$ is actually contained in the horizontal subspaces of the connection $A$, and therefore the transversality condition of Theorem 3.1.1 does not hold anywhere, but that of Theorem 3.1.2 holds everywhere and so the resulting manifold has a nonsingular quaternionic structure.

### 3.2.1. First example

Let $M$ be a compact simply-connected quaternionic manifold, and suppose that $P$ is a non-trivial, primitive $U(1)$- bundle (and so, has simply-connected total space) carrying a quaternionic connection $A$. For instance, $M$ could be $\mathbb{CP}^2$ and the instanton could be the one with curvature form the Kähler form of the Fubini-Study metric; this generalizes to the higher-dimensional symmetric spaces $SU(n+2)/S(U(n)\times U(2))$. Or $M$ could be a self-dual metric on $n\mathbb{CP}^2$ and the instanton that one arising from the harmonic form representing any integral, primitive, non-zero two-dimensional cohomology class.

Then the associated bundle $\mathcal{U}(M)$ introduced in §1.2 is hypercomplex, but not yet compact. Let $r$ be a positive real constant. Then the integers $\mathbb{Z}$ act on $\mathcal{U}(M)$ by multiplication by $e^{rn}, n\in\mathbb{Z}$, and dividing by this action gives a compact hypercomplex manifold $\mathcal{U}(M)/\mathbb{Z}$ that is not simply-connected, and fibres over $M$ with fibre the Hopf surface (divided by $\{\pm 1\}$).

Let $\Psi$ be the action of $U(1)$ on $\mathcal{U}(\mathcal{M})/\mathbb{Z}$ of dilation on the fibres, that is, let $e^{i\theta}$ act by multiplication by $e^{\frac{r\theta}{2\pi}}$. This action preserves the fibration over $M$ and thus the lift of the instanton $A$ to $\mathcal{U}(M)$. Now applying Theorem 3.1.2 we get a new hypercomplex manifold that is $\mathcal{U}(M)/\mathbb{Z}$ twisted by the non-trivial

$U(1)$- bundle $P$. The new manifold $N$ is compact and simply-connected (though we may need to take a double cover if $M$ is spin, as then $\mathcal{U}(M)$ has a double cover), because twisting by $P$ kills the fundamental group as $P$ is primitive. It also fibres over $M$ with fibre the Hopf surface (divided by $\{\pm 1\}$), but the $U(1)$-component of the fibration is now non-trivial.

Hopf surfaces appear in most of the examples we shall give. An interesting point about the case when $M$ is $\mathbb{CP}^2$ is that the resulting manifold $N$ is homogeneous, that is, has transitive symmetry group. In fact $N$ is $SU(3)$, and has symmetry group $SU(3) \times U(1)$. Homogeneous hypercomplex and quaternionic manifolds will be the topic of Chapter 5.

A variation on the above is, instead of working with the standard $\mathbb{R}^*_+ \subset \mathbb{H}^*$, to choose a more arbitrary one-parameter subgroup of $\mathbb{H}^*$. Let this act on $\mathcal{U}(M)$ by multiplication, and divide $\mathcal{U}(M)$ by a sublattice of this subgroup. One ends up with a $U(1)$- action upon the quotient of $\mathcal{U}(M)$ by a different action of $\mathbb{Z}$.

The difference in this case is that the actions of $\mathbb{R}$ and $\mathbb{Z}$ do not have to preserve the individual complex structures but only the family, so we are effectively regarding $\mathcal{U}(M)$ as a quaternionic manifold instead of as a hypercomplex manifold. The manifold that is then constructed is compact, simply-connected (up to a double cover) and quaternionic but not (for general one-parameter subgroups) hypercomplex, and will fibre over $M$ with fibre a Hopf surface over $\{\pm 1\}$, but this time a Hopf surface that is not the quotient of $\mathbb{H} \setminus \{0\}$ by a group of dilations, but by a group generated by left multiplication by a general non-unit quaternion.

### 3.2.2. Second example

Let $M_1$ and $M_2$ be compact, simply-connected quaternionic manifolds. Now the product of two quaternionic manifolds is not quaternionic, but there is a notion analogous to a product for quaternionic manifolds, called the *join* ([Sw], §5, p. 23). The join $M_1 * M_2$ of $M_1, M_2$ is defined to be the quaternionic manifold with associated bundle $\mathcal{U}(M_1) \times \mathcal{U}(M_2)$. This is an associated bundle because the product of the two hypercomplex manifolds $\mathcal{U}(M_j)$ is hypercomplex, and has an $\mathbb{H}^*$- action given by combining the $\mathbb{H}^*$-actions on the factors.

Then $M_1 * M_2$ is not compact, but fibres over $M_1 \times M_2$ with fibre $(\mathbb{H} \setminus \{0\})/\{\pm 1\}$. Note that $\dim M_1 * M_2 = \dim M_1 + \dim M_2 + 4$. Let $r > 0$ be a real number. We may make $M_1 * M_2$ compact by dividing it by the integers, acting by dilation by $e^{rn}$ in the fibres $(\mathbb{H} \setminus \{0\})/\{\pm 1\}$ to get a bundle over $M_1 \times M_2$ with fibre the Hopf surface over $\{\pm 1\}$. This action of the integers is given on the associated bundle $\mathcal{U}(M_1) \times \mathcal{U}(M_2)$ by multiplication of the first factor by $e^{\frac{rn}{2}}$ and of the second factor by $e^{-\frac{rn}{2}}$.

Thus there are simple examples of compact nonsingular quaternionic manifolds involving Hopf surfaces; we exclude these because they are not simply-connected, and also because they are locally the joins of two lower-dimensional manifolds. However, both of these disadvantages may be removed by taking a non-trivial, primitive $U(1)$- instanton on $M_1$ or $M_2$ or both, lifting to get a $U(1)$- instanton on $M_1 * M_2$, and applying Theorem 3.1.2 as in §3.2.1. The action $\Psi$ of $U(1)$ on the associated bundle is that $\Psi(e^{i\theta})$ acts by multiplication of the first factor by $e^{\frac{r\theta}{4\pi}}$, and of the second by $e^{-\frac{r\theta}{4\pi}}$.

As in §3.2.1, the transversality condition holds everywhere, and the result is a nonsingular, compact quaternionic manifold, which is simply-connected (because the instanton was chosen to be primitive, and again up to a double cover) and fibres over $M_1 \times M_2$ with fibre the Hopf surface over $\{\pm 1\}$. It is not locally the join of two manifolds, because if it were then by simple-connectedness it would be globally so as well, and so non-compact.

As examples of suitable pairs $M_1, M_2$, one could take $M_1$ and the instanton to be any of the possibilities given in §3.2.1, and $M_2$ to be a quaternionic Kähler Riemannian symmetric space, say, or any compact self-dual 4-manifold.

### 3.3. A more general twisting construction

The method of Theorem 3.1.1 actually admits some generalization. This is so far only of theoretical value as I have not found any particularly interesting applications, but it will be given for completeness.

Let $M$ be a quaternionic manifold, $G$ a Lie group, and $\Psi$ a free $G$- action upon $M$ preserving the quaternionic structure. Let $F$ be a manifold of the same dimension as $G$, and $K$ be a Lie subgroup of $\mathrm{Diff}(F)$. For simplicity we suppose $K$ to be finite-dimensional. It can easily be shown that nonsingularity of the quaternionic structures we produce below requires $K$ to act transitively on $F$, so $F$ should be a homogeneous space of a finite-dimensional group — a reasonably restrictive condition. Let $P$ be a principal $K$- bundle over $M$, and let a lifting of $\Psi$ to $P$, also to be denoted $\Psi$, be given, that preserves the principal bundle structure.

As $K$ acts on $F$, there is an $F$- bundle over $M$ naturally associated to $P$ — call this bundle $Q$, say. The action $\Psi$ on $P$ descends to $Q$. Moreover, if $A$ is a connection on $P$, then naturally associated to $A$ there is a field of horizontal subspaces in $Q$, which will be called $B$. After all these definitions, we can state the result:

**Theorem 3.3.1.** *Let $M$, $G$, $P$, $F$, $K$ and $\Psi$ be as above, and let $A$ be a quaternionic connection on $P$. Then the manifold $N = Q/\Psi(G)$ has a natural (possibly singular) quaternionic structure, which is nonsingular wherever the Lie algebra of $\Psi(G)$ is transverse to the horizontal subspaces of $B$ in $Q$.*

*Proof.* The proof of this theorem is identical in spirit to that of Theorem 3.1.1. In order to use the method of proof of §3.1 we need to know that $F$ is a real analytic manifold with a complexification $F^c$ upon which $K^c$ acts, but if $F$ is really a homogeneous space for a finite-dimensional group this is not a problem. In brief, the argument runs as follows; to fill in the details and notation, refer to the proof of Theorem 3.1.1.

By the Ward correspondence, $\tilde{P}^c$ is a real holomorphic principal $K^c$ bundle over the twistor space $Z$, and so we may associate a real holomorphic $F^c$ bundle to it, $\tilde{Q}^c$. Defining $W = \tilde{Q}^c/\tilde{\Psi}^c(G^c)$ gives a complex manifold, which can be identified with $\tilde{Q}/\tilde{\Psi}(G)$ giving a real structure and a fibration over $N$ with fibre $\mathbb{CP}^1$. The normal bundle of the fibres turns out to be $n\mathcal{O}(1)$, and so $W$ is a twistor space and gives a quaternionic structure on $N$, as required. $\square$

It would be very interesting if this theorem led to some good examples of self-dual 4-manifolds. The most obvious starting point is to put $G = U(1)$, and $F = \mathcal{S}^1$. Then as a finite-dimensional group of diffeomorphisms of $F$ we can choose $K = PSL(2,\mathbb{R})$ acting upon $F$ viewed as $\mathbb{RP}^1$, or any finite cover of $PSL(2,\mathbb{R})$ acting upon a finite cover of $\mathbb{RP}^1$, for instance the double cover $SU(1,1)$ ($SL(2,\mathbb{R})$). So Theorem 3.3.1 tells us that we may twist a $U(1)$- invariant self-dual manifold by a $U(1)$- invariant $SU(1,1)$- instanton to get a new self-dual manifold. This new self-dual manifold will probably not have a $U(1)$- action, but it will have a *conformal retraction* onto the 3-manifold over which it is a bundle.

Now $U(1)$- invariant $SU(1,1)$- instantons may be easily produced upon open sets in $\mathcal{S}^4$ using monads, so we may certainly locally produce new self-dual metrics by this idea. However, I have so far failed to produce compact examples. For instance one might look for an $SU(1,1)$ monopole with singularities modelled upon those of $U(1)$- monopoles for copies of $U(1)$ embedded in $SU(1,1)$, and make a self-dual metric upon $n\mathbb{CP}^2$ similar to LeBrun's examples, but more general in that there were a conformal retraction onto hyperbolic space but no $U(1)$- symmetry.

I suspect that there are probably gauge-theoretic reasons why any such global monopole must be a $U(1)$- monopole in disguise, because $SU(1,1)$ retracts onto $U(1)$ and hence $SU(1,1)$- bundles cannot have global topological invariants other than those of the associated $U(1)$- bundle. This could perhaps be proved by twistor means through showing that a $\mathbb{CP}^1$- bundle over a compact complex manifold must under certain conditions admit a global holomorphic section.

### 3.3.1. Another construction

Here is a completely different way of using instantons to make hypercomplex and quaternionic manifolds. It is well known that the moduli spaces of instantons on a hyperkähler 4-manifold are hyperkähler, and that one way of looking at this is to regard the moduli spaces as infinite-dimensional hyperkähler quotients of the space of all smooth connections by the gauge group, with the self-duality equations as the moment maps. In fact the moduli spaces of instantons on a hypercomplex 4-manifold may be regarded as hypercomplex quotients in the same way, and so the moduli spaces will be (singular, noncompact) hypercomplex manifolds.

Let $X$ be a hypercomplex Hopf surface. Then $U(2)$ acts transitively on $X$ permuting the complex structures, in the same way that $\mathbb{H}^*$ acts upon its own complex structures by left multiplication. Let $\mathcal{M}$ be a moduli space of instantons over $X$; then $\mathcal{M}$ is hypercomplex. The action of $U(2)$ on $X$ induces an action of $U(2)$ upon $\mathcal{M}$ that permutes the complex structures of $\mathcal{M}$ in the same way. It is not difficult to see that the quotient of $\mathcal{M}$ by this action of $U(2)$ will in fact be a quaternionic manifold, where it is nonsingular. I do not know if any of the quaternionic manifolds arising in this fashion can be made compact.

# Chapter 4: Examples. Self-Dual Four-Manifolds

Now we shall apply the results of Chapters 2 and 3 to construct quaternionic 4-manifolds, which are just self-dual conformal 4-manifolds. We begin in §4.1 with an introduction to what is currently known about the existence of compact, self-dual 4-manifolds. The results in this field are divided into two types: abstract existence results, which use analysis or complex geometry to show that self-dual metrics exist on a particular 4-manifold, and explicit constructions, that write down formulae for special families of self-dual metrics.

Important examples of the second type are the metrics defined by Poon [P] on $2\mathbb{CP}^2$ and, more generally, by LeBrun [L2] on $n\mathbb{CP}^2$. (This notation means the $n$- fold connected sum of $\mathbb{CP}^2$, which will be explained in §4.1.) Explicit examples are valuable as a guide to the behaviour of self-dual metrics and moduli spaces, and as a pattern for the development of further theory, even if it is already known from abstract existence results that the manifolds admit self-dual metrics.

This chapter gives explicit constructions for some self-dual metrics. We offer a new quotient perspective upon Poon's and LeBrun's metrics, and we find incidentally some new (I believe) families of explicit self-dual metrics on $n\mathbb{CP}^2$ for $n \geq 4$. The advantages of the quotient picture are that it becomes obvious that the twistor spaces have algebraic dimension 3, it gives a way of understanding the edges of the family, and also it enables the construction of monads for instantons on the self-dual metrics. (I have done this for weighted projective spaces and for LeBrun's metrics, but I shall not even put it into an appendix, as if I did, the laser printer would never talk to me again.)

Since Chapter 2 did not include any examples of the hypercomplex quotient or the quaternionic quotient, in §4.2 a very basic, carefully explained example will be given to work through the theoretical material. The example we give is that of quaternionic structures on complex 2-dimensional weighted projective spaces, some of which have been looked at before from the quaternionic Kähler point of view by Galicki and Lawson. The quaternionic quotient, however, reaches projective spaces other quotients cannot reach, giving quaternionic structures on all of the weighted projective spaces.

In Appendix B we shall describe a method for building up quotients from the building blocks of these weighted projective space quotients, to make quotients for nonsingular quaternionic structures on 4-manifolds that have been built up out of the corresponding weighted projective spaces. The results of this appendix are summarized in §4.3. I have put the material in an appendix because I feel it is rather too complicated and involved for the general reader to be interested in it; for this reason I do not intend to submit it for publication. It is going in the thesis because I want it to appear somewhere, as it took a lot of work and I like it.

As a more complicated example of the quaternionic quotient, §4.4 gives a quotient version of Poon's metrics on $2\mathbb{CP}^2$ [P], and gives an explicit correspondence between the quotient representation and Poon's description of the twistor spaces. The quotient version is in some ways a simplification, but I have great admiration for the way that Poon's metrics appear from almost nowhere in a puff of algebraic geometry.

The chapter ends in §4.5 with a construction of some new explicit families of self-dual metrics on $n\mathbb{CP}^2$ for $n \geq 4$. They were found using the quotient methods of Appendix B, but the presentation largely copies LeBrun's construction in [L2], because it is neater, and to provide an example of the twisting methods of Chapter 3.

## 4.1. Some facts about self-dual 4-manifolds

We shall first explain what the connected sum of two manifolds of the same dimension is, and indicate a generalization to orbifolds. Then we will discuss some of the general results upon existence and properties of families of self-dual metrics upon compact 4-manifolds, and finally we will look at the families of self-dual metrics that have been explicitly written down.

Suppose that $M_1$ and $M_2$ are connected, oriented, smooth manifolds of the same dimension $n$. The connected sum gives a way of combining $M_1$ and $M_2$ to give another connected, oriented, smooth manifold of dimension $n$, which is called the connected sum of $M_1$ and $M_2$, and written $M_1 \# M_2$. To define $M_1 \# M_2$, let $m_1, m_2$ be points in $M_1, M_2$ respectively; as $M_i$ are smooth manifolds for $i = 1, 2$, the points $m_i$ have open neighbourhoods $B_i$ and diffeomorphisms $\Phi_i$ from the open ball $B_1(0)$ of radius 1 about the origin in $\mathbb{R}^n$ to $B_i$, such that $\Phi_i(0) = m_i$. To make the orientations come out right, we ask in addition that $\Phi_1$ should be orientation-preserving, and $\Phi_2$ should be orientation-reversing.

Define the connected sum of $M_1$ and $M_2$ to be

$$M_1 \# M_2 = \left(M_1 \setminus \Phi_1\left[\overline{B}_{\frac{1}{2}}(0)\right]\right) \amalg \left(M_2 \setminus \Phi_2\left[\overline{B}_{\frac{1}{2}}(0)\right]\right) \Big/ \sim,$$

where $\sim$ is the equivalence relation defined by

$$\Phi_1[v] \quad \frown \quad \Phi_2\left[\frac{v}{2|v|^2}\right] \qquad\qquad \text{whenever } v \in \mathbb{R}^n \text{ and } \tfrac{1}{2} < |v| < 1.$$

The orientation of $M_1 \# M_2$ may be induced from that of $M_1$ or $M_2$. Note that the connected sum is independent, as a smooth manifold, of the choice of points $m_1, m_2$.

We comment briefly on an extension of this idea to orbifolds. An orbifold is a smooth manifold that is allowed, in addition, to contain certain mild sorts of singular points. To be more precise, whereas a smooth manifold is a Hausdorff topological space in which every point has an open neighbourhood homeomorphic to the unit ball in $\mathbb{R}^n$, and the transition functions between the neighbourhoods are smooth, for an orbifold the open neighbourhoods may be homeomorphic to the unit ball divided by a finite subgroup $\Gamma$ of $SO(n)$, that acts freely away from the origin. Thus orbifolds can contain isolated singular points modelled upon the singular point in $\mathbb{R}^n/\Gamma$ for such a group $\Gamma$, which is called the *orbifold group* of the point.

The natural extension of the connected sum to orbifolds will be called the *generalized connected sum*, and the extension is that as well as allowing $m_1, m_2$ to be nonsingular points of the orbifolds, we may allow them to be orbifold points with the same orbifold group $\Gamma$, which acts with opposite orientation around the two points. In this case the generalized connected sum $M_1 \# M_2$ of $M_1$ and $M_2$ at $m_1$ and $m_2$ is $M_1$ and $M_2$ joined by a 'tube' that has cross-section $\mathcal{S}^{n-1}/\Gamma$ rather than $\mathcal{S}^{n-1}$.

Let us now return to the subject of self-dual manifolds. A self-dual conformal structure on a compact 4-manifold must lie in a possibly singular, finite-dimensional family of self-dual conformal structures on the manifold, called a moduli space; the properties of these moduli spaces are explored by King and Kotschick in [KK], who develop a theory analogous to that of Atiyah et al. [AHS] for the case of instantons.

The existence of many self-dual metrics, that have not been written down explicitly, has been shown using analysis and complex geometry. Firstly, it was proved that K3 surfaces carry families of hyperkähler metrics, by Yau's proof of the Calabi conjecture. Then Floer [F] gave an analytical proof of the existence of self-dual metrics upon $n\mathbb{CP}^2$, the connected sum of $n$ copies of $\mathbb{CP}^2$, and more generally Donaldson and Friedman [DF] gave an existence theory for self-dual metrics on connected sums of self-dual manifolds, this time using twistor ideas.

I believe that it should be possible without great difficulty to generalize Donaldson and Friedman's theory to an existence theory for self-dual metrics on generalized connected sums of self-dual orbifolds, in the sense discussed above. This would form the theoretical underpinning for the construction of metrics by generalized connected sums of orbifolds that we discuss in §4.3. The last abstract existence result we shall mention is a very recent analysis proof announced by Taubes, showing that the connected sum of every compact 4-manifold with sufficiently many copies of $\overline{\mathbb{CP}}^2$ admits a self-dual metric. This implies that compact self-dual 4-manifolds are actually very numerous; it also gives hope that results on self-dual 4-manifolds might lead to a better understanding of smooth 4-manifolds in general.

The self-dual conformal structures on compact 4-manifolds that can at present be written down explicitly, are the conformally flat examples on $\mathcal{S}^4$, tori and Hopf surfaces and so on, the Fubini-Study metric on $\mathbb{CP}^2$, some families of self-dual metrics on connected sums of $\mathbb{CP}^2$'s, and one or two other examples in a similar vein. The first of the families of metrics on connected sums of $\mathbb{CP}^2$'s to be discovered is for $2\mathbb{CP}^2$, and was written down by Poon [P]. Poon's family is locally complete, and it is known that these are the only self-dual metrics of positive scalar curvature on $2\mathbb{CP}^2$.

These were followed by LeBrun's families of metrics on $n\mathbb{CP}^2$ [L2], which were also found independently by the author using the quotient methods of Chapter 2. For $n > 2$, LeBrun's families are not locally complete, so that they are a subfamily of the whole moduli space of self-dual metrics on $n\mathbb{CP}^2$ characterized by some special property. One special property that they have is that their twistor spaces are *Moishezon*, which is a concept from algebraic geometry, meaning having full algebraic dimension.

Moishezon twistor spaces have been much studied from the point of view of algebraic geometry by authors such as Poon, LeBrun, and Campana, and it is known that a Moishezon twistor space must represent a metric of positive scalar curvature on a manifold homeomorphic, at least, to $n\mathbb{CP}^2$. The classification of all Moishezon twistor spaces may be an achievable goal, and is related to the classification of Kähler twistor spaces by Hitchin.

## 4.2. Quaternionic structures on weighted projective spaces

This section and §4.4 provide examples of the quaternionic quotient. Here we will construct using the quaternionic quotient a quaternionic metric on each of the two-dimensional weighted projective spaces $\mathbb{CP}^2_{p,q,r}$. These provide the building blocks for the generalized connected sums considered in Appendix B. They are essentially the examples considered by Galicki and Lawson in [GaL], but using the quaternionic quotient instead of the quaternionic Kähler quotient simplifies and generalizes the exposition.

Let $p, q, r$ be positive integers. Then the *weighted projective space* $\mathbb{CP}^2_{p,q,r}$ is the possibly singular complex manifold defined as the quotient of $\mathbb{C}^3 \setminus \{(0,0,0)\}$ by an action of $\mathbb{C}^*$, given by

$$(f,g,h)\stackrel{u}{\longmapsto}(u^p f, u^q g, u^r h), \qquad u\in\mathbb{C}^*. \tag{9}$$

Thus $\mathbb{CP}^2$ is $\mathbb{CP}^2_{1,1,1}$. Now $p,q,r$ may always be divided through by their highest common factor, but they do not have to be pairwise coprime. Suppose that they are. Then if none of $p,q$ or $r$ is equal to 1, $\mathbb{CP}^2_{p,q,r}$ has three orbifold points, at $[1,0,0],[0,1,0]$ and $[0,0,1]$. If one, say $r$, is equal to 1, then $\mathbb{CP}^2_{p,q,1}$ has two orbifold points, at $[1,0,0]$ and $[0,1,0]$. If two, say $q$ and $r$, then $\mathbb{CP}^2_{1,1,r}$ has one orbifold point at $[1,0,0]$. If on the other hand two of $p,q,r$ have a common factor, say $q$ and $r$, then the whole line $\{[0,g,h]\in\mathbb{CP}^2_{p,q,r}\}$ will be singular.

Choose such a triple $p,q,r$. Then there exist integers $a,b,c$ such that $p=b+c, q=c+a$ and $r=a+b$ if $p+q+r$ is even, and $2p=b+c, 2q=c+a, 2r=a+b$ if $p+q+r$ is odd. Because $p,q,r>0$ it is clear that at most one of $a,b,c$ can be non-positive, and if one is, say $a$, then $b,c>-a$.

Now a quaternionic structure on $\mathbb{CP}^2_{p,q,r}$ will be given as a quaternionic quotient of $\mathbb{HP}^2$ by the group $U(1)$. For convenience we single out the complex structure $I_1$ and write everything in coordinates that are complex with respect to $I_1$. The other complex structures are then given by the action of $I_2$. The associated bundle of $\mathbb{HP}^2$ is $\mathbb{H}^3\setminus\{0\}$, which will be represented by complex coordinates $(x,y,z,l,m,n)$, with the action of the second complex structure being

$$I_2\big((x,y,z,l,m,n)\big)=(\bar{l},\overline{m},\overline{n},-\overline{x},-\overline{y},-\overline{z}), \tag{10}$$

and the action of the group is

$$(x,y,z,l,m,n)\stackrel{u}{\longmapsto}(u^a x,u^b y,u^c z,u^{-a}l,u^{-b}m,u^{-c}n) \qquad u\in U(1). \tag{11}$$

The moment maps we choose are

$$\mu_1=|x|^2+|y|^2+|z|^2-|l|^2-|m|^2-|n|^2 \tag{12}$$

$$\text{and} \qquad \mu_2+i\mu_3=2i(xl+ym+zn). \tag{13}$$

It will be shown that the transversality condition for the quaternionic quotient (§2.2, condition $(ii)$) is satisfied. The generator of the Lie algebra of $U(1)$ is $\frac{\partial}{\partial u}$, and this induces the vector field

$$\begin{aligned} iax\frac{\partial}{\partial x}&+iby\frac{\partial}{\partial y}+icz\frac{\partial}{\partial z}-ial\frac{\partial}{\partial l}-ibm\frac{\partial}{\partial m}-icn\frac{\partial}{\partial n}\\ &-ia\overline{x}\frac{\partial}{\partial\overline{x}}-ib\overline{y}\frac{\partial}{\partial\overline{y}}-ic\overline{z}\frac{\partial}{\partial\overline{z}}+ia\overline{l}\frac{\partial}{\partial\overline{l}}+ib\overline{m}\frac{\partial}{\partial\overline{m}}+ic\overline{n}\frac{\partial}{\partial\overline{n}} \end{aligned} \tag{14}$$

on the associated bundle of $\mathbb{HP}^2$. But the 1-form $I_1 d\mu_1$ is given by

$$ixd\overline{x}-i\overline{x}dx+iyd\overline{y}-i\overline{y}dy+izd\overline{z}-i\overline{z}dz-ild\overline{l}+i\overline{l}dl-imd\overline{m}+i\overline{m}dm-ind\overline{n}+i\overline{n}dn. \tag{15}$$

The contraction of (14) and (15) is

$$2a(|x|^2+|l|^2)+2b(|y|^2+|m|^2)+2c(|z|^2+|n|^2). \tag{16}$$

Now the transversality condition requires that this scalar field should not vanish anywhere on the solution set of the moment maps. Recall that at most one of $a, b, c$ may be less than or equal to zero. When $a, b, c > 0$ the condition holds trivially because (16) is positive definite on $\mathbb{H}^3\backslash\{0\}$. Suppose therefore that $a \le 0$. From (12), (13) we calculate that on the zero set of the moment maps,

$$\begin{aligned} (|x|^2+|l|^2)^2 &= (|x|^2-|l|^2)^2+|2ixl|^2\\ &= (|y|^2+|z|^2-|m|^2-|n|^2)^2+|2i(ym+zn)|^2\\ &\le (|y|^2+|z|^2+|m|^2+|n|^2)^2, \end{aligned}$$

and therefore, since $b, c > -a$ we have $-2a(|x|^2+|l|^2) < 2b(|y|^2+|m|^2)+2c(|z|^2+|n|^2)$. Thus (16) is positive definite on the zero set of the moment maps (12), (13), which is what we set out to prove.

To show that the quotient is indeed $\mathbb{CP}^2_{p,q,r}$, a map to it from the twistor space will be given that is constant on real lines.

Consider the vector product of the three-dimensional complex vectors $(x, y, z), (\overline{l}, \overline{m}, \overline{n})$. This induces a map from $\mathbb{H}^3$ to $\mathbb{C}^3$. The map is fixed by complex multiplication by $I_1$ and $I_2$. So up to multiplication by positive real constants, the map from $\mathbb{H}^3$ to $\mathbb{C}^3$ is constant on quaternionic lines in $\mathbb{H}^3$, and thus induces a map $\phi : U \subset \mathbb{HP}^2 \longrightarrow \mathcal{S}^5$, where $U$ is the set of points in $\mathbb{HP}^2$ for which $(x, y, z) \wedge (\overline{l}, \overline{m}, \overline{n}) \neq 0$.

Clearly the zero set of the moment maps in $\mathbb{HP}^2$ lies inside $U$, for if $(x, y, z, l, m, n)$ in $\mathbb{HP}^2$ is not in $U$ then $(x, y, z)$ is proportional to $(\overline{l}, \overline{m}, \overline{n})$, and the moment maps then force $x = \ldots = n = 0$, which is a contradiction.

Now consider the effect of the $U(1)$ action on the image under $\phi$ of a point of $\mathbb{HP}^2$. It is

$$(f, g, h) \stackrel{u}{\longmapsto} (u^{b+c}f, u^{c+a}g, u^{a+b}h), \qquad u \in U(1). \tag{17}$$

Pushing $\phi$ down to the quotients of both spaces gives a map $\pi(\phi)$. Thus the image under $\pi(\phi)$ of a point in the quotient of $\mathbb{HP}^2$ by $U(1)$ lies in the quotient of $\mathcal{S}^5 \subset \mathbb{C}^3$ by the action (17) of $U(1)$ on $\mathbb{C}^3$, i.e. in the weighted projective space $\mathbb{CP}^2_{p,q,r}$.

Only the weighted projective spaces $\mathbb{CP}^2_{b+c,c+a,a+b}$ with $a, b, c > 0$ are quaternionic Kähler quotients of $\mathbb{HP}^2$, and these are the examples considered by Galicki and Lawson. Open sets of the other weighted projective spaces can be constructed by a hyperkähler quotient of $\mathbb{H}^2$ or a quaternionic Kähler quotient of the non-compact dual of $\mathbb{HP}^2$.

### 4.2.1. LeBrun's metrics on line bundles over $\mathbb{CP}^1$

As an example of the quaternionic complex quotient, let us consider the two dimensional weighted projective spaces with at most one singular point. There is just one family that has at most one singular point, those of the form $\mathbb{CP}^2_{n,1,1}$, and they have symmetry group $U(2)$. Using the results of §2.4, the Kähler metrics of zero scalar curvature conformal to these manifolds can be simply described.

There is an obvious twistor function on these weighted projective spaces, given on the associated bundle by $\mu_1 = |x|^2 - |l|^2$, $\mu_2 + i\mu_3 = 2ixl$. As this is a moment map for a $U(1)$- action it clearly satisfies the twistor equation, and it vanishes only at the orbifold point $x = l = 0$, so it represents a Kähler metric which is ALE at the orbifold point and has no other poles. It is up to a constant the unique twistor function preserved by the symmetry group $U(2)$ of the weighted projective space, so the Kähler metric it represents has symmetry group $U(2)$.

Thus the quaternionic metric on $\mathbb{CP}^2_{n,1,1}$ is conformal to a $U(2)$- symmetric, nonsingular, complete Kähler metric of zero scalar curvature that is ALE near the orbifold point. But all such metrics have been classified by LeBrun in his paper [L1]. He finds that for each $n > 0$, the total space of the line bundle $L^{-n}$ over $\mathbb{CP}^1$ admits a Kähler metric with zero scalar curvature that is ALE, $U(2)$- symmetric and unique up to homothety, and that these comprise all the cases. Here $L$ is the line bundle over $\mathbb{CP}^1$ that has Chern class $+1$.

It is clear on topological grounds how the two descriptions correspond, for $\mathbb{CP}^2_{n,1,1}$ is the compactification of the total space of the line bundle $L^{-n}$ over $\mathbb{CP}^1$. The first two cases are $n = 1$, where the quaternionic manifold is $\mathbb{CP}^2$ and the zero-scalar-curvature Kähler metric is the Burns metric, and $n = 2$, which is the familiar Eguchi-Hanson space with its hyperkähler metric.

### 4.3. More general self-dual manifolds produced by the quaternionic quotient

This section summarizes the results of Appendix B, so that people don't need to read it. It also acts as an advertisement for the quaternionic quotient, as it partly answers the questions 'What does the quaternionic quotient do?', and 'What is it for?'. I have not yet found any way to make compact quaternionic manifolds in higher dimensions using a quaternionic quotient of $\mathbb{HP}^n$. This is not through want of trying; but always the tricks and subterfuges I tried still ended up being singular somewhere. It seemed to happen in such a regular way that I suspect there may be some reason behind, making it so. But in four dimensions, the orbifold singularities that spring up in the simplest quotients of the previous sections, can be removed neatly by gluing on other quotients with singularities that are complementary, leaving something with no singularities at all.

The main theme of the appendix is that, given two quotients of $\mathbb{HP}^{n+1}$ by $U(1)^n$ of a certain form, that have orbifold points that are complementary in the sense that they may be identified under a change of orientation, one can write down a new quotient made out of the two old ones. This new quotient is the two old quotients joined by a narrow neck of cross-section $\mathcal{S}^3/\Gamma$, replacing the orbifold points, where $\Gamma$ is the orbifold group. Thus, starting with a collection of quotients for weighted projective spaces as in the previous section, one may stick them together at their orbifold points, getting larger and larger quotients, until at last one ends up with a quotient for a nonsingular 4-manifold.

The manifolds one ends up with are always connected sums of $\mathbb{CP}^2$'s, one $\mathbb{CP}^2$ for each weighted projective space appearing in the original collection. This is clear, at least up to homeomorphism, from the homology and intersection form of the resulting manifold, using classification results of Freedman and of Donaldson. LeBrun's metrics [L2] can be made up in this way from weighted projective spaces, as can many other families of smaller dimension, that are described in §4.5.

We may also describe lots of orbifold objects, of no use to anyone, and some possibly interesting noncompact objects: zero scalar curvature Kähler manifolds, that may be made Asymptotically Euclidean on a multiple blow-up of $\mathbb{C}^2$ (these are conformal to metrics on $n\mathbb{CP}^2$), Asymptotically Locally Euclidean (by being conformal to a manifold with a single orbifold point), Asymptotically Flat, or Asymptotically Locally Flat. These last two are generalizations of Hawking's Ricci-flat, ALF metrics of [Ha].

For me personally, the interesting questions that these ideas raise are to do with the relation between decomposition of $n\mathbb{CP}^2$ into generalized connected sums of orbifolds, and the algebraic decomposition of the homology and intersection form. There are lots of questions one can ask. Is every decomposition of $n\mathbb{CP}^2$ into orbifolds a decomposition into self-dual orbifolds? If so, are all the orbifold decompositions of $n\mathbb{CP}^2$ realized at the edge of the moduli space of self-dual structures on $n\mathbb{CP}^2$? Do these decompositions represent the only type of boundary of this moduli space? How can we tell when a given decomposition of the homology of $n\mathbb{CP}^2$ corresponds to a division into self-dual orbifolds, can one calculate the orbifold group from the algebraic data, and does this help us to describe the boundary of the moduli space of self-dual structures on $n\mathbb{CP}^2$?

It is possible that each quotient for a family of self-dual metrics on $n\mathbb{CP}^2$ is specified by some simple algebraic structure on the homology that I can't quite grasp yet, and that by understanding those structures it might be possible to learn how to model heuristically the moduli spaces of self-dual metrics on $n\mathbb{CP}^2$. Even on $3\mathbb{CP}^2$, though, the orbifold decompositions are rather more varied and complicated than one would suppose, and in this case one can say that there are some decompositions into self-dual orbifolds at the boundary of the moduli space that do not occur in our quotient picture; thus the scheme of Appendix B does not capture all of the orbifold decompositions.

However, these are the musings of one too involved with his subject. A question that might interest other people too is the classification problem for Moishezon twistor spaces. The bearing that this work has upon the problem is that it could be possible under fairly general conditions on a Moishezon twistor space for a self-dual metric on $n\mathbb{CP}^2$, to show that it comes from a quaternionic quotient by reconstructing the quaternionic quotient from which it came, using algebraic geometry, sections of line bundles and so on, on the twistor space itself. It might be true, for instance, that all Moishezon twistor spaces for metrics on $n\mathbb{CP}^2$ for $n \geq 4$ arise from a quaternionic quotient, which would mean that they are exactly LeBrun's twistor spaces, together with those of the metrics of §4.5.

### 4.4. Poon's metrics on $\mathbb{CP}^2\#\mathbb{CP}^2$

In [P], Poon describes a family of self-dual metrics on $\mathbb{CP}^2\#\mathbb{CP}^2$ parameterized by an open interval of the real line. This is done by showing that the intersection of two quadrics in $\mathbb{CP}^5$ can be given a real structure and desingularized so that it is the twistor space of a nonsingular manifold. Poon's description of the twistor space is (from p. 114 of [P]) a small resolution of the intersection of the two quadrics

$$2(z_0^2 + z_1^2) + \lambda z_2^2 + \frac{3}{2} z_3^2 + (z_4^2 + z_5^2) = 0 \qquad (Q_0)$$

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0 \tag{$Q_\infty$}$$

in $\mathbb{CP}^5$, with the real structure

$$(z_0, z_1, z_2, z_3, z_4, z_5) \longmapsto (\overline{z}_0, \overline{z}_1, \overline{z}_2, -\overline{z}_3, -\overline{z}_4, -\overline{z}_5). \tag{$R$}$$

In the previous section we claimed that there is a method to write down a quaternionic quotient for a self-dual metric on any 4-manifold or orbifold that is built up out of weighted projective spaces in a certain way. The simplest example of this method, self-dual metrics on $\mathbb{CP}^2\#\mathbb{CP}^2$ , is brought in now as an example of the quaternionic quotient. The method is described at length in Appendix B, and will be applied in §B.5 to construct the quotient used in this section, and thus justify its appearance here.

Consider the quaternionic quotient of $\mathbb{HP}^3$ by the group $U(1)\times U(1)$. For convenience in what follows the complex structure $I_1$ will be singled out, and everything will be written in complex coordinates with respect to $I_1$. We choose complex coordinates $(x_1, l_1, x_2, l_2, x_3, l_3, x_4, l_4)$ on $\mathbb{H}^4$, the associated bundle of $\mathbb{HP}^3$, with the other complex structures given by the antilinear action of $I_2$:

$$I_2\big((x_1, l_1, x_2, l_2, x_3, l_3, x_4, l_4)\big) = (\bar{l}_1, -\overline{x}_1, \bar{l}_2, -\overline{x}_2, \bar{l}_3, -\overline{x}_3, \bar{l}_4, -\overline{x}_4). \tag{18}$$

The action of $U(1)\times U(1)$ is

$$\begin{aligned}(x_1, l_1, x_2, l_2, x_3, l_3, x_4, l_4) \overset{(u,v)}{\longmapsto} (ux_1, u^{-1}l_1, ux_2, u^{-1}l_2, vx_3, v^{-1}l_3, vx_4, v^{-1}l_4),\\ (u,v)\in U(1)\times U(1),\end{aligned} \tag{19}$$

which preserves $I_1, I_2$ and $I_3 = I_1 I_2$, and the quaternionic moment maps we choose are

$$\mu_1 = \begin{pmatrix} |x_1|^2 - |l_1|^2 + |x_2|^2 - |l_2|^2 + \alpha(|x_3|^2 - |l_3|^2 - |x_4|^2 + |l_4|^2) \\ |x_3|^2 - |l_3|^2 + |x_4|^2 - |l_4|^2 + \alpha(|x_1|^2 - |l_1|^2 - |x_2|^2 + |l_2|^2) \end{pmatrix} \tag{20}$$

$$\mu_2 + i\mu_3 = 2i \begin{pmatrix} x_1 l_1 + x_2 l_2 + \alpha(x_3 l_3 - x_4 l_4) \\ x_3 l_3 + x_4 l_4 + \alpha(x_1 l_1 - x_2 l_2) \end{pmatrix}, \tag{21}$$

in coordinates on the associated bundle, where $\alpha$ is a real parameter which lies in the interval $(0,1)$.

We will prove that the twistor spaces described by Poon and the twistor spaces of the quaternionic quotients above are isomorphic.

**Proposition 4.4.1.** *The twistor space of the quaternionic quotient above with parameter $\alpha$ is biholomorphic to Poon's description of the twistor space with parameter $\lambda = \frac{2+\alpha^4}{1+\alpha^4}$ in a way that identifies the real structures and real lines.*

*Proof.* A holomorphic map will be defined from the twistor space of the quaternionic quotient, which is the projectivization of the associated bundle with respect to $I_1$, to $\mathbb{CP}^5$ with homogeneous coordinates $(z_0, \ldots, z_5)$. The image will be seen to satisfy $(Q_0)$ and $(Q_\infty)$ and the real structure on the quotient twistor space will induce the real structure $(R)$ on $\mathbb{CP}^5$. This map is the required biholomorphism.

We make the following string of definitions:

let $w_0 = x_1 l_2, w_1 = -x_2 l_1, w_2 = \alpha x_3 l_4, w_3 = -\alpha x_4 l_3$, $z_0 = \frac{w_0+w_1}{2}, z_1 = \frac{w_0-w_1}{2i}, z_4 = \frac{w_2-w_3}{2}$ and $z_5 = \frac{w_2+w_3}{2i}$.

This gives $w_0 w_1 = -(x_1 l_2)(x_2 l_1) = z_0^2 + z_1^2$, $-w_2 w_3 = \alpha^2 (x_3 l_4)(x_4 l_3) = z_4^2 + z_5^2$, and the action of $I_2 : (z_0, z_1, z_4, z_5) \longmapsto (\overline{z}_0, \overline{z}_1, -\overline{z}_4, -\overline{z}_5)$.

Define $z_2 = \frac{i}{2}(1+\alpha^4)^{\frac{1}{2}} \cdot (x_1 l_1 - x_2 l_2)$, $z_3 = \frac{1}{\sqrt{2}}(x_1 l_1 + x_2 l_2)$.

Then $I_2 : (z_2, z_3) \longmapsto (\overline{z}_2, -\overline{z}_3)$. Let $\lambda = \frac{2+\alpha^4}{1+\alpha^4}$. Note that all the new variables above are not acted upon by the quotient variables of the quaternionic quotient, and thus descend to functions on the associated bundle of the quotient that are holomorphic w.r.t. $I_1$.

Then

$$
\begin{aligned}
(z_0^2 + z_1^2) + (\lambda - 1) z_2^2 + \frac{1}{2} z_3^2 &= -(x_1 l_1)(x_2 l_2) + \frac{1}{1+\alpha^4}\left(\frac{i}{2}(1+\alpha^4)^{\frac{1}{2}} \cdot (x_1 l_1 - x_2 l_2)\right)^2 \\
&\quad + \frac{1}{2}\left(\frac{1}{\sqrt{2}}(x_1 l_1 + x_2 l_2)\right)^2 \\
&= 0
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
(2-\lambda) z_2^2 + \frac{1}{2} z_3^2 + (z_4^2 + z_5^2) &= \frac{\alpha^4}{1+\alpha^4}\left(\frac{i}{2}(1+\alpha^4)^{\frac{1}{2}} \cdot (x_1 l_1 - x_2 l_2)\right)^2 \\
&\quad + \frac{1}{2}\left(\frac{1}{\sqrt{2}}(x_1 l_1 + x_2 l_2)\right)^2 + \alpha^2 (x_3 l_3)(x_4 l_4) \\
&= -\frac{\alpha^4}{4}(x_1 l_1 - x_2 l_2)^2 + \frac{1}{4}(x_1 l_1 + x_2 l_2)^2 + \alpha^2 (x_3 l_3)(x_4 l_4).
\end{aligned} \tag{23}
$$

Now, from the quaternionic moment map equations,

$$
x_3 l_3 = -\frac{(1+\alpha^2) x_1 l_1 + (1-\alpha^2) x_2 l_2}{2\alpha}
$$

$$x_4 l_4 = \frac{(1-\alpha^2)x_1 l_1 + (1+\alpha^2)x_2 l_2}{2\alpha},$$

so $\alpha^2(x_3 l_3)(x_4 l_4) = -\frac{1}{4}\big((x_1 l_1 + x_2 l_2)^2 - \alpha^4(x_1 l_1 - x_2 l_2)^2\big)$. Therefore by (23),

$$\begin{aligned}(2-\lambda)z_2^2 + \frac{1}{2}z_3^2 + (z_4^2+z_5^2) &= -\frac{1}{4}\alpha^4(x_1 l_1 - x_2 l_2)^2 + \frac{1}{4}(x_1 l_1 + x_2 l_2)^2 \\ &\quad - \frac{1}{4}\big((x_1 l_1 + x_2 l_2)^2 - \alpha^4(x_1 l_1 - x_2 l_2)^2\big) \\ &= 0.\end{aligned} \tag{24}$$

Collecting all the above information together, we find that there is a map from the twistor space of the quaternionic quotient to $\mathbb{CP}^5$ with homogeneous coordinates $(z_0,\ldots,z_5)$, such that the action of $I_2$ induces the involution

$$(z_0,z_1,z_2,z_3,z_4,z_5) \longmapsto (\overline{z}_0,\overline{z}_1,\overline{z}_2,-\overline{z}_3,-\overline{z}_4,-\overline{z}_5),$$

that is, equation $(R)$, and such that the image of the twistor space in $\mathbb{CP}^5$ satisfies (22) and (24); equivalently, the image satisfies 2(22) + (24) and (22) + (24), which are

$$2(z_0^2+z_1^2) + \lambda z_2^2 + \frac{3}{2}z_3^2 + (z_4^2+z_5^2) = 0$$

and

$$z_0^2+z_1^2+z_2^2+z_3^2+z_4^2+z_5^2 = 0,$$

that is, $(Q_0)$ and $(Q_\infty)$.

So far we have shown that our quotient twistor space can be mapped onto Poon's singular model so as to identify the real structures. There is more to be proved, however: there may be many ways of resolving the singularities of a complex space, so we must show that we have the same resolution of the singular model as does Poon. Also, a twistor space might admit several distinct families of real lines, giving different self-dual metrics. Both of these problems will be handled together by showing that away from the singularities of the singular model, the fibrations by real lines given above and by Poon are the same. This shows that the corresponding self-dual metrics agree on an open set, and as they are real analytic and simply-connected, they must therefore agree on all of $\mathbb{CP}^2\#\mathbb{CP}^2$ .

In §4 of [P], Poon shows that the real conics in $\mathbb{CP}^5$ lying in the nonsingular part of his singular model are divided into two families, the $\alpha$- conics and the $\beta$- conics. These two families are interchanged by a coordinate change of $\mathbb{CP}^5$, and either family gives a fibration by real lines generating his self-dual conformal class. Now by parametrizing a real line in the quotient and mapping it into the singular model, it is easy to see that it becomes a conic. So it is a real line in one of Poon's two possible fibrations; and by twistor geometry, all nearby real lines in the two twistor spaces must be identified as well.

Thus it has been shown that the quotient model of the twistor space of $\mathbb{CP}^2\#\mathbb{CP}^2$ is isomorphic as a twistor space to Poon's model of the twistor space with the value $\lambda = \frac{2+\alpha^4}{1+\alpha^4}$. Also, the interval of the parameter $\alpha$, which is $(0,1)$, is mapped bijectively to the interval of Poon's parameter $\lambda$, which is $(\frac{3}{2},2)$. This completes the proof. $\square$

## 4.5. New self-dual metrics on $n\mathbb{CP}^2$

In §4.3 we claimed that the quotient methods of Chapter 2 produce not only LeBrun's metrics on $n\mathbb{CP}^2$ but also other families of self-dual metrics on $n\mathbb{CP}^2$ that are not yet explicitly known. We now use the twisting methods of Chapter 3 to describe these other families in a very similar manner to LeBrun's description of his metrics, the difference being that whereas LeBrun's metrics are constructed by twisting $\mathcal{S}^4$ by a singular $U(1)$- invariant $U(1)$ instanton, the new ones are constructed by twisting $\mathcal{S}^4$ by a singular $U(1)^2$- invariant $U(1)^2$ instanton.

We proceed by first defining the action of $U(1)^2$ on $\mathcal{S}^4$ and writing down the conformal metric in a suitable coordinate system, then writing down two singular $U(1)^2$- invariant $U(1)$ instantons, the product of which is the required $U(1)^2$ instanton, and finally by giving the twisted action and so writing down the twisted conformal class explicitly in coordinates.

LeBrun's metrics on $n\mathbb{CP}^2$ are made by twisting $\mathcal{S}^4$ by $n$ $U(1)$- invariant $U(1)$ instantons, each of which is undefined upon a single $U(1)$- orbit. Thus there is only one family for each $n$. Our new metrics are made by having two $U(1)$ actions, and twisting one action by $k$ $U(1)^2$- invariant $U(1)$ instantons undefined upon a single $U(1)$- orbit, and the other action by $l$ $U(1)^2$- invariant $U(1)$ instantons undefined upon one orbit, to get a self-dual metric upon $(k+l)\mathbb{CP}^2$.

Thus upon $n\mathbb{CP}^2$ we construct $[n/2]+1$ families of self-dual metrics by this method. The first of these ($k=0$ or $l=0$) is a subfamily of LeBrun's family in an obvious way. However, by studying the quotient picture in detail I can show that the second family, with $k=1$ or $l=1$, is isomorphic to the first family in a non-obvious way, but that none of the further families are isomorphic to these two.

This is because for $k=0$ and $k=1$ the group action of the quotient that is constructed agrees upon two variable pairs, and an isomorphism of the quotients can be constructed, but when neither $k$ nor $l$ is 0 or 1, the group action is different on every variable pair. So there are in fact $[n/2]$ families of self-dual metrics, as the first two can be identified. I believe that there are no further identifications between the families. There are thus no new families on $2\mathbb{CP}^2$ or $3\mathbb{CP}^2$, one new family (in addition to LeBrun's metrics) on $4\mathbb{CP}^2$ and $5\mathbb{CP}^2$, and so on.

### 4.5.1. An action of $U(1)^2$ on $\mathcal{S}^4$

Let us regard $\mathcal{S}^4$ as the conformal compactification of $\mathbb{C}^2$ with coordinates $(z_1, z_2)$ and metric $ds^2 = |dz_1|^2 + |dz_2|^2$. The extra point will be denoted $\infty$, and $(0,0)$ will be denoted 0. Define an action $\Psi$ of $U(1)^2$ on $\mathcal{S}^4$ by

$$\Psi(u_1, u_2) : (z_1, z_2) \longmapsto (u_1 z_1, u_2 z_2), \qquad (u_1, u_2) \in U(1)^2. \tag{25}$$

Then $\Psi$ is an isometry, and a fortiori a conformal isometry.

Interchangeably the coordinates $r_i, \theta_i$ will be used in place of $z_i$, where as usual $r_i = |z_i|$ and $\theta_i = \arg z_i$. So the metric may be written $ds^2 = dr_1^2 + r_1^2 d\theta_1^2 + |dz_2|^2$, and conformally rescaling by $1/r_1^2$ gives

$$ds^2 = d\theta_1^2 + \frac{dr_1^2 + |dz_2|^2}{r_1^2}. \tag{26}$$

This is a conformal isometry between $\mathcal{S}^4 \setminus \mathcal{S}^2$ and $\mathcal{S}^1 \times \mathcal{H}^3$, as the two terms are the trivial $\mathcal{S}^1$ metric and the upper half-space model for $\mathcal{H}^3$. Now LeBrun proceeds by constructing magnetic monopoles on $\mathcal{H}^3$, which are really singular $U(1)$- invariant $U(1)$ instantons on $\mathcal{S}^4$. In our situation there are two $U(1)$ actions and hence two different conformal isometries with $\mathcal{S}^1 \times \mathcal{H}^3$, the other one being

$$ds^2 = d\theta_2^2 + \frac{|dz_1|^2 + dr_2^2}{r_2^2}. \tag{27}$$

### 4.5.2. Explicit expressions for $U(1)$ instantons

We now quote the explicit expressions for $U(1)$ instantons in this situation worked out by LeBrun in [L2], §§3,4 and 5. Equation (26) gives the relation between $\mathcal{S}^4$ which is central in our point of view, and $\mathcal{H}^3$ which is central in his. To construct a $U(1)$ instanton that is invariant under the $U(1)$ action $\Psi(u_1, 1) : u_1 \in U(1)$ and undefined upon the $U(1)$- orbit $r_1 = a, z_2 = c$, for some $a \in \mathbb{R}_{>0}, c \in \mathbb{C}$, he defines in §4 a function $V_1 = 1 + G$, where $G$ is the nonnegative function

$$G(r_1, \theta_1, z_2) = \frac{1}{2}\left(\frac{r_1^2 + a^2 + |z_2 - c|^2}{\left((r_1^2 + a^2 + |z_2 - c|^2)^2 - 4r_1^2 a^2\right)^{1/2}} - 1\right). \tag{28}$$

Then he defines a 2-form $\alpha_1$ upon $\mathcal{H}^3$ by $\alpha_1 = *dV_1$. In four dimensions rather than three it is necessary to define

$$\alpha_1 = *(dV_1 \wedge d\theta_1); \tag{29}$$

then this $\alpha_1$ is the lift to $\mathcal{S}^1 \times \mathcal{H}^3$ of LeBrun's 2-form $\alpha_1$ on $\mathcal{H}^3$. Note that as $*$ is conformally invariant on 2-forms this is an invariant definition.

LeBrun then shows that with this choice of $V_1$, $\alpha_1$ is closed and represents an integral cohomology class upon the complement of the singular point in $\mathcal{H}^3$, and so $\alpha_1$ is the curvature form for a connection form $\omega_1$ for a suitable circle bundle over the complement of the singular point. This circle bundle and connection lifts to $\mathcal{S}^1 \times \mathcal{H}^3$, where the curvature form is (29), and in fact extends smoothly over the fixed point set of $\Psi(u_1, 1) : u_1 \in U(1)$, which corresponds to the sphere at infinity in $\mathcal{H}^3$.

So from the four-dimensional point of view, $\alpha_1$ is closed, defined everywhere in $\mathcal{S}^4$ except a single $U(1)$- orbit, and represents an integral cohomology class of the complement of this orbit. Thus there is a circle bundle, $P_1$ say, over the complement of this orbit with a connection form $\omega_1$, with curvature $\alpha_1$. Also, the $U(1)$ action $\Psi(u_1, 1) : u_1 \in U(1)$ lifts to the circle bundle, and in fact the bundle and connection are lifted from a bundle and connection on the quotient by the $U(1)$ action.

Now we may construct a singular $U(1)$ instanton from the data we have. Given a connection $\omega_1$ upon a circle bundle $P_1$, one may add a 1-form to get another connection upon the circle bundle. Thus $\omega_1' = \omega_1 + V_1 d\theta_1$ is a connection upon $P_1$. The curvature of $\omega_1'$ is

$$\Omega_1' = d\omega_1' = d\omega_1 + dV_1 \wedge d\theta_1 = \alpha_1 + dV_1 \wedge d\theta_1. \tag{30}$$

But from (29) and the fact that $*^2 = 1$ on 2-forms it is clear that $*\Omega_1' = \Omega_1'$, i.e. $\omega_1'$ is a self-dual connection, a $U(1)$ instanton.

To define more general $U(1)$- invariant $U(1)$ instantons that are undefined upon the finite collection of $n$ orbits $r_1 = a_j, z_2 = c_j$ for some finite set $\{(a_j, c_j) : j = 1, \ldots, n\}$ in $\mathbb{R}_{>0} \times \mathbb{C}$, in §5 of [L2] LeBrun defines $V_1 = 1 + \Sigma_j G_j$, where $G_j$ is defined by (28) using $a_j, c_j$ in place of $a, c$, and continues as above.

All this subsection so far has been the work of LeBrun taken from [L2], with changes of notation and concepts to fit the four-dimensional rather than the three-dimensional picture. Now it is time to introduce some new material by considering singular $U(1)^2$- invariant instantons.

What is the condition for one of the $U(1)$ instantons defined above to be invariant under the other $U(1)$ factor $\Psi(1, u_2) : u_2 \in U(1)$? As this action takes $(z_1, z_2)$ to $(z_1, u_2 z_2)$ it is clear that the effect of $\Psi(1, u_2)$ is to take the instanton defined by the set $\{(a_j, c_j) : j = 1, \ldots, n\}$ to the instanton defined by the set $\{(a_j, u_2 c_j) : j = 1, \ldots, n\}$, and the condition for invariance is that $c_j = 0$ for each $j$. In the $\mathcal{H}^3$ picture this simply says that the $U(1)$ action is rotation about a line in $\mathcal{H}^3$, and for invariance the $n$ points should all lie on the line.

However, there is a catch: there is no unique lifting of $\Psi(1, u_2) : u_2 \in U(1)$ to $P_1$, as a lifting of the action may be composed with the action of rotation in the fibres raised to any integer power, to give another lifting. It will be explained later which lifting to choose in the situation we consider. For the present, simply suppose some lifting of $\Psi(1, u_2) : u_2 \in U(1)$ to $P_1$ that preserves $\omega_1'$ has been selected. As $\Psi(u_1, 1) : u_1 \in U(1)$ already acts upon $P_1$ this means that the whole action $\Psi$ lifts to $P_1$; the lifted action will also be called $\Psi$.

Differentiating $\Psi$ gives a map from the Lie algebra of $U(1)^2$ into the vector fields on $P_1$. Denote this map by $\psi_1$. Then $\psi_1(1, 0)$ is a vector field that exponentiates to give the 1-parameter group $\Psi(u_1, 1) : u_1 \in U(1)$ acting on $P_1$, and similarly $\psi_1(0, 1)$ exponentiates to give $\Psi(1, u_2) : u_2 \in U(1)$. Now $V_1$ is just the function $\omega_1'(\psi_1(1, 0))$ formed by contracting together the 1-form $\omega_1'$ and the vector field $\psi_1(1, 0)$. In the same way we form the function $W_1 = \omega_1'(\psi_1(0, 1))$. Note that this depends upon the particular lifting of $\Psi$ to $P_1$ chosen, and choosing a different lifting has the effect of changing $W_1$ by an integer.

An interesting fact is that $V_1$ and $W_1$ together, as functions, actually define the bundle $P_1$ and the connection $\omega_1'$ explicitly. This is shown as follows. Choose locally a $\Psi$- invariant trivialization for $P_1$. Then the connection $\omega_1'$ is represented by a 1-form $\beta_1$ in this trivialization. The definitions of $V_1, W_1$ imply $\beta_1$ takes the form

$$\beta_1 = V_1(r_1, r_2) d\theta_1 + W_1(r_1, r_2) d\theta_2 + A(r_1, r_2) dr_1 + B(r_1, r_2) dr_2. \tag{31}$$

Since $\omega_1'$ is an instanton, its curvature $d\beta_1$ is self-dual. But this forces $d(A(r_1, r_2)dr_1 + B(r_1, r_2)dr_2) = 0$, as this is the $dr_1 \wedge dr_2$ component of $d\beta_1$, and the dual $r_1 r_2 d\theta_1 \wedge d\theta_2$ component is zero. So by changing the trivialization one can locally make $A = B = 0$, and $V_1 d\theta_1 + W_1 d\theta_2$ is a connection matrix for $\omega_1'$.

To express $W_1$ explicitly in terms of the $a_i$'s, consider first the case $n = 1, a_1 = a$. The Hodge star acts by $*dr_1 \wedge (r_1 d\theta_1) = dr_2 \wedge (r_2 d\theta_2)$, $*dr_2 \wedge (r_1 d\theta_1) = -dr_1 \wedge (r_2 d\theta_2)$, and so from (28) and (29) we calculate

$$\alpha_1 = \frac{\left(2a^2(r_2^2 - r_1^2 + a^2) r_2 dr_2 + 4 r_1 a^2 r_2^2 dr_1\right) \wedge d\theta_2}{((r_1^2 + r_2^2 + a^2)^2 - 4 r_1^2 a^2)^{\frac{3}{2}}}. \tag{32}$$

But $\alpha_1 = dW_1 \wedge d\theta_2$, and so as $dW_1$ has no $d\theta_2$ component this gives us $dW_1$. Integrating gives

$$W_1 = \frac{r_1^2 + r_2^2 - a^2}{2\left((r_1^2 + r_2^2 + a^2)^2 - 4 r_1^2 a^2\right)^{\frac{1}{2}}} + C, \tag{33}$$

for some constant $C$.

To understand the rôle of $C$, consider what happens when $r_2 = 0$. Then

$$W_1 = \frac{r_1^2 - a^2}{2\left((r_1^2 - a^2)^2\right)^{\frac{1}{2}}} + C.$$

Now when $r_2 > 0$, the expression in the root in (33) is always positive, so the root cannot change sign, and by continuity the sign must be constant – say, positive – when $r_2 = 0$ as well. Thus when $r_2 = 0$, $W_1 = C - 1/2$ when $r_1 < a$ and $W_1 = C + 1/2$ when $r_1 > a$.

What is happening here is that we are dealing with a coordinate singularity: when $r_2 = 0$, $\theta_2$ is not defined because the point is a fixed point in $\mathcal{S}^4$ of the $U(1)$ action $\Psi(1, u_2) : u_2 \in U(1)$. But because the point is fixed, $\Psi(1, u_2) : u_2 \in U(1)$ acts upon the fibre $U(1)$ of $P_1$ over the point. Actions of $U(1)$ on $U(1)$ are classified by the integers, as in $e^{i\theta} \mapsto u_2^n e^{i\theta}$ for some $n \in \mathbb{Z}$. The connection 1-form $\omega_1'$ restricted to a fibre is just $d\theta$, and this gives $W_1 = \omega_1'(\psi_1(0,1)) = n$.

So it has been shown that when $r_2 = 0$, $W_1$ must take an integer value. Thus $C + 1/2$ and $C - 1/2$ must both be integers, and $W_1$ is determined up to an integer rather than just a constant. It will be determined entirely by the choice of lifting of $\Psi(1, u_2) : u_2 \in U(1)$ to $P_1$, as remarked above.

This explains how we have managed to give a global connection matrix (choice of gauge), and thus apparently a global trivialization, for a topologically nontrivial bundle. When $r_2 = 0$, $W_1$ takes an integer value, and if that integer is nonzero then the trivialization does not extend over that portion of the set $r_2 = 0$ — it has a coordinate singularity because it involves $d\theta_2$. The set $r_2 = 0$ will be divided up into disconnected subsets by the collection of singular orbits, and $W_1$ will take a different integer value on each.

### 4.5.3. Construction of the metrics

The key idea introduced next is that because the situation has been set up so that the rôles of $z_1$ and $z_2$ are interchangeable, one can also consider $\Psi$- invariant $U(1)$ instantons that are undefined upon orbits of the other $U(1)$ factor, $z_1 = 0, r_2 = b_j$ for some finite set of positive reals $b_j$. Let $k, l$ be nonnegative integers, and let $a_1, \ldots, a_k, b_1, \ldots, b_l$ be positive real numbers such that $a_i \neq a_j$ and $b_i \neq b_j$ when $i \neq j$. Define

$$V_1(r_1, \theta_1, r_2, \theta_2) = 1 + \frac{1}{2}\sum_{j=1}^{k}\left(\frac{r_1^2 + r_2^2 + a_j^2}{\left((r_1^2 + r_2^2 + a_j^2)^2 - 4r_1^2 a_j^2\right)^{1/2}} - 1\right) \tag{34}$$

and

$$V_2(r_1, \theta_1, r_2, \theta_2) = 1 + \frac{1}{2}\sum_{j=1}^{l}\left(\frac{r_1^2 + r_2^2 + b_j^2}{\left((r_1^2 + r_2^2 + b_j^2)^2 - 4r_2^2 b_j^2\right)^{1/2}} - 1\right). \tag{35}$$

Let $\alpha_1 = *(dV_1 \wedge d\theta_1)$ and $\alpha_2 = *(dV_2 \wedge d\theta_2)$. As above, $\alpha_1, \alpha_2$ are closed and define integral cohomology classes upon the complements of the respective collections of $U(1)$- orbits, so there are circle bundles $P_1, P_2$ with connections $\omega_1, \omega_2$ such that $d\omega_i = \alpha_i$.

Define a connection $\omega_1'$ upon $P_1$ by $\omega_1' = \omega_1 + V_1 d\theta_1$ and a connection $\omega_2'$ on $P_2$ by $\omega_2' = \omega_2 + V_2 d\theta_2$. Then $A = (\omega_1', \omega_2')$ is a self-dual $U(1)^2$ connection on the $U(1)^2$ bundle $P = P_1 \times P_2$, which is undefined on the set $X = \{(z_1, z_2) : |z_1| = a_j, z_2 = 0 \text{ or } z_1 = 0, |z_2| = b_j\} \subset \mathcal{S}^4$.

To apply Theorem 3.1.1 we need to choose a lifting of $\Psi$ to $P_1 \times P_2$, as in the §4.5.2. To get a manifold after the twisting and not an orbifold or worse, the lifted action $\Psi$ must be free on $P_1 \times P_2$. The action will in fact automatically be free except on the fibres of $P_1 \times P_2$ over 0 and $\infty$ in $\mathcal{S}^4$. Now to get a free action on the fibre at 0, a sufficient (nearly a necessary) condition is that either $\Psi(1, u_2) : u_2 \in U(1)$ should act trivially on the fibre of $P_1$ at 0 or $\Psi(u_1, 1) : u_1 \in U(1)$ should act trivially on the fibre of $P_2$ at 0.

So to ensure the action is free on both fibres, let us define the lifting of $\Psi$ to $P_1$ by requiring that $\Psi(1, u_2) : u_2 \in U(1)$ should act trivially upon the fibre of $P_1$ at 0, and the lifting of $\Psi$ to $P_2$ by requiring that $\Psi(u_1, 1) : u_1 \in U(1)$ should act trivially upon the fibre of $P_2$ at $\infty$.

Let $\psi_1, \psi_2$ be the derivatives at $(1, 1)$ of $\Psi$ acting on $P_1, P_2$. As in §4.5.2, define $W_1 = \omega_1'(\psi_1(0, 1))$ and $W_2 = \omega_2'(\psi_2(1, 0))$. The work above enables us to write down $W_1, W_2$ explicitly up to integer constants. But as $\Psi(1, u_2) : u_2 \in U(1)$ acts trivially upon the fibre of $P_1$ at 0, $W_1 = 0$ at 0, and since $\Psi(u_1, 1)$ acts trivially upon the fibre of $P_2$ at $\infty$, $W_2 = 0$ at $\infty$. This gives the following explicit expressions for $W_1$ and $W_2$:

$$W_1 = \frac{1}{2}\sum_{j=1}^{k}\left(\frac{r_1^2+r_2^2-a_j^2}{\left((r_1^2+r_2^2+a_j^2)^2-4r_1^2a_j^2\right)^{\frac{1}{2}}}+1\right), \tag{36}$$

$$W_2 = \frac{1}{2}\sum_{j=1}^{l}\left(\frac{r_1^2+r_2^2-b_j^2}{\left((r_1^2+r_2^2+b_j^2)^2-4r_2^2b_j^2\right)^{\frac{1}{2}}}-1\right). \tag{37}$$

There is another constraint, which is this: the transversality condition requires that the vector subspace $\langle(\psi_1(1,0),\psi_2(1,0)),(\psi_1(0,1),\psi_2(0,1))\rangle$ should be transverse to the horizontal subspaces of $\omega'$. Using the explicit form of the connection matrix, this boils down to

$$\begin{vmatrix} V_1 & W_1 \\ W_2 & V_2 \end{vmatrix} \neq 0. \tag{38}$$

But this is trivially true, for $V_1, V_2 > 0$ everywhere, and $W_1 \geq 0, W_2 \leq 0$ everywhere since

$$-1 \leq \frac{r_1^2+r_2^2-a_j^2}{\left((r_1^2+r_2^2+a^2)^2-4r_1^2a_j^2\right)^{\frac{1}{2}}} \leq 1,$$

which implies that $W_1 \geq 0$, and similarly for $W_2$. The chosen lifting of $\Psi$ to $P_1 \times P_2$ thus acts freely, and the transversality condition holds everywhere.

Now apply Theorem 3.1.1, with $G = U(1)^2$, $M = \mathcal{S}^4 \setminus X$, and $\Psi, P, A$ as defined above. The theorem shows that the conformal metric on $P/\Psi(U(1)^2)$ defined by twisting is self-dual (where it is nonsingular). In the dense open set $0 < r_1, r_2 < \infty$ an explicit expression for the conformal metric in coordinates will be given, as follows. The effect of dividing by $\Psi(G)$ is to get rid of the coordinates $\theta_1, \theta_2$, so that $P_1, P_2$ become circle bundles over $\mathbb{R}^2_{>0}$, a set with coordinates $r_1, r_2$. The quotient of $P_1 \times P_2$ by $U(1)^2$ can be identified with the restriction of $P_1 \times P_2$ to $\theta_1 = \theta_2 = 0$.

Let $\lambda_1, \lambda_2$ be $U(1)$ coordinates for the fibres of $P_1, P_2$ using the 'global trivialization' constructed above. Then the $\mathfrak{u}(1)+\mathfrak{u}(1)$- valued connection 1-form on $P_1 \times P_2$ is $(d\lambda_1 + V_1 d\theta_1 + W_1 d\theta_2, d\lambda_2 + W_2 d\theta_1 + V_2 d\theta_2)$. The vanishing of this 1-form can thus be written as a pair of equations, and in fact as a matrix transformation:

$$\begin{pmatrix} d\lambda_1 \\ d\lambda_2 \end{pmatrix} = -\begin{pmatrix} V_1 & W_1 \\ W_2 & V_2 \end{pmatrix}\begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix}. \tag{39}$$

To write down the twisted conformal metric, we substitute this transformation into the expression for the original metric, eliminating $d\theta_1, d\theta_2$. What you get is the new metric, in terms of $r_1, r_2, \lambda_1, \lambda_2$ which are the coordinates. The original metric can be written

$$ds^2 = dr_1^2 + dr_2^2 + \begin{pmatrix} d\theta_1 & d\theta_2 \end{pmatrix} \begin{pmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{pmatrix} \begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix}.$$

So inverting (39) and substituting in gives

$$\begin{aligned} ds^2 =& dr_1^2 + dr_2^2 + (V_1V_2 - W_1W_2)^{-2} \cdot \\ & \begin{pmatrix} d\lambda_1 & d\lambda_2 \end{pmatrix} \begin{pmatrix} V_2 & -W_2 \\ -W_1 & V_1 \end{pmatrix} \begin{pmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{pmatrix} \begin{pmatrix} V_2 & -W_1 \\ -W_2 & V_1 \end{pmatrix} \begin{pmatrix} d\lambda_1 \\ d\lambda_2 \end{pmatrix}, \end{aligned}$$

and multiplying out and conformally rescaling gives

$$\begin{aligned} ds^2 = (V_1V_2 - W_1W_2)^2(dr_1^2 + dr_2^2) + (r_1^2V_2^2 + r_2^2W_2^2)d\lambda_1^2 + (r_1^2W_1^2 + r_2^2V_1^2)d\lambda_2^2 \\ - 2(r_1^2V_2W_1 + r_2^2V_1W_2)d\lambda_1 d\lambda_2, \end{aligned} \tag{40}$$

the final form of the conformal metric on the open set $0 < r_1, r_2 < \infty$.

The work above shows that this conformal structure is nonsingular on the twisted 4-manifold. However, the 4-manifold is noncompact. The final stage of LeBrun's argument is to show that his conformal 4-manifold can be compactified by adding $n$ points in place of the $n$ removed $U(1)$ orbits, to give a nonsingular self-dual structure on $n\mathbb{CP}^2$. The same holds in this case, and for exactly the same reason. To compactify the new conformal metric we must add $k+l$ points in place of the $(k+l)$ $U(1)$ orbits originally removed from $\mathcal{S}^4$, i.e. where $P_1 \times P_2$ was undefined.

In a neighbourhood of each of these orbits one of the two $U(1)$- instantons is singular and one nonsingular. Imagining the twisting process as happening by first twisting by the nonsingular instanton and then by the singular, one can see that twisting by the nonsingular instanton does not actually change the situation, and so the behaviour of the twisted conformal metric close to the removed orbit is basically $U(1)$- twisting by the singular $U(1)$ instanton. But the singularities of each $U(1)$ instanton are of LeBrun's type, and therefore are such that the twisted conformal structure can be locally compactified in a smooth way by adding a single point.

So adding $k+l$ points gives a compact 4-manifold with a self-dual conformal structure. The 4-manifold is $(k+l)\mathbb{CP}^2$, as it arises by twisting $\mathcal{S}^4$ by $(k+l)$ $U(1)$ instantons, each of which is singular upon a single $\mathcal{S}^1$ in $\mathcal{S}^4$, and as in LeBrun's work each of these twistings adds another $\mathbb{CP}^2$.

This can be seen explicitly by letting one of the $a_j$ approach $\infty$ or one of the $b_j$ approach 0, whilst keeping the others constant. As this happens the conformal metric decays into a connected sum of the self-dual metric constructed from the remaining $a_j$ and $b_j$, and a copy of $\mathbb{CP}^2$, joined by a very long neck. So performing this process in reverse shows that adding another $a_j$ or $b_j$ means taking a connected sum with another $\mathbb{CP}^2$. (It is necessary to let $a_j \to \infty$ and $b_j \to 0$, rather than the other way round, because of the way the lifting of $\Psi$ to $P_1, P_2$ involved 0 and $\infty$. When $a_j \to 0$ or $b_j \to \infty$ with the others constant, the metric decays into the generalized connected sum of orbifolds.)

# Chapter 5: Homogeneous Hypercomplex and Quaternionic Manifolds

In the fifth chapter of this thesis we will explore the idea of a homogeneous hypercomplex or quaternionic structure using the structure theory of Lie groups. It will be shown that, given any compact Lie group $G$, there exists $k$ with $0 \leq k \leq \max(3, \operatorname{rk} G)$ such that $U(1)^k \times G$ admits a homogeneous hypercomplex structure (Theorem 5.1.2). An analogous statement holds for general compact homogeneous hypercomplex manifolds and for quaternionic manifolds.

The theory of homogeneous complex structures on compact manifolds was given in the 1950's by Wang [W] and Samelson [Sm], and most of what follows is a straightforward adaptation of material in these papers; the problem is to find three homogeneous complex structures satisfying the quaternion relations. In Chapter 6, several sorts of noncompact homogeneous complex structures will be brought in to make four or more anticommuting complex structures, and we shall see that the compact case is much simpler than the noncompact case.

Samelson describes homogeneous complex structures on groups, which are relatively straightforward. Wang describes general homogeneous complex manifolds, at the expense of losing the simplicity of Samelson's approach. We will follow this pattern, first finding hypercomplex structures on groups to make the basic idea clear, and then extending to homogeneous spaces.

We now briefly summarize Samelson's results. In [Sm] it is shown that every compact Lie group $G$ of even dimension has a complex structure such that left translations are holomorphic mappings. This is an extension of the well-known theorem of Borel which states that the quotient of a compact Lie group by its maximal torus always has a homogeneous complex structure.

Suppose $G$ is a compact Lie group and $H$ is a maximal torus of $G$, with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ respectively. Now as $G$ is compact it has a finite cover $G'$ that is the product $T \times S$ of a torus and a semisimple group. Then lifting $H$ to $H' \subset G'$, it is clear that $H' = T \times C$, where $C$ is a maximal torus of $S$. Thus for our purposes we may treat $G$ as though it were semisimple, and $H$ as though it were the maximal torus of a semisimple group, and perform the usual structure theory decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$.

When $\mathfrak{g}$ is a Lie algebra, denote its complexification by $\tilde{\mathfrak{g}}$. From the structure theory of Lie algebras ([V], §4.3), the complexified Lie algebra $\tilde{\mathfrak{g}}$ of $G$ is decomposed into root subspaces:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha, \tag{41}$$

where $\Delta$ is a finite subset of non-zero elements of $\tilde{\mathfrak{h}}^*$ (the *roots*), and each $\mathfrak{g}_\alpha$ is the one-dimensional subspace of $\mathfrak{g}$ defined by

$$\mathfrak{g}_\alpha = \big\{x : x \in \mathfrak{g}, [h, x] = \alpha(h)x \quad \forall\ h \in \mathfrak{h}\big\}. \tag{42}$$

Samelson defines a complex structure on $G$ by choosing a *positive system* of roots ([V], p. 280), which is a set $P \subseteq \Delta$ satisfying $P \cap (-P) = \emptyset$; $P \cup (-P) = \Delta$; $\alpha, \beta \in P, \alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in P$. For let $I'$ be a complex structure on $\mathfrak{h}$. Then if $W$ is the set of (1,0)-forms in $\tilde{\mathfrak{h}}$ with respect to $I'$, we can define $\mathfrak{m}$ as a subspace of $\tilde{\mathfrak{g}}$ by

$$\mathfrak{m} = W + \sum_{\alpha \in P} \mathfrak{g}_\alpha. \tag{43}$$

From structure theory we see that $\mathfrak{m}$ is closed under the complexified Lie bracket, and thus generates a complex subgroup $M$ of the complexified group $\tilde{G}$ with Lie algebra $\mathfrak{m}$. Samelson shows [Sm] that $\tilde{G}/M$ is diffeomorphic to $G$, and as $\tilde{G}, M$ are both complex groups, this makes $G$ a complex manifold. The complex structure on $\mathfrak{g}$ is easily described: as real vector spaces $\tilde{\mathfrak{g}} = \mathfrak{g} + \mathfrak{m}$, and this gives an identification $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{m}$, which is a quotient of complex vector spaces and so gives a complex structure on $\mathfrak{g}$. It is clear that $\mathfrak{m}$ is simply the (1,0)-forms for the complex structure on $\mathfrak{g}$.

We note that the results of Theorem 5.1.2 are already known – they appear in a Physics paper by Spindel et al. ([SSTV], see p. 685, Table 1). The proof we give is different to theirs, and is needed as an introduction to Theorem 5.2.1 and the following sections. They approach the problem from the point of view of supersymmetry, and restrict their attention to hypercomplex structures on groups. Our results go quite a lot further, for we define hypercomplex structures on general homogeneous spaces as well as groups, and we consider the question of homogeneous quaternionic manifolds, which they do not touch. Also, our presentation should be much easier for mathematical readers to understand, as [SSTV] is written in the language of physics. I am grateful to Professor Galicki for drawing this paper to my attention.

### 5.1. Homogeneous hypercomplex structures on groups

In Theorem 5.1.2 an analogue of Samelson's result for the hypercomplex case will be given. First we prove a preparatory Lemma.

**Lemma 5.1.1.** *Let $G$ be a compact Lie group, with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ can be decomposed as*

$$\mathfrak{g} = \mathfrak{b} + \sum_{k=1}^{n} \mathfrak{d}_k + \sum_{k=1}^{n} \mathfrak{f}_k, \tag{44}$$

*where $\mathfrak{b}$ is abelian, $\mathfrak{d}_k$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$, $\mathfrak{b} + \Sigma_k \mathfrak{d}_k$ contains the Lie algebra of a maximal torus of $G$ and $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ are (possibly empty) vector subspaces of $\mathfrak{g}$, such that for each $k = 1, 2, \ldots, n$, $\mathfrak{f}_k$ satisfies the following two conditions:*

*(i) $[\mathfrak{d}_l, \mathfrak{f}_k] = \{0\}$ whenever $l < k$, and*

*(ii) $\mathfrak{f}_k$ is closed under the Lie bracket with $\mathfrak{d}_k$, and the Lie bracket action of $\mathfrak{d}_k$ on $\mathfrak{f}_k$ is isomorphic to the sum of $m$ copies of the action of $\mathfrak{su}(2)$ on $\mathbb{C}^2$ by left multiplication, for some integer $m$.*

*Proof.* Let $H$ be a maximal torus in $G$ with Lie algebra $\mathfrak{h}$, and $\Delta_1$ the set of roots of $\tilde{\mathfrak{g}}$ relative to $\mathfrak{h}$. Let $\mathfrak{b}_0 = \mathfrak{g}$. Choose a highest root $\alpha_1$ in $\Delta_1$. Then the three-dimensional subspace $\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{-\alpha_1} + [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}]$ of $\tilde{\mathfrak{g}}$ is in fact a complex subalgebra of $\tilde{\mathfrak{g}}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, and its intersection with $\mathfrak{g}$ is a subalgebra isomorphic to $\mathfrak{su}(2)$. So let $\mathfrak{d}_1 = \mathfrak{g} \cap \left(\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{-\alpha_1} + [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}]\right)$; then $\mathfrak{d}_1$ is a subalgebra of $\mathfrak{b}_0$ isomorphic to $\mathfrak{su}(2)$.

Define $\mathfrak{b}_1$ to be the centralizer of $\mathfrak{d}_1$ in $\mathfrak{b}_0$; $\mathfrak{b}_1$ is also a subalgebra. Define $\mathfrak{f}_1$ by

$$\mathfrak{f}_1 = \mathfrak{b}_0 \cap \sum_{\substack{\beta \in \Delta_1: \\ \beta + \alpha_1 \in \Delta_1}} \mathfrak{g}_\beta + \mathfrak{g}_{\beta + \alpha_1}, \tag{45}$$

where $\mathfrak{g}_\beta$ is the root subspace for the root $\beta$. Then $\mathfrak{b}_0$ decomposes as $\mathfrak{b}_0 = \mathfrak{b}_1 + \mathfrak{d}_1 + \mathfrak{f}_1$. This is because for every root $\beta \neq \pm\alpha_1$, $\mathfrak{g}_\beta$ appears as a summand in either $\tilde{\mathfrak{b}}_1$ or $\tilde{\mathfrak{f}}_1$, but not both, depending on whether $[\mathfrak{d}_1, \mathfrak{g}_\beta]$ is zero or non-zero respectively. Also, the $\mathfrak{g}_{\pm\alpha_1}$ appear as summands in $\tilde{\mathfrak{d}}_1$, and $\tilde{\mathfrak{h}}$ splits as $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}] + (\alpha_1)^\circ$, of which the first summand comes from $\tilde{\mathfrak{d}}_1$ and the second summand from $\tilde{\mathfrak{b}}_1$. Thus from (41) it follows that $\tilde{\mathfrak{b}}_0 = \tilde{\mathfrak{b}}_1 + \tilde{\mathfrak{d}}_1 + \tilde{\mathfrak{f}}_1$, and hence the result.

Now $\mathfrak{h} \cap \mathfrak{b}_1$ is the Lie algebra of a maximal torus for the subgroup of $G$ generated by $\mathfrak{b}_1$, and the roots $\Delta_2$ of $\tilde{\mathfrak{b}}_1$ relative to this subalgebra are just the roots in $\Delta_1$ that are zero on $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}]$. Either $\mathfrak{b}_1$ is abelian (and hence contained in $\mathfrak{h}$) or else we may in exactly the same way choose a highest root $\alpha_2$ and decompose $\mathfrak{b}_1$ as $\mathfrak{b}_1 = \mathfrak{b}_2 + \mathfrak{d}_2 + \mathfrak{f}_2$.

By repeating this process, we obtain a succession of subalgebras $\mathfrak{b}_0, \ldots, \mathfrak{b}_n$, $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ and subspaces $\mathfrak{f}_1, \ldots \mathfrak{f}_n$ such that $\mathfrak{b}_{i-1} = \mathfrak{b}_i + \mathfrak{d}_i + \mathfrak{f}_i$, $\mathfrak{d}_i$ is isomorphic to $\mathfrak{su}(2)$, $\mathfrak{b}_i$ is the centralizer of $\mathfrak{d}_i$ in $\mathfrak{b}_{i-1}$, and $\mathfrak{b}_n$ is abelian. Then, recalling that $\mathfrak{b}_0 = \mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{b}_n + \sum_{i=1}^{n} \mathfrak{d}_i + \sum_{i=1}^{n} \mathfrak{f}_i. \tag{46}$$

Putting $\mathfrak{b} = \mathfrak{b}_n$ gives the decomposition (44). Condition $(i)$ is satisfied because if $l < k$ then $\mathfrak{f}_k \subset \mathfrak{b}_l$, which is the centralizer of $\mathfrak{d}_l$; thus $[\mathfrak{d}_l, \mathfrak{f}_k] = 0$.

Condition $(ii)$ is satisfied because $\alpha_i$ is a highest root in $\mathfrak{b}_{i-1}$, and so by structure theory the roots of $\mathfrak{b}_{i-1}$ that do not commute with $\pm\alpha_i$ are split into pairs $\beta, \beta + \alpha_i$. It is then easy to see that $\mathfrak{f}_i$ splits as the sum of vector subspaces $\mathfrak{b}_{i-1} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{-\beta} + \mathfrak{g}_{\beta+\alpha_i} + \mathfrak{g}_{-\beta-\alpha_i})$, and each of these is a representation of $\mathfrak{d}_i$ of the required form. □

It will now be shown that a decomposition of this form is just what is needed to define a hypercomplex structure on $U(1)^k \times G$ for some $k$.

**Theorem 5.1.2 [SSTV].** *Let $G$ be a compact Lie group. Then there exists an integer $k$ with $0 \leq k \leq \max(3, \operatorname{rk} G)$ such that $U(1)^k \times G$ admits a homogeneous hypercomplex structure.*

*Proof.* By Lemma 5.1.1, the Lie algebra $\mathfrak{g}$ of $G$ admits a decomposition

$$\mathfrak{g} = \mathfrak{b} + \sum_{k=1}^{n} \mathfrak{d}_k + \sum_{k=1}^{n} \mathfrak{f}_k$$

satisfying certain conditions. Now either $\dim \mathfrak{b} \leq n$ or $\dim \mathfrak{b} > n$. If $\dim \mathfrak{b} \leq n$, define $k = n - \dim \mathfrak{b}$, and $0 \leq k < \operatorname{rk} G$; let $m = 0$. Otherwise choose $k = 0, 1, 2$ or $3$ such that $\dim \mathfrak{b} + k = n + 4m$ for $m$ some positive integer.

The Lie algebra of $U(1)^k \times G$ is $k\mathfrak{u}(1) + \mathfrak{g}$. We will define a hypercomplex structure on this Lie algebra, which gives an almost hypercomplex structure on the group by left translation, and use Samelson's characterization of homogeneous complex structures on groups to show that the complex structures are integrable, and thus that the almost hypercomplex structure is hypercomplex.

Choose an identification (of real vector spaces) of $k\mathfrak{u}(1) + \mathfrak{b}$ with $\mathbb{H}^m + \mathbb{R}^n$; by abuse of notation we will write $k\mathfrak{u}(1) + \mathfrak{b} = \mathbb{H}^m + \mathbb{R}^n$. Note that there is a freedom in doing this of $(n + 4m)^2$ parameters. In general this will mean that there are infinitely many non-isomorphic hypercomplex structures on $U(1)^k \times G$. Let $(e_1, \ldots, e_n)$ be the standard basis for $\mathbb{R}^n$.

For each $k$, choose an isomorphism $\phi_k$ from $\mathfrak{su}(2)$ to $\mathfrak{d}_k$. (There are $3n$ parameters of freedom in doing this, but the different ways will lead to hypercomplex structures isomorphic up to conjugacy.)

Now the Lie algebra $\mathfrak{su}(2)$ may be written as $\langle i_1, i_2, i_3 \rangle$, where $i_1, i_2$ and $i_3$ satisfy $[i_1, i_2] = 2i_3$, $[i_2, i_3] = 2i_1$ and $[i_3, i_1] = 2i_2$. Define complex structures $I_1, I_2, I_3$ on $\mathfrak{g}$ by components as follows:

$(a)$ Let the actions of $I_1, I_2, I_3$ on $\mathbb{H}^m$ be as usual.

$(b)$ Let the actions of $I_1, I_2, I_3$ on $\mathbb{R}^n + \Sigma_j \mathfrak{d}_j$ be given by

$$I_a(e_j) = \phi_j(i_a), \qquad I_a(\phi_j(i_a)) = -e_j, \qquad \text{and} \qquad I_a(\phi_j(i_b)) = \phi_j(i_c),$$

$$I_a(\phi_j(i_c)) = -\phi_j(i_b) \text{ whenever } (abc) \text{ is an even permutation of } (123).$$

$(c)$ Let the actions of $I_1, I_2, I_3$ on $\mathfrak{f}_j$ be given by

$$I_a(v) = [v, \phi_j(i_a)], \qquad \text{for each } v \in \mathfrak{f}_j.$$

The proof of Theorem 5.1.2 will be completed by the following

**Lemma 5.1.3.** *The $I_1, I_2, I_3$ defined above are complex structures on $k\mathfrak{u}(1) + \mathfrak{g}$ satisfying $I_1 I_2 = I_3$, and the almost complex structures on $U(1)^k \times G$ generated by left translation are integrable.*

*Proof.* For the first part, it is clear that parts $(a)$ and $(b)$ lead to complex structures $I_1, I_2$ and $I_3$ satisfying $I_1 I_2 = I_3$ on their respective components, so it remains only to verify this for part $(c)$, that is, for $\mathfrak{f}_j$. But from condition $(ii)$ of Lemma 5.1.1 it can be seen that the action of $\mathfrak{d}_j$ on $\mathfrak{f}_j$ by conjugation is isomorphic to the action of $\operatorname{Im}\mathbb{H}$ on $\mathbb{H}^l$ for some $l$, and $(c)$ is just a way of writing down this isomorphism.

So $I_1, I_2, I_3$ do form a hypercomplex structure on $k\mathfrak{u}(1) + \mathfrak{g}$. Using Samelson's results it will now be shown that they generate homogeneous complex structures by left translation.

Let $a$ be 1,2 or 3. Define $\mathfrak{t}$ by

$$\mathfrak{t} = \mathbb{H}^m + \mathbb{R}^n + \langle \phi_1(i_a), \ldots, \phi_n(i_a) \rangle; \tag{47}$$

then $\mathfrak{t}$ is the Lie algebra of a maximal torus $T$ of $U(1)^k \times G$. Let $V \subset k\widetilde{\mathfrak{u}(1)} + \mathfrak{g}$ be the vector subspace of $(1,0)$- forms of $I_a$ in $k\widetilde{\mathfrak{u}(1)} + \mathfrak{g}$. We will construct a basis for $V$ involving a positive system of roots for $k\widetilde{\mathfrak{u}(1)} + \mathfrak{g}$ relative to $\mathfrak{t}$, and hence by Samelson's results show that $I_a$ gives an integrable complex structure on $U(1)^k \times G$.

We describe $V$ by components $(a),(b),(c)$ as above.

$(a)$ The $(1,0)$- forms of $\mathbb{H}^m$ with respect to $I_a$ are as usual.

$(b)$ The $(1,0)$- forms of $\mathbb{R}^n + \Sigma_j \mathfrak{d}_j$ are

$$\langle e_1 + i\phi_1(i_a), \dots, e_n + i\phi_n(i_a), \phi_1(i_b) + i\phi_1(i_c), \dots, \phi_n(i_b) + i\phi_n(i_c)\rangle,$$

where $(abc)$ is an even permutation of (123). Now $e_j + i\phi_j(i_a)$ is an element of $\tilde{\mathfrak{t}}$, and $\phi_j(i_b) + i\phi_j(i_c)$ is a root vector of $\tilde{\mathfrak{g}}$ relative to $\mathfrak{t}$. Let $\alpha_j$ be the root corresponding to the root vector $\phi_j(i_b) + i\phi_j(i_c)$. Also

$$\begin{aligned}[\phi_j(i_a), \phi_j(i_b) + i\phi_j(i_c)] &= -2i(\phi_j(i_b) + i\phi_j(i_c)) \\ &= \alpha_j(\phi_j(i_a))(\phi_j(i_b) + i\phi_j(i_c)),\end{aligned} \tag{48}$$

and so $\alpha_j(\phi_j(i_a)) = -2i$. Thus $\alpha_j(i\phi_j(i_a)) > 0$. (Note that $\beta(i\phi_j(i_a))$ is real for all roots $\beta$.)

$(c)$ Now we claim that the $(1,0)$- forms of $\tilde{\mathfrak{f}}_j$ are given by

$$V \cap \tilde{\mathfrak{f}}_j = \sum_{\substack{\beta\in\Delta_j:\beta\neq\alpha_j,\\ \beta(i\phi_j(i_a))>0}} \mathfrak{g}_\beta, \tag{49}$$

in other words, the sum of all root subspaces of $\mathfrak{b}_{j-1}$ corresponding to roots $\beta$ other than $\alpha_j$ that have $\beta(i\phi_j(i_a)) > 0$. Recall that in the proof of Lemma 5.1.1 it was shown that $\tilde{\mathfrak{f}}_j$ splits into subspaces of the form $\mathfrak{g}_\beta + \mathfrak{g}_{\beta+\alpha_j} + \mathfrak{g}_{-\beta} + \mathfrak{g}_{-\beta-\alpha_j}$, upon which the representation of $\mathfrak{d}_j$ is the complexification of the standard representation of $\mathfrak{su}(2)$ upon $\mathbb{C}^2$. It is an easy calculation to show that the $(1,0)$- forms of this subspace are $\mathfrak{g}_{\beta+\alpha_j} + \mathfrak{g}_{-\beta}$, which verifies the claim, as by structure theory we have $\alpha_j(i\phi_j(i_a)) = -2\beta(i\phi_j(i_a))$.

Now the roots of $\tilde{\mathfrak{b}}_j$ are exactly the roots of $\tilde{\mathfrak{g}}$ that give zero when evaluated upon $\phi_1(i_a), \dots, \phi_j(i_a)$, because these are the roots that centralize $\mathfrak{d}_1, \dots, \mathfrak{d}_j$. So define a subset $P$ of $\Delta$, the set of roots of $\tilde{\mathfrak{g}}$, by

$$\begin{aligned}P = \big\{\alpha \in \Delta : \alpha(\phi_1(i_a)) = \cdots = \alpha(\phi_{j-1}(i_a)) = 0,& \\ \alpha(i\phi_j(i_a)) > 0 \quad \text{for some } j \in \{1, 2, \dots, n\}\big\}.&\end{aligned} \tag{50}$$

Then $P$ is a positive system (as defined above). But by examination we see that

$$V = V \cap \tilde{\mathfrak{t}} + \sum_{\alpha\in P} \mathfrak{g}_\alpha. \tag{51}$$

So $V$, the $(1,0)$- forms of $I_a$, are the sum of the $(1,0)$- forms of some complex structure on $\mathfrak{t}$ together with a positive system of roots. Therefore by [Sm], the left translation of $I_a$ gives a homogeneous complex structure on $U(1)^k \times G$.

□

## 5.2. General homogeneous hypercomplex manifolds

The previous section extended Samelson's result on existence of homogeneous complex structures on even-dimensional groups to the hypercomplex case. In this section we extend some of Wang's results on existence of homogeneous complex structures on general homogeneous manifolds to the hypercomplex case. His Theorem II ([W], p. 15) states:

*Let $X$ be a C-subgroup of a simply-connected compact semisimple Lie group $K$. If $K/X$ is even-dimensional, then $K/X$ has a homogeneous complex structure.*

Here a *C-subgroup* of $K$ is a closed and connected subgroup whose semisimple part coincides with the semisimple part of the centralizer of a toral subgroup of $K$.

This theorem of Wang generalizes Samelson's result. An extension to the hypercomplex case will now be given; it will be seen that the restrictions on the subgroup $X$ are quite severe.

First we make some definitions. Let $G$ be a compact Lie group. We may choose a maximal torus $H$, and decompose $\tilde{\mathfrak{g}}$ into weight spaces with respect to $\mathfrak{h}$. If $\alpha$ is any highest root, then there is a subalgebra of $\tilde{\mathfrak{g}}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ generated by $\mathfrak{g}_{\pm\alpha}$, and the intersection of this with $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$. Define a *D-subgroup* of $G$ to be the centralizer in $G$ of any such $\mathfrak{su}(2)$ embedded in $\mathfrak{g}$ that comes from a highest root in this way.

Now we define an *E-subgroup* of $G$ to be any subgroup $E$ of $G$ such that there is a chain of subgroups and inclusions

$$G = G_0 \supset G_1 \supset \ldots \supset G_j = E, \tag{52}$$

such that $G_{i+1}$ is a D-subgroup of $G_i$. We call $j$ the *length* of $E$; it is well defined.

The hypercomplex version of Wang's result quoted above is:

**Theorem 5.2.1.** *Let $G$ be a compact Lie group, and let $E$ be an E-subgroup of $G$ of length $j$. Let $F$ be the semisimple part of $E$, and let $X$ be any closed subgroup of $G$ such that $F \subseteq X \subseteq E$. Then there exists an integer $k$ with $0 \leq k \leq \max(3,j)$ such that $U(1)^k \times G/X$ admits a homogeneous hypercomplex structure, that is, one that is preserved by left translations in $U(1)^k \times G$.*

*Proof.* The proof is very similar to the proof of Theorem 5.1.2, but using, where appropriate, ideas from [W] instead of ideas from [Sm], so it will only be briefly sketched. Using the definition of the E-subgroup

$E$, one may carry out a decomposition of $\mathfrak{g}$ into subalgebras $\mathfrak{b}_i, \mathfrak{d}_i$ and subspaces $\mathfrak{f}_i$ as in Lemma 5.1.1, but instead of stopping when $\mathfrak{b}_n$ is abelian, we stop at $\mathfrak{b}_j = \mathfrak{e}$, where $j$ is the length of $E$ and $\mathfrak{e}$ is the Lie algebra of $E$. Now $X$ lies between $E$ and $F$, so $\mathfrak{e}$ is just $\mathfrak{x}$ plus the Lie algebra of some torus. As in the proof of Theorem 5.1.2, choose a suitable $k$ and define a hypercomplex structure on $k\mathfrak{u}(1) + \mathfrak{g}/\mathfrak{x}$. Then a similar analysis to that of Lemma 5.1.3 shows that the complex structures $I_1, I_2, I_3$ give integrable complex structures when extended over the space by left translation, using methods of Wang [W].

(Note that to be able to define the left translation of the complex structure it is necessary that it should be invariant under conjugation by $X$. This is true because the complex structures are defined using a sequence of highest roots, and $X$ is a subgroup of the centralizer of these roots.) $\square$

### 5.2.1. Examples

As by Theorems 5.1.2 and 5.2.1 every compact Lie group provides examples of homogeneous hypercomplex spaces, just a few interesting cases will be given. The hypercomplex structures on $U(2)$ and $SU(3)$ are the first examples of hypercomplex structures on the families $U(2n)$ and $SU(2n+1)$, and more generally on $U(2k+l)/U(l)$. Also, inclusions of groups can lead to inclusions of hypercomplex manifolds; for instance, $U(1)^n \times Sp(n)$ can appear as a hypercomplex submanifold in $U(1)^{2n} \times SO(4n)$, if the sequences of highest roots are chosen in a suitable way.

We will give $U(1) \times SO(6)$ as a worked example of Theorem 5.1.2, and as a worked example of Theorem 5.2.1 a pretty, compact, simply-connected hypercomplex 12-manifold: if $SU(2)$ is embedded in $U(3) \subset SO(6)$, it will be shown that $SO(6)/SU(2)$ is hypercomplex. Let $G$ be $SO(6)$, and $H$, a maximal torus, be the diagonal matrices in $U(3) \subset SO(6)$. The Lie algebra $\mathfrak{h}$ of $H$ is then the set of matrices in $\mathfrak{u}(3) \subset \mathfrak{so}(6)$ of the form $\mathrm{diag}(i\lambda_1, i\lambda_2, i\lambda_3)$, $\lambda_j \in \mathbb{R}$. Define coordinates $(x_1, x_2, x_3)$ on $\tilde{\mathfrak{h}}^*$ such that $(x_1, x_2, x_3)$ is the element of $\tilde{\mathfrak{h}}^*$ taking $\mathrm{diag}(i\lambda_1, i\lambda_2, i\lambda_3)$ to $2(x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3)$.

In these coordinates the twelve roots of $SO(6)$ are given by $(\pm i, \pm i, 0)$, $(\pm i, 0, \pm i)$, $(0, \pm i, \pm i)$. These roots are all equivalent under automorphisms of $G$ preserving $H$, so every root is a highest root. Choose $(i, i, 0)$ as a highest root to generate $\mathfrak{d}_1$. This gives

$$\tilde{\mathfrak{d}}_1 = \langle \mathrm{diag}(i, i, 0) \rangle + \mathfrak{g}_{(i,i,0)} + \mathfrak{g}_{(-i,-i,0)},$$

$\tilde{\mathfrak{f}}_1$ turns out to be the sum of the eight root spaces of the roots $(\pm i, 0, \pm i), (0, \pm i, \pm i)$, and $\tilde{\mathfrak{b}}_1$ is

$$\tilde{\mathfrak{b}}_1 = \langle \mathrm{diag}(i, -i, 0), \mathrm{diag}(0, 0, i)\rangle + \mathfrak{g}_{(i,-i,0)} + \mathfrak{g}_{(-i,i,0)}.$$

There is then only one choice for $\mathfrak{d}_2$:

$$\tilde{\mathfrak{d}}_2 = \langle \mathrm{diag}(i, -i, 0)\rangle + \mathfrak{g}_{(i,-i,0)} + \mathfrak{g}_{(-i,i,0)},$$

and we have $\mathfrak{f}_2 = 0$ and $\mathfrak{b}_2 = \langle \mathrm{diag}(0, 0, i)\rangle$, which is abelian. So $n = 2$, and this completes the decomposition of Lemma 5.1.1. To apply Theorem 5.1.2, we must have $k$ such that $\dim \mathfrak{b}_n + k = n + 4m$ for some $m$; here $n = 2$ and $\dim \mathfrak{b}_2 = 1$, so $k = 1$ and $m = 0$ will do. Thus by Theorem 5.1.2 $U(1) \times SO(6)$ is hypercomplex; the freedom in the hypercomplex structure is the freedom to choose a basis $(e_1, e_2)$ for $\mathfrak{u}(1) + \mathfrak{b}_2$, and so is of four real parameters.

To apply Theorem 5.2.1, we follow the decomposition of $\mathfrak{so}(6)$ above, but stop at $\mathfrak{e} = \mathfrak{b}_1$. The semisimple part of $\tilde{\mathfrak{e}}$ is

$$\tilde{\mathfrak{f}} = \langle \mathrm{diag}(i, -i, 0)\rangle + \mathfrak{g}_{(i,-i,0)} + \mathfrak{g}_{(-i,i,0)},$$

and to apply Theorem 5.2.1 we must choose $X$ such that $F \subseteq X \subseteq E$. Let $X$=$F$; then $X$ is given by

$$X = \left\{ \begin{pmatrix} A & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \begin{smallmatrix} 0\,0 \end{smallmatrix} & 1 \end{pmatrix} : A \in SU(2) \right\} \subset U(3) \subset SO(6), \tag{53}$$

and $\mathfrak{e} = \mathfrak{x} + \langle \mathrm{diag}(0, 0, i)\rangle$ is the splitting of $\mathfrak{e}$ into $\mathfrak{x}$ and the Lie algebra of a torus. To make a hypercomplex structure we now need to take the product with $U(1)^k$ for suitable $k$. But because the length of $E$ is 1 and there is one dimension left over of the maximal torus, generated by $\mathrm{diag}(0, 0, i)$, we can take $k = 0$. The freedom in making the hypercomplex structure is the freedom in choosing a basis $(e_1)$ for $\langle \mathrm{diag}(0, 0, i)\rangle$, and so is of one real parameter. So by Theorem 5.2.1, $G/X = SO(6)/SU(2)$ is a homogeneous hypercomplex manifold.

## 5.3. Homogeneous quaternionic manifolds

There is one obvious source of homogeneous quaternionic manifolds: if a quaternionic manifold has a homogeneous associated bundle, then it will be homogeneous. As the associated bundle of a quaternionic manifold is hypercomplex, we can construct homogeneous quaternionic manifolds from homogeneous hypercomplex manifolds.

In general, this sort of homogeneous quaternionic manifold will be of the form $G/U(2)X$, where $G/X$ is a compact homogeneous hypercomplex manifold, and $U(2)$ embedded in $G$ centralizes $X$, descends to a hypercomplex submanifold in $G/X$, and the action of $U(2)$ on the right on $G/X$ permutes the complex structures in the way that $\mathbb{H}^*$ does on itself by left multiplication. The problem, then, given a homogeneous hypercomplex manifold $G/X$ as from the last section, is to find a suitable embedded (or immersed) $U(2)$. Let the embedding be $\Phi$, so that in terms of Lie algebras we seek a Lie algebra endomorphism $\Phi : \mathfrak{u}(2) \to \mathfrak{g}$.

This can be done using the method of construction of the last two sections, which involves a sequence of highest roots. Using the notation of Theorem 5.1.2, $\Phi(\mathfrak{su}(2))$ must be

$$\Phi(\mathfrak{su}(2)) = \langle \phi_1(i_1) + \cdots + \phi_n(i_1), \phi_1(i_2) + \cdots + \phi_n(i_2), \phi_1(i_3) + \cdots + \phi_n(i_3) \rangle. \tag{54}$$

This is because the hypercomplex structure is defined using $\phi_i(\mathfrak{su}(2))$, and so to permute the complex structures in the necessary way, the Lie bracket with $\Phi(\mathfrak{su}(2))$ must act on $\phi_i(\mathfrak{su}(2))$ as the Lie bracket with itself.

In order that $\Phi(U(2))$ should be a hypercomplex submanifold of $G/X$, $\Phi(\mathfrak{u}(2))$ must be closed under $I_1, I_2, I_3$. This requirement determines the fourth basis vector for $\Phi(\mathfrak{u}(2))$: it is $e_1 + \cdots + e_n$. So put

$$\Phi(\mathfrak{u}(2)) = \langle e_1 + \cdots + e_n, \phi_1(i_1) + \cdots + \phi_n(i_1), \phi_1(i_2) + \cdots + \phi_n(i_2), \phi_1(i_3) + \cdots + \phi_n(i_3) \rangle. \tag{55}$$

Then $\Phi(\mathfrak{u}(2))$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{u}(2)$, and we can form the subgroup of $G$ generated by it. But this subgroup may not be an embedding, or even an immersion, of $U(2)$, because it may not be closed. When $n > 1$ this is a non-trivial condition upon the hypercomplex structure chosen on $G/X$ in §5.2, and is a *rationality condition*, as it simply says that the centre of the embedded $U(2)$ (generated by $e_1 + \cdots + e_n$) should be a closed subgroup of the maximal torus of $G$. The condition therefore holds for a dense subset of the homogeneous hypercomplex structures on $G/X$ constructed in §5.2.

Suppose that this rationality condition holds for the choice of hypercomplex structure on $G/X$. Then the Lie algebra endomorphism $\Phi$ lifts to give a group homomorphism $\Phi : U(2) \to G$ which is an embedding or an immersion.

**Proposition 5.3.1.** *$G/\Phi(U(2))X$ is a compact, homogeneous quaternionic manifold.*

*Proof.* Let $U(1) \times U(1) \subset U(2)$ be the subgroup of $U(2)$ preserving the complex structure $I_1$ on $G/X$. Define $Z = G/\Phi(U(1) \times U(1))X$. Then $Z$ is complex with complex structure $I_1$, as it is the quotient of

$G/X$ by $\Phi(U(1)\times U(1))$, which is a complex group with respect to $I_1$. Also, $Z$ fibres over $G/\Phi(U(2))X$ with fibre $U(2)/(U(1)\times U(1)) = \mathbb{CP}^1$, and if $x$ is any element of $SU(2)$ that anti-commutes with the fixed $U(1)\subset SU(2)$, then $\Phi(x)$ induces an involution $\sigma$ that preserves the fibres and is independent of $x$. This $\sigma$ is antiholomorphic because, acting on the right on $G/X$, it takes $I_1$ to $-I_1$.

So to see that $Z$ is a twistor space for a quaternionic structure on $G/\Phi(U(2))X$, we only need to show that the normal bundle of the fibres is $2a\mathcal{O}(1)$ for some integer $a$. By homogeneity it is enough to show this for the identity fibre $\Phi(U(2))X/X$. Let $\nu$ be the normal bundle of $\Phi(U(2))X/X$ in $G/X$. As $\Phi(U(2))X/X$ is a hypercomplex submanifold of $G/X$, which is hypercomplex, the total space of $\nu$ is hypercomplex. The left action of $\Phi(U(2))$ on $\nu$ preserves this hypercomplex structure and identifies all of the fibres; thus it gives a trivialization of $\nu$.

This does not trivialize $\nu$ as a holomorphic bundle, as the flat connection on $U(2)$ it is associated with is not torsion-free. However, it can be seen that as a hypercomplex manifold, the total space of $\nu$ only depends upon $a$ and the hypercomplex structure of $\Phi(U(2))X/X$: the Lie algebra structure in the normal directions does not affect the hypercomplex structure of $\nu$. So the normal bundle of $\Phi(U(2))X/X$ in $G/X$ is isomorphic as a hypercomplex manifold to some standard example.

As this standard example, let $G'$ be $GL(a+1,\mathbb{H})$. Then $G'$ acts transitively on $\mathbb{H}^{a+1}/\mathbb{Z}$, where $\mathbb{Z}$ acts by dilation; we choose the action of $\mathbb{Z}$ so that $\mathbb{H}/\mathbb{Z}\subset\mathbb{H}^{a+1}/\mathbb{Z}$ and $\Phi(U(2))X/X$ are isomorphic as hypercomplex manifolds. Let $X'$ be the stabilizer of a point. From above, the bundle $\nu$ over $\Phi(U(2))X/X$ is isomorphic to the normal bundle of $\mathbb{H}/\mathbb{Z}$ in $G'/X' = \mathbb{H}^{a+1}/\mathbb{Z}$. Dividing by $U(1)\times U(1)$ shows that the normal bundle of the identity fibre in $Z$ is isomorphic to the normal bundle of $\mathbb{CP}^1$ in $\mathbb{CP}^{2a+1}$, which is $2a\mathcal{O}(1)$. $\square$

We should point out the connection between $G/X$ and the associated bundle of $G/\Phi(U(2))X$: in general $G/X$ can be constructed from the quotient of the associated bundle of $G/\Phi(U(2))X$ by a dilation action of $\mathbb{Z}$, by twisting by some homogeneous quaternionic $U(1)$-connection on $G/\Phi(U(2))X$, as in Chapter 3.

### 5.3.1. An example

We consider the case of $G = SU(5)$, $X$ trivial, which shows what can happen when $n>1$. Choose as highest weights the $SU(2)$'s embedded as $2\times 2$ matrices in the first, second and third, fourth diagonal positions of the $5\times 5$ matrix. The construction gives that $SU(5)/\Phi(U(2))$ is quaternionic, where

$$\Phi(SU(2)) = \left\{ \begin{pmatrix} A & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & A & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 0 \end{smallmatrix} & \begin{smallmatrix} 0 & 0 \end{smallmatrix} & 1 \end{pmatrix} : A \in SU(2) \right\}, \tag{56}$$

and $\Phi(U(1))$ is some closed subgroup of

$$U(1) \times U(1) = \left\{ \begin{pmatrix} \theta I & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \eta I & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 0 \end{smallmatrix} & \begin{smallmatrix} 0 & 0 \end{smallmatrix} & \theta^{-2}\eta^{-2} \end{pmatrix} : \theta, \eta \in U(1) \right\}; \tag{57}$$

here $I$ is the $2 \times 2$ identity matrix.

Now the important point is that for different choices of closed subgroup $\Phi(U(1))$, $SU(5)/\Phi(U(2))$ will have different topology. This one example, then, provides us with an infinite collection of distinct, compact, simply-connected quaternionic manifolds, and in fact each of these manifolds has infinitely many distinct quaternionic structures.

### 5.3.2. Different types of homogeneity

Above we have given a way of making homogeneous quaternionic manifolds. Perhaps all compact homogeneous quaternionic manifolds with homogeneous associated bundle are constructed in this manner. But what about homogeneous quaternionic manifolds for which the group is not big enough to act transitively on the associated bundle? In Chapter 3 an example of this phenomenon was given: a quaternionic structure on $SU(3)$ was made with symmetry group $U(3)$, which is of too small dimension to act transitively on the associated bundle.

The quaternionic structure on such a homogeneous space $G/X$ is given by a hypercomplex structure upon $\mathfrak{g}/\mathfrak{x}$. One approach to finding integrability conditions for a quaternionic structure specified in this way is to define, using linear functionals, first-order sections of the bundle of complex structures, and require that the Nijenhuis tensor of these sections should vanish. In this way I have shown that each of the homogeneous hypercomplex manifolds defined in §§5.1, 5.2 also admits homogeneous quaternionic structures that are not hypercomplex. The method is to construct a hypercomplex structure on the quotient of the Lie algebras using a sequence of highest roots as before, but instead of putting a standard hypercomplex structure on each of the embedded $\mathfrak{u}(2)$'s, to choose a hypercomplex structure corresponding to a quaternionic structure on $U(2)$ that is not hypercomplex. The calculation mentioned above then shows that the almost quaternionic structure defined by left translation is quaternionic, but not hypercomplex.

# Chapter 6: Many Anticommuting Complex Structures

To finish off this part of the thesis, some rather unpolished ideas are presented on how to extend the work in what is, as far as I can tell, a new direction. We do not give anything like a complete picture, we just show that nontrivial examples of certain sorts of geometrical structure do exist, a fact that seemed very surprising to me when I first found it out.

Whether the ideas are ever fleshed out into a complete theory depends on whether anyone else is interested in them, and thinks them worthy of study. There comes a point when it is silly to make more and more abstruse generalizations just for the sake of it. The structures seem to come up in supersymmetry, at least, as Spindel et al. [SSTV] tried to construct examples of this sort in a Physics paper. Their effort failed, in effect because they restricted themselves to compact semisimple groups.

Hypercomplex manifolds have three anticommuting complex structures. This chapter explains some methods of making non-flat manifolds with four, five or arbitrarily many integrable, anticommuting complex structures. Such a manifold is therefore a hypercomplex manifold in lots of different ways. To understand how these complex structures could act upon a tangent space of the manifold takes us into the realm of Clifford algebras and their modules, which are treated in detail by Atiyah et al. in [ABS], and also summarized in [SSTV], Appendix C. We shall define them now.

Consider the normed vector space $V = \mathbb{R}^k$ with the usual distance $|.|$, and let $T_k$ be the graded algebra $T_k = \bigoplus_{i=0}^{\infty} \otimes^i V$, where $\otimes^0 V = \mathbb{R}$, and multiplication is by tensor products in the obvious way. Let $I_k$ be the two-sided ideal of $T_k$ generated by elements of the form $x \otimes x + |x|^2 \cdot 1$ for $x \in V$. Define $C_k$ to be the quotient algebra $T_k/I_k$. Then $C_k$ is the $k^{\text{th}}$ Clifford algebra, as defined in [ABS], §2.

Suppose $(e_1, \ldots, e_k)$ is an orthonormal basis of $V$. Then $e_a$ also represent elements of $C_k$, by inclusion, and it is easily shown that they satisfy $e_a^2 = -1$ and $e_a e_b = -e_b e_a$ for all $a, b = 1, \ldots, k$ with $a \neq b$. These elements therefore behave like a set of anticommuting complex structures. The first three Clifford algebras are $C_0 \cong \mathbb{R}$, $C_1 \cong \mathbb{C}$ and $C_2 \cong \mathbb{H}$, and from [ABS], Table 1, the sequence continues $\mathbb{H} \oplus \mathbb{H}$, $M(2, \mathbb{H})$, $M(4, \mathbb{C})$, $M(8, \mathbb{R})$, $M(8, \mathbb{R}) \oplus M(8, \mathbb{R})$, $M(16, \mathbb{R})$, and so on, where $M(k, A)$ is the algebra of $k \times k$ matrices with entries in the algebra $A$. In §5, Atiyah et al. also classify modules over Clifford algebras. A module over $C_k$ corresponds precisely to a real vector space with $k$ anticommuting complex structures. Thus the classification, which we will not go into, tells us exactly what the possible actions are of $k$ anticommuting complex structures upon a single tangent space.

This is a geometry that is intrinsically without metrics; in the hypercomplex case hyperkähler manifolds are metric analogues, but from the classification [S2] of holonomy groups of Riemannian manifolds, any structure consisting of more than three anticommuting structures all of which are Kähler w.r.t. some metric, must be flat, as the group preserving this structure is not a possible holonomy group for a curved metric. Moreover, the structures do not fit within the theory of holonomy groups of torsion-free connections either, because there is no corresponding holonomy group in the classification [Br] of holonomy groups of torsion-free connections.

However, given a set of anticommuting complex structures on a manifold, every pair of them defines a hypercomplex structure, which is preserved by a unique torsion-free connection $\nabla$, the Obata connection. So if the structure is not flat, these Obata connections cannot all be the same, and we have a collection of torsion-free connections that each preserve a part of the structure, and not the whole.

In §§6.1-6.3 we shall show that there do exist non-flat examples with arbitrarily many anticommuting structures, and that a few of them can be made compact. The structures of §6.1 use the ideas on twisting from Chapter 3, and those of §6.2 are homogeneous, as in Chapter 5. In §6.3 we consider a more pedestrian sort of homogeneous structure, that can be made compact by dividing by a discrete subgroup. The resulting manifolds are nontrivial torus bundles over tori.

It is also possible by the same means to construct non-flat examples of structures on manifolds that stand in the same relation to the collections of anticommuting complex structures as quaternionic structures do to hypercomplex structures. For these new structures there are complex twistor spaces fibred by complex manifolds; the fibre is the set of complex structures at a point, and usually has quite a high dimension. We shall not deal with these ideas here.

### 6.1. Making many anticommuting complex structures by twisting

We begin with the simplest, most explicit example, that of four anticommuting complex structures on $\mathbb{R}^8$. Let us view $\mathbb{R}^8$ as $\mathbb{C}^4$ with coordinates $(w, x, y, z)$. Define four complex structures $I_1, I_2, I_3, I_4$ by their effect upon the vectors $\partial/\partial w, \ldots, \partial/\partial z$ as follows:

$$I_1 \begin{pmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} i\frac{\partial}{\partial w} \\ i\frac{\partial}{\partial x} \\ -i\frac{\partial}{\partial y} \\ -i\frac{\partial}{\partial z} \end{pmatrix}, \quad I_2 \begin{pmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \overline{x}} \\ -\frac{\partial}{\partial \overline{w}} \\ -\frac{\partial}{\partial \overline{z}} \\ \frac{\partial}{\partial \overline{y}} \end{pmatrix}, \quad I_3 \begin{pmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} i\frac{\partial}{\partial \overline{x}} \\ -i\frac{\partial}{\partial \overline{w}} \\ -i\frac{\partial}{\partial \overline{z}} \\ i\frac{\partial}{\partial \overline{y}} \end{pmatrix}, \quad I_4 \begin{pmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ -\frac{\partial}{\partial w} \\ -\frac{\partial}{\partial x} \end{pmatrix}. \tag{58}$$

Taking real and imaginary parts in these equations gives the actions of $I_1, \ldots, I_4$ upon eight basis vectors of $T\mathbb{C}^4$ at each point, and so four complex structures are defined. It can easily be verified that $I_j I_k = -I_k I_j$ for $j \neq k$. Notice also that $I_3 \neq I_1 I_2$ here, so $(I_1, I_2, I_3)$ is not our usual hypercomplex structure; this is because if $I_3 = I_1 I_2$ and $I_4$ anticommutes with $I_1, I_2$, then $I_3$ and $I_4$ commute, and $I_1, \ldots, I_4$ do not form an anticommuting set.

We shall use the ideas of Chapter 3 to twist this structure by an 'instanton' to get another structure $(I_1, \ldots, I_4)$ that is not flat. From §3.1, the first thing that is required is a principal bundle $P$ over $\mathbb{R}^8$ with Lie group $G$, and a connection $A$ on $P$ with curvature $\Omega$ that is of type $(1,1)$ with respect to each complex structure. Therefore let us consider the 2-forms on $\mathbb{R}^8$ that are of type $(1,1)$ with respect to each of $I_1, \ldots, I_4$. It is an easy calculation to show that there is a real 3-dimensional space generated by $f_1, f_2, f_3$, where

$$f_1 = idw \wedge d\overline{w} - idx \wedge d\overline{x} + idy \wedge d\overline{y} - idz \wedge d\overline{z}, \quad f_2 = \mathrm{Re}(dw \wedge d\overline{x} + dy \wedge d\overline{z}), \quad f_3 = \mathrm{Im}(dw \wedge d\overline{x} + dy \wedge d\overline{z}). \tag{59}$$

The problem, then, is to find a bundle $P$ with connection $A$, such that the 2-form part of the curvature $\Omega$ is made up only of the 2-forms $f_1, f_2$ and $f_3$.

Let $G$ be the group $\mathbb{R}$ under addition. Then because $G$ is abelian, locally $\Omega$ is the curvature of a connection on a $G$- bundle if and only if it is closed. So any combination of $f_1, f_2, f_3$ that is closed will lift to a suitable connection on a $G$- bundle. I do not know if there are any non-constant combinations of $f_1, f_2, f_3$ that are closed, but it is clear that any constant 2-form is closed. So let $\Omega$ be the constant 2-form $2f_1$. As $\Omega$ is closed and $H^2(\mathbb{R}^8) = 0$, there is a 1-form $\alpha$ such that $d\alpha = \Omega$; and in fact we may choose

$$\alpha = iwd\overline{w} - i\overline{w}dw - ixd\overline{x} + i\overline{x}dx + iyd\overline{y} - i\overline{y}dy - izd\overline{z} + i\overline{z}dz. \tag{60}$$

Then on the trivial bundle $P = \mathbb{R} \times \mathbb{R}^8$, with coordinates $(u, w, x, y, z)$, where $u \in G = \mathbb{R}$ and $(w, x, y, z) \in \mathbb{C}^4 \simeq \mathbb{R}^8$, the 1-form $\omega = du + \alpha$ is a connection 1-form with curvature $d\omega = d\alpha = \Omega = 2f_1$ as we want.

The next thing that is needed is an action of $G$ on $P$ preserving the bundle structure, the complex structures, and the connection; then, provided that this action satisfies a transversality condition, dividing $P$ by this action gives a new $(I_1, I_2, I_3, I_4)$- structure on $N = P/G$ that may not be flat. Therefore let us consider the automorphisms of the whole structure $P$. There is a trivial action of $\mathbb{R} = G$ by translation in the fibres, taking $u$ to $u + t$ and fixing $w, x, y$ and $z$. But as $\Omega$ is constant and so invariant under translations in $\mathbb{R}^8$, we also expect to be able to lift translations in $\mathbb{R}^8$ to $P$.

To try and do this, suppose $w, x, y$ and $z$ transform as $(w, x, y, z) \mapsto (\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}) = (w + p, x + q, y + r, z + s)$, and let us consider how $u$ must transform to fix $\omega = du + \alpha$. From (60), $\alpha$ transforms to

$$\tilde{\alpha} = \alpha + ipd\overline{w} - i\overline{p}dw - iqd\overline{x} + i\overline{q}dx + ird\overline{y} - i\overline{r}dy - isd\overline{z} + i\overline{s}dz,$$

so for $\tilde{\omega} = \omega$, we must have

$$d\tilde{u} = du - ipd\overline{w} + i\overline{p}dw + iqd\overline{x} - i\overline{q}dx - ird\overline{y} + i\overline{r}dy + isd\overline{z} - i\overline{s}dz,$$

and thus

$$\tilde{u} = u + t - ip\overline{w} + i\overline{p}w + iq\overline{x} - i\overline{q}x - ir\overline{y} + i\overline{r}y + is\overline{z} - i\overline{s}z. \tag{61}$$

With this definition, the transformation preserves the complex structures, the bundle structure and the connection and is given by

$$(u, w, x, y, z) \mapsto (u + t - ip\overline{w} + i\overline{p}w + iq\overline{x} - i\overline{q}x - ir\overline{y} + i\overline{r}y + is\overline{z} - i\overline{s}z, w + p, x + q, y + r, z + s). \tag{62}$$

The set of all transformations of this type forms a group $K$, say. Denoting the transformation (62) by $(t, p, q, r, s)$, the composition law is

$$\begin{aligned}(t_1, p_1, q_1, r_1, s_1) \circ (t_2, p_2, q_2, r_2, s_2) = (t_1{+}t_2 &- ip_1\overline{p}_2 + i\overline{p}_1p_2 + iq_1\overline{q}_2 - i\overline{q}_1q_2 - ir_1\overline{r}_2 + i\overline{r}_1r_2 \\ &+ is_1\overline{s}_2 - i\overline{s}_1s_2, p_1 + p_2, q_1 + q_2, r_1 + r_2, s_1 + s_2).\end{aligned} \tag{63}$$

It is a nonabelian group, and is a group extension fitting into the sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow K \longrightarrow \mathbb{R}^8 \longrightarrow 0. \tag{64}$$

Note that $K$ is not the full automorphism group of the structures on $P$: there are also automorphisms lifting affine transformations of $\mathbb{R}^8$ rather than just translations. But $K$ is large enough for our purposes.

Choose a subgroup $H$ of $K$ isomorphic to $\mathbb{R}$, generated by some nonzero $(t, p, q, r, s)$. Then the transversality condition is that the vector field generating this subgroup, contracted with $\omega$, should be nonzero. Clearly this vector field is

$$\begin{aligned} v = (t - ip\overline{w} + i\overline{p}w + iq\overline{x} - i\overline{q}x - ir\overline{y} + i\overline{r}y + is\overline{z} - i\overline{s}z)\frac{\partial}{\partial u} \\ + p\frac{\partial}{\partial w} + \overline{p}\frac{\partial}{\partial \overline{w}} + q\frac{\partial}{\partial x} + \overline{q}\frac{\partial}{\partial \overline{x}} + r\frac{\partial}{\partial y} + \overline{r}\frac{\partial}{\partial \overline{y}} + s\frac{\partial}{\partial z} + \overline{s}\frac{\partial}{\partial \overline{z}}, \end{aligned} \tag{65}$$

and its contraction with $\omega$ is

$$\omega(v) = t + 2(-ip\overline{w} + i\overline{p}w + iq\overline{x} - i\overline{q}x - ir\overline{y} + i\overline{r}y + is\overline{z} - i\overline{s}z). \tag{66}$$

When $t \neq 0$ and $p = q = r = s = 0$, $\omega(v)$ is nonzero everywhere, but this is just the principal bundle action, and dividing by it gives the flat structure back again.

If $p, q, r$ and $s$ are not all zero, $\omega(v)$ vanishes upon a hyperplane in $P$, so that the twisted structures become singular on this hyperplane. Away from this hyperplane, though, twisting by the action of $H$ in the manner of §3.1 gives a quadruple $(I_1, I_2, I_3, I_4)$ of complex structures on $\mathbb{R}^8$ that is not flat, in the sense that there is no coordinate system in which all four are constant in coordinates. One can write these complex structures down explicitly in coordinates, by choosing a $(t, p, q, r, s)$ for which $t$ is nonzero, and then using $u = 0$ as a transversal for the orbits when $u, w, x, y$ and $z$ are small, a technique used in Appendix A.

The centralizer $C$ of $H$ in $K$ is an eight-dimensional group containing $H$ as a normal subgroup, and it is $C$ that commutes with the action of $H$ and so descends to $N = \big(\mathbb{R} \times \mathbb{R}^8\big)/H$. But $H \subset C$ acts trivially on $N$, so the actual group acting on $N$ is $C/H$, which has seven dimensions and is abelian, whilst $N$ is eight-dimensional. Therefore $N$ is not homogeneous, which is not surprising, as the structure $(I_1, I_2, I_3, I_4)$ becomes singular on a hyperplane in $N$.

We have seen that on an open set in $\mathbb{R}^8$ there exist anticommuting, integrable complex structures $I_1, \ldots, I_4$, that together make up a structure that is not homogeneous or flat. The same idea of taking a vector space with an arrangement of complex structures, choosing an $\mathbb{R}$- connection of constant curvature, and twisting by a group action, is one that will work whenever the arrangement of complex structures allows a 2-form of type $(1, 1)$ w.r.t. each complex structure. Usually it can easily be shown that the resulting structure is not flat, though there are some simple cases such as a single complex structure or a collection of commuting complex structures, where the result may be flat for an injudicious choice of the 2-form.

However, for some collections of complex structures on a vector space there are no 2-forms of type $(1,1)$ with respect to each complex structure. Consider again the case of $\mathbb{R}^8$, following Appendix C of [SSTV]. The algebra generated by the above four complex structures is $M(2,\mathbb{H})$, acting upon $\mathbb{R}^8 \simeq \mathbb{H}^2$, and there is a commuting action of $\mathbb{H}$; we may think of the three forms $f_1, f_2, f_3$ as corresponding to $\operatorname{Im}\mathbb{H}$. Now five anticommuting complex structures can also act on $\mathbb{R}^8$, and the algebra they generate is $M(4,\mathbb{C})$, acting upon $\mathbb{R}^8 \simeq \mathbb{C}^4$. In this case there is only one form of type $(1,1)$ under all five complex structures, as the additional complex structure kills of the other two forms; we may think of this form as corresponding to $\operatorname{Im}\mathbb{C}$, where $\mathbb{C}$ is the algebra acting on $\mathbb{R}^8$ commuting with $M(4,\mathbb{C})$.

Going up the final stage, we can have seven complex structures acting on $\mathbb{R}^8$ (as the product of six anticommuting complex structures is a seventh anticommuting complex structure). The algebra they generate is $M(8,\mathbb{R})$, and the commuting algebra is $\mathbb{R}$, which has no imaginary part. This corresponds to the fact that there are no 2-forms on $\mathbb{R}^8$ invariant under all seven anticommuting complex structures, and therefore the twisting game we played above does not work. I conjecture this means that seven anticommuting complex structures on an eight-dimensional manifold must necessarily be flat.

By increasing the dimension, though, we may show that there exist patches of manifold with arbitrarily many anticommuting complex structures which taken together are not flat, provided the dimension of the manifold is high enough. For given any integer $n > 1$, there exists an integer $k > 1$ such that there are $n$ anticommuting elements in $U(2^k)$ that square to $-1$. These give $n$ anticommuting complex structures upon $\mathbb{C}^{2^k}$, and as they are in the unitary group they preserve the Kähler form of the metric, which is of type $(1,1)$ with respect to each one. This is the form we need to make a nontrivial connection to twist by.

### 6.2. Homogeneous manifolds with many anticommuting complex structures

In Chapter 5, homogeneous hypercomplex manifolds were constructed, effectively by embedding a series of copies of the algebra $\mathbb{H}$ into the Lie algebra of a group. If the embeddings were chosen correctly, then every element of $\mathbb{H}$ with square $-1$ gave rise to a homogeneous complex structure on the group. We shall now show that this is a general principle applying to any finite-dimensional algebra $A$ over $\mathbb{R}$: given a suitable embedding of a collection of copies of $A$ into the Lie algebra $\mathfrak{g}$ of a group $G$ (perhaps with a subgroup $H$), then each element in $A$ with square $-1$ gives rise to a homogeneous complex structure on the group $G$, or homogeneous space $G/H$. In particular, if $A$ is the Clifford algebra $C_k$ defined in the introduction to this chapter, then $G$ or $G/H$ has $k$ anticommuting complex structures.

The difference is that the trick of constructing a suitable embedding of Lie algebras by choosing a sequence of highest roots now no longer works, so we must suppose the embedding is given to begin with. This means that examples for more complicated algebras $A$ are somewhat sparse. By an embedding of an algebra into a Lie algebra we mean an embedding in the sense of vector spaces, such that the commutator $[a, b] = (ab - ba)/2$ of the algebra agrees with the Lie bracket of the Lie algebra.

The result we need generalizes Theorem 5.1.2, and the hypotheses come from Lemma 5.1.1. There is also a similar result for general homogeneous spaces $G/H$, corresponding to Theorem 5.2.1.

**Proposition 6.2.1.** *Let $A$ be a finite-dimensional algebra over $\mathbb{R}$ that is generated by its elements of square $-1$, and let $G$ be a Lie group, with Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g}$ can be decomposed as*

$$\mathfrak{g} = \sum_{k=1}^{n} \mathfrak{d}_k + \sum_{k=1}^{n} \mathfrak{f}_k, \tag{67}$$

*where $\mathfrak{d}_k$ is a subalgebra of $\mathfrak{g}$ isomorphic to $A$ under the isomorphism $\iota_k$, and $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ are (possibly empty) vector subspaces of $\mathfrak{g}$, such that for each $k = 1, 2, \ldots, n$, $\mathfrak{f}_k$ satisfies the following two conditions:*

*(i) $[\mathfrak{d}_l, \mathfrak{f}_k] = \{0\}$ whenever $l < k$, and*

*(ii) $\mathfrak{f}_k$ is closed under the Lie bracket with $\mathfrak{d}_k$, and for each $e \in A$ such that $e^2 = -1$, $\iota_k(e)$ acts as a complex structure on $\mathfrak{f}_k$ under the Lie bracket.*

*Then this decomposition defines an $A$- module structure on $\mathfrak{g}$, such that each element of $A$ with square $-1$ exponentiates to give a homogeneous, integrable complex structure on $G$.*

*Proof.* The underlying principles are the same as those of the proof of Theorem 5.1.2. With this as a rough guide, the proof will be left to the reader. □

Examples of suitable algebras $A$ include $M(k, \mathbb{H})$, $M(k, \mathbb{C})$ and $M(2k, \mathbb{R})$. The only algebras $A$ that can be included in the Lie algebras of compact groups $G$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, so any structures arising in this fashion that are more complicated than hypercomplex structures must occur on noncompact groups. There is a reason for this. In [SSTV], Spindel et al. prove as their main result that nontrivial homogeneous structures of four or more anticommuting complex structures on semisimple groups do not exist. I believe that their argument is correct for compact groups, but false for noncompact groups. The problem comes on p. 676, the sentence after (4.37), in which they reduce the noncompact case to the compact case: this step is wrong.

To justify this claim we shall delve briefly into the differences between the theory of compact and noncompact semisimple Lie groups. Recall the summary of the structure theory of Lie groups in the introduction to Chapter 5. The difference between the compact and noncompact cases comes in here. For in the compact case, the set of roots $\Delta \subset \tilde{\mathfrak{h}}^*$ is actually a subset of $i\mathfrak{h}^*$, where $\mathfrak{h}$ is the Lie algebra of a maximal torus, and $\tilde{\mathfrak{h}}$ denotes its complexification. But in the noncompact case, $\Delta$ need not lie in this subspace. Thus in the compact case, complex conjugation of a root just changes its sign, but in the noncompact case this need not apply.

In Chapter 5, defining a complex structure on $G$ was done by dividing the roots up into positive and negative roots, so that $\Delta = P \cup (-P)$. This is the compact case; to deal with the noncompact case too, this analysis must be changed, for what is needed is a splitting of $\Delta$ into $P$ and $\overline{P}$, rather than $P$ and $-P$. The new concept of positive system of roots, which covers the noncompact case too, is a set $P \subseteq \Delta$ satisfying $P \cap \overline{P} = \emptyset$; $P \cup \overline{P} = \Delta$; $\alpha, \beta \in P, \alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in P$. In the compact case, complex conjugation coincides with change of sign, so this is the same as the old definition.

This marks a divide between the compact and noncompact cases, for it means that all the old arguments about choosing a highest root, and then a sequence of highest roots, do not work in the noncompact case. It also means that more bizarre sorts of positive system exist, and it is these that get put together to form larger sets of anticommuting complex structures.

Finally we give some examples of many complex structures on noncompact groups and homogeneous manifolds. For the case of four anticommuting structures the relevant Clifford algebra is $M(2, \mathbb{H})$, and this may be embedded $k$ times in $2 \times 2$ diagonal squares in $M(2k, \mathbb{H})$, which gives a structure of four anticommuting complex structures on $GL(2k, \mathbb{H})$. Interestingly, for $k > 1$ this is not the obvious flat structure on the same group, but is curved. This generalizes to $GL(2k + l, \mathbb{H})/GL(l, \mathbb{H})$.

For five anticommuting complex structures the algebra is $M(4, \mathbb{C})$, which may be suitably embedded $k$ times in the Lie algebras of $GL(4k, \mathbb{C})$ $(k > 1)$ and $SL(4k + 1, \mathbb{C})$, also giving non-flat structures. The same tricks work with matrices over the reals too. More interesting examples occur when we embed general linear groups into orthogonal groups of mixed sign; for instance, $GL(k, \mathbb{R})$ may be embedded into $SO(k, k)$ by allowing it to act on a maximal null subspace of $\mathbb{R}^{2k}$ with its indefinite metric. Then the algebras may be embedded into the Lie algebras of these general linear subgroups, as before.

### 6.3. Compact examples

In this section we shall construct compact manifolds with many anticommuting complex structures. We shall do it using groups that are not semisimple, and hence are not dealt with by the last section, but instead are nilpotent. Indeed, the groups $G$ we consider are in a sense as close to being abelian as a Lie group can get, for if $x, y, z$ are elements of the Lie algebra $\mathfrak{g}$ of $G$, they will satisfy $[x,[y,z]] = 0$.

Let $A, B$ be nonzero real vector spaces, and let $f : A \times A \to B$ be a nonzero, bilinear, antisymmetric map. Let $E = A \oplus B$. We define a group operation on $E$ by

$$(a_1, b_1) \circ (a_2, b_2) = \big(a_1 + a_2, b_1 + b_2 + f(a_1, a_2)\big). \tag{68}$$

It is easily verified that with this operation, $E$ is a nonabelian Lie group with Lie algebra $\mathfrak{e} = A \oplus B$, such that $[e_1,[e_2,e_3]] = 0$ for any $e_1, e_2, e_3 \in \mathfrak{e}$. Also, $E$ fits into the exact sequence of groups

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0.$$

Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ on which anticommuting complex structures $I_1, \ldots, I_k$ act linearly. Let $\{\,,\,\} : V \times V \to V$ be a symmetric bilinear operation, and suppose that $\{\,,\,\}$ satisfies $\{v, w\} = \{I_j v, I_j w\}$ for all $v, w \in V$ and $j = 1, \ldots, k$. Define $\mathfrak{g}$ to be the tensor product $\mathfrak{e} \otimes V$, and define a bilinear bracket operation on $\mathfrak{g}$ by

$$[e_1 \otimes v_1, e_2 \otimes v_2] = [e_1, e_2] \otimes \{v_1, v_2\}. \tag{69}$$

We claim that this operation makes $\mathfrak{g}$ into a Lie algebra. To verify this it must be shown that $[\,,\,]$ is antisymmetric and satisfies the identity $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$. The first holds because $[\,,\,]$ on $\mathfrak{e}$ is antisymmetric and $\{\,,\,\}$ on $V$ is symmetric, and the second holds trivially because we have arranged that any triple bracket vanishes in $\mathfrak{e}$, and therefore each term in the identity vanishes individually.

So $\mathfrak{g}$ is a Lie algebra. Let $G$ be the connected Lie group with Lie algebra $\mathfrak{g}$. We may in fact explicitly identify $G$ with $\mathfrak{g}$, and write the group operation $\cdot$ as $x \cdot y = x + y + [x, y]/2$, where $[\,,\,]$ is the Lie bracket of $\mathfrak{g}$. It may easily be shown that as $[x,[y,z]] = 0$ for $x, y, z \in \mathfrak{g}$, this defines a group operation, which has the correct Lie bracket. Now the complex structures $I_1, \ldots, I_k$ on $V$ induce anticommuting complex structures, also denoted $I_1, \ldots, I_k$, on $\mathfrak{g}$. In the next lemma we shall show that these complex structures induce integrable homogeneous complex structures on $G$.

**Lemma 6.3.1.** *The anticommuting complex structures $I_1, \ldots, I_k$ of $\mathfrak{g}$ induce right-invariant, integrable, anticommuting complex structures on $G$.*

*Proof.* Let $\tilde{\mathfrak{g}}$ denote the complexification of $\mathfrak{g}$. To show that $I_j$ defines an integrable complex structure on $G$, it must be shown that the $(1,0)$- forms of $I_j$ in $\tilde{\mathfrak{g}}$ are a Lie subalgebra of $\tilde{\mathfrak{g}}$, so that $G$ can be regarded as the quotient of $G^c$ by a complex group, as in Chapter 5. The $(1,0)$- forms of $I_j$ are those of the form $(1 - iI_j)x$ for $x \in \mathfrak{g}$.

Therefore we have to prove that for all $x, y \in \mathfrak{g}$, there exists $z \in \mathfrak{g}$ such that $\big[(1-iI_j)x, (1-iI_j)y\big] = (1 - iI_j)z$. But

$$\big[(1-iI_j)x, (1-iI_j)y\big] = \big([x,y] - [I_jx, I_jy]\big) - i\big([I_jx, y] + [x, I_jy]\big),$$

and from our assumption that $\{v, w\} = \{I_jv, I_jw\}$ for $v, w \in V$, it follows that $[x,y] = [I_jx, I_jy]$ for $x, y \in \mathfrak{g}$, and both terms on the right hand side vanish separately. Thus we may take $z = 0$, and the $(1,0)$-forms of $I_j$ are an abelian subalgebra of $\tilde{\mathfrak{g}}$, so $I_j$ defines a right-invariant, integrable complex structure. $\square$

Now $G$ will be nonabelian provided $E$ is nonabelian and $\{\,,\,\}$ is nonzero. To be able to define a suitable operation $\{\,,\,\}$ on a vector space $V$ equipped with anticommuting complex structures $I_1, \ldots, I_k$ — thus, a $C_k$- module — we need $V$ to possess some nonzero symmetric bilinear forms that are invariant under each $I_j$. This can easily be arranged for arbitrarily large $k$; for instance, we may choose $V = \mathbb{R}^{2^a}$ and $I_1, \ldots, I_k \in SO(2^a)$ for some large enough positive integer $a$, and then $I_1, \ldots, I_k$ leave invariant the usual metric on $V = \mathbb{R}^{2^a}$. Then we choose an arbitrary nonzero map from the dual of this space of forms into $V$, and this defines the operation $\{\,,\,\}$.

When this holds, $G$ is a nontrivial group extension appearing in the exact sequence

$$0 \longrightarrow B \otimes V \longrightarrow G \longrightarrow A \otimes V \longrightarrow 0. \tag{70}$$

Our goal is to produce compact manifolds with anticommuting structures. To achieve this we shall find a discrete subgroup $H$ of $G$ such that $G/H$ is compact. Because the complex structures on $G$ have been chosen to be right-invariant, they descend to the coset space $G/H$, and so this will provide an example of a compact manifold with non-flat, integrable, anticommuting complex structures $I_1, \ldots, I_k$.

To choose such a subgroup $H$ we need to put extra conditions upon the elements used in the construction of $G$. The conditions we need are that $A$ should admit a basis $(a_1, \ldots, a_k)$ such that the elements $f(a_i, a_j)$ of $B$ generate a discrete subgroup of $B$ over $\mathbb{Z}$, and that $V$ admits a basis $(v_1, \ldots, v_l)$ such that the elements $\{v_i, v_j\}$ generate a discrete subgroup of $V$ over $\mathbb{Z}$. These conditions mean that the subgroups of the vector spaces generated over $\mathbb{Z}$ by the elements given should be discrete, and not dense in any nonzero vector subspaces.

Both of these conditions can easily be achieved. For instance, for the first we may set $B = \Lambda^2 A$ and $f$ to be the map $f(a, b) = a \wedge b$, so that the first condition is fulfilled. To satisfy the second condition, for any $k$ we may choose $I_1, \ldots, I_k$ in $SO(2^a)$ for some suitable $a$, and then the condition is satisfied by putting $V = \mathbb{R}^{2^a}$, $(v_1, \ldots, v^{2^a})$ to be an orthonormal basis of $V$, and letting $\{x, y\} = (x \cdot y)v$ for some fixed nonzero $v \in V$, where $x \cdot y$ is the inner product in $V$.

Suppose that the conditions are satisfied and bases $(a_1, \ldots, a_k)$ and $(v_1, \ldots, v_l)$ have been chosen as above. Then the elements $a_i \otimes v_j$ generate a discrete subgroup $P$ of $A \otimes V$, such that $A \otimes V/P$ is a torus. Using (70) we may pull $P$ back to $G$ and take the commutator of the resulting (nondiscrete) group to get a subgroup $Q$ of $B \otimes V$. The two conditions above imply that $Q$ is a discrete subgroup of $B \otimes V$.

Choose a discrete subgroup $R$ of $B \otimes V$ that contains $Q$, such that $B \otimes V/R$ is a compact torus. Define $H$ to be the subgroup of $G$ generated by the elements of $R$ and by arbitrary lifts to $G$ of a basis over $\mathbb{Z}$ of $P$. Then $H$ is a discrete subgroup of $G$; its projection to $A \otimes V$ is $P$; and its intersection with $B \otimes V$ is $R$. Therefore $G/H$ is a compact manifold, being a nontrivial fibre bundle over the torus $A \otimes V/P$, with fibre the torus $B \otimes V/R$. So we have proved the following lemma:

**Lemma 6.3.2.** *For arbitrarily large $k$, there exist compact, nontrivial torus bundles over tori that admit $k$ anticommuting, integrable complex structures, that considered as a single structure are not locally flat.*

However, these examples are really as close to being flat as they could be. To better understand the incidence of compact examples, I would like to be able to answer the next question.

**Question 6.3.3.** *Do there exist compact, simply-connected manifolds admitting four or more anticommuting complex structures?*

# Part II: Constant Scalar Curvature Metrics on Connected Sums and the Yamabe Problem

## Chapter 7: Background Material for Part II

The background we shall need for this part of the thesis is a working knowledge of the Yamabe problem, and some of the mathematical techniques used in its solution. The problem is posed in §7.1 as a geometric problem, and the first steps in the solution are followed to show how it reduces to a problem in Analysis, the task being to construct a smooth and positive solution to a certain equation (the Yamabe equation) on a Riemannian manifold. In §7.2 we give the necessary analytic preliminaries, defining Sobolev spaces and norms, quoting the Sobolev embedding theorem, and proving a regularity result stating that a weak solution of the Yamabe equation in a certain Sobolev space must be smooth.

Section 7.3 goes into greater detail on a part of the solution of the Yamabe problem, the contribution of Schoen and Yau to the proof of the Yamabe problem on the subject of asymptotically flat manifolds and positive mass. In doing this stereoscopic projections are introduced, which are a means of making an asymptotically flat manifold of zero scalar curvature from a compact manifold of positive scalar curvature, and which are very important in the following chapters.

### 7.1. The Yamabe problem

Given a Riemannian manifold $M$ with metric $g$, one may form the Riemann curvature tensor $R^i{}_{jkl}$, the Ricci curvature $R_{ij} = R^k{}_{ikj}$, and the scalar curvature $S = g^{ij}R_{ij}$. The scalar curvature is thus a real-valued function on $M$. Now given a smooth positive function $\gamma$ on $M$, $\tilde{g} = \gamma g$ is a Riemannian metric on $M$ conformal to $g$, and will have its own scalar curvature $\tilde{S}$. So one scalar function, $\gamma$, gives rise to another scalar function, $\tilde{S}$. The Yamabe problem is to find $\gamma$ such that $\tilde{S}$ is constant.

**The Yamabe Problem:** *Given a compact Riemannian manifold $M$ with metric $g$ of dimension $\geq 3$, find a metric conformal to $g$ with constant scalar curvature.*

In 1960, Yamabe [Y] claimed to have solved it. However, his proof contained an error, discovered in 1968 by Trüdinger [T]. Trüdinger found that Yamabe's proof could be repaired, but only with a rather restrictive assumption on the manifold $M$. The problem has now been completed by Aubin, Schoen and Yau. Yamabe's approach is a variational one, and will now be briefly described; for a more thorough introduction to the Yamabe problem, with references, we recommend the survey paper [LP] by Lee and Parker.

Suppose $M$ is a compact, connected Riemannian manifold of dimension $n \geq 3$, with metric $g$. Any metric conformal to $g$ can be written $\tilde{g} = e^{2f}g$, for some $f \in C^\infty(M)$. Then by [Bs], Theorem 1.159, the Ricci curvatures $R_{ij}$ and $\tilde{R}_{ij}$ of $g$ and $\tilde{g}$ are related by

$$\tilde{R}_{ij} = R_{ij} - (n-2)\big(\nabla_i\nabla_j f - (\nabla_i f)(\nabla_j f)\big) + \big(\Delta f - (n-2)|\nabla f|^2\big)g_{ij}, \tag{71}$$

where $\nabla$ is the covariant derivative with respect to $g$, and $\Delta f$ is the Laplacian $\Delta = -\nabla^i\nabla_i$ of $f$.

Let $S$ and $\tilde{S}$ denote the scalar curvatures of $g$ and $\tilde{g}$, respectively. Then $S = R_{ij}g^{ij}$ and $\tilde{S} = \tilde{R}_{ij}\tilde{g}^{ij}$, so contracting (71) with $\tilde{g}^{ij}$ gives

$$\tilde{S} = e^{-2f}\left(S + 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2\right). \tag{72}$$

Now if $k$ is a real number and $\psi$ a smooth function, $\Delta\psi^k = k\psi^{k-1}\Delta\psi - k(k-1)\psi^{k-2}|\nabla\psi|^2$. Making the substitution $e^{2f} = \psi^k$ in (72), for one particular value of $k$ the term in $|\nabla\psi|^2$ vanishes, and the formula simplifies. This value of $k$ is $k = 4/(n-2)$, so put $e^{2f} = \psi^{p-2}$, with $p = 2n/(n-2)$ and $\tilde{g} = \psi^{p-2}g$. Then (72) becomes

$$\tilde{S} = \psi^{1-p}\left(4\frac{n-1}{n-2}\Delta\psi + S\psi\right). \tag{73}$$

**Definition 7.1.1.** *In line with [LP], we shall use the definitions $n = \dim M \geq 3$, $p = 2n/(n-2)$, $a = 4(n-1)/(n-2)$ throughout.*

So $\tilde{g} = \psi^{p-2}g$ has constant scalar curvature $\nu$ if and only if $\psi$ satisfies the *Yamabe equation*:

$$a\Delta\psi + S\psi = \nu|\psi|^{p-1}. \tag{74}$$

Consider now the functional $Q$ defined upon the set of Riemannian metrics on $M$ by

$$Q(g) = \frac{\int_M S dV_g}{\mathrm{vol}(M)^{2/p}}. \tag{75}$$

This functional is known as the total scalar curvature, or *Hilbert action*, and the purpose of the power of the volume in the denominator is to normalize it so that it is invariant under homotheties. We wish to apply the calculus of variations to $Q$ and find its stationary points. Therefore let $g$ be a fixed metric on $M$, and $h$ a smooth symmetric $(2,0)$ tensor that is small compared to $g$.

The scalar curvature of $g+h$ is given by $S_{g+h} = S + \Delta(g^{ij}h_{ij}) + \nabla^i\nabla^j h_{ij} - R^{ij}h_{ij} + O(h^2)$, and as by Stokes' Theorem the second and third terms on the right integrate over $M$ to give zero, we find that

$$Q(g+h) = Q(g) + \frac{1}{\mathrm{vol}(M)^{2/p}} \cdot \int_M \left(\frac{S}{2}g^{ij} - R^{ij}\right) h_{ij}dV_g - \frac{Q(g)}{p\,\mathrm{vol}(M)} \cdot \int_M g^{ij}h_{ij}dV_g + O(h^2). \tag{76}$$

Thus $g$ is a stationary point of $Q$ if

$$\frac{S}{2}g^{ij} - R^{ij} - \frac{n-2}{2n}\frac{\left(\int_M SdV_g\right)}{\mathrm{vol}(M)}g^{ij} = 0, \tag{77}$$

so that $g$ is a stationary point of $Q$ if and only if $R_{ij} = g_{ij} \cdot \int_M SdV_g / \big(n\,\mathrm{vol}(M)\big)$, that is, if $g$ is Einstein with constant scalar curvature; this calculation has been taken from [Bs], §4C.

Now consider the stationary points of $Q$ within a single conformal class. This means restricting to variations $h_{ij} = \theta g_{ij}$ for small scalar functions $\theta$, and (76) shows that $g$ is a stationary point of $Q$ within its conformal class if and only if $S \equiv \int_M SdV_g/\,\mathrm{vol}(M)$, i.e. if $S$ is constant.

It can easily be shown using Hölder's inequality that $Q$ is bounded below in its conformal class. So we may define

$$\lambda(M) = \inf\big\{Q(\tilde{g}) : \tilde{g} \text{ conformal to } g\big\}. \tag{78}$$

This constant $\lambda(M)$ is an invariant of the conformal class of $g$ on $M$, called the *Yamabe invariant*. It plays an important rôle in the solution of the Yamabe problem.

Yamabe's method aims to show that the infimum $\lambda(M)$ is actually achieved by some smooth, nonsingular $\tilde{g}$ conformal to $g$, and that as $\tilde{g}$ is a stationary point of the Hilbert action $Q$, it must have constant scalar curvature. Such a minimizing metric $\tilde{g}$ is called a *Yamabe metric*. It is now known that Yamabe metrics always exist, although Yamabe's original proof was flawed. The corrected proof can be summarized in two theorems:

**Theorem A (Yamabe, Trüdinger, Aubin).** *The Yamabe problem can be solved on any compact Riemannian manifold $M$ with $\lambda(M) < \lambda(\mathcal{S}^n)$, where $\mathcal{S}^n$ is the sphere with its standard metric.*

**Theorem B (Aubin, Schoen, Yau).** *If $M$ is a compact Riemannian manifold, then $\lambda(M) < \lambda(\mathcal{S}^n)$ unless $M$ is $\mathcal{S}^n$ and its metric is conformal to the standard metric on $\mathcal{S}^n$.*

In §7.3 we will go into Theorem B in greater detail, by examining the rôles of test functions and positive mass. To prepare for this, and give an intuitive idea of why the problem is split up into these two theorems, we will try to explain how the condition $\lambda(M) < \lambda(\mathcal{S}^n)$ arises.

Let $M$ be a Riemannian manifold, and $m$ a point in $M$. Using geodesic coordinates we may identify a small ball around $M$ with a small ball about the origin in $\mathbb{R}^n$ in such a way that, sufficiently close to $m$, the agreement between the metric on $M$ and the standard metric on $\mathbb{R}^n$ is very good. Now $\mathbb{R}^n$ has a whole family of conformal identifications with the complement of a point in $\mathcal{S}^n$, and some of these take a very small ball about the origin in $\mathbb{R}^n$ to almost the whole of $\mathcal{S}^n$, whilst crushing the rest of $\mathbb{R}^n$ down to a very small size.

We may choose a conformal rescaling of $M$ that is like this, in that the conformal factor is very small except on a small ball about $m$, and on this small ball it looks like the conformal factor corresponding to a conformal identification of a small ball about the origin in $\mathbb{R}^n$ with almost all of $\mathcal{S}^n$. The conformally rescaled metric then looks quite like the standard metric on $\mathcal{S}^n$, with a small hole cut out and a very small copy of $M$ glued in its place.

A simple argument shows that the Hilbert action of this conformal rescaling is quite close to that of the standard metric on $\mathcal{S}^n$. So taking a limit, it can be seen that $\lambda(M) \le \lambda(\mathcal{S}^n)$ for all Riemannian manifolds $M$. Now suppose that $\lambda(M) = \lambda(\mathcal{S}^n)$ for some Riemannian manifold $M$. Then as a minimizing sequence for the Hilbert action we can choose a sequence of conformal factors as above, which become progressively more and more concentrated around a point. These clearly have no subsequence converging to a nonsingular metric on $M$. Thus, if $\lambda(M) = \lambda(\mathcal{S}^n)$, then minimizing sequences for the Hilbert action need not converge to a Yamabe metric on $M$.

However, if $\lambda(M) < \lambda(\mathcal{S}^n)$, it is clear that a minimizing sequence for the Hilbert action cannot become concentrated around single points in the above fashion. This makes it intuitively credible, at least, that the condition $\lambda(M) < \lambda(\mathcal{S}^n)$ implies that a minimizing sequence must have some convergent subsequence.

### 7.2. Sobolev spaces, embedding theorems, and elliptic regularity

We begin by fixing our notation for Lesbesgue spaces and Sobolev spaces, which will be used a lot in the next chapters. Then we explain the Sobolev embedding theorem, and go on to prove a result on smoothness of solutions to the Yamabe equation that lie in a Sobolev space.

Let $M$ be a Riemannian manifold with metric $g$. For $q \geq 1$, define the *Lesbesgue space* $L^q(M)$ to be the set of locally integrable functions $u$ on $M$ for which the norm

$$\|u\|_q = \left(\int_M |u|^q dV_g\right)^{1/q} \tag{79}$$

is finite. Here $dV_g$ is the volume form of the metric $g$. Suppose that $r, s, t \geq 1$ and that $1/r = 1/s + 1/t$. If $\phi \in L^s(M)$, $\psi \in L^t(M)$, then $\phi\psi \in L^r(M)$, and $\|\phi\psi\|_r \leq \|\phi\|_s \|\psi\|_t$; this is *Hölder's inequality*.

Let $q \geq 1$ and let $k$ be a nonnegative integer. Define the *Sobolev space* $L^q_k(M)$ to be the set of $u \in L^q(M)$ such that $u$ is $k$ times weakly differentiable and $|\nabla^i u| \in L^q(M)$ for $i \leq k$. Define the *Sobolev norm* on $L^q_k(M)$ to be

$$\|u\|_{q,k} = \left(\sum_{i=0}^{k} \int_M |\nabla^i u|^q dV_g\right)^{1/q}. \tag{80}$$

Then $L^q_k(M)$ is a Banach space w.r.t. the Sobolev norm. Furthermore, $L^2_k(M)$ is a Hilbert space.

Here weak differentiation is defined as follows. If $\nabla$ is a connection on a bundle $E$, we say that $f$ is the weak derivative of $u$, written $f = \nabla u$, if $u, f$ are locally integrable sections of $E, E \otimes T^*$ respectively, and for every compactly supported smooth vector field $v$ on $M$, $u$ and $f$ satisfy the identity $\int_M (f \cdot v - u\nabla^* v) dV_g = 0$. If $u$ is differentiable in the ordinary sense, the first term integrates by parts to give $\int_M v \cdot (f - \nabla u) dV_g = 0$, so the weak derivative is equal to the usual derivative.

For each integer $r \geq 0$, define the space $C^r(M)$ to be the space of continuous, bounded functions that have $r$ continuous, bounded derivatives, and define the norm $\|.\|_{C^r}$ on $C^r(M)$ by $\|u\|_{C^r} = \max_{0 \leq l \leq r} \sup\left|\nabla^l u\right|$.

An important tool in problems involving Sobolev spaces is the Sobolev embedding theorem, which includes one Sobolev space inside another.

**Theorem 7.2.1 (Sobolev embedding theorem for compact manifolds).** *Suppose $M$ is a compact Riemannian manifold of dimension $n$. If*

$$\frac{1}{r} \geq \frac{1}{q} - \frac{k}{n}, \tag{81}$$

*then $L^q_k(M)$ is continuously embedded in $L^r(M)$ by inclusion. If*

$$-\frac{r}{n} > \frac{1}{q} - \frac{k}{n}, \tag{82}$$

*then $L^q_k(M)$ is continuously embedded in $C^r(M)$ by inclusion.*

*Proof.* Sobolev embedding theorems are dealt with at length by Aubin in [Au], §§2.3-2.9. This version is a partial statement of Theorem 2.20, p. 44. □

In Chapters 9 and 10, solutions in $L^2_1(M)$ to the Yamabe equation (74) will be constructed for certain special compact manifolds $M$. For these solutions to give Riemannian metrics of constant scalar curvature, it is necessary that they be not just $L^2_1$ solutions, but $C^\infty$ solutions. We will now prove a result showing that this is the case. It relies upon a result of Trüdinger, and is part of the much more general theory of elliptic operators, which we will not go into.

**Proposition 7.2.2.** *Let $M$ be a compact Riemannian manifold and $u$ be an $L^2_1(M)$ solution of the equation*

$$a\Delta u + Su = \tilde{S}|u|^{\frac{n+2}{n-2}}, \tag{83}$$

*for $\tilde{S}$ some smooth function on $M$. Then $u \in C^2(M)$, and where $u$ is nonzero it is $C^\infty$.*

*Proof.* In the proof of his Theorem 3 ([T], p. 271), Trüdinger shows that if the hypotheses of the proposition hold, then $u \in L^r(M)$ for some $r > p$. We will not go into the proof of this as it is quite long, but a crude idea of how it works is that using (83) to locally bound $\left\| |u|^{2/(n-2)} \nabla u \right\|_2$ in terms of $\|u\|_p$, and then using a Sobolev embedding theorem to bound $\|u\|_r$ in terms of $\left\| |u|^{2/(n-2)} \nabla u \right\|_2 + \left\| u \right\|_p$, one can show that $\|u\|_r$ exists in small neighbourhoods and is controlled by $\|u\|_p$.

This is the crucial step in the proof, which is otherwise quite standard, and an exercise in the application of the Sobolev embedding theorem. We quote the following local elliptic regularity result, which is a partial statement of [LP], Theorem 2.4:

**Theorem 7.2.3 (Local elliptic regularity).** *Suppose that $K$ is a compact subset of $\mathbb{R}^n$, $g$ is any Riemannian metric on an open set of $\mathbb{R}^n$ containing $K$, and that $f, u$ are locally integrable functions on this set that satisfy $\Delta u = f$ weakly. If $f \in L^q_k(K)$, then $u \in L^q_{k+2}(K)$.*

We have seen that $u \in L^r(M)$ for some $r > p$. This implies that $\tilde{S}|u|^{\frac{n+2}{n-2}} - Su \in L^{(n-2)r/(n+2)}(M)$, as $S, \tilde{S}$ are bounded. Therefore by Theorem 7.2.3, $u \in L_2^{(n-2)r/(n+2)}$ locally, and so $u \in L_2^{(n-2)r/(n+2)}(M)$

as $M$ is compact. But then by the Sobolev embedding theorem (Theorem 7.2.1), $u \in L^{r_1}(M)$, where $r_1$ is defined by

$$\frac{1}{r_1} = \frac{n+2}{(n-2)r} - \frac{2}{n}.$$

Since $r > p$, it follows that $r_1 > r$, and by repeated applications of this process, we see that $u \in L^q_2(M)$ for arbitrarily large $q$. In particular, this holds for some $q > n$, and applying the second part of Theorem 7.2.1, $u \in C^1(M)$. Therefore $u \in L^q_1(M)$ for arbitrarily large $q$. But if $u \in L^q_1(M)$, then $\tilde{S}|u|^{\frac{n+2}{n-2}} - Su \in L_1^{(n-2)q/(n+2)}(M)$, as before. Applying Theorem 7.2.3 again gives that $u \in L_3^{(n-2)q/(n+2)}(M)$. Choosing $q$ such that $(n-2)q/(n+2) > n$ and applying the second part of Theorem 7.2.1, we conclude that $u \in C^2(M)$, which is one of the conclusions of the proposition.

Now there is a difficulty in continuing this process further, which is that as $|u|^{(n+2)/(n-2)}$ is no more than a $C^1$ function of $u$ at zero, we cannot conclude that if $u \in L^q_2(M)$, then $|u|^{(n+2)/(n-2)} \in L_2^{(n-2)q/(n+2)}(M)$. The next step, of proving that $u \in C^3(M)$, therefore fails. What remains to be shown, however, is that $u$ is $C^\infty$ wherever it is nonzero; this avoids the difficulty, as $|u|^{(n+2)/(n-2)}$ is a $C^\infty$ function where $u$ is nonzero.

Let $K$ be a compact subset of $M$ upon which $u$ does not vanish. We will show by induction that $u \in C^k(K)$ for every $k$, and hence that $u$ is $C^\infty$ in $K$. It has been shown that $u \in C^1(K)$, which is the first step. Suppose by induction that $u \in C^k(K)$. Then $u \in L^q_k(K)$ for arbitrarily large $q$. As $u$ is nonzero in $K$, it follows that $\tilde{S}|u|^{\frac{n+2}{n-2}} - Su \in L_k^{(n-2)q/(n+2)}(K)$, since $S, \tilde{S}$ are $C^\infty$. Therefore by Theorem 7.2.3, $u \in L_{k+2}^{(n-2)q/(n+2)}(K)$ for arbitrarily large $q$. Choosing $q$ so large that $(n-2)q/(n+2) > n$, we may apply the second part of Theorem 7.2.1 to show that $u \in C^{k+1}(K)$. This proves the inductive step, and so by induction $u \in C^\infty(K)$, and the proposition is complete. □

## 7.3. Asymptotically flat manifolds and the positive mass theorem

Let $M$ be a compact Riemannian manifold with metric $g$ and scalar curvature $S$. Then from §7.1, the condition for $\tilde{g} = \psi^{p-2}g$ to have zero scalar curvature is $a\Delta\psi + S\psi = 0$. This is a linear equation, and so is quite easy to treat. In particular, we may apply standard analytic results on the existence of Green's functions. (For any point $m \in M$, a Green's function for $a\Delta + S$ is a function $\Gamma_m$ on $M \setminus \{m\}$ such that $(a\Delta + S)\Gamma_m = \delta_m$ in the sense of distributions.) The condition for existence and positivity of these functions turns out to be the positivity of the Yamabe constant $\lambda(M)$ of §7.1.

**Proposition 7.3.1.** *Suppose $\lambda(M) > 0$. Then for each $m \in M$ the Green's function $\Gamma_m$ for $a\Delta + S$ exists, and is unique, smooth and strictly positive.*

*Proof.* This is [LP], Lemma 6.1, p. 63. □

In line with [LP], Definition 6.2, we make the following definition.

**Definition 7.3.2.** *Suppose $(M, g)$ is a compact Riemannian manifold with $\lambda(M) > 0$. For $m \in M$ define the metric $\hat{g} = \Gamma_m^{p-2} g$ on $\hat{M} = M \setminus \{m\}$. The manifold $(\hat{M}, \hat{g})$ together with the natural map from $M \setminus \{m\}$ to $\hat{M}$ is called the* **stereographic projection** *of $M$ from $m$.*

The stereographic projection of $\mathcal{S}^n$ is $\mathbb{R}^n$. The general case is similar: $\hat{M}$ is noncompact, and its noncompact end asymptotically resembles $\mathbb{R}^n$. Let us adopt the notation (used by Lee and Parker in [LP], p. 64) that $\phi = O'(|v|^q)$ means $\phi = O(|v|^q)$ and $\nabla\phi = O(|v|^{q-1})$, and $\phi = O''(|v|^q)$ means $\phi = O(|v|^q)$, $\nabla\phi = O(|v|^{q-1})$, and $\nabla\nabla\phi = O(|v|^{q-2})$. With this notation we make precise what we mean by asymptotically resembling $\mathbb{R}^n$ in the next definition, which is [LP], Definition 6.3.

**Definition 7.3.3.** *A Riemannian manifold $N$ with $C^\infty$ metric $\hat{g}$ is called* **asymptotically flat** *of order $s > 0$ if there exists a decomposition $N = N_0 \cup N_\infty$ with $N_0$ compact, and a diffeomorphism $\Phi$ from $\mathbb{R}^n \setminus \overline{B}_R(0)$ to $N_\infty$ for some $R > 0$, satisfying $\hat{g}_{ij}(x) = \delta_{ij} + O''(|x|^{-s})$ for large $x$. Here $x^i$ are the coordinates induced on $N_\infty$ by the standard coordinates on $\mathbb{R}^n$, called* **asymptotic coordinates***, and $\hat{g}_{ij}$ are the components of $\hat{g}$ with respect to $\{x^i\}$.*

The reason for making this definition is that in the next chapters, stereographic projections will be used to construct metrics of nearly constant scalar curvature on the connected sum $M' \# M''$ of two Riemannian manifolds $M', M''$. The device used is to take a stereographic projection of $M''$, choose some large ball within this that is close at its edge the metric of a ball in $\mathbb{R}^n$, shrink it very small with a homothety, and then 'glue' it into a small hole in $M'$ with its given metric. The result is a copy of $M'$ with a very tiny copy of $M''$ glued in.

To make the definition of connected sum easier, and avoid having many error terms, we shall suppose that the manifolds $M', M''$ are conformally flat around the connected sum points. In this case the asymptotic expansion of the metric $\hat{g}$ is particularly simple.

**Proposition 7.3.4.** *Suppose $(M, g)$ is a compact Riemannian manifold of dimension $n \geq 3$ with $\lambda(M) > 0$, and $m \in M$ has a neighbourhood that is conformally flat. Then the stereographic projection $(\hat{M}, \hat{g})$ of $M$ from $m$ has zero scalar curvature, is asymptotically flat of order $n-2$, and there is some real constant $\mu$ and asymptotic coordinates $\{x^i\}$, with respect to which $\hat{g}_{ij}$ satisfies*

$$\hat{g}_{ij}(x) = \psi^{p-2}(x)\delta_{ij}, \qquad \textit{where} \qquad \psi(v) = 1 + \mu|x|^{2-n} + O'(|x|^{1-n}) \qquad \textit{for large } |x|. \tag{84}$$

*Proof.* The metric $\hat{g}$ has zero scalar curvature as explained at the beginning of this section. The rest of the proposition is case (a) of Theorem 6.5 of [LP], with a little bit of extra thought to see that in the conformally flat case the error $O''(\rho^{-2})$ of their equation (6.4) is zero. □

The constant $\mu$ appearing in the proposition is proportional to an important invariant of asymptotically flat manifolds called the *mass*. From [LP], p. 79, the constant $\mu$ of Proposition 7.3.4 (called $A$ in [LP]) is related to the usual definition of mass by $4(n-1)\mu = m(\hat{g})$. The concept arose from general relativity, where one reasons that any 3-dimensional asymptotically flat gravitational system, when viewed from a long way off, should look like a point concentration of mass, that is, a Schwarzschild metric.

Thus the highest-order deviation from flatness of an asymptotically flat metric satisfying some field equation, should equal the highest order deviation from flatness of a Schwarzschild metric. But associated to the Schwarzschild metric is a number, the mass of the black hole. So by comparing highest-order terms, an invariant of asymptotically flat gravitational systems, called the mass, is defined. The definition generalizes easily to dimensions higher than three, and is given in [LP], Definition 8.2.

From the physical considerations that gave rise to the concept of mass, it was reasonable to conjecture that the mass of an asymptotically flat manifold of nonnegative scalar curvature should be nonnegative, and zero only if the manifold is isometric to $\mathbb{R}^n$ with its Euclidean metric; much effort went into proving results along these lines because of their importance in physics.

The relevance of positive mass to the Yamabe conjecture is as follows. As we saw in §7.1, given a compact Riemannian manifold $M^n$, it is easy to produce conformally equivalent metrics on $M$ with Hilbert action $Q$ close to $\lambda(\mathcal{S}^n)$, but to show the Yamabe conjecture holds for $M$ we need to prove that $\lambda(M) < \lambda(\mathcal{S}^n)$. A method developed for doing this is the *method of test functions*, and involves choosing a conformal factor, called a test function, that is concentrated around a point in the manner of §7.1, evaluating its Hilbert action and showing that it is less than $\lambda(\mathcal{S}^n)$.

This test function is constructed by taking the Green's function of a point used to make a stereographic projection, and smoothing off the function around the pole, so that the asymptotically flat end is replaced, approximately, by a large round sphere. The Hilbert action of the test function is approximately $\lambda(\mathcal{S}^n)$, and the difference is given, to highest order, in terms of geometric invariants of the stereographic projection of $M$. Thus it is that the sign of the geometric invariants is of crucial importance: with one sign, the Hilbert action of test functions with only small degrees of smoothing will be smaller than $\lambda(\mathcal{S}^n)$, and the Yamabe conjecture holds for $M$, but with the other sign, the Hilbert action will be larger than $\lambda(\mathcal{S}^n)$, and nothing is proved.

It turns out ([LP], §7 and Lemma 6.4) that if $n \geq 6$, and $M$ is not conformally flat at $m$, then the highest order term in the expansion of the Hilbert action of the test function involves the Weyl conformal curvature of $M$ at $m$, and has a favourable sign, so that the Yamabe conjecture holds for manifolds of dimension $n \geq 6$ that are not conformally flat. But in the other cases $n = 3, 4, 5$ and $n \geq 6$, $M$ conformally flat, the highest order term involves the mass, and its sign is favourable provided the mass is positive. Therefore the remaining cases of the Yamabe conjecture would be implied by a general $n$- dimensional positive mass theorem, as discussed above.

Such a theorem has been proved by Schoen and Yau, and was announced [Sc] by Schoen in 1984; one version is given in [LP] as Theorem 10.1. We shall only need the case when the metric $g$ of $M$ is conformally flat about $m$, which we state as follows:

**Theorem 7.3.5 (Schoen, Yau).** *In the situation of Proposition 7.3.4, the constant $\mu$ is greater than or equal to zero, with equality if and only if $M$ is $\mathcal{S}^n$, and its metric is conformal to the round metric.*

To finish off this section we give an analogue of the Sobolev embedding theorem, Theorem 7.2.1, for asymptotically flat manifolds.

**Theorem 7.3.6 (Sobolev embedding theorem for asymptotically flat manifolds).** *Suppose that $N$ is an asymptotically flat Riemannian manifold of dimension $n$, and that*

$$\frac{1}{r} = \frac{1}{q} - \frac{k}{n}. \tag{85}$$

*Then $L^q_k(N)$ is continuously embedded in $L^r(N)$.*

*Proof.* This can be deduced from Theorem 2.21 of [Au], p. 45. To apply the theorem, it is necessary to show that $N$ has bounded curvature and injectivity radius $\delta > 0$. But these are obvious as the asymptotic conditions ensure that the curvature becomes small and the local injectivity radius becomes large towards infinity in $N$. □

In the special case of $q = 2, k = 1$ there is a stronger, homothety-invariant statement which reads

**Theorem 7.3.7.** *Suppose that $(N, g_N)$ is a connected, asymptotically flat Riemannian manifold of dimension $n$. Then there is a constant $A$ such that*

$$\|\phi\|_p \leq A\left(\int_N |\nabla\phi|^2 dV_{g_N}\right)^{\frac{1}{2}} \qquad \textit{for } \phi \in L^2_1(N). \tag{86}$$

This is an *isoperimetric inequality*.

*Proof.* I cannot find a proof of this in the literature, but it is not really relevant to the exposition, so it has been relegated to Appendix C. □

# Chapter 8: Approximate Metrics on Connected Sums

The central results of this part of the thesis come in Chapters 9 and 10, and they will say that if a Riemannian manifold $(M, g)$ has scalar curvature $S$ sufficiently close to a constant function, where 'sufficiently close' depends upon other geometrical invariants of $M$, then $M$ admits a small conformal change giving a metric of constant scalar curvature. In this chapter, we shall consider the connected sum $M$ of two constant scalar curvature Riemannian manifolds $(M', g')$ and $(M'', g'')$, and define families of metrics $g_t$ upon $M$ with scalar curvature sufficiently close to constant, in the above sense.

Two families will be considered, defined in §§8.1 and 8.2 respectively. The first family, in §8.1, is made by taking a manifold of constant scalar curvature $\nu$, cutting out a small ball, and then gluing in an asymptotically flat manifold, homothetically shrunk very small. Now shrinking a manifold by a homothety makes its scalar curvature large, unless the scalar curvature is zero in the first place. Therefore, to control the scalar curvature on the resulting manifold, the asymptotically flat manifold to be glued in should have zero scalar curvature. Stereographic projections of positive scalar manifolds, defined in §7.3, are thus suitable candidates (indeed, the only candidates) for the asymptotically flat manifold, and so these are used.

The second family, in §8.2, is made by joining together two manifolds with metrics of constant scalar curvature $\nu$. They are joined by a small 'neck', to form a connected sum. The problem is to ensure that the scalar curvature is controlled on this neck. We do this by modelling the neck on a Riemannian manifold $N$ of zero scalar curvature, that has two asymptotically flat ends. Gluing these two ends into the two manifolds that form the connected sum reduces this case to the case of §8.1.

The chapter is finished off with two results, Lemma 8.3.1 and Lemma 8.4.1, about the families of metrics. The first gives explicit bounds for their scalar curvature, to determine how good an approximation to constant scalar curvature they are. The second shows that the Sobolev constant for a certain Sobolev embedding of function spaces can be given a bound independent of the width of the neck, for small values of this parameter; this gives an inequality relating two Sobolev norms that will be needed many times in the analysis of the next chapter.

Some diagrams giving a visual picture of the manifolds $(M, g_t)$ defined in this chapter will be found in §11.1; they have been put there for easy reference, rather than scattering them through the text. We recommend that the reader should study the diagrams before reading §§8.1 and 8.2, and at any point later in the text when it is necessary to recall what the metrics look like.

### 8.1. Combining a metric of constant and a metric of positive scalar curvature

Let $(M', g')$ be a compact Riemannian manifold of dimension $n \geq 3$ with constant scalar curvature $\nu$; for instance, $g'$ could be a Yamabe metric for some conformal class. Applying a homothety to $M'$ if necessary, we may assume that $\nu = 1, 0$ or $-1$. To make it easier to define metrics on connected sums with $M'$, assume $M'$ contains a point $m'$ that has a neighbourhood in which $g'$ is conformally flat. Then $M'$ contains a ball $B'$ about $m'$, with a diffeomorphism $\Phi'$ from $B_\delta(0) \subset \mathbb{R}^n$ to $B'$ for some $\delta$ with $0 < \delta < 1$, such that $\Phi'(0) = m'$ and $(\Phi')^*(g') = (\psi')^{p-2}h$ for some function $\psi'$ on $B_\delta(0)$, where $h$ is the standard metric on $\mathbb{R}^n$. By choosing a different conformal identification with $B_\delta(0)$ if necessary, we may suppose $\psi'(0) = 1$ and $d\psi'(0) = 0$, so that $\psi'(v) = 1 + O'(|v|^2)$ in the notation of §7.3.

Let $M''$ be a compact Riemannian manifold of the same dimension $n$ with positive scalar curvature, that is, with $\lambda(M'') > 0$. As with $M'$, suppose $M''$ contains a point $m''$, with a neighbourhood in which the metric of $M''$ is conformally flat. From Proposition 7.3.4, there is an asymptotically flat metric $g''$ of zero scalar curvature in the conformal class of $M'' \setminus \{m''\}$. There is a subset $N''$ of $M'' \setminus \{m''\}$ that is the complement of a compact set in $M'' \setminus \{m''\}$, and a diffeomorphism $\mathrm{X}'' : \mathbb{R}^n \setminus \overline{B}_R(0) \to N''$ for some $R > 0$, such that $(\mathrm{X}'')^*(g'') = \xi^{p-2}h$, where $\xi$ is a smooth function on $\mathbb{R}^n \setminus \overline{B}_R(0)$ satisfying $\xi(v) = 1 + O'(|v|^{2-n})$. By making $\delta$ smaller if necessary, we set $R = \delta^{-4}$.

A family of metrics $\{g_t : t \in (0, \delta)\}$ on $M = M' \# M''$ will now be written down. For any $t \in (0, \delta)$, define $M$ and the conformal class of $g_t$ by

$$M = \Big(M' \setminus \Phi'[\overline{B}_{t^2}(0)]\Big) \amalg \Big(M'' \setminus \big(\{m''\} \cup \mathrm{X}''[\mathbb{R}^n \setminus B_{t^{-5}}(0)]\big)\Big) \Big/ \frown_t, \tag{87}$$

where $\frown_t$ is the equivalence relation defined by

$$\Phi'[v] \quad \frown_t \quad \mathrm{X}''[t^{-6}v] \qquad \text{whenever } v \in \mathbb{R}^n \text{ and } t^2 < |v| < t. \tag{88}$$

The conformal class $[g_t]$ of $g_t$ is then given by the restriction of the conformal classes of $g'$ and $g''$ to the open sets of $M', M''$ that make up $M$; this definition makes sense because the conformal classes agree on the annulus of overlap where the two open sets are glued by $\sim_t$. Let $A_t$ be this annulus in $M$. Then $A_t$ is diffeomorphic via $\Phi'$ to the annulus $\{v \in \mathbb{R}^n : t^2 < |v| < t\}$ in $\mathbb{R}^n$.

To define a metric $g_t$ within the conformal class just given, we shall let $g_t = g'$ on the component of $M \setminus A_t$ coming from $M'$, and $g_t = t^{12}g''$ on the component of $M \setminus A_t$ coming from $M''$. So it remains to choose a conformal factor on $A_t$ itself. Using $\Phi'$, this is the same as choosing a conformal factor for the subset $\{v \in \mathbb{R}^n : t^2 < |v| < t\}$ of $\mathbb{R}^n$. On this subset, we let $(\Phi')^*(g_t) = \psi_t^{p-2}h$, where $\psi_t$ is a positive real-valued function that will shortly be defined.

The conditions for smoothness at the edges of the annulus $A_t$ are that $\psi_t(v)$ should join smoothly onto $\psi'(v)$ at $|v| = t$, and that $\psi_t(v)$ should join smoothly onto $\xi(t^{-6}v)$ at $|v| = t^2$. Such a function $\psi_t(v)$ may easily be defined using a smooth partition of unity. Choose a $C^\infty$ function $\sigma : \mathbb{R} \to [0,1]$, that is 0 for $x \geq 2$ and 1 for $x \leq 1$, and that is strictly decreasing in the interval $[1,2]$. Then define smooth functions $\beta_1, \beta_2$ by $\beta_1(v) = \sigma(\log|v|/\log t)$ and $\beta_2(v) = 1 - \beta_1(v)$ for all $v \in \mathbb{R}^n$ such that $t^2 < |v| < t$.

Finally, define the function $\psi_t$ by

$$\psi_t(v) = \beta_1(v)\psi'(v) + \beta_2(v)\xi(t^{-6}v). \tag{89}$$

This finishes the definition of the metric $g_t$ for $t \in (0,\delta)$. The reasons for defining the metrics this way — why $M'' \setminus \{m''\}$ is shrunk by a factor of $t^6$, but the cut-off functions change between radii $t$ and $t^2$, for instance — will emerge in §8.3, where we show that for this definition the scalar curvature of $g_t$ is close to the constant function $\nu$ in the $L^{n/2}$ norm.

### 8.2. Combining two metrics of constant scalar curvature $\nu$

The previous section showed how to shrink an asymptotically flat Riemannian manifold of zero scalar curvature by a homothety, and then glue it into a Riemannian manifold of constant scalar curvature $\nu$, to get a Riemannian metric on the connected sum. There we dealt with the case of gluing a manifold with a single asymptotically flat end into another manifold at a single point, but the definition clearly works just as well for gluing a collection of asymptotically flat manifolds, or a single manifold with several asymptotically flat ends, into a collection of manifolds of constant scalar curvature.

In this section a family of metrics will be defined on the connected sum of two Riemannian manifolds with constant scalar curvature $\nu$. We shall do this by considering a zero scalar curvature Riemannian manifold that has two asymptotically flat ends, and gluing one end into each of the constant scalar curvature manifolds; the new manifold will form the 'neck' in between.

Let $N$ be the Riemannian manifold $\mathbb{R}^n \setminus \{0\}$ with the metric $g_N = (1+|v|^{-(n-2)})^{p-2}h$. Then, by Definition 7.3.3, $N$ is clearly asymptotically flat of order $n-2$ (neglecting the condition $N_0$ compact), taking the diffeomorphism $\Phi$ to be the identity upon $\mathbb{R}^n \setminus \overline{B}_R(0) \subset N$, for some $R > 1$. But the involution $v \mapsto v/|v|^2$ of $N$ may easily be shown to preserve the metric. Therefore the other noncompact end of $N$, close to zero in $\mathbb{R}^n$, is also asymptotically flat of order $n-2$, because it is identified with the first end by this involution. Also, as $\Delta|v|^{-(n-2)} = 0$, equation (73) shows that $N$ has zero scalar curvature.

We shall use the gluing method of the previous section to glue each of the two asymptotically flat ends of $N$ into a manifold of constant scalar curvature. Let $(M', g')$ and $(M'', g'')$ be two Riemannian manifolds of dimension $n$, with constant scalar curvature $\nu$; applying homotheties if necessary we shall assume that $\nu = 1, 0$ or $-1$. Let $M = M'\#M''$. A family of metrics $\{g_t : t \in (0,\delta)\}$ on $M$ will now be defined, such that when $t$ is small, $g_t$ resembles the union of $M'$ and $M''$ with their metrics $g'$ and $g''$, joined by a small 'neck' of approximate radius $t^6$, which is modelled upon $N$ with metric $t^{12}g_N$. It will be done quite briefly, because the treatment simply generalizes the definition of the previous section.

As in §8.1, suppose that $M', M''$ contain points $m', m''$ respectively, having neighbourhoods in which $g', g''$ are conformally flat. Thus $M', M''$ contain open balls $B', B''$ with diffeomorphisms $\Phi', \Phi''$ from $B_\delta(0)$ in $\mathbb{R}^n$ to $B', B''$, such that $\Phi'(0) = m', \Phi''(0) = m''$ and $(\Phi')^*(g') = (\psi')^{p-2}h$, $(\Phi'')^*(g'') = (\psi'')^{p-2}h$, for some functions $\psi', \psi''$ on $B_\delta(0)$. By choosing different conformal identifications with $B_\delta(0)$ if necessary, we may suppose that $\psi'(0) = \psi''(0) = 1$ and $d\psi'(0) = d\psi''(0) = 0$, so that $\psi'(v) = 1 + O'(|v|^2)$ and $\psi''(v) = 1 + O'(|v|^2)$.

For any $t \in (0,\delta)$, define $M$ and the conformal class of $g_t$ by

$$M = \left(M' \setminus \Phi'[\overline{B}_{t^2}(0)]\right) \amalg \left(M'' \setminus \Phi''[\overline{B}_{t^2}(0)]\right) \amalg \left\{v \in N : t^5 < |v| < t^{-5}\right\} \Big/ \frown_t, \tag{90}$$

where $\frown_t$ is the equivalence relation defined by

$$\begin{aligned} \Phi'\left[t^6 v\right] \;&\frown_t\; v \qquad \text{if } v \in N \text{ and } t^{-4} < |v| < t^{-5}, \text{ and} \\ \Phi''\left[\frac{t^6 v}{|v|^2}\right] \;&\frown_t\; v \qquad \text{if } v \in N \text{ and } t^5 < |v| < t^4. \end{aligned} \tag{91}$$

The conformal class $[g_t]$ of $g_t$ is then given by the restriction of the conformal classes of $g', g''$ and $g_N$ to the open sets of $M', M''$ and $N$ that make up $M$; this definition makes sense because the conformal classes agree on the annuli of overlap where the four open sets are glued by $\sim_t$. Let $A_t$ be this region of gluing in $M$. Then $A_t$ is diffeomorphic via $\Phi'$ and $\Phi''$ to two copies of the annulus $\{v \in \mathbb{R}^n : t^2 < |v| < t\}$.

To define a metric $g_t$ within this conformal class, let $g_t = g'$ on the component of $M \setminus A_t$ coming from $M'$, $g_t = g''$ on the component of $M \setminus A_t$ coming from $M''$, and $g_t = t^{12} g_N$ on the component on $M \setminus A_t$ coming from $N$. So it remains to choose a conformal factor on $A_t$ itself. Using $\Phi'$ and $\Phi''$, this is the same as choosing a conformal factor for two copies of the subset $\{v \in \mathbb{R}^n : t^2 < |v| < t\}$ of $\mathbb{R}^n$.

As in §8.1, define the smooth partition of unity $\beta_1, \beta_2$ on $A_t$, and define the function $\psi_t$ by $\psi_t(v) = \beta_1(v)\psi'(v) + \beta_2(v)(1 + t^{6(n-2)}|v|^{-(n-2)})$ on the component of $A_t$ coming from $M'$, and $\psi_t(v) = \beta_1(v)\psi''(v) + \beta_2(v)(1 + t^{6(n-2)}|v|^{-(n-2)})$ on the component coming from $M''$. Here $\psi_t$ is thought of as a function on $A_t$, which by abuse of notation is identified by $\Phi'$ and $\Phi''$ with two disjoint copies of $\{v \in \mathbb{R}^n : t^2 < |v| < t\}$. Now let $g_t$ be equal to $\psi_t^{p-2} h$ in $A_t$, where $h$ is the push-forward to $A_t$ by $\Phi', \Phi''$ of the standard metric on $\{v \in \mathbb{R}^n : t^2 < |v| < t\}$. This completes the definition of $g_t$.

### 8.3. Estimating the scalar curvature of the metrics $g_t$

Our goal in the last two sections in defining the metrics $g_t$, was to make the scalar curvature of $g_t$ approach the constant value $\nu$ in the $L^{n/2}$ norm as $t \to 0$. In Chapters 9 and 10 it will be shown that when the scalar curvature is sufficiently close to $\nu$, a small conformal change exists to make the scalar curvature constant (and equal to $\nu$, unless $\nu = 0$).

So the important thing in defining the metrics $g_t$ is to control their scalar curvatures. This is the point of the next lemma.

**Lemma 8.3.1.** *Let $\{g_t : t \in (0, \delta)\}$ be one of the families of metrics defined on the manifold $M = M' \# M''$ in §8.1 and §8.2. Let the scalar curvature of the metric $g_t$ be $\nu - \epsilon_t$. Then there exist constants $Y, Z > 0$ such that $|\epsilon_t| \le Y$ and $\|\epsilon_t\|_{n/2} \le Z t^2$.*

*Proof.* The proof will be given for the metrics $g_t$ of §8.1 only, the (trivial) modifications for the case of §8.2 being left to the reader. We first derive an expression for $\epsilon_t$. Outside the annulus $A_t$, the metric $g_t$ has scalar curvature $\nu$ and 0 on the regions coming from $M'$ and $M''$ respectively. So it is only necessary to evaluate the scalar curvature in $A_t$. Calculating with (73) gives

$$\nu - \epsilon_t(v) = \nu\beta_1(v)(\psi'(v))^{(n+2)/(n-2)}\psi_t^{-(n+2)/(n-2)}(v) + \psi_t^{-(n+2)/(n-2)}(v)\big(\Delta\beta_1(v)\big)\big(\psi'(v) - \xi(t^{-6}v)\big)$$
$$-2\psi_t^{-(n+2)/(n-2)}(v)\big(\nabla\beta_1(v)\big)\cdot\big(\nabla\big(\psi'(v) - \xi(t^{-6}v)\big)\big), \tag{92}$$

using the fact that $\beta_1 + \beta_2 = 1$.

As $\psi'(v) = 1+O'(|v|^2)$, $\xi(v) = 1+O'(|v|^{2-n})$, and $t^2 \leq |v| \leq 1$, it follows easily that $\psi'(v)-\xi(t^{-6}v) = O'(|v|^2)$. The reason for choosing to scale $g''$ by a factor of $t^{12}$ whilst making $\beta_1$ change between $t^2$ and $t$ is to make this estimate work — the first power has to be as high as 12 to work in dimension 3. Substituting it into (92) gives that

$$|\epsilon_t| \leq |\nu|\cdot\Big|\beta_1(\psi')^{(n+2)/(n-2)}\psi_t^{-(n+2)/(n-2)} - 1\Big| + \psi_t^{-(n+2)/(n-2)}\cdot\Big\{\big|\nabla\beta_1\big|O(|v|) + \big|\Delta\beta_1\big|O\big(|v|^2\big)\Big\}.$$

Moreover, using a lower bound for $\psi'$, we can easily show that on the subannulus $t^2 \leq |v| \leq t$, the estimate $|\psi_t(v)| \geq C_0 > 0$ holds for some constant $C_0$ and any $t \in (0,\delta)$. Using this expression to get rid of the $\psi_t$ terms on the right hand side, and an upper bound on $\psi'$, it can be seen that

$$|\epsilon_t| = O(1) + |\nabla\beta_1|O(|v|) + |\Delta\beta_1|O\big(|v|^2\big) \tag{93}$$

on the subannulus $t^2 \leq |v| \leq t$. But

$$|\nabla\beta_1| = \frac{\big|\frac{d\tau_1}{dx}\big|}{|v|\log t} = O\big(|v|^{-1}\big), \tag{94}$$

and in a similar way $|\Delta\beta_1| = O\big(|v|^{-2}\big)$. Substituting into (93), we find that for all $t$ with $0 < t < \delta$, $|\epsilon_t| \leq Y$ on the subannulus $t^2 \leq |v| \leq t$, for some constant $Y \geq |\nu|$.

We conclude that $|\epsilon_t|$ is bounded by $Y$ on the annulus $A_t$ embedded in $M$, and outside the annulus $A_t$, $\epsilon_t = 0$ on the component coming from $M'$, and $|\epsilon_t| = |\nu| \leq Y$ on the component coming from $M''$. Therefore $|\epsilon_t| \leq Y$, which establishes the first part of the lemma. To prove the second part, observe that using the estimates on $\psi_t$ above, the volume of the support of $\epsilon_t$ is easily shown to be $\leq C_1 t^n$, for some constant $C_1 > 0$. So $\|\epsilon\|_{n/2} \leq Zt^2$, where $Z = YC_1^{2/n}$. □

## 8.4. A uniform bound for a Sobolev embedding

If $M$ is a compact Riemannian manifold of dimension $n$, then by the Sobolev embedding theorem Theorem 7.2.1, $L_1^2(M)$ is continuously embedded in $L^{\frac{2n}{n-2}}(M) = L^p(M)$. (Notice that this is a borderline case of the theorem.) This means that $L^2_1(M) \subset L^p(M)$, and there exists a real constant $A$ such that $\|\phi\|_p \le A\|\phi\|_{2,1}$ for all $\phi \in L^2_1(M)$. The constant $A$ depends on the metric on $M$.

Now for the work that follows this inequality will be needed for the families of metrics $\{g_t : t \in (0,\delta)\}$ of §8.1 and §8.2, but for a constant $A$ that is independent of $t$. Therefore we must prove the following lemma:

**Lemma 8.4.1.** *Let $\{g_t : t \in (0,\delta)\}$ be one of the families of metrics defined on the manifold $M = M'\#M''$ in §8.1 and §8.2. Then there exist constants $A > 0$ and $\zeta$, $0 < \zeta < \delta$, such that ${}^t\|\phi\|_p \le A\cdot{}^t\|\phi\|_{2,1}$ whenever $\phi \in L^2_1(M)$ and $0 < t \le \zeta$. Here norms are taken with respect to $g_t$.*

*Proof.* The proof is by 'gluing' inequalities upon the component manifolds that were used to make $M$. By the Sobolev embedding theorem (Theorem 7.2.1), there exists a constant $D_0$ such that on $M'$, $\|\phi\|_p \le D_0\|\phi\|_{2,1}$ for $\phi \in L^2_1(M')$, where norms are with respect to $g'$, and for the case of §8.2, the inequality $\|\phi\|_p \le D_0\|\phi\|_{2,1}$ for $\phi \in L^2_1(M'')$ also holds on $M''$, where norms are with respect to $g''$.

We also need a Sobolev-type inequality applying to the asymptotically-flat manifold glued into $M'$, or $M'$ and $M''$, to control functions upon the corresponding part of $M$. For the case of §8.1, this asymptotically-flat manifold is $(M'' \setminus \{m''\}, g'')$, and by Theorem 7.3.7 the following Sobolev inequality holds:

$$\|\phi\|_p \le D_1\left(\int_{M''\setminus\{m''\}} |\nabla\phi|^2 dV_{g''}\right)^{\frac{1}{2}} \qquad \text{for } \phi \in L^2_1\big(M'' \setminus \{m''\}\big). \tag{95}$$

For the case of §8.2, this asymptotically-flat manifold is the 'neck' manifold $N$, and by Theorem 7.3.7 a Sobolev inequality of the form

$$\|\phi\|_p \le D_1\left(\int_N |\nabla\phi|^2 dV_{g_N}\right)^{\frac{1}{2}} \qquad \text{for } \phi \in L^2_1(N) \tag{96}$$

must hold on $N$. It can easily be verified that these inequalities are invariant under homotheties. So (95) and (96) apply to $L^2_1$ functions on the rescaled metric used to construct $g_t$. Let $D_2 = \max(D_0, D_1)$. Then $D_2$ is a bound for the embedding of $L^2_1$ into $L^p$ for each of the Riemannian manifolds that were glued together with a partition of unity to make $(M, g_t)$.

The lemma will now be completed for the case of the metrics of §8.1 only; for the case of §8.2, the method is just the same from this point, and is left to the reader. The volume form of $(M, g_t)$ on the subset $A_t$ is $\psi_t^{2n/(n-2)} dV_h$, and for $X, Y \ge 0$, the inequality $(X+Y)^{2n/(n-2)} \le 2^{(n+2)/(n-2)}(X^{2n/(n-2)}+Y^{2n/(n-2)})$

holds. Substituting (89) into the volume form on the annulus, and using this inequality to divide it into terms upon the two component manifolds, we conclude that

$$\text{vol. form of } (M, g_t) \leq 2^{\frac{n+2}{n-2}} \big(\beta_1^p \cdot \text{vol. form of } (M', g') + \beta_2^p \cdot \text{vol. form of } (M'' \setminus \{m''\}, g'')\big),$$

in which we have abused notation by identifying open sets of $M$ with the open sets of $M', M''$ used to define $M$ and $g_t$.

Therefore

$$\int_M |\phi|^p dV_{g_t} \leq 2^{\frac{n+2}{n-2}} \left( \int_{M'} \beta_1^p |\phi|^p dV_{g'} + \int_{M''\setminus\{m''\}} \beta_2^p |\phi|^p dV_{g''} \right) \qquad \text{for } \phi \in L^p(M),$$

where we have abused notation in a similar way, by identifying functions on $M$ with functions on open sets of $M'$ and $M''$, that must be extended by zero to the rest of the manifolds. Taking $p^{\text{th}}$ roots gives that

$${}^t\|\phi\|_p \leq 2^{\frac{n+2}{2n}} \big( {}^{M'}\|\beta_1\phi\|_p + {}^{M''\setminus\{m''\}}\|\beta_2\phi\|_p \big);$$

here ${}^{M'}\|.\|$ is a shorthand for norms on $M'$ taken with respect to $g'$, and so on.

But we have already shown that $D_2$ bounds the embedding of $L^2_1$ into $L^p$ for the two manifolds and metrics. So for $\phi \in L^2_1(M)$,

$$\begin{aligned} {}^t\|\phi\|_p &\leq 2^{\frac{n+2}{2n}} D_2 \big( {}^{M'}\|\beta_1\phi\|_{2,1} + {}^{M''\setminus\{m''\}}\|\beta_2\phi\|_{2,1} \big) \\ &\leq 2^{\frac{n+2}{2n}} D_2 \big( D_3 \cdot {}^t\|\phi\|_{2,1} + {}^t\|\phi|\nabla\beta_1|\|_2 + {}^t\|\phi|\nabla\beta_2|\|_2 \big), \end{aligned} \tag{97}$$

where $D_3 \geq 1$ is some constant such that $\beta_1^2 dV_{g'} + \beta_2^2 dV_{g''} \leq D_3 dV_{g_t}$ in the identified subsets of $M', M''$ and $M$ for all $t \in (0, \delta)$. Such a $D_3$ can be shown to exist by bounding $dV_{g'}$ and $dV_{g''}$ above and below by constant multiples of $dV_h$ independently of $t$, using estimates on $\psi'$ and $\psi''$.

By Hölder's inequality, ${}^t\|\phi|\nabla\beta_i|\|_2 \leq {}^t\|\phi\|_{2n/(n-2)} \cdot {}^t\big\||\nabla\beta_i|\big\|_n$. Substituting this into (97) and rearranging gives

$${}^t\|\phi\|_p \Big( 1 - 2^{\frac{n+2}{2n}} D_2 \big( {}^t\big\||\nabla\beta_1|\big\|_n + {}^t\big\||\nabla\beta_2|\big\|_n \big) \Big) \leq 2^{\frac{n+2}{2n}} D_2 D_3 \cdot {}^t\|\phi\|_{2,1}. \tag{98}$$

Now if the $\beta_i$ satisfy $\big\||\nabla\beta_i|\big\|_n \leq (4 \cdot 2^{(n+2)/2n} \cdot D_2)^{-1}$, then (98) implies that

$$
{}^t\|\phi\|_p \le 2^{\frac{3n+2}{2n}} D_2 D_3 \cdot {}^t\|\phi\|_{2,1} \qquad \text{for all } \phi \in L^2_1(M), \tag{99}
$$

and setting $A = 2^{(3n+2)/2n} \cdot D_2 D_3$, the lemma is complete.

Therefore it remains only to show that $\big\||\nabla\beta_i|\big\|_n \le (4 \cdot 2^{(n+2)/2n} \cdot D_2)^{-1}$ for $t$ sufficiently small. As the $L^n$ norm is conformally invariant on 1-forms it can be evaluated with respect to the standard metric on $\mathbb{R}^n$, and since $\beta_i$ depends only on $r = |v|$ we have

$$
\begin{aligned}
\big\||\nabla\beta_i|\big\|_n &= \left(D_4 \cdot \int_{t^2}^{t} r^{n-1} \left|\frac{d\beta_i}{dr}\right|^n dr\right)^{\frac{1}{n}} \\
&= \left(D_4 \cdot \int_{2\log t}^{\log t} \left|\frac{d\beta_i}{d(\log r)}\right|^n d(\log r)\right)^{\frac{1}{n}} \\
&= \left(D_4 \cdot \int_{1}^{2} \left|\frac{1}{\log t} \cdot \frac{d\tau_i}{dx}\right|^n |\log t| dx\right)^{\frac{1}{n}} \\
&= |\log t|^{(1-n)/n} \cdot \left(D_4 \cdot \int_{1}^{2} \left|\frac{d\tau_i}{dx}\right|^n dx\right)^{\frac{1}{n}},
\end{aligned} \tag{100}
$$

where $D_4$ is the volume of the $(n-1)$- dimensional sphere of radius 1. So $\big\||\nabla\beta_i|\big\|_n$ is proportional to $|\log t|^{(1-n)/n}$ with constant of proportionality depending only on $n$, and there exists a constant $\zeta$ with $0 < \zeta < \delta$, such that $\big\||\nabla\beta_i|\big\|_n \le (4 \cdot 2^{(n+2)/2n} \cdot D_2)^{-1}$ when $0 < t \le \zeta$. This completes the proof of the lemma. □

# Chapter 9: Constant Positive and Negative Scalar Curvature on Connected Sums

Let $M$ be the manifold of §8.1 or §8.2 with one of the metrics $g_t$ defined there, and denote its scalar curvature by $S$. As in §7.1, a conformal change to $\tilde{g}_t = \psi^{p-2} g_t$ may be made, and the condition for $\tilde{g}_t$ to have constant scalar curvature $\nu$ is the Yamabe equation

$$a\Delta\psi + S\psi = \nu|\psi|^{p-1}. \tag{101}$$

Now the metrics $g_t$ have scalar curvature close to $\nu$, so let $S = \nu - \epsilon$; then by Lemma 8.3.1, $\|\epsilon\|_{n/2} \leq Zt^2$. Also we would like the conformal change to be close to 1, so put $\psi = 1 + \phi$, where we aim to make $\phi$ small. Substituting both of these changes into (101) gives

$$a\Delta\phi - \nu b\phi = \epsilon + \epsilon\phi + \nu f(\phi), \tag{102}$$

where $b = 4/(n-2)$ and $f(t) = |1+t|^{(n+2)/(n-2)} - 1 - (n+2)t/(n-2)$.

In this chapter, we shall suppose that $\nu = \pm 1$, as the zero scalar curvature case requires different analytic treatment and will be considered in Chapter 10. Equation (102) has been written so that on the left is a linear operator $a\Delta - \nu b$ applied to $\phi$, and on the right are the 'error terms'. We think of them as error terms because our approach to them will be simply to try and ensure that they are small, and the precise form of the functions on the right hand side will matter less than crude limits on their size.

The method of §9.1 is to define by induction a sequence $\{\phi_i\}_{i=0}^{\infty}$ of functions in $L_1^2(M)$ by $\phi_0 = 0$, and

$$a\Delta\phi_i - \nu b\phi_i = \epsilon + \epsilon\phi_{i-1} + \nu f(\phi_{i-1}). \tag{103}$$

This definition depends upon being able to invert the operator $a\Delta - \nu b$. The question of invertibility and the size of the inverse is deferred to §9.2 and §9.3; the case $\nu = -1$ is easy, but the case $\nu = 1$ requires some discussion. Given this invertibility, the argument proceeds by showing that provided $\epsilon$ is sufficiently small, the sequence $\{\phi_i\}_{i=0}^{\infty}$ converges in $L_1^2(M)$; the limit $\phi$ is therefore a weak $L_1^2$ solution to (102).

The condition that $\epsilon$ should be small is because of the nonlinearity of the equation, as when $\phi_{i-1}$ is small, $f(\phi_{i-1})$ is very small, but when $\phi_{i-1}$ is large, $f(\phi_{i-1})$ is very large and the sequence cannot be controlled. The argument is finished off by showing that $\phi$ is in fact smooth and $\psi = 1+\phi$ is positive, and therefore the conformally changed metric $\tilde{g}_t$, with which we began, is a Riemannian metric with constant scalar curvature $\nu$.

In §§9.2 and 9.3 we state the existence theorems for constant positive and negative scalar curvature respectively on connected sums, the main results of this chapter. The work done in these sections is to apply the abstract result of §9.1 to the specific cases of the metrics of §§8.1 and 8.2, for scalar curvature $-1$ and 1. Note that §9.3 produces three distinct metrics of scalar curvature 1 in the conformal class of each suitable connected sum of positive scalar curvature manifolds, in contrast to the negative scalar curvature case, where any metric of scalar curvature $-1$ is unique in its conformal class.

## 9.1. The main result

Fix $\nu$ equal to 1 or $-1$, and let $M$ be a compact Riemannian manifold of dimension $n$, with metric $g$. Let $A, B, X$ and $Y$ be positive constants, to be chosen later. We shall now write down four properties, which $(M, g)$ may or may not satisfy:

**Property 1.** *The volume of $M$ satisfies $X/2 \leq \mathrm{vol}(M) \leq X$.*

**Property 2.** *Let the scalar curvature of $g$ be $\nu - \epsilon$. Then $|\epsilon| \leq Y$.*

**Property 3.** *Whenever $\phi \in L^2_1(M)$, $\phi \in L^p(M)$, and $\|\phi\|_p \leq A\|\phi\|_{2,1}$.*

**Property 4.** *For every $\xi \in L^{2n/(n+2)}(M)$, there exists a unique $\phi \in L^2_1(M)$ such that $a\Delta\phi - \nu b\phi = \xi$ (that is, $\Delta\phi$ exists in the weak sense, and satisfies the equation). Moreover, $\|\phi\|_{2,1} \leq B\|\xi\|_{2n/(n+2)}$.*

We of course think of $(M, g)$ as being the manifold $M$ of §8.1 or §8.2, with one of the metrics $g_t$ defined there. Then Property 1 is clear from the definitions, Property 2 comes from Lemma 8.3.1, Property 3 from Lemma 8.4.1, and Property 4 remains to be proved. In terms of these properties, we state the next result, which is the core of the analysis of this chapter.

**Theorem 9.1.1.** *Let $A, B, X$ and $Y$ be given positive constants, and $n \geq 3$ a fixed dimension. Then there exist positive constants $W, c$ depending only upon $A, B, X, Y$ and $n$, such that if $(M, g)$ satisfies Properties 1–4 above and $\|\epsilon\|_{n/2} \leq c$, then the metric $g$ admits a smooth conformal rescaling to $\tilde{g} = (1+\phi)^{p-2}g$,* *which is a nonsingular Riemannian metric of constant scalar curvature $\nu$. Moreover, $\phi$ satisfies $\|\phi\|_{2,1} \le W\|\epsilon\|_{n/2}$.*

*Proof.* Suppose that $(M, g)$ satisfies Properties 1–4 above. Define a map $T : L^2_1(M) \to L^2_1(M)$ by $T\eta = \xi$, where

$$a\Delta\xi - \nu b\xi = \epsilon + \epsilon\eta + \nu f(\eta). \tag{104}$$

By Property 4, $\xi$ exists and is unique, provided that the right hand side is in $L^{2n/(n+2)}(M)$. So it must be shown that if $\eta \in L^2_1(M)$, then $\epsilon + \epsilon\eta + \nu f(\eta) \in L^{2n/(n+2)}(M)$. Now $\epsilon \in L^{n/2}(M)$ implies that $\epsilon \in L^{2n/(n+2)}(M)$; by the Sobolev embedding theorem $\eta \in L^{2n/(n-2)}(M)$, and as $\epsilon \in L^{n/2}(M)$ it follows that $\epsilon\eta \in L^{2n/(n+2)}(M)$. Thus the first two terms are in $L^{2n/(n+2)}(M)$. For the third term, as $\eta \in L^{2n/(n-2)}(M)$, $1 + \eta$ is too, and $(1 + \eta)^{(n+2)/(n-2)} \in L^{2n/(n+2)}(M)$. This deals with the first part of $f(\eta)$, and the last two parts are trivially in $L^{2n/(n+2)}(M)$. Therefore the right hand of (104) is in $L^{2n/(n+2)}(M)$, and the map $T$ is well defined.

Now define a sequence $\{\phi_i\}_{i=0}^{\infty}$ of elements of $L^2_1(M)$ by $\phi_i = T^i(0)$. Our first goal is to prove that if $\|\epsilon\|_{n/2}$ is sufficiently small, then this sequence converges in $L^2_1(M)$. Setting $\phi$ as the limit of the sequence, (104) implies that $\phi$ will satisfy (102), as we would like. This will be achieved via the next lemma.

**Lemma 9.1.2.** *Suppose $L$ is a Banach space with norm $\|.\|$, and $T : L \to L$ is a map satisfying $\|T(0)\| \le F_0 s$ and*

$$\|T(v) - T(w)\| \le \|v - w\|\Big\{F_1 s + F_2\big(\|v\| + \|w\|\big) + F_3\big(\|v\|^{\frac{4}{n-2}} + \|w\|^{\frac{4}{n-2}}\big)\Big\} \qquad \textit{for all } v, w \in L, \tag{105}$$

*where $F_0, F_1, F_2, F_3$ and $s$ are positive constants. Then there exists a constant $W > 0$ depending only on $F_0, F_1, F_2, F_3$ and $n$, such that if $s$ is sufficiently small, then the sequence $\{\phi_i\}_{i=0}^{\infty}$ defined by $\phi_i = T^i(0)$ converges to a limit $\phi$, satisfying $\|\phi\| \le Ws$.*

*Proof.* Putting $v = \phi_{i-1}$ and $w = 0$ into (105) gives $\|\phi_i - T(0)\| \le F_1 s\|\phi_{i-1}\| + F_2\|\phi_{i-1}\|^2 + F_3\|\phi_{i-1}\|^{\frac{n+2}{n-2}}$, and as $\|T(0)\| \le F_0 s$ this implies that $\|\phi_i\| \le \chi(\|\phi_{i-1}\|)$, where $\chi(x) = F_0 s + F_1 s x + F_2 x^2 + F_3 x^{(n+2)/(n-2)}$.

From the form of this equation, it is clear after a little thought, or by drawing a picture, that there exists $W > 0$ such that whenever $s$ is small enough, there exists an $x$ for which $0 < x \le Ws$ and $2\chi(x) = x$. This constant $W$ obviously depends only on $F_0, F_1, F_2, F_3$ and $n$. Suppose that $s$ is small enough, so that such an $x$ exists. Now $\phi_0 = 0$, so that $\|\phi_0\| \le x$, and if $\|\phi_{i-1}\| \le x$ then $\|\phi_i\| \le \chi(\|\phi_{i-1}\|) \le \chi(x) \le x$. Thus by induction, $\|\phi_i\| \le x$ for all $i$.

Put $v = \phi_i$ and $w = \phi_{i-1}$ in (105). This gives $\|\phi_{i+1} - \phi_i\| \le \|\phi_i - \phi_{i-1}\| \cdot \big(F_1 s + 2F_2 x + 2F_3 x^{4/(n-2)}\big)$, using the inequality $\|\phi_i\| \le x$ that we have just proved. Dividing the equation $x = 2\chi(x)$ by $x$ and subtracting some terms it follows that $1 > F_1 s + 2F_2 x + 2F_3 x^{4/(n-2)} > 0$, and so $\{\phi_i\}_{i=0}^\infty$ converges, by comparison with a geometric series. Let the limit of the sequence be $\phi$. Then as $\|\phi_i\| \le x \le Ws$ for all $i$, by continuity $\phi$ also satisfies $\|\phi\| \le Ws$. □

To apply this lemma we must show that the operator $T : L^2_1(M) \to L^2_1(M)$ defined above satisfies the hypotheses of the lemma. Let $s = \|\epsilon\|_{n/2}$. We will now define constants $F_0, F_1, F_2, F_3$ depending only upon $A, B, X$ and $n$, such that these hypotheses are satisfied.

Putting $\eta = 0$ in (104), and applying Properties 1 and 4, we see that $\|T(0)\|_{2,1} \le B\|\epsilon\|_{2n/(n+2)} \le B\,\mathrm{vol}(M)^{(n-2)/2n}\|\epsilon\|_{n/2} \le BX^{(n-2)/2n}s$, so let $F_0 = BX^{(n-2)/2n}$. From the definition of $f$, it can be seen that

$$\big|f(x) - f(y)\big| \le \big|x - y\big| \cdot \Big(F_4\big(|x| + |y|\big) + F_5\big(|x|^{\frac{4}{n-2}} + |y|^{\frac{4}{n-2}}\big)\Big), \tag{106}$$

where $F_4, F_5$ are constants depending only on $n$, and $F_4 = 0$ if $n \ge 6$. Let $\eta_1, \eta_2 \in L^2_1(M)$, and let $T(\eta_i) = \xi_i$. Then taking the difference of (104) with itself for $i = 1, 2$ we get

$$(a\Delta - \nu b)(\xi_1 - \xi_2) = \epsilon \cdot (\eta_1 - \eta_2) + \nu\big(f(\eta_1) - f(\eta_2)\big).$$

Applying Property 4 gives

$$\begin{aligned}
\|\xi_1 - \xi_2\|_{2,1} &\le B\Big(\|\epsilon \cdot (\eta_1 - \eta_2)\|_{2n/(n+2)} + |\nu|\big\|f(\eta_1) - f(\eta_2)\big\|_{2n/(n+2)}\Big) \\
&\le B\|\eta_1 - \eta_2\|_{2n/(n-2)} \cdot \Big(\|\epsilon\|_{n/2} + F_4\big(\|\eta_1\|_{n/2} + \|\eta_2\|_{n/2}\big) \\
&\qquad + F_5\big(\|\eta_1\|^{4/(n-2)}_{2n/(n-2)} + \|\eta_2\|^{4/(n-2)}_{2n/(n-2)}\big)\Big) \\
&\le \|\eta_1 - \eta_2\|_{2,1} \cdot \Big(ABs + ABF_4\,\mathrm{vol}(M)^{(6-n)/2n}\big(\|\eta_1\|_{2n/(n-2)} + \|\eta_2\|_{2n/(n-2)}\big) \\
&\qquad + A^{(n+2)/(n-2)}BF_5\big(\|\eta_1\|^{4/(n-2)}_{2,1} + \|\eta_2\|^{4/(n-2)}_{2,1}\big)\Big) \\
&\le \|\eta_1 - \eta_2\|_{2,1} \cdot \Big(ABs + A^2BF_4X^{(6-n)/2n}\big(\|\eta_1\|_{2,1} + \|\eta_2\|_{2,1}\big) \\
&\qquad + A^{(n+2)/(n-2)}BF_5\big(\|\eta_1\|^{4/(n-2)}_{2,1} + \|\eta_2\|^{4/(n-2)}_{2,1}\big)\Big) \\
&\le \|\eta_1 - \eta_2\|_{2,1} \cdot \Big(F_1 s + F_2\big(\|\eta_1\|_{2,1} + \|\eta_2\|_{2,1}\big) + F_3\big(\|\eta_1\|^{4/(n-2)}_{2,1} + \|\eta_2\|^{4/(n-2)}_{2,1}\big)\Big),
\end{aligned} \tag{107}$$

where $F_1 = AB$, $F_2 = A^2BF_4X^{(6-n)/2n}$ and $F_3 = A^{(n+2)/(n-2)}BF_5$. Here we have freely used Hölder's inequality, Property 1, and Property 3, between the first and second lines we have used (106), and between the second and third lines we have used the expression $\|\eta\|_r \leq \|\eta\|_s(\mathrm{vol}\, M)^{\frac{s-r}{rs}}$ when $1 \leq r < s$ and $\eta \in L^r(M) \subset L^s(M)$, and the fact that $F_4 = 0$ if $n \geq 6$ to ensure that $n/2 < 2n/(n-2)$ whenever the term involving $F_4$ does not vanish.

This inequality is (105) for the operator $T : L^2_1(M) \to L^2_1(M)$. So putting $L = L^2_1(M)$ and applying Lemma 9.1.2, there is a constant $W > 0$ depending only on $F_0, F_1, F_2, F_3$ and $n$, such that if $\|\epsilon\|_{n/2}$ is sufficiently small, then the sequence $\{\phi_i\}_{i=0}^{\infty}$ defined by $\phi_i = T^i(0)$ converges to a limit $\phi$, satisfying $\|\phi\| \leq W\|\epsilon\|_{n/2}$. Now $W$ depends only on $F_0, \ldots, F_3$ and $n$, and these depend only on $n, A, B$ and $X$, so $W$ depends only on $n, A, B$ and $X$. Since $\phi_i = T(\phi_{i-1})$ and $T$ is continuous, taking the limit gives $\phi = T(\phi)$, so (104) shows that $\phi$ satisfies (102) weakly. Thus we have proved the following lemma:

**Lemma 9.1.3.** *There is a constant $W > 0$ depending only on $n, A, B$ and $X$, such that if $\|\epsilon\|_{n/2}$ is sufficiently small, then there exists $\phi \in L^2_1(M)$ satisfying (102) weakly, with $\|\phi\|_{2,1} \leq W\|\epsilon\|_{n/2}$.*

It has been shown that weak solutions $\phi$ of (102) do exist for small $\|\epsilon\|_{n/2}$, and for these $\psi = 1 + \phi$ gives a weak solution of (74). However, for $\tilde{g} = \psi^{p-2}g$ to be a sensible metric, it is essential that $\psi$ be a smooth, positive function. As $\psi \in L^2_1(M)$ and satisfies (74) weakly, by Proposition 7.2.2 it follows that $\psi \in C^2(M)$, and is $C^\infty$ wherever it is nonzero. Therefore it remains only to prove that $\psi$ is strictly positive.

Examples of manifolds (with negative scalar curvature) can be produced for which the Yamabe equation (74) does admit solutions that change sign. It is therefore an actual problem to show that solutions $\psi$ to the Yamabe equation produced by some analytic means are in fact positive — but the difficulty does not really arise in the proof of the Yamabe problem, as there the function $\psi$ is produced as a limit of a minimizing sequence of functions which are already nonnegative.

We deal with this problem in the following proposition, by showing that if $\psi = 1 + \phi$ is a solution to (74) that is negative somewhere, then $\|\phi\|_{2,1}$ must be at least a certain size. So if $\phi$ is small in $L^2_1(M)$, then $\psi = 1 + \phi$ will be nonnegative. It will then easily follow that $\psi$ must be strictly positive, using a maximum principle.

**Proposition 9.1.4.** *If $\|\phi\|_{2,1}$ is sufficiently small, then $\psi \geq 0$.*

*Proof.* If $\xi \in L^2_1(M)$, then by Property 3, $\xi \in L^{2n/(n-2)}(M)$ and $\|\xi\|_{2n/(n-2)} \leq A\|\xi\|_{2,1}$. Therefore

$$\int_M |\nabla\xi|^2 dV_g + \int_M \xi^2 dV_g \geq A^{-2}\cdot\left(\int_M \xi^{\frac{2n}{(n-2)}} dV_g\right)^{\frac{n-2}{n}} \tag{108}$$
$$\geq A^{-2}\cdot \mathrm{vol}(\mathrm{supp}\,\xi)^{-\frac{2}{n}} \int_M \xi^2 dV_g,$$

so $\int_M |\nabla\xi|^2 dV_g \geq \big(A^{-2}\cdot \mathrm{vol}(\mathrm{supp}\,\xi)^{-\frac{2}{n}} - 1\big)\cdot\int_M \xi^2 dV_g$, and there are constants $F_6, F_7 > 0$ depending only on $A$ and $n$, such that if $\mathrm{vol}(\mathrm{supp}\,\xi) \leq F_6$, then $\int_M |\nabla\xi|^2 dV_g \geq F_7\,\mathrm{vol}(\mathrm{supp}\,\xi)^{-\frac{2}{n}} \int_M \xi^2 dV_g$.

Now let $\xi = \min(\psi, 0)$. Then $\xi$ satisfies $\int_M \xi\Delta\psi dV_g = \int_M |\nabla\xi|^2 dV_g$, since $\xi\Delta\psi = |\nabla\xi|^2 + \frac{1}{2}\Delta\xi^2$, and $\int_M \Delta\xi^2 dV_g = 0$. This makes sense because $\Delta\xi^2$ does exist weakly provided $\Delta\psi$ and $\Delta\psi^2$ do; there are no problems at the points where $\psi = 0$ because the function taking $\psi$ to $\xi^2$ has two bounded derivatives. So multiplying (74) by $\xi$ and integrating over $M$ gives

$$\int_M (a|\nabla\xi|^2 + S\xi^2 + \nu|\xi|^p) dV_g = 0,$$

as $\xi = -|\xi|$. But $S \geq \nu - Y$, and $|\nu| = 1$. Therefore

$$\int_M |\nabla\xi|^2 dV_g \leq a^{-1} \int_M \big((1+Y)\xi^2 + |\xi|^p\big) dV_g.$$

Suppose for the moment that $0 < \mathrm{vol}(\mathrm{supp}\,\xi) \leq F_6$. Then applying the inequality derived above,

$$a^{-1} \int_M \big((1+Y)\xi^2 + |\xi|^p\big) dV_g \geq F_7\,\mathrm{vol}(\mathrm{supp}\,\xi)^{-\frac{2}{n}} \int_M \xi^2 dV_g,$$

and thus

$$\int_M |\xi|^p dV_g \geq \Big(aF_7\,\mathrm{vol}(\mathrm{supp}\,\xi)^{-\frac{2}{n}} - (1+Y)\Big) \int_M \xi^2 dV_g.$$

By making $F_6$ smaller if necessary, we may assume that $aF_7\,F_6^{-2/n} \geq 2(1+Y)$, and so

$$\int_M |\xi|^p dV_g \geq F_8\,\mathrm{vol}(\mathrm{supp}\,\xi)^{-\frac{2}{n}} \int_M \xi^2 dV_g, \tag{109}$$

where $F_8 = aF_7/2$.

But $\int_M |\xi|^{2n/(n-2)} dV_g \leq \big(\int_M \xi^2 dV_g\big)^{n/(n-2)} \cdot \mathrm{vol}(\mathrm{supp}\,\xi)^{-2/(n-2)}$, and substituting in we may cancel to get

$$\left(\int_M \xi^2 dV_g\right)^{\frac{2}{n-2}} \geq F_8\,\mathrm{vol}(\mathrm{supp}\,\xi)^{\frac{4}{n(n-2)}}. \tag{110}$$

Therefore $\int_M \xi^2 dV_g \geq F_8^{(n-2)/2}\,\mathrm{vol}(\mathrm{supp}\,\xi)^{\frac{2}{n}}$. Substituting this into (109) we find that

$$\int_M |\xi|^p dV_g \geq F_8^{n/2}. \tag{111}$$

So if $0 < \mathrm{vol}(\mathrm{supp}\,\xi) \leq F_6$, then (111) holds. Thus if $\psi$ is negative anywhere, then either $\mathrm{vol}(\mathrm{supp}\,\xi) > F_6$, or else (111) holds. But in the first case, $\|\phi\|_{2,1} > F_6^{1/2}$, and in the second case, $A\|\phi\|_{2,1} \geq \|\phi\|_p > \|\xi\|_p \geq F_8^{(n-2)/4}$. In either case, $\|\phi\|_{2,1} > \min(F_6^{1/2}, F_8^{(n-2)/4}A^{-1})$. Conversely, if $\|\phi\|_{2,1}$ is smaller than this, then $\psi$ is nonnegative. □

We are now ready to define the constant $c$ of the statement of the Theorem. Let $c$ be sufficiently small that three conditions hold: firstly, $\|\epsilon\|_{n/2} \leq c$ implies $\|\epsilon\|_{n/2}$ is sufficiently small to satisfy the hypothesis of Lemma 9.1.3, so by this lemma $\phi$ exists and satisfies $\|\phi\|_{2,1} \leq cW$; secondly, that $\|\phi\|_{2,1} \leq cW$ implies $\|\phi\|_{2,1}$ is sufficiently small to satisfy the hypothesis of Proposition 9.1.4, so that $\psi = 1 + \phi$ is nonnegative; and thirdly, that $\|\phi\|_{2,1} \leq cW$ implies $\|\phi\|_{2,1}$ is sufficiently small that $\phi$ cannot be the constant $-1$. (As by Property 1, $X/2 \leq \mathrm{vol}(M)$, this is a condition depending only on $X$.)

Then $c$ depends only on $n, A, B, X$ and $Y$, as the three conditions each separately do. Thus if $\|\epsilon\|_{n/2} \leq c$, then there exists $\phi$ with $\|\phi\|_{2,1} \leq W\|\epsilon\|_{n/2}$, such that $\psi = 1 + \phi$ is nonnegative and satisfies (74). Moreover, by the third condition on $c$, $\psi$ is not identically zero. By Proposition 7.2.2, $\psi \in C^2(M)$, and is $C^\infty$ wherever it is nonzero. Therefore it remains only to show $\psi$ is strictly positive, for the rescaled metric $\tilde{g} = \psi^{p-2}g$ to be nonsingular and have constant scalar curvature $\nu$. This we achieve using the *strong maximum principle*, which is [LP], Theorem 2.6:

**Theorem [LP].** *Suppose $h$ is a nonnegative, smooth function on a connected manifold $M$, and $u \in C^2(M)$ satisfies $(\Delta + h)u \geq 0$. If $u$ attains its minimum $m \leq 0$, then $u$ is constant on $M$.*

As $M$ is compact and $S$ and $\psi$ are continuous, they are bounded on $M$, and there is a constant $h \geq 0$ such that $S - \nu\psi^{p-2} \leq h$ on $M$. Now $M$ is connected, and $\psi \in C^2(M)$ satisfies (74) and is nonnegative, so $\psi$ satisfies $a\Delta\psi + h\psi \geq 0$. Thus by the strong maximum principle, if $\psi$ attains the minimum value zero, then $\psi$ is identically zero on $M$. But it has already been shown that this is not the case, so $\psi$ cannot be zero anywhere and must be strictly positive. The proof of Theorem 9.1.1 is therefore complete. □

Throughout the preceding proofs we chose to use particular Sobolev spaces and work with particular norms, with very little explanation of why these spaces were chosen and whether others would have worked. In fact in most cases the Sobolev spaces were the only working possibilities, and we chose them deliberately to be conformally invariant in a certain sense. The reason is that the exponent $p - 1$ in the

Yamabe equation (74) is exactly the critical value for this type of equation, in that the analytic behaviour of similar equations with smaller and larger values of the exponent are quite different.

This is reflected in the fact that the applications of the Sobolev embedding theorem that were made were usually borderline cases. If the exponent in (74) were a little larger, (81) would not hold and the method of proof used above would not work. So choosing the exponents for Sobolev spaces was easy — borderline cases of the embedding theorem were used whenever possible, and it all worked out neatly.

### 9.2. Constant negative scalar curvature metrics

We now construct metrics of scalar curvature $-1$ on connected sums using the results of the previous section. To apply Theorem 9.1.1 to the metrics $g_t$ defined in §8.1 and §8.2, it must be shown that Properties 1-4 of §9.1 hold and that $\|\epsilon_t\|_{n/2}$ is small when $t$ is small. As Properties 1-3 have already been dealt with, it only remains to prove that Property 4 holds for the metrics. Fixing $\nu = -1$, we see that Property 4 is about the invertibility of $a\Delta + b$; this is quite easy to demonstrate, as the eigenvalues of $\Delta$ are all nonnegative.

**Lemma 9.2.1.** *Let $\{g_t : t \in (0, \delta)\}$ be one of the families of metrics defined on the manifold $M = M'\#M''$ in §8.1 or §8.2. Let $A, \zeta$ be the constants constructed for this family in Lemma 8.4.1. Then for every $t \in (0, \zeta]$ and every $\xi \in L^{2n/(n+2)}(M)$, there exists a unique $\phi \in L^2_1(M)$ such that $a\Delta\phi + b\phi = \xi$ (that is, $\Delta\phi$ exists in the weak sense, and satisfies the equation). Moreover, $\|\phi\|_{2,1} \leq B\|\xi\|_{2n/(n+2)}$, where $B = A/b$.*

*Proof.* As $\Delta$ is self-adjoint and all its eigenvalues are nonnegative, and as $a, b > 0$, by some well-known analysis $a\Delta + b$ has a right inverse, $T$ say, from $L^2(M) \to L^2(M)$. Now $M$ is compact, and so $L^2(M) \subset L^{2n/(n+2)}(M)$. Let $\xi \in L^2(M)$. We may define $\phi \in L^2(M)$ by $\phi = T\xi$, and $a\Delta\phi + b\phi = \xi$ will hold weakly.

It must first be shown that $\phi \in L^2_1(M)$ and that it satisfies the inequality. Since $\phi, \xi \in L^2(M)$, $\int_M \phi\xi dV_{g_t}$ exists, and by subtraction $\int_M \phi\Delta\phi dV_{g_t}$ exists as well. This is weakly equal to $\int_M |\nabla\phi|^2 dV_{g_t}$, and so $\phi \in L^2_1(M)$ by definition.

Multiplying the expression above by $\phi$ and integrating gives $a\int_M |\nabla\phi|^2 dV_{g_t} + b\|\phi\|_2^2 = \int_M \phi\xi dV_{g_t}$. As $a > b$, the left hand side is at least $b\|\phi\|^2_{2,1}$, and the right hand side is at most $\|\phi\|_{2n/(n-2)}\|\xi\|_{2n/(n+2)}$ by Hölder's inequality, since $\phi \in L^{2n/(n-2)}(M)$ by the Sobolev embedding theorem. But by Lemma 8.4.1,

$\|\phi\|_{2n/(n-2)} \leq A\|\phi\|_{2,1}$. Putting all this together gives $b\|\phi\|_{2,1}^2 \leq A\|\xi\|_{2n/(n+2)}\|\phi\|_{2,1}$, and so dividing by $b\|\phi\|_{2,1}$ shows that $\|\phi\|_{2,1} \leq B\|\xi\|_{2n/(n+2)}$.

So far we have worked with $\xi \in L^2(M)$ rather than $L^{2n/(n+2)}(M)$. It has been shown that the operator $T : L^2(M) \subset L^{2n/(n+2)}(M) \to L_1^2(M)$ is linear and continuous with respect to the $L^{2n/(n+2)}$ norm on $L^2(M)$, and bounded by $B$. But therefore, by elementary functional analysis, the operator $T$ extends uniquely to a continuous operator on the closure of $L^2(M)$ in $L^{2n/(n+2)}(M)$, that is, $L^{2n/(n+2)}(M)$ itself. Call this extended operator $\overline{T}$. Then for $\xi \in L^{2n/(n+2)}(M)$, $\phi = \overline{T}\xi$ is a well-defined element of $L_1^2(M)$, satisfies $\|\phi\|_{2,1} \leq B\|\xi\|_{2n/(n+2)}$, and $a\Delta\phi + b\phi = \xi$ holds in the weak sense, by continuity. This concludes the proof. □

All the previous work now comes together to prove the following two existence theorems for metrics of scalar curvature $-1$:

**Theorem 9.2.2.** *Let $(M', g')$ be a compact Riemannian manifold of dimension $n \geq 3$ with scalar curvature $-1$, and let $M''$ be a compact Riemannian manifold of the same dimension $n$ with positive scalar curvature. Suppose that $M'$ and $M''$ contain points $m', m''$ respectively, with neighbourhoods in which the metrics of $M'$ and $M''$ are conformally flat.*

*As in §8.1, define the family $\{g_t : t \in (0, \delta)\}$ of metrics on the connected sum $M = M' \# M''$. Then there exists a constant $C$ such that for sufficiently small $t$, the metric $g_t$ admits a smooth conformal rescaling to $\tilde{g}_t = (1+\phi)^{p-2} g_t$, which is a nonsingular Riemannian metric of scalar curvature $-1$, and $\phi$ satisfies ${}^t\|\phi\|_{2,1} \leq Ct^2$. Here ${}^t\|.\|_{2,1}$ is the $L_1^2$ norm induced by $g_t$.*

**Theorem 9.2.3.** *Let $(M', g')$ and $(M'', g'')$ be compact Riemannian manifolds of dimension $n \geq 3$ with scalar curvature $-1$. Suppose that $M'$ and $M''$ contain points $m', m''$ respectively, with neighbourhoods in which the metrics of $M'$ and $M''$ are conformally flat.*

*As in §8.2, define the family $\{g_t : t \in (0, \delta)\}$ of metrics on the connected sum $M = M' \# M''$. Then there exists a constant $C$ such that for sufficiently small $t$, the metric $g_t$ admits a smooth conformal rescaling to $\tilde{g}_t = (1+\phi)^{p-2} g_t$, which is a nonsingular Riemannian metric of scalar curvature $-1$, and $\phi$ satisfies ${}^t\|\phi\|_{2,1} \leq Ct^2$. Here ${}^t\|.\|_{2,1}$ is the $L_1^2$ norm induced by $g_t$.*

The proofs of the theorems are nearly the same, so only the first will be given. To get the second proof, change $\mathrm{vol}(M')$ to $\mathrm{vol}(M') + \mathrm{vol}(M'')$ in the definition of $X$.

*Proof of Theorem 9.2.2.* Applying Lemmas 8.3.1 and 8.4.1 to the family $\{g_t : t \in (0, \delta)\}$ gives a constant $Y$ for Property 2 of §9.1, and constants $A, \zeta$ such that if $t \leq \zeta$ then Property 3 holds for $g_t$ with constant $A$. By Lemma 9.2.1, there is a constant $B$ such that Property 4 also holds for $g_t$ when $t \leq \zeta$.

It is clear that as $t \to 0$, $\mathrm{vol}(M, g_t) \to \mathrm{vol}(M') > 0$. So there is a constant $X > 0$ such that $X/2 \leq \mathrm{vol}(M, g_t) \leq X$ for small enough $t$. This gives Property 1. Thus there are constants $n, A, B, X, Y$ such that Properties 1-4 of §9.1 hold for $(M, g_t)$ when $t$ is sufficiently small. Theorem 9.1.1 therefore gives a constant $c$ such that if $\|\epsilon_t\|_{n/2} \leq c$, we have the smooth conformal rescaling to a constant scalar curvature metric that we want.

But by Lemma 8.3.1, $\|\epsilon_t\|_{n/2} \leq Zt^2$. So for small enough $t$, $\|\epsilon_t\|_{n/2} \leq c$, and there exists a smooth conformal rescaling to a Riemannian metric $\tilde{g}_t = (1 + \phi)^{p-2} g_t$ which has scalar curvature $-1$. Moreover, $\|\phi\|_{2,1} \leq W \|\epsilon_t\|_{n/2} \leq WZt^2$, where $W$ is the constant given by Theorem 9.1.1. Therefore putting $C = WZ$ completes the theorem. □

The case of negative scalar curvature is the simplest case of the Yamabe problem, which is why it has been tackled first. To give one reason why the negative case is easier, we give a proof taken from [Au], p. 135 showing that any metric of scalar curvature $-1$ is unique in its conformal class:

**Lemma 9.2.4 [Au].** *Suppose that $M$ is a compact Riemannian manifold, and that $g$ and $\tilde{g} = \psi^{p-2} g$ are Riemannian metrics on $M$ which both have scalar curvature $-1$. Then $\psi \equiv 1$, so that $g = \tilde{g}$.*

*Proof.* By (73), $\psi$ satisfies $a\Delta\psi + \psi^{(n+2)/(n-2)} = \psi$. Now at a point of $M$ where $\psi$ is maximum, $\Delta\psi \geq 0$ and so $\psi^{(n+2)/(n-2)} \leq \psi$ and $\psi \leq 1$. Similarly, at a point where $\psi$ is minimum, $\psi \geq 1$. But since $M$ is compact, the maximum and minimum of $\psi$ are achieved and $1 \leq \psi \leq 1$, so $\psi \equiv 1$ and $g = \tilde{g}$. □

This lemma shows that the constant scalar curvature metrics $\tilde{g}_t$ produced by Theorems 9.2.2 and 9.2.3 are, up to homothety, the unique metrics of constant scalar curvature in their conformal classes, and are therefore Yamabe metrics. The proof works because, in the case of negative scalar curvature, the signs are right. In the positive case there is indeed no uniqueness, and a metric may have several conformal equivalents of scalar curvature 1, which may or not be minimal for the Hilbert action.

### 9.3. Constant positive scalar curvature metrics

Next we construct metrics of scalar curvature 1 on connected sums. The problems we encounter are in proving Property 4 of §9.1, which now deals with the invertibility of $a\Delta - b$, and they arise because $a\Delta$ may have eigenvalues close to $b$. Our strategy is to show that if $a\Delta$ has no eigenvalues in a fixed neighbourhood of $b$ on the component manifolds of the connected sum, then for small $t$, $a\Delta$ has no eigenvalues in a subneighbourhood of $b$ on $(M, g_t)$.

This is the content of the next theorem. We shall indicate here why the theorem holds, but the proof we leave until Appendix D, because it forms a rather long and involved diversion from the main thread of this part of the thesis, and few readers will want to look at it.

**Theorem 9.3.1.** *Let $\{g_t : t \in (0, \delta)\}$ be one of the families of metrics defined on $M = M'\#M''$ in §8.1 or §8.2, and suppose that for some $\gamma > 0$, $a\Delta$ has no eigenvalues in the interval $(b - 2\gamma, b + 2\gamma)$ on $M'$ in the case of §8.1, and on both $M'$ and $M''$ in the case of §8.2. Then for sufficiently small $t$, $a\Delta$ has no eigenvalues in the interval $(b - \gamma, b + \gamma)$ on $(M, g_t)$.*

*Proof: see Appendix D.*

We offer the following explanation of why the theorem is true. Suppose that $\phi$ is an eigenvector of $a\Delta$ on $(M, g_t)$ for small $t$. Then restricting $\phi$ to the portions of $M$ coming from $M'$ and $M''$ and smoothing off gives functions on $M', M''$; we may try to show that at least one of these is close to an eigenvector of $a\Delta$ on $M'$ or $M''$. This can be done except when $\phi$ is large on the neck compared to the rest of the manifold.

But as the neck is a small region when $t$ is small, for $\phi$ to be large there and small elsewhere means that $\phi$ must change quickly around the neck, so that $\int_M |\nabla\phi|^2 dV_{g_t}$ has to be large compared to $\int_M \phi^2 dV_{g_t}$. If this is the case then the eigenvalue associated to $\phi$ must be large. Conversely, if $\phi$ is associated to an eigenvalue of $a\Delta$ close to $b$, it cannot be large on the neck compared to the rest of the manifold, and therefore either $M'$ or $M''$ must also have an eigenvalue close to $b$.

Using this result, Property 4 of §9.1 can be proved for the metrics:

**Lemma 9.3.2.** *Let $\{g_t : t \in (0, \delta)\}$ be one of the families of metrics defined on $M = M'\#M''$ in §8.1 or §8.2, and suppose that $b$ is not an eigenvalue of $a\Delta$ on $M'$ in the case of §8.1, and not an eigenvalue on either of $M'$ or $M''$ in the case of §8.2. Then there is a constant $B > 0$ such that for sufficiently small $t$, whenever $\xi \in L^{2n/(n+2)}(M)$ there exists a unique $\phi \in L^2_1(M)$ satisfying $a\Delta\phi - b\phi = \xi$ on the Riemannian manifold $(M, g_t)$, and moreover $\|\phi\|_{2,1} \leq B\|\xi\|_{2n/(n+2)}$.*

*Proof.* The spectrum of $a\Delta$ on a compact manifold is discrete, so if $b$ is not an eigenvalue of $a\Delta$, then $a\Delta$ has no eigenvalues in a neighbourhood of $b$. Suppose that $b$ is not an eigenvalue of $a\Delta$ on $M'$ in the case of §8.1, and not on either of $M'$ or $M''$ in the case of §8.2. Then there exists $\gamma > 0$ such that $a\Delta$ has no eigenvalues in the interval $(b-2\gamma, b+2\gamma)$ on these manifolds. So by Theorem 9.3.1, for small enough $t$, $a\Delta$ has no eigenvalues in the interval $(b-\gamma, b+\gamma)$ on the manifold $(M, g_t)$.

Thus easy analytical facts about the Laplacian imply that $a\Delta - b$ has a right inverse $T : L^2(M) \to L^2(M)$. As $M$ is compact, $L^2(M) \subset L^{2n/(n+2)}(M)$. Let $\xi \in L^2(M)$. Then $\phi = T\xi \in L^2(M)$ exists and satisfies the equation $a\Delta\phi - b\phi = \xi$ in the weak sense, and as $a\Delta - b$ has no kernel, $\phi$ is unique. Since $\phi, \xi \in L^2(M)$, multiplying this equation by $\phi$ and integrating gives a convergent integral, so by subtraction, $\int_M |\nabla\phi|^2 dV_{g_t}$ converges, and $\phi \in L^2_1(M)$.

It remains to bound $\phi$ in $L^2_1(M)$. Let $\phi_1$ be the part of $\phi$ made up of eigenvectors of $a\Delta$ associated to eigenvalues less than $b$, and $\phi_2$ the part associated to eigenvalues greater than $b$. Multiplying the equation $a\Delta\phi - b\phi = \xi$ by $\phi_2 - \phi_1$ and integrating gives

$$\int_M \bigl(a|\nabla\phi_2|^2 - b\phi_2^2\bigr)dV_{g_t} - \int_M \bigl(a|\nabla\phi_1|^2 - b\phi_1^2\bigr)dV_{g_t} = \int_M (\phi_2 - \phi_1)\xi dV_{g_t}. \tag{112}$$

But the restriction on the eigenvalues of $a\Delta$ means that

$$\int_M a|\nabla\phi_1|^2 dV_{g_t} \le (b-\gamma)\int_M \phi_1^2 dV_{g_t} \qquad \text{and} \qquad \int_M a|\nabla\phi_2|^2 dV_{g_t} \ge (b+\gamma)\int_M \phi_2^2 dV_{g_t},$$

and these together with (112) and Hölder's inequality imply

$$\frac{\gamma a}{a+b+\gamma}\int_M \bigl(|\nabla\phi_1|^2 + |\nabla\phi_2|^2 + \phi_1^2 + \phi_2^2\bigr)dV_{g_t} \le \|\phi_2 - \phi_1\|_{2n/(n-2)}\|\xi\|_{2n/(n+2)}. \tag{113}$$

For $t \le \zeta$, we may apply Lemma 8.4.1 to $\phi_2 - \phi_1$ to give $\|\phi_2 - \phi_1\|_{2n/(n-2)} \le A\|\phi_2 - \phi_1\|_{2,1}$. But $\phi_1, \phi_2$ are orthogonal in $L^2_1(M)$, so the right hand side of this is equal to $A\|\phi\|_{2,1}$; similarly, the integral on the left hand side of (113) is $\|\phi\|^2_{2,1}$. Therefore

$$\frac{\gamma a}{a+b+\gamma}\|\phi\|^2_{2,1} \le A\|\phi\|_{2,1}\|\xi\|_{2n/(n+2)}. \tag{114}$$

Dividing this by $\gamma a\|\phi\|_{2,1}/(a+b+\gamma)$ then gives that $\|\phi\|_{2,1} \le B\|\xi\|_{2n/(n+2)}$ whenever $t$ is small enough, where $B = (a+b+\gamma)A/a\gamma$. So the lemma holds for $\xi \in L^2(M)$. This may easily be extended to $\xi \in L^{2n/(n+2)}(M)$ as in the proof of Lemma 9.2.1, and the argument is complete. □

We shall now prove the following two existence theorems for metrics of scalar curvature 1:

**Theorem 9.3.3.** *Let $(M', g')$ be a compact Riemannian manifold of dimension $n \geq 3$ with scalar curvature 1, and let $M''$ be a compact Riemannian manifold of the same dimension $n$ with positive scalar curvature. Suppose that $b$ is not an eigenvalue of $a\Delta$ on $(M', g')$, and that $M'$ and $M''$ contain points $m', m''$ respectively, with neighbourhoods in which the metrics of $M'$ and $M''$ are conformally flat.*

*As in §8.1, define the family $\{g_t : t \in (0, \delta)\}$ of metrics on the connected sum $M = M' \# M''$. Then there exists a constant $C$ such that for sufficiently small $t$, the metric $g_t$ admits a smooth conformal rescaling to $\tilde{g}_t = (1+\phi)^{p-2} g_t$, which is a nonsingular Riemannian metric of scalar curvature 1, and $\phi$ satisfies ${}^t\|\phi\|_{2,1} \leq Ct^2$. Here ${}^t\|.\|_{2,1}$ is the $L^2_1$ norm induced by $g_t$.*

**Theorem 9.3.4.** *Let $(M', g')$ and $(M'', g'')$ be compact Riemannian manifolds of dimension $n \geq 3$ with scalar curvature 1. Suppose that $b$ is not an eigenvalue of $a\Delta$ on $(M', g')$ or $(M'', g'')$, and that $M'$ and $M''$ contain points $m', m''$ respectively, with neighbourhoods in which the metrics of $M'$ and $M''$ are conformally flat.*

*As in §8.2, define the family $\{g_t : t \in (0, \delta)\}$ of metrics on the connected sum $M = M' \# M''$. Then there exists a constant $C$ such that for sufficiently small $t$, the metric $g_t$ admits a smooth conformal rescaling to $\tilde{g}_t = (1+\phi)^{p-2} g_t$, which is a nonsingular Riemannian metric of scalar curvature 1, and $\phi$ satisfies ${}^t\|\phi\|_{2,1} \leq Ct^2$. Here ${}^t\|.\|_{2,1}$ is the $L^2_1$ norm induced by $g_t$.*

*Proof of Theorems 9.3.3 and 9.3.4.* These are the same as the proofs of Theorems 9.2.2 and 9.2.3, except that Lemma 9.3.2 should be applied in place of Lemma 9.2.1, and where the proofs of Theorems 9.2.2 and 9.2.3 mention scalar curvature $-1$, these proofs should have scalar curvature 1. □

In the case of negative scalar curvature, Lemma 9.2.4 shows that any metric of constant scalar curvature $-1$ is unique in its conformal class. This uniqueness does not extend to the positive case, for Theorems 9.3.3 and 9.3.4 show that given two manifolds $M', M''$ with scalar curvature 1 and the same dimension $n \geq 3$, we will usually be able to define *three* metrics of scalar curvature 1 in the conformal class of their connected sum, when the 'neck' parameter $t$ is sufficiently small.

The first of the three metrics resembles $M'$ with metric of scalar curvature 1, and with a small, asymptotically flat copy of $M''$ glued in at one point, as in §8.1. The second is like the first, but reverses the rôles of $M'$ and $M''$. The third metric resembles the union of $M'$ and $M''$, each with their metric of scalar curvature 1, and joined by a small neck, as in §8.2.

Despite having constant scalar curvature, these metrics may not be Yamabe metrics because they may not be minima of the Hilbert action, and in fact, for sufficiently small $t$, the third metric is never minimal.

Suppose that $(M', g')$ and $(M'', g'')$ are Yamabe metrics of scalar curvature 1; suppose moreover that these Yamabe metrics are unique. In this case I believe that generically either the first or the second metric will be a Yamabe metric on the connected sum, depending on whether $\lambda(M') < \lambda(M'')$ or $\lambda(M'') < \lambda(M')$ respectively.

In a codimension 1 set of cases, when $\lambda(M')$ and $\lambda(M'')$ are nearly equal, the Hilbert actions of the first and second metrics will agree, and in this case I believe that *both* are (distinct) Yamabe metrics, so that Yamabe metrics may not be unique. At the moment, however, I have no proof that any of the metrics I construct in the positive scalar curvature case are Yamabe metrics.

Let us now consider the condition that $b$ is not an eigenvalue of $a\Delta$ on a manifold of scalar curvature 1, since if this never holds, then Theorems 9.3.3 and 9.3.4 are useless. Intuitively we expect that for generic metrics the condition will hold. It is an open condition, as if $g$ is a metric for which $b$ is not an eigenvalue of $a\Delta$, then every conformal class in a small neighbourhood of $g$ also contains a metric close to $g$ with scalar curvature 1, for which $b$ is not an eigenvalue of $a\Delta$.

The condition has already been studied, for it comes up in connection with the local behaviour of the family of metrics with constant scalar curvature ([Bs], §4F). By 4.46 Remark ii) of [Bs], if $g$ is any Einstein metric on $M$ with scalar curvature 1, then $b$ is only an eigenvalue of $a\Delta$ if $M$ is $\mathcal{S}^n$ and $g$ the round metric. So Einstein metrics, and constant scalar curvature metrics close to them, provide some examples of Riemannian manifolds for which the condition holds. In the next proposition we show that all manifolds admitting a metric of positive scalar curvature have some metric of scalar curvature 1 satisfying the condition.

**Proposition 9.3.5.** *Let $M$ be a compact manifold admitting a metric of positive scalar curvature. Then there exists a metric $g$ on $M$ with scalar curvature 1, such that $b$ is not an eigenvalue of $a\Delta$ on $(M, g)$.*

*Proof.* In the family of metrics on $M$, there are metrics with positive scalar curvature by the assumption of the proposition, and there are metrics of negative scalar curvature, because there are on every manifold. Let $g_0$ be a metric with positive scalar curvature, and $g_1$ be a metric with negative scalar curvature. Thus $\lambda(g_0) > 0$ and $\lambda(g_1) < 0$.

Now by [Bs], Proposition 4.31, $\lambda$ is a continuous function on the space of metrics on $M$ with the $C^2$ topology, and thus $\lambda\big((1-t)g_0 + tg_1\big)$ is a continuous function of $t$. As this function is positive at 0 and negative at 1, there is at least one $t \in (0,1)$ for which it vanishes. Let $t$ be the smallest such zero of the function, and define $g'$ to be a metric of zero scalar curvature conformal to $(1-t)g_0 + tg_1$; such a metric must exist by the solution to the Yamabe problem.

The scalar curvature is an operator from the set of Riemannian metrics on $M$ to the set of smooth functions on $M$ that is differentiable when viewed as a map between infinite-dimensional manifolds, so we may consider the linearization of the scalar curvature operator at $g'$. A result of Bourguignon ([Bs], Proposition 4.37) states that this linearization is surjective unless either $g'$ is Ricci-flat, or $g'$ has constant positive scalar curvature $s_g$ and $s_g/(n-1)$ is an eigenvalue of $\Delta$. As $g'$ has zero scalar curvature, we find that the linearization is surjective unless $g'$ is Ricci-flat.

Suppose for the moment that $g'$ is not Ricci-flat. Our basic idea is to use the surjectivity of the linearized scalar curvature together with an 'implicit function theorem', to show that close to $g'$ there exist metrics with scalar curvature equal to a small positive constant, and to rescale one of these by a homothety to provide a metric of scalar curvature 1 satisfying the condition we want. The 'implicit function theorem' is a theorem of Koiso ([Bs], Theorem 4.44, rewritten):

**Theorem [Bs].** *Let $C$ be the set of metrics $g$ on $M$ with constant scalar curvature $s_g$ and volume 1, and let $g \in C$ be such that $s_g/(n-1)$ is not a positive eigenvalue of $\Delta_g$. Then, in a neighbourhood of $g$, $C$ is an ILH- submanifold of the infinite-dimensional manifold of Riemannian metrics on $M$, and its tangent space is the inverse image of the constants under the linearization of the scalar curvature operator on metrics of volume 1 at $g$, provided this is surjective.*

Here an ILH- manifold is a particular type of infinite-dimensional manifold, and the thrust of the theorem is that the set of constant scalar curvature metrics is a well-behaved infinite-dimensional manifold, and does have the tangent space we expect when the linearization is surjective.

Going back to $g'$, as it is supposed not Ricci-flat, the linearization of the scalar curvature operator is surjective and there is a vector $h$ taken to the constant function 1 by the linearization. Applying the Theorem, $h$ is a tangent vector to $C$, and thus we may move on $C$ a little way away from $g'$ in the direction of $h$, to get a point $g''$ of $C$ with constant positive scalar curvature, and satisfying the condition on the eigenvalues of $\Delta$. Rescale $g''$ by a homothety to get a metric of scalar curvature 1, and call this new metric $g$. The condition on the eigenvalues of the Laplacian implies that $b$ is not an eigenvalue of $a\Delta$ on $(M, g)$, so that $g$ satisfies the requirements of the Proposition.

To finish the proof of the Proposition, it therefore only remains to consider the case when $g'$ is Ricci-flat. As we have taken care that there are metrics of positive scalar curvature arbitrarily close to $g'$, we may eliminate the case that $g'$ is a local maximum of $\lambda$, and with this information we may show, by a closer examination of Koiso's result, that the set of metrics of zero scalar curvature close to $g'$, modulo diffeomorphisms, is non-empty and infinite-dimensional. As Ricci-flatness is an elliptic condition, the set of such metrics is finite-dimensional, so there must be metrics of zero scalar curvature close to $g'$ in $C$, that are not Ricci-flat. Replacing $g'$ by one of these and arguing as above, the proof is complete. $\square$

Finally we note that for any *Yamabe metric* of scalar curvature 1, it may be shown that $b$ is the minimum possible eigenvalue of $a\Delta$. For if $g$ is a Yamabe metric then it minimizes the Hilbert functional $Q$ in the conformal class, and so $Q\big((1+x\phi)^{2/p}g\big) \geq Q(g)$ for any smooth function $\phi$ and sufficiently small $x$. Looking at the second order term in $x$ and manipulating, it can be shown that if $\phi$ is a non-constant eigenvector of $a\Delta$, then its associated eigenvalue is at least $b$.

# Chapter 10: Connected Sums involving Manifolds with Zero Scalar Curvature

In this chapter the methods of Chapters 8 and 9 will be adapted to study zero scalar curvature manifolds. We have three cases to consider, the connected sum of a zero scalar curvature manifold and a positive scalar curvature manifold, the connected sum of two manifolds of zero scalar curvature, and the connected sum of a zero scalar curvature manifold and a negative scalar curvature manifold. Each of these cases introduces specific difficulties, and each needs some additional methods to prove the existence of constant scalar curvature metrics.

The first two cases fit into a common analytic framework, and will be handled together. We proceed as before by defining metrics with nearly constant scalar curvature on the connected sum, and then using analytic methods to prove the existence of a small conformal change, giving a metric of constant scalar curvature. Here, however, the approximate metrics must be defined rather more carefully to control the errors sufficiently, and the constant scalar curvature that results is in fact negative and small, depending upon the 'neck' parameter $t$ of the gluing. These approximate metrics are defined in §10.1 and §10.2.

In §10.3 we prove some estimates and inequalities on the new metrics, which take the place of those of Chapter 8. Section 10.4 details the modifications needed to make the proof of §9.1 apply in the situation of this chapter. The new feature of the analysis is that the operator $a\Delta - \nu b$ now has one or two small eigenvalues, and so when the sequence $\{\phi_i\}_{i=0}^{\infty}$ is defined inductively, using the inverse of this operator, the components in the directions of the corresponding eigenvectors have to be carefully controlled, to prevent the sequence diverging.

The third case is discussed in §10.5. We shall not actually prove any explicit existence results, but we will explain how they could be proved, what the results are, and why the resulting metrics should look like they do. This is because no new analytic ideas are required to complete the solution, but instead the work is in producing metrics $g_t$ that are very good approximations to constant scalar curvature, and this is not very interesting. The picture is that the zero scalar curvature manifold gets homothetically shrunk by a factor of $t^{(n-2)/n}$, and then is glued into the negative scalar curvature manifold using a neck of approximate radius $t$.

In §11.1 some diagrams are collected which give a mental picture of the Riemannian manifolds $(M, g_t)$ defined in §§10.1 and 10.2, and also a diagram of what the metrics of §10.5 should look like. The reader will probably find it helpful to look at the diagrams when reading each of these sections, to aid understanding of the text.

## 10.1. Combining a metric of zero and a metric of positive scalar curvature

Let $(M', g')$ be a compact Riemannian manifold of dimension $n \geq 3$ with zero scalar curvature. As in the last two chapters, suppose $M'$ contains a point $m'$ with a neighbourhood in which $g'$ is conformally flat. Then $M'$ contains a ball $B'$ about $m'$, with a diffeomorphism $\Phi'$ from $B_r(0) \subset \mathbb{R}^n$ to $B'$ for some $r < 1$, such that $\Phi'(0) = m'$ and $(\Phi')^*(g') = (\psi')^{p-2}h$ for some function $\psi'$ on $B_r(0)$, where $h$ is the standard metric on $\mathbb{R}^n$. As $g'$ has zero scalar curvature, by (73) $\psi'$ satisfies $\Delta\psi' = 0$. By choosing a different conformal identification with $B_r(0)$ if necessary, we may suppose that $\psi'(0) = 1$ and $d\psi'(0) = 0$, so that $\psi'(v) = 1 + O'(|v|^2)$, in the notation of §7.3.

Let $M''$ be a compact Riemannian manifold of the same dimension $n$ with positive scalar curvature, that is not conformal to $\mathcal{S}^n$ with its round metric. Suppose $M''$ contains a point $m''$, with a neighbourhood in which the metric of $M''$ is conformally flat. Then by Proposition 7.3.4, there is an asymptotically flat metric $g''$ of zero scalar curvature in the conformal class of $M'' \setminus \{m''\}$. There is a subset $N''$ of $M'' \setminus \{m''\}$ that is the complement of a compact set, and a diffeomorphism $\mathrm{X}'' : \mathbb{R}^n \setminus \overline{B}_R(0) \to N''$ for some $R > 0$, such that $(\mathrm{X}'')^*(g'') = \xi^{p-2}h$, for $\xi$ a smooth function on $\mathbb{R}^n \setminus \overline{B}_R(0)$. Also, there is a real constant $\mu$ such that

$$\xi(v) = 1 + \mu|v|^{2-n} + O'(|v|^{1-n}). \tag{115}$$

Moreover, as $M''$ is not conformal to $\mathcal{S}^n$ with its round metric, $\mu > 0$ by Theorem 7.3.5.

Choose a real constant $k$ with $(n-2)(n+2)/2(n+1) < k < (n-2)(n+2)/2n$, which will remain fixed throughout this chapter. Choose another constant $\delta \in (0,1)$ such that $\delta^{2k/(n+2)} \leq r$ and $\delta^{-2/n} \geq R$. A family of metrics $\{g_t : t \in (0, \delta)\}$ on $M = M' \# M''$ will now be written down, in a similar way to §8.1. For any $t \in (0, \delta)$, define $M$ and the conformal class of $g_t$ by

$$M = \Big(M' \setminus \Phi'[\overline{B}_{t^{(n-2)/n}}(0)]\Big) \amalg \Big(M'' \setminus \big(\{m''\} \cup \mathrm{X}''[\mathbb{R}^n \setminus B_{t^{2k/(n+2)-1}}(0)]\big)\Big) \Big/ \sim_t, \tag{116}$$

where $\underset{t}{\frown}$ is the equivalence relation defined by

$$\Phi'[v] \quad \underset{t}{\frown} \quad \mathrm{X}''[t^{-1}v] \qquad\qquad \text{whenever } v \in \mathbb{R}^n \text{ and } t^{(n-2)/n} < |v| < t^{2k/(n+2)}. \tag{117}$$

As in §8.1, the conformal class $[g_t]$ of $g_t$ is the restriction of the conformal classes of $g'$ and $g''$ to the open sets of $M', M''$ making up $M$, and is well-defined because the conformal classes agree on the annulus of overlap $A_t$, where the two open sets are glued by $\underset{t}{\frown}$. Define $g_t$ within this conformal class by $g_t = g'$ on the component of $M \setminus A_t$ coming from $M'$, and $g_t = t^2 g''$ on the component of $M \setminus A_t$ coming from $M''$. It remains only to choose a conformal factor on $A_t$ itself. This is done just as in §8.1, except that the annulus $\{v \in \mathbb{R}^n : t^{(n-2)/n} < |v| < t^{2k/(n+2)}\}$ in $\mathbb{R}^n$ replaces the annulus $\{v \in \mathbb{R}^n : t^2 < |v| < t\}$ in $\mathbb{R}^n$ in the definition of the partition of unity. This completes the definition of the metric $g_t$ for $t \in (0, \delta)$.

The difference between this definition and that given in §8.1 is that $g''$ has been shrunk by a factor of $t^2$ rather than $t^{12}$, and all the nice powers of $t$ such as $t$ and $t^2$ appearing there have been replaced by nasty and apparently quite arbitrary powers; they will be needed to fine-tune the scalar curvature estimates for the metrics $g_t$. These estimates are given in the next lemmas.

**Lemma 10.1.1.** *Let the scalar curvature of the metric $g_t$ be $-\epsilon_t$. Then $\epsilon_t$ is zero outside $A_t$. There exists a constant $Y$ such that for any $t \in (0, \delta)$, $\epsilon_t$ satisfies $|\epsilon_t| \leq Y$, and the volume of $A_t$ with respect to $g_t$ satisfies* $\mathrm{vol}(A_t) = O(t^{2nk/(n+2)})$. *These trivially imply that* ${}^t\|\epsilon_t\|_{2n/(n+2)} = O(t^k)$ *and* ${}^t\|\epsilon_t\|_{n/2} = O(t^{4k/(n+2)})$.

*Proof.* Outside $A_t$, the metric $g_t$ is equal to $g'$ or homothetic to $g''$, and so has zero scalar curvature, verifying the first claim of the lemma. The proof that $|\epsilon_t| \leq Y$ is the same as that for the corresponding statement in Lemma 8.3.1, setting $\nu = 0$. The estimate on the volume of $A_t$ also easily follows by the method used in Lemma 8.3.1. And as the lemma says, the last two statements are trivial corollaries of the facts that $\epsilon_t$ is supported in $A_t$, $|\epsilon_t| \leq Y$, and the estimate on the volume of $A_t$. □

Note that the use of the constant $k$ in the definition of the metrics $g_t$ was in order to make the estimate on ${}^t\|\epsilon\|_{2n/(n+2)}$ easy to write down. In the next lemma we estimate the integral of $\epsilon_t$ over $M$.

**Lemma 10.1.2.** *For small $t$,*

$$\int_M \epsilon_t dV_{g_t} = (n-2)\omega_{n-1}\mu t^{n-2} + O(t^{n-2+\alpha}), \tag{118}$$

*where $\mu$ is the constant of (115), $\omega_{n-1}$ is the volume of the $(n-1)$- dimensional sphere $\mathcal{S}^{n-1}$ of radius 1, and $\alpha > 0$ is given by* $\alpha = \min\big(2/n, 2k(n+1)/(n+2) - (n-2)\big)$.

*Proof.* Calculating with (73) gives

$$\epsilon_t(v) = \psi_t^{-(n+2)/(n-2)}(v)\Big(2\big(\nabla\beta_1(v)\big)\cdot\big(\nabla\big(\psi'(v)-\xi(t^{-1}v)\big)\big) - \big(\Delta\beta_1(v)\big)\big(\psi'(v)-\xi(t^{-1}v)\big)\Big).$$

Let $F$ be the quadratic form on $\mathbb{R}^n$ given by the second derivatives of $\psi'$; then $\psi' = 1 + F + O'(|v|^3)$. Now $dV_{g_t} = \psi_t^{2n/(n-2)} dV_h$. Multiplying through by this equation and making various estimates gives that

$$\begin{aligned}\epsilon_t(v)dV_{g_t} = \psi_t(v)\Big(&2\big(\nabla\beta_1(v)\big)\cdot\big(\nabla F + O(|v|^2) - \mu t^{n-2}\nabla(|v|^{2-n}) - t^{n-1}O(|v|^{-n})\big)\\ &- \big(\Delta\beta_1(v)\big)\big(F + O(|v|^3) - \mu t^{n-2}|v|^{2-n} - t^{n-1}O(|v|^{1-n})\big)\Big)dV_h.\end{aligned} \tag{119}$$

Let us integrate this equation over $A_t$. By its definition, $\beta_1$ may be written $\beta_1(v) = \beta(|v|)$ for a smooth function $\beta$ depending on $t$; in fact $\beta(|v|) = \sigma(\log|v|/\log t)$, from §8.1. It follows that $(\nabla\beta_1(v))\cdot(\nabla F) = 2|v|^{-1}F\frac{d\beta}{dx}$, as $F$ is a quadratic form, and similarly $(\nabla\beta_1(v))\cdot(-\mu t^{n-2}\nabla(|v|^{2-n})) = (n-2)\mu t^{n-2}|v|^{1-n}\frac{d\beta}{dx}$. Also $\Delta\beta_1 = -\frac{d^2\beta}{dx^2} + (1-n)|v|^{-1}\frac{d\beta}{dx}$. Therefore

$$\begin{aligned}\int_{A_t}\epsilon_t(v)dV_{g_t} = \int_{A_t}\psi_t(v)\Bigg(&\big(4|v|^{-1}F + 2(n-2)\mu t^{n-2}|v|^{1-n}\big)\frac{d\beta}{dx} + 2\big(\nabla\beta_1(v)\big)\cdot\big(O(|v|^2) - t^{n-1}O(|v|^{-n})\big)\\ &+\left(\frac{d^2\beta}{dx^2} + (n-1)|v|^{-1}\frac{d\beta}{dx}\right)\big(F - \mu t^{n-2}|v|^{2-n} + O(|v|^3) - t^{n-1}O(|v|^{1-n})\big)\Bigg)dV_h.\end{aligned} \tag{120}$$

But $\psi_t(v) = 1 + O(|v|^2) + t^{n-2}O(|v|^{2-n})$, and substituting this in gives

$$\begin{aligned}\int_{A_t}\epsilon_t(v)dV_{g_t} = \int_{A_t}\Bigg(&\frac{d\beta}{dx}\big((n+3)|v|^{-1}F + (n-3)\mu t^{n-2}|v|^{1-n}\big)\\ &+ |v|\frac{d^2\beta}{dx^2}\big(|v|^{-1}F - \mu t^{n-2}|v|^{1-n}\big)\Bigg)dV_h + \text{ error terms.}\end{aligned} \tag{121}$$

Using a Fubini theorem, we may write the integral on the right hand side as a double integral by

$$\int_{A_t}(\ldots)dV_h = \int_{t^{(n-2)/n}}^{t^{2k/(n+2)}}\int_{\mathcal{S}^{n-1}}(\ldots)|v|^{n-1}d\Omega d|v|,$$

where $\Omega$ is the volume form of the metric on a round sphere $\mathcal{S}^{n-1}$ of radius 1. But $F$ is the quadratic form on $\mathbb{R}^n$ given by the second derivatives of $\psi'$ at $m'$, and $\Delta\psi' = 0$, as $M'$ has zero scalar curvature. This implies that the trace of $F$ with respect to $h$ is zero, and so $\int_{\mathcal{S}^{n-1}} F d\Omega = 0$ and the terms on the right hand side of (121) involving $F$ vanish. So viewing (121) as a double integral and integrating over $\mathcal{S}^{n-1}$ gives

$$\int_{A_t}\epsilon_t(v)dV_{g_t} = \omega_{n-1}\int_{t^{(n-2)/n}}^{t^{2k/(n+2)}}\left((n-3)\mu t^{n-2}\frac{d\beta}{dx} - \mu t^{n-2}x\frac{d^2\beta}{dx^2}\right)dx + \text{ error terms,}$$

where $\omega_{n-1}$ is the volume of $\mathcal{S}^{n-1}$.

The integral on the right is an exact integral, for the second term integrates by parts. From the definition of $\beta$, we have $\beta(t^{(n-2)/n}) = 0$, $\beta(t^{2k/(n+2)}) = 1$ and $\frac{d\beta}{dx}(t^{(n-2)/n}) = 0$, $\frac{d\beta}{dx}(t^{2k/(n+2)}) = 0$, and so the answer is

$$\int_{A_t} \epsilon_t(v) dV_{g_t} = (n-2)\omega_{n-1}\mu t^{n-2} + \text{ error terms,} \tag{122}$$

which is nearly the conclusion of the lemma; it remains only to show that the 'error terms' are of order $t^{n-2+\alpha}$.

This is a simple calculation and will be left to the reader, the necessary ingredients being that as $|v|$ lies between $t^{(n-2)/n}$ and $t^{2k/(n+2)}$, $O(|v|)$ may be replaced by $O\big(t^{2k/(n+2)}\big)$ and $tO\big(|v|^{-1}\big)$ may be replaced by $O\big(t^{2/n}\big)$, $\frac{d\beta}{dx} = O\big(|v|^{-1}\big)$ and $\frac{d^2\beta}{dx^2} = O\big(|v|^{-2}\big)$. The error term that is usually the biggest is $O\big(|v|^{n+1}\big)$, and in order to ensure that this error term is smaller than the leading term calculated above, that is, to ensure $\alpha > 0$, $k$ must satisfy $k > (n+2)(n-2)/2(n+1)$, which was one of the conditions in the definition of $k$ above. □

Lemma 10.1.2 implies that the average scalar curvature of $(M, g_t)$ is close to $-(n-2)\omega_{n-1}\mu t^{n-2}\,\mathrm{vol}(M')^{-1}$. This is why, in §10.4, we shall choose this value for the scalar curvature of the metric that we construct in the conformal class of $g_t$.

### 10.2. Combining two metrics of zero scalar curvature

In this section we define metrics on the connected sum of two zero scalar curvature manifolds using the method of the previous section; the metrics of this section bear the same relation to those of §10.1 as do the metrics of §8.2 to those of §8.1. Let $(M', g')$ and $(M'', g'')$ be two Riemannian manifolds of dimension $n$, with zero scalar curvature. Suppose that $M', M''$ contain points $m', m''$ respectively, having neighbourhoods in which $g', g''$ are conformally flat, and let $M = M' \# M''$.

As in §8.2, we may define a family of conformal classes on $M$ depending on $t$. Now applying a homothety to a manifold of zero scalar curvature gives another manifold of zero scalar curvature, so to define a metric in one of these conformal classes that has scalar curvature close to zero, we may start from the metrics on $M'$ and $M''$ scaled homothetically by arbitrary factors, before choosing the metric on the 'neck' with a partition of unity. This gives a whole family of metrics in the conformal class that have scalar curvature close to zero, and as in general we expect metrics of constant scalar curvature in a conformal class to be isolated, we need some way to determine the relative sizes $M'$ and $M''$ should be for the glued metric to be close to one of constant scalar curvature.

The necessary condition is that the volumes of $M'$ and $M''$ should be the same; the reason for this will become clear in the next two sections. For the moment let us suppose, by applying a homothety to $M'$ or $M''$ if necessary, that the volumes of $M'$ and $M''$ are equal. A family of metrics $\{g_t : t \in (0,\delta)\}$ on $M$ will be defined, such that when $t$ is small, $g_t$ resembles the union of $M'$ and $M''$ with their metrics $g'$ and $g''$, joined by a small 'neck' of approximate radius $t$, which is modelled upon the manifold $N$ of §8.2, with metric $t^2 g_N$.

To make the definition, choose a constant $k$ with $(n-2)(n+2)/2(n+1) < k < (n-2)(n+2)/2n$, and apply the gluing method of §10.1 twice, once to glue one asymptotically flat end of $(N, t^2 g_N)$ into $M'$ at $m'$, and once to glue the other asymptotically flat end into $M''$ at $m''$. The constant $\delta$ must be chosen to be the lesser of two constants $\delta', \delta''$ giving the range of $t$ for each of these gluings.

The rôle of $A_t$ in §10.1 is played by $A_t = A'_t \cup A''_t$, the union of an annulus $A'_t$ at the junction between $N$ and $M'$ and a second annulus $A''_t$ at the junction of $N$ and $M''$; $A'_t$ and $A''_t$ are defined to be the annuli $A_t$ of §10.1 for the two gluings. With this definition we may state the next two lemmas, which are analogues of Lemmas 10.1.1 and 10.1.2.

**Lemma 10.2.1.** *Let the scalar curvature of the metric $g_t$ be $-\epsilon_t$. Then $\epsilon_t$ is zero outside $A_t$. There is a constant $Y$ such that for any $t \in (0,\delta)$, $\epsilon_t$ satisfies $|\epsilon_t| \le Y$, and the volume of $A_t$ with respect to $g_t$ satisfies* $\mathrm{vol}(A_t) = O(t^{2nk/(n+2)})$. *These trivially imply that ${}^t\|\epsilon_t\|_{2n/(n+2)} = O(t^k)$ and ${}^t\|\epsilon_t\|_{n/2} = O(t^{4k/(n+2)})$.*

*Proof.* This is identical to Lemma 10.1.1, and its proof is the same, except that $g_t$ may also be homothetic to $g_N$ in the first sentence. □

**Lemma 10.2.2.** *For all small $t$,*

$$\int_{A'_t} \epsilon_t dV_{g_t} = (n-2)\omega_{n-1} t^{n-2} + O(t^{n-2+\alpha}) \quad \textit{and} \quad \int_{A''_t} \epsilon_t dV_{g_t} = (n-2)\omega_{n-1} t^{n-2} + O(t^{n-2+\alpha}), \tag{123}$$

*where $\omega_{n-1}$ is the volume of the $(n-1)$- dimensional sphere $\mathcal{S}^{n-1}$ of radius 1, and $\alpha > 0$ is given by* $\alpha = \min\bigl(2/n, 2k(n+1)/(n+2) - (n-2)\bigr)$.

*Proof.* This is merely Lemma 10.1.2 applied twice, firstly to the gluing of $N$ into $M'$ and secondly to the gluing of $N$ into $M''$. We have also used the observation that for both asymptotically flat ends of $N$, the constant $\mu$ of §10.1 takes the value 1. To see this, compare the definition of $\mu$ in Proposition 7.3.4 with the definition of $(N, g_N)$ in §8.2. $\square$

## 10.3. Inequalities on the connected sum manifolds

In this section we derive the analytic inequalities needed for the analysis of the next. First of all, observe that Lemma 8.4.1 applies to the metrics of §§10.1 and 10.2:

**Lemma 10.3.1.** *Let $\{g_t : t \in (0, \delta)\}$ be one of the families of metrics defined on the manifold $M = M' \# M''$ in §10.1 and §10.2. Then there exist constants $A > 0$ and $\zeta$, $0 < \zeta < \delta$, such that ${}^t\|\phi\|_p \leq A \cdot {}^t\|\phi\|_{2,1}$ whenever $\phi \in L^2_1(M)$ and $0 < t \leq \zeta$. Here norms are taken with respect to $g_t$.*

*Proof.* The proof follows that of Lemma 8.4.1, applied to the metrics of §§10.1 and 10.2 rather than §§8.1 and 8.2, except for some simple changes to take into account the different powers of $t$ used to define the new metrics. $\square$

As in §9.3, we shall need some results about the spectrum of $a\Delta$ on $(M, g_t)$ in order to calculate with the inverse of $a\Delta - \nu b$, and we are again going to relegate the proofs of these results to Appendix D rather than giving them in full here. For the metrics $g_t$ of §10.1, the situation is quite simple: the eigenvalues of $a\Delta$ on $M'$ are zero (associated to the constants), and otherwise are positive and bounded below, say by $2\gamma > 0$. Then for sufficiently small $t$, the eigenvalues of $a\Delta$ on $M$ with the metric $g_t$ are zero (associated to the constants) and otherwise are positive and bounded below by $\gamma$. This is the content of the next theorem, which is very similar to Theorem 9.3.1.

**Theorem 10.3.2.** *Let $\{g_t : t \in (0, \delta)\}$ be the family of metrics defined on the manifold $M = M' \# M''$ in §10.1. Choose $\gamma > 0$ such that all positive eigenvalues of $a\Delta$ on $M'$ are greater than or equal to $2\gamma$. Then for all sufficiently small $t$, zero is an eigenvalue of $a\Delta$ on $(M, g_t)$ associated to the constant functions, and all other eigenvalues of $a\Delta$ on $(M, g_t)$ are greater than or equal to $\gamma$.*

*Proof: see Appendix D.*

For the metrics $g_t$ of §10.2, though, the situation is more complicated. For small $t$ we expect the eigenvectors of $a\Delta$ on $(M, g_t)$ with small eigenvalues to be close to eigenvectors of $a\Delta$ on $M'$ or $M''$ with small eigenvalues, that is, to constant functions on $M'$ and $M''$. So we expect two eigenvectors on $(M, g_t)$ associated to small eigenvalues, one of which will certainly be a constant function, but the other one of which will be close to one constant value on the $M'$ part of $M$, and to a different constant value on the

$M''$ part of $M$. This is in fact what happens, and in the following proposition we give some information about the eigenvector.

**Proposition 10.3.3.** *Let $\{g_t : t \in (0,\delta)\}$ be the family of metrics defined on the manifold $M = M'\#M''$ in §10.2. Then for sufficiently small $t$, there exists a positive number $\lambda$ and a function $\beta \in C^\infty(M)$ such that $a\Delta\beta = \lambda\beta$. Here $\lambda = O(t^{n-2})$, and $\beta$ satisfies*

$$\beta = \begin{cases} \ \ 1 + O(t^{n-2}) & \text{on } M' \setminus B' \\ \ \ 1 + O(t^{n-2}|v|^{2-n}) & \text{on } \{v : t \le |v| < \delta\} \subset B' \\ -1 + O(t^{n-2}) & \text{on } M'' \setminus B'' \\ -1 + O(t^{n-2}|v|^{2-n}) & \text{on } \{v : t \le |v| < \delta\} \subset B'', \end{cases} \tag{124}$$

*identifying subsets of $M', M''$ with subsets of $M$, by abuse of notation.*

*Proof: see Appendix D.*

The proposition is proved by a series method, starting with a function that is 1 on the part of $M$ coming from $M'$ and $-1$ on the part coming from $M''$, and then adding small corrections to get to an eigenvector of $a\Delta$. Note that $\beta$ takes the approximate values $\pm 1$ on the two halves because $\mathrm{vol}(M') = \mathrm{vol}(M'')$ by assumption; if the volumes were different, then the approximate values would have to be adjusted so that $\int_M \beta dV_{g_t} = 0$.

Having constructed this eigenvector, we may now state the analogue of Theorem 10.3.2 for the metrics of §10.2, which guarantees that 0 and $\lambda$ are the only small eigenvalues of $a\Delta$ on $(M, g_t)$ for small $t$.

**Theorem 10.3.4.** *Let $\{g_t : t \in (0,\delta)\}$ be the family of metrics defined on the manifold $M = M'\#M''$ in §10.2. Choose $\gamma > 0$ such that all positive eigenvalues of $a\Delta$ on $M'$ and $M''$ are greater than or equal to $2\gamma$. Then for all sufficiently small $t$, zero is an eigenvalue of $a\Delta$ associated to the constant functions, $\lambda$ is an eigenvalue of $a\Delta$ associated to $\beta$ as in Proposition 10.3.3, and all other eigenvalues of $a\Delta$ are greater than or equal to $\gamma$.*

*Proof: see Appendix D.*

Theorems 10.3.2 and 10.3.4 will fit into the existence proofs of the next section in the same way as Theorem 9.3.1 does into that of §9.3. The one or two small eigenvalues mean that components of functions in the direction of the corresponding eigenvectors will have to be carefully controlled, to ensure that inverting an operator with a small eigenvalue does not give a large result. This is the purpose of the last result of this section, which shows that the $\beta$- component of $\epsilon_t$ is small.

**Lemma 10.3.5.** *Let $\{g_t : t \in (0, \delta)\}$ be the family of metrics defined on the manifold $M = M' \# M''$ in §10.2. Then for sufficiently small $t$,*

$$\int_M \beta\epsilon_t dV_{g_t} = O(t^{n-2+\alpha}), \tag{125}$$

*where $\beta$ is the function described in Proposition 10.3.3 and $\alpha$ is the constant of Lemma 10.2.2.*

*Proof.* By Proposition 10.3.3, $\beta = 1 + O(t^{2(n-2)/n})$ on $A'_t$ and $\beta = -1 + O(t^{2(n-2)/n})$ on $A''_t$, as these are annuli in which $t^{(n-2)/n} < |v| < t^{2k/(n+2)}$. Applying these estimates and Lemmas 10.2.1 and 10.2.2 to the integral of $\beta\epsilon_t$ over $M$, we get

$$\int_M \beta\epsilon_t dV_{g_t} = O(t^{n-2+\alpha}) + Y\,\mathrm{vol}(A_t) \cdot O(t^{2(n-2)/n}),$$

and as $\mathrm{vol}(A_t) = O(t^{2nk/(n+2)})$, the second term is $O(t^{2nk/(n+2)+2(n-2)/n})$. But by the definitions of $k$ and $\alpha$, it is easily shown that $n - 2 + \alpha < 2nk/(n + 2) + 2(n - 2)/n$, and so the first error term is larger and subsumes the second, as required. □

We note that this lemma is the reason for requiring that $\mathrm{vol}(M') = \mathrm{vol}(M'')$. For if the two are not equal, then Lemma 10.2.2 still shows that $\int_{A'_t} \epsilon_t dV_{g_t}$ and $\int_{A''_t} \epsilon_t dV_{g_t}$ are equal to highest order, but $\beta$ takes values approximately proportional to $\mathrm{vol}(M')^{-1}$ on $A'_t$, and to $\mathrm{vol}(M'')^{-1}$ on $A''_t$. Thus in this case the integral of $\beta\epsilon_t$ does not cancel out to highest order, but is $O(t^{n-2})$.

In the next section we will see that the error estimate of Lemma 10.3.5 is crucial for the existence of a small conformal rescaling to constant scalar curvature. There is a clear geometrical reason for this: $\lambda^{-1} \int_M \beta\epsilon_t dV_{g_t}$ is a measure of the relative rescaling of $M'$, $M''$ required to define a metric of constant scalar curvature on the connected sum, as this is the $\beta$- component of $\big(a\Delta\big)^{-1}(\epsilon_t - \mathrm{const.})$, which is a sort of first approximation to $\phi$, where $(1 + \phi)^{p-2}$ is the rescaling factor. Thus if $\int_M \beta\epsilon_t dV_{g_t} = O(t^{n-2})$ then this component is $O(1)$, and the rescaling factor need not be small, but if the integral is $O(t^{n-2+\alpha})$ then the component is $O(t^\alpha)$, and is small for small $t$.

## 10.4. Existence of constant scalar curvature metrics

Now we give the existence results for constant scalar curvature metrics on the connected sums of §10.1 and §10.2.

**Theorem 10.4.1.** *Let $\{g_t : t \in (0, \delta)\}$ be one of the families of metrics on the connected sum $M = M' \# M''$ defined in §10.1 or §10.2. Then there exists a constant $C$ such that for sufficiently small $t$, the metric $g_t$ admits a smooth conformal rescaling to $\tilde{g}_t = (1+\phi)^{p-2} g_t$, which is a nonsingular Riemannian metric of scalar curvature $-(n-2)\omega_{n-1}\mu t^{n-2} \operatorname{vol}(M')^{-1}$ in the case of §10.1, and of scalar curvature $-(n-2)\omega_{n-1} t^{n-2} \operatorname{vol}(M')^{-1}$ in the case of §10.2, and $\phi$ satisfies ${}^t\|\phi\|_{2,1} \leq Ct^{\alpha}$. Here $\mu$ and $\alpha$ are the constants of §10.1, $\omega_{n-1}$ is the volume of the $(n-1)$- dimensional sphere of radius 1, and ${}^t\|.\|_{2,1}$ is the $L^2_1$ norm induced by $g_t$.*

*Proof.* Let $D_0$ be equal to $(n-2)\omega_{n-1}\mu \operatorname{vol}(M')^{-1}$ in the case of §10.1 and to $(n-2)\omega_{n-1} \operatorname{vol}(M')^{-1}$ in the case of §10.2. Define a function $\eta$ on $(M, g_t)$ by $\eta = \epsilon_t - D_0 t^{n-2}$. Then $-\eta$ represents the deviation of the scalar curvature of $g_t$ from the constant value $-D_0 t^{n-2}$. By the same reasoning as in Chapter 9, the condition for $\tilde{g}_t = (1+\rho+\tau)^{p-2} g_t$ to have scalar curvature $-D_0 t^{n-2}$ is

$$a\Delta(\rho+\tau) + bD_0 t^{n-2}(\rho+\tau) = \eta + \eta \cdot (\rho+\tau) - D_0 t^{n-2} f(\rho+\tau). \tag{126}$$

Let $P$ be the constant functions on $M$ for the case of §10.1, and the sum of the constant functions and multiples of $\beta$ (described in Proposition 10.3.3), for the case of §10.2. We shall construct $\rho$ and $\tau$ satisfying (126), with $\rho \in P$ and $\tau \in P^{\perp}$ with respect to the $L^2_1$ inner product.

Define inductively sequences $\{\rho_i\}_{i=0}^{\infty}$ of elements of $P$ and $\{\tau_i\}_{i=0}^{\infty}$ of elements of $P^{\perp} \subset L^2_1(M)$ by $\rho_0 = \tau_0 = 0$, and having defined the sequences up to $i-1$, let $\rho_i$ and $\tau_i$ be the unique elements of $P$ and $P^{\perp}$ satisfying

$$a\Delta(\rho_i+\tau_i) + bD_0 t^{n-2}(\rho_i+\tau_i) = \eta + \eta \cdot (\rho_{i-1}+\tau_{i-1}) - D_0 t^{n-2} f(\rho_{i-1}+\tau_{i-1}). \tag{127}$$

If we can show that these sequences converge to $\rho \in P$ and $\tau \in P^{\perp}$ that are small when $t$ is small, then the arguments of Chapter 9 complete the theorem.

The difficulty lies in inverting the operator $a\Delta + bD_0 t^{n-2}$: by Theorems 10.3.2 and 10.3.4, the operator is invertible on $P^{\perp}$ with inverse bounded by $\gamma^{-1}$, as the Lemmas show that all the eigenvectors of $a\Delta$ in $P^{\perp}$ have eigenvalues at least $\gamma$. But on $P$, the inverse is of order $t^{2-n}$, which is large; so $\rho_i$ may be quite large even if the right hand side of (127) is quite small.

The solution is to ensure that the $P$ components of $\eta$ are smaller even than $t^{n-2}$, so that after applying the inverse of $a\Delta + bD_0 t^{n-2}$ to them, they are still small. Let $\pi$ denote orthogonal projection onto $P$;

both the $L^2$ and the $L^2_1$ inner product give the same answer, and in fact the projection makes sense even in $L^1(M)$. Then from (127) we make the estimates

$$\|\rho_i\|_{2,1} \leq D_1 t^{2-n}\Big(\|\pi(\eta)\|_1 + \|\pi(\eta\rho_{i-1})\|_1 + \|\pi(\eta\tau_{i-1})\|_1\Big) + D_2\big\|\pi\big(f(\rho_{i-1}+\tau_{i-1})\big)\big\|_1, \tag{128}$$

$$\|\tau_i\|_{2,1} \leq D_3\Big(\|\eta\|_{2n/(n+2)} + \|\eta\rho_{i-1}\|_{2n/(n+2)} + \|\eta\tau_{i-1}\|_{2n/(n+2)} + D_0 t^{n-2}\|f(\rho_{i-1}+\tau_{i-1})\|_{2n/(n+2)}\Big), \tag{129}$$

for some constants $D_1, D_2, D_3$ independent of $t$. The norms on the right hand side of (128) would normally be $L^{2n/(n+2)}$ norms, but as $P$ is a finite-dimensional space all norms are equivalent, and we may use the $L^1$ norm.

Our strategy is to show that if $\|\rho_{i-1}\|_{2,1} \leq D_4 t^\alpha$ and $\|\tau_{i-1}\|_{2,1} \leq D_5 t^k$ for large enough constants $D_4, D_5$, then $\|\rho_i\|_{2,1} \leq D_4 t^\alpha$ and $\|\tau_i\|_{2,1} \leq D_5 t^k$ also hold for sufficiently small $t$, so by induction the sequences are bounded; convergence for small $t$ easily follows by a similar argument to that used in Lemma 9.1.2.

From Lemmas 10.1.2, 10.2.2 and 10.3.5 we may deduce that $\|\pi(\eta)\|_1 = O(t^{n-2+\alpha})$, so the first term on the right of (128) contributes $O(t^\alpha)$ to $\|\rho_i\|_{2,1}$, consistent with $\|\rho_i\|_{2,1} \leq D_4 t^\alpha$ if $D_4$ is chosen large enough. The third term $\|\pi(\eta\tau_{i-1})\|_1$ is bounded by $A\|\eta\|_{2n/(n+2)}\|\tau_{i-1}\|_{2,1}$, and by Lemmas 10.1.1 and 10.2.1, $\|\eta\|_{2n/(n+2)} = O(t^k)$; the third term therefore contributes $O(t^{2k+2-n})$ to $\|\rho_i\|_{2,1}$, and by the definition of $\alpha$, this error term is strictly smaller than $O(t^\alpha)$. The fourth error term is also easily shown to be smaller than $O(t^\alpha)$.

Thus the only problem term in (128) is the second term, and the only reason it is a problem is that the $P^\perp$ component of $\eta$, multiplied by $\rho_{i-1}$, may have an appreciable component in $P$. We get round this as follows. Suppose $\xi \in P^\perp$ and $\rho \in P$, and consider the $P$ component of $\xi\rho$. In the case of §10.1 this component is zero, and there is no problem; in the case of §10.2 there may be a component in the direction of $\beta$, and it is measured by $\int_M \xi\beta^2 dV_{g_t}$. But by the description of $\beta$ in Proposition 10.3.3, $\beta^2$ is close to 1, and $\xi$ is orthogonal to the constants, and so in general the $P$ component of $\xi\rho$ will be small compared to the sizes of $\xi$ and $\rho$. Taking this into account, it is easy to get a good bound on $\|\pi(\eta\rho_{i-1})\|_1$.

The rest of the proof will be left to the reader. What remains to be done is to prove inductively that bounds $\|\rho_i\|_{2,1} \leq D_4 t^\alpha$, $\|\tau_i\|_{2,1} \leq D_5 t^k$ hold for small enough $t$, and then to prove the convergence of the sequences, and these may both be done using the methods of Lemma 9.1.2, working from (128) and

(129). Setting $\phi = \rho + \tau$, where $\rho, \tau$ are the limits of the sequences, the reader may then rejoin the proof of Theorem 9.1.1 after Lemma 9.1.3. $\square$

Now as the metrics constructed have negative scalar curvature, they are unique in their conformal classes by Lemma 9.2.4, and are Yamabe metrics. The theorem thus tells us that the Yamabe metric on the connected sum, with small neck, of two zero scalar curvature manifolds, balances the volumes of the two component manifolds so that they are equal, a fact which seems to me to be rather appealing.

## 10.5. Combining a metric of zero and a metric of negative scalar curvature

We shall not handle this case in very much detail, partly in order not to tire the reader with another rehash of the material of Chapter 9, and partly for a reason that will be explained later. Consider the connected sum $M$ of a manifold $M'$ with scalar curvature $-1$, and a manifold $M''$ with scalar curvature 0. What do we expect constant scalar curvature metrics in the conformal class to look like?

Well, the negative scalar curvature will win, because negative scalar curvature is always dominant, so we expect a metric of scalar curvature $-1$, which will be unique by Lemma 9.2.4. Imagining $M''$ to have very small positive or negative scalar curvature instead, and extrapolating from the results of §9.2, we expect $M$ to look like a copy of $M'$ with its metric nearly unchanged, but with a small copy of $M''$ glued in at one point.

It is the nature of this gluing that interests us. What actually happens is that the metric $g''$ of $M''$ is homothetically shrunk by multiplying it by $t^{2(n-2)/n}$, and then it is joined to $M'$ by a 'neck' of approximate radius $t$, for small $t$. So the diameter of $M''$ gets multiplied by $t^{(n-2)/n}$, but the radius of the neck is $t$. This is a sort of interpolation between the case that $M''$ has positive scalar curvature, when in effect the diameter of $M''$ is multiplied by $t$ when the radius of the neck is $t$, and the case that $M''$ has negative scalar curvature, when the diameter of $M''$ gets multiplied by 1 when the radius of the neck is $t$.

One may see this as follows. Let $M''$ be a compact manifold of zero scalar curvature, let $m''$ be a point of $M''$ with a conformally flat neighbourhood, and let $\xi$ be the Green's function of $a\Delta$ satisfying $a\Delta\xi = \delta_{m''} - \mathrm{vol}(M'')^{-1}$ in the sense of distributions. Since $\xi$ is only defined up to the addition of a constant, choose $\xi$ to have minimum value 0. Then $\xi$ is a $C^\infty$ function on $M'' \setminus \{m''\}$ with a pole at $m''$, of the form $F_0|v|^{2-n} + O'(|v|^{1-n})$ in the usual coordinates. Here $F_0 > 0$, and actually takes the value $(n-2)\omega_{n-1}^{-1}$.

Consider the metric $g''_t = \bigl(t^{(n-2)^2/2n} + t^{(n-2)(n+2)/2n} \operatorname{vol}(M'')\xi\bigr)^{p-2} g''$ on $M'' \setminus \{m''\}$. Calculating its scalar curvature $S_t$ using (73) gives

$$S_t = -\bigl(t^{(n-2)^2/2n} + t^{(n-2)(n+2)/2n} \operatorname{vol}(M'')\xi\bigr)^{-\frac{n+2}{n-2}} \cdot t^{(n-2)(n+2)/2n} = -\bigl(1 + t^{2(n-2)/n} \operatorname{vol}(M'')\xi\bigr)^{-\frac{n+2}{n-2}}, \tag{130}$$

so that $-1 \le S_t < 0$ for $t > 0$, and when $t$ is small, $S_t$ is close to $-1$ everywhere in $M''$ except very close to $m''$.

How shall we describe this new metric $g''_t$? For small $t$, outside a small neighbourhood of $m''$ in $M''$ the metric $g_t$ is approximately rescaled by $t^{2(n-2)/n}$, so that the diameter of $M''$ is multiplied by $t^{(n-2)/n}$ as we said above. But in a small neighbourhood of $m''$, the metric resembles

$$g''_t \sim \bigl(t^{(n-2)^2/2n} + F_0 \operatorname{vol}(M'') t^{(n-2)(n+2)/2n} |v|^{2-n}\bigr)^{p-2} h = \bigl(1 + F_0 \operatorname{vol}(M'') t^{n-2} |v|^{2-n}\bigr)^{p-2} h, \tag{131}$$

changing variables from $v$ to $t^{(n-2)/n} v$ to simplify the expression. The right hand side is an expression familiar to us already, and represents a metric with a 'neck' of radius proportional to $t$. So the metric $g''_t$ looks like $M''$ rescaled by $t^{(n-2)/n}$, and with a 'neck' of radius proportional to $t$, opening out to an asymptotically flat end. On the $M''$ side of the neck, the scalar curvature is close to $-1$, and on the asymptotically flat side of the neck, the scalar curvature approaches zero.

We may now form a family of metrics $\{g_t : t \in (0,\delta)\}$ on $M$ by gluing the metrics $g''_t$ into $g'$ at $m'$, and get a good approximation to constant scalar curvature $-1$. Therefore, the next step is to consider whether the results of §9.2 may be applied to this family without change. They cannot be, and the reason they fail is that Lemma 8.4.1 does not hold for this family of metrics. For consider a function $\phi$ that is equal to 0 on the $M'$ side of the neck and to 1 on the $M''$ side of the neck, and changes over the neck in a smooth fashion. It may easily be calculated that $\|\phi\|_{2n/(n-2)} = O\bigl(t^{(n-2)^2/2n}\bigr)$, but that $\|\phi\|_{2,1} = O\bigl(t^{(n-2)/2}\bigr)$, so that $\|\phi\|_{2n/(n-2)}/\|\phi\|_{2,1} = O\bigl(t^{(2-n)/n}\bigr)$. Thus the best result we can hope for to replace Lemma 8.4.1 is one that says ${}^t\|\phi\|_{2n/(n-2)} \le A t^{(2-n)/n} \cdot {}^t\|\phi\|_{2,1}$ for $\phi \in L^2_1(M)$, where the norms are taken with respect to $g_t$.

Such a result can indeed be proved, and then we may go on to prove existence results for small conformal changes of the metrics $g_t$ metrics giving scalar curvature $-1$. The proof roughly follows that of §§9.1 and 9.2, but constants such as $A$ in the formulae must be replaced by multiples of powers of $t$, and a subterfuge must be used to cut down the number of applications of the modified Lemma 8.4.1. The difference is that, because the constants of the proof now contain unfavourable powers of $t$, the estimates on the difference $\epsilon_t$ between the scalar curvature of $g_t$, and the constant value $-1$, must be much better in order to compensate. So the metrics $g_t$ need to be really good approximations to constant scalar curvature to begin with.

The way we define such good approximations $g_t$ is to take the metrics $g''_t$ with a neck of radius proportional to $t$, define another family $g'_t$ of metrics on $M'$ with a neck of the same radius by using a Green's function for $a\Delta$ in a similar way, and then glue these two metrics $g'_t, g''_t$ together at the neck itself. One can then make estimates of the scalar curvature of the resulting metric, and we find that the scalar curvature is close enough to constant, for small $t$, for the proof to work. This provides the second reason for the lack of detail in this section: to treat it properly requires not some interesting analysis, but a lot of very careful definitions of the metrics $g_t$, and delicate estimates for their scalar curvature.

# Chapter 11: Summing Up

This chapter begins with some diagrams to give the reader a mental picture of what all the families of metrics $\{g_t : t \in (0, \delta)\}$ defined in Chapters 8 and 10 are actually like. We recommend that they are studied in conjunction with §§8.1, 8.2, 10.1, 10.2 and 10.5, and referred to later if the reader needs to be able to picture what is going on.

In §11.2 we consider how the results of the last few chapters would be changed if we removed the assumption of conformal flatness in neighbourhoods of the points $m', m''$. Our conclusion is that the basic picture presented there remains unchanged except for minor details. We finish off in §11.3 with a few interesting questions.

### 11.1. A pictorial guide to the connected sum metrics

The diagrams that follow are intended to represent $n$- dimensional manifolds, but have been drawn as two-dimensional. Below is a diagram of the metrics described in §10.5. The next two pages summarize the construction of the metrics of §§8.1 and 10.1, and of §§8.2 and 10.2, respectively.

**11.2. Doing without the assumption of conformal flatness**

In the definitions of families of metrics $\{g_t : t \in (0, \delta)\}$ in §§8.1, 8.2, 10.1 and 10.2 above, we invariably assumed that $M'$ and $M''$ were conformally flat in neighbourhoods of the points $m', m''$ at which the connected sum was performed. There is a lot to be said for this assumption: it gives canonical coordinate systems about $m', m''$ and a simple, natural family of conformal classes on the connected sum, it brings in the positive mass theorem in Chapter 10, and the topic of constant scalar curvature metrics on conformally flat manifolds seems to me to be potentially very interesting. But it restricts the application of the results and excludes, for instance, application to families of self-dual metrics on 4-manifolds, which is one reason why this work was undertaken.

So in this section we consider how the assumption may be relaxed. There are (at least) two obvious ways to try and improve the results. The first is to prove that given two Riemannian manifolds $M', M''$ containing points $m', m''$, there exists a family of metrics on the connected sum $M$, depending on a parameter $t$ and of constant scalar curvature, that approach the connected sum form we expect for small values of the parameter. The second way is to prove that given $M', M''$ and a family of conformal classes on $M$ depending upon $t$ that are close to the connected sum conformal classes when $t$ is small, we can choose constant scalar curvature metrics in the conformal classes that are close to the connected sum form we expect, for sufficiently small $t$.

Of these, the second way seems to me to be much more interesting and general than the first, and would for instance apply to families of self-dual 4-manifolds. So we shall consider how to extend the results of this part of the thesis following the second way. The first problem in following this programme is exactly how to define what it means for a family of conformal classes depending on $t$ to approach the connected sum, as the limit we suppose they approach is singular (an 'infinitely small neck', basically the two manifolds joined at a point). I can't see a particularly nice way of doing this; perhaps a method comparing $[g_t]$ on $M$ with the obvious metric on $M' \setminus B_t(m') \amalg M'' \setminus B_t(m'')$ would work.

Having done this, the proof we envisage would have two stages. The first stage is to show that the given conformal classes $[g_t]$ contain metrics of close to constant scalar curvature, with the difference bounded in suitable norms by powers of $t$ – as in Chapter 8 and §§10.1 and 10.2. Then the second stage is to follow Chapter 9 and §§10.3-10.5 to prove that constant scalar curvature metrics exist when the approximate metrics have scalar curvature close enough to constant.

After some thought, it becomes clear that the results and proofs of Chapter 9 are going to work for this more general case. The only modifications the statements will need is in the estimates on $\|\phi\|_{2,1}$ in terms of $t$; it should be estimated by the $L^{n/2}$ norm of $\epsilon$, the deviation from constant of the scalar curvature, and whether this is bounded by a multiple of $t^2$, or some other function, depends entirely upon the new set-up. The reason that the Chapter 9 work extends so readily is that the only important condition the family of conformal classes must satisfy is that they contain metrics looking quite like the metrics $g_t$ of §8.1 or §8.2, with scalar curvature bounded and close to a constant in the $L^{n/2}$ norm. This is quite an easy condition to satisfy, and if it doesn't hold, it is clear that the definition chosen for a family of metrics approaching the connected sum is an inadequate one.

However, the results of Chapter 10 will require significant modification from the conformally flat case. For §10.1 makes essential use of the constant $\mu$ proportional to the mass, and in the non-conformally flat case, the mass term is dominated by other error terms and may not even be defined. Therefore let us first consider what results we expect in generalizing Chapter 10 to the non-conformally-flat case.

For the case of §10.1, the value of the constant scalar curvature should be determined by the highest order term in the deviation from flatness of the stereographic projection of $M''$. Studying [LP], Theorem 6.5 and Proposition 7.1, it turns out that it is natural to expect scalar curvature proportional to $-\mu t^{2-n}$ when $n = 3, 4, 5$ as well as when $M''$ is conformally flat close to $m''$, proportional to $-\left|W(m'')\right|^2 t^{-4} \log t$ when $n = 6$ and $M''$ is not conformally flat at $m''$, and proportional to $-\left|W(m'')\right|^2 t^{-4}$ when $n > 6$ and $M''$ is not conformally flat at $m''$. Here $W$ is the Weyl conformal curvature and $t$ is a parameter measuring the rescaling of $M''$ as before, so that $g_t$ agrees with $t^2 g''$ on the $M''$ part of $M$.

The form of the constant scalar curvature metrics for the generalizations of the metrics of §10.2 to the non-conformally-flat case will depend very much upon what the metrics $g_t$ look like at the 'neck'. If they approach our standard conformally flat form as $t$ approaches zero, then we may expect the same behaviour as in Chapter 10, with the volumes of the two halves equalized, and scalar curvature of order $t^{n-2}$ when the width of the neck is $t$; this is quite a reasonable condition to impose, and probably holds for many applications. The proofs of Chapter 10 should extend to this case. If, however, the neck is nowhere near being conformally flat for small $t$ — say, the neck is modelled upon some conformally curved manifold with two asymptotically flat ends — then the behaviour is much more nasty, and we cannot say very much about it.

I believe the picture presented in §10.5 to hold good in the non-conformally-flat case, but I doubt if the methods of proof suggested there are strong enough to cover this case, and extra work would be required to prove anything about it.

### 11.3. Final comments

First we ask some questions on the subject matter of Part II of the thesis.

- Does the Yamabe problem fail for orbifolds? I do not know whether it does or not, but it can be seen that the techniques used in the solution of the manifold case do not cover the orbifold case as well. For suppose that $(M, g)$ has an orbifold point $m$ with finite orbifold group $\Gamma$. Then by using test functions concentrated about $m$, we can show that $\lambda(M) \le \lambda(\mathcal{S}^n)/|\Gamma|$. The test function method applied at any nonsingular point only shows that $\lambda(M) < \lambda(\mathcal{S}^n)$, so that $\lambda(M) = \lambda(\mathcal{S}^n)/|\Gamma|$ is not excluded by this.

  To solve the generalized Yamabe problem using the old methods, it must be shown that $\lambda(M) < \lambda(\mathcal{S}^n)/|\Gamma|$. But this cannot be guaranteed (at least in four dimensions) because of the failure of the generalized positive mass theorem for 4-orbifolds [L1]. So the case $\lambda(M) = \lambda(\mathcal{S}^n)/|\Gamma|$ could conceivably arise, and in this case the known methods tell us nothing about the existence of Yamabe metrics.

- Does the constant scalar curvature condition mix well with self-duality of 4-manifolds? That is, does choosing a constant scalar curvature metric within the conformal class of a self-dual manifold help in any way with the study of moduli spaces of self-dual metrics, for instance in looking at behaviour at the edge of the moduli space? In connection with this, King and Kotschick have already found that the Yamabe invariant plays a rôle in the theory of the moduli spaces ([KK], §3).

- On a compact manifold with positive scalar curvature, is the condition that a conformal class containing metrics of positive scalar curvature should contain an element of scalar curvature 1 for which $b$ is not an eigenvalue of $a\Delta$, a generic condition on such conformal classes? I firmly believe that it is, but cannot quite prove it.

- In the case of positive scalar curvature, what are the possible ways in which critical points of the Hilbert action can vanish as the underlying conformal class changes in a smooth, nonsingular way? As we shall shortly explain in more detail, by using the results of §9.3 for the connected sum of a manifold of positive scalar curvature with a sphere with a non-round metric, and then deforming this connected sum conformal class back to the original conformal class, it can be seen that critical points may annihilate each other in pairs. But are there any other possibilities; can isolated critical points just vanish, for instance?

Following on from the last question, we shall now discuss the critical points of the Hilbert action $Q$, to lead up to formulating a conjecture about them. A conformal class of metrics on a manifold is contractible, as is a conformal class of metrics modulo homotheties. Trying to use the Hilbert action as a sort of Morse function on such a conformal class will therefore not yield any interesting information on the topology of the class, but it might on the other hand yield some information upon the stationary points of the Hilbert action on the class (modulo homotheties), which are just the metrics of constant scalar curvature in the class. However, this analogy cannot necessarily be pushed very far, as a conformal class is noncompact, and the behaviour of the Hilbert action towards the 'edge' of it is fairly horrible.

For negative scalar curvature manifolds we know by Lemma 9.2.4 that there is a unique stationary point of $Q$ that is a minimum, and so has index zero. So we shall consider the positive scalar curvature case. Let us suppose that critical points of the Hilbert action on a compact conformal class of positive scalar curvature are well behaved under smooth changes of the conformal class, so that critical points do not appear or disappear from the 'edge' of the conformal class during such changes. Suppose also that there are only finitely many critical points of $Q$, that they are all suitably generic, and that they all have finite index. Can we make any sort of topological invariant from the critical points?

In this simplest situation, the only sort of change in the number and nature of the critical points one expects during a change of the conformal class, is for pairs of critical points with indices differing by one to appear or disappear together. Thus the integer $r = p - q$ would be conserved, where $p$ is the number of critical points of $Q$ on the conformal class of even index, and $q$ is the number of critical points of odd index. Then $r$ would be an integer invariant of compact smooth manifolds of positive scalar curvature, as it could be evaluated by choosing a generic conformal class of positive scalar curvature on the manifold, and counting the stationary points of $Q$ on it.

These are perhaps far-fetched suppositions; on a good day, with the wind behind us, it might be possible to define this invariant $r$ for some limited class of compact manifolds of some dimension. But what value would $r$ take? We will now argue heuristically that $r$ should take the value 1 for all positive scalar curvature manifolds. Let $M'$ and $M''$ be compact manifolds of the same dimension $n$, with suitably generic metrics of positive scalar curvature. Using the results of §9.3, we can construct the metrics of scalar curvature 1 in the conformal class of the connected sum $M'\#M''$, given the metrics of scalar curvature 1 in the chosen conformal classes on $M'$ and $M''$. These lead us to the formula

$$r(M'\#M'') = r(M') + r(M'') - r(M')r(M''). \tag{132}$$

Here the first two terms on the right come from the metrics modelled on those of §8.1, and the last term comes from the metrics modelled on those of §8.2. The last term has a negative sign because it can be shown that the index of a critical point contributing to this term is 1 plus the sum of the indices of the relevant critical points on $M'$ and $M''$.

Now (132) may be rewritten in the multiplicative form

$$1 - r(M'\#M'') = \big(1 - r(M')\big) \cdot \big(1 - r(M'')\big). \tag{133}$$

It follows that $r(\mathcal{S}^n)$ is either 0 or 1. Here is an argument to show that $r(\mathcal{S}^n)$ is odd: it can easily be shown that the round metric on $\mathcal{S}^n/\{\pm 1\}$ is the unique metric of scalar curvature 1 in its conformal class. As this metric is 'generic' in the sense that $b$ is not an eigenvalue of $a\Delta$ on it, every nearby conformal class must also contain exactly one metric of scalar curvature 1.

Taking the double cover, any metrics of scalar curvature 1 in the conformal class of the double cover that do not come from metrics on $\mathcal{S}^n/\{\pm 1\}$ must come in pairs, and so make no contribution modulo 2 to $r(\mathcal{S}^n)$. Thus $r(\mathcal{S}^n) \equiv 1$ modulo 2, and $r(\mathcal{S}^n) = 1$. Putting $M''$ equal to $\mathcal{S}^n$ in (133) then gives $r(M') = 1$ for any $M'$ of dimension $n$ and positive scalar curvature.

The informal argument above suggests, rather than proves, that when the integer $r$ exists it ought to take the value 1. This is the motivation for the following conjecture:

**Conjecture 11.3.1.** *Suppose that $M$ is a compact manifold admitting metrics of positive scalar curvature. Then on a generic conformal class of positive scalar curvature, the Hilbert action $Q$ has finitely many stationary points each of finite index, and the difference between the number $p$ of stationary points of even index and the number $q$ of odd index is $p - q = 1$.*

One might ask whether a proof of this conjecture would lead to an alternative proof of the Yamabe conjecture in the positive scalar curvature case. I think that it would not; that is, though logically speaking it would, it would be necessary to use most of the difficult material in the present proof to make the new proof, so the new proof would simply be the old proof made harder. My reason for thinking this is that to show that a critical point cannot disappear during a smooth change of the underlying conformal class by developing a point concentration of volume — equivalent to 'budding off' a copy of $\mathcal{S}^n$ with its round metric by connected sum — a 'Nonzero mass theorem' would be required, which is probably just as difficult to prove as the positive mass theorem.

# Appendix A: Another Proof of the Hypercomplex Case of Theorem 3.1.1

In this appendix we give an alternative proof of the hypercomplex case of Theorem 3.1.1 that does not involve complexifying group actions, and does not invoke the Ward correspondence; in fact the methods used are more-or-less a proof of the Ward correspondence. The proof works by defining the new almost complex structures and showing that the Nijenhuis tensor of each vanishes, and thus that the almost complex structures are integrable. The quaternionic case of the theorem may be deduced by applying the hypercomplex case to the associated bundle.

**Theorem 3.1.1, hypercomplex case.** *Let $M$ be hypercomplex and $P$, $\Phi$ and $\Psi$ be as in §3.1, and let $A$ be a $\Psi$- invariant quaternionic connection on $P$. Suppose $\Psi(G)$ acts freely on $P$. Then the manifold $N = P/\Psi(G)$ has a natural (possibly singular) hypercomplex structure, which is nonsingular wherever the Lie algebra of $\Psi(G)$ is transverse to the horizontal subspaces of $A$ in $P$.*

*Proof.* Let the projections of $P$ to $M$ be $\pi$ and to $N$ be $\rho$, and let the field of horizontal subspaces of $A$ in $P$ be $H$. Where the Lie algebra of $\Psi(G)$ is transverse to $H$, projection to $N$ induces an isomorphism between $H$ and $\rho^*(TN)$. On the other hand, projection to $M$ gives an isomorphism between the $H$ and $\pi^*(TM)$. But $M$ is hypercomplex, and so $I_1, I_2, I_3$ act on $TM$. The lifting of these almost complex structures to $H$ is $\Psi$- invariant, and induces an almost hypercomplex structure $(I_1, I_2, I_3)$ on $N$.

Thus it is sufficient to prove that each of these almost complex structures $I_j$ on $N$ is integrable, except at points where $\Psi(G)$ is not transverse to $H$. As this is a local property, it need only be proved for arbitrarily small patches of $N$. This will be done by calculating the Nijenhuis tensors of the almost complex structures.

If $U$ is a patch of manifold, and $i$ is a smooth almost complex structure on $U$, then the Nijenhuis tensor $N_i$ of $i$ is defined by

$$N_i(x, y) = [x, y] + i[ix, y] + i[x, iy] - [ix, iy], \tag{134}$$

where $x, y$ are vector fields on $U$ and $[\,,\,]$ is the Lie bracket. It can be shown that $N_i$ is a tensor (so it is bilinear with respect to multiplication of $x, y$ by scalar fields), and that $i$ is integrable if and only if $N_i$ vanishes identically.

Let $U$ be a small open patch of $M$ on which $\Psi(G)$ is transverse to $H$. If $U$ is sufficiently small, we may choose a trivialization $P|_U = U \times G$, such that the action of $\Phi$ on $P$ is given as usual by $\Phi(g)\big((g',u)\big) = (gg',u)$, and the action of $\Psi$ is given by $\Psi(g)\big((g',u)\big) = (g'g^{-1}, \Psi(g)u)$.

Let $\omega$ be the $\mathfrak{g}$-valued 1-form on $U$ that represents $A$ with respect to this trivialization. Regarding $P|_U$ as $U \times G$, the submanifold $U \times \{1\}$ is transverse to the action of $\Psi(G)$, and so is isomorphic to a small open set $V$ in $N$. (Note that $U \times \{1\}$ would not be transverse to $\Psi(G)$ if the trivialization had been chosen $\Psi$- invariant.) We shall identify $U$ and $V$ in the obvious way.

From above, it is sufficient to show that $I_1, I_2, I_3$ (acting on TN) are integrable on $V$. Choose $j = 1, 2$ or 3. Denote the complex structure $I_j$ on $U$ (coming from $M$) by $i$, and the almost complex structure $I_j$ on $V$ (coming from $N$) by $J$; under the identification of $U, V$ this gives two almost complex structures $i, J$ on $U$.

We shall give a formula for $J$ in terms of $i$. Now the action $\Psi$ of $G$ on $M$ induces a map $\psi : \mathfrak{g} \to \Gamma(TU)$ that is a Lie algebra homomorphism (where $\Gamma(TU)$ is a Lie algebra with operation the Lie bracket of vector fields). The isomorphism of $TU$ with $H|_{U\times\{1\}}$ is $x \mapsto (x, -\omega(x))$ (so that $\omega$ vanishes on $H$), and under the identification of $U$ and $V$, the isomorphism of $H|_{U\times\{1\}}$ with $TV$ is $(x, -\omega(x)) \mapsto x - \psi(\omega(x))$. (Since in the chosen trivialization the action of the Lie algebra of $\Psi(G)$ on $TP|_{U\times\{1\}}$ is $g' : (x, g) \mapsto (x + \psi(g'), g - g')$, for $x$ a vector on $U$ and $g, g' \in \mathfrak{g}$.)

Thus, although we have identified $U$ and $V$, the identification of $TU$ and $TV$ coming from the double identification of $\pi^*(TU)$ and $\rho^*(TV)$ with $H|_{U\times G}$ is $x \mapsto x - \psi(\omega(x))$. Let $A = \psi \circ \omega$. Then $J$ is defined by the formula

$$J = (1 - A)i(1 - A)^{-1}, \tag{135}$$

where 1 denotes the identity on $TU$ (to avoid confusion with almost complex structures.) Note that $1 - A$ is invertible exactly when $\Psi(G)$ is transverse to $H$, which by assumption holds on $U$.

We calculate $N_J(x - Ax, y - Ay)$ for arbitrary vector fields $x, y$:

$$\begin{aligned} N_J(x - Ax, y - Ay) &= [x - Ax, y - Ay] + J[J(x - Ax), y - Ay] + J[x - Ax, J(y - Ay)] \\ &\quad - [J(x - Ax), J(y - Ay)] \\ &= [x - Ax, y - Ay] + J[ix - Aix, y - Ay] + J[x - Ax, iy - Aiy] \\ &\quad - [ix - Aix, iy - Aiy]. \end{aligned} \tag{136}$$

As $1-A$ is invertible, $N_J(x,y)$ vanishes for all $x,y$ if $N_J(x-Ax,y-Ay)$ does, and the proof of Theorem 3.1.1 will be completed by the next lemma.

**Lemma A.1.** *Under the hypotheses above,*

$$\begin{aligned} N_J(x-Ax,y-Ay) =&[x-Ax,y-Ay]+J[ix-Aix,y-Ay] \\ &+J[x-Ax,iy-Aiy]-[ix-Aix,iy-Aiy]=0. \end{aligned} \tag{137}$$

*Proof.* Let $\Omega$ be the curvature of $A$ on $U$. As by assumption $A$ is of type $(1,1)$ with respect to $i$, for vector fields $x,y$ we have $\Omega(x,y)-\Omega(ix,iy)=0$. Applying $\psi$ to this equation gives

$$\psi(\Omega(x,y))-\psi(\Omega(ix,iy))=0,$$

and also

$$\psi(\Omega(ix,y))+\psi(\Omega(x,iy)=0.$$

Therefore

$$\psi(\Omega(x,y))+J\psi(\Omega(ix,y))+J\psi(\Omega(x,iy))-\psi(\Omega(ix,iy))=0. \tag{138}$$

This condition will be expressed in terms of $\omega$, and it will be seen that it differs from (137) by a term that vanishes because $\Psi$ preserves $J$.

First we find an expression for $\psi(\Omega(x,y))$. Now the curvature $\Omega$ is usually written $d\omega+\frac{1}{2}[\omega\wedge\omega]$. In terms of vector fields $x,y$,

$$d\omega(x,y)=\mathcal{L}_x(\omega(y))-\mathcal{L}_y(\omega(x))-\omega([x,y]) \qquad \text{and} \qquad \frac{1}{2}[\omega\wedge\omega](x,y)=[\omega(x),\omega(y)]_{\mathfrak{g}},$$

where $\mathcal{L}_x$ is the Lie derivative with respect to the vector field $x$, and $[\,,\,]_{\mathfrak{g}}$ is the Lie bracket in $\mathfrak{g}$ (to distinguish it from the Lie bracket of vector fields). Thus

$$\Omega(x,y)=\mathcal{L}_x(\omega(y))-\mathcal{L}_y(\omega(x))-\omega([x,y])+[\omega(x),\omega(y)]_{\mathfrak{g}}. \tag{139}$$

So

$$\begin{aligned}\psi(\Omega(x,y)) &= \psi\big(\mathcal{L}_x(\omega(y))\big) - \psi\big(\mathcal{L}_y(\omega(x)\big) - \psi\circ\omega([x,y]) + \psi([\omega(x),\omega(y)]_{\mathfrak{g}}) \\ &= [x,Ay] - [y,Ax] - A([x,y]) - \big(\mathcal{L}_x\psi\big)\omega(y) + \big(\mathcal{L}_y\psi\big)\omega(x) \\ &\quad + \big(\mathcal{L}_{Ax}\psi\big)\omega(y) - \big(\mathcal{L}_{Ay}\psi\big)\omega(x) - [Ax,Ay],\end{aligned} \tag{140}$$

where we justify the substitution for $\psi([\omega(x),\omega(y)]_{\mathfrak{g}})$ as follows: choose a connection $\nabla$ on $TU$. Then

$$\begin{aligned}\psi([\omega(x),\omega(y)]_{\mathfrak{g}}) &= \big(\nabla_{Ax}\psi\big)\omega(y) - \big(\nabla_{Ay}\psi\big)\omega(x) \\ &= \big(\mathcal{L}_{Ax}\psi\big)\omega(y) - \big(\mathcal{L}_{Ay}\psi\big)\omega(x) + \nabla_{Ay}(Ax) - \nabla_{Ax}(Ay) \\ &= \big(\mathcal{L}_{Ax}\psi\big)\omega(y) - \big(\mathcal{L}_{Ay}\psi\big)\omega(x) + [Ay,Ax];\end{aligned}$$

here the first line holds because $\psi$ is a Lie algebra homomorphism, and so relates $[\,,\,]_{\mathfrak{g}}$ and $[\,,\,]$.

We substitute (140) into (138). Rearranging the terms gives

$$\begin{aligned}&[x-Ax,y-Ay] - (1-A)[x,y] + J\Big\{[ix-Aix,y-Ay] - (1-A)[ix,y] \\ &+[x-Ax,iy-Aiy] - (1-A)[x,iy]\Big\} - [ix-Aix,iy-Aiy] + (1-A)[ix,iy] \\ &= -\big(\mathcal{L}_{x-Ax}\psi\big)\omega(y) + \big(\mathcal{L}_{y-Ay}\psi\big)\omega(x) - J\Big\{\big(\mathcal{L}_{ix-Aix}\psi\big)\omega(y) - \big(\mathcal{L}_{y-Ay}\psi\big)\omega(ix) \\ &+\big(\mathcal{L}_{x-Ax}\psi\big)\omega(iy) - \big(\mathcal{L}_{iy-Aiy}\psi\big)\omega(x)\Big\} + \big(\mathcal{L}_{ix-Aix}\psi\big)\omega(iy) - \big(\mathcal{L}_{iy-Aiy}\psi\big)\omega(ix)\end{aligned} \tag{141}$$

But the left hand side of (141) is $N_J(x-Ax,y-Ay)-(1-A)N_i(x,y)$, from (137) and using the fact that $J=(1-A)i(1-A)^{-1}$. As the almost complex structure $i$ is integrable, $N_i$ is zero and so the left hand side is just $N_J(x-Ax,y-Ay)$. So Lemma A.1 will be completed by showing that the right hand side of (141) vanishes.

Now $i$ is $\Psi$- invariant, and $\psi$ is $\Psi$- equivariant. A special property of the trivialization chosen for $P$ is that $\omega$ has the opposite sort of $\Psi$- equivariance to $\psi$, so that $A=\psi\circ\omega$ is $\Psi$- invariant. Thus $J$ is $\Psi$- invariant. This reflects the fact that $\Phi$ induces an action of $G$ on $N$ preserving $J$. So $\mathcal{L}_{\psi(v)}J{=}0$ for each $v$ in $\mathfrak{g}$. Thus if $x$ is a vector field on $U$, then $\mathcal{L}_{\psi(v)}(Jx)=J\mathcal{L}_{\psi(v)}x$. But this is equivalent to $\mathcal{L}_{Jx}(\psi(v))=J\mathcal{L}_x(\psi(v))$. As this holds for all $v$ in $\mathfrak{g}$, we have

$$\mathcal{L}_{Jx}\psi = J\mathcal{L}_x\psi.$$

So for instance,

$$J\big(\mathcal{L}_{ix-Aix}\psi\big)\omega(y) = -\big(\mathcal{L}_{x-Ax}\psi\big)\omega(y). \tag{142}$$

But this shows that two of the terms from the right hand side of (141) cancel out. In the same way, all the terms on the right hand side of (141) cancel out in pairs, and so $N_J(x - Ax, y - Ay) = 0$. $\square$

A similar proof could be given for the hypercomplex case of Theorem 3.1.2. It would be different to the above in detail because for general $G$ the diagonal is not a normal subgroup of $G \times G$, so $N = P/G$ does not have a natural $G$- action and the last part of the proof, that uses the invariance of $J$ by $\psi$, cannot be used. For abelian $G$, this proof is sufficient.

# Appendix B: Quotient Constructions for Compact Self-Dual Four-Manifolds

In Chapter 2 quotients for quaternionic and hypercomplex manifolds were described. In this appendix some families of four-dimensional self-dual conformal manifolds and orbifolds will be constructed using the quaternionic quotient. Most of the appendix is actually about the notation and organization necessary to write down the quotients, rather than being actual mathematics.

The connected sum of two 4-manifolds $M_1, M_2$ at the points $m_1, m_2$ is made by removing small balls around $m_1$ and $m_2$ and then gluing small $\mathcal{S}^3 \times [-1,1]$- neighbourhoods of the holes by an orientation-preserving map, identifying the inside of one neighbourhood and the outside of the other. To form the generalized connected sum at $m_1, m_2$ of two 4-orbifolds $M_1, M_2$, the points $m_1, m_2$ must have neighbourhoods isomorphic to $\mathcal{B}^4/\Gamma_1$, $\mathcal{B}^4/\Gamma_2$ with $\Gamma_1, \Gamma_2$ subgroups of $SO(4)$ (possibly trivial), such that there is an orientation-reversing automorphism $D$ of $\mathbb{R}^4$ identifying $\Gamma_1$ and $\Gamma_2$. Then $D$ acts on $\mathcal{S}^3$ and the generalized connected sum of $M_1$ and $M_2$ is defined by cutting out small $\mathcal{B}^4/\Gamma_i$ neighbourhoods of $m_i$ from $M_i$ and gluing $(\mathcal{S}^3/\Gamma_i) \times [-1,1]$ neighbourhoods of the holes by the map $(s,x) \mapsto (D(s), -x)$, where $D$ maps from $\mathcal{S}^3/\Gamma_1$ to $\mathcal{S}^3/\Gamma_2$.

The building blocks of the family are the weighted projective spaces described in §4.2. In each weighted projected space three special points called *corners* will be defined, which are usually orbifold points. Two subgroups $\Gamma_1, \Gamma_2$ of $SO(4)$ will be called *complementary* if they can be identified by an orientation-reversing automorphism $D$ of $\mathbb{R}^4$. Similarly, if two weighted projective spaces $M_1, M_2$ have corners $m_1, m_2$ modelled on $\mathcal{B}^4/\Gamma_i$, then $m_1, m_2$ are called complementary if $\Gamma_1$ and $\Gamma_2$ are complementary.

Given a collection of $n+1$ weighted projective spaces with $n$ identified pairs of complementary corners forming a connected whole (so that loops in the tree of weighted projective spaces are not allowed), we shall present a method for writing down a quaternionic quotient for self-dual conformal metrics on the generalized connected sum of the collection of weighted projective spaces at the $n$ pairs of identified, complementary corners.

The quotients that result are quaternionic quotients of $\mathbb{HP}^{n+1}$ by $U(1)^n$. This method yields many examples of non-singular self-dual conformal manifolds, which are invariably connected sums of $\mathbb{CP}^2$'s.

However, as $n$ increases the method yields many distinct families of metrics on $n\mathbb{CP}^2$, because for large $n$, $n\mathbb{CP}^2$ decays into weighted projective spaces in lots of different ways which give inequivalent quotients.

The simplest family of metrics on $n\mathbb{CP}^2$ made by this method have symmetry groups with identity component $U(1)$ for $n > 2$, and are the same as the metrics given by LeBrun in [L2]. The other families all have identity component $U(1) \times U(1)$, and are described in an alternative way in §4.5.

The method also gives self-dual conformal orbifolds with a single orbifold point, which can be viewed as asymptotically flat, zero-scalar-curvature Kähler resolutions of the quotient of $\mathbb{C}^2$ by a subgroup of $U(2)$. Examples are the ALE spaces for cyclic groups [Hi], [Kr] and LeBrun's metrics on line bundles over $\mathbb{CP}^1$ [L1]. Such resolutions exist for every cyclic subgroup $\Gamma$ of the group $U(1) \times U(1)$ of diagonal elements of $U(2)$ that acts freely on $\mathcal{S}^3$. This might be useful to someone wishing to desingularize a general self-dual orbifold by extensions of methods in [DF].

Using the results of §2.4, the Kähler metrics in the conformal class of the quotients can be described in detail. Thus we have many examples of Kähler metrics with zero scalar curvature. Viewed as Kähler manifolds these examples are never compact, because the conformal rescaling always becomes infinite somewhere, but they can usually be chosen to be asymptotically Euclidean or locally Euclidean. A method for including Asymptotically Locally Flat Kähler metrics of zero scalar curvature into our general scheme will also be given.

## B.1. Singular points of weighted projective spaces

In §4.2 we defined quaternionic structures on weighted projective spaces $\mathbb{CP}^2_{p,q,r}$ using the quaternionic quotient, and we saw that if $p, q, r$ are pairwise coprime then the space has at most three singular points, the orbifold points $[1,0,0], [0,1,0], [0,0,1]$. These three points in each weighted projective space will be called *corners*, and are the points at which we will do generalized connected sums. In the associated bundle they are given by the equations $x = l = 0, y = m = 0$ and $z = n = 0$ respectively. In this section we shall explain how to find the group $\Gamma$ for an orbifold point of a weighted projective space $\mathbb{CP}^2_{p,q,r}$. It will then be possible to decide when two corners of two weighted projected spaces are complementary.

Consider the weighted projective space $\mathbb{CP}^2_{p,q,r}$ near the point $[1,0,0]$. Every point $[f,g,h]$ in a neighbourhood of $[1,0,0]$ can be written as $[1,g,h]$, uniquely up to a transformation $(g,h) \mapsto (u^q g, u^r h)$ with $u^p = 1$. Therefore a neighbourhood of $[1,0,0]$ in $\mathbb{CP}^2_{p,q,r}$ is biholomorphic to a neighbourhood of the origin in $\mathbb{C}^2/\Gamma$, where

$$\Gamma = \left\{ \begin{pmatrix} u^q & 0 \\ 0 & u^r \end{pmatrix} : u^p = 1, u \in \mathbb{C}^* \right\}. \tag{143}$$

If $p$ is coprime to $q$ and $r$ then $\Gamma$ has $p$ elements. To give a criterion for when two such groups $\Gamma_1, \Gamma_2$ are complementary, observe that $(x, y) \mapsto (\overline{x}, y)$ is orientation-reversing. Thus

$$\Gamma' = \left\{ \begin{pmatrix} u^{-q} & 0 \\ 0 & u^r \end{pmatrix} : u^p = 1, u \in \mathbb{C}^* \right\}$$

and

$$\Gamma'' = \left\{ \begin{pmatrix} u^{-r} & 0 \\ 0 & u^q \end{pmatrix} : u^p = 1, u \in \mathbb{C}^* \right\}$$

are both complementary to $\Gamma$, and these are in fact the only groups of this form complementary to $\Gamma$.

So suppose that $(p_1, q_1, r_1)$ and $(p_2, q_2, r_2)$ are two triples of pairwise coprime positive integers. Then $\mathbb{CP}^2_{p_1,q_1,r_1}$ and $\mathbb{CP}^2_{p_2,q_2,r_2}$ will be complementary at $[1, 0, 0]$ and $[1, 0, 0]$ if and only if either

(i) $p_1 = p_2$ and there is some integer $s$ such that

$$sq_1 \equiv -q_2 \pmod{p_1}, \qquad sr_1 \equiv r_2 \pmod{p_1}, \tag{144}$$

or

(ii) $p_1 = p_2$ and there is some integer $s$ such that

$$sq_1 \equiv -r_2 \pmod{p_1}, \qquad sr_1 \equiv q_2 \pmod{p_1}. \tag{145}$$

These tests may be used to build up clusters of weighted projective spaces with selected pairs of complementary corners.

## B.2. Definitions and notation

In the following sections it will be shown how, given two quotients of $\mathbb{HP}^{n+1}$ by $U(1)^n$ and selected, complementary corners of each, one can define a quotient that is the generalized connected sum of the two quotients at the two corners.

First we establish some notation. A quaternionic quotient of $\mathbb{HP}^{n+1}$ by $U(1)^n$ will be denoted by $Q$. So $Q$ represents the quadruple *(space $\mathbb{HP}^{n+1}$, group $U(1)^n$, action $\phi, \chi$ or $\psi$ of $U(1)^n$ on $\mathbb{HP}^{n+1}$, quaternionic moment maps for action)*. It will frequently be necessary to distinguish between the quotient, which is this quadruple, and the quaternionic orbifold that is the result of the quotient. Therefore the quotient quaternionic orbifold will be denoted $\mathbf{Q}$. The associated bundle $\mathbb{H}^{n+2} \setminus \{0\}$ of $\mathbb{HP}^{n+1}$ will be described by $n+2$ pairs of complex coordinates, labelled $(x_i, l_i), (y_i, m_i)$ or $(z_i, n_i)$ $(i = 1, \ldots, n+2)$, which are complex with respect to $I_1$, and upon which $I_2$ acts as $(x_i, l_i) \mapsto (\bar{l}_i, -\overline{x}_i)$.

It would be possible to write everything in quaternionic coordinates, i.e. to have one quaternion-valued coordinate instead of each pair of complex coordinates. This would make the moment maps appear somewhat simpler. We have opted not to do this for three reasons. Firstly, to write out the moment maps in complex form makes it clear that $\mu_2 + i\mu_3$ is holomorphic in $I_1$, which is one of the conditions, and would not be obvious from a quaternionic formula. Secondly, complex coordinates make it easier to deal with twistor spaces, and so the computations in §2.3 are much easier, for instance. Thirdly, the standard sorts of action of $U(1)^n$ we consider are easily written out in complex coordinates.

These pairs of coordinates will be called quaternionic pairs. In general the equations $x_i = 0, l_i = 0$ for a particular $i$ will determine a unique point in the quaternionic quotient, and this will be called the *corner associated to* $(x_i, l_i)$.

The group action $\phi$ (or $\chi$, or $\psi$) will be of a standard form. It will be given by an endomorphism $\phi : U(1)^n \to U(1)^{n+2}$, where $\phi = (\phi_1, \ldots, \phi_{n+2})$ and each $\phi_i$ is a homomorphism $U(1)^n \to U(1)$. Then the action $\phi$ is defined by $\phi(g)\big((x_1, l_1, \ldots, x_{n+2}, l_{n+2})\big) = (\phi_1(g)x_1, \phi_1^{-1}(g)l_1, \ldots, \phi_{n+2}(g)x_{n+2}, \phi_{n+2}^{-1}(g)l_{n+2})$ for $g \in U(1)^n$.

The quaternionic moment maps are given by an $n$- dimensional vector space $V$ of twistor functions that are invariant under the action of the group (as $U(1)^n$ is commutative, this is the same as being equivariant) and suitably transverse. As each $\mu_i$ determines the other two, a twistor function $\mu$ on the associated bundle is defined by giving $\mu_1$.

Now $V$ is an $n$-dimensional subspace of the vector space $W$ of invariant twistor functions, and $W$ is fairly easily described. If neither $\phi_i = \phi_j$ nor $\phi_i = \phi_j^{-1}$ holds for any distinct $i, j$, and for no $i$ is $\phi_i = 1$, then $W = \langle |x_1|^2 - |l_1|^2, \ldots, |x_{n+2}|^2 - |l_{n+2}|^2 \rangle$ (these are the functions $\mu_1$). If on the other hand $\phi_i = \phi_j$ or $\phi_i = \phi_j^{-1}$ then one must add other quadratic basis elements that mix up $x_i, l_i, x_j, l_j$, and if $\phi_i = 1$ then one must add the basis elements $\operatorname{Re} x_i\bar{l}_i, \operatorname{Im} x_i\bar{l}_i$.

Let $Q_1, Q_2$ be quotients of $\mathbb{HP}^{r+1}$ by $U(1)^r$ and $\mathbb{HP}^{s+1}$ by $U(1)^s$ respectively. The coordinates on the associated bundle of $\mathbb{HP}^{r+1}$ will be $(x_1, l_1, \ldots, x_{r+2}, l_{r+2})$ and the group action $\phi = (\phi_1, \ldots, \phi_{r+2})$, and the coordinates on the associated bundle of $\mathbb{HP}^{s+1}$ will be $(y_1, m_1, \ldots, y_{s+2}, m_{s+2})$ with group action

$\chi = (\chi_1, \ldots, \chi_{s+2})$. The quaternionic moment maps will be $r$- and $s$- dimensional vector subspaces $V_1, V_2$ of the vector spaces $W_1, W_2$ of $\phi$- and $\chi$- invariant twistor functions on $\mathbb{HP}^{r+1}$ and $\mathbb{HP}^{s+1}$.

Now the extra basis elements for $W_1, W_2$ that are introduced if the actions of $\phi, \chi$ are not sufficiently general will cause complications in the construction of quotients for connected sums. We shall therefore suppose until after the proof of Theorem B.4.1 that $V_1, V_2$ are vector subspaces of $\langle |x_1|^2 - |l_1|^2, \ldots, |x_{r+2}|^2 - |l_{r+2}|^2 \rangle$, $\langle |y_1|^2 - |m_1|^2, \ldots, |y_{s+2}|^2 - |m_{s+2}|^2 \rangle$ respectively. After the proof some comments will be made about the more general case.

Let $q_1$ and $q_2$ be complementary corners in $\mathbf{Q}_1, \mathbf{Q}_2$; for convenience suppose that these are associated to the quaternionic pairs $(x_1, l_1)$ and $(y_1, m_1)$. In the next two sections a quaternionic quotient for the generalized connected sum of $\mathbf{Q}_1, \mathbf{Q}_2$ at $q_1$ and $q_2$ will be described.

## B.3. Data for the generalized connected sum

To make the generalized connected sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ at $q_1, q_2$, some information is needed on exactly how to join the two orbifolds together. In this section the necessary information will be described, and some group actions required later will be defined. The connected sum $M_1 \# M_2$ of two conformal manifolds $M_1, M_2$ at the points $m_1, m_2$ can be thought of as the union of $M_1, M_2$ with small holes cut out about $m_1, m_2$, joined by a thin neck. Making the holes very small and the neck very thin, the connected sum starts to resemble the union of $M_1$ and $M_2$, joined in some way at $m_1$ and $m_2$.

The appropriate way to join $M_1$, $M_2$ at $m_1$, $m_2$ is to identify $T_{m_1}M_1 \setminus \{0\}$ and $T_{m_2}M_2 \setminus \{0\}$. This is done by a map $C : T_{m_1}M_1 \setminus \{0\} \longrightarrow T_{m_2}M_2 \setminus \{0\}$ defined by $C(v) = D(v)/\|v\|^2$, where $D : T_{m_1}M_1 \longrightarrow T_{m_2}M_2$ is an orientation-reversing isomorphism of conformal vector spaces and $\|.\|$ is some norm in the conformal class of $T_{m_1}M_1$. Then by choosing identifications of neighbourhoods of zero in the tangent spaces with neighbourhoods of $m_i$, $C$ can be used to identify small annuli about $m_i$, and thus to define a connected sum. In fact in the quaternionic case the most convenient way of describing $C$ is as an isomorphism of the associated bundles of $T_{m_1}M_1 \setminus \{0\}$ and $T_{m_2}M_2 \setminus \{0\}$. These are the normal bundles of the fibres of the associated bundles of $M_1, M_2$ over $m_1$ and $m_2$ respectively, and are isomorphic to $\mathbb{H}^2 \setminus (\mathbb{H} \times \{0\} \cup \{0\} \times \mathbb{H})$.

Now when dealing with generalized connected sums, $q_1, q_2$ can be orbifold points, and $T_{q_i}\mathbf{Q}_i$ may have to be replaced by the quotient of a vector space by a finite group. These will be called *first-order neighbourhoods* as they are not strictly tangent spaces. We give first-order neighbourhoods of the points

$q_1, q_2$ in $\mathbf{Q}_1, \mathbf{Q}_2$, and choose an identification $C$ between them which is the data necessary to form the generalized connected sum.

Firstly, coordinates $(x, l)$ will be defined on the fibre of the associated bundle over $q_1$, and similarly coordinates $(y, m)$ over the fibre of the associated bundle over $q_2$. Now $q_1$ is the point in $\mathbf{Q}_1$ where $x_1 = l_1 = 0$. But $V_1$ cannot contain $|x_1|^2 - |l_1|^2 + \alpha(|x_i|^2 - |l_i|^2)$ for any $i > 0$ and real constant $\alpha$, because it can be shown that this would contradict the transversality of $V_1$. Thus $V_1$ restricted to $x_1 = l_1 = 0$ is an $r$- dimensional subspace of the $r+1$- dimensional vector space $\langle |x_2|^2 - |l_2|^2, \ldots, |x_{r+2}|^2 - |l_{r+2}|^2 \rangle$ containing no elements of the form $|x_i|^2 - |l_i|^2$. So there exist non-zero real constants $\lambda_3, \ldots \lambda_{r+2}$ such that the restricted solutions $(x_2, l_2, \ldots, x_{r+2}, l_{r+2})$ of $V_1$ satisfy $|x_i|^2 - |l_i|^2 = \lambda_i(|x_2|^2 - |l_2|^2)$, $i > 2$.

The transversality condition for $V_1$ ensures that given any such solution $(x_2, \ldots, l_{r+2})$, it is possible to choose a point $(x'_2, \ldots, l'_{r+2})$ in its orbit of $U(1)^r$ such that $x'_i = \sqrt{\lambda_i} x'_2, l'_i = \sqrt{\lambda_i} l'_2$ if $i > 2$ and $\lambda_i > 0$, and $x'_i = \sqrt{-\lambda_i} l'_2, l'_i = -\sqrt{-\lambda_i} x'_2$ if $i > 2$ and $\lambda_i < 0$. Then as coordinates on the fibre of the associated bundle of $\mathbf{Q}_1$ over $q_1$, either $x = x'_2$, $l = l'_2$ or $x = l'_2$, $l = -x'_2$ will be chosen. Later one of these two choices will be selected. They are quaternionic coordinates, that is, they are triholomorphic in the usual way with respect to the quotient hypercomplex structure.

The point $(x'_2, \ldots, l'_{r+2})$ is not necessarily the only point in the orbit of $(x_2, \ldots, l_{r+2})$ under the group $U(1)^r$ satisfying the conditions above. There will in fact be a finite subgroup $\Gamma_1$ (possibly trivial) of $U(1)^r$ that preserves the conditions, and the different possible choices for $(x'_2, \ldots, l'_{r+2})$ are exactly the orbit under $\Gamma_1$ of any given possible $(x'_2, \ldots, l'_{r+2})$. So as $q_1$ may be an orbifold point, the fibre of the associated bundle over $q_1$ may not be $\mathbb{H}/\{\pm 1\}$ but instead $\mathbb{H}/(\{\pm 1\} \times \Gamma_1)$. Therefore one should regard the coordinates $(x, l)$ as lying not in $\mathbb{H}$ but in $\mathbb{H}/(\{\pm 1\} \times \Gamma_1)$. As $\Gamma_1$ is a subgroup of $U(1)^r$ it also acts on $(x_1, l_1)$ with action $\phi_1$. This gives an action of $\Gamma_1$ on $\mathbb{H}^2$ with coordinates $(x, l, x_1, l_1)$.

In practice, to apply the method we describe, it is necessary first to work out the finite subgroup $\Gamma_1$ of $U(1)^r$ that fixes elements $(x'_2, \ldots, l'_{r+2})$ in the special form above, and then to find the action of $\Gamma_1$ on $(x, l)$ and $(x_1, l_1)$. The end product of this calculation is a finite (cyclic) group $\Gamma_1$, and an action of $\Gamma_1$ upon $\mathbb{H}^2$ with coordinates $(x, l, x_1, l_1)$.

The relevance of this information is that $\mathbb{H}^2/(\{\pm 1\} \times \Gamma_1)$ is the normal bundle of the fibre of the associated bundle of $\mathbf{Q}_1$ over $q_1$, which is the associated bundle of the first-order neighbourhood of $q_1$, and $(x, l, x_1, l_1)$ are the natural coordinates on it. In the same way, one finds a finite group $\Gamma_2$ acting on $\mathbb{H}^2$ with coordinates $(y, m, y_1, m_1)$. As by assumption $\Gamma_1$ and $\Gamma_2$ are compatible, there is an isomorphism of the two copies of $\mathbb{H}^2$ that identifies the actions of $\{\pm 1\} \times \Gamma_1$ and $\{\pm 1\} \times \Gamma_2$. This is the isomorphism $C$ above that gives the data for the generalized connected sum.

The factor $\{\pm 1\}$ comes in because the fibre of associated bundles is $\mathbb{H}/\{\pm 1\}$ and not $\mathbb{H}$, so that it is only necessary to identify the actions of $\Gamma_i$ on $\mathbb{H}^2/\{\pm 1\}$, not on $\mathbb{H}^2$. If the manifolds $\mathbf{Q}_1, \mathbf{Q}_2$ are spin then the associated bundles have double covers with fibre $\mathbb{H}$, and the generalized connected sum will be spin if $C$ identifies the actions of $\Gamma_1, \Gamma_2$. In §B.5 an example is given in which $C$ does not identify the actions of $\Gamma_1, \Gamma_2$ on $\mathbb{H}^2$, and so inserting $\{\pm 1\}$ is necessary.

To achieve the effect of inverting about the unit sphere in the definition of $C$, this isomorphism must identify the quaternionic pairs $(x, l)$ and $(y_1, m_1)$, and also the quaternionic pairs $(x_1, l_1)$ and $(y, m)$. Consider how one might identify the first pair. Up to the action of complex constants acting like $(x, l) \mapsto (wx, \overline{w}l)$ $(w \in \mathbb{C}^*)$, there are two ways of identifying the two quaternionic pairs compatible with the standard $U(1)$ actions used here. These are $x = y_1, l = m_1$ and $x = m_1, l = -y_1$.

There are two identifications of pairs to make, so there are four possibilities. Define the involution $\sigma$ of $Q_1$ by $\sigma(x_i) = l_i, \sigma(l_i) = -x_i, \sigma(\phi) = \phi^{-1}, \sigma(V_1) = V_1$; it is an automorphism of $Q_1$. Applying $\sigma$ shows that the four possible identifications of $(x, l), (y_1, m_1)$ and $(x_1, l_1), (y, m)$ are isomorphic in pairs, so there are only two real choices. For general groups $\Gamma_1$ and $\Gamma_2$ only one of these will identify the actions of the groups $\{\pm 1\} \times \Gamma_1, \{\pm 1\} \times \Gamma_2$. But for groups $\Gamma_i$ which are represented at most by $\pm 1$ on one of the quaternionic pairs (so, for instance, a connected sum rather than a generalized one) both will identify the group actions.

Thus for general $\Gamma_i$ there is only one distinct way to glue the neighbourhoods together to make a quotient, but for $\Gamma_i$ contained in either $SU(2) \subset SO(4)$ there are two distinct ways that can lead to two different quotients for the generalized connected sum $\mathbf{Q}_1 \# \mathbf{Q}_2$. Let $C$ be one of the identifications above identifying the action of $\{\pm 1\} \times \Gamma_1$ and $\{\pm 1\} \times \Gamma_2$. Recall that in defining the quaternionic pairs $(x, l)$ and $(y, m)$ there were two possibilities for the definition of each. We now stipulate that $(x, l)$ and $(y, m)$ should be defined such that the identification $C$ is given by $(y, m, y_1, m_1) = C\big((x, l, x_1, l_1)\big) = (x_1, l_1, x, l)$.

Now the definition of $(x, l)$ implies that for $i = 2, \ldots, r+2$ there is a non-zero real constant $a_i$ such that either $x_i' = a_i x$, $l_i' = a_i l$ or $x_i' = a_i l$, $l_i' = -a_i x$. Define $c_i$ to be 1 in the first case, and $-1$ in the second. Similarly, define $b_i, d_i$ so that when $d_i = 1$, $y_i' = b_i y$ and $m_i' = b_i m$, and when $d_i = -1$, $y_i' = b_i m$ and $m_i' = -b_i y$.

Consider $U(1)$ acting in the standard way on $(x,l)$ and $(y,m)$ by $(x,l) \mapsto (ux, u^{-1}l), (y,m) \mapsto (uy, u^{-1}m)$. As $x,l,y,m$ are defined in terms of $x_2,\ldots,l_{r+2}$ and $y_2,\ldots,m_{s+2}$, we will give natural actions $\tau, \upsilon$ of $U(1)$ upon $\mathbb{H}^{r+1}$ and $\mathbb{H}^{s+1}$ respectively that induce these actions upon $(x,l)$ and $(y,m)$.

For $i = 2,\ldots,r+2$, let $\tau_i(u) = u^{c_i}$. Then define the action $\tau$ of $U(1)$ on $\mathbb{H}^{r+1}$ by $\tau(u)\big((x_2,l_2,\ldots,x_{r+2}, l_{r+2})\big) = (\tau_2(u)x_2, \tau_2^{-1}(u)l_2, \ldots, \tau_{r+2}(u)x_{r+2}, \tau_{r+2}^{-1}(u)l_{r+2})$. The special property of this action $\tau$ is that not only does it commute with $\phi$, it also commutes with the map taking $(x_2,\ldots,l_{r+2})$ to the $\Gamma_1$ coset of $(x'_2,\ldots,l'_{r+2})$ and hence with the action of $U(1)$ on $(x,l)$.

Similarly define $\upsilon_i(u) = u^{d_i}$ and the corresponding action $\upsilon$ of $U(1)$ on $\mathbb{H}^{s+1}$. An important point about the definitions of $\tau, \upsilon$ is that they depend upon discrete information from $V_1, V_2$ respectively. The sign of the action of $U(1)$ upon each variable pair is decided by $c_i, d_i$, and the transversality conditions ensure that these are well-defined. To define the group action needs discrete information, and here is an example of how some of that information is taken from the moment maps, which are continuous information.

To summarize this section, for each $i = 2,\ldots,r+2$ there is a non-zero real number $a_i$, and $c_i$ which is $\pm 1$, such that whenever $(x_2,l_2,\ldots,x_{r+2},l_{r+2})$ satisfies $V_1$ restricted to $x_1 = l_1 = 0$ there is an element $(x'_2,\ldots,l'_{r+2})$ in its orbit under $U(1)^r$ of the form $x'_i = a_i x, l'_i = a_i l$ if $c_i = 1$ and $x'_i = a_i l, l'_i = -a_i x$ otherwise, where $(x,l)$ is an element of $\mathbb{H}$, and moreover that $(x,l)$ is unique up to the action of a finite subgroup $\Gamma_1$ of $U(1)^r$. A similar statement holds for $(y_2,\ldots,m_{s+2})$ with $(y,m) \in \mathbb{H}$.

Then $(x,l)$, $(y,m)$ are quaternionic coordinates on the fibres of the associated bundles over $q_1$ and $q_2$ defined up to the actions of $\Gamma_1, \Gamma_2$, such that the identification $C$ given on the associated bundles by $(y,m,y_1,m_1) = C\big((x,l,x_1,l_1)\big) = (x_1,l_1,x,l)$ identifies the quotient groups $\{\pm 1\} \times \Gamma_1, \{\pm 1\} \times \Gamma_2$ and is the data for the generalized connected sum. Actions $\tau, \upsilon$ on $\mathbb{H}^{r+1}$ and $\mathbb{H}^{s+1}$ have also been defined, that are in a certain sense natural liftings of the standard actions of $U(1)$ upon $(x,l)$ and $(y,m)$, and will be used in the definition of the quotient $Q_{1\#2}$ for the generalized connected sum $\mathbf{Q}_1\#\mathbf{Q}_2$.

### B.4. A quotient for the generalized connected sum

In this section a quaternionic quotient will be given for the generalized connected sum of two quaternionic quotients $\mathbf{Q}_1, \mathbf{Q}_2$ considered in the last two sections. Let $e_2,\ldots,e_{r+2}$ be real numbers such that $e_2 c_2 a_2^2 + \ldots + e_{r+2} c_{r+2} a_{r+2}^2 = 1$ and $f_2,\ldots,f_{s+2}$ also be real numbers satisfying $f_2 d_2 b_2^2 + \ldots +$

$f_{s+2}d_{s+2}b_{s+2}^2 = 1$. Define $w_1, w_2$ as twistor functions on $\mathbb{H}^{r+1}, \mathbb{H}^{s+1}$ by $w_1 = e_2(|x_2|^2 - |l_2|^2) + \ldots + e_{r+2}(|x_{r+2}|^2 - |l_{r+2}|^2)$ and $w_2 = f_2(|y_2|^2 - |m_2|^2) + \ldots + f_{s+2}(|y_{s+2}|^2 - |m_{s+2}|^2)$.

Now from the definition of the coordinates $(x, l)$, when $(x_2, \ldots, l_{r+2})$ is a point descending to $(x, l)$ in the fibre of the associated bundle over $q_1$, then $|x_i|^2 - |l_i|^2 = c_i a_i^2(|x|^2 - |l|^2)$ and so $w_1(x_2, \ldots, l_{r+2}) = (e_2c_2a_2^2 + \ldots + e_{r+2}c_{r+2}a_{r+2}^2)(|x|^2 - |l|^2) = |x|^2 - |l|^2$, because of the condition on $e_2, \ldots, e_{r+2}$. Thus $w_1$ is an arbitrary twistor function that descends to give $|x|^2 - |l|^2$ on the fibre over $q_1$. Similarly $w_2$ gives $|y|^2 - |m|^2$ on the fibre over $q_2$. The twistor functions $w_1, w_2$ will be used in defining the moment maps for the new quotient.

Let $\epsilon$ be a real number. Now $V_1$ is a real vector space of functions of $x_2, \ldots, l_{r+2}$ with linear terms in $|x_1|^2 - |l_1|^2$. Define $V_1^\epsilon$ to be the real vector space of functions of $x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2}$ given by substituting $\epsilon^2 w_1$ for every occurrence of $|x_1|^2 - |l_1|^2$ in $V_1$. Similarly define $V_2^\epsilon$ by substituting $\epsilon^2 w_1$ in place of $|y_1|^2 - |m_1|^2$ in $V_2$. Define $V_{1\#2}^\epsilon$ by $V_{1\#2}^\epsilon = V_1^\epsilon \oplus V_2^\epsilon$. Then $V_{1\#2}^\epsilon$ is a vector space of $U(1)^{r+s}$-invariant twistor functions on $\mathbb{H}^{r+s+2}$, of dimension $r + s$. It will be the space of moment maps for the new quotient.

We shall prove the following theorem:

**Theorem B.4.1.** *Define the quaternionic quotient $Q_{1\#2}$ to be the quaternionic quotient of $\mathbb{HP}^{r+s+1}$ with coordinates $(x_2, l_2, \ldots, x_{r+2}, l_{r+2},\ y_2, m_2, \ldots, y_{s+2}, m_{s+2})$ on the associated bundle $\mathbb{H}^{r+s+2}$ by $U(1)^{r+s}$, with the action*

$$\begin{matrix} (x_2, l_2, \ldots, x_{r+2}, l_{r+2}) \\ (y_2, m_2, \ldots, y_{s+2}, m_{s+2}) \end{matrix} \overset{(g_1,g_2)}{\longmapsto} \begin{matrix} \big(\tau_2(\chi_1(g_2))\phi_2(g_1)x_2, \tau_2^{-1}(\chi_1(g_2))\phi_2^{-1}(g_1)l_2, \ldots, \\ \tau_{r+2}(\chi_1(g_2))\phi_{r+2}(g_1)x_{r+2}, \tau_{r+2}^{-1}(\chi_1(g_2))\phi_{r+2}^{-1}(g_1)l_{r+2}\big) \\ \big(\upsilon_2(\phi_1(g_1))\chi_2(g_2)y_2, \upsilon_2^{-1}(\phi_1(g_1))\chi_2^{-1}(g_2)m_2, \ldots, \\ \upsilon_{s+2}(\phi_1(g_1))\chi_{s+2}(g_2)y_{s+2}, \upsilon_{s+2}^{-1}(\phi_1(g_1))\chi_{s+2}^{-1}(g_2)m_{s+2}\big) \\ (g_1, g_2) \in U(1)^r \times U(1)^s, \end{matrix} \tag{146}$$

*and $V_{1\#2}^\epsilon$ the vector space of moment maps. Then for sufficiently small $\epsilon \neq 0$, $\mathbf{Q}_{1\#2}$ is as an orbifold the generalized connected sum of the quaternionic quotients $\mathbf{Q}_1, \mathbf{Q}_2$ at the points $q_1, q_2$, and is non-singular except for other orbifold points present in $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Moreover, as $\epsilon \to 0$ there is a well-defined limiting process whereby the quotient approaches the singular union of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, joined together at $q_1, q_2$ by $C$.*

*Proof.* Putting $\epsilon = 0$, the zero set in $\mathbb{H}^{r+s+2}$ of $V_{1\#2}^0$ is the Cartesian product of the solutions of $V_1$ restricted to $x_1 = l_1 = 0$ and the solutions of $V_2$ restricted to $y_1 = m_1 = 0$. The quotient by $U(1)^{r+s}$ is not the associated bundle of the union of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ as one might hope, but is in fact $\mathbb{H}^2/\Gamma_i$. This can be viewed as the identification of the associated bundles of first-order neighbourhoods of $q_1$ and $q_2$. To obtain

$\mathbf{Q}_1 \cup \mathbf{Q}_2$ as a limit requires a different viewpoint. It will be shown that the quotient of the topological limit set

$$S_1 = \lim_{\epsilon \to 0} \big\{ (x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2}) : (\epsilon x_2, \ldots, \epsilon l_{r+2}, y_2, \ldots, m_{s+2}) \text{ is a zero of } V^{\epsilon}_{1\#2} \big\} \tag{147}$$

by $U(1)^{r+s}$ can be identified with the associated bundle of the quotient $\mathbf{Q}_1$. (Here the topological limit means that $s$ is in the limit set if it is a limit of a sequence of points from the sequence of sets.) Suppose that $s = (x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2})$ is in $S_1$. Now because the powers of $\epsilon$ cancel out, the vector space of functions $V^1_1$ vanishes on each set in the sequence, and thus on the limit $S_1$. Thus $s$ is a zero of $V^1_1$.

On the other hand, $(y_2, \ldots, m_{s+2})$ is in the zero set of $V_2$ restricted to $y_1 = m_1 = 0$, because the extra powers of $\epsilon$ make the $w_1$ substituted into $V_2$ vanish. Therefore the quotient of $(y_2, \ldots, m_{s+2})$ by $U(1)^s$ lies in the fibre of the associated bundle of $\mathbf{Q}_2$ over $q_2$.

Recall that the coordinates $(y, m)$ (up to the action of $\Gamma_2$) were identified with $(x_1, l_1)$ by $x_1 = y$, $l_1 = m$. Let this be the definition of $x_1, l_1$ in the limiting situation. Then $w_2 = |x_1|^2 - |l_1|^2$ by its definition, so $V^1_1 = V$. It remains only to observe that although $(y, m)$ is defined only up to an orbit of $\Gamma_2$, $\Gamma_2$ acts on $x_2, \ldots, l_{r+2}$ as well through $\tau$ and that the definitions of $\tau$ and $\upsilon$ ensure that $\Gamma_1$ and $\Gamma_2$ actually have the same action on $(x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2})$. So one may equally regard $(y, m)$ as being defined only up to an orbit of $\Gamma_1$.

This means that the quotient of $S_1$ by $U(1)^s$ is naturally identified with the quotient of the zero set of $V_1$ in $\mathbb{H}^{r+2}$ by the finite subgroup $\Gamma_1$ of $U(1)^r$. So dividing by $U(1)^r$ gives the quotient of the zero set of $V_1$ in $\mathbb{H}^{r+2}$ by $U(1)^r$, because first dividing by $\Gamma_1 \subset U(1)^r$ makes no difference. Thus $\mathbf{Q}_1$ is a limit of $\mathbf{Q}_{1\#2}$ in the sense that its associated bundle is the quotient of the limit set $S_1$ by $U(1)^{r+s}$. In the same way the associated bundle of $\mathbf{Q}_2$ appears as a limit as it is the quotient of the set $S_2$ by $U(1)^{r+s}$, where $S_2$ has the obvious definition.

Note: We can now see the point of the careful defining of $\tau$ and $\upsilon$ to identify the actions of $\Gamma_1, \Gamma_2$. If $\tau, \upsilon$ had been defined in a way that did not identify $\Gamma_1, \Gamma_2$ then the quotient of $S_1$ by $U(1)^s$ would be the zero set of $V_1$ divided by $\Gamma_2$, not $\Gamma_1$, and as the action of $\Gamma_2$ on $\mathbb{H}^{r+2}$ would not then need to be contained in the action of $U(1)^r$, the result of further dividing by $U(1)^r$ would not be the quotient $\mathbf{Q}_1$ but instead $\mathbf{Q}_1$ divided by an action of $\Gamma_2$, not necessarily trivial.

Therefore the result of trying to make a generalized connected sum of $\mathbf{Q}_1, \mathbf{Q}_2$ at $q_1, q_2$ with respective orbifold groups $\Gamma_1, \Gamma_2$ which have not been chosen to be complementary, is in fact the generalized connected sum of $\mathbf{Q}_1/\Gamma_2$ and $\mathbf{Q}_2/\Gamma_1$, and the orbifold groups will both be some quotient group of $\Gamma_1 \times \Gamma_2$. Only if $\Gamma_1, \Gamma_2$ are complementary do we get the generalized connected sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

To continue with the proof, observe that heuristically it has been shown that as $\epsilon \to 0$, the zeros $(x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2})$ of $V^\epsilon_{1\#2}$ in which $(y_2, \ldots, m_{s+2})$ is large compared with $(x_2, \ldots, l_{r+2})$ approximate the quotient $Q_1$, and the zeros $(x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2})$ in which $(y_2, \ldots, m_{s+2})$ is small compared with $(x_2, \ldots, l_{r+2})$ approximate $Q_2$. And we saw at the beginning that the solutions with $(y_2, \ldots, m_{s+2})$ about the same size as $(x_2, \ldots, l_{r+2})$ approximate the identified neighbourhoods of $q_1$ and $q_2$.

To make these statements precise it is necessary to clarify what is meant by approximation. The argument above can be rewritten like this: let the map $\alpha^\epsilon : \mathbf{Q}^\epsilon_{1\#2} \longrightarrow \mathbb{HP}^{r+s+1}/U(1)^{r+s}$ be the quotient by $U(1)^{r+s}$ of the map given on the associated bundles by mapping a zero $(x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2})$ of $V^\epsilon_{1\#2}$ to $(x_2, \ldots, l_{r+2}, \epsilon y_2, \ldots, \epsilon m_{s+2})$. Then one shows that the topological limit of $\alpha^\epsilon[\mathbf{Q}^\epsilon_{1\#2}]$ is a subset of $\mathbb{HP}^{r+s+1}/U(1)^{r+s}$ that has an explicit isomorphism $\beta$ with $\mathbf{Q}_1$.

Now by differential topology, where $\beta$ is well-behaved there is a diffeomorphism $\gamma$ of a neighbourhood of $\beta[\mathbf{Q}_1]$ in $\mathbb{HP}^{r+s+1}$ and a neighbourhood of the zero section in the normal bundle of $\beta[\mathbf{Q}_1]$. So $\gamma$ gives a smooth retraction $\delta$ of a neighbourhood of $\beta[\mathbf{Q}_1]$ onto $\beta[\mathbf{Q}_1]$. But $\beta$ is well-behaved everywhere except at $q_1$, so by cutting out the part of $\mathbf{Q}_{1\#2}$ sent to a small neighbourhood of $q_1$ the rest of $\mathbf{Q}_{1\#2}$ may be mapped to $\mathbf{Q}_1$.

We assume that for sufficiently small $\epsilon$ the subset $U_1$ of $\mathbf{Q}_{1\#2}$ coming from the set

$$T_1 = \left\{(x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2}) : |x_2|^2 + |l_2|^2 + \ldots + |l_{r+2}|^2 < 2(|y_2|^2 + |m_2|^2 + \ldots + |m_{s+2}|^2)\right\} \quad (148)$$

is mapped by $\alpha^\epsilon$ into the range of $\delta$, and that the smooth map $\rho^\epsilon = \beta^{-1} \circ \delta \circ \alpha^\epsilon : U_1 \longrightarrow \mathbf{Q}_1$ is a diffeomorphism between $U_1$ and its image in $\mathbf{Q}_1$. We also assume that for sufficiently small $\epsilon$ the subset $U_2$ of $\mathbf{Q}_{1\#2}$ coming from the set

$$T_2 = \left\{(x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2}) : |y_2|^2 + |m_2|^2 + \ldots + |m_{s+2}|^2 < 2(|x_2|^2 + |l_2|^2 + \ldots + |l_{r+2}|^2)\right\} \quad (149)$$

can be mapped diffeomorphically to its image in $\mathbf{Q}_2$ under the map $\sigma^\epsilon$, constructed in the same way as $\rho^\epsilon$. These are reasonable assumptions because they mean that away from $q_1$, the limit of half of $\mathbf{Q}_{1\#2}$ to $\mathbf{Q}_1$ is well-behaved, and similarly for $\mathbf{Q}_2$.

By applying the same arguments to the associated bundles, $\rho^\epsilon, \sigma^\epsilon$ may be lifted to maps of bundles, also denoted $\rho^\epsilon, \sigma^\epsilon$, between the associated bundles of $U_1, U_2$ and $\mathbf{Q}_1, \mathbf{Q}_2$ that fibre over the first maps. The proof will be completed by studying the join between $U_1$ and $U_2$ when $\epsilon$ is small. It will be seen that in the quotient it forms a 'neck' as in the generalized connected sum of the quotients $\mathbf{Q}_1, \mathbf{Q}_2$ at $q_1$ and $q_2$. Thus the quotient manifold is indeed the generalized connected sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

Suppose that $\epsilon$ is small. Let $A$ be the region in $\mathbb{H}^{r+s+2}$ defined by

$$A = \big\{(x_2, \ldots, l_{r+2}, y_2, \ldots, m_{s+2}) : \frac{1}{2}(|x_2|^2 + |l_2|^2 + \ldots + |l_{r+2}|^2) < |y_2|^2 + |m_2|^2 + \ldots + |m_{s+2}|^2 \\ < 2(|x_2|^2 + |l_2|^2 + \ldots + |l_{r+2}|^2)\big\}; \tag{150}$$

then $A = T_1 \cap T_2$ and the subset of $\mathbf{Q}_{1\#2}$ coming from $A$ is $U_1 \cap U_2$. So $\rho^\epsilon[U_1 \cap U_2]$ is defined by the condition $\frac{1}{2}\rho^\epsilon(|x_2|^2 + |l_2|^2 + \ldots + |l_{r+2}|^2) < \rho^\epsilon(|y_2|^2 + |m_2|^2 + \ldots + |m_{s+2}|^2) < 2\rho^\epsilon(|x_2|^2 + |l_2|^2 + \ldots + |l_{r+2}|^2)$. Now as $\epsilon \to 0$, $\rho^\epsilon(|x_2|^2 + \ldots + |l_{r+2}|^2) \to |x_2|^2 + \ldots + |l_{r+2}|^2$, and $\epsilon^2 \rho^\epsilon(|y_2|^2 + \ldots + |m_{s+2}|^2) \to K_1(|x_1|^2 + |l_1|^2)$, for some positive constant $K_1$. Thus for small $\epsilon$, $\rho^\epsilon[U_1 \cap U_2]$ is approximately the subset of $\mathbf{Q}_1$ coming from

$$A_1 = \big\{(x_1, l_1, x_2, \ldots, l_{r+2}) \in Q_1 : \frac{\epsilon^2}{2}(|x_2|^2 + |l_2|^2 + \ldots + |l_{r+2}|^2) < K_1(|x_1|^2 + |l_1|^2) \\ < 2\epsilon^2(|x_2|^2 + |l_2|^2 + \ldots + |l_{r+2}|^2)\big\}. \tag{151}$$

It is easy to see that the subset of $\mathbf{Q}_1$ coming from $A_1$ is a small 'annulus' about $q_1$, for $q_1$ is the point $x_1 = l_1 = 0$, and the condition on $A_1$ is that $x_1$ and $l_1$ should be small compared to the other quotient variables, but not too small. Similarly, $\sigma^\epsilon[U_1 \cap U_2]$ is approximately the subset of $\mathbf{Q}_2$ coming from

$$A_2 = \big\{(y_1, m_1, y_2, \ldots, m_{s+2}) \in Q_2 : \frac{\epsilon^2}{2}(|y_2|^2 + |m_2|^2 + \ldots + |m_{s+2}|^2) < K_2(|y_1|^2 + |m_1|^2) \\ < 2\epsilon^2(|y_2|^2 + |m_2|^2 + \ldots + |m_{s+2}|^2)\big\}. \tag{152}$$

Now the assumptions about $\rho^\epsilon, \sigma^\epsilon$ above imply that when $\epsilon$ is sufficiently small, $\mathbf{Q}_{1\#2}$ is the union of two open sets $U_1, U_2$, of which $U_1$ is diffeomorphic to $\mathbf{Q}_1$ minus a small ball about $q_1$ and $U_2$ diffeomorphic to $\mathbf{Q}_2$ minus a small ball about $q_2$, and moreover that the overlap between $U_1$ and $U_2$ is mapped to a small annulus of size about $\epsilon$ in each of $\mathbf{Q}_1, \mathbf{Q}_2$. Therefore, once it is shown that the identification between the two annuli is the one required, the proof that when $\epsilon$ is small $\mathbf{Q}_{1\#2}$ is the generalized connected sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ at $q_1, q_2$ will be complete.

Recall from §B.3 that $(x, l, x_1, l_1)$ are coordinates for the normal bundle of the fibre of the associated bundle of $\mathbf{Q}_1$ over $q_1$, and similarly $(y, m, y_1, m_1)$ for $\mathbf{Q}_2$. The techniques used to make $\rho^\epsilon, \sigma^\epsilon$ above give a smooth coordinate system $(x, l, x_1, l_1)$ on a neighbourhood of the fibre of the associated bundle of $\mathbf{Q}_1$ over $q_1$ inducing the old $x, l, x_1, l_1$ on the normal bundle, and similarly for $\mathbf{Q}_2$. For small $\epsilon$ these coordinate systems can be pulled back using $\rho^\epsilon$ and $\sigma^\epsilon$ to the associated bundle of $U_1 \cap U_2$.

Comparing the two on the associated bundle of $U_1 \cap U_2$ gives a description of the identification of the annuli in $\mathbf{Q}_1$ and $\mathbf{Q}_2$. In the approximation of $\mathbf{Q}_{1\#2}$ to $\mathbf{Q}_1$, $x_1$ and $l_1$ were defined using $y$ and $m$. In fact the factors of $\epsilon$ used in the definition of $S_1$ imply that when $\epsilon$ is small, $(\rho^\epsilon)^{-1}(x_1) \approx \epsilon(\sigma^\epsilon)^{-1}(y)$ and $(\rho^\epsilon)^{-1}(l_1) \approx \epsilon(\sigma^\epsilon)^{-1}(m)$. Similarly $(\sigma^\epsilon)^{-1}(y_1) \approx \epsilon(\rho^\epsilon)^{-1}(x)$ and $(\sigma^\epsilon)^{-1}(m_1) \approx \epsilon(\rho^\epsilon)^{-1}(l)$.

Thus when $\epsilon$ is small, the identification between the annuli in $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is given approximately in coordinates on the associated bundles by $x_1 = \epsilon y, l_1 = \epsilon m, \epsilon x = y_1, \epsilon l = m_1$. But this is just the identification $C$ chosen in §B.3 for the generalized connected sum, up to factors of $\epsilon$. Therefore the quotient $\mathbf{Q}_{1\#2}$ is the generalized connected sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ at $q_1$ and $q_2$. $\square$

Recall that in §B.2 the assumption was made that $V_1, V_2$ should be respectively contained in $\langle |x_1|^2 - |l_1|^2, \ldots, |x_{r+2}|^2 - |l_{r+2}|^2 \rangle$ and $\langle |y_1|^2 - |m_1|^2, \ldots, |y_{s+2}|^2 - |m_{s+2}|^2 \rangle$, and that this will be true anyway unless the actions $\phi, \chi$ have extra symmetries. Now we consider whether this assumption can be removed. The problem with allowing $V_i$ to be more general is that it may no longer be possible to define the actions $\tau, \upsilon$, firstly because $\tau, \upsilon$ would have to preserve $V_1$ and $V_2$, and secondly because $V_1, V_2$ are used to define $\tau$ and $\upsilon$ and the definitions no longer make sense. Thus the construction cannot be carried out for more general moment maps $V_i$.

However, there is another possibility, which is that if the actions $\phi, \chi$ are not sufficiently general – that is, some $\phi_i$ or $\chi_i$ is trivial, or for some distinct $i, j$ either $\phi_i = \phi_j$, $\phi_i = \phi_j^{-1}$, $\chi_i = \chi_j$ or $\chi_i = \chi_j^{-1}$ – then the new action of $U(1)^{r+s}$ on $\mathbb{HP}^{r+s+1}$ may be in the same sense not sufficiently general. This happens in the quotients given as examples in §§B.5 and B.6. In this case, we may consider allowing the moment maps $V_{1\#2}$ to be a subspace of the larger vector space $W_{1\#2}$ not contained in $\langle |x_2|^2 - |l_2|^2, \ldots, |y_{s+2}|^2 - |m_{s+2}|^2 \rangle$.

It is clear that for such $V_{1\#2}$ sufficiently close to one of the nonsingular quotients produced above, the quotient with moment maps $V_{1\#2}$ will also be nonsingular. Thus the moduli spaces of nonsingular quotients will be of larger dimension in the case when the group action is not general than it would otherwise be, because of the extra freedom to vary the moment maps. An example of this will appear in §B.6.

### B.5. Poon's metrics on $\mathbb{CP}^2\#\mathbb{CP}^2$

As an example in §4.4, it was shown that a certain quaternionic quotient of $\mathbb{HP}^3$ by $U(1)^2$ gives Poon's self-dual metrics on the connected sum of two $\mathbb{CP}^2$'s [P]. In this section it will be explained why this particular quotient was chosen: as the simplest non-trivial examples of the application of Theorem B.4.1 we shall consider the connected sum of two $\mathbb{CP}^2$'s, and the generalized connected sum of two $\mathbb{CP}^2_{2,1,1}$'s with the quaternionic structures of §4.2. These will both turn out to be the quotient given in §4.4, but represent different ends of the open interval $(0,1)$ of self-dual conformal structures on $\mathbb{CP}^2\#\mathbb{CP}^2$.

We begin with the connected sum of two $\mathbb{CP}^2$'s. Let $Q_1$ be the quotient of $\mathbb{HP}^2$ with coordinates $(x_1,l_1,x_2,l_2,x_3,l_3)$ on the associated bundle by the action $\phi:(x_1,l_1,x_2,l_2,x_3,l_3)\longmapsto(ux_1,u^{-1}l_1,$ $ux_2,u^{-1}l_2,\,ux_3,u^{-1}l_3)$ of $U(1)$ and moment maps $V_1=\langle|x_1|^2+|x_2|^2+|x_3|^2-|l_1|^2-|l_2|^2-|l_3|^2\rangle$. Let $Q_2$ be the same quotient but with coordinates $(y_1,m_1,y_2,m_2,y_3,m_3)$ and action $\chi$. Then $\mathbf{Q}_1,\mathbf{Q}_2$ are $\mathbb{CP}^2$. Let $q_1,q_2$ be the corners associated to $(x_1,l_1)$ and $(y_1,m_1)$.

Now $\Gamma_1,\Gamma_2$ are just $\{\pm1\}$, and thus give no constraints on the definitions of $x,l,y,m$. Examining the solutions of $V_1$ when $x_1=l_1=0$ shows that $c_2=-c_3$. So set $c_2=1,c_3=-1$, and define coordinates $(x,l)$ on the fibre of the associated bundle over $q_1$ by $x_2'=\frac{1}{\sqrt2}x,l_2'=\frac{1}{\sqrt2}l,x_3'=\frac{1}{\sqrt2}l,l_3'=-\frac{1}{\sqrt2}x$. Then $c_2,c_3$ define the action $\tau$ of $U(1)$ upon $x_2,l_2,x_3,l_3$. Because of the '$\sqrt2$'s in the definitions of $x,l$ we can put $e_2=1,e_3=-1$ and thus get the twistor function $w_1=|x_2|^2-|l_2|^2-|x_3|^2+|l_3|^2$.

Similarly let $d_2=1,d_3=-1$, and define $y,m,\upsilon$ and $w_2$ as $x,l,\tau,w_1$ so that $w_2=|y_2|^2-|m_2|^2-|y_3|^2+|m_3|^2$. Now apply Theorem B.4.1 to get a new quaternionic quotient $Q_{1\#2}$ which is the connected sum of $\mathbf{Q}_1,\mathbf{Q}_2$ at $q_1,q_2$.

The theorem states that if $Q_{1\#2}$ is defined as the quaternionic quotient of $\mathbb{HP}^3$ with coordinates $(x_2,l_2,x_3,l_3,y_2,m_2,y_3,m_3)$ on the associated bundle by $U(1)^2$ with action

$$\begin{matrix}(x_2,l_2,x_3,l_3,\\ y_2,m_2,y_3,m_3)\end{matrix}\quad\overset{(u_1,u_2)}{\longmapsto}\quad\begin{matrix}(u_1u_2x_2,u_1^{-1}u_2^{-1}l_2,u_1u_2^{-1}x_3,u_1^{-1}u_2l_3,\\ u_1u_2y_2,u_1^{-1}u_2^{-1}m_2,u_1^{-1}u_2y_3,u_1u_2^{-1}m_3)\end{matrix},\qquad(u_1,u_2)\in U(1)\times U(1),\tag{153}$$

and moment maps $V_{1\#2} = \langle \epsilon^2(|y_2|^2 - |m_2|^2 - |y_3|^2 + |m_3|^2) + |x_2|^2 - |l_2|^2 + |x_3|^2 - |l_3|^2, \epsilon^2(|x_2|^2 - |l_2|^2 - |x_3|^2 + |l_3|^2) + |y_2|^2 - |m_2|^2 + |y_3|^2 - |m_3|^2\rangle$, then for small $\epsilon$ the quotient $\mathbf{Q}_{1\#2}$ is a non-singular quaternionic structure on $\mathbb{CP}^2\#\mathbb{CP}^2$.

To simplify this, make the following definitions:

$$\begin{aligned} &z_1 = x_2, \quad n_1 = l_2, \quad z_2 = y_2, \quad n_2 = m_2, \\ &z_3 = x_3, \quad n_3 = l_3, \quad z_4 = m_3, \quad n_4 = -y_3, \\ &u = u_1u_2, \quad v = u_1u_2^{-1}. \end{aligned} \tag{154}$$

Then the quotient can be written as the quaternionic quotient of $\mathbb{HP}^3$ with coordinates $(z_1, n_1, \dots, z_4, n_4)$ on the associated bundle by $U(1)^2$ with action

$$\begin{matrix} (z_1, n_1, z_2, n_2, \\ z_3, n_3, z_4, n_4) \end{matrix} \xrightarrow{(u,v)} \begin{matrix} (uz_1, u^{-1}n_1, uz_2, u^{-1}n_2, \\ vz_3, v^{-1}n_3, vz_4, v^{-1}n_4) \end{matrix}, \qquad (u,v) \in U(1) \times U(1), \tag{155}$$

and moment maps $V_{1\#2} = \langle (1+\epsilon^2)(|z_1|^2 - |n_1|^2 + |z_2|^2 - |n_2|^2) + (1-\epsilon^2)(|z_3|^2 - |n_3|^2 - |z_4|^2 + |n_4|^2), (1+\epsilon^2)(|z_3|^2 - |n_3|^2 + |z_4|^2 - |n_4|^2) + (1-\epsilon^2)(|z_1|^2 - |n_1|^2 - |z_2|^2 + |n_2|^2)\rangle$, where we have changed basis in $V_{1\#2}$.

Now by inspection this is the same as the quaternionic quotient of §4.4, with parameter $\alpha = \frac{1-\epsilon^2}{1+\epsilon^2}$. Thus when $\epsilon$ is small, $\alpha$ is in $(0,1)$ and very close to 1. So $\alpha = 1$ corresponds to the metric decaying into two $\mathbb{CP}^2$'s.

Our second application of Theorem B.4.1 is to the generalized connected sum of two $\mathbb{CP}^2_{2,1,1}$'s. This time let $Q_1$ be the quotient of $\mathbb{HP}^2$ with coordinates $(x_1, l_1, x_2, l_2, x_3, l_3)$ on the associated bundle by the action $\phi : (x_1, l_2, x_2, l_2, x_3, l_3) \longmapsto (x_1, l_1, ux_2, u^{-1}l_2, ux_3, u^{-1}l_3)$ of $U(1)$ and moment maps $V_1 = \langle |x_1|^2 + |x_2|^2 + |x_3|^2 - |l_1|^2 - |l_2|^2 - |l_3|^2\rangle$. Let $Q_2$ be the same quotient but with coordinates $(y_1, m_1, y_2, m_2, y_3, m_3)$ and action $\chi$. Then from §4.2, $\mathbf{Q}_1, \mathbf{Q}_2$ are quaternionic structures on $\mathbb{CP}^2_{2,1,1}$. Let $q_1, q_2$ be the corners associated to $(x_1, l_1)$ and $(y_1, m_1)$. Then $q_1, q_2$ are the orbifold points of $\mathbf{Q}_1, \mathbf{Q}_2$.

Then $\Gamma_1, \Gamma_2$ are again equal to $\{\pm 1\}$, but the actions on $\mathbb{H}^2$ are different, since $-1$ acts by $(x, l, x_1, l_1) \mapsto (x, l, -x_1, -l_1)$ and $(y, m, y_1, m_1) \mapsto (y, m, -y_1, -m_1)$. The identification $C$ does not actually identify the actions of $\Gamma_1$ and $\Gamma_2$, but only the actions of $\{\pm 1\} \times \Gamma_1$ and $\{\pm 1\} \times \Gamma_2$, as discussed in §B.3. The disagreement arises because $\mathbb{CP}^2_{2,1,1}$ is spin, but $\mathbb{CP}^2\#\mathbb{CP}^2$ is not.

Let $x, l, y, m, \tau, \upsilon, w_1$ and $w_2$ be as in the previous case. Actually, the definitions of $\tau, \upsilon$ don't matter, since $\phi_1 = \chi_1 = 1$ and so $\tau, \upsilon$ act trivially. Theorem B.4.1 then says that if $Q_{1\#2}$ is defined to be the quaternionic quotient of $\mathbb{HP}^3$ with coordinates $(x_2, l_2, x_3, l_3, y_2, m_2, y_3, m_3)$ on the associated bundle, by $U(1)^2$ with action

$$\begin{matrix} (x_2, l_2, x_3, l_3, \\ y_2, m_2, y_3, m_3) \end{matrix} \overset{(u_1,u_2)}{\longmapsto} \begin{matrix} (u_1 x_2, u_1^{-1} l_2, u_1 x_3, u_1^{-1} l_3, \\ u_2 y_2, u_2^{-1} m_2, u_2 y_3, u_2^{-1} m_3) \end{matrix}, \qquad (u_1, u_2) \in U(1) \times U(1), \tag{156}$$

and moment maps $V_{1\#2} = \langle \epsilon^2(|y_2|^2 - |m_2|^2 - |y_3|^2 + |m_3|^2) + |x_2|^2 - |l_2|^2 + |x_3|^2 - |l_3|^2, \epsilon^2(|x_2|^2 - |l_2|^2 - |x_3|^2 + |l_3|^2) + |y_2|^2 - |m_2|^2 + |y_3|^2 - |m_3|^2 \rangle$, then for small $\epsilon$ the quotient $\mathbf{Q}_{1\#2}$ is a non-singular quaternionic structure on the generalized connected sum of $\mathbb{CP}^2_{2,1,1}$ with itself at the orbifold points.

This will again be rewritten by defining

$$\begin{aligned} &z_1 = x_2, \quad n_1 = l_2, \quad z_2 = x_3, \quad n_2 = l_3, \\ &z_3 = y_2, \quad n_3 = m_2, \quad z_4 = y_3, \quad n_4 = m_3, \\ &u = u_1, \quad v = u_2. \end{aligned} \tag{157}$$

Then the quotient can be written as the quaternionic quotient of $\mathbb{HP}^3$ with coordinates $(z_1, n_1, \ldots, z_4, n_4)$ on the associated bundle, by $U(1)^2$ with action

$$\begin{matrix} (z_1, n_1, z_2, n_2, \\ z_3, n_3, z_4, n_4) \end{matrix} \overset{(u,v)}{\longmapsto} \begin{matrix} (u z_1, u^{-1} n_1, u z_2, u^{-1} n_2, \\ v z_3, v^{-1} n_3, v z_4, v^{-1} n_4) \end{matrix}, \qquad (u, v) \in U(1) \times U(1), \tag{158}$$

and moment maps $V_{1\#2} = \langle (|z_1|^2 - |n_1|^2 + |z_2|^2 - |n_2|^2) + \epsilon^2(|z_3|^2 - |n_3|^2 - |z_4|^2 + |n_4|^2), (|z_3|^2 - |n_3|^2 + |z_4|^2 - |n_4|^2) + \epsilon^2(|z_1|^2 - |n_1|^2 - |z_2|^2 + |n_2|^2) \rangle$.

By inspection this is the quotient of §4.4 with parameter $\alpha = \epsilon^2$, and also of course the same as the previous quotient except that $\epsilon^2$ replaces $\frac{1-\epsilon^2}{1+\epsilon^2}$. Note that when $\epsilon$ is small, $\alpha$ is in $(0,1)$ and very close to 0. Therefore $\alpha = 0$ corresponds to the metric decaying into two $\mathbb{CP}^2_{2,1,1}$'s (which are ALE spaces associated to the group $\{\pm 1\}$) glued at their orbifold points.

Theorem B.4.1 has been applied in two different situations to give the same quotient, but we found that small $\epsilon$ parameterized different regions of the space of moment maps in each case. This means that the space of self-dual conformal structures on $\mathbb{CP}^2 \# \mathbb{CP}^2$ is an open interval, and the two cases construct the same quotient, but start at opposite ends of the interval.

## B.6. LeBrun's metrics on $n\mathbb{CP}^2$

To each finite subgroup $\Gamma$ of $SU(2)$ there is associated a family of four-dimensional, nonsingular hyperkähler manifolds called ALE spaces which approach the flat metric on $\mathbb{C}^2/\Gamma$ at infinity [Hi]. A construction of these metrics as hyperkähler quotients has been given by Kronheimer in [Kr]. The simplest

ALE spaces are those associated to cyclic subgroups of $SU(2)$, and these are hyperkähler quotients of $\mathbb{H}^{n+1}$ by $U(1)^n$, and so are also quaternionic quotients of $\mathbb{HP}^{n+1}$ by $U(1)^n$.

Thus the ALE spaces associated to finite cyclic groups are the type of quotient considered in §B.2. They can in fact be constructed from weighted projective spaces using the methods of §B.4, as will be explained in §B.7.

Recall that weighted projective spaces with only one orbifold point were studied in §4.2. They are exactly the weighted projective spaces $\mathbb{CP}^2_{n,1,1}$ with $n > 1$. The orbifold group at $[1, 0, 0]$ in $\mathbb{CP}^2_{n,1,1}$ is

$$\Gamma = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : u^n = 1 \right\}, \tag{159}$$

which with our orientation conventions is complementary to the orbifold group of the orbifold point of an ALE space associated to the cyclic group of order $n$.

Using these two ingredients and the results of §B.4, a quaternionic quotient will be constructed for the generalized connected sum of these two orbifolds, which will be shown in Theorem B.6.1 to be one of LeBrun's self-dual metrics on $n\mathbb{CP}^2$ [L2].

Let $Q_1$ be the quaternionic quotient of $\mathbb{HP}^n$ with coordinates $(x_1, l_1, \ldots, x_{n+1}, l_{n+1})$ on the associated bundle by $U(1)^{n-1}$, with action

$$\begin{gathered}(x_1, l_1, \ldots, x_i, l_i, \ldots, x_n, l_n, x_{n+1}, l_{n+1}) \overset{(u_1, \ldots, u_{n-1})}{\longmapsto} \\ (u_1 x_1, u_1^{-1} l_1, \ldots, u_i u_{i-1}^{-1} x_i, u_i^{-1} u_{i-1} l_i, \ldots, u_{n-1}^{-1} x_n, u_{n-1} l_n, x_{n+1}, l_{n+1}), \\ (u_1, \ldots, u_{n-1}) \in U(1)^{n-1},\end{gathered} \tag{160}$$

and moment maps

$$\begin{aligned} V_1 = \langle &|x_1|^2 - |l_1|^2 - |x_2|^2 + |l_2|^2 + \zeta_1(|x_{n+1}|^2 - |l_{n+1}|^2), \ldots, \\ &|x_{n-1}|^2 - |l_{n-1}|^2 - |x_n|^2 + |l_n|^2 + \zeta_{n-1}(|x_{n+1}|^2 - |l_{n+1}|^2)\rangle; \end{aligned} \tag{161}$$

then $Q_1$ is in fact the quotient given in [Kr] for the ALE space associated to the cyclic group of order $n$, rewritten in quaternionic form and with moment maps $\mu_2 = \mu_3 = 0$ and $\mu_1$ given by $\zeta_1, \ldots, \zeta_{n-1}$, which are some real constants. Let $\zeta_1, \ldots, \zeta_{n-1}$ be generic; then $\mathbf{Q}_1$ is nonsingular except for the orbifold point $x_{n+1} = l_{n+1} = 0$.

Let $Q_2$ be the quotient given in §4.2 for $\mathbb{CP}^2_{n,1,1}$, so that $Q_2$ is the quaternionic quotient of $\mathbb{HP}^2$ with coordinates $(y_1, m_1, y_2, m_2, y_3, m_3)$ on the associated bundle, by $U(1)$ with action

$$(y_1, m_1, y_2, m_2, y_3, m_3) \stackrel{u}{\longmapsto} (u^{2-n}y_1, u^{n-2}m_1, u^n y_2, u^{-n}m_2, u^n y_3, u^{-n}m_3), \qquad u \in U(1), \tag{162}$$

and moment maps $V_2 = \langle |y_1|^2 + |y_2|^2 + |y_3|^2 - |m_1|^2 - |m_2|^2 - |m_3|^2 \rangle$. Then the corners $q_1, q_2$ associated to $(x_{n+1}, l_{n+1})$ in $\mathbf{Q}_1$ and $(y_1, m_1)$ in $\mathbf{Q}_2$ are complementary, and Theorem B.4.1 may be applied to get a quotient $Q_{1\#2}$, such that $\mathbf{Q}_{1\#2}$ is the generalized connected sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ at $q_1, q_2$, and is thus a compact, non-singular 4-manifold.

In defining $x, l, y, m$, the moment maps give $c_1 = \ldots = c_n$ and $d_2 = -d_3$; as the orbifold group is contained in $SU(2) \subset SO(4)$ it does not matter which signs are chosen. Choose $c_1 = \ldots = c_n = 1$ and define the coordinates $(x, l)$ by $x_i' = \frac{1}{\sqrt{n}}x, l_i' = \frac{1}{\sqrt{n}}l$. Put $e_i = 1$, so that $w_1 = |x_1|^2 - |l_1|^2 + \ldots + |x_n|^2 - |l_n|^2$. Also choose $d_2 = 1, d_3 = -1$ and define $(y, m)$ by $y_2' = \frac{1}{\sqrt{2}}y, m_2' = \frac{1}{\sqrt{2}}m, y_3' = \frac{1}{\sqrt{2}}m, m_3' = -\frac{1}{\sqrt{2}}y$. Then we may put $f_2 = 1, f_3 = -1$ so that $w_2 = |y_2|^2 - |m_2|^2 - |y_3|^2 + |m_3|^2$.

So by Theorem B.4.1, if $Q_{1\#2}$ is defined as the quaternionic quotient of $\mathbb{HP}^{n+1}$ with coordinates $(x_1, \ldots, l_n, y_2, \ldots, m_3)$ on the associated bundle, by $U(1)^n$ with action

$$\begin{aligned}(x_1, l_1, \ldots, x_i, l_i, \ldots, &\stackrel{(u_1,\ldots,u_{n-1},u)}{\longmapsto} (u^{2-n}u_1x_1, u^{n-2}u_1^{-1}l_1, \ldots, u^{2-n}u_iu_{i-1}^{-1}x_i, u^{n-2}u_i^{-1}u_{i-1}l_i, \ldots, \\ x_n, l_n, y_2, m_2, y_3, m_3) \qquad & \qquad u^{2-n}u_{n-1}^{-1}x_n, u^{n-2}u_{n-1}l_n, u^n y_2, u^{-n}m_2, u^n y_3, u^{-n}m_3), \\ & \qquad (u_1, \ldots, u_{n-1}, u) \in U(1)^n,\end{aligned} \tag{163}$$

and moment maps

$$\begin{aligned} V_{1\#2} = \langle &|x_1|^2 - |l_1|^2 - |x_2|^2 + |l_2|^2 + \zeta_1\epsilon^2(|y_2|^2 - |m_2|^2 - |y_3|^2 + |m_3|^2), \ldots, \\ &|x_{n-1}|^2 - |l_{n-1}|^2 - |x_n|^2 + |l_n|^2 + \zeta_{n-1}\epsilon^2(|y_2|^2 - |m_2|^2 - |y_3|^2 + |m_3|^2), \\ &\epsilon^2(|x_1|^2 - |l_1|^2 + \ldots + |x_n|^2 - |y_n|^2) + |y_2|^2 - |m_2|^2 + |y_3|^2 - |m_3|^2 \rangle, \end{aligned} \tag{164}$$

then $\mathbf{Q}_{1\#2}$ is the nonsingular generalized connected sum of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ at $q_1, q_2$ for $\epsilon$ sufficiently small.

One very special feature of the quotient $Q_{1\#2}$ is that the group action upon the quaternionic pairs $(y_2, m_2)$ and $(y_3, m_3)$ is the same. This is a consequence of the fact that $\mathbf{Q}_1$ is hypercomplex rather than simply quaternionic, which causes $\phi_{n+1}$ to be trivial. This fact will be used in the proof of Theorem B.6.1; it also means that $\mathrm{Re}(y_2\overline{y}_3 - \overline{m}_2m_3), \mathrm{Im}(y_2\overline{y}_3 - \overline{m}_2m_3)$ are $U(1)^n$- invariant twistor functions and can thus be used to make deformations of $V_{1\#2}$ to get different metrics, as at the end of §B.4.

Starting from a quotient known to be nonsingular, such as $Q_{1\#2}$ with $\epsilon$ small, it is clear that for sufficiently small changes to the vector space $V_{1\#2}$ of moment maps in the vector space $W_{1\#2}$ of $U(1)^n$- invariant twistor functions, the new quotient will also be nonsingular, for nonsingularity is an open condition. (We think of fixing $\epsilon$ small, and then making changes much smaller than $\epsilon$.)

A parameter count reveals that allowing $V_{1\#2}$ to vary in $W_{1\#2}$ gives $3n-6$ parameters of distinct quaternionic structures on $W_{1\#2}$ for $n>2$. This is the same as the number of parameters in LeBrun's construction, and it will be proved below that the two constructions yield exactly the same families of quaternionic structures.

For the rest of this section, we shall suppose that $Q_{1\#2}$ is the quotient produced above, but that the vector space of moment maps $V_{1\#2}$ is any $n$- dimensional subspace of $W_{1\#2}$ such that $\mathbf{Q}_{1\#2}$ is compact and nonsingular.

**Theorem B.6.1.** *The nonsingular quotient $\mathbf{Q}_{1\#2}$ defined above is one of LeBrun's metrics on $n\mathbb{CP}^2$ [L2]. Moreover, so is any nonsingular quotient given by choosing a different vector space of moment maps in $W_{1\#2}$. All of LeBrun's metrics on $n\mathbb{CP}^2$ are examples of this construction.*

*Proof.* Each self-dual conformal 4-manifold $M$ has a twistor space $Z$ fibring over it that is a 3-dimensional complex manifold with a real structure. If $M$ has a conformally isometric action of $U(1)$, this action can be lifted to $Z$ and then complexified, to give an action of $\mathbb{C}^*$. Quotienting a suitable open set of $Z$ by this action gives a 2-dimensional complex manifold $T$ with a real structure, called a mini-twistor space. Mini-twistor spaces correspond to 3-manifolds with a structure on them called an Einstein-Weyl geometry, in the same way as twistor spaces correspond to 4-manifolds with a self-dual conformal structure.

In twistor language, LeBrun's construction of self-dual metrics on $n\mathbb{CP}^2$ is the reverse of this process: one begins with a mini-twistor space $T$ and defines the twistor space $Z$ as a line bundle $L$ over $T$; if $L$ is chosen carefully, its total space can be compactified to give the twistor space of a compact manifold. The extra information of the scalar-flat Kähler metric in the conformal class is given by a real section of $K_T^{-\frac{1}{2}}$. (For all the above material, see [L2], §6.)

What we will do is produce a conformally isometric action of $U(1)$ on the quotient $\mathbf{Q}_{1\#2}$, and show that the mini-twistor space $T_{1\#2}$ constructed from the twistor space $Z_{1\#2}$ is the mini-twistor space of the hyperbolic 3-plane $\mathcal{H}^3$. It will then follow that the quaternionic structure on $\mathbf{Q}_{1\#2}$ is constructed by LeBrun's 'Hyperbolic Ansatz' ([L2], §4, Proposition 2).

Examining the fixed points of the $U(1)$ action then show us the 'nuts and bolts' that must be glued in to compactify the metric, and one can tell exactly which of LeBrun's metrics $\mathbf{Q}_{1\#2}$ is by projecting the fibres of the isolated fixed points to the mini-twistor space $T_{1\#2}$; under the isomorphism of $T_{1\#2}$ with the mini-twistor space of $\mathcal{H}^3$ these correspond to the points $\{p_1, \ldots, p_n\}$ that are the data LeBrun uses in §5 of [L2] to construct his metrics.

Define an action $\theta$ of $U(1)$ on $Q_{1\#2}$ by

$$\theta : (x_1, l_1, \ldots, x_n, l_n, y_2, m_2, y_3, m_3) \overset{v}{\longmapsto} (vx_1, v^{-1}l_1, \ldots, vx_n, v^{-1}l_n, y_2, m_2, y_3, m_3), \quad v \in U(1); \tag{165}$$

then $\theta$ induces actions of $U(1)$ on $\mathbf{Q}_{1\#2}$ and the twistor space $Z_{1\#2}$ that will also be called $\theta$. Let the complexification of $\theta$ on $Z_{1\#2}$ be $\theta^c$.

Now the mini-twistor space of $\mathcal{H}^3$ is the quotient $T$ of the stable points of the twistor space $\mathbb{CP}^3$ of $\mathcal{S}^4$ by a complexified action of $U(1)$; in homogeneous coordinates $[z_0, z_1, z_2, z_3]$ on $\mathbb{CP}^3$ with real structure $[z_0, z_1, z_2, z_3] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2]$, the action is

$$[z_0, z_1, z_2, z_3] \overset{v}{\longmapsto} [vz_0, v^{-1}z_1, vz_2, v^{-1}z_3], \qquad v \in \mathbb{C}^*. \tag{166}$$

Using the fact that $Z_{1\#2}$ may equally be defined using the real group $U(1)^n$ and all three moment maps, or the complexification $(\mathbb{C}^*)^n$ and the complex moment map $\mu_2 + i\mu_3$, the quotient $T_{1\#2}$ of the stable points of $Z_{1\#2}$ by $\theta^c$ may be mapped biholomorphically to $T$ by the map

$$[x_1, l_1, \ldots, x_n, l_n, y_2, m_2, y_3, m_3]/(\mathbb{C}^*)^{n+1} \longmapsto [y_2, m_2, y_3, m_3]/\mathbb{C}^*. \tag{167}$$

This maps into the right space because of the actions (163) and (165), and it is a biholomorphism because given any $[z_0, z_1, z_2, z_3]$ in $T$, the identifications $y_2 = z_0$, $m_2 = z_1$, $y_3 = z_2$, $m_3 = z_3$ do define the terms in $y_2, m_2, y_3, m_3$ in the complex moment maps exactly, and then the space of complex moment maps corresponding to $V_{1\#2}$ defines the values of $x_1 l_1, \ldots, x_n l_n$.

But these values are sufficient to define the orbit of $(x_1, l_1, \ldots, x_n, l_n)$ under $(\mathbb{C}^*)^n$. So given an element of $T$, one can define the orbit under $(\mathbb{C}^*)^{n+1}$ of a solution of the complex moment maps corresponding to $V_{1\#2}$, in other words, an element of $T_{1\#2}$. This gives a holomorphic inverse to the holomorphic map $T_{1\#2} \to T$, so the two are biholomorphic; moreover, the map identifies the real structures of $T$ and $T_{1\#2}$.

It is easy to verify that the isolated fixed points of the action $\theta$ on $\mathbf{Q}_{1\#2}$ are the corners associated to the quaternionic pairs $(x_1, l_1)$,…,$(x_n, l_n)$, and that the non-isolated fixed points can be represented by points $(x_1, \ldots, m_3)$ with $y_2 = y_3 = 0$ (alternatively, $m_2 = m_3 = 0$). Now $x_i = l_i = 0$ defines a real line in $T_{1\#2}$ that is identified with the fibre over some point $p_i$ in $\mathcal{H}^3$ under the identification with $T$. In this way one defines points $p_1, \ldots, p_n$ in $\mathcal{H}^3$ corresponding to isolated fixed points of the action $\theta$ of $U(1)$ on $\mathbf{Q}_{1\#2}$, and it is clear that $\mathbf{Q}_{1\#2}$ is isomorphic to the metric constructed from $p_1, \ldots, p_n$ by LeBrun in §5 of [L2].

The correspondence can be made explicit: there is a unique basis for $V_{1\#2}$ with $i^{\text{th}}$ element $|x_i|^2 - |l_i|^2 + \alpha_i(|y_2|^2 - |m_2|^2) + \beta_i(|y_3|^2 - |m_3|^2) + \gamma_i \operatorname{Re}(y_2\overline{y}_3 - \overline{m}_2 m_3) + \delta_i \operatorname{Im}(y_2\overline{y}_3 - \overline{m}_2 m_3)$. Rescaling $(x_i, l_i)$ by a real factor rescales $(\alpha_i, \ldots, \delta_i)$ by a positive real factor, so only the equivalence class $[\alpha_i, \beta_i, \gamma_i, \delta_i]$ in $\mathcal{S}^3$ of $(\alpha_i, \ldots, \delta_i)$ matters.

Then $[\alpha_i, \beta_i, \gamma_i, \delta_i] \in \mathcal{S}^3$ defines the point $p_i$ through a 1-1 correspondence between an open set of $\mathcal{S}^3$ and $\mathcal{H}^3$. (The rest of $\mathcal{S}^3$ leads to a singular quotient.) Conversely, $p_1, \ldots, p_n$ define $\alpha_1, \ldots, \delta_n$ up to positive real factors and choosing the factors gives $V_{1\#2}$, and then the quotient $\mathbf{Q}_{1\#2}$ is isomorphic to LeBrun's metric on $n\mathbb{CP}^2$ constructed from $p_1, \ldots, p_n$. This concludes the proof of Theorem B.6.1. $\square$

For $n > 2$ and generic moment maps $V_{1\#2}$, $\theta$ gives the entire symmetry group of $\mathbf{Q}_{1\#2}$, since other $U(1)$ actions on $\mathbb{HP}^{n+1}$ will not preserve $V_{1\#2}$. The vector space of twistor functions on $\mathbf{Q}_{1\#2}$ is simply $W_{1\#2}/V_{1\#2}$, so its dimension is four. From §2.4, these represent the Kähler metrics of zero scalar curvature in the conformal class of $\mathbf{Q}_{1\#2}$.

The metrics that are asymptotically Euclidean are represented by twistor functions that vanish at only one point in $\mathbf{Q}_{1\#2}$, and it is easy to see that there is (up to homothety) exactly one of these for every fixed point of $\theta$; the twistor function $|x_i|^2 - |l_i|^2$ is an asymptotically Euclidean Kähler metric on the complement of the corner associated to $(x_i, l_i)$ in $\mathbf{Q}_{1\#2}$ for $i = 1, \ldots, n$, and there is also a two-sphere's worth of metrics that are asymptotically Euclidean close to a point on the two-sphere fixed by $\theta$.

## B.7. Generalized connected sums of weighted projective spaces

We now have at our disposal a collection of orbifolds – the weighted projective spaces – and a method which gives quaternionic structures on generalized connected sums of them at their corners, which are usually their orbifold points. In this section it will be shown how to build up nonsingular spaces from these elements.

Firstly we will show that every orbifold singularity of a weighted projective space can be desingularized by making a generalized connected sum with a collection of weighted projective spaces. In fact for each orbifold group, a particular collection of weighted projective spaces will be built up in an inductive way, that will be called the *standard resolution* of the orbifold group.

Consider resolving the corner $[1,0,0]$ of $\mathbb{CP}^2_{p,q,r}$ with $p,q,r$ pairwise coprime. (Actually $q,r$ do not need to be coprime.) Recall from §B.1 that $\mathbb{CP}^2_{p,q,r}$ and $\mathbb{CP}^2_{p,q',r'}$ have complementary singularities at $[1,0,0]$ if either there is an integer $s$ such that $sq' \equiv -q(\mathrm{mod}\, p)$ and $sr' \equiv r(\mathrm{mod}\, p)$ or there is an integer $s$ such that $sq' \equiv -r(\mathrm{mod}\, p)$ and $sr' \equiv q(\mathrm{mod}\, p)$.

There is a unique integer $q'$ such that $rq' \equiv -q(\mathrm{mod}\, p)$ and $1 \leq q' < p$, and a unique integer $q''$ such that $qq'' \equiv -r(\mathrm{mod}\, p)$ and $1 \leq q'' < p$. Then the points $[1,0,0]$ in $\mathbb{CP}^2_{p,q',1}$ and $\mathbb{CP}^2_{p,q'',1}$ are both complementary to the point $[1,0,0]$ in $\mathbb{CP}^2_{p,q,r}$. Define $p_0 = p$ and $p_1 = \min\{q', q''\}$. Then $p_0, p_1$ are coprime, $p_1 < p_0$ and $\mathbb{CP}^2_{p_0,p_1,1}$ has only two singular points at $[1,0,0]$ and $[0,1,0]$, of which $[1,0,0]$ is complementary to the point $[1,0,0]$ in $\mathbb{CP}^2_{p,q,r}$.

Construct a finite sequence $p_0 = p > p_1 > \ldots > p_k = 1$ inductively, as follows. Make $p_0, p_1$ as above. Having just constructed $p_l$, if $p_l = 1$ then set $k = l$ and terminate the sequence. Otherwise make $p_{l+1}$ in the same way as $p_1$, so that $p_l, p_{l+1}$ are coprime, $1 \leq p_{l+1} < p_l$ and $[1,0,0]$ in $\mathbb{CP}^2_{p_l,p_{l+1},1}$ is complementary to $[0,1,0]$ in $\mathbb{CP}^2_{p_{l-1},p_l,1}$. The sequence must terminate because it is a decreasing sequence of positive integers.

The point $[1,0,0]$ in $\mathbb{CP}^2_{p,q,r}$ may now be desingularized by doing a chain of connected sums: take the generalized connected sum of $\mathbb{CP}^2_{p,q,r}$ at $[1,0,0]$ with $\mathbb{CP}^2_{p_0,p_1,1}$ at $[1,0,0]$, and then for $l = 2,\ldots,k$ take the generalized connected sum of $\mathbb{CP}^2_{p_{l-1},p_l,1}$ at $[0,1,0]$ with $\mathbb{CP}^2_{p_l,p_{l+1},1}$ at $[1,0,0]$. The last element of the chain is $\mathbb{CP}^2_{p_{k-1},1,1}$ which has only the one singular point $[1,0,0]$.

So given a quotient $Q$ and a corner $q$ such that the orbifold singularity at $q$ in $\mathbf{Q}$ is that of $\mathbb{CP}^2_{p,q,r}$ at $[1,0,0]$, using the sequence $\mathbb{CP}^2_{p_0,p_1,1}, \ldots, \mathbb{CP}^2_{p_{k-1},1,1}$ and the method of §B.4 a quotient $Q'$ may be built up, such that for small values of the parameters in the moment maps, $\mathbf{Q}'$ is this chain of generalized connected sums. Then $\mathbf{Q}'$ is $\mathbf{Q}$ with the orbifold point $q$ desingularized by gluing in a collection of weighted projective spaces. This will be called the standard resolution of $\mathbf{Q}$ at $q$.

For each orbifold group of a point in a weighted projective space, this method also yields quaternionic structures upon a compact manifold with one orbifold point complementary to the orbifold group, by just taking the generalized connected sum of the chain $\mathbb{CP}^2_{p_0,p_1,1}, \ldots, \mathbb{CP}^2_{p_{k-1},1,1}$. It can be shown that for finite cyclic subgroups of $U(1)$ acting by scalar multiplication, these spaces are the corresponding ALE spaces which appeared in §B.6.

The simplest way to build up non-singular spaces using the techniques of §B.4 is just to take the connected sum of several $\mathbb{CP}^2$'s at their corners. The discussion above gives another method: take any weighted projective space or generalized connected sum of a collection of weighted projective spaces, and then desingularize by performing the standard resolution of each remaining orbifold point. For example, LeBrun's metrics of §B.6 are just the standard resolution of $\mathbb{CP}^2_{n,1,1}$.

We shall now see that these methods yield more than one different quaternionic quotient for $n\mathbb{CP}^2$ when $n > 3$. There are in fact quotients for $n\mathbb{CP}^2$ for which the group action is not the same for any two quaternionic pairs, and which cannot therefore be isomorphic as quotients to the quaternionic quotient for $n\mathbb{CP}^2$ in §B.6.

The simplest such example is the connected sum of 4 $\mathbb{CP}^2$'s with three of the four joined on at the three corners of the fourth. Applying the methods of §B.4 yields a quaternionic quotient of $\mathbb{HP}^5$ by $U(1)^4$ with its space of moment maps depending on three small parameters $\epsilon_1, \epsilon_2, \epsilon_3$, and by inspection all six quaternionic pairs in the quotient have distinct group actions. So the quotient is not isomorphic to the quotient for LeBrun's metrics on $4\mathbb{CP}^2$. It can be shown by induction that for $n > 4$ there are similar connected sums yielding quotients not isomorphic to the corresponding quotient for LeBrun's metrics.

To show that these other quotients do not nevertheless yield subfamilies of the LeBrun metrics, observe that we understand the boundary of the moduli space of LeBrun's metrics well enough to say that they cannot decay into connected sums in certain ways. Thus, as well as LeBrun's metrics on $n\mathbb{CP}^2$ there are for $n > 3$ other distinct families of quaternionic structures on $n\mathbb{CP}^2$ that can be constructed by the quaternionic quotient.

All of these families are quaternionic quotients of $\mathbb{HP}^{n+1}$ by $U(1)^n$, such that the group actions on all $n+2$ quaternionic pairs are non-trivial and distinct. These facts imply that the vector space $W$ of $U(1)^n$-invariant twistor functions is just $\langle |x_1|^2 - |l_1|^2, \ldots, |x_{n+2}|^2 - |l_{n+2}|^2 \rangle$ if $(x_1, \ldots, l_{n+2})$ are quaternionic coordinates on the associated bundle of $\mathbb{HP}^{n+1}$; there are $2n$ parameters for the choice of the moment map vector space $V$, but these are acted upon by an $n+1$- dimensional group of isomorphisms coming from rescaling the quaternionic pairs, and so the dimension of the moduli space is $n - 1$.

Thus each of these other families of self-dual metrics on $n\mathbb{CP}^2$ for $n > 3$ should have dimension $n - 1$, which fits nicely with the observation that repeated applications of Theorem B.4.1 to build up a connected sum of $n$ $\mathbb{CP}^2$'s result in $n-1$ small parameters $\epsilon_1, \ldots, \epsilon_{n-1}$ for the quotient. In contrast, for $n > 2$ LeBrun's family of metrics on $n\mathbb{CP}^2$ has $3n-6$ parameters. Also the space of twistor functions $W/V$ on the quotients has two dimensions, there are up to homothety $n+2$ different asymptotically flat Kähler metrics with zero scalar curvature in the conformal class of each, and the identity component of the symmetry group is generically $U(1)^2$, rather than $U(1)$ as in LeBrun's case.

A concrete description of these families of metrics, similar to LeBrun's description of his metrics, has been given in §4.5 of Chapter 4.

## B.8. Quotients for Asymptotically Locally Flat metrics

In the study of 'gravitational instantons' initiated by Gibbons and Hawking, attention has focussed on complete self-dual solutions of Einstein's vacuum equations with two sorts of asymptotic behaviour at infinity, called Asymptotically Locally Euclidean (ALE) and Asymptotically Locally Flat (ALF). ALE metrics are metrics which at infinity asymptotically resemble $\mathbb{R}^4/\Gamma$, where $\Gamma$ is some finite subgroup of $SO(4)$ (usually also of $SU(2)$) acting freely on $\mathbb{R}^4 \setminus \{0\}$. Thus the conformal compactification of an ALE end of some metric on a manifold is just an orbifold, with orbifold group $\Gamma$ at the extra point.

A space is ALF if outside a compact set the metric approaches the metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\sigma_1^2 + \sigma_2^2) + \sigma_3^2, \tag{168}$$

where $\sigma_i$ are the left invariant 1-forms on $\mathcal{S}^3/\Gamma$, and $\Gamma$ is a finite isometry group of this $U(2)$- invariant metric. So the topology of an ALF end of a Riemannian manifold may be the same as the topology of such a manifold with an ALE end; it is just the asymptotic properties of the metric which differ. Note that the metric does not actually approach a flat model at infinity, but does have finite action. Some simple Ricci-flat ALF spaces have been written down by Hawking [Ha], for cyclic groups $\Gamma$.

In this section we will write down a quotient for the basic hyperkähler ALF metric on $\mathbb{R}^4$, and then see how to include it into the general scheme of the last few sections of taking connected sums of quotients. The first corollary of this is a quotient construction for Hawking's ALF spaces; I would like to note that this quotient was already known to P.B. Kronheimer (personal communication, 1991), who also has a construction of hyperkähler ALF spaces for dihedral groups $\Gamma$, using a monopole moduli space. The second corollary is the ability to stick ALF ends onto self-dual manifolds that are already known as quotients.

In particular, there is a large supply of complete ALF Kähler manifolds with zero scalar curvature, which are perhaps interesting.

### B.8.1. An ALF metric on $\mathbb{R}^4$

Consider the following quaternionic quotient of $\mathbb{HP}^2$ by $\mathbb{R}$. Let $(x, y, z, l, m, n)$ be complex coordinates on $\mathbb{H}^3$, the associated bundle of $\mathbb{HP}^2$, in the usual way, and let $\mathbb{R}$ act by

$$(x, y, z, l, m, n) \overset{t}{\longmapsto} (e^{it}x, y + tz, z, e^{-it}l, m + tn, n), \qquad t \in \mathbb{R}, \tag{169}$$

with moment maps

$$\mu_1 = |x|^2 - |l|^2 - i\overline{y}z + iy\overline{z} + i\overline{m}n - im\overline{n} \tag{170}$$

$$\text{and} \qquad \mu_2 + i\mu_3 = 2ixl - 2yn + 2zm. \tag{171}$$

This quotient has only one 'corner' that can be used to form a connected sum in the sense of the previous sections, since the nonstandard action means that the other two variable pairs cannot be incorporated into their scheme. The corner is associated to $(x, l)$ and is a nonsingular point of the quotient, corresponding to 0 in $\mathbb{R}^4$. The point $z = 0, n = 0$ of the quotient is the singular point where the ALF end is; when $z = n = 0$ the action of $\mathbb{R}$ is no longer free, as $2\pi i\mathbb{Z} \subset \mathbb{R}$ acts trivially.

The quotient can be seen as a hyperkähler quotient, as fixing z=1 and n=0 gives the quotient of $\mathbb{H}^2$ with coordinates $x, y, l, m$ by the action

$$(x, y, l, m) \overset{t}{\longmapsto} (e^{it}x, y + t, e^{-it}l, m), \qquad t \in \mathbb{R}, \tag{172}$$

with moment maps

$$\mu_1 = |x|^2 - |l|^2 - 2\,\mathrm{Im}\,y \tag{173}$$

$$\text{and} \qquad \mu_2 + i\mu_3 = 2ixl + 2m, \tag{174}$$

which is the hyperkähler quotient of $\mathbb{H}^2$ by the given action.

This hyperkähler quotient is isometric to the metric

$$\mathrm{d}s^2 = \frac{r+1}{r}\mathrm{d}r^2 + r(r+1)(\sigma_1^2+\sigma_2^2) + \frac{r}{r+1}\sigma_3^2, \tag{175}$$

which is the simplest example of the ALF metrics in [Ha]. To see this, divide the solution set of the moment maps by $2\pi i\mathbb{Z} \subset \mathbb{R}$ to get a $U(1)$- bundle over $\mathbb{R}^3 \times \mathcal{S}^1$ with coordinates $(y + 2\pi i\mathbb{Z}, m)$. The $U(1)$-bundle comes from a monopole on $\mathbb{R}^3$, derived from a Green's function for the origin. The metric thus comes from 'twisting' $\mathbb{R}^3 \times \mathcal{S}^1$ by this monopole, which is how Hawking constructs his metrics; these ideas were explained in Chapter 3.

### B.8.2. Including ALF ends in more general quotients

A quotient has been given with a single corner associated to $(x, l)$ – call this quotient $Q_1$. To make a connected sum with another quotient $Q_2$ the method of the previous sections may be followed, the only nonstandard part being the definition of the action $\tau$, which should be

$$\tau(u) : (y, z, m, n) \longmapsto (uy, uz, u^{-1}m, u^{-1}n), \qquad u \in U(1). \tag{176}$$

One can also make a generalized connected sum with an orbifold corner of $Q_2$. If the orbifold group is $\Gamma$ then the result is a self-dual metric with an ALF end with group $\Gamma$.

Let $Q_2$ be the hyperkähler quotient for an ALE space associated to a cyclic group $\Gamma$, and the chosen corner in $Q_2$ be the orbifold point. (To write $Q_2$ in our standard quaternionic quotient form involves introducing an extra variable pair as coordinates for the quaternionic lines of the associated bundle, and this variable pair is the 'corner' associated to the orbifold point. Functions of this variable pair replace the constants in the hyperkähler moment map equations, and the quotient group acts trivially upon the extra variables.)

The considerations above show that the resulting connected sum will be self-dual, with the topology of the chosen ALE space but with an ALF end. We claim that it is in fact one of the metrics of [Ha]. Again, this is shown by the method of twisting by monopoles mentioned above, but here we will just show that the quotient is hyperkähler.

This can be seen by displaying the quotient as the quaternionic form of a hyperkähler quotient, but also follows from a more general principle: the quotient will certainly be hypercomplex, as the quotient group of the new quotient $Q_{1\#2}$ acts trivially upon the variable pair $(z, n)$ from $Q_1$, which is the condition for a quaternionic quotient to be hypercomplex. But for general reasons a hypercomplex 4-manifold that is compact or has suitable asymptotic ends will be locally hyperkähler, for the connection induced by $\nabla$ upon the real line bundle of volume forms is self-dual.

A self-dual connection on a trivial bundle that is either over a compact 4-manifold, or else over a non-compact manifold but which is asymptotically trivial at the ends, must be flat by the usual topological argument. So the holonomy locally reduces from $GL(1,\mathbb{H})$ to $SL(1,\mathbb{H})$, i.e. the manifold is locally hyperkähler. This argument is essentially that given by Boyer ([Bo], top of p. 163).

Finally we remark that the twistor function $\mu_1 = |z|^2 - |n|^2$, $\mu_2 + i\mu_3 = 2izn$ transfers to the new quotient $Q_{1\#2}$, and defines a Kähler metric of zero scalar curvature on $\mathbf{Q}_{1\#2}$ that is complete and ALF. So, for instance, taking $Q_2$ to be a quotient for $n\mathbb{CP}^2$, it can be seen that $\mathbb{C}^2$ blown up at $n$ points (in some sufficiently special configuration) admits an ALF zero-scalar-curvature Kähler metric.

# Appendix C: A Sobolev Embedding Theorem for Asymptotically Flat Manifolds

This appendix contains a sketch proof of Theorem 7.3.7, which is a strengthened particular case of the Sobolev embedding theorem for asymptotically flat manifolds.

**Theorem C.** *Suppose that $(N, g_N)$ is a connected, asymptotically flat Riemannian manifold of dimension $n$. Then there is a constant $A$ such that*

$$\|\phi\|_p \leq A\biggl(\int_N |\nabla\phi|^2 dV_{g_N}\biggr)^{\frac{1}{2}} \qquad \textit{for } \phi \in L^2_1(N). \tag{177}$$

*Proof.* By [Au], Theorem 2.28, the theorem holds for $N$ equal to $\mathbb{R}^n$ with its flat metric. Now for any other asymptotically flat metric on $\mathbb{R}^n$ (with the standard coordinates as asymptotic coordinates), there are uniform estimates relating volume forms and length of vectors of this new metric and the standard metric, and so by increasing the constant $A$, the Theorem holds for any asymptotically flat metric on $\mathbb{R}^n$.

Using a gluing argument, we may construct a smooth, asymptotically flat metric $\hat{g}$ on $\mathbb{R}^n$, that outside some ball in $\mathbb{R}^n$ is isometric to the metric on $N$, outside some compact set. We may also make a Riemannian metric $\tilde{g}$ on $\tilde{N}$, the one-point compactification of $N$, that agrees with the metric on $N$ outside some ball about the added point. Let these metrics $\hat{g}, \tilde{g}$ be chosen such that the regions of $N$ on which they agree with $g_N$ have union $N$, and intersection $A_N$, an annulus in $N$.

Choose a smooth partition of unity $(\beta_1, \beta_2)$ for $N$, that is identically $(0, 1)$ and $(1, 0)$ respectively on the two components of $N \setminus A_N$. Now given a function $\phi \in L^2_1(N)$, $\phi = \beta_1\phi + \beta_2\phi$; these two functions may be regarded as functions on $\mathbb{R}^n$ and $\tilde{N}$ in the obvious way. But it has already been shown that the theorem applies to functions on $\mathbb{R}^n$ with respect to $\hat{g}$, and by Theorem 7.2.1 a suitable Sobolev embedding theorem applies on $\tilde{N}$. Using these two inequalities and manipulating, it can be shown that

$$\|\phi\|_p \leq K\biggl(\int_N |\nabla\phi|^2 dV_{g_N}\biggr)^{\frac{1}{2}} + \int_N u|\phi| dV_{g_N} \qquad \text{for } \phi \in L^2_1(N), \tag{178}$$

where $u$ is some fixed nonnegative $C^\infty$ function, of compact support in $N$.

Because $N$ is connected and asymptotically flat, it can be shown using the method of Green's functions that there exists a $C^\infty$ function $f$ with $|\nabla f| = O(|v|^{1-n})$ for large $|v|$, such that $\Delta f = u$ on $N$. Integrating by parts, we find that

$$\int_N u|\phi| dV_{g_N} = \int_N \nabla f \cdot \nabla|\phi| dV_{g_N} \leq \left( \int_N |\nabla f|^2 dV_{g_N} \right)^{\frac{1}{2}} \cdot \left( \int_N |\nabla \phi|^2 dV_{g_N} \right)^{\frac{1}{2}}, \tag{179}$$

by Hölder's inequality, and as $\int_N |\nabla f|^2 dV_{g_N}$ exists, combining this with (178) proves the theorem. $\square$

# Appendix D: The Spectrum of $a\Delta$ on Connected Sum Manifolds

In this appendix we prove some results about the eigenvalues and eigenvectors of the operator $a\Delta$ on the manifold $M$ with the metrics $g_t$ defined in §§8.1, 8.2, 10.1 and 10.2. They were quoted in §9.3 and §10.3, and appear here and not in the main text because the proofs are fairly long, rather unenlightening calculations that probably will not interest most readers of this thesis.

The first result, which takes up §§D.1 and D.2, is about the metrics of §§8.1 and 8.2. The metrics $g_t$ of §8.1 consist of a small asymptotically flat manifold $M''$ glued into a constant scalar curvature manifold $M'$. For these, the first result says that if $a\Delta$ on $M'$ has no eigenvalues within $2\gamma$ of $b$, then $a\Delta$ on $M$ with the metric $g_t$ has no eigenvalues within $\gamma$ of $b$ for small $t$. It also gives a similar statement for the metrics of §8.2.

It is natural to divide the problem up into the eigenvalues smaller than $b$ and the eigenvalues larger than $b$. The eigenvectors with eigenvalues smaller than $b$ form a finite dimensional space $E$, say, and when $t$ is small, $E$ approximates $E'$ in the case of §8.1 and $E' \oplus E''$ in the case of §8.2. So one way to approach the proof would be to construct the space $E$ on $(M, g_t)$ for small $t$, starting from $E'$ or $E' \oplus E''$ as appropriate, and then to show that all eigenvalues for eigenvectors in $E$ are at most $b - \gamma$, and all eigenvalues for eigenvectors in $E^{\perp}$ are at most $b + \gamma$.

Our proof does not do this, as providing an explicit construction of $E$, say using a sequence method, would be a lot of work. Instead we define a space $E_t$ that is quite a good approximation to $E$. The extra idea we need is this. For any nonzero $\phi \in L^2_1(M)$, we may define a sort of 'average eigenvalue' of $a\Delta$ on $\phi$ by $\lambda = \int_M a|\nabla\phi|^2 dV_{g_t} / \bigl(\int_M \phi^2 dV_{g_t}\bigr)$. Then we show that for sufficiently small $t$ and any nonzero $\phi \in E_t$, this 'average eigenvalue' is at most $b - \gamma$, and for any nonzero $\phi \in (E_t)^{\perp}$ it is at least $b + \gamma$. With these two statements we are able to prove, by contradiction, that $a\Delta$ has no eigenvalues in $(b - \gamma, b + \gamma)$.

The first part of the argument, the construction of the space $E_t$ and the proof that the average eigenvalues of $E_t$ are at most $b - \gamma$, is carried out in §D.1, and the second part of the argument, showing that the average eigenvalues of $(E_t)^{\perp}$ are at least $b + \gamma$, is carried out in §D.2.

In the third section §D.3, we prove similar results for use in the zero scalar curvature material of Chapter 10. Most of the work needed to prove them has already been done in §§D.1 and D.2, and the main problem is the construction of an eigenvector $\beta$ of $a\Delta$ with a small eigenvalue $\lambda$. This is done by a sequence method, the basic idea being to start with an approximation to $\beta$ and repeatedly invert $a\Delta$ upon it; as $\lambda$ is the smallest positive eigenvalue, the $\beta$- component of the resulting sequence grows much faster than any other and so comes to dominate.

### D.1. Eigenvalues of $a\Delta$ smaller than $b$

In this section and the next we prove Theorem 9.3.1, which is reproduced here.

**Theorem 9.3.1.** *Let $\{g_t : t \in (0,\delta)\}$ be one of the families of metrics defined on $M = M'\#M''$ in §8.1 or §8.2, and suppose that for some $\gamma > 0$, $a\Delta$ has no eigenvalues in the interval $(b-2\gamma, b+2\gamma)$ on $M'$ in the case of §8.1, and on both $M'$ and $M''$ in the case of §8.2. Then for sufficiently small $t$, $a\Delta$ has no eigenvalues in the interval $(b-\gamma, b+\gamma)$ on $(M, g_t)$.*

*Proof.* It is well known that the spectrum of the Laplacian on a compact Riemannian manifold is discrete and nonnegative, and that the eigenspaces are finite-dimensional. Therefore on $M'$ and $M''$ there are only finitely many eigenvalues of $a\Delta$ smaller than $b$, and to each is associated a finite-dimensional space of eigenfunctions.

Let $E'$ be the finite-dimensional vector space of smooth functions on $M'$ generated by eigenfunctions of $a\Delta$ on $M'$ associated to eigenvalues less than $b$; we think of $E'$ as a subspace of $L^2_1(M')$. For the case of §8.2, define $E''$ on $M''$ in the same way. From the statement of the theorem, we assume that $a\Delta$ has no eigenvalues in the interval $(b-2\gamma, b+2\gamma)$. This implies the following two statements:

$$\text{if } \phi \in E', \text{ then } \int_{M'} a\big|\nabla\phi\big|^2 dV_{g'} \leq \int_{M'} (b-2\gamma)\phi^2 dV_{g'}, \tag{180}$$

$$\text{if } \phi \in (E')^\perp \subset L^2_1(M'), \text{ then } \int_{M'} a\big|\nabla\phi\big|^2 dV_{g'} \geq \int_{M'} (b+2\gamma)\phi^2 dV_{g'}, \tag{181}$$

and also two analogous inequalities for $M''$ in the case of §8.2. The perpendicular subspace $(E')^\perp$ of (181) may be taken with respect to the inner product of $L^2_1(M')$ or with respect to that of $L^2(M')$ — both give the same space, as $E'$ is a sum of eigenspaces of $a\Delta$.

Now if we have two statements like (180) and (181) but applying to $M$ rather than $M'$, then we can prove the result. This is the content of the next lemma.

**Lemma D.1.1.** *Suppose that for small enough $t$ there is a subspace $E_t$ of $L^2_1(M)$ satisfying the following two conditions:*

$$\textit{if } \phi \in E_t, \textit{ then } \int_M a\big|\nabla\phi\big|^2 dV_{g_t} \le \int_M (b-\gamma)\phi^2 dV_{g_t}, \tag{182}$$

$$\textit{if } \phi \in (E_t)^\perp \subset L^2_1(M), \textit{ then } \int_M a\big|\nabla\phi\big|^2 dV_{g_t} \ge \int_M (b+\gamma)\phi^2 dV_{g_t}, \tag{183}$$

*where the inner product used to construct $(E_t)^\perp$ is that of $L^2_1(M)$. Then Theorem 9.3.1 holds.*

*Proof.* Let $t$ be sufficiently small that such a space $E_t$ exists. We must show that $l \in (b-\gamma, b+\gamma)$ cannot be an eigenvalue of $a\Delta$ on $(M, g_t)$. Suppose for a contradiction that $\phi$ is an eigenfunction of $a\Delta$ for this eigenvalue $l$. Let $\phi_1$ and $\phi_2$ be the components of $\phi$ in $E_t$ and $(E_t)^\perp$ respectively. Then, as $a\Delta\phi - l\phi = 0$,

$$\begin{aligned} 0 &= \int_M (\phi_2 - \phi_1)\big(a\Delta(\phi_1+\phi_2) - l(\phi_1+\phi_2)\big)dV_{g_t} \\ &= \int_M \big(a|\nabla\phi_2|^2 - l\phi_2^2\big)dV_{g_t} - \int_M \big(a|\nabla\phi_1|^2 - l\phi_1^2\big)dV_{g_t} \\ &\ge \int_M \big((\gamma+l-b)\phi_1^2 + (\gamma+b-l)\phi_2^2\big)dV_{g_t}, \end{aligned}$$

using (182) and (183) in the last line. But as $\gamma+l-b, \gamma+b-l > 0$, this shows that $\phi_1 = \phi_2 = \phi = 0$, which is a contradiction. □

To complete the proof of the theorem, we therefore need to produce some spaces $E_t$ of functions on $M$ satisfying (182) and (183). The space $E_t$ approximates the vector space of eigenfunctions of $a\Delta$ associated to eigenvalues smaller than $b$. Let $\eta$ be such an eigenfunction. In the case of §8.1, we intuitively recognize two possibilities: firstly, that $\eta$ is of order 1 on the parts of $M$ coming from both $M'$ and $M''$, so that $\eta$ approximates an eigenfunction of $a\Delta$ on $M'$, and secondly, that $\eta$ is small on the part coming from $M'$ and large on that from $M''$.

The second possibility we may exclude, for as $\eta$ is large on only a very small volume, $\|\nabla\eta\|_2$ is large compared to $\|\eta\|_2$, and thus $\eta$ must be associated to a large eigenvalue of $a\Delta$, and not one smaller than $b$. So in the case §8.1, it is natural to model the space $E_t$ upon $E'$. A similar heuristic argument shows that for the case of §8.2, it is natural to model $E_t$ upon $E' \oplus E''$.

As a first step towards constructing the spaces $E_t$, a space $\tilde{E}'$ of functions on $M'$ will be made that is close to $E'$, but the functions of which vanish on a small ball around $m'$; in the case of §8.2, a similar space $\tilde{E}''$ will also be made. Let $\sigma'$ be a $C^\infty$ function on $M'$ that is identically 1 on the complement of a small ball about $m'$, identically 0 on a smaller ball about $m'$, and otherwise taking values in $[0,1]$. Now define $\tilde{E}'$ by

$$\tilde{E}' = \sigma' E' = \{\sigma' v : v \in E'\}. \tag{184}$$

By choosing the ball outside of which $\sigma'$ is identically 1 to be sufficiently small, and making sure that $\sigma'$ is reasonably well behaved on that ball, we can ensure that $\tilde{E}'$ is close to $E'$ in $L^2_1(M')$ in the following sense: the two have the same dimension, and any $\tilde{v} \in \tilde{E}'$ may be written as $\tilde{v} = v_1 + v_2$, where $v_1, v_2 \in E', (E')^\perp$ respectively, and satisfy

$$\|v_2\|^2_{2,1} \le \frac{\gamma}{2(a+b+2\gamma)} \|v_1\|^2_{2,1}. \tag{185}$$

Suppose that $\sigma'$ has been chosen so that these hold. Then two statements similar to (180) and (181) hold for $\tilde{E}'$, as we shall see in the next lemma.

**Lemma D.1.2.** *The subspace $\tilde{E}'$ satisfies the following two conditions:*

$$\text{if } \phi \in \tilde{E}', \text{ then } \int_{M'} a\left|\nabla\phi\right|^2 dV_{g'} \le \int_{M'} (b - \tfrac{3}{2}\gamma)\phi^2 dV_{g'}, \tag{186}$$

$$\text{if } \phi \in (\tilde{E}')^\perp \subset L^2_1(M'), \text{ then } \int_{M'} a\left|\nabla\phi\right|^2 dV_{g'} \ge \int_{M'} (b + \tfrac{3}{2}\gamma)\phi^2 dV_{g'}, \tag{187}$$

*where the inner product used to construct $(\tilde{E}')^\perp$ is that of $L^2_1(M')$.*

*Proof.* First we prove (186). Let $\phi \in \tilde{E}'$; then $\phi = v_1 + v_2$, with $v_1 \in E'$ and $v_2 \in (E')^\perp$. Because $v_1$ and $v_2$ are orthogonal in both $L^2$ and $L^2_1$,

$$\begin{aligned}
a\int_{M'} \left|\nabla\phi\right|^2 dV_{g'} &= a\int_{M'} \left|\nabla v_1\right|^2 dV_{g'} + a\int_{M'} \left|\nabla v_2\right|^2 dV_{g'} \\
&\le a\int_{M'} \left|\nabla v_1\right|^2 dV_{g'} + a\|v_2\|^2_{2,1} \\
&\le a\int_{M'} \left|\nabla v_1\right|^2 dV_{g'} + \frac{a\gamma}{2(a+b+2\gamma)}\|v_1\|^2_{2,1} \\
&= a\left(1 + \frac{\gamma}{2(a+b+2\gamma)}\right)\int_{M'} \left|\nabla v_1\right|^2 dV_{g'} + \frac{a\gamma}{2(a+b+2\gamma)}\int_{M'} v_1^2 dV_{g'} \\
&\le \left((b-2\gamma)\left(1 + \frac{\gamma}{2(a+b+2\gamma)}\right) + \frac{a\gamma}{2(a+b+2\gamma)}\right)\int_{M'} v_1^2 dV_{g'} \\
&\le (b - \tfrac{3}{2}\gamma)\int_{M'} \phi^2 dV_{g'}.
\end{aligned} \tag{188}$$

Here between the second and third lines we have used (185), between the fourth and fifth lines we have used (180), and between the last two we have used the $L^2$- orthogonality of $v_1$ and $v_2$ and the trivial inequality $(b-2\gamma)[1+\gamma/2(a+b+2\gamma)] + a\gamma/2(a+b+2\gamma) \le b - 3\gamma/2$. This proves (186).

To prove (187), observe that by (185), orthogonal projection from $\tilde{E}'$ to $E'$ is injective, and as they have the same (finite) dimension, it must also be surjective. Let $\tilde{v}_2 \in (\tilde{E}')^\perp$. Then $\tilde{v}_2 = v_1 + v_2$ with $v_1 \in E'$ and $v_2 \in (E')^\perp$. By this surjectivity, there exists $\tilde{v}_1 \in \tilde{E}'$ such that $\tilde{v}_1 = v_1 + v_3$ with $v_3 \in (E')^\perp$, that is, the $E'$- component of $\tilde{v}_1$ is $v_1$, the same as that of $\tilde{v}_2$. But $\tilde{v}_1$ and $\tilde{v}_2$ are orthogonal in $L^2_1(M')$, so taking their inner product gives that $\|v_1\|^2_{2,1} = -v_2 \cdot v_3 \le \|v_3\|_{2,1}\|v_2\|_{2,1}$.

As $\tilde{v}_1 = v_1 + v_3 \in \tilde{E}'$, we may square this inequality, substitute in for $\|v_3\|^2_{2,1}$ using (185), and divide through by $\|v_1\|^2_{2,1}$. The result is that

$$\|v_1\|^2_{2,1} \le \frac{\gamma}{2(a+b+2\gamma)}\|v_2\|^2_{2,1}, \qquad \text{for } \tilde{v}_2 = v_1 + v_2 \in (\tilde{E}')^\perp, \tag{189}$$

which is the analogue of (185) for $(\tilde{E}')^\perp$ instead of $\tilde{E}'$. This is the ingredient needed to prove (187) by the method used above for (186), and the remainder of the proof will be left to the reader. $\square$

For the case of §8.2, a subspace $\tilde{E}''$ of functions on $M''$ is created in exactly the same way, and Lemma D.1.2 clearly applies to this space too. The point of defining the modified spaces $\tilde{E}', \tilde{E}''$ is that, since they vanish on neighbourhoods of $m', m''$, the functions may easily be transferred to functions on $(M, g_t)$ for sufficiently small $t$. This is because subsets of $(M, g_t)$ are identified as Riemannian manifolds with the complements of balls about $m', m''$ in $M', M''$; the functions are then naturally extended by zero outside the subsets on which they are defined. The transferred spaces will give the function spaces $E_t$ needed to apply Lemma D.1.1.

Define the space of functions $E_t$ on $M$ as follows. For the case of §8.1, let $E_t$ be the space of functions that are equal to some function in $\tilde{E}'$ on the subset of $M$ naturally identified with $M' \setminus \Phi'[\overline{B}_t(0)]$, and are zero outside this subset. For the case of §8.2, let $E_t$ be the direct sum of this space of functions, and the space of functions made in the same way, but with the rôles of $M'$ and $M''$ reversed.

For sufficiently small $t$, the functions of $\tilde{E}'$ vanish upon $\Phi'[\overline{B}_t(0)]$ (and similarly for $\tilde{E}''$), and so the functions in $E_t$ are $C^\infty$. Also, in this case the metric $g_t$ actually agrees with that of $M'$ ($M''$) on the support of the functions in $E_t$. Thus (186) applies to $E_t$.

**Lemma D.1.3.** *For small enough $t$, the subspace $E_t$ of $L^2_1(M)$ satisfies*

$$\textit{if } \phi \in E_t, \textit{ then } \int_M a\left|\nabla\phi\right|^2 dV_{g_t} \le \int_M (b - \tfrac{3}{2}\gamma)\phi^2 dV_{g_t}. \tag{190}$$

*A fortiori, it satisfies the inequality (182) of Lemma D.1.1.*

*Proof.* In the case of §8.1, this follows immediately from (186), because of the preceding remark that the metrics of $M$ and $M'$ agree upon the support of the functions of $E_t$. In the case of §8.2, the function $\phi$ is the sum of a function from $\tilde{E}'$ and a function from $\tilde{E}''$; both sides of (190) split into two terms, each involving one function. So (190) is the sum of two inequalities, which follow immediately from (186) as before, and from the counterpart of (186) applying to $\tilde{E}''$. □

The previous lemma showed that the space of functions $E_t$ upon $M$ satisfies inequality (182) of Lemma D.1.1. In the next proposition we show that the inequality (183) is satisfied too.

**Proposition D.1.4.** *For sufficiently small $t$, inequality (183) of Lemma D.1.1 holds.*

*Proof: see §D.2.*

The proof of this proposition is somewhat messy, and is the subject of the next section. Suppose for the moment that the proposition holds. Then a space of functions $E_t$ upon $M$ has been constructed, satisfying inequality (182) by Lemma D.1.3, and inequality (183) by Proposition D.1.4. So by Lemma D.1.1, the proof of Theorem 9.3.1 is finished. □

## D.2. Eigenvalues of $a\Delta$ larger than $b$

This section is devoted to the proof of the last proposition of the previous section, which will finish the proof of Theorem 9.3.1. The idea of the proof is as follows. Given a function $\phi$ in $(E_t)^\perp$, we want to show that its 'average eigenvalue' of $a\Delta$ is at least $b+\gamma$. We may restrict $\phi$ to $M'$ or both $M'$ and $M''$ and use facts about the average eigenvalues of $a\Delta$ on these manifolds, but the process of restriction does not use the values of $\phi$ upon the small copy of $M''$ or the 'neck', and thus this method only works if the proportion of $\int \phi^2 dV_{g_t}$ in these regions is small enough.

So we are left with the case of $\phi \in (E_t)^\perp$ with $\phi^2$ concentrated in a small region. When $t$ is small, this implies that $\phi$ is large on this region compared with its values elsewhere, and therefore $\phi$ must change a lot in a neighbourhood of the small region. This enables us to give a lower bound for $\int_M a\left|\nabla\phi\right|^2 dV_{g_t}$ in terms of $\int_M \phi^2 dV_{g_t}$, and hence a lower bound for the 'average eigenvalue'.

**Proposition D.1.4.** *Let $M$, $g_t$ and $E_t$ be as in §D.1. Then for sufficiently small $t$, if $\phi \in (E_t)^\perp \subset L^2_1(M)$, then*

$$\int_M a\left|\nabla\phi\right|^2 dV_{g_t} \geq \int_M (b+\gamma)\phi^2 dV_{g_t}. \tag{191}$$

*Proof.* For simplicity we shall prove the proposition for the case of the metrics of §8.1 only, and the fairly easy modifications to get the case of §8.2 will be left to the reader. We will start from (187) of Lemma D.1.2. The constants in (191) and (187) are different — the first has $b+\gamma$, the second $b+3\gamma/2$. This difference will be needed in the course of the proof to absorb several error terms into. In order to avoid lots of numerical constants, let $b_0 = b+3\gamma/2$, let $b_5 = b+\gamma$, and choose constants $b_1, b_2, b_3, b_4$ such that $b_0 > b_1 > b_2 > b_3 > b_4 > b_5$. These will be used to contain four separate error terms.

Shortly we shall define three constants $r_1, r_2, r_3$ such that $r_1 < r_2 < r_3$. They are be independent of $t$, and define three compact Riemannian submanifolds $R_t \subset S_t \subset T_t$ of $(M, g_t)$, with boundaries, which will be the subsets of $M$ coming from subsets $R, S$ and $T$ of $M''$ respectively, where

$$R = M'' \setminus \Phi''\big[\mathbb{R}^n \setminus \overline{B}_{r_1}(0)\big], \quad S = M'' \setminus \Phi''\big[\mathbb{R}^n \setminus \overline{B}_{r_2}(0)\big] \quad \text{and} \quad T = M'' \setminus \Phi''\big[\mathbb{R}^n \setminus \overline{B}_{r_3}(0)\big]. \tag{192}$$

When $t$ is sufficiently small, $R_t, S_t$ and $T_t$ lie in the region of $M$ in which the function $\beta_2$, used in §8.1 to define $g_t$, is equal to 1. Then $R_t, S_t, T_t$ are homothetic to $R, S, T$ respectively, by a homothety multiplying their metrics by $t^{12}$.

The idea is this. A diffeomorphism $\Psi'_t$ from $M' \setminus \{m'\}$ onto $M \setminus R_t$ will be constructed, which will be the identity (with the natural identifications) outside the set $T_t$. Using the diffeomorphism, any function in $L^2_1(M)$ will define a function in $L^2_1(M')$. Applying (187) of Lemma D.1.2 therefore induces an inequality upon functions in $L^2_1(M)$. We will be able to show that for functions that are not in some sense too large in $S_t$, this inequality implies (191) as we require. Then only the case of functions that are large in $S_t$ will remain.

Suppose, for the moment, that $r_1, r_2, r_3$ are fixed with $r_1 < r_2 < r_3$. For the set $R_t$ to be well defined, $r_1$ must satisfy $r_1 > \delta^{-4}$. For $T_t$ to be well defined, $t$ must be sufficiently small that $t^6 r_3 < \delta$; let us also suppose that $t$ is small enough that the functions of $E_t$ vanish on $T_t$.

Let the map $\Psi'_t : M' \backslash \{m'\} \to M$ be the identity (under the natural identification) outside $\Phi'\big[B_{t^6 r_3}(0)\big]$ in $M'$, and on $\Phi'\big[B_{t^6 r_3}(0)\big]$ in $M'$ define $\Psi'_t$ by

$$\Psi'_t\left(\Phi'(v)\right) = \Phi'\left(\frac{t^6 r_1 v}{|v|} + \frac{(r_3 - r_1)v}{r_3}\right). \tag{193}$$

Let $\phi \in L^2_1(M)$, and define $\phi'$ by $\phi'(x) = \phi(\Psi'_t(x))$. Then, as we shall see, $\phi' \in L^2_1(M')$.

An easy calculation shows that

$$b_0 \int_{M'} (\phi')^2 dV_{g'} = b_0 \int_M \phi^2 \cdot F_t dV_{g_t}, \tag{194}$$

where $F_t$ is a function on $M$ that is 1 on that part of $M$ coming from $M' \setminus \Phi'\big[B_{t^6 r_3}(0)\big]$, is 0 on that part of $M$ not coming from $M' \setminus \Phi'\big[B_{t^6 r_1}(0)\big]$, and in the intermediate annulus is given by

$$F_t\left((\phi')^{-1}(v)\right) = \frac{(|v| - t^6 r_1)^{n-1} r_3^n}{|v|^{n-1}(r_3 - r_1)^n} \cdot \psi'(v)^p \psi_t(v)^{-p}. \tag{195}$$

Similarly, we may easily show that

$$a \int_{M'} \big|\nabla\phi'\big|^2 dV_{g'} \le a \int_M \big|\nabla\phi\big|^2 \cdot G_t dV_{g_t}, \tag{196}$$

where $G_t$ is a function on $M$ that is 1 on that part of $M$ coming from $M' \setminus \Phi'\big[B_{t^6 r_3}(0)\big]$, is 0 on that part of $M$ not coming from $M' \setminus \Phi'\big[B_{t^6 r_1}(0)\big]$, and in the intermediate annulus is given by

$$G_t\left((\phi')^{-1}(v)\right) = \max\left(\frac{(|v| - t^6 r_1)^{n-1} r_3^{n+2}}{|v|^{n-1}(r_3 - r_1)^{n+2}}, \frac{(|v| - t^6 r_1)^{n-3} r_3^{n-2}}{|v|^{n-3}(r_3 - r_1)^{n-2}}\right) \cdot \psi'(v)\psi_t(v)^{-1}. \tag{197}$$

Here, the first term in the max(...) is the multiplier for the radial component of $\nabla\phi$, and the second term is the multiplier for the nonradial components. As $F_t, G_t$ are bounded, we see from (194) and (196) that $\phi' \in L^2_1(M')$, as was stated above.

Suppose now that $\phi \in (E_t)^\perp \subset L^2_1(M)$. For small enough $t$ this implies that $\phi' \in (\tilde{E}')^\perp$, and so (187) applies by Lemma D.1.2. Combining this with (194) and (196) gives that

$$a \int_M \big|\nabla\phi\big|^2 \cdot G_t dV_{g_t} \ge b_0 \int_M \phi^2 \cdot F_t dV_{g_t}. \tag{198}$$

Now by the definition of $\psi_t$, $\psi'(v)\psi_t(v)^{-1}$ approaches 1 as $t \to 0$. In fact it may be shown that

$$\big|\psi'(v)\psi_t(v)^{-1} - 1\big| \le C_0 t^{6(n-2)} |v|^{-(n-2)} \qquad \text{when } t^6 \le |v| \le t^{6-2/(n-2)},$$

for some constant $C_0$. For $t$ small enough this certainly holds in the region $t^6 r_1 \le |v| \le t^6 r_3$, and in this region we have $\big|\psi'(v)\psi_t(v)^{-1} - 1\big| \le C_0 r_1^{-(n-2)}$.

Choose $r_1$ greater than 1 or $\delta^{-4}$ as appropriate, and sufficiently large that $b_1(1 + C_0 r_1^{-(n-2)})^p \le b_0$ and $b_2 \le b_1(1 - C_0 r_1^{-(n-2)})$. Then for small $t$, the $\psi'\psi_t^{-1}$ terms in $F_t$ and $G_t$ can be absorbed by putting $b_2$ in place of $b_0$. Next, $r_2$ is defined in terms of $r_3$ to be the unique constant satisfying $r_1 < r_2 < r_3$ and

$b_3(r_2-r_1)^{n-1}r_3^n r_2^{1-n}(r_3-r_1)^{-n}=b_4$. Then $b_3(|v|-t^6r_1)^{n-1}r_3^n|v|^{1-n}(r_3-r_1)^{-n}\geq b_4$ when $t^6r_2\leq|v|\leq t^6r_3$. This is to bound the function $F_t$ below on the region $|v|\geq t^6r_2$.

Finally, we define $r_3$. Choose $r_3>r_1$ sufficiently large that two conditions hold. The first is that $b_3\cdot\max\bigl((|v|-t^6r_1)^{n-1}r_3^{n+2}|v|^{1-n}(r_3-r_1)^{-(n+2)},(|v|-t^6r_1)^{n-3}r_3^{n-2}|v|^{3-n}(r_3-r_1)^{2-n}\bigr)\leq b_2$ when $t^6r_1\leq|v|\leq t^6r_3$; combining this with one of the inequalities used to define $r_1$ shows that $b_3G_t\leq b_1$. The second condition is that

$$\frac{\operatorname{vol}(S)}{\operatorname{vol}(T)-\operatorname{vol}(S)}\leq\frac{b_4-b_5}{4b_5}. \tag{199}$$

This condition will be needed later.

The last two definitions are somewhat circular, as $r_2$ is defined as a function of $r_3$, and it also enters into the second condition defining $r_3$ because $S_t$ depends on $r_2$. However, manipulating the definition of $r_2$ reveals that however large $r_3$ is, $r_2$ must satisfy $r_2\leq r_1(1-b_3^{1/(n-1)}b_2^{-1/(n-1)})^{-1}$, and so $\operatorname{vol}(S)$ is bounded in terms of $r_1$, whereas $\operatorname{vol}(T)$ can grow arbitrarily large. Therefore the second condition does hold for $r_3$ sufficiently large.

The above estimates show that $G_t\leq b_1/b_3$ and $F_t\geq b_1b_4/b_0b_3$ on $M\setminus S_t$ for small $t$. Substituting these into (198) gives that when $t$ is sufficiently small,

$$a\int_M\bigl|\nabla\phi\bigr|^2dV_{g_t}\geq b_4\int_{M\setminus S_t}\phi^2dV_{g_t}. \tag{200}$$

Suppose that $\int_{S_t}\phi^2dV_{g_t}\leq(b_4/b_5-1)\cdot\int_{M\setminus S_t}\phi^2dV_{g_t}$. Then $b_4\int_{M\setminus S_t}\phi^2dV_{g_t}\geq b_5\int_M\phi^2dV_{g_t}$, and from (200) we see that (191) holds for $\phi$, which is what we have to prove. Therefore it remains only to deal with the case that $\int_{S_t}\phi^2dV_{g_t}>(b_4/b_5-1)\cdot\int_{M\setminus S_t}\phi^2dV_{g_t}$.

Suppose that this inequality holds. The basic idea of the remainder of the proof is that when $t$ is small, the volume of $S_t$ is also small, and this forces $\phi$ to be large on $S_t$ compared to its average value elsewhere. Therefore $\phi$ must change substantially in the neighbourhood $T_t$ of $S_t$, and this forces $\nabla\phi$ to be large in $T_t$.

Restrict $t$ further, to be small enough that $t^6r_3\leq t^2$. Then $T_t$ is contained in the region of gluing in which $\beta_2=1$. So the pair $(S_t,T_t)$ is homothetic to a pair $(S,T)$ of compact manifolds with $C^\infty$ boundaries and with $S$ contained in the interior of $T$; the metrics on $(S_t,T_t)$ are the metrics on $(S,T)$ multiplied by $t^{12}$. For these $S,T$ the following lemma holds.

**Lemma D.2.1.** *Let $S, T$ be compact, connected Riemannian manifolds of dimension $n$ with smooth boundaries, such that $S \subset T$ but $S \neq T$. Then there exists a constant $C_1$ such that for all $\phi \in L^2_1(T)$,*

$$\left(\frac{\int_S \phi^2 dV_g}{\mathrm{vol}(S)}\right)^{\frac{1}{2}} - \left(\frac{\int_{T\setminus S} \phi^2 dV_g}{\mathrm{vol}(T) - \mathrm{vol}(S)}\right)^{\frac{1}{2}} \leq C_1 \left(\int_T |\nabla\phi|^2 dV_g\right)^{\frac{1}{2}}. \tag{201}$$

*Proof.* We begin by quoting a theorem on the existence of solutions of the equation $\Delta u = f$ on a manifold with smooth boundary.

**Theorem [Hö].** *Suppose that $T$ is a compact manifold with smooth boundary, and that $f \in L^2(T)$ and satisfies $\int_T f dV_g = 0$. Then there exists $\xi \in L^2_2(T)$, unique up to the addition of a constant, such that $\Delta\xi = f$, and in addition $\mathbf{n} \cdot \nabla\xi$ vanishes at the boundary, where $\mathbf{n}$ is the unit outward normal to the boundary.*

*Proof.* This is a partial statement of Example 2 on p. 265 of [Hö]. Hörmander's example is only stated for $C^\infty$ functions $f$ and $\xi$, but the proof works for $f \in \mathcal{H}_{(0)}(T)$ and $\xi \in \mathcal{H}_{(2)}(T)$ in his notation, which are $L^2(T)$ and $L^2_2(T)$ in ours. □

Now put $f = \mathrm{vol}(S)^{-1}$ in $S$ and $f = (\mathrm{vol}(S) - \mathrm{vol}(T))^{-1}$ in $T \setminus S$. Then $\int_T f dV_g = 0$, so by the theorem, there exists a function $\xi \in L^2_2(M)$ satisfying $\Delta\xi = f$, and that $\nabla\xi$ vanishes normal to the boundary. Because of this vanishing, the boundary term has dropped out of the following integration by parts equation:

$$\int_T \phi\Delta\xi = -\int_T (\nabla\phi) \cdot (\nabla\xi) dV_g. \tag{202}$$

Substituting in for $\Delta\xi$ and using Hölder's inequality gives

$$\frac{1}{\mathrm{vol}(S)} \left|\int_S \phi dV_g\right| - \frac{1}{\mathrm{vol}(T) - \mathrm{vol}(S)} \left|\int_{T\setminus S} \phi dV_g\right| \leq \left(\int_T |\nabla\xi|^2 dV_g\right)^{\frac{1}{2}} \cdot \left(\int_T |\nabla\phi|^2 dV_g\right)^{\frac{1}{2}}. \tag{203}$$

Now $S$ is connected, so the constants are the only eigenvectors of $\Delta$ on $S$ with eigenvalue 0 and derivative vanishing normal to the boundary. By the discreteness of the spectrum of $\Delta$ on $S$ with these boundary conditions, there is a positive constant $K_S$ less than or equal to all the positive eigenvalues. It easily follows that for $\phi \in L^2_1(S)$,

$$\left(\frac{\int_S \phi^2 dV_g}{\mathrm{vol}(S)}\right)^{\frac{1}{2}} \leq \frac{1}{\mathrm{vol}(S)} \left|\int_S \phi dV_g\right| + \left(\frac{\int_S |\nabla\phi|^2 dV_g}{K_S \cdot \mathrm{vol}(S)}\right)^{\frac{1}{2}}. \tag{204}$$

Also, a simple application of Hölder's inequality yields

$$\frac{1}{\mathrm{vol}(T)-\mathrm{vol}(S)}\left|\int_{T\setminus S}\phi dV_g\right| \leq \left(\frac{\int_{T\setminus S}\phi^2 dV_g}{\mathrm{vol}(T)-\mathrm{vol}(S)}\right)^{\frac{1}{2}}. \tag{205}$$

Adding together (203), (204) and (205) gives (201), as we want, with constant $C_1 = \left(\int_T |\nabla\xi|^2 dV_g\right)^{1/2} + (K_S \cdot \mathrm{vol}(S))^{-1/2}$. $\square$

The point of this calculation is that because $(S_t, T_t)$ are homothetic to $(S,T)$ by the constant factor $t^{12}$, Lemma D.2.1 implies that for all $\phi \in L^2_1(T_t)$,

$$\left(\frac{\int_{S_t}\phi^2 dV_{g_t}}{\mathrm{vol}(S)}\right)^{\frac{1}{2}} - \left(\frac{\int_{T_t\setminus S_t}\phi^2 dV_{g_t}}{\mathrm{vol}(T)-\mathrm{vol}(S)}\right)^{\frac{1}{2}} \leq C_1 t^6 \left(\int_{T_t}\left|\nabla\phi\right|^2 dV_{g_t}\right)^{\frac{1}{2}}. \tag{206}$$

Now, using the mysterious condition (199) that appeared in the definition of $r_3$, it follows that

$$\left(\int_{S_t}\phi^2 dV_{g_t}\right)^{\frac{1}{2}} - \left(\frac{b_4-b_5}{4b_5}\int_{M\setminus S_t}\phi^2 dV_{g_t}\right)^{\frac{1}{2}} \leq \mathrm{vol}(S)^{\frac{1}{2}} C_1 t^6 \left(\int_M \left|\nabla\phi\right|^2 dV_{g_t}\right)^{\frac{1}{2}}. \tag{207}$$

But, because we are dealing only with the case that $\int_{S_t}\phi^2 dV_{g_t} > (b_4/b_5 - 1)\cdot\int_{M\setminus S_t}\phi^2 dV_{g_t}$, substituting this into (207), squaring and manipulating gives that

$$b_5\int_M \phi^2 dV_{g_t} < \frac{b_4 b_5}{b_4-b_5}\cdot\int_{S_t}\phi^2 dV_{g_t} < \frac{4b_4 b_5 t^{12} C_1^2\,\mathrm{vol}(S)}{b_4-b_5}\cdot\int_M\left|\nabla\phi\right|^2 dV_{g_t}. \tag{208}$$

Therefore, if $t$ is sufficiently small, then inequality (191) holds. This completes the proof of the proposition. $\square$

## D.3. The spectrum of $a\Delta$ in the zero scalar curvature case

Now we prove Proposition 10.3.3 and Theorems 10.3.2 and 10.3.4, whose proofs were deferred until this appendix. We shall start with a preliminary version of Theorems 10.3.2 and 10.3.4. Suppose as in these two theorems that we are given $\gamma > 0$ such that every positive eigenvalue of $a\Delta$ on $M'$ is greater or equal to $2\gamma$, and that in the case of §10.2 the same holds on $M''$. Then

$$\text{if } \phi \in L^2_1(M') \text{ and } \textstyle\int_{M'}\phi dV_{g'} = 0, \text{ then } \displaystyle\int_{M'} a\left|\nabla\phi\right|^2 dV_{g'} \geq \int_{M'} 2\gamma\phi^2 dV_{g'}, \tag{209}$$

and the same for $M''$ in the case of §10.2. This is an analogue of (181) of §D.1, and the analogues of $E'$ and $E''$ are the spaces of constant functions on $M'$ and $M''$.

Define spaces of functions $E_t$ on $M$ as in §D.1. This involves choosing a function $\sigma'$ on $M'$ (done just before Lemma D.1.2) that is equal to 1 outside a small ball around $m'$, and a similar function $\sigma''$ on $M''$. For the proof of Proposition 10.3.3 later, we must choose $\sigma', \sigma''$ to vary with $t$ rather than being fixed as they were before. Let $\sigma', \sigma''$ be functions on $M', M''$ defined in the same way as the functions $\beta_1, \beta_2$ of §10.1, but that are equal to 0 when $|v| \leq t^{2k/(n+2)}$ and equal to 1 when $|v| \geq t^{(n-2)/(n+1)}$ and outside $B', B''$.

This definition makes sense for small enough $t$, because the definition of $k$ in §10.1 implies that $2k/(n+2) > (n-2)/(n+1)$ so that $t^{2k/(n+2)} < t^{(n-2)/(n+1)} < \delta$ for small $t$. Note also that the choice of the inner radius $t^{2k/(n+2)}$ means (from §10.1) that $g_t$ is identified with $g'$ and $g''$ on the support of $\sigma'$ and $\sigma''$ respectively, so that the functions $E_t$ are supported in the parts of $M$ with metric equal to $g'$ or $g''$. Using the spaces $E_t$, we may state the following first approximation to Theorems 10.3.2 and 10.3.4:

**Lemma D.3.1.** *Let $\{g_t : t \in (0, \delta)\}$ be one of the families of metrics defined in §§10.1 or 10.2. Then for all sufficiently small $t$,*

$$\text{if } \phi \in (E_t)^{\perp} \subset L^2_1(M), \text{ then } \int_M a\big|\nabla\phi\big|^2 dV_{g_t} \geq \gamma \int_M \phi^2 dV_{g_t}. \tag{210}$$

*Here the orthogonal space is taken with respect to the $L^2$ inner product.*

*Proof.* This is proved just as is Proposition D.1.4, except that we use the inner product in $L^2(M)$ rather than that in $L^2_1(M)$, and instead of choosing a series of constants interpolating between $b + 2\gamma$ and $b + \gamma$, we choose a series of constants interpolating between $2\gamma$ and $\gamma$, and some simple changes must be made to the proof because the powers of $t$ used in defining the metrics of §§10.1 and 10.2 are different to those used in §§8.1 and 8.2. □

Now as $E_t$ is modelled on $E'$ for the case of §10.1 and on $E' \oplus E''$ for the case of §10.2, it is close to the constant functions in the first case, and to functions taking one constant value on the $M'$ part of $M$ and another constant value on the $M''$ part in the second case.

These spaces $E_t$ are not quite good enough for the purposes of this section, for we shall need spaces that contain the constants. We therefore produce modified spaces $\tilde{E}_t$, and prove a similar lemma for them. In the case of §10.1, let $\tilde{E}_t$ be the constant functions. For the case of §10.2, let $e \in E_t$ be the unique element that is nonnegative on the part of $M$ coming from $M'$ and satisfies $\int_M e\, dV_{g_t} = 0$ and $\int_M e^2 dV_{g_t} = 2\,\mathrm{vol}(M')$, and let $\tilde{E}_t$ be the two-dimensional vector space generated by $e$ and the constants.

**Lemma D.3.2.** *Let $\{g_t : t \in (0,\delta)\}$ be one of the families of metrics defined in §§10.1 or 10.2. Then for all sufficiently small $t$,*

$$\text{if } \phi \in (\tilde{E}_t)^\perp \subset L^2_1(M), \text{ then } \int_M a|\nabla\phi|^2 dV_{g_t} \geq \gamma \int_M \phi^2 dV_{g_t}. \tag{211}$$

*Here the orthogonal space is taken with respect to the $L^2$ inner product.*

*Proof.* Let $\xi$ be the unique element of $E_t$ satisfying $\int_M \xi dV_{g_t} = 1$, and $\int_M e\xi dV_{g_t} = 0$ in the case of §10.2. If $\phi \in (\tilde{E}_t)^\perp$, then $\phi - \langle\phi,\xi\rangle \in E_t^\perp$, where $\langle\, .\, ,\, .\rangle$ is the inner product of $L^2(M)$. So by Lemma D.3.1,

$$\begin{aligned}\int_M a|\nabla\phi|^2 dV_{g_t} &\geq \gamma \int_M \big(\phi - \langle\phi,\xi\rangle\big)^2 dV_{g_t}\\ &= \gamma \int_M \big(\phi^2 + \langle\phi,\xi\rangle^2\big) dV_{g_t}\\ &\geq \gamma \int_M \phi^2 dV_{g_t}.\end{aligned}$$

Here between the first and second lines we have used the fact that $\phi \in (\tilde{E}_t)^\perp$, and thus $\int_M \phi dV_{g_t} = 0$, as the constants lie in $\tilde{E}_t$. □

As $\tilde{E}_t$ is just the constant functions in the case of §8.1, we immediately deduce

**Corollary D.3.3.** *Theorem 10.3.2 is true.*

We will now construct the eigenvector $\beta$ of Proposition 10.3.3, using a sequence method. The proposition is stated again here.

**Proposition 10.3.3.** *Let $\{g_t : t \in (0,\delta)\}$ be the family of metrics defined on the manifold $M = M'\#M''$ in §10.2. Then for sufficiently small $t$, there exists a positive number $\lambda$ and a function $\beta \in C^\infty(M)$ such that $a\Delta\beta = \lambda\beta$. Here $\lambda = O(t^{n-2})$, and $\beta$ satisfies*

$$\beta = \begin{cases} 1 + O(t^{n-2}) & \text{on } M' \setminus B' \\ 1 + O(t^{n-2}|v|^{2-n}) & \text{on } \{v : t \leq |v| < \delta\} \subset B' \\ -1 + O(t^{n-2}) & \text{on } M'' \setminus B'' \\ -1 + O(t^{n-2}|v|^{2-n}) & \text{on } \{v : t \leq |v| < \delta\} \subset B'', \end{cases} \tag{212}$$

*identifying subsets of $M', M''$ with subsets of $M$, by abuse of notation.*

*Proof.* Let $y$ be the unique element of $C^\infty(M)$ satisfying $\int_M y\, dV_{g_t} = 0$ and $a\Delta y = e$. To make $\beta = e + w$ and $a\Delta\beta = \lambda\beta$, we must find $w$ and $\lambda$ such that $a\Delta w = \lambda(e+w) - a\Delta e$. Define inductively a sequence of real numbers $\{\lambda_i\}_{i=0}^\infty$ and a sequence $\{w_i\}_{i=0}^\infty$ of elements of $C^\infty(M)$ beginning with $\lambda_0 = w_0 = 0$. Let

$$\lambda_i = 2\,\mathrm{vol}(M') \left(\int_M y(e + w_{i-1}) dV_{g_t}\right)^{-1}, \tag{213}$$

and let $w_i$ be the unique element of $C^\infty(M)$ satisfying $\int_M w_i dV_{g_t} = 0$ and

$$a\Delta w_i = \lambda_i(e + w_{i-1}) - a\Delta e. \tag{214}$$

Note that as $\int_M e\, dV_{g_t} = \int_M w_{i-1} dV_{g_t} = 0$, the right hand side has integral zero over $M$, and so $w_i$ exists. Thus the sequences $\{\lambda_i\}_{i=0}^\infty$ and $\{w_i\}_{i=0}^\infty$ are well-defined, provided only that the integral on the right hand side of (213) is nonzero; we will prove later that it is bounded below by a positive constant.

If both sequences converge to $\lambda$ and $w$ respectively, say, then (214) implies that $a\Delta w = \lambda(e+w) - a\Delta e$, so that $\beta = e + w$ is an eigenvector of $a\Delta$ associated to $\lambda$. The rôle of the slightly mysterious (213) is as follows: multiply (214) by $y$ and integrate over $M$. Integrating by parts gives

$$\int_M w_i a\Delta y\, dV_{g_t} = \lambda_i \int_M y(e + w_{i-1}) dV_{g_t} - \int_M ea\Delta y\, dV_{g_t}.$$

Substituting $e$ for $a\Delta y$ and recalling that $\int_M e^2 dV_{g_t} = 2\,\mathrm{vol}(M')$, we get

$$\int_M ew_i dV_{g_t} = \lambda_i \int_M y(e + w_{i-1}) dV_{g_t} - 2\,\mathrm{vol}(M') = 0, \tag{215}$$

so that $\int_M ew_i dV_{g_t} = 0$, and if $w$ is the limit of the sequence then $\int_M ew\, dV_{g_t} = 0$.

Now as $\int_M w_i dV_{g_t} = \int_M ew_i dV_{g_t} = 0$, $w_i \in (\tilde{E}_t)^\perp$ and Lemma D.3.2 implies that

$$a\big\|\nabla w_i\big\|_2^2 \geq \gamma \|w_i\|_2^2. \tag{216}$$

Multiplying (214) by $w_i$ and integrating over $M$ by parts gives

$$\begin{aligned} a\big\|\nabla w_i\big\|_2^2 = \int_M a\big|\nabla w_i\big|^2 dV_{g_t} &= \lambda_i \int_M w_i(e + w_{i-1}) dV_{g_t} - a\int_M \nabla w_i \cdot \nabla e\, dV_{g_t} \\ &\leq |\lambda_i| \|w_i\|_2 \big(\|e\|_2 + \|w_{i-1}\|_2\big) + a\big\|\nabla w_i\big\|_2 \big\|\nabla e\big\|_2 \\ &\leq |\lambda_i| a^{\frac{1}{2}} \gamma^{-\frac{1}{2}} \big\|\nabla w_i\big\|_2 \big(\|e\|_2 + \|w_{i-1}\|_2\big) + a\big\|\nabla w_i\big\|_2 \big\|\nabla e\big\|_2, \end{aligned}$$

applying (216) between the second and third lines, and Hölder's inequality. Dividing by $a^{1/2}\gamma^{-1/2}\big\|\nabla w_i\big\|_2$ and using (216) on the left hand side gives

$$\gamma \|w_i\|_2 \leq |\lambda_i| \big(\|e\|_2 + \|w_{i-1}\|_2\big) + D_0, \tag{217}$$

where $D_0 = (a\gamma)^{1/2}\big\|\nabla e\big\|_2$.

Define $D_1 = \int_M ye\,dV_{g_t}$. Then $y = D_1 e + z$, where $\int_M z\,dV_{g_t} = \int_M ze\,dV_{g_t} = 0$. So $z \in (\tilde{E}_t)^\perp$, and by Lemma D.3.2

$$a\big\|\nabla z\big\|_2^2 \geq \gamma\|z\|_2^2. \tag{218}$$

As $\int_M ze\,dV_{g_t} = 0$ and $e = a\Delta y = aD_1\Delta e + a\Delta z$, we find that $\int_M z\big(D_1\Delta e + \Delta z\big)dV_{g_t} = 0$, so

$$\big\|\nabla z\big\|_2^2 = -D_1\int_M \nabla z\cdot\nabla e\,dV_{g_t} \leq D_1\big\|\nabla z\big\|_2\big\|\nabla e\big\|_2.$$

Multiplying by $(a\gamma)^{1/2}\big\|\nabla z\big\|_2^{-1}$ and substituting (218) into the left hand side then gives $\gamma\|z\|_2 \leq D_0 D_1$.

As $y = D_1 e + z$ and $\int_M e w_{i-1} dV_{g_t} = 0$, from (213) we calculate

$$\frac{|\lambda_i|}{2\operatorname{vol}(M')} = \left(D_1 + \int_M z w_{i-1} dV_{g_t}\right)^{-1} \leq \left(D_1 - \|z\|_2\|w_{i-1}\|_2\right)^{-1} \leq \big(D_1 - D_0 D_1\|w_{i-1}\|_2/\gamma\big)^{-1}, \tag{219}$$

provided $D_0\|w_{i-1}\|_2 < \gamma$.

Now (217) and (219) are what we need to prove that the sequences $\{\lambda_i\}_{i=0}^\infty$ and $\{w_i\}_{i=0}^\infty$ are well-defined and convergent, provided the constants $D_0, D_1$ satisfy suitable inequalities, that is, that $D_0$ should be sufficiently small and $D_1$ sufficiently large. In fact it can be shown that if $2D_0 \leq \gamma$ and $D_1$ is large enough, then the two sequences converge to $\lambda$ and $w$ respectively satisfying $a\Delta w = \lambda(e+w) - a\Delta e$, where $\|w\|_2 \leq 2D_0/\gamma$, and $|\lambda| \leq 2\operatorname{vol}(M')D_1^{-1}(1 - 2D_0^2/\gamma^2) \leq 4D_1^{-1}\operatorname{vol}(M')$. The proof uses the same sort of reasoning as Lemma 9.1.2, and will be left to the reader.

Let us now look more closely at the constants $D_0$ and $D_1$. Firstly, $D_0 = (a\gamma)^{1/2}\big\|\nabla e\big\|_2$, and $e$ is defined by the functions $\sigma'$ and $\sigma''$ chosen just before Lemma D.3.1. In fact, $e = c'\sigma' - c''\sigma''$, where $c'$ and $c''$ are close to 1, as $\operatorname{vol}(M') = \operatorname{vol}(M'')$. But $\sigma', \sigma''$ are defined in the same way as $\beta_1, \beta_2$ of §8.1, and an estimate analogous to (94) of §8.3 applies to them, from which it may easily be shown that $D_0 = O(t^{(n-2)^2/(n+1)})$ for sufficiently small $t$. Then because $\|w\|_2 \leq 2D_0/\gamma$, this gives that $\|w\|_2 = O(t^{(n-2)^2/(n+1)})$.

Secondly, we need to estimate $D_1$. Let $\xi'$ be the Green's function of $a\Delta$ on $M'$, satisfying $a\Delta\xi' = \delta_{m'} - 1/\operatorname{vol}(M')$ in the sense of distributions. Then $\xi'$ has a pole of the form $D_2|v|^{2-n} + O''(|v|^{1-n})$ at $m'$, where $D_2$ is a constant, in fact equal to $(n-2)^{-1}\omega_{n-1}^{-1}$. Similarly, we may define the Green's function $\xi''$ at $m''$ of $a\Delta$ on $M''$.

The application of this is in modelling the function $y$ on $M$. We may view $M$, approximately, as being made up of the unions of $M'$ and $M''$, each with a small ball of radius $t$ cut out. To get a function $\xi$ on $M$

with $\Delta\xi$ close to $\mathrm{vol}(M')^{-1}$ on the part coming from $M'$, and close to $-\mathrm{vol}(M'')^{-1}$ on the part coming from $M''$, we try $\xi$ equal approximately to $d'-\xi'$ on $M'$ and $\xi''-d''$ on the part coming from $M''$, for constants $d'$ and $d''$. To join these two functions together on the neck we must have $d'+d''=2D_2t^{2-n}+O(t^{1-n})$, and for $\int_M \xi dV_{g_t}=0$ we must have $d'\,\mathrm{vol}(M')=d''\,\mathrm{vol}(M'')+O(t^{1-n})$. As $\mathrm{vol}(M')=\mathrm{vol}(M'')$ this gives $d'=D_2t^{2-n}+O(t^{1-n})=d''$.

But $e$ is approximately equal to 1 on $M'$ and to $-1$ on $M''$, so that $e\sim\mathrm{vol}(M')a\Delta\xi$. Therefore $y\sim\mathrm{vol}(M')\xi$, and $D_1=\int_M ye\,dV_{g_t}\sim 2d'\,\mathrm{vol}(M')^2$. So finally, we conclude that $D_1=2D_2\,\mathrm{vol}(M')^2t^{2-n}+O(t^{1-n})$. This validates the claim that $D_1$ is large for sufficiently small $t$, that was used earlier to ensure convergence.

Taking the limit over $i$ in (213), we find that

$$\lambda = 2\,\mathrm{vol}(M')\left(D_1+\int_M zw\,dV_{g_t}\right)^{-1}, \tag{220}$$

so using estimates on $D_1$, $z$ and $w$ gives that $\lambda=O(t^{n-2})$ for sufficiently small $t$, one of the conclusions of the proposition. Also, if $e$ is a first approximation to $\beta$, then $\lambda y$ is the second, and the model of $y$ above gives a model of $\beta$. It can therefore be seen that (212) holds for $\beta$, which is the only remaining claim of the proposition. $\square$

Finally, we show that Lemma D.3.2 may be modified further, to apply to functions orthogonal to both 1 and the eigenvector $\beta$ constructed in the last proposition:

**Lemma D.3.4.** *Let $\{g_t : t\in(0,\delta)\}$ be the family of metrics defined on the manifold $M=M'\#M''$ in §10.2. Then for all sufficiently small $t$, if $\phi\in L^2_1(M)$ satisfies $\langle\phi,1\rangle=\langle\phi,\beta\rangle=0$ in either the $L^2$ or the $L^2_1$ inner product, then*

$$\int_M a\left|\nabla\phi\right|^2 dV_{g_t}\geq\gamma\int_M\phi^2 dV_{g_t}. \tag{221}$$

*Here $\beta$ is the eigenvector of $a\Delta$ constructed in Proposition 10.3.3.*

*Proof.* The proof is almost the same as that of Lemma D.3.2. Note that as $1,\beta$ are eigenvectors of $\Delta$, orthogonality to them with respect to the $L^2$ norm and the $L^2_1$ norm is equivalent, and so we may suppose that $\langle\,.\,,\,.\,\rangle$ is the inner product of $L^2(M)$. Let $\xi$ be the unique element of $\tilde{E}_t$ satisfying $\langle\xi,\beta\rangle=1$ and $\int_M\xi dV_{g_t}=0$. If $\phi\in L^2_1(M)$ satisfies $\langle\phi,1\rangle=\langle\phi,\beta\rangle=0$, then $\phi-\langle\phi,\xi\rangle\beta\in\tilde{E}_t^{\perp}$, taken with respect to the $L^2$ norm. So by Lemma D.3.2,

$$\int_M a\big|\nabla\phi - \langle\phi,\xi\rangle\nabla\beta\big|^2 dV_{g_t} \geq \gamma \int_M \big(\phi - \langle\phi,\xi\rangle\beta\big)^2 dV_{g_t}.$$

But as $\beta$ is an e-value of $\Delta$, it is orthogonal to $\phi$ in both $L^2$ and $L^2_1$ norms, so this equation becomes

$$\int_M a\Big(\big|\nabla\phi\big|^2 + \langle\phi,\xi\rangle^2\big|\nabla\beta\big|^2\Big)dV_{g_t} \geq \gamma \int_M \big(\phi^2 + \langle\phi,\xi\rangle^2\beta^2\big)dV_{g_t}. \tag{222}$$

Now for sufficiently small $t$, the eigenvalue $\lambda$ is smaller than $\gamma$, so that $\int_M a\big|\nabla\beta\big|^2 dV_{g_t} \leq \gamma \int_M \beta^2 dV_{g_t}$, and subtracting this multiplied by $\langle\phi,\xi\rangle^2$ from (222) gives (221). □

**Corollary D.3.5.** *Theorem 10.3.4 is true.*

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# Contents