Vertex algebras, moduli stacks, cohomological Hall algebras and quantum groups

The Doctoral Thesis of

Alexei Latyntsev

University of Oxford

Trinity Term 2022

Abstract

The content of this Thesis comes in three parts. Firstly, it shows a compatibility between two structures on the homology of moduli stacks of certain codimension one categories: Joyce's *vertex algebra* structure and the *cohomological Hall algebra* (CoHA) structure. Our Theorem 3.10.1 can be thought of as saying the homology $\mathrm{H}_\bullet(\mathcal{M})$ is a vertex analogue of a "quantum group" (i.e. triangular bialgebra).

Secondly, the main technical work of this thesis builds up the machinery to let us compute cohomological Hall algebras using torus localisation. To begin with, we construct a "bivariant" Euler class and use it to get a clean formulation of torus localisation for singular stacks. We then explain how combining this, with stratifications of the stacks under consideration, allows us to compute their CoHA products. We finish by using these techniques to give new formulae for CoHA products, and a new interpretation of existing ones.

Thirdly, we turn to the question of q deforming Joyce's vertex algebra structure. We interpret the well known (q deformed) Frenkel-Segal-Kac free field realisation in terms of homology of moduli stacks, then make steps to interpreting it as a map of q deformed vertex algebras.

The appendices include the categorical axiomatics needed to talk about vertex analogues of quasitriangular bialgebras and related structures, as well as the construction of the "cohomological" exponential map for algebraic stacks, which is needed to "linearise" closed embeddings by replacing them with the associated normal bundle/complex.

Acknowledgements

This page is to thank everyone who helped make the past years interesting and happy, mathematically or otherwise.

I would like to thank Adeel Khan for organising Derived Office Hours; André Henriques for sharing so much maths generously and friendlily; Andrés Ibáñez Núñez for livening the Geometry group mathematically and socially; Anna Keszthelyi for making Oxford much more interesting; Atul Sharma for patiently explaining physics on long Pandemic walks; Balázs Szendrői; Carlo Attubato for interesting kitchen chats; Charanya Ravi; Christoph Dorn for teaching me ∞-categories; Christopher Douglas; Christoph Weiss; Dhyan Aranya; Dominic Joyce for being an excellent supervisor to me over the past four years: for his extreme generosity with his time reading over my drafts and in meetings, which his sharing of ideas and openness to discussion made really enjoyable, for keeping me accountable but also giving me freedom mathematically, and for teaching me how to do research; Dylan Butson; Eloïse Hamilton; George Cooper; George Fortune for many nice Cambridge visits; Fabian Haiden; my family for the happiness and affection from the start, especially Babushka, Baba Vicky, Dedushka, Hen, Mama and Papa; Filip Zivanovic; Frances Kirwan for supervising the first part of my PhD, for all the useful advice and the interesting maths taught in our meetings; Henry Liu; Hyeonjun Park for catching a big error; Ian Grojnowski for his generously given time, mathematical energy and inspiration; James Kouhot for teaching me octopush and interesting chats on the way to Caversham; Jan Steinebrunner; Jay Swar; Jack Coulter; Joar Skalse for his liveliness and late night chats and walks; Josh de Gromoboy; Kevin Costello; Kevin Lin for his friendship and our eight years' long maths conversation; Kobi Kremnitzer for answering my many questions; Lisa Lokteva for her sunniness and being a de facto second sister; Lukas Brantner for teaching me Koszul duality; Minhyong Kim for guiding me mathematically early on in my PhD; Mykolas Blazonis for starting the Pandemic film club; Nadav Gropper; Netan Dogra; Søren Gammelgård; Sujay Nair for teaching me some physics; The cats of Oxford; Valerio Toledano-Laredo; Vikram Giri for his friendliness and openness; and Zarko Randjelovic.

Contents

Chapter 1

Introduction

Vertex algebra is a rigorous definition of the "holomorphic part" of two dimensional conformal field theories from physics [BPV]. They were discovered in the 90's by Borcherds, Beilinson and Drinfeld [Bo1, BD2] and are still at the forefront of our rigorous understanding of quantum field theory. Today these ideas are spreading ever wider in mathematics [Ga4, GL, Groj].

Let $\mathcal{M}$ be the moduli stack of objects in an abelian or triangulated category $\mathcal{A}$. Under some mild assumptions, Joyce [Jo2] discovered that its homology $\mathrm{H}_\bullet(\mathcal{M}, \mathbf{Q})$ is naturally a vertex algebra, related to enumerative invariants of that category [GJT, Jo3].

Cohomological Hall algebras are a rigorous definition of "algebra of BPS states" from physics [HM]. The idea of Kontsevich and Soibelman [KS] is that, for certain abelian categories $\mathcal{A}$, extensions in $\mathcal{A}$ should put an associative algebra structure on something like the cohomology of $\mathcal{M}$.

Thus, we have moduli stacks $\mathcal{M}$ whose cohomology $\mathrm{H}^\bullet(\mathcal{M}, \mathbf{Q})$ carry two structures, Joyce's vertex coalgebra and the cohomological Hall algebra. How are they related? Our first goal is to show that they form a *vertex bialgebra*, twisted by a braiding element $S(z)$ solving the Yang-Baxter equation. Our proof method is an application of the torus localisation formula, and gives new formulas for cohomological Hall algebras, as well as new interpretations of results in [KS, Da].

Quantum groups are q-deformations of universal enveloping algebras discovered by Drinfeld [Dr], and have since touched many areas of mathematics [EFK, RT, Wit]. There have been many attempts to define similar q-deformed vertex algebras [EK, FR, Li2]. Our second goal is to show how Joyce's vertex algebras fit into this, giving new interpretations for old results [FJW1, FJW2] as well as many new examples.

Summary of contents

Chapter 2 introduces the main background concepts for readers who are unfamiliar with them. To begin with we introduce vertex algebras through the lens of chiral algebras, which is more or less the same as a vertex algebra but much closer to the relevant physics (which we also briefly sketch). Then, we explain how to build a vertex algebra structure on the homology $\mathrm{H}_\bullet(\mathcal{M}, \mathbf{Q})$ of moduli spaces of abelian or triangulated categories $\mathcal{A}$. Our approach is a little different from [Jo2]. Finally, since many of our results will be insensitive to whether our spaces are for instance topological spaces or Artin stacks, it makes sense to introduce the minimal structure (a sheaf theory with the six functors) needed to define notions like "(co)homology" and "Borel Moore homology" with the correct functoriality properties.

Chapter 3 builds up the tools to prove our main Theorem in which $\mathcal{A}$ is coherent sheaves on a smooth proper curve or representations of a finite quiver. In the *symmetric* case when $\mathcal{A}$ is representations of a finite symmetric quiver it states that

Theorem (Theorem 3.10.1)**.** *In the symmetric case, the cohomology of its moduli stack of objects* $\mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})$ *is a* vertex bialgebra *under Joyce's vertex coalgebra* $Y^\vee(-, z)$ *and the cohomological Hall algebra* $\cdot$ *structures, i.e.*

$$Y^\vee(\alpha, z) \cdot Y^\vee(\beta, z) \;=\; Y^\vee(\alpha \cdot \beta, z).$$

To start Chapter 3 we introduce the different types of cohomological Hall algebra that exist in the wild. We then define the *bivariant Euler class*, which is the "correct" extension of the Euler class to singular analogues of vector bundles. To justify this assertion we prove a number of properties: a singular Whitney sum formula, compatibility with the fundamental class and compatibility with deformation to the normal cone.

We then we turn to abelian localisation. After giving the proof for schemes in terms of the bivariant Euler class, we explain how to think about abelian localisation for Artin stacks (Theorem 3.5.16). The statement is more subtle than for schemes because Artin stacks can have cohomology in infinitely many degrees, and so it is no longer enough to just invert equivariant cohomology classes on the point $\mathrm{H}^\bullet_T(\mathrm{pt})$: we need to invert equivariant classes on the *whole space* which satisfy conditions called *concentration* and *specialisation*.

Then we explain a method to compute CoHA products using abelian localisation. This recovers and explains the explicit combinatorial formulas for CoHAs in the literature, e.g. [Da, KS]. As an example, towards the end of the chapter we use it to give a new formula for the CoHA of a curve

(Theorem 3.13.16).

We then check that the spaces appearing in our CoHAs satisfy the conditions which allow us to use abelian localisation. We do this by proving a general result: for any space with a Białynicki-Birula type stratification, if these conditions hold stratawise then they hold on the whole space.

The general case of Theorem 3.10.1 is more complicated when $\mathcal{A}$ is coherent sheaves on an arbitrary curve or representations of an arbitrary quiver. It says we get a vertex bialgebra *twisted* by an element $S(z)$:

$$Y^\vee(\alpha, z) \cdot_{S(z)} Y^\vee(\beta, z) \;=\; Y^\vee(\alpha \cdot \beta, z).$$

This can be viewed as defining an alternative vertex coalgebra structure on $\mathrm{H}^\bullet(\mathcal{M}_\mathcal{A}) \otimes \mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})$. We then build the categorical theory to show this is the same as a vertex bialgebra in a twist of the underlying vertex symmetric monoidal category. This is analogous to the story for twists of ordinary graded bialgebras.

The proof of Theorem 3.10.1 then proceeds by giving explicit formulas for both structures. Joyce's vertex coalgebra can be written in terms of Euler classes essentially by definition, which can be compared to the Euler classes in our explicit formula for the CoHA which appear because the formulas come from applying abelian localisation. Briefly, we pull back to the locus where the middle term of the short exact sequence splits as a direct sum

$$\begin{tikzcd} & \mathrm{Ext}_\mathcal{A} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_\mathcal{A}^2 & & \mathcal{M}_\mathcal{A} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & \mathrm{Ext}_\mathcal{A} \times_{\mathcal{M}_\mathcal{A}} (\mathcal{M}_\mathcal{A} \times \mathcal{M}_\mathcal{A}) \arrow[dl] \arrow[dr] & \\ \mathcal{M}_\mathcal{A}^2 & & \mathcal{M}_\mathcal{A} \times \mathcal{M}_\mathcal{A} \end{tikzcd}$$

then notice a torus action on $\mathrm{Ext}_\mathcal{A} \times_{\mathcal{M}_\mathcal{A}} (\mathcal{M}_\mathcal{A} \times \mathcal{M}_\mathcal{A})$ coming from scaling the factors of the direct sum. We prove in section 3.7 that the conditions of abelian localisation are met, then we use our method of computing CoHAs by abelian localisation to give a formula for push-pull along the above diagram on the right.

Finally, we use our technique to recover formulas of Kontsevich and Soibelman for the CoHA of a quiver and give new formulas for the CoHA of a curve.

Chapter 4 is an account on the progress of a project to q deform constructions of vertex algebra structures on the homology of moduli stacks, with a view to relating them to quantum groups in the future. After some introductory remarks about quantum groups and quantum affine algebras, in section 4.4 we explain the main crutch we will use to connect homology of moduli spaces and (quantum) affine algebras: the FKS isomorphism. More precisely, taking the category $\mathcal{A} = \mathrm{Rep}Q$

of representations of an ADE quiver, in section 4.5 and Theorem 4.5.21 we express the FKS isomorphism explicitly in terms of simple operations on homology, and in sections 4.6 and 4.7 we q deform these calculations. Then, in section 4.8 we give a definition of a q deformation of the vertex algebra structure on $\mathrm{H}^\bullet(\mathcal{M}_{\mathcal{A}})$ (see Theorem 4.8.10). Finally, in 4.9 we discuss future directions, in particular how we expect to be able to axiomatise this q deformed structure to a q deformed vertex algebra such that the FKS isomorphism is a map of such structures.

Note for examiners: Chapter 3 is largely based on the paper [La], chapter 4 will soon be turned into a paper, and the collaboration in [AKLPR] is related to the work of this Thesis but does not appear here.

Chapter 2

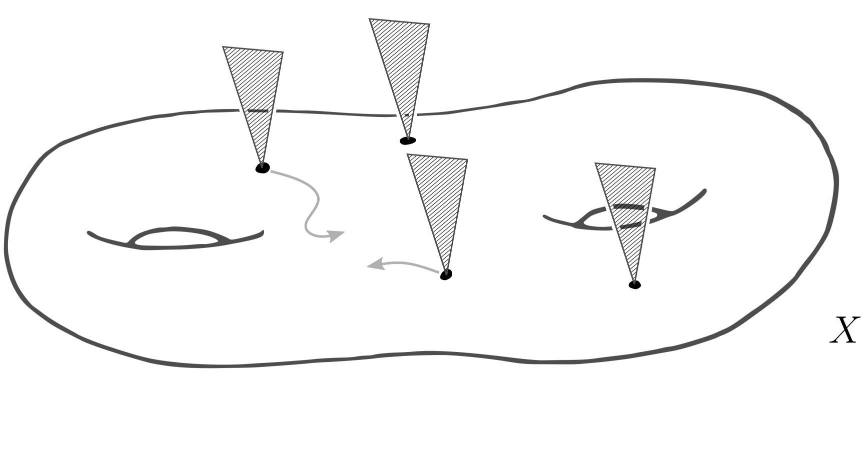
General background

2.1 Definitions: vertex and factorisation algebras

One can understand or motivate vertex algebras through

1. the corresponding notion in physics of the holomorphic part of a $2d$ conformal field theory (section 2.1.25),
2. Beilinson and Drinfeld's *factorisation* (or *chiral*) *algebras* (section 2.1.1), which more closely resemble the physics but require more machinery, or
3. just reading the definition (Definition 2.1.9).

2.1.1. **Factorisation algebras (sketch).** Loosely speaking, a factorisation algebra over an algebraic curve X is a vector spaces $V_{x_1}, ..., V_{x_n}$ living above finitely many points of X, which we are entitled to parallel transport. The interesting part of the structure is what happens when parallel transporting V_{x_1} and V_{x_2} if we *collide* the points x_1 and x_2.



Before giving the definition we work through an example.

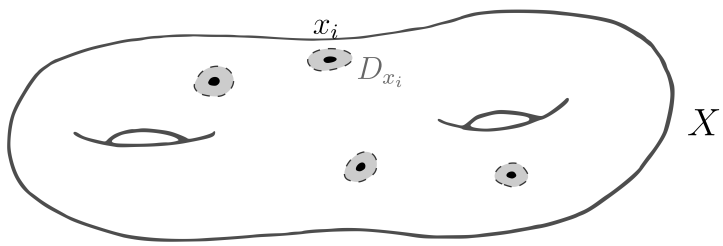
2.1.2. Let X/k be a smooth algebraic curve over a field of characteristic zero. Line bundles $\mathcal{L}$ on X can be expressed in terms of a divisor

$$\mathcal{L} \;\simeq\; \mathcal{O}(n_1x_1 + \cdots + n_kx_k) \qquad \text{for } x \in X,\; n_i \in \mathbf{Z}. \tag{2.1}$$

This divisor is unique after choosing a meromorphic section φ, which gives a trivialisation

$$\varphi \;:\; \mathcal{L}|_{X\backslash\{x_1,\ldots,x_k\}} \;\simeq\; \text{triv}. \tag{2.2}$$

$\mathcal{L}$ can be recovered from this trivial line bundle by gluing trivial line bundles on D_{x_i}, along transition functions z^{n_i} on $D_{x_i}^{\times} = D_{x_i}\backslash\{x_i\}$. Here $D_{x_i} \simeq \operatorname{Spec} k[[z]]$ is the formal disk at x_i.



Thus the data of $(\mathcal{L}, \varphi)$ canonically "factors" onto finitely many points of X. Indeed, the above gives a functor

$$\{\text{Line bundles on } X \text{ with meromorphic section}\}$$
$$\to \{\text{finite subset } \{x_1, \ldots, x_n\} \subseteq X \text{ with line bundles on } D_{x_i} \text{ and trivialisations on } D_{x_i}^{\times}\}$$

which by the Beauville Laszlo theorem is an equivalence of categories.

2.1.3. The data at each point is then a line bundle on D_{x_i} with a trivialisation along $D_{x_i}^{\times}$. This arranges into a space (prestack) Gr_{x_i} called the *affine Grassmannian*, whose functor of points is

$$\operatorname{Maps}(S, \mathrm{Gr}_{x_i}) \;=\; \{\mathcal{L}_i \in \operatorname{Pic}(S \hat{\times} D_{x_i}),\; \varphi_i : \mathcal{L}|_{S\hat{\times} D_{x_i}^{\times}} \xrightarrow{\sim} \text{triv}\}/\sim .$$

By [BD1, Thm 4.5.1] it is an ind scheme over k, with a group structure given by tensor product of line bundles. Its k points are

$$\mathrm{Gr}_{x_i}(k) \;=\; \mathcal{O}^{\times}(D_{x_i}^{\times})/\mathcal{O}^{\times}(D_{x_i}) \;\simeq\; k((t))^{\times}/k[[t]]^{\times} \;\simeq\; \mathbf{Z}$$

which corresponds to the integers $n_i \in \mathbf{Z}$ appearing in the divisor above, and one can show that $\mathrm{Gr}_{x_i} \simeq \mathbf{Z} \times \exp(k((t))/k[[t]])$ as ind schemes.[1]

[1] Here the ind vector space $k((t))/k[[t]] = \operatorname{colim} t^{-n}k[[t]]/k[[t]]$ is viewed as an ind scheme, by viewing the finite dimensional vector spaces $t^{-n}k[[t]]/k[[t]]$ as schemes.

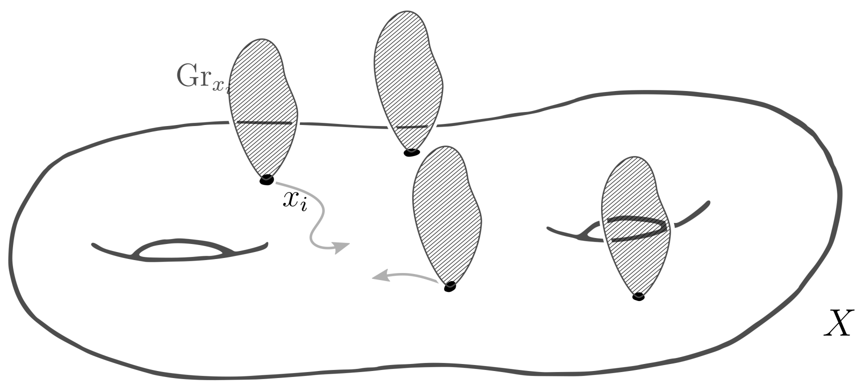
2.1.4. The global data also arranges into a space: the *Beilinson Drinfeld Grassmannian* Gr_X

$$\mathrm{Maps}(S, \mathrm{Gr}_X) \;=\; \{\mathcal{L} \in \mathrm{Pic}(S \times X),\ \{x_1, ..., x_n\} \subseteq X(S),\ \varphi : \mathcal{L}_{S\times X\backslash\{x_1,...,x_n\}} \xrightarrow{\sim} \mathrm{triv}\}/\sim$$

parametrising a line bundle and a trivialisation away from a finite subset. Forgetting the line bundle gives a flat map to the space (prestack) of nonempty finite subsets of X

$$\mathrm{Gr}_X \;\twoheadrightarrow\; \mathrm{Ran}X$$

and its fibre above $\{x_1, ..., x_n\}$ is the product of the Gr_{x_i}. The *Ran space* $\mathrm{Ran}X$ is defined as the colimit of X^n for $n \geqslant 0$ over all diagonal embeddings (corresponding to surjective maps of finite sets).



For general reasons (see section 2.4.7) it follows from the above that $\mathrm{Gr}_X \to \mathrm{Ran}X$ admits a flat connection. See [BD1, MV] for more on Gr_X.

2.1.5. To get a factorisation algebra, we take the distributions supported on

$$0 \;\hookrightarrow\; \mathrm{Gr}_{x_i} \;\simeq\; \mathbf{Z} \times \exp\left(k((t))/k[[t]]\right)$$

which is a vector space V_{x_i} with basis the delta function at the origin and its derivatives

$$V_{x_i} \;\simeq\; k\left\{\partial_{-1}^{n_1} \cdots \partial_{-r}^{n_r}\delta\right\}$$

in the various normal directions $N_{0/\mathrm{Gr}_{x_i}} \simeq k\{t^{-1}, t^{-2}, \dots\}$. Here n_i and r vary over all nonnegative integers.

2.1.6. This is the vector space, what is the structure on it? The above construction globalises, by taking distributions supported on

$$\mathrm{Ran}X \;\overset{\mathrm{triv}}{\hookrightarrow}\; \mathrm{Gr}_X$$

to give a quasicoherent sheaf V on Ran$X$. Precisely, we take $\mathcal{D}$ module pushforward triv$_*\mathcal{O}$ then take its $\mathcal{O}$ module sections along $\mathrm{Gr}_X \twoheadrightarrow \mathrm{Ran}X$. The connection on Gr_X endows V with a $\mathcal{D}$ module structure, and the factorisation structure implies

$$\mathrm{Gr}_{\{x_1,\dots,x_n\}} \simeq \mathrm{Gr}_{x_1} \times \cdots \times \mathrm{Gr}_{x_n} \quad \Rightarrow \quad V_{\{x_1,\dots,x_n\}} \simeq V_{x_1} \otimes \cdots \otimes V_{x_n}$$

if the x_i are distinct.

2.1.7. To translate this into a structure on V_x, we restrict to the case of two points. Writing $V_n = V|_{X^n}$, we get a $\mathcal{D}$ module X^2 whose fibres (as an $\mathcal{O}$ module) above the diagonal and its complement are

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d] & V_2 \arrow[d] & V_1 \boxtimes V_1 \arrow[l] \arrow[d] \\ X \arrow[r, "\Delta"] & X^2 & X^2 \backslash X \arrow[l, "j"'] \end{tikzcd}$$

If X is a *curve*, we have the Mayer Vietoris sequence

$$V_2 \ \to \ j_*j^\bullet V_2 \ \xrightarrow{\partial} \ \Delta_*\Delta^\bullet V_2$$

where $f^\bullet$ denotes the $\mathcal{O}$ module pullback.[2] Thus we get

$$j_*(V_1 \boxtimes V_1) \ \xrightarrow{\partial} \ \Delta_* V_1. \tag{2.3}$$

When $X = \mathbf{A}^1$, taking global sections gives a map

$$V_0 \otimes V_0 \otimes k[x, y, (x-y)^{-1}] \ \xrightarrow{\partial} \ V_0 \otimes k[x, y, (x-y)^{-1}]/\langle x - y \rangle.$$

If the $\mathcal{D}$ module is weakly $\mathbf{G}_a$ equivariant, one can show (e.g. [Bu]) that the restriction of ∂ to $V_0 \otimes V_0 \otimes k[x, y]$ is uniquely determined by a map

$$Y \ : \ V_0 \otimes V_0 \ \to \ V_0 \otimes k((x-y))$$

which endows V_0 with the structure of a vertex algebra.

2.1.8. **Vertex algebras.** We say what a vertex algebra is then describe the one (called the *level zero Heisenberg*) corresponding to the above factorisation algebra. The comparative advantages of vertex algebras over factorisation algebras are that examples are much easier to construct (indeed, most known examples do not arise "naturally" as factorisation algebras) and it is easier to make explicit computations.

[2]This follows since $f^! = f^\bullet[d]$ for any $f : X \to Y$, where $d = \dim X - \dim Y$ is the dimension of f.

Definition 2.1.9. A *vertex algebra* is a vector space V with a distiguished vector $|0\rangle$, a map

$$Y(-,z)(-) \ : \ V \otimes V \ \rightarrow \ V((z)),$$

such that the $Y(\alpha, z)$ for $\alpha \in V$ weakly commute (Definition 2.1.13) and

$$Y(|0\rangle, z) \ = \ \mathrm{id}, \qquad\qquad Y(\alpha, z)|0\rangle \ = \ \alpha \quad \mod zV[[z]],$$

as well as an endomorphism T satisfying $T|0\rangle = 0$ and $Y(T\alpha, z) = \partial_z Y(\alpha, z)$.

Elements of V are called *states*, $|0\rangle$ the *vacuum*, T *translation* and $Y(\alpha, z)$ the *field* of α.

2.1.10. The fibre of a $\mathbf{G}_a$ equivariant factorisation algebra on $X = \mathbf{A}^1$ is a vertex algebra. The translation operator T comes from the $\mathbf{G}_a$ equivariance, and weak commutativity comes from the $\mathfrak{S}_2$ equivariance of the factorisation algebra.[3]

2.1.11. The vertex algebra corresponding to the above factorisation algebra should have underlying vector space as in section 2.1.5

$$V \ \simeq \ k\{b_{-1}^{n_1} \cdots b_{-r}^{n_r}|0\rangle \ : \ r, n_i \geqslant 0\}.$$

We can identify this with functions on the *jet space* of $\mathbf{A}^1$, the space of maps from the formal disk $D \to \mathbf{A}^1$, see section 2.3.4. As we discuss there, infinitesimal translation in D endows V with a vector field T given by $T(b_{-r}) = b_{-r-1}$ and the Liebniz rule

$$T(b_{-1}^{n_1} \cdots b_{-r}^{n_r}|0\rangle) \ = \ \sum n_i b_{-1}^{n_1} \cdots b_{-i-1}^{n_{i+1}+1} b_{-i}^{n_i-1} \cdots b_{-r}^{n_r}|0\rangle.$$

We then define

$$Y(b_{-1}|0\rangle, z) \ = \ \sum_{n \geqslant 0} b_{-n-1} z^n,$$

and it follows from the axioms that its derivatives give $Y(b_{-n}|0\rangle, z)$. Similarly, we define

$$Y(b_{-1}^{n_1} \cdots b_{-r}^{n_r}|0\rangle, z) \ = \ \prod Y(b_{-i}|0\rangle)^{n_i}.$$

This is a particularly simple example of a vertex algebra because the fields literally commute, not just weakly. To get a more representative example of what a vertex algebra looks like we need to introduce twists (see section 2.5.4).

[3] Note that if V is any sheaf over $\mathrm{Ran}X$ then its restriction to X^n is automatically $\mathfrak{S}_n$ equivariant. This is because to map $X^n \to \mathrm{Ran}X$ lifts to $X^n/\mathfrak{S}_n = \mathrm{colim}_{\sigma \in \mathfrak{S}_n}(X^n \to X^n) \hookrightarrow \mathrm{Ran}X$.

2.1.12. Let V be any vector space and $\alpha(z), \beta(z) : V \to V((z))$ linear maps. The compositions $\alpha(z)\beta(w)$ and $\alpha(w)\beta(z)$ may be viewed as elements of a common vector space

$$(\mathrm{End}V)((z))((w)) \hookrightarrow (\mathrm{End}V)[[z^{\pm 1}, w^{\pm 1}]] \hookleftarrow (\mathrm{End}V)((w))((z)).$$

These inclusions are $k[[z,w]]$ linear so may be viewed as maps of quasicoherent sheaves over the product of two formal disks $D_z \times D_w$. Weak commutativity says that the commutator $[\alpha(z), \beta(w)]$ is only supported on the diagonal.

Definition 2.1.13. Linear maps $\alpha(z), \beta(z) : V \to V((z))$ *weakly commute* if

$$(z-w)^N [\alpha(z), \beta(w)] \;=\; 0$$

for some $N \gg 0$.

2.1.14. **Factorisation algebras.** Currently, chiral and factorisation (co)algebras (Definitions 2.1.24 and 2.1.22) are perhaps the most successful and conceptual attempt at mathematically defining aspects of two dimensional conformal field theory.

Whereas other attempts start with a vector space and introduce extra structures by hand to mimic those in the physics literature, often leaving it unclear why the definitions are one way and not another, the definition of factorisation coalgebras is remarkably simple: they are factorisable coalgebras in a certain symmetric monoidal category. This captures the idea of states "living on a curve", which can move around and collide to form new states.

2.1.15. We first define an algebraic geometric analogue of the collection of finite subsets of X, which are allowed to "collide". Let X be any prestack, and take the functor defined on the category FSet^{surj} of nonempty finite sets with surjections

$$X^{(-)} \;:\; \mathrm{FSet}^{surj} \;\to\; \mathrm{PreStk}$$

sending I to X^I, and a surjection $I \to J$ to the associated diagonal map $\Delta_{I/J} : X^J \to X^I$. The *Ran space* of X is the colimit of this diagram

$$\mathrm{Ran}X \;=\; \mathrm{colim}_{I \in \mathrm{FSet}^{surj,op}} X^I.$$

Thus $\mathrm{Maps}(S, \mathrm{Ran}X)$ is the set of nonempty finite subsets of $\mathrm{Maps}(S, X)$, see e.g. [CF, §10].

2.1.16. The Ran space is a (nonunital) commutative monoid in PreStk^{corr} in two different ways, meaning that it admits correspondences as below satisfying an associativity condition. The first comes from taking union of finite sets

$$\begin{tikzcd} & \mathrm{Ran}X \times \mathrm{Ran}X \arrow[dl, equal] \arrow[dr, "\pi"] & \\ \mathrm{Ran}X \times \mathrm{Ran}X & & \mathrm{Ran}X \end{tikzcd} \tag{2.4}$$

and the second from taking unions on the locus of *disjoint* finite subsets

$$\begin{tikzcd} & (\mathrm{Ran}X \times \mathrm{Ran}X)_{disj} \arrow[dl, "j"'] \arrow[dr, "\pi j"] & \\ \mathrm{Ran}X \times \mathrm{Ran}X & & \mathrm{Ran}X \end{tikzcd} \tag{2.5}$$

The fibre of π over a nonempty finite subset $I \subseteq X$ are the pairs of nonempty finite subsets I_1, I_2 with $I = I_1 \cup I_2$. Likewise for πj, except the subsets I_i are disjoint.

2.1.17. One can also define a *unital* Ran space $\mathrm{Ran}_{un}X$ (see [Ga4]), a lax prestack which should be thought of as parametrising all finite subsets of X (including the empty one).

2.1.18. From now on, assume that X is a seperated scheme of finite type over a field k. It follows from the definition of the Ran space as a colimit that its category of $\mathcal{D}$ modules (see section 2.7.6) is

$$\mathcal{D}(\mathrm{Ran}X) \;=\; \lim_{I \in \mathrm{FSet}^{surj}} \mathcal{D}(X^I),$$

meaning a $V \in \mathcal{D}(\mathrm{Ran}X)$ corresponds to a collection of $V_I \in \mathcal{D}(X^I)$ with compatible isomorphisms $V_J \simeq \Delta^!_{I/J} V_I$ for all surjections of (nonempty) finite sets $I \twoheadrightarrow J$. To give a $\mathcal{D}$ module on the *unital* Ran space is to in addition supply compatible maps $\Delta^!_{I/J}\mathcal{F}_I \to \mathcal{F}_J$ for all maps of finite sets $I \to J$. For instance, this gives a map $V_\varnothing \otimes \omega_{X^I} \to V_I$ for all I.

2.1.19. By smooth base change, each (nonunital) commutative monoid structure on $\mathrm{Ran}X$ as an object in PreStk^{corr} where the rightwards map to $\mathrm{Ran}X$ is an open immersion induces a (nonunital) symmetric monoidal structure on $\mathrm{Sh}(\mathrm{Ran}X)$. Applying this to the above monoidal structures, we get the $*$ and *chiral* tensor products

$$\mathcal{A} \otimes^* \mathcal{B} \;=\; \pi_*(\mathcal{A} \boxtimes \mathcal{B}), \qquad\qquad \mathcal{A} \otimes^{ch} \mathcal{B} \;=\; \pi_* j_* j^!(\mathcal{A} \boxtimes \mathcal{B}).$$

2.1.20. It is easy to describe these tensor products explicitly [FG, §2.3], first

$$(\mathcal{A} \otimes^* \mathcal{B})_I \;=\; \bigoplus_{I = I_1 \cup I_2} \Delta^!_{I_1 \amalg I_2 / I}(\mathcal{A}_{I_1} \boxtimes \mathcal{B}_{I_2}),$$

where direct sum is over all two nonempty subsets I_1, I_2 with $I = I_1 \cup I_2$, not necessarily disjoint. To describe the chiral tensor product, we write $j : (X^{I_1} \times X^{I_2})_{disj} \hookrightarrow X^I$ for the open locus where the first I_1 and last I_2 points are disjoint. Since $j^! = j^*$, we have

$$(\mathcal{A} \otimes^{ch} \mathcal{B})_I \;=\; (\pi_* j_* j^* \mathcal{A} \boxtimes \mathcal{B})_I \;=\; \bigoplus_{I = I_1 \amalg I_2} j_{I*} j_I^* (\mathcal{A}_{I_1} \boxtimes \mathcal{B}_{I_2}),$$

where direct sum is over partitions $I = I_1 \amalg I_2$ into disjoint nonempty subsets.

2.1.21. We now define a factorisation algebra over a scheme X of finite type over a field of characteristic 0.

Definition 2.1.22. [BD2, FG] A *factorisation algebra* is a (chiral) cocommutative coalgebra

$$\mathcal{B} \;\in\; \mathrm{commCoAlg}\left(\mathcal{D}(\mathrm{Ran}X), \otimes^{ch}\right)$$

which *factorises*: considering the coproduct $\mathcal{B} \to \mathcal{B} \otimes^{ch} \mathcal{B}$, each component

$$\mathcal{B}_I \;\to\; j_{I*} j_I^* \mathcal{B}_{I_1} \boxtimes \mathcal{B}_{I_2} \qquad\qquad I \;=\; I_1 \amalg I_2$$

becomes an equivalence when restricted to the open locus (i.e. after applying j_I^*).

2.1.23. We can apply the construction of (2.1.7) to a general factorisation algebra. What is the structure that we get?

Definition 2.1.24. [BD2, FG] A *chiral algebra* is a (chiral) Lie algebra

$$\mathcal{A} \;\in\; \mathrm{Lie}\left(\mathcal{D}(\mathrm{Ran}X), \otimes^{ch}\right)$$

lying in the image of $\Delta_* : \mathcal{D}(X) \to \mathcal{D}(\mathrm{Ran}X)$.

Francis and Gaitsgory [FG, Thm. 1.2.4] have constructed an equivalence between the categories of chiral and factorisation algebras by interpreting the construction as in (2.1.7) as an instance of Koszul duality.

2.1.25. **Physics motivations.** Factorisation algebras on curves and vertex algebras both attempt to formalise what physicists mean by holomorphic part of a two dimensional conformal field theory. Generally speaking, trying to put aspects of quantum field theory (QFT) on a mathematical footing has been a very fruitful source of new mathematics over the past few decades.

2.1.26. Most of this section will not be rigorous mathematics. Rather, the point is to sketch some aspects of physicists' points of view on vertex algebras.

2.1.27. First following [Ta], we sketch part of what physicists expect to attach to a quantum field theory. A quantum field theory Q is defined on a class of manifolds with extra structure $\mathcal{S}$, for instance a Riemannian manifold or conformal manifold, and dimension d. At minimum it should assign:

1. an element $\mathcal{Z}_Q(N) \in k$ for every d dimension $\mathcal{S}$-manifold,
2. a vector space $\mathcal{H}_Q(M)$ of *states* to any $d-1$ dimensional $\mathcal{S}$-manifold M.

Moreover, it should interact interestingly with manifolds with boundary, assigning

1'. a map

$$\mathcal{Z}_Q(B) \;:\; \mathcal{H}_Q(M_1) \;\to\; \mathcal{H}_Q(M_2)$$

to any d dimensional $\mathcal{S}$-manifold with corners B with boundary $M_1 \amalg M_2$.

This data should give a symmetric monoidal functor from some sort of cobordism category

$$\mathrm{Cob}^d_{\mathcal{S}} \;\to\; \mathrm{Vect}_k \tag{2.6}$$

which should satisfy the Atiyah-Segal axioms [At2, Se3], with both the category and axioms suitably modified according to $\mathcal{S}$. In particular, $\mathcal{H}(\varnothing) = k$ so the two notions of $\mathcal{Z}_Q$ both give a number for manifolds without boundary.

Q should also attach data to manifolds of dimensions $d-2$ and lower (this is called an *extended* QFT), lifting the above functor (2.6) to a map of the associated d-categories. Thus it assigns

3. a k linear category to any $d-2$ dimensional manifold,
4. a k linear $(n-1)$-category to any $d-n$ dimensional manifold, for $n \geqslant 1$.

It should also assign data to extended cobordisms, similarly to the unextended situation (1').

Separate from this, Q should come with

A. a vector space $\mathcal{V}^0_Q$ of *point operators*,

B. a tensor category $\mathcal{V}^1_Q$ of *line operators*,

C. a certain n-category $\mathcal{V}^n_Q$ for every $0 \leqslant n \leqslant d$.

This data should interact with the extended structure above, e.g. given point operators $\varphi_1, ..., \varphi_n \in \mathcal{V}_Q^0$ and distinct points $x_1, ..., x_n$ in a d dimensional $\mathcal{S}$ manifold N, we get a number denoted

$$\mathcal{Z}_Q(N, \varphi_1(x_1) \cdots \varphi_n(x_n)) \ \in \ k$$

called a *correlation function* in the $x_1, ..., x_n$. There are two more relevant expectations about point operators: $\mathcal{V}_Q$ should carry a kind of algebra structure, and in the example of conformal field theories, $\mathcal{V}_Q^0 \simeq \mathcal{H}_Q(S^{d-1})$. Thus for $d = 2$ dimensional CFTs we expect an algebra structure on $\mathcal{H}_Q(S^1)$, and its "holomorphic" subspace is expected to carry a vertex algebra structure.

2.1.28. QFTs often (but not always) come up in the physics literature through path integrals. First one defines an auxiliary vector space $\mathcal{H}_Q(B)$ of *fields* and an *action* function $S : \mathcal{H}_Q(B) \to \mathbf{C}$ for every d-dimensional $\mathcal{S}$-manifold B. It is claimed that this vector space carries a measure, denoted $d\psi$, and if B has no boundary one can symbolically write

$$\mathcal{Z}_Q(B) \ = \ \int_{\mathcal{H}_Q(B)} e^{-S(\psi)} d\psi.$$

If B has a boundary it is expected that there should be a *restriction* map on fields $|_{\partial B} : \mathcal{H}_Q(B) \to \mathcal{H}_Q(\partial B)$. Then if B is a bordism from M_1 to M_2, one can symbolically write

$$\mathcal{Z}_Q(B) \ : \ \mathcal{H}_Q(M_1) \ \to \ \mathcal{H}_Q(M_2), \qquad\qquad \varphi \ \mapsto \ \int_{\psi|_{M_1} = \varphi} e^{-S(\psi)} \psi|_{M_2} d\psi.$$

2.1.29. To be more explicit, we sketch the best understood example to illustrate some of the above data: topological field theories (TQFTs). Loosely, TQFTs are QFTs which only depend on the topological structure of the manifold. To be precise, a TQFT is *defined* to be a symmetric monoidal functor

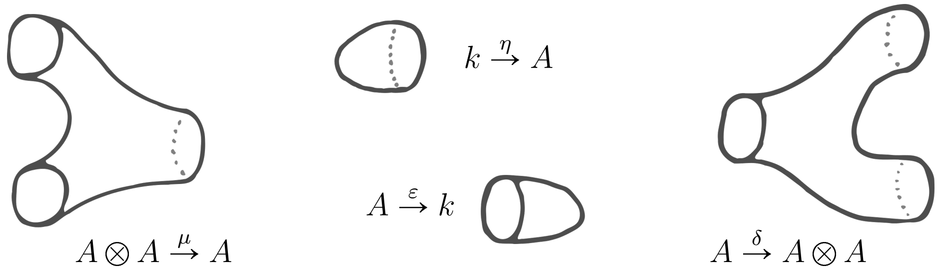
$$\mathrm{Cob}^d \ \to \ \mathrm{Vect}$$

from the category of closed oriented $d-1$ dimensional manifolds with morphisms cobordisms, satisfying the Atiyah-Segal axioms (see [Koc, 1.2.23]).

Let us now take the example of $2d$ TQFTs, which might help us understand $2d$ CFTs and so vertex algebras. It assigns:

- a vector space $A = \mathcal{H}_Q(S^1)$ to the only connected one manifold S^1,
- a map $A^{\otimes n} \to A^{\otimes m}$ for every oriented 2-manifold giving a bordism from $\sqcup^n S^1$ to $\sqcup^m S^1$ (so it has boundary $\sqcup^n S^1 \coprod \sqcup^m S^1$).

In particular we get maps



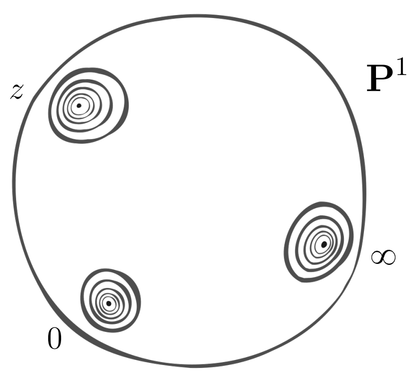
One can then show that these maps arrange into a Frobenius algebra, and that the categories of Frobenius algebras and $2d$ TQFTs are equivalent [Koc, 2.3.24].

2.1.30. Thus vertex algebras should be thought of as the analogue of this Frobenius algebra structure in the case of $d = 2$ dimensional conformal field theories Q. Note that 2 dimensional conformal manifolds are the same thing as Riemann surfaces, which is why there is a hope of linking the subject with algebraic geometry.

Take as the underlying vector space

$$V \;=\; \lim \mathcal{H}_Q(S^1_r)$$

where $S^1_r \subseteq \mathbf{C}$ is the circle of radius $r > 0$ centred at the origin. Now take three points $\{0, z, \infty\} \subseteq \mathbf{P}^1$ along with three families of circles with origins $\{0, z, \infty\}$ and radii tending to zero. This data is conformally invariant. In particular, we should get maps depending nontrivially on $z \in \mathbf{P}^1 \backslash \{0, \infty\}$:



Similarly, we get the vacuum $|0\rangle \to V$ just as we got the unit in the $2d$ TQFT case.

We reiterate that it is expected that $\mathcal{H}_Q(S^1_r)$ should agree with the vector space of point operators: this is called the *state-operator correspondence*. In particular, given any Riemann surface X and elements $\alpha_1, ..., \alpha_n \in V$ we should expect *correlation functions*, in this context usually denoted as

$$\langle \alpha_1(-) \cdots \alpha_n(-) \rangle \ : \ X^n \ \to \ \mathbf{C}.$$

2.1.31. A common class of examples of QFTs are σ *models*. Loosely speaking, fixing a target manifold T, a σ model is a d dimensional QFT acting on d manifolds by

$$M \ \mapsto \ \mathrm{Maps}(M, T).$$

Strictly speaking the right hand side is not a vector space, so it is necessary to suitably linearise it, e.g. by taking the vector space of functions. Thus we should expect to get examples of vertex algebras from something like functions on loop spaces

$$LT \ = \ \mathrm{Maps}(S^1, T).$$

Something like this works for any scheme T (section 2.3.4) producing a vertex algebra structure on $\mathcal{O}(J_\infty T)$, but it is fairly uninteresting. To get more interesting examples, we need to *quantise* the σ model, which on the mathematics side corresponds to producing a filtered vertex algebra whose associated graded is $\mathcal{O}(J_\infty T)$.

2.1.32. **Remarks.** There is a closely related notion of topological factorisation algebra due to Lurie and developed among others by Costello and Gwilliam, see [Lur1, CG1, CG2]. This point of view is sometimes taken in the algebraic geometry literature, e.g. [KV2]. In another direction, Segal [Se2] has a different formalisation of $2d$ CFTs. On the physics side, $2d$ CFT is a very large subject, see e.g. [FMS] for an introduction.

2.2 Properties of vertex algebras

In a vertex algebra, its fields $Y(\alpha, z) = \sum_{n \in \mathbf{Z}} \alpha_n z^{-n-1}$ behave quite similarly to elements in a commutative algebra. To justify this claim, see the following properties of vertex algebras.

2.2.1. **Normally ordered product of fields.** It is not possible to compose two linear maps $\alpha(z), \beta(z) : V \to V((z))$, since the z coefficients of

$$\text{“}\alpha(z)\beta(z)\text{”} \ = \ \sum_{n,m \in \mathbf{Z}} \alpha_n \beta_m z^{-n-m-2}$$

are infinite sums and do not define endomorphisms of V. However, since $\alpha_n\gamma = \beta_n\gamma = 0$ for $n \gg 0$, as an ad hoc fix we may salvage this by defining their *normally ordered product* to be

$$:\alpha(z)\beta(z): \;=\; \sum_{n,m\in\mathbf{Z}} :\alpha_n\beta_m: z^{-n-m-2} \qquad\qquad :\alpha_n\beta_m: \;=\; \begin{cases} \alpha_n\beta_m & \text{if } n<0 \\ \beta_m\alpha_n & \text{if } n \geqslant 0. \end{cases}$$

This product is neither commutative nor associative. The reason this product is worth considering is Dong's lemma [FBZ, 2.3.4], which says that $:\alpha(z)\beta(z):$ weakly commute with any field which $\alpha(z)$ and $\beta(z)$ both weakly commute with.

If V is a vertex algebra then one can show that

$$:Y(\alpha,z)Y(\beta,z): \;=\; Y(\alpha_{-1}\beta_{-1}|0\rangle, z)$$

for any $\alpha, \beta \in V$.

Note that $\alpha = \alpha_{-1}|0\rangle$, so compare this to $Y(\alpha,z) = Y(\alpha_{-1}|0\rangle, z)$.

2.2.2. Note that $\alpha(z)\beta(w) = \sum \alpha_n\beta_m z^{-n-1}w^{-m-1}$ gives a well defined map $V \to V((w))((z))$, and also we can define their normally ordered product $V \to V[[z,w]][z^{-1},w^{-1}]$ given by

$$:\alpha(z)\beta(w): \;=\; \sum_{n,m\in\mathbf{Z}} :\alpha_n\beta_m: z^{-n-1}w^{-m-1}.$$

2.2.3. **Reconstruction.** Vertex algebras can be described in terms of generators, just like algebras. Notice that the algebra structure on a commutative algebra A is uniquely determined by the multiplication maps

$$a_i \;:\; A \;\to\; A$$

for $\{a_i\}_{i\in I}$ a generating set. Conversely, to give a commutative algebra structure on the vector space A, it is enough to give a nonzero element $1 \in A$ and commuting linear maps $a_i : A \to A$ so that a_i1 are distinct and

$$A \;=\; \mathrm{span}\{a_{i_1}\cdots a_{i_r}1\}_{r\in\mathbf{N}, i_k\in I}.$$

The analogue for vertex algebras is the *reconstruction theorem*. Let V be a vector space with endomorphism T and nonzero element $|0\rangle \in V$.

Proposition 2.2.4. (Reconstruction theorem [FBZ, Thm. 4.4.1]) *A vertex algebra structure on V is specified by weakly commuting maps $\alpha_i(z) = \sum_{n\in\mathbf{Z}} \alpha_{i,n}z^{-n-1} : V \to V((z))$ for $i \in I$ so that $\alpha_i(z)|0\rangle \in V[[z]]$ and their constant terms are distinct, $[T, \alpha_i(z)] = \partial_z\alpha_i(z)$, and*

$$V \;=\; \mathrm{span}\{\alpha_{i_1,n_1}\cdots\alpha_{i_r,n_r}|0\rangle\}_{r\in\mathbf{N}, i_k\in I, n_i\in\mathbf{Z}}.$$

The vertex algebra structure is

$$Y(\alpha_{i_1,n_1}\cdots\alpha_{i_r,n_r}|0\rangle, z) \;=\; \frac{1}{(-n_1-1)!\cdots(-n_r-1)!} : \partial_z^{-n_1-1}\alpha_{i_1}(z)\cdots\partial_z^{-n_r-1}\alpha_{i_r}(z) : .$$

Taking a vertex algebra given by commutative algebra gives back the statement for commutative algebras.

2.2.5. **Operator product expansion.** We have explained generators, we now explain what are the analogues of relations between generating fields in a vertex algebra.

Let V be a vector space and $\alpha(z), \beta(z) : V \to V((z))$ two linear maps. An equivalent formulation of weak commutativity (Definition 2.1.13) is that their commutator is a finite sum of the delta function $\delta(z-w) := \sum_{n\in\mathbf{Z}} z^n w^{-n-1}$ and its derivatives:

$$[\alpha(z), \beta(w)] \;=\; \sum_{k=0}^{N} \frac{1}{k!}\gamma_k(w)\partial_w^k\delta(z-w) \tag{2.7}$$

for some $\gamma_k(w) : V \to V((w))$. Alternatively, their composition is a finite sum

$$\alpha(z)\beta(w) \;=\; \sum_{k=0}^{N} \frac{\gamma_k(w)}{(z-w)^{k+1}} \;+\; :\alpha(z)\beta(w): \tag{2.8}$$

where $1/(z-w)$ is expanded in positive powers of w/z,[4] and $\beta(w)\alpha(z)$ is equal to the same expression with $1/(z-w)$ expanded in negative powers of w/z.

For vertex algebras the coefficients in (2.7) and (2.8) have an explicit description

Proposition 2.2.6. (Operator product expansion, e.g. [FBZ, §3]) *If V is a vertex algebra*

$$Y(\alpha, z)Y(\beta, w) \;=\; Y(Y(\alpha, z-w)\beta, w) \;=\; \sum_{k\in\mathbf{Z}} \frac{Y(\alpha_k\beta, w)}{(z-w)^{k+1}}.$$

2.2.7. Thus if $\alpha(z), \beta(z)$ weakly commute, we get maps

$$\alpha(z)\beta(w) \;:\; V \;\to\; V[[z,w]][z^{-1}, (z-w)^{-1}, w^{-1}]$$

$$:\alpha(z)\beta(w): \;:\; V \;\to\; V[[z,w]][z^{-1}, w^{-1}].$$

We write $\sim$ for the equivalence relation on $V[[z,w]][z^{-1},(z-w)^{-1},w^{-1}]$ given by quotienting out by $V[[z,w]][z^{-1},w^{-1}]$, so e.g.

$$\alpha(z)\beta(w) \;\sim\; \sum_{k=0}^{N} \frac{\gamma_k(w)}{(z-w)^k}.$$

[4]Here,

$$:\alpha(z)\beta(w): \;=\; \alpha(z)_+\beta(w) + \beta(w)\alpha(z)_-$$

denotes the *normally ordered product*, where if $\alpha(z) = \sum_{n\in\mathbf{Z}}\alpha_n z^n$ we have written $\alpha_+(z) = \sum_{n\geqslant 0}\alpha_n z^n$ and $\alpha_-(z) = \sum_{n<0}\alpha_n z^n$.

2.2.8. **Modules over vertex algebras.** A *module* over vertex algebra V is a vector space M with a map

$$Y_M(-,z)(-) \;:\; V \otimes M \;\to\; M((z))$$

satisfying properties analogous to vertex algebra, see [FBZ, §5.1.1]. It should correspond to the notion of factorisation comodule of a factorisation coalgebra.

In [Y.Zh], Y. Zhu defined an associative algebra $\mathrm{Zhu}(V)$ attached to any graded vertex algebra V, which controls the representations of V:

Definition 2.2.9. The *Zhu algebra* of a graded vertex algebra V is the quotient space

$$\mathrm{Zhu}(V) \;=\; V/O(V)$$

where $O(V)$ is spanned by elements of the form

$$\alpha \circ \beta \;=\; \mathrm{Res}_z \left(\frac{(1+z)^{\deg \alpha}}{z^2} Y(\alpha, z)\beta \right)$$

where α is homogeneous.

Proposition 2.2.10 ([Y.Zh])**.** $\mathrm{Zhu}(V)$ *is an associative algebra, with unit* $|0\rangle$ *and product*

$$\alpha \cdot \beta \;=\; \mathrm{Res}_z \left(\frac{(1+z)^{\deg \alpha}}{z} Y(\alpha, z)\beta \right).$$

Moreover, there is an equivalence of categories

$$V\text{-}\mathrm{Mod}_{\mathbf{N}} \;\xrightarrow{\sim}\; \mathrm{Zhu}(V)\text{-}\mathrm{Mod} \qquad\qquad M \;=\; \bigoplus_{n=0}^{\infty} M_n \;\mapsto\; M_0$$

between the category of $\mathbf{N}$*-graded* V *modules and of* $\mathrm{Zhu}(V)$ *modules. A homogeneous representative* α *of an element in* $\mathrm{Zhu}(V)$ *acts on* M_0 *by* $\alpha_{\deg \alpha - 1}$.

For instance, Frenkel and Zhu [FZ, §3] computed the Zhu algebra of the affine vertex algebra attached to any finite dimensional simple Lie algebra $\mathfrak{g}$ and any level k:

$$\mathrm{Zhu}(V_k(\mathfrak{g})) \;=\; U(\mathfrak{g}),$$

and if k is a positive integer then $\mathrm{Zhu}(L_k(\mathfrak{g})) = U(\mathfrak{g})/\langle e_\theta^{k+1}\rangle$, where e_θ generates the root space of the highest root θ.

2.3 Examples of vertex algebras

2.3.1. **Algebras.** Every commutative algebra A is a vertex algebra, with field map given by the algebra product

$$Y \; : \; A \otimes A \;\dot{\to}\; A \;\hookrightarrow\; A((z))$$

so it is independent of z, translation operator $T = 0$ and vacuum $|0\rangle = 1$.

2.3.2. **Differential algebras.** Every commutative algebra with derivation (A, ∂) is a vertex algebra with field map

$$Y(\alpha, z)\beta \;:=\; (e^{z\partial}\alpha) \cdot \beta, \tag{2.9}$$

translation operator $T = \partial$ and vacuum $|0\rangle = 1$.

Definition 2.3.3. A vertex algebra is called *holomorphic* if the negative modes of the fields vanish, i.e. the field map factors as $Y \; : \; V \otimes V \to V[[z]]$, as is the case above.

In a holomorphic vertex algebra the fields $Y(\alpha, z)$ commute, since $(z - w)$ is not a zero divisor in $V[[z, w]]$. We can define a product on V by setting

$$\alpha \cdot \beta \;=\; \text{constant coefficient of } Y(\alpha, z)Y(\beta, z)|0\rangle$$

and show that T defines a derivation, showing that the category of holomorphic vertex algebras and commutative algebras with derivation are equivalent [FBZ, 1.4].

In a holomorphic vertex algebra, $Y(\alpha, z)Y(\beta, w) \sim 0$, i.e. product and normally ordered product of fields coincide. Thus one should think about the singular terms in the OPE as being the most interesting part of the vertex algebra structure.

2.3.4. **Jet spaces.** Recall from section 2.1.31 the physics heuristic that vertex algebras are meant to have something to do with loop spaces LT.

To make this precise, let T be a scheme of finite type over a field k and $D = \mathrm{Spec}k[[t]]$ the formal disk. Instead of loops into T, we should actually consider the *arc* (or *jet*) *space* of T, which is the (completed) mapping space $J_\infty T = \mathrm{Maps}(D, T)$,[5] which one can show is a scheme. It is usually of infinite type.

Note that D carries two vector fields: the *Euler* vector field coming from scaling

$$t\frac{\partial}{\partial t} \; : \; \sum a_n t^n \;\mapsto\; \sum n a_n t^n$$

[5]Its S-valued points are $\mathrm{Maps}(S \hat{\times} D, T)$, where $\hat{\times}$ is the completion of $S \times D$ along $S \times 0$.

and the *translation* vector field

$$\frac{\partial}{\partial t} \ : \ \sum a_n t^n \ \mapsto \ \sum n a_n t^{n-1}.$$

This induces two vector fields on $J_\infty T$, also denoted by $t\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t}$; we consider the latter. It follows that $\big(\mathcal{O}(J_\infty T), \frac{\partial}{\partial t}\big)$ is a commutative vertex algebra. If T is in addition Poisson, then $\mathcal{O}(J_\infty T)$ is a Poisson vertex algebra.

Jet spaces can be described explicitly, for instance

$$J_\infty \mathbf{A}^1 \ = \ \mathrm{Spec}\mathbf{C}[x_{-1}, x_{-2}, \dots]$$

a point of which should be thought of as a power series

$$t \ \mapsto \ \sum_{n \geqslant 0} a_n t^n \qquad\qquad a_n \in \mathbf{A}^1.$$

Then taking a generator x of $\mathcal{O}(\mathbf{A}^1)$, the value of x_{-n} at this point is $x(a_n)$, so that the derivation acts as $\frac{\partial}{\partial t} x_{-n} = (n+1)x_{-n-1}$. Likewise we have that for any finitely presented algebra A with generators x_i and relations f_j, the holomorphic vertex algebra $\mathcal{O}(J_\infty \mathrm{Spec} A)$ is

$$\mathbf{C}[x_{i,-n}]_{i \in I, n \geqslant 0} / (f_{j,-m})_{j \in J, m \geqslant 0}.$$

Here, we identify $x_{i,-n-1} = T^n x_{i,-1}$ to compute $f_{j,-m-1} = T^m f_{j,-1}$. See [AMo] for more.

2.3.5. **Aside: quantisation.** What is the correct notion of quantisation of a holomorphic vertex algebra?

In [Li1], Li constructed a canonical decreasing filtration on the underlying vector space of any vertex algebra V, given by

$$V_k \ = \ \mathrm{span}\{ \alpha_{1,-n_1-1} \cdots \alpha_{r,-n_r-1} |0\rangle \}_{n_i \geqslant 0, n_1 + \cdots + n_r \geqslant k}$$

so that $V_0 = V$. This filtration satisfies

$$T V_k \ \subseteq \ V_{k+1} \qquad\qquad (V_k)_n V_\ell \ \subseteq \ \begin{cases} V_{k+\ell-n-1} & \text{if } n < 0 \\ V_{k+\ell-n} & \text{if } n \geqslant 0 \end{cases}$$

and so we get a holomorphic vertex algebra structure on $\mathrm{gr}V$. V is called a *chiral quantisation* of scheme T if $\mathrm{gr}V \simeq \mathcal{O}(J_\infty T)$ as vertex algebras.

2.3.6. However, this associated graded actually carries more structure, a *Poisson vertex algebra* structure (see [AMo, §4]). This should be thought of as analogous to the fact that being Poisson is a smoking gun that a variety or algebra might be quantisable (e.g. deformation quantised). To be precise, if the scheme T has a Poisson structure then this endows $\mathcal{O}(J_\infty T)$ with a Poisson vertex algebra structure.

2.3.7. **Affine vertex algebras.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$ be an ad-invariant bilinear form. The affine vertex algebra $V_k(\mathfrak{g})$ which we will define shortly will be generated in the sense of the Reconstruction Theorem by fields $\alpha(z)$ depending linearly on $\alpha \in \mathfrak{g}$, subject to the OPEs

$$\alpha(z)\beta(w) \ \sim \ \frac{\kappa(\alpha,\beta)}{(z-w)^2}\mathrm{id} \ + \ \frac{[\alpha,\beta](w)}{z-w}.$$

These OPEs imply that the coefficients α_n of $\alpha(z) = \sum \alpha_n z^{-n-1}$ satisfy the commutation relations of the affine Lie algebra $\widehat{\mathfrak{g}}$, and the vertex algebra is a highest weight representation of $\widehat{\mathfrak{g}}$ of level κ (see Appendix B) and highest weight vector $|0\rangle$.

We now define the *affine vertex algebra* (or *current algebra*) to be the maximal highest weight representation of level κ: the Verma module of level κ

$$V_\kappa(\mathfrak{g}) \ = \ \mathrm{Ind}_{\mathfrak{g}[t]\oplus kc}^{\widehat{\mathfrak{g}}} k$$

which admits a PBW basis in terms of a basis $\alpha_1, ..., \alpha_r$ of $\mathfrak{g}$:

$$V_\kappa(\mathfrak{g}) \ \simeq \ k[\alpha_{1,-n}, ..., \alpha_{r,-n}]_{n\geqslant 1}.$$

The field of $\alpha_{i,-1}|0\rangle$ is the power series valued endomorphism $\alpha_i(z) = \sum \alpha_n z^{-n-1}$, which together generate the vertex algebra; thus the field map is determined by the Reconstruction Theorem. Moreover, T is uniquely determined by the axiom $[T, \alpha_i(z)] = \partial_z \alpha_i(z)$. It is a chiral quantisation of the Poisson space $\mathfrak{g}^*$:

$$\mathrm{gr} V_\kappa(\mathfrak{g}) \ \simeq \ \mathcal{O}(J_\infty \mathfrak{g}^*).$$

To contrast jet space examples, we emphasise that for positive n the operators $\alpha_{k,n}$ will *not* all act trivially so long as κ is not zero. This is true even when $\mathfrak{g}$ is abelian.

2.3.8. By the reconstruction theorem, any highest weight representation of $\widehat{\mathfrak{g}}$ carries a vertex algebra structure. An important example is the maximal quotient $L_\kappa(\mathfrak{g})$ of $V_\kappa(\mathfrak{g})$, called the *simple* affine vertex algebra.

2.3.9. If $\kappa = k \cdot \kappa_0/2h^\vee$ is a multiple of the normalised Killing form $\kappa_0/2h^\vee$ (with κ_0 the Killing form and $h^\vee$ is the dual Coxeter number), then we often write $V_k(\mathfrak{g}), L_k(\mathfrak{g})$ in place of $V_\kappa(\mathfrak{g}), L_\kappa(\mathfrak{g})$.

2.3.10. **Lattice vertex algebras.** Let $\Lambda \subseteq \mathbf{C}^n$ be a lattice and κ be a $2\mathbf{Z}$ valued bilinear form. The lattice vertex algebra, first defined by Borcherds [Bo1], has associated graded

$$\mathcal{O}(J_\infty \mathbf{C}^n) \otimes \mathbf{C}[\Lambda].$$

One should think of this as being functions on space of formal loops $D^\times \to \mathbf{C}^n/\Lambda$. Indeed, there are Λ many homotopy classes of loops, and the space of contractible maps form the jet space $J_\infty \mathbf{C}^n = \mathrm{Maps}(D, \mathbf{C}^n)$.

As a vector space, the lattice vertex algebra is

$$V_\Lambda \;=\; V_1(\mathbf{C}^n) \otimes \mathbf{C}[\Lambda]$$

where $\mathfrak{t} = \mathbf{C}^n$ is viewed as an abelian Lie algebra. To make it into a vertex algebra, we first put on $V_1(\mathbf{C}^n) \otimes e^\lambda$ the structure of Verma module of $\hat{\mathfrak{t}}$ of level one and weight $\kappa(\lambda, -) \in \mathfrak{t}^*$.

This uniquely determines the rest of the fields. For any $x \in \mathfrak{t}$ and $\lambda \in \Lambda$, writing $x(z)$ and $e^\lambda(z)$ for the fields of $x_{-1}|0\rangle \otimes 1$ and $1 \otimes e^\lambda$, the OPE formula gives

$$x(z)e^\lambda(w) \;\sim\; \kappa(\lambda, x)\frac{e^\lambda(w)}{z-w}$$

or equivalently, $[x_n, e^\lambda(w)] = \kappa(\lambda, x) w^n e^\lambda(w)$, which forces

$$e^\lambda(z) \;=\; \pm e^\lambda \cdot z^{\lambda_0} e^{\sum_{k<0} \frac{z^{-k}}{k}\lambda_k} e^{-\sum_{k>0} \frac{z^{-k}}{k}\lambda_k}.$$

Here we have written e^λ for the group algebra action and $\lambda(z) = \sum \lambda_n z^{-n-1}$ as the field given by viewing $\Lambda \subseteq \mathfrak{t}$. Finally, the sign on $V_\kappa(\mathfrak{g}) \otimes e^\mu$ is given by the component $c_{\lambda,\mu}$ of any two cocycle $c : \Lambda \times \Lambda \to \{\pm 1\}$, i.e.

$$c_{\lambda,0} \;=\; c_{0,\mu} \;=\; 1, \qquad\qquad c_{\lambda,\mu}c_{\lambda+\mu,\nu} \;=\; c_{\lambda,\mu+\nu}c_{\mu,\nu},$$

which satisfies in addition

$$c_{\lambda,\mu}c_{\mu,\lambda} \;=\; (-1)^{\kappa(\lambda,\lambda)\kappa(\mu,\mu)+\kappa(\lambda,\mu)}.$$

2.3.11. To put these formulae into context, consider the logarithmic power series

$$\int \lambda(z) \;=\; \sum_{n\neq 0} \frac{1}{n}\lambda_n z^{-n} + \log z \lambda_{-1} + \lambda.$$

Then up to signs, we have that $\lambda(z) = \partial(\int \lambda(z))$ and $e^\lambda(z) \;=\; \exp(\int \lambda(z))$. One might expect that this can be make precise using the framework of logarithmic vertex algebras, see [BVi].

2.3.12. **Borcherds' bicharacter construction.** We have seen that differential algebras A give holomorphic vertex algebras. Borcherds noticed that if there is additionally a cocommutative coproduct Δ, this construction can be *twisted*. By what? A *bicharacter* of a commutative, cocommutative bialgebra A is a linear map $r : A \otimes A \to k((z^{-1}))$ with

$$r(a \otimes 1) \;=\; r(1 \otimes a) \;=\; 1,$$

$$r(ab \otimes c) \;=\; r(a \otimes c) r(b \otimes c), \qquad r(a \otimes bc) \;=\; r(a \otimes b) r(a \otimes c),$$
$$r(\partial a \otimes c) \;=\; \partial_z r(a \otimes b), \qquad r(a \otimes \partial c) \;=\; -\partial_z r(a \otimes b)$$

and the *symmetry* axiom $r(a \otimes b, z) = r(b \otimes a, -z)$.

Theorem 2.3.13. [Bo2] *Let A be a cocommutative, commutative bialgebra with a symmetric bicharacter r. The formula*

$$Y(\alpha, z)\beta \;=\; m\left((e^{z\partial} \otimes \mathrm{id}) \cdot r_{24} \cdot \Delta\alpha \otimes \Delta\beta\right) \;=\; \sum (e^{z\partial}\alpha_{(1)})\beta_{(1)} r(\alpha_{(2)} \otimes \beta_{(2)}) \tag{2.10}$$

defines a vertex algebra structure on A.

The proof proceeds very similarly to the proof of Theorem 2.6.21 about the vertex algebra structure on the homology of moduli spaces. In that case, the bialgebra A is the homology of moduli space with product $\oplus_*$ given by the direct sum map and coproduct the dual of the cup product. The bicharacter is cap product (see below) with a cohomology valued power series $\Psi \in A^\vee \otimes A^\vee((z^{-1}))$.

2.3.14. This formula (2.10) involves what should be called the *cap product* action of r on $A^{\otimes 2}$:

$$r\cdot \;:\; A \otimes A \overset{\Delta\otimes\Delta}{\to} A \otimes A \otimes A \otimes A \overset{r_{24}}{\to} A \otimes A((z^{-1})).$$

Recall that for any cocommutative coalgebra C, the *cap product* action of $C^\vee$ is

$$C^\vee \otimes C \overset{\mathrm{id}\otimes\Delta}{\to} C^\vee \otimes C \otimes C \overset{\mathrm{ev}\otimes\mathrm{id}}{\to} C,$$

which agrees with the usual definition in topology when C is the homology of some space.

2.3.15. **Aside: graded, super, … vertex algebras.** One can define a vertex algebra in any k linear symmetric monoidal category $\mathcal{C}$ (see Appendix A), and taking $\mathcal{C} = \mathrm{Vect}$ gives back the usual definition. Some other variants:

1. $\mathbf{Z}$ *graded vertex algebras* are vertex algebras in the category of $\mathbf{Z}$ graded vector spaces $\mathcal{C} = \mathrm{Vect}_{\mathbf{Z}}$, where we grade $V((z))$ by setting $|z| = -2$.

2. *Vertex superalgebras* are vertex algebras in the category of super vector spaces $\mathcal{C} = \mathrm{Vect}_{\mathbf{Z}/2}$, as before setting $|z| = 0 \mod 2$.

In either case, whenever α is homogeneous of degree $|\alpha|$, since Y is grading preserving, its modes have degree

$$|\alpha_n| \;=\; |\alpha| - 2n - 1.$$

It follows that $|0\rangle$ has degree zero and T has degree one. Finally, weak commutativity translates in this case

$$(z-w)^n \big(Y(\alpha,z)Y(\beta,w) - (-1)^{|\alpha|\cdot|\beta|}Y(\beta,w)Y(\alpha,z)\big) \;=\; 0$$

for homogeneous elements α, β and $n \gg 0$.

2.3.16. We can define graded and super analogues of the examples in this section, for instance affine Lie superalgebras attached to a finite dimensional super Lie algebra with an ad-invariant bilinear form, super lattice vertex algebras attached to a super lattice with bilinear form, etc.

2.3.17. **Other examples.** There are other examples which we will not touch in this Thesis. Some of the most important examples are:

- *Virasoro.* The Virasoro Vir_c attached to $c \in k$ is generated by a single field $T(z)$ satisfying OPE

$$T(z)T(w) \;\sim\; \frac{c/2}{(z-w)^4} \;+\; \frac{2T(w)}{(z-w)^2} \;+\; \frac{\partial_w T(w)}{z-w}$$

  which implies that the coefficients of $T(z)$ define a representation of the Virasoro algebra of charge c. In particular, any highest weight reprentation gives a vertex algebra, like the Verma module Vir_c and its irreducible quotient $L(c)$. See [DR, Wil] for more.

- *W algebras.* The Heisenberg and Virasoro are the $\mathfrak{g} = \mathfrak{gl}_1, \mathfrak{sl}_2$ examples of a general construction of the *W vertex algebra*

$$\mathcal{W}^\kappa(\mathfrak{g}, f)$$

  attached to a finite dimensional Lie algebra $\mathfrak{g}$ and nilpotent element $f \in \mathfrak{g}$. It admits a grading such that as a vertex algebra $\mathrm{gr}\mathcal{W}^\kappa(\mathfrak{g}, f) \simeq \mathcal{O}(J_\infty \mathcal{S}_f)$ is functions on the jet space of the Slodowy slice $\mathcal{S}_f \subseteq \mathfrak{g}^*$ associated to f. When $f = 0$ it reduces to the affine vertex algebra $V_\kappa(\mathfrak{g})$. It is meant to be a vertex analogue of the "quantum Hamiltonian reduction" definition of finite W algebras. See [Ar1, Ar2] for more.

- *Vertex algebras from 2d SCFTs.* New vertex algebras are constantly coming out of the physics literature. One of the most interesting in recent years are the vertex algebras arising from $2d$ SCFTs studied by Beem, Rastelli and others (see for instance [BLL]). Another interesting example is the triplet W algebra, studied by Adamović and Milas in [AMi].

2.4 Properties of factorisation algebras

Continuing section 2.1.1, we give more information about factorisation algebras. This helps motivate the vertex algebraic constructions in this Thesis, but the vertex algebras we cover in this Thesis have not (yet) been found naturally as factorisation algebras, so the reader may skip this section if they want.

2.4.1. **Unital versions.** One disadvantage of the usual Ran space is that the chiral monoidal structure on $\mathrm{Sh}(\mathrm{Ran}X)$ is not unital. Note that for any non-unital algebra A we only have maps

$$\otimes_{i\in I}A \;\to\; \otimes_{j\in J}A$$

for any surjection $I \twoheadrightarrow J$, whereas if it is unital, by inserting the unit we also get maps

$$\otimes_{i\in I_1}A \;\to\; \otimes_{i\in I_2}A$$

for any subset $I_1 \subseteq I_2$. In this way we get a map $\otimes_{i\in I}A \to \otimes_{j\in J}A$ for any map $I \to J$.

We define the *unital Ran space* as the lax colimit

$$\mathrm{Ran}_{un}X \;=\; \mathrm{colim}_{\mathrm{FSet}^{op},\mathrm{FSet}^{surj,op}}X^{(-)}$$

where FSet is the category of all finite sets (not necessarily empty) and all maps. Its S points are the category with objects finite subsets of $\mathrm{Maps}(S,X)$ and morphisms inclusions of sets. See [Ras, §4.9] or [CF, Def. 10.3.3]. The objects in the category

$$\mathrm{Sh}(\mathrm{Ran}_{un}X) \;=\; \lim_{\mathrm{FSet},\mathrm{FSet}^{surj}}\mathrm{Sh}(X^I)$$

consist of a sheaf $V_I \in \mathrm{Sh}(X^I)$ for every nonempty set I and compatible maps $\Delta^!_{I/J}V_I \to V_J$ for every map $I \to J$ (which induces $\Delta_{I/J} : X^J \to X^I$), such that they are isomorphisms $\Delta^!_{I/J}V_J \xrightarrow{\sim} V_I$ for surjections $I \twoheadrightarrow J$.

2.4.2. We can define unital analogues of the $*$ and chiral correspondences (2.4) and (2.5), and hence define unital factorisation and chiral algebras. Explicitly, a *unital factorisation algebra* is a factorisation algebra $V \in \mathrm{Sh}(\mathrm{Ran}X)$ along with compatible maps $\Delta^!_{I_1/I_2}V_{I_1} \to V_{I_2}$ for all inclusions $I_1 \subseteq I_2$ of finite subsets. Thus for instance if $V_\varnothing = k \in \mathrm{Sh}(\mathrm{pt})$ we get a map

$$\omega_{X^I} \;=\; \Delta^!_{X^I/\mathrm{pt}}k \;\to\; V_I.$$

2.4.3. **Factorisation spaces.** We similarly get notions of a prestack or quasicoherent sheaf over $\mathrm{Ran}X$ and $\mathrm{Ran}_{un}X$. For instance,

$$\mathrm{QCoh}(\mathrm{Ran}X) \;=\; \lim_{I\in\mathrm{FSet}^{surj}}\mathrm{QCoh}(X^I) \qquad\qquad \mathrm{PreStk}_{/\mathrm{Ran}X} \;=\; \lim_{I\in\mathrm{FSet}^{surj}}\mathrm{PreStk}_{/X^I}$$

the limit taken over the pullback maps $\Delta^*_{I/J}$. Thus, by base change $\pi_*j_*j^*$ endows these categories with the *chiral* symmetric monoidal product, denoted $\otimes^{ch}$ and $\times^{ch}$ respectively.[6] As before, (co)commutative (co)algebras with respect to these symmetric monoidal structures whose structure maps are isomorphisms when restricted to $(X^{I_1}\times X^{I_2})_{disj} \hookrightarrow X^I$ are called *factorisation (co)algebras*. See [CP, Ras].

2.4.4. Factorisation algebras in $\mathrm{PreStk}_{/\mathrm{Ran}X}$ are called *factorisation spaces*. For instance, $\mathrm{Ran}X$ is itself a unital factorisation space. For an algebra $Y \in (\mathrm{PreStk}_{/\mathrm{Ran}X}, \times^{ch})$, the factorisation condition is equivalent to

$$j^*(Y\times Y) \;\to\; j^*\pi^*\pi_*j_*j^*(Y\times Y) \;=\; j^*\pi^*(Y\times^{ch} Y) \;\to\; j^*\pi^*Y$$

being an equivalence, where the first map comes from applying the unit of the adjunction. In particular, for any factorisation space Y this means that we get a pullback

$$\begin{tikzcd}[column sep=small]
 & & (Y\times Y)_{disj} \arrow[dll, "\bar{\jmath}"'] \arrow[d, "\mathring{f}"] \arrow[drr, "\overline{\pi\jmath}"] & & \\
Y\times Y \arrow[d, "f\times f"] & & (\mathrm{Ran}X\times\mathrm{Ran}X)_{disj} \arrow[dll, "j"'] \arrow[drr, "\pi j"] & & Y \arrow[d, "f"] \\
\mathrm{Ran}X\times\mathrm{Ran}X & & & & \mathrm{Ran}X
\end{tikzcd} \tag{2.11}$$

Using this correspondence we can thus repeat all the above with Y in place of $\mathrm{Ran}X$, giving definitions of factorisation algebras, spaces, etc. over Y.

2.4.5. For instance, assume the map f is ind-schematic on reduced prestacks, so that f_* is defined on $\mathrm{Sh}(-)$. Then given factorisation coalgebra $\mathcal{B}\in\mathrm{Sh}(Y)$, by applying f_* to the structure map

$$\mathcal{B} \;\to\; \overline{\pi}_*\bar{\jmath}_*\bar{\jmath}^!(\mathcal{B}\boxtimes\mathcal{B})$$

and applying base change, we see that we get a factorisation algebra $f_*\mathcal{B}$. Similarly, in the quasicoherent case for any f, if we have a factorisation coalgebra $\mathcal{E}\in\mathrm{QCoh}(Y)$, then base change we get a factorisation algebra structure on $f_*\mathcal{E}$.

[6]For a map $f: S\to T$ we write $f^*:\mathrm{PreStk}_{/T}\to\mathrm{PreStk}_{/S}$ for the pullback map on stacks, which has left adjoint the forgetful functor f_*.

2.4.6. All of this can be more conceptually described in the language of factorisation categories, see [Ras].

2.4.7. **Flat connection.** We notice the following remark made by Lurie. A unital factorisation space whose structure map $Y \to \mathrm{Ran}_{un}X$ is flat admits a flat connection, i.e. is the pullback of a map to $\mathrm{Ran}_{un}X_{dR}$. It is enough to give compatible isomorphisms between fibres of Y over infinitesimally close points $\mathrm{Ran}_{un}X$. Given two tuples of points $x, y : S \to X^I$ inducing the space S_{red} tuple, the map of spaces over S given by the unit

$$Y_x \;\to\; Y_{(x,y)} \;\leftarrow\; Y_y$$

are isomorphisms. Here Y_S denotes the fibre of Y above finite subset $S \subseteq X$. Indeed, they are maps of flat spaces which become isomorphisms after reducing the base, so are isomorphisms.

2.4.8. **Factorisation homology.** One advantage of factorisation algebras is that it gives a more conceptual definition of conformal block. The *factorisation* (or *chiral*) *homology* of $\mathcal{A} \in \mathrm{Sh}(\mathrm{Ran}X)$ is

$$\mathrm{H}^{ch}_{\bullet}(X, \mathcal{A}) \;=\; p_! \mathcal{A} \;=\; \mathrm{H}^{\bullet}_{c}(\mathrm{Ran}X, \mathcal{A})$$

where $p : \mathrm{Ran}X \to \mathrm{pt}$. If X is proper, then so is $\mathrm{Ran}X$, i.e. $p_! = p_*$ preserves colimits and factorisation homology can be computed as $\mathrm{colim}_{I \in \mathrm{FSet}^{surj}} p_{X^I *} \mathcal{A}_I$.

2.4.9. $\mathrm{H}^{ch}_0(X, \mathcal{B})$ is usually what is called the space of conformal blocks, see [BD2].

2.4.10. In the classical definition [FBZ, Def. 9.2.7], the space of *conformal blocks* of a vertex algebra V at a point of a curve $x \in X$ is the dual to a space of coinvariants

$$C(X, x, V) \;=\; (V / U_{X \backslash x} V)^{\vee}$$

and so admits a map $C(X, x, V) \to V^{\vee}$. If the vertex algebra $V = \mathcal{A}_x$ is the fibre of a factorisation algebra $\mathcal{A}$ on a smooth curve X, applying adjunction to the proper map $i : x \hookrightarrow \mathrm{Ran}X$ gives

$$\mathcal{A}_x[-2] \;=\; i^! \mathcal{A} \;=\; p_! i_! i^! \mathcal{A} \;\to\; p_! \mathcal{A} \;=\; \mathrm{H}^{ch}_{\bullet}(X, \mathcal{A}).$$

Thus for each $x \in X$ we get a map $\mathcal{A}_x \to \mathrm{H}^{ch}_{\bullet+2}(X, \mathcal{A})$, which one expects to factor through taking coinvariants. Likewise, taking a collection of n distinct points $i : (x_1, ..., x_n) \hookrightarrow \mathrm{Ran}X$ gives a map

$$\mathrm{H}^{ch}_{\bullet+2n}(X, \mathcal{A})^{\vee} \;\to\; (\mathcal{A}_{x_1} \otimes \cdots \otimes \mathcal{A}_{x_n})^{\vee}$$

which one expects might factor through a map to conformal blocks $\mathrm{H}^{ch}_{\bullet+2n}(X, \mathcal{A}) \to C(X, x_1, ..., x_n, V, ..., V)$.

2.5 Examples of factorisation algebras

2.5.1. **Affine factorisation algebras.** Let X be a curve and G a reductive group over k. The *Beilinson Drinfeld Grassmannian* $\mathrm{Gr}_{G,X}$ is the prestack given by functor of points

$$\mathrm{Gr}_{G,X}(S) \;=\; \big\{(x_1,...,x_n,P,\varphi) \;:\; n \geqslant 0,\; x_i \in X(S),\; P \in \mathrm{Bun}_G(X_S),\; \varphi : P|_{X_S \backslash \{x_1,\ldots,x_n\}} \simeq \mathrm{triv}\big\}$$

where X_S is the base change to S and $\mathrm{Bun}_G(X_S)$ is the groupoid of G torsors on X_S. It admits a map to $\mathrm{Ran}_{un}X$ by forgetting everything but the subset of X. Then $\mathrm{Gr}_{G,X}$ is a factorisation space with factorisation structure[7]

$$j_*j^*(\mathrm{Gr}_{G,X^{I_1}} \times \mathrm{Gr}_{G,X^{I_2}}) \;\to\; \mathrm{Gr}_{G,X^I}$$

taking two G bundles with trivialisations along $X\backslash\{x_{i_1}\}_{i_1 \in I_1}$ and $X\backslash\{x_{i_2}\}_{i_2\in I_2}$ respectively and using the trivialisations to glue them along $X\backslash\{x_i\}_{i\in I}$, where $I = I_1 \cup I_2$.[8]

2.5.2. This factorisation space structure is unital, with unit

$$\mathrm{triv} \;:\; \mathrm{Ran}X \;\to\; \mathrm{Gr}_{G,X}$$

given by the trivial G bundle. Write $\pi : \mathrm{Gr}_{G,X} \to \mathrm{Ran}X$ for the retraction.

2.5.3. One can show that the map $\mathrm{Gr}_{G,X} \;\to\; \mathrm{Ran}_{un}X$ is ind-schematic [BD1], so the fibres $\mathrm{Gr}_{G,x_1,\ldots,x_n}$ above a finite subset of $X(k)$ form an ind-scheme.

2.5.4. To form the affine factorisation algebra, take as category of sheaves $\mathrm{Sh}(-)$ the category of holonomic D modules. In particular, it admits a forgetful functor to $\mathrm{QCoh}(-)$, and so we can take

$$\mathrm{Sh}(\mathrm{Ran}X) \;\overset{\mathrm{triv}_*}{\to}\; \mathrm{Sh}(\mathrm{Gr}_{G,X}) \;\to\; \mathrm{QCoh}(\mathrm{Gr}_{G,X}) \;\overset{\otimes\mathcal{L}}{\to}\; \mathrm{QCoh}(\mathrm{Gr}_{G,X}) \;\overset{\pi_*}{\to}\; \mathrm{QCoh}(\mathrm{Ran}X) \tag{2.12}$$

where $\mathcal{L} \in \mathrm{QCoh}(\mathrm{Gr}_{G,X})$. If $\mathcal{L}$ is factorisable, meaning we have compatible isomorphisms $j^*(\mathcal{L} \boxtimes \mathcal{L}) \simeq j^*\pi^*\mathcal{L}$, or equivalently $j^*(\mathcal{L}_{I_1} \boxtimes \mathcal{L}_{I_2}) \simeq j^*\mathcal{L}_I$, then for any factorisation coalgebra $\mathcal{A} \in \mathrm{QCoh}(\mathrm{Gr}_{G,X})$ we get a map

$$j^*\pi^*(\mathcal{A}\otimes\mathcal{L}) \;\simeq\; j^*\pi^*\mathcal{A}\otimes j^*\pi^*\mathcal{L} \;\to\; j^*(A\boxtimes\mathcal{A})\otimes j^*(\mathcal{L}\boxtimes\mathcal{L}) \;\simeq\; j^*(\mathcal{A}\otimes\mathcal{L})^{\boxtimes 2}$$

which by adjunction is the same as

$$\mathcal{A}\otimes\mathcal{L} \;\to\; \pi_*j_*j^*(\mathcal{A}\otimes\mathcal{L})^{\boxtimes 2}.$$

Thus: if $\mathcal{A}$ is a factorisation coalgebra, so too is $\mathcal{A}\otimes\mathcal{L}$.

[7] A factorisation structure is an algebra map $\pi_*j_*j^*(\mathrm{Gr}_{G,X}\times\mathrm{Gr}_{G,X}) \to \mathrm{Gr}_{G,X}$, which by adjunction is the same as a map $j_*j^*(\mathrm{Gr}_{G,X}\times\mathrm{Gr}_{G,X}) \to \pi^*\mathrm{Gr}_{G,X}$.

[8] To be precise, we take these two G bundles P_1, P_2 to $P = P_1 \amalg_{\mathrm{triv}|_{X\backslash\{x_i\}_{i\in I}}} P_2$.

2.5.5. We finally note that $\mathrm{Gr}_{G,X}$ is unital so the projection π comes from a pullback of a factorisation space $\mathrm{Gr}^{\nabla}_{G,X} \to \mathrm{Ran}X_{dR}$. Thus if $\mathcal{L}$ is also unital then by section (2.4.7) we can lift the above to a sequence

$$\mathrm{Sh}(\mathrm{Ran}X_{dR}) \overset{\mathrm{triv}_*}{\to} \mathrm{Sh}(\mathrm{Gr}^{\nabla}_{G,X}) \to \mathrm{QCoh}(\mathrm{Gr}^{\nabla}_{G,X}) \overset{\otimes\mathcal{L}}{\to} \mathrm{QCoh}(\mathrm{Gr}^{\nabla}_{G,X}) \overset{\pi_*}{\to} \mathrm{QCoh}(\mathrm{Ran}X_{dR}). \quad (2.13)$$

2.5.6. There is a distinguished factorisable line bundle $\mathcal{L}_G \in \mathrm{Pic}(\mathrm{Gr}_{G,X})$ called the *determinant bundle*, see [FBZ]. Taking the constant holonomic D module $k \in \mathrm{Sh}(\mathrm{Ran}X_{dR})$ and pushing it forward to give $\delta = \mathrm{triv}_* k$, we get the *affine factorisation algebra* of level $k \in \mathbf{Z}$:

$$\mathcal{A}_G \; = \; \pi_*(\delta \otimes \mathcal{L}_G^{\otimes k}).$$

Since δ is supported on $\mathrm{Ran}X_{dR} \subseteq \mathrm{Gr}^{\nabla}_{G,X}$, as an element of $\mathcal{D}\text{-Mod}(\mathrm{Ran}X)$ this does not depend on k: only its factorisation algebra structure is affected by the twist by $\mathcal{L}_G$.

2.6 Homology of moduli spaces

2.6.1. In this Thesis we will focus on a new class of vertex algebras discovered by Joyce [Jo2]. The idea is that if $\mathcal{A}$ is a (abelian, triangulated or dg) category, the extra structure this imposes on its moduli stack of objects corresponds (after taking homology) to a vertex algebra structure. For instance, the singularities in the operator product expansions and correlation functions are controlled by *extensions* in $\mathcal{A}$.

Instead of worrying about what it means for a space to be a "moduli spaces of objects" in an (abelian, triangulated or dg) category, we will instead just say what sort of space admits a vertex algebra structure on its homology, this includes all standard examples of such moduli spaces. For more detail on the first question, see [TV].

2.6.2. We will build up the vertex algebra structure piece by piece: by adding more structure each step, we will build

1. A commutative algebra (Proposition 2.6.4).
2. A commutative algebra with derivation (Definition 2.6.13).
3. A vertex algebra (Theorem 2.6.17).

Note that, by section 2.3, these are all examples of vertex algebras.

2.6.3. **Algebra.** What should taking direct sums of objects correspond to on the level of spaces? Let $\mathcal{M}$ be a space with marked point $0 : \mathrm{pt} \to \mathcal{M}$ and a map

$$\oplus \;:\; \mathcal{M} \times \mathcal{M} \;\to\; \mathcal{M}$$

making it into a commutative monoid in the category of pointed spaces. We will sometimes call $\oplus$ the *direct sum* map. It is easy to see that

Proposition 2.6.4. *If* $(\mathcal{M}, 0)$ *is a commutative monoid, its cohomology* $\mathrm{H}^\bullet(\mathcal{M})$ *is a supercommutative, cocommutative graded Hopf algebra. Its algebra and coalgebra structure are given by cup product and* $\oplus^*$*, its unit and counit are* 1 *and* 0^**, and its antipode is* $S = (-1)^{\deg}$.

Corollary 2.6.5. *If* $(\mathcal{M}, 0)$ *is a commutative monoid, its homology* $\mathrm{H}_\bullet(\mathcal{M}) = \mathrm{H}^\bullet(\mathcal{M})^\vee$ *is a commutative, supercocommutative graded Hopf algebra.*

In particular, $\mathrm{H}_\bullet(\mathcal{M})$ (and $\mathrm{H}^\bullet(\mathcal{M})$) are trivial examples of vertex (co)algebras.

2.6.6. If $(\mathcal{M}, 0)$ is a commutative monoid, write ${}_{\mathcal{M}}\mathrm{Rep}$ for the symmetric monoidal category of left modules over $\mathrm{H}^\bullet(\mathcal{M})$, similarly $\mathrm{Rep}_{\mathcal{M}}$ and ${}_{\mathcal{M}}\mathrm{Rep}_{\mathcal{M}}$. Their symmetric monoidal structures are given by the cocommutative coproduct.

2.6.7. An example of a commutative monoid in the category of pointed Artin stacks is $\mathrm{B}\mathbf{G}_m = (\mathrm{B}\mathbf{G}_m, \mathrm{triv})$, the classifying space of line bundles with marked point the trivial line bundle. Its monoid structure is given by tensor product of line bundles

$$\otimes \;:\; \mathrm{B}\mathbf{G}_m \times \mathrm{B}\mathbf{G}_m \;\to\; \mathrm{B}\mathbf{G}_m \qquad\qquad (\mathcal{L}, \mathcal{L}') \;\mapsto\; \mathcal{L} \otimes \mathcal{L}'$$

so $\mathrm{B}\mathbf{G}_m$ is even an abelian group object. As a Hopf algebra its cohomology is the universal enveloping algebra of a one dimensional Lie algebra $\mathfrak{t} = k \cdot \tau$ in degree two:

$$\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m) \;=\; U(\mathfrak{t}) \;\simeq\; k[\tau].$$

The generator τ is the first chern class of the tautological line bundle γ on $\mathrm{B}\mathbf{G}_m$.[9] Dually,

$$\mathrm{H}_\bullet(\mathrm{B}\mathbf{G}_m) \;=\; U(\mathfrak{t}^\vee) \;\simeq\; k[\check{\tau}]$$

where $\mathfrak{t}^\vee = k \cdot \check{\tau}$ has $\check{\tau}(\tau) = 1$.

[9] A map into $\mathrm{B}\mathbf{G}_m$ is defined by what the pullback of γ is, so we can *define* $\otimes$ by

$$\otimes^*\gamma \;=\; \gamma \boxtimes \gamma.$$

It follows that $\otimes^*\tau = \tau \otimes 1 + 1 \otimes \tau$, and multiples of τ are the only primitive elements.

2.6.8. **Algebra with derivation.** What should the categories being k linear over some field k correspond to on the level of spaces? It should give the commutative monoid $(\mathcal{M}, 0)$ an action by the group object $\mathrm{B}\mathbf{G}_m$

$$\mathrm{act} \ : \ \mathrm{B}\mathbf{G}_m \times \mathcal{M} \ \to \ \mathcal{M}.$$

This action then defines for us a derivation

Proposition 2.6.9. *Let* $(\mathcal{M}, 0)$ *be a commutative monoid with an action of* $\mathrm{B}\mathbf{G}_m$*. Then*

$$t \ = \ \mathrm{act}_*(\tau \otimes \mathrm{id})$$

defines a derivation on $\mathrm{H}_\bullet(\mathcal{M})$*. Dually,*

$$t \ = \ (\tau \otimes \mathrm{id})\mathrm{act}^*$$

defines a coderivation on $\mathrm{H}^\bullet(\mathcal{M})$*.*

This follows from the following Lemma about Hopf algebras, because the element t is primitive. Notice that $\mathrm{H}^\bullet(\mathcal{M})$ is a Hopf algebra internal to $\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)$-coMod. Likewise, $\mathrm{H}_\bullet(\mathcal{M})$ is a Hopf algebra internal to $\mathrm{H}_\bullet(\mathrm{B}\mathbf{G}_m)$-Mod.

Lemma 2.6.10. *Let A be a graded Hopf algebra with finite dimensional graded pieces, so that its contragredient dual $A^\vee$ is also a Hopf algebra. We have a functor*

$$A\text{-coMod} \ \to \ \text{Mod-}A^\vee \qquad\qquad M \ \mapsto \ M$$

acting trivially on the underlying vector space. If M is an algebra internal to A-coMod*, the primitive elements of $A^\vee$ act on M as derivations.*

Proof. If M is a left A comodule, we get a right $A^\vee$ module structure by

$$A^\vee \otimes M \ \overset{\mathrm{id}\otimes\Delta}{\to} \ A^\vee \otimes A \otimes M \ \overset{\mathrm{ev}\otimes\mathrm{id}}{\to} \ k \otimes M \ \simeq \ M.$$

For the second part, we claim that if $f \in A^\vee$ and $m, m' \in M$, then

$$f(m \cdot m') \ = \ \sum f_{(1)}(m) f_{(2)}(m')$$

using Sweedler notation.[10] This implies that f acts as a derivation if and only if it is primitive. To show the claim, we follow Grinberg and Reiner [GrR], and write

$$f(m \cdot m') \ = \ (f \otimes \mathrm{id})\Delta(m \cdot m') \ = \ (f \otimes \mathrm{id}) \sum a_{(1)}a'_{(1)} \otimes m_{(2)}m'_{(2)} \ = \ \sum f(a_{(1)}a'_{(1)}) \otimes m_{(2)}m'_{(2)}.$$

[10]That is, we write $\Delta f = \sum f_{(1)} \otimes f_{(2)}$ and $\Delta m = \sum a_{(1)} \otimes m_{(2)}$.

The right hand side is

$$\sum(\Delta f)(a_{(1)} \otimes a'_{(1)}) \otimes m_{(2)}m'_{(2)} \;=\; \sum f_{(1)}(a_{(1)})f_{(2)}(a'_{(1)}) \otimes m_{(2)}m'_{(2)} \;=\; \sum f_{(1)}(m)f_{(2)}(m').$$

$\square$

Note that in our case $A = \mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)$ is cocommutative, so the distinction between left and right (co)modules in the above disappears.

2.6.11. It follows that $\mathrm{H}_\bullet(\mathcal{M})$ (and $\mathrm{H}^\bullet(\mathcal{M})$) are commutative (co)algebras with derivation, so define holomorphic vertex (co)algebras. We will now give a more explicit formula.

Lemma 2.6.12. *We have* $\mathrm{act}^* = \exp(\check{\tau} \otimes t)$ *as maps* $\mathrm{H}^\bullet(\mathcal{M}) \to \mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m) \otimes \mathrm{H}^\bullet(\mathcal{M})$.

Proof. Note that $\check{\tau}^n(\tau^n) = n!$, where the product on cohomology and homology comes from cup product and tensor product, respectively. To prove this, writing

$$\otimes_n \;:\; \mathrm{B}\mathbf{G}_m^n \;\to\; \mathrm{B}\mathbf{G}_m$$

for the n fold tensor product, we have $\oplus_n^* \gamma = \gamma_1 \otimes \cdots \otimes \gamma_n$ where γ_i is the pullback of γ along the ith projection $\mathrm{B}\mathbf{G}_m^n \to \mathrm{B}\mathbf{G}_m$. Thus,

$$\begin{aligned} \check{\tau}^n(\tau^n) \;&=\; \otimes_{n,*}(\tau \otimes \cdots \otimes \tau)(c_1(\gamma)^n) \;=\; (\tau \otimes \cdots \otimes \tau)(c_1(\gamma \boxtimes \cdots \boxtimes \gamma)^n) \\ &=\; (\tau \otimes \cdots \otimes \tau)(c_1(\gamma_1) + \cdots + c_1(\gamma_n))^n \;=\; n! \end{aligned}$$

since $\tau(c_1(\gamma)) = \tau(\check{\tau}) = 1$. We now prove the Lemma. We have

$$\exp(\check{\tau} \otimes t)\alpha \;=\; \sum_{n \geqslant 0} \check{\tau}^n \otimes \frac{t^n}{n!}\alpha$$

so all that we need to show is that the $\check{\tau}^n$ coefficient of $\mathrm{act}^*\alpha$ is $t^n\alpha/n!$, or equivalently

$$(\tau^n \otimes \mathrm{id})\mathrm{act}^* \;=\; t^n.$$

That the dual endomorphisms on homology are equal is clear: the dual of the right side is action by t applied n times, and the dual of the left side side is multiplication by τ^n, thus they are equal because $\mathrm{act} : \mathrm{B}\mathbf{G}_m \times \mathcal{M} \to \mathcal{M}$ is a group action. $\square$

Thus if we identify

$$\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m) \;\xrightarrow{\sim}\; k[z] \qquad\qquad \check{\tau} \;\mapsto\; z$$

we get that $\mathrm{act}^* = e^{zt}$. Repeating section 2.3.2,

Definition 2.6.13. Let $(\mathcal{M}, 0)$ be a commutative monoid with an action of $\mathrm{B}\mathbf{G}_m$. *Joyce's holomorphic vertex coalgebra* structure on cohomology is

$$\Delta(z^{-1}) \;:\; \mathrm{H}^\bullet(\mathcal{M}) \;\to\; \mathrm{H}^\bullet(\mathcal{M}) \otimes \mathrm{H}^\bullet(\mathcal{M})[z] \qquad\qquad \alpha \;\mapsto\; \mathrm{act}_1^* \oplus^* \alpha$$

where $\mathrm{act}_1 : \mathrm{B}\mathbf{G}_m \times \mathcal{M}^2 \to \mathcal{M}^2$ is induced by $\mathrm{B}\mathbf{G}_m$ acting on the first copy of $\mathcal{M}$.

Thus the vertex coalgebra formula simplifies in a way that the dual vertex algebra formula $Y(\sigma, z) = \oplus_*(e^{zt}\sigma \otimes \mathrm{id})$ on $\mathrm{H}_\bullet(\mathcal{M})$ does not.

2.6.14. **Vertex algebra.** Finally, consider the (derived) hom space $\mathrm{Hom}_{\mathcal{A}}(a, a')$, which is compatible with direct sum like

$$\mathrm{Hom}_{\mathcal{A}}(a_1 \oplus a_2, a') \;=\; \mathrm{Hom}_{\mathcal{A}}(a_1, a') \oplus \mathrm{Hom}_{\mathcal{A}}(a_2, a') \tag{2.14}$$

$$\mathrm{Hom}_{\mathcal{A}}(a, a_1' \oplus a_2') \;=\; \mathrm{Hom}_{\mathcal{A}}(a, a_1') \oplus \mathrm{Hom}_{\mathcal{A}}(a, a_2'), \tag{2.15}$$

and compatible k linearity in that the left and right action of $k^\times$ on $\mathrm{Hom}_{\mathcal{A}}(a, a')$ induced by its action on a and a' is a representation of weight 1 and -1, respectively. What should this structure correspond to on the level of moduli spaces?

It corresponds to a perfect complex

$$\theta \;\in\; \mathrm{Perf}(\mathcal{M} \times \mathcal{M})$$

(whose fibre above (a, a') should be thought of as being $\mathrm{Hom}_{\mathcal{A}}(a, a')$), which is *compatible* with respect to monoidal structure, meaning

$$(\oplus \times \mathrm{id})^*\theta \;=\; \theta_{13} \oplus \theta_{23} \tag{2.16}$$

$$(\mathrm{id} \times \oplus)^*\theta \;=\; \theta_{12} \oplus \theta_{13}, \tag{2.17}$$

and compatible with the $\mathrm{B}\mathbf{G}_m$ action in that it has *weight* 1 and -1 the left and right $\mathrm{B}\mathbf{G}_m$ action on $\mathcal{M} \times \mathcal{M}$, respectively, meaning

$$\mathrm{act}_1^*\theta \;=\; \gamma \boxtimes \theta, \qquad\qquad \mathrm{act}_2^*\theta \;=\; \gamma^{-1} \boxtimes \theta. \tag{2.18}$$

Here, $\theta_{ij} = \pi_{ij}^*\theta$ is the pullback by the projection $\pi_{ij} : \mathcal{M}^3 \to \mathcal{M}^2$ to the ith and jth factors, and act_i is the map $\mathrm{B}\mathbf{G}_m \times \mathcal{M}^2 \to \mathcal{M}^2$ given by acting on the ith factor.

2.6.15. We can combine all this structure using the bicharacter

$$\Psi(\theta, z) \ := \ \sum_{k \geqslant 0} z^{\mathrm{rk}\theta - k} c_k(\theta). \tag{2.19}$$

To understand this better, consider for the moment the case when θ is a vector bundle. Writing $x_1, ..., x_n$ for its chern roots, the above is $(z + x_1) \cdots (z + x_n)$. Identifying $\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m) \simeq k[z]$, we have

$$\Psi(\theta) \ = \ e(\gamma \boxtimes \theta) \ = \ e(\mathrm{act}_1^* \theta).$$

This clearly remains true for any weight one vector bundle (over a base with a $\mathrm{B}\mathbf{G}_m$ action so that the notion of weight makes sense).

One can show that $\Psi(\theta)$ defines a bicharacter on $\mathrm{H}_\bullet(\mathcal{M})$, which is a commutative vertex algebra by Definition 2.6.13, so can be Borcherds twisted by $\Psi(\theta)$. If θ is *symmetric*:

$$c_k(\sigma^* \theta) \ = \ c_k(\theta^\vee)$$

where $\sigma : \mathcal{M}^2 \to \mathcal{M}^2$ is the swap map, and the rank of θ is everywhere even, this bicharacter is symmetric,

$$\sigma^* \Psi(\theta, z) \ = \ \Psi(\sigma^* \theta, z) \ = \ \Psi(\theta^\vee, z) \ = \ \Psi(\theta, -z)$$

and so the Borcherds twist by a symmetric bicharacter and so defines a genuine vertex algebra. This is the starting observation of [Jo2].

2.6.16. We collect everything that we have discussed so far.

Theorem 2.6.17 (Joyce)**.** *Let* $(\mathcal{M}, 0)$ *be a pointed space with a commutative monoid structure* $\oplus : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$*, and a compatible action* $\mathrm{act} : \mathrm{B}\mathbf{G}_m \times \mathcal{M} \to \mathcal{M}$*. Let* $\theta \in \mathrm{Perf}(\mathcal{M} \times \mathcal{M})$ *be symmetric* ($c_k(\theta) \simeq c_k(\sigma^* \theta^\vee)$)*, compatible with* $\oplus$ *and have weights* 1 *and* -1 *with respect to the left and right* $\mathrm{B}\mathbf{G}_m$ *actions on* $\mathcal{M} \times \mathcal{M}$*.*

If $\mathrm{rk}\theta$ *is everywhere even, then*

$$Y(\alpha, z)\beta \ = \ \oplus_* \left(e^{zt} \otimes \mathrm{id} \cdot \Psi(\theta) \alpha \otimes \beta \right) \tag{2.20}$$

defines a vertex algebra structure on $\mathrm{H}_\bullet(\mathcal{M})$*.*

A small modification of the proof of Theorem 2.6.17 below gives

Theorem 2.6.18. (Joyce) *Keep the notation of Theorem 2.6.17. If we drop the condition that* θ *is symmetric and* $\mathrm{rk}\theta$ *even, then the same formula (2.20) gives a defines a* nonlocal *vertex algebra structure on* $\mathrm{H}_\bullet(\mathcal{M})$ *(see Definition A.2.2).*

Proof of Theorem 2.6.17. The nontrivial part of the Theorem is weak commutativity. So we begin by noting

$$\begin{aligned} & Y(\alpha,z)Y(\beta,w)\gamma \\ &\qquad = \oplus_* \big((e^{zt}\otimes \mathrm{id}\otimes \mathrm{id})\cdot \Psi((\mathrm{id}\times\oplus)^*\theta, z)\cdot(\mathrm{id}\otimes e^{wt}\otimes\mathrm{id})\cdot\Psi(\theta_{23},w)\cdot\alpha\otimes\beta\otimes\gamma\big) \\ &\qquad = \oplus_* \big((e^{zt}\otimes \mathrm{id}\otimes \mathrm{id})\cdot \Psi(\theta_{12}, z)\cdot(\mathrm{id}\otimes e^{wt}\otimes\mathrm{id})\cdot\Psi(\theta_{13},z)\cdot\Psi(\theta_{23},w)\cdot\alpha\otimes\beta\otimes\gamma\big) \end{aligned}$$

where we have written $\oplus : \mathcal{M}\times\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ for the three way direct sum map. Thus to finish this computation we will need to understand how to commute the middle e^{wt} and $\Psi(\theta,z)$ terms past each other.

Lemma 2.6.19 (Commutation Lemma)**.** *Let X be any space with an action of $\mathrm{B}\mathbf{G}_m$ and θ be a perfect complex on X weight n with respect to the $\mathrm{B}\mathbf{G}_m$ action. Then*

$$\Psi(\theta,z)e^{wt} \;=\; e^{wt}\Psi(\theta,z+nw). \tag{2.21}$$

Here $\Psi(\theta,z)$ is as in (2.19) and t the derivation defined in Proposition 2.6.9.

Proof. To begin, we claim that

$$[t,\mathrm{ch}_k(\theta)] \;=\; n\mathrm{ch}_{k-1}(\theta) \tag{2.22}$$

for $k\geqslant 1$. Indeed, we have

$$\mathrm{act}^*\mathrm{ch}_k(\theta) \;=\; \mathrm{ch}_k(\gamma^n\boxtimes\theta) \;=\; 1\otimes\mathrm{ch}_k(\theta)+n\tau\otimes\mathrm{ch}_{k-1}(\theta)+\cdots$$

so $t\mathrm{ch}_k(\theta) = n\mathrm{ch}_{k-1}(\theta)$. Thus since t is a derivation on cohomology, $t(\mathrm{ch}_k(\theta)\alpha) = \mathrm{ch}_k(\theta)t\alpha + n\mathrm{ch}_{k-1}(\theta)\alpha$, which proves the claim.

Before continuing, we note that

$$\Psi(\theta) \;=\; \sum_{n\geqslant 0} z^{\mathrm{rk}\theta-k}c_k(\theta) \;=\; z^{\mathrm{rk}\theta}\exp\left(-\sum_{k\geqslant 1}(-z)^{-k}(k-1)!\mathrm{ch}_k(\theta)\right),$$

which follows from the definition of chern classes and characters of a perfect complexes as pullbacks of certain classes in $\mathrm{H}^\bullet(\mathrm{Perf})$.

Writing $B = -\sum_{k\geqslant 1}(-z)^{-k}(k-1)!\mathrm{ch}_k(\theta)$, we have by the Baker Campbell Hausdorff formula and

(2.22) that

$$\begin{aligned}
e^{wt}e^{B}e^{-wt} &= \exp\left(\sum_{k\geqslant 0}\frac{(\mathrm{ad}wt)^k}{k!}B\right)\\
&= \exp\left(\sum_{r\geqslant 1}\sum_{k\geqslant 0}\frac{(nw)^k}{k!}(-z)^{-r}(r-1)!\mathrm{ch}_{r-k}(\theta)\right)\\
&= \exp\left(\sum_{r\geqslant 1}\sum_{k\geqslant 0}\frac{(-z)^{-(r-k)}(-nw/z)^k}{k!}(r-1)!\mathrm{ch}_{r-k}(\theta)\right)\\
&= \exp\left(-\sum_{l\geqslant 0}\sum_{k\geqslant 0}(-z)^{-l}(-nw/z)^k\frac{(l+k-1)!}{k!}\mathrm{ch}_l(\theta)\right)\\
&= \exp\left(-\sum_{k\geqslant 1}\frac{(-nw/z)^k}{k}\mathrm{ch}_0(\theta)\right)\exp\left(-\sum_{l\geqslant 1}\sum_{k\geqslant 0}(-z)^{-l}(nw/z)^k\frac{(l+k-1)!}{k!}\mathrm{ch}_l(\theta)\right)\\
&= \exp\left(-\mathrm{rk}\theta\sum_{k\geqslant 1}\frac{(-nw/z)^k}{k}\right)\exp\left(-\sum_{l\geqslant 1}\sum_{k\geqslant 0}(-z)^{-l}(nw/z)^k\binom{-l}{k}(l-1)!\mathrm{ch}_l(\theta)\right)\\
&= (1+nw/z)^{\mathrm{rk}\theta}\exp\left(-\sum_{l\geqslant 1}(-z+nw)^{-l}(l-1)!\mathrm{ch}_l(\theta)\right),
\end{aligned}$$

where we have set $(-1)! = 0$ for ease of notation. Multiplying both sides by $z^{\mathrm{rk}\theta}$ then gives the Commutation Lemma. □

Compare this Lemma with [FBZ, Lem. 3.2.3]. Now we can continue our computation: because θ_{12} has weight -1 in the second factor,

$$Y(\alpha,z)Y(\beta,w)\gamma \;=\; \oplus_*\left(e^{zt}\otimes e^{wt}\otimes \mathrm{id}\cdot\Psi(\theta_{12},z-w)\cdot\Psi(\theta_{13},z)\cdot\Psi(\theta_{23},w)\cdot\alpha\otimes\beta\otimes\gamma\right) \tag{2.23}$$

which we can compare to

$$\begin{aligned}
&Y(\beta,w)Y(\alpha,z)\gamma\\
&\quad= \oplus_*\left(e^{wt}\otimes e^{zt}\otimes \mathrm{id}\cdot\Psi(\theta_{12},w-z)\cdot\Psi(\theta_{13},w)\cdot\Psi(\theta_{23},z)\cdot\beta\otimes\alpha\otimes\gamma\right) \qquad (2.24)\\
&\quad= \oplus_*\left(e^{zt}\otimes e^{wt}\otimes \mathrm{id}\cdot\Psi(\theta_{21},w-z)\cdot\Psi(\theta_{23},w)\cdot\Psi(\theta_{13},z)\cdot\alpha\otimes\beta\otimes\gamma\right)
\end{aligned}$$

As $\Psi(\theta,z)$ defines a symmetric bicharacter (as θ is symmetric with $\mathrm{rk}\theta$ even) we have

$$\Psi(\theta_{21},w-z) \;=\; \sigma_{12}^*\Psi(\theta_{12},w-z) \;=\; \Psi(\theta_{12},z-w),$$

hence (2.23) and (2.24) are equal, proving weak commutativity. It is then easy to show that letting $T = t$ and $|0\rangle$ be the image of 1 under $\mathrm{H}_\bullet(\mathrm{pt})\to\mathrm{H}_\bullet(\mathcal{M})$, the homology is endowed with the structure of a vertex algebra, proving Theorem 2.6.17. □

2.6.20. **Orientations.** For those who do not like the fact in 2.6.18 that the resulting structure is a *nonlocal* vertex algebra, we give an alternative way to remove the condition that $\mathrm{rk}\theta$ be even from Theorem 2.6.17 whilst still remaining a vertex algebra. This requires the introduction of sign corrections similar to those in the definition of a lattice vertex algebra, which were not unique but depended on a choice of two cocycle (section 2.3.10).

In our situation these functions are called orientations in [Jo2]. Note that the commutative monoid structure on $(\mathcal{M}, 0)$ makes $\pi_0(\mathcal{M})$ into a commutative monoid with unit 0 given by the image of $\pi_0(0) \to \pi_0(\mathcal{M})$. An *orientation* is then a biadditive function

$$\varepsilon \; : \; \pi_0(\mathcal{M}) \times \pi_0(\mathcal{M}) \;\simeq\; \pi_0(\mathcal{M} \times \mathcal{M}) \;\to\; \{\pm 1\}$$

which satisfies

$$\varepsilon_{0,a} \;=\; \varepsilon_{a,0} \;=\; 1 \qquad\qquad \varepsilon_{a,b}\varepsilon_{b,a} \;=\; (-1)^{\mathrm{rk}\theta_{a,b} + \mathrm{rk}\theta_{a,a}\cdot \mathrm{rk}\theta_{b,b}}$$

$$\varepsilon_{a,b}\varepsilon_{a+b,c} \;=\; \varepsilon_{a,b+c}\cdot\varepsilon_{b,c}$$

where $\theta_{a,b} = \theta|_{\mathcal{M}_a \times \mathcal{M}_b}$. If $\pi_0(\mathcal{M})$ is in fact a group (as in most examples when $\mathcal{M}$ is the moduli space of objects in a *derived* category), this defines a two cocycle, i.e. a central extension

$$0 \;\to\; \mathbf{Z}/2 \;\to\; \widehat{\pi_0(\mathcal{M}^2)} \;\to\; \pi_0(\mathcal{M}^2) \;\to\; 0.$$

In particular, there are potentially many choices of orientation, and when the rank of θ is everywhere even $\varepsilon = 1$ is one such choice. Joyce notes in [Jo2] that geometrically ε often comes from a trivialisation of an *orientation bundle*, a principal $\mathbf{Z}/2$ bundle $\mathcal{O} \to \mathcal{M}$.

Given an orientation ε, we define the operator $\check{\varepsilon}$ on $\mathrm{H}_\bullet(\mathcal{M} \times \mathcal{M})$, which acts on $\mathrm{H}_d(\mathcal{M}_a) \otimes \mathrm{H}_\bullet(\mathcal{M}_b)$ with eigenvalue

$$\check{\varepsilon}_{a,b} \;=\; (-1)^{d\mathrm{rk}\theta_{b,b}}\varepsilon_{a,b}. \tag{2.25}$$

Theorem 2.6.21 (Joyce)**.** *With notation as in Theorem 2.6.17, without the condition that* $\mathrm{rk}\theta$ *be everywhere even. If* ε *is an orientation (section 2.6.20), then*

$$Y(\alpha, z)\beta \;=\; \check{\varepsilon} \oplus_* \left(e^{zt} \otimes \mathrm{id} \cdot \Psi(\theta)\alpha \otimes \beta\right) \tag{2.26}$$

defines a vertex algebra structure on $\mathrm{H}_\bullet(\mathcal{M})$*. If we also drop the condition that* θ *be symmetric* ($c_k(\theta) = c_k(\sigma^*\theta^\vee)$) *then this defines a nonlocal vertex algebra (Definition A.2.2).*

2.6.22. If the assignment $(a, b) \mapsto \mathrm{rk}\theta_{a,b}$ is symmetric then the above admits a $\mathbf{Z}$ grading by

$$\deg \mathrm{H}_d(\mathcal{M}_a) \;=\; d \;+\; \mathrm{rk}\theta_{a,a}.$$

Indeed, if $\alpha \in \mathrm{H}_d(\mathcal{M}_a)$ and $\beta \in \mathrm{H}_e(\mathcal{M}_b)$, then

$$\oplus_*(\alpha \otimes \beta) \;\in\; \mathrm{H}_{d+e}(\mathcal{M}_{a+b})$$

has degree $d + e + \mathrm{rk}\theta_{a+b,a+b} = d + e + \mathrm{rk}\theta_{a,a} + 2\mathrm{rk}\theta_{a,b} + \mathrm{rk}\theta_{b,b}$ and so

$$|\oplus_* (\alpha \otimes \beta)| \;=\; |a| + |b| + 2\mathrm{rk}\theta_{a,b}.$$

The vertex algebra is then degree preserving if we set $|z| = -2$ because $(e^{zt} \otimes \mathrm{id})$ has degree zero and $\Psi(\theta, z) = z^{\mathrm{rk}\theta_{a,b}} \sum z^{-k} c_k(\theta_{a,b})$ has degree $-2\mathrm{rk}\theta_{a,b}$.

2.6.23. Note that for any θ as in section 2.6.14, the perfect complex $\theta \oplus \sigma^*\theta^\vee$ is symmetric. Note however that its rank is not necessarily even.

2.6.24. **Examples.** The typical example is

$$\mathcal{M} \;=\; \text{moduli stack of objects in } \mathcal{A}$$

where $\mathcal{A}$ is an abelian, triangulated or dg category, and $\theta = \mathrm{Ext}(\;,\;)$ the vector bundle or perfect complex whose fibre above the pair of objects (a, a') is $\mathrm{Ext}^\bullet(a, a')$.

When the category is $2n$ Calabi Yau, we have

$$\mathrm{Ext}^\bullet(a, a') \;\simeq\; \mathrm{Ext}^\bullet(a', a)^\vee[2n]$$

and so we have $c_k(\theta) = c_k(\sigma^*\theta^\vee)$, thus θ is symmetric. Whe

2.6.25. **Example.** As the simplest example, take the category $\mathcal{A}_0 = D^b(\mathrm{Coh}(\mathrm{pt}))_0$ of bounded rank zero complexes of vector spaces up to quasiisomorphism

$$\cdots \to E_{-1} \to E_0 \to E_1 \to \cdots \qquad\qquad \textstyle\sum(-1)^i \mathrm{rk} E_i \;=\; 0.$$

Its moduli stack of objects $\mathcal{M}_{\mathcal{A}_0} = \mathrm{Perf}_0$ parametrises *families* of such structures. Maps into it correspond to rank zero complexes of *vector bundles* up to quasiisomorphism. Its cohomology is thus generated by chern characters:

$$\mathrm{H}^\bullet(\mathcal{M}_{\mathcal{A}_0}) \;=\; k[\mathrm{ch}_1, \mathrm{ch}_2, \dots\,].$$

Readers might have noticed that as a vector space this is just (dual to) the Heisenberg vertex algebra. In this case Joyce's construction gives a geometric construction of the Heisenberg algebra structure on homology[11]

$$\mathrm{H}_\bullet(\mathcal{M}_{\mathcal{A}_0}) \;=\; k[\mathrm{ch}_1^\vee, \mathrm{ch}_2^\vee, \dots\,].$$

2.6.26. Indeed, consider the field of the vector $\mathrm{ch}_1^\vee$. Writing $\theta = \gamma^\vee \boxtimes \gamma$ where γ is the (rank zero) tautological perfect complex over $\mathcal{M}$, we get for $n > 0$ that

$$\mathrm{ch}_n(\theta) \cdot (\mathrm{ch}_1^\vee \otimes \beta) \;=\; -1 \otimes \mathrm{ch}_{n-1}(\gamma) \cdot \beta.$$

All higher degree polynomials in the chern characters act by zero, so separating $\Psi(\theta) = 1 + (\Psi(\theta) - 1)$ gives

$$Y(\mathrm{ch}_1^\vee, z)\beta \;=\; \oplus_* \left(e^{zt} c_1^\vee \otimes \beta\right) \;-\; \left(\sum_{k\geqslant 1} \frac{(-z)^{-k}}{k} k!\mathrm{ch}_{k-1}(\gamma)\right) \cdot \beta.$$

Finally, we can rewrite the first term as (and this is where it is important that we are using chern characters rather than classes)

$$Y(\mathrm{ch}_1^\vee, z)\beta \;=\; \sum_{k\geqslant 0} z^n \mathrm{ch}_n^\vee \beta \;+\; \left(\sum_{k\geqslant 1} (-z)^{-k+1}(k-1)!\mathrm{ch}_{k-1}(\gamma)\right) \cdot \beta.$$

Thus as operators, we set

$$b_n \;\mapsto\; \begin{cases} \mathrm{ch}_{-n}^\vee & \text{if } n \geqslant 0 \\ (-1)^{n-1}(n-1)!\mathrm{ch}_{n-1} & \text{if } n > 0 \end{cases}$$

which one can show satisfy the Heisenberg algebra relations at level zero. This means the isomorphism of vector spaces to the Heisenberg vertex algebra preserves the field $Y(\mathrm{ch}_1^\vee, z)$. Because this is a generating field of the Heisenberg vertex algebra, by the reconstruction Theorem 2.2.4 this map of vector spaces is actually a vertex algebra isomorphism.

2.6.27. **Vector spaces.** Moving in the lattice direction, let $\mathcal{A} = \mathrm{Vect}_K^{f.d.}$ be the abelian category of finite dimensional vector spaces over K. This is a zero dimensional Calabi Yau category, and its moduli space of objects is

$$X \;=\; \textstyle\coprod_{n\geqslant 0} \mathrm{BGL}_n.$$

[11]Here we have taken duals with respect to the monomial basis in the ch_n, and the algebra structure is the one given by $\oplus_*$.

A map into X is uniquely specified by what the pullback is of the tautological vector bundle γ. Therefore the structure maps $\oplus$ and *act* are *defined* by requiring

$$\oplus^*\gamma \;=\; \gamma \boxplus \gamma, \qquad \text{and} \qquad \mathrm{act}^*\gamma \;=\; \gamma_1 \boxtimes \gamma,$$

where γ_1 is the tautological line bundle on $\mathrm{B}\mathbf{G}_m$. The perfect complex here is simply $\theta = \gamma^\vee \boxtimes \gamma$.

2.6.28. To describe the action of $\oplus$ and act on cohomology, pick a maximal torus $\mathrm{T}_n \subseteq \mathrm{GL}_n$, so that the map $\mathrm{BT}_n \to \mathrm{BGL}_n$ identifies

$$\mathrm{H}^\bullet(\mathrm{BGL}_n) = k[t_1, ..., t_n]^{\mathfrak{S}_n} \;\to\; k[t_1, ..., t_n] \;=\; \mathrm{H}^\bullet(\mathrm{BT}_n).$$

The maps $\oplus$ and *act* can be lifted to BT_n in a manner similar to the above, and they induce maps on cohomology

$$\oplus^* \;:\; k[s_1, ..., s_n, t_1, ..., t_m] \;\xrightarrow{\sim}\; k[s_1, ..., s_n] \otimes k[t_1, ..., t_m],$$

and

$$\mathrm{act}^* \;:\; k[t_1, ..., t_n] \;\to\; k[t] \otimes k[t_1, ..., t_n]$$

which sends $t_i \mapsto t \otimes t_i$. Taking symmetric group invariants then recovers these maps for X.

2.6.29. To explicitly describe the vertex algebra we get, it is easier instead to work with the derived category $\mathcal{C}$ of finite dimensional vector spaces, which has $\mathcal{C}^\heartsuit = \mathcal{A}$. In this case,

$$\mathcal{M} \;=\; \textstyle\coprod_{n \in \mathbf{Z}} \mathrm{Perf}_n$$

where Perf_n classifies perfect complexes of rank n. Again we have $\theta = \gamma^\vee \boxtimes \gamma$ where γ is the tautological perfect complex on $\mathcal{M}$. It is not hard to show that $\mathrm{H}^\bullet(\mathcal{M}) \;\simeq\; V_{\mathbf{Z}}$ is the one dimensional lattice vertex algebra. Thus the vertex subalgebra corresponding to the abelian category has basis

$$\{b_{-n}^{a_n} \cdots b_{-1}^{a_1}|n\rangle \;:\; n \geqslant 0,\; a_n \geqslant 0\}.$$

2.6.30. At the moment there is no satisfying explanation of Joyce's constructions at the level of chiral algebras. It would be interesting to relate these constructions to [KV2].

2.6.31. **Variants.** Note that we may replace

$$\theta \;\rightsquigarrow\; \theta^n \oplus (\sigma^*\theta^\vee)^{\oplus m} \qquad\qquad n, m \geqslant 0$$

and still get a (nonlocal) vertex algebra. The holomorphic case corresponds to setting both integers to zero. More generally, for any $\lambda \in k$ we may replace $\Psi(\theta)$ with

$$\Psi(\theta^{\oplus\lambda}) \;:=\; z^{\lambda \mathrm{rk}\theta} \sum_{r \geqslant 0} \lambda c_r(\theta) z^{-r}.$$

As before, if k has characteristic zero we have

$$\Psi(\theta^{\oplus\lambda}) \;=\; z^{\lambda \mathrm{rk}\theta} \exp\left(-\lambda \sum_{r \geqslant 1} (-z)^{-r} (r-1)! \mathrm{ch}_r(\theta) \right).$$

Indeed, this is true for all nonnegative integral λ, and as on each cohomologically graded piece both sides are polynomials in λ agreeing on the positive integers, they are equal. We can repeat the proof of the Commutation Lemma 2.6.19 to give

Lemma 2.6.32. *Let X be a space with a $\mathrm{B}\mathbf{G}_m$ action. If θ is a perfect complex on X with weight n with respect to the $\mathrm{B}\mathbf{G}_m$ action, then*

$$e^{wt}\Psi(\theta^{\lambda}, z) \;=\; \Psi(\theta^{\lambda}, z + nw)e^{wt}. \tag{2.27}$$

It follows that in Theorems 2.6.17 and 2.6.21 if $\mathrm{char} k = 0$ and we replace

$$\Psi(\theta, z) \;\rightsquigarrow\; \Psi(\theta^{\oplus\lambda}) \qquad\qquad \lambda \in k$$

we get a (nonlocal) vertex algebra structure.

2.6.33. Moreover, we may let λ be a variable, and replacing $\Psi(\theta)$ with $\Psi(\theta^{\oplus\lambda})$ gives a vertex algebra over the ring $k[\lambda]$. In particular, if θ_1 and θ_2 are any two such perfect complexes then $\Psi(\theta_1^{1-\lambda} \oplus \theta_2^{\lambda})$ interpolates between one vertex algebra structure at $\lambda = 0$ and the other at $\lambda = 1$.

2.7 Review of the six functors

In this section we will review what we mean by *space* and *sheaf* in this thesis.

2.7.1. Grothendieck's *six functor formalism* is an extremely useful enhancement of the notion of cohomology. Standard properties of cohomology are lifted to the category $\mathrm{Sh}(X)$ of *sheaves* on the space X. There are many examples of cohomology, likewise, there are many examples of sheaf theories with the six functors:

1. Topological spaces with $\mathrm{Sh}(X)$ the bounded below derived category of sheaves of abelian groups on X, see [Iv]. Recovers singular cohomology.

2. Schemes (or more generally, higher Artin stacks [LZ1, LZ2]) X over a field of characteristic prime to ℓ, with $\mathrm{Sh}(X)$ the bounded derived category of constructible ℓ adic sheaves. Recovers ℓ adic cohomology.

3. Schemes X over a field of characteristic 0 with $\mathrm{Sh}(X)$ the derived category of holonomic D modules, see [Ber, HTT]. Recovers de Rham cohomology.

4. Complex varieties X (separated and reduced) with $\mathrm{Sh}(X)$ the bounded derived category of mixed Hodge modules, see [Sa]. Recovers polarisable $\mathbf{Q}$ mixed Hodge cohomology.

5. Schemes X over k and $\mathrm{Sh}(X)$ the category of Beilinson motives, see [CD].

This list is far from exhaustive. In the below we will assume that $\mathrm{Sh}(X)$ is a triangulated category since this is all we will need in the Thesis, however most six functor formalisms do admit enhancements to stable ∞ categories. Finally, we warn that one needs to impose additional finiteness assumptions in the above examples to define the ! pullback and pushforward functors. For a general account, see [CD].

2.7.2. **What we mean by *space* and *sheaf*.** We fix a (∞-)category Sp of *spaces*. A *sheaf theory* with the six functors means an assignment to every space of a triangulated (stable ∞-)category

$$X \in \mathrm{Sp} \quad \rightsquigarrow \quad \mathrm{Sh}(X) \in \mathrm{Triang}$$

and to every map of spaces two adjoint pairs of triangulated functors (f^*, f_*) and $(f_!, f^!)$:

$$X \xrightarrow{f} Y \quad \rightsquigarrow \quad \mathrm{Sh}(X) \underset{f^*, f^!}{\overset{f_*, f_!}{\rightleftarrows}} \mathrm{Sh}(Y)$$

We require that $\mathrm{Sh}(X)$ comes equipped with a closed symmetric monoidal structure $(\otimes, \mathcal{H}\mathrm{om})$,[12] and each of the four functors attached to f induce 2-functors $\mathrm{Sp}^{(op)} \to \mathrm{Triang}$, such that

1. f^* is monoidal.

2. Given a pullback in Sp

$$\begin{tikzcd} X' \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

   there are natural *base change* isomorphisms

$$g^* f_! \overset{\sim}{\Rightarrow} \overline{f}_! \overline{g}^*, \qquad \overline{g}_* \overline{f}^! \overset{\sim}{\Rightarrow} f^! g_*.$$

[12] Recall that this means a symmetric monoidal structure $\otimes$ and an internal Hom functor $\mathcal{H}\mathrm{om}$, satisfying tensor-Hom adjunction.

3. There are natural *projection formula* isomorphisms

$$(f_!\mathcal{F}) \otimes_Y \mathcal{G} \ \tilde{\Rightarrow}\ f_!(\mathcal{F} \otimes_X f^*\mathcal{G}), \qquad \mathcal{H}\mathrm{om}_Y(f_!\mathcal{F}, \mathcal{G}) \ \tilde{\Rightarrow}\ f_*\mathcal{H}\mathrm{om}_X(\mathcal{F}, f^!\mathcal{G}),$$

$$f^!\mathcal{H}\mathrm{om}_Y(\mathcal{G}, \mathcal{G}') \ \tilde{\Rightarrow}\ \mathcal{H}\mathrm{om}_X(f^*\mathcal{G}, f^!\mathcal{G}').$$

We also require that the structure interact well with maps which are open, closed, proper, smooth, ... Rather than axiomatise the meaning of these adjectives in Sp, we simply give an example:

Theorem 2.7.3. [LZ1, LZ2] *Let* Sp *be the category of dg higher Artin stacks locally of finite type over a field whose characteristic is prime to ℓ. Then the triangulated (stable ∞-)category* $\mathrm{Sh}(X)$ *of constructible ℓ adic sheaves satisfies the above, and also satisfies*

4. *For any schematic map f we have a natural transformation $f_! \Rightarrow f_*$, which is an equivalence if f is proper.*

5. *If i and j are complementary closed and open embeddings, there are distinguished triangles (fibre sequences)*

$$i_*i^! \ \Rightarrow\ \mathrm{id} \ \Rightarrow\ j_*j^! \ \overset{+1}{\Rightarrow}$$

$$j_*j^! \ \Rightarrow\ \mathrm{id} \ \Rightarrow\ i_*i^! \ \overset{+1}{\Rightarrow}$$

called the Mayer Vietoris sequence*.*

6. *If i is a closed embedding, the counit $i^*i_* \tilde{\Rightarrow} \mathrm{id}$ is an equivalence.*

7. $\mathrm{Sh}(\mathrm{pt}) = D^b(\mathrm{Vect}_{\mathbf{Q}_\ell})$*.*

8. *If f is smooth of dimension d then there is an equivalence $f^! \tilde{\Rightarrow} f^*\langle 2d\rangle$.*

Here $\langle 2d\rangle = [2d](d)$ where (d) denotes the dth Tate twist. The adjunction (f^*, f_*) can be extended to arbitrary maps of dg higher Artin stacks.

2.7.4. **Conventions.** We will use $\mathrm{Sh}(X)$ to denote the triangulated stable ∞ category of a sheaf theory with the six functors. In particular, its homotopy category is a triangulated category; we will often abuse notation by also denoting it by $\mathrm{Sh}(X)$. This will not cause confusion because invariants like cohomology of a space (see below) are built from $\mathrm{Sh}(X)$ or from its homotopy category in identical ways.

We will use as our category Sp of spaces the category of derived Artin stacks over a field of characteristic zero. This will include all moduli stacks we will be covering. The reason for considering *derived* Artin stacks is that this is the correct framework when talking about fundamental classes (see e.g. [**?**] or Appendix C).

2.7.5. **Consequent notions.** From a sheaf theory with the six functors, a number of different notions can be defined.

1. *Cohomology.* Writing $p_X : X \to \mathrm{pt}$ for the projection to a point, the *cohomology* (*with compact support*) of a sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is

$$\mathrm{H}^\bullet(X, \mathcal{F}) \;=\; p_{X,*}\mathcal{F}, \qquad\qquad \mathrm{H}^\bullet_c(X, \mathcal{F}) \;=\; p_{X,!}\mathcal{F}.$$

   By adjunction cohomology is equivalently $\mathrm{Hom}_{\mathrm{Sh}(X)}(k_X, \mathcal{F})$. The *constant sheaf* with value $A \in \mathrm{Sh}(\mathrm{pt}) = D^b(\mathrm{Vect}_k)$ is $A_X = p_X^* A$, and the *cohomology* (*with compact support*) of X is the cohomology (with compact support) of the constant sheaf k_X.

2. *Homology.* The *dualising sheaf* of a space X is $\omega_X = p_X^! k$. The (Borel Moore) homology of a space X is[13]

$$\mathrm{H}_\bullet(X) \;=\; p_{X!}\omega_X, \qquad\qquad \mathrm{H}^{\mathrm{BM}}_\bullet(X) \;=\; p_{X*}\omega_X.$$

3. *Gysin sequence.* If i and j are complementary closed and open embeddings, the *Gysin sequence* is the distinguished triangle

$$i^! \;\Rightarrow\; i^* \;\Rightarrow\; i^*j_*j^* \;\overset{+1}{\Rightarrow}$$

   formed by applying i^* to the Mayer Vietoris sequence.

4. *Cup product.* Because f^* is a monoidal functor, its right adjoint f_* is lax monoidal and its lax monoidal structure

$$f_*\mathcal{F} \otimes f_*\mathcal{G} \;\to\; f_*f^*(f_*\mathcal{F} \otimes f_*\mathcal{G}) \;=\; f_*f^*(f_*\mathcal{F} \otimes f_*\mathcal{G}) \;\to f_*(\mathcal{F} \otimes \mathcal{G})$$

   takes commutative monoids to commutative monoids, see [GL, Prop. 3.2.3.1]. In particular, applying this to the commutative monoid k_X (which is the unit in $\mathrm{Sh}(X)$), we get an algebra map

$$\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \;\to\; \mathrm{H}^\bullet(X)$$

   called the *cup product.*

[13] Sometimes we also denote Borel Moore homology by $\mathrm{H}^\bullet_{\mathrm{BM}}$.

5. *Pullback.* By applying the unit $\mathrm{id} \Rightarrow f_* f^*$ and counit $f_! f^! \Rightarrow \mathrm{id}$ we get a map

$$f^* \;:\; \mathrm{H}^\bullet(Y, \mathcal{G}) \;\to\; \mathrm{H}^\bullet(X, f^*\mathcal{G}), \qquad\qquad f_* \;:\; \mathrm{H}^\bullet_c(X, f^!\mathcal{G}) \;\to\; \mathrm{H}^\bullet_c(Y, \mathcal{G})$$

and if $\mathcal{G}$ is an commutative monoid f^* is a map of algebras. This shows that (co)homology of spaces is (contravariantly) functorial.

6. *Poincaré duality.* The *Verdier dual* endofunctor is

$$\mathbf{D}_X \;=\; \mathcal{H}\mathrm{om}_X(-, \omega_X).$$

When f is a finite type map of schemes, the natural transformation $\mathrm{id} \overset{\sim}{\Rightarrow} \mathbf{D}_X^2$ is an equivalence, exchanging

$$f_! \mathbf{D}_X \;\overset{\sim}{\Rightarrow}\; \mathbf{D}_Y f_* \qquad\qquad f^! \mathbf{D}_Y \;\overset{\sim}{\Rightarrow}\; \mathbf{D}_X f^*$$

see [SGA5, Ex. I]. Thus the two Mayer Vietoris sequences above are Verdier dual. Noting that $\mathbf{D}_{\mathrm{pt}}$ is nothing but taking the dual vector space, we get the *Poincaré duality* isomorphism for X a smooth scheme of finite type

$$\mathrm{H}^\bullet(X, \mathcal{F}) \;=\; \mathrm{H}^\bullet_c(X, \mathbf{D}_X \mathcal{F})^\vee.$$

7. *Cohomology of classifying spaces.* Let G be a smooth connected algebraic group. To describe the category of sheaves on the classifying space $\mathrm{B}G$, we use the fact that $\pi^!$ is conservative and apply Lurie Barr Beck as in [DGai, §7.2]. Consider the pullback

$$\begin{tikzcd} G \arrow[r, "\sigma"] \arrow[d, "\sigma"] & \mathrm{pt} \arrow[d, "\pi"] \\ \mathrm{pt} \arrow[r, "\pi"] & \mathrm{B}G \end{tikzcd}$$

so that by smooth base change, $\pi_! \pi_! k = \sigma_! \sigma^! k = \mathrm{H}^\bullet(G)^\vee$. Thus by Lurie Barr Beck,

$$\mathrm{Sh}(\mathrm{B}G) \;\simeq\; \mathrm{H}^\bullet(G)^\vee\text{-Mod}(\mathrm{Sh}(\mathrm{pt})) \;=\; \mathrm{H}^\bullet(G)^\vee\text{-Mod}$$

is the category of modules over the dg algebra $B = \mathrm{H}^\bullet(G)^\vee$. For instance, since $k_{\mathrm{B}G}$ corresponds to the trivial B module, taking cohomology corresponds to

$$\mathrm{H}^\bullet(\mathrm{B}G, -) \;=\; \mathrm{Hom}_{\mathrm{Sh}(\mathrm{B}G)}(k_{\mathrm{B}G}, -) \;\simeq\; \mathrm{Hom}_B(k, -).$$

In particular, the cohomology $\mathrm{H}^\bullet(\mathrm{B}G) \;=\; \mathrm{Hom}_B(k, k)$ is the Koszul dual of B. By taking an explicit projective resolution, we see that if $\mathrm{H}^\bullet(G)$ is freely generated in degrees $2d_i - 1$, then the cohomology $\mathrm{H}^\bullet(\mathrm{B}G)$ is freely generated in degrees $2d_i$.

8. *Bivariant homology.* The *bivariant homology* (sometimes also called *relative Borel Moore homology*) of the map f is

$$\mathrm{H}^\bullet(X/Y) = \mathrm{H}^\bullet(X, f^! k_Y).$$

Moreover $\mathrm{H}^\bullet(X/Y)$ is a *bivariant theory* in the sense of Fulton and MacPherson [FM, §7.4], i.e.

(a) there is a product map

$$\mathrm{H}^\bullet(X/Y) \otimes \mathrm{H}^\bullet(Y/Z) \ \dot{\to}\ \mathrm{H}^\bullet(X/Z) \qquad \text{for} \qquad X \xrightarrow{f} Y \xrightarrow{g} Z,$$

(b) if f is proper there is a pushforward map $f_* : \mathrm{H}^\bullet(X/Z) \to \mathrm{H}^\bullet(Y/Z)$,

(c) and there is a pullback map

$$g^* : \mathrm{H}^\bullet(X/Y) \to \mathrm{H}^\bullet(X'/Y') \qquad \text{for} \qquad \begin{tikzcd} X' \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The product takes two bivariant classes $k_X \to f^! k_Y$ and $k_Y \to g^! k_Z$ to their composition $k_X \to f^! k_Y \to f^! g^! k_Z$, proper pushforward is given by $f_* f^! = f_! f^! \Rightarrow \mathrm{id}$, and pullback is given by base change $f^! \Rightarrow f^! g_* g^* \simeq \overline{g}_* \overline{f}^! \overline{g}^*$.

These three structures together satisfy:

A_1) The product is associative: $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.

A_2) Pushforward is functorial: if f_1, f_2 are composable proper maps, $f_{1*}(f_{2*}(\alpha)) = (f_1 f_2)_*(\alpha)$.

A_3) Pullback is functorial: if g_1, g_2 are composable maps, $g_1^*(g_2^*(\alpha)) = (g_1 g_2)^*(\alpha)$.

A_{13}) Product and pullback commute: $g^*(\alpha \cdot \beta) = g^*(\alpha) \cdot g^*(\beta)$.

A_{23}) Pushforward and pullback commute: $g^* f_* \alpha = f'_* g^* \alpha$ for any Cartesian diagrams

$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r] \arrow[d] & X' \arrow[d, "g"] \\ Z \arrow[r, "f"] & Y \arrow[r] & X \end{tikzcd}$$

where f is proper, and any class $\alpha \in \mathrm{H}^\cdot(Z/X)$.

A_{123}) The *projection formula* for any Cartesian diagram: $\beta \cdot f_* \alpha = f'_*(g'^* \beta \cdot \alpha)$

$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "g'"] & \\ Z \arrow[r, "f"] & Y \arrow[r] & X \end{tikzcd}$$

where f is proper, and classes $\alpha \in \mathrm{H}^{\cdot}(Z/X)$ and $\beta \in \mathrm{H}^{\cdot}(Y'/Y)$.

C) Skew-commutativity: $g^*\alpha \cdot \beta = (-1)^{\deg(\alpha)\cdot\deg(\beta)} f^*(\beta) \cdot \alpha$ for any Cartesian diagram

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

and classes $\alpha \in \mathrm{H}^{\cdot}(Y/X)$ and $\beta \in \mathrm{H}^{\cdot}(X'/X)$.

There is also a notion of *virtual fundamental class* attached to any quasismooth map between derived Artin stacks, see section C.4.

2.7.6. **Extending sheaf theories.** We can define sheaves on more general categories of spaces, at the cost of losing the six functors, see e.g. [Ga3]. Consider just the $f^!$ functor

$$\mathrm{Sh}^!(-) \ : \ \mathrm{Sch}^{op} \ \to \ \mathrm{Triang}.$$

Then if $f^!$ preserves limits (as is the case when it has a left adjoint $f_!$), we can extend this functor to prestacks Y by continuity:

$$\mathrm{Sh}^!(Y) \ = \ \lim_{\substack{S \to Y \\ S \in \mathrm{Sch}}} \mathrm{Sh}^!(S)$$

and similarly for lax prestacks, see [Ga3, §1,§2]. Moreover,

1. In many cases, e.g. $\mathcal{D}$ modules, then by [GaR, Thm 2.1.2] for any ind schematic map of prestacks $f : X \to Y$ we get two functors

$$f_* \ : \ \mathrm{Sh}(X) \ \to \ \mathrm{Sh}(Y) \qquad\qquad f^! \ : \ \mathrm{Sh}(Y) \ \to \ \mathrm{Sh}(X)$$

satisfying base change. Moreover, $(f_*, f^!)$ are adjoint if f is proper and $(f^!, f_*)$ are adjoint if f is an open embedding.

2. In the case of constructible sheaves over topological spaces, ℓ adic sheaves over a base of characteristic prime to ℓ or holonomic $\mathcal{D}$ modules over a base of characteristic 0, then if $f : X \to Y$ is *any* map of prestacks then we get functors

$$f_! \ : \ \mathrm{Sh}(X) \ \to \ \mathrm{Sh}(Y) \qquad\qquad f^! \ : \ \mathrm{Sh}(Y) \ \to \ \mathrm{Sh}(X)$$

such that $(f_!, f^!)$ is an adjunction, see [Ga3, Cor. 1.4.2] for details.

Chapter 3

Cohomological Hall algebras

3.1 Cohomological Hall algebras

3.1.1. Cohomological Hall algebra is the catch-all name for some algebras associated to the moduli space of objects in abelian categories $\mathcal{A}$, formed by turning the abelian category structure into an algebra structure, e.g. by taking cohomology.

The modern definition was discovered by Kontsevich and Soibelman in [KS] for $\mathcal{A} = \mathrm{Rep}(Q, W)$ the representations of a quiver with potential, drawing on analogies with the string theory notion of *algebra of BPS states* due to Harvey and Moore [HM], and the earlier notion of *Hall algebra* of a finitary category due to Ringel and Hall, see [Sc].

3.1.2. **Hall algebras.** If $\mathcal{A}$ is an abelian category with a finiteness condition, Ringel and Hall gave

$$\mathbf{H}_{\mathcal{A}} \; = \; \mathbf{C}[\pi_0(\mathcal{A})]$$

the structure of an associative algebra, by using extensions in the category $\mathcal{A}$. The condition is *finitary*, meaning $\mathrm{Hom}(a, a')$ and $\mathrm{Ext}^1(a, a')$ are finite for all objects a, a'. Examples include representations of a quiver over $\mathbf{F}_q$, or coherent sheaves on a scheme defined over $\mathbf{F}_q$.

Theorem 3.1.3 (Ringel [Rin]). *If $\mathcal{A}$ is a finitary abelian category, and $a, a' \in \mathcal{A}$,*

$$a \cdot a' \; = \; \langle a, a' \rangle \sum_{a \to e \to a'} e \tag{3.1}$$

defines an algebra structure on $\mathbf{H}_{\mathcal{A}}$, where we sum over all short exact sequences, and

$$\langle a, a' \rangle \; = \; \frac{1}{|\mathrm{Aut} a| \cdot |\mathrm{Aut} a'|} \sqrt{\prod |\mathrm{Ext}^i(a, a')|^{(-1)^i}}.$$

The term $\chi(a,a') = \prod |\mathrm{Ext}^i(a,a')|^{(-1)^i}$ is called the *multiplicative Euler form*, and defines a homomorphism $\mathrm{K}(\mathcal{A}) \times \mathrm{K}(\mathcal{A}) \to \mathbf{Q}^\times$. Similarly, $\overline{\chi}(a,a') = \sqrt{\chi(a,a')\chi(a',a)}$ is called the *symmetrised* multiplicative Euler form.

3.1.4. Green discovered that this can be extended to a bialgebra structure. Note that $\mathbf{H}_\mathcal{A}$ is graded by $\mathrm{K}(\mathcal{A})$, allowing us to take the completed tensor product with itself. Green defined the map

$$\Delta \; : \; \mathbf{H}_\mathcal{A} \; \to \; \mathbf{H}_\mathcal{A} \widehat{\otimes} \mathbf{H}_\mathcal{A} \qquad\qquad e \; \mapsto \; \frac{1}{|\mathrm{Aut} e|} \sum_{a \to e \to a'} \sqrt{\chi(a,a')} a \otimes a' \tag{3.2}$$

where we sum over all short exact sequences, and showed that it is a topological[1] coalgebra. However, even ignoring convergence issues, we do not literally have a bialgebra structure. The coproduct will only be compatible with the *twisted* product

$$\Delta(fg) \;=\; \Delta(f) \cdot_\chi \Delta(g)$$

defined on homogenous elements by

$$(a \otimes b) \cdot_\chi (c \otimes d) \;=\; \overline{\chi}(b,c)(ac \otimes bd).$$

Note that both product (3.1) and coproduct (3.2) preserve the grading by $\mathrm{K}(\mathcal{A})$, and twisted bialgebras are just bialgebras for a certain symmetric monoidal structure τ_S on the category $\mathrm{Vect}_{\mathrm{K}(\mathcal{A})}$ of $\mathrm{K}(\mathcal{A})$-graded vector spaces (see 3.1.6 below).

Theorem 3.1.5 ([Gre])**.** *Let* $\dim \mathcal{A} \leqslant 1$. *Then* $\mathbf{H}_\mathcal{A}$ *is a topological bialgebra in* $(\mathrm{Vect}_{\mathrm{K}(\mathcal{A})}, \tau_S)$, *with product (3.1), coproduct (3.2), and (co)unit (evaluation at) the zero object* $0 \in \mathrm{Obj}\mathcal{A}$.

3.1.6. The isomorphisms

$$S_{\lambda,\mu} \; : \; \mathbf{C}_\lambda \otimes \mathbf{C}_\mu \overset{\overline{\chi}(\lambda,\mu)\cdot}{\to} \mathbf{C}_\lambda \otimes \mathbf{C}_\mu \; \tilde{\to} \; \mathbf{C}_\mu \otimes \mathbf{C}_\lambda$$

can be extended by cocontinuity to $\mathrm{K}(\mathcal{A})$-graded isomorphisms

$$\tau_{S,V,W} \; : \; V \otimes W \; \tilde{\to} \; W \otimes V$$

for all graded vector spaces V, W. Here $\lambda, \mu \in \mathrm{K}(\mathcal{A})$ and $\mathbf{C}_\lambda$ is the one dimensional vector space with grading λ. Then τ_S defines a symmetric monoidal structure (see section A.1.3) on $\mathrm{Vect}_{\mathrm{K}(\mathcal{A})}$, as the conditions are implied by $\overline{\chi}$ being a homomorphism and symmetric in both factors.

[1] i.e. not only are all the coalgebra axioms satisfied, but all terms are well defined, which is not a priori clear due to convergence issues (see the discussion after [Sc, Prop 1.4]).

3.1.7. The Hall algebra has an interpretation in terms of the moduli stack $\mathcal{M}_\mathcal{A}$ parametrising objects in $\mathcal{A}$: in all the relevant examples, it exists as an algebraic stack over a field k with points $\mathcal{M}_\mathcal{A}(k) \simeq \pi_0(\mathcal{A})$. There is also a stack $\mathrm{Ext}_\mathcal{A}$ parametrising short exact sequences in $\mathcal{A}$. Thus we have maps

$$\begin{tikzcd} & \mathrm{Ext}_\mathcal{A} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{M}_\mathcal{A} \times \mathcal{M}_\mathcal{A} & & \mathcal{M}_\mathcal{A} \end{tikzcd} \tag{3.3}$$

which on k points sends

$$\begin{tikzcd} & {[0 \to a \to e \to a' \to 0]} \arrow[dl, mapsto, "q"'] \arrow[dr, mapsto, "p"] & \\ (a, a') & & e \end{tikzcd} \tag{3.4}$$

The Hall algebra $\mathbf{H}_\mathcal{A}$ can then be interpreted as *constructible functions* on the Artin stack $\mathcal{M}_\mathcal{A}$, see [Jo1]. Informally, one interprets the Hall algebra as pulling back constructible functions by q then pushing forward by p.

3.1.8. **Cohomological Hall algebras.** Cohomological Hall algebras take the correspondence (3.3), but instead of applying constructible functions to get the Hall algebra, apply cohomology or similar invariant like Borel Moore homology. For a review of cohomology, see section 2.7.

3.1.9. Let $X \in \mathrm{Alg}(\mathrm{Art}^{corr})$ be an Artin stack which is an associative algebra in the category of Artin stacks with morphisms correspondences. This means that there is a map

$$1 \ : \ \mathrm{pt} \ \to \ X$$

and a correspondence

$$\begin{tikzcd} & C \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X \times X & & X \end{tikzcd} \tag{3.5}$$

satisfying an associativity condition, and 1 is a unit.[2]

[2]This means that the two pullbacks

$$\begin{tikzcd}[column sep=small] & & C_{(12)3} \arrow[dl] \arrow[dr] & & \\ & X \times C \arrow[dl] \arrow[dr] & & C \arrow[dl] \arrow[dr] & \\ (X \times X) \times X & & X \times X & & X \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & C_{1(23)} \arrow[dl] \arrow[dr] & & \\ & C \times X \arrow[dl] \arrow[dr] & & C \arrow[dl] \arrow[dr] & \\ X \times (X \times X) & & X \times X & & X \end{tikzcd} \tag{3.6}$$

are isomorphic correspondences, and that $X \simeq C \times_{X \times X} (\mathrm{pt} \times X) \simeq C \times_{X \times X} (X \times \mathrm{pt})$.

Definition 3.1.10. Let X be an Artin stack as in 3.1.9. Assume p is proper. The following structures are all called *cohomological Hall algebras* (or *CoHAs*):

1. If p quasismooth, $(\mathrm{H}^\bullet(X), p_*q^*)$.
2. If q is quasismooth, $(\mathrm{H}^{\mathrm{BM}}_\bullet(X), p_*q^*)$.
3. If p is quasismooth, $(\mathrm{H}^\bullet(X, \mathcal{F}), p_*q^*)$ for any sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ with a map $q^*(\mathcal{F} \boxtimes \mathcal{F}) \to p^*\mathcal{F}$ satisfying an associativity condition.

Recall that in cohomology, there are all pullbacks and quasismooth proper pushforwards. In Borel Moore homology, there are quasismooth pullbacks and proper pushforwards. The reason is that all these maps are constructed using fundamental classes and proper pushforwards in bivariant homology, see Appendix C.5. The third map is constructed by

$$\begin{aligned} \mathrm{H}^\bullet(X,\mathcal{F}) \otimes \mathrm{H}^\bullet(X,\mathcal{F}) \;\simeq\; \mathrm{H}^\bullet(X \times X, \mathcal{F} \boxtimes \mathcal{F}) \;&\xrightarrow{q^*}\; \mathrm{H}^\bullet(C, q^*(\mathcal{F} \boxtimes \mathcal{F})) \;\to\; \mathrm{H}^\bullet(C, p^*\mathcal{F}) \\ &\xrightarrow{[C/X]}\; \mathrm{H}^{\bullet-2d}(C, p^!\mathcal{F}) \;\xrightarrow{p_*}\; \mathrm{H}^{\bullet-2d}(X, \mathcal{F}) \end{aligned}$$

where $[C/X]$ is the fundamental class and p_* is the (bivariant) pushforward by p.

3.1.11. We list some examples. Let $\mathcal{A}$ be an abelian category. In all relevant cases there is a moduli stack of objects $\mathcal{M}_\mathcal{A}$ which fits into a correspondence (3.3), and a quasicoherent sheaf $\mathrm{Ext}_\mathcal{A}(-,-)$ defined on $\mathcal{A} \times \mathcal{A}$ whose fibre above (a, a') is the dg vector space $\mathrm{Ext}_\mathcal{A}(a, a')$, such that

$$q^*\mathrm{Ext}_\mathcal{A}(-,-) \;\simeq\; \mathbf{T}_p.$$

Thus we should expect $\mathbf{T}_p$ to Tor amplitude in $(-\infty, \dim \mathcal{A}]$, in particular p should be quasismooth when $\mathcal{A}$ has dimension at most one: in examples this is clear since $\mathcal{M}_\mathcal{A}$ and $\mathrm{Ext}_\mathcal{A}$ are smooth.

1. $\dim \mathcal{A} \leqslant 1$. $\mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})$ is a cohomological Hall algebra when

   $$\mathcal{A} \;=\; \mathrm{Rep}Q,\; \mathrm{Coh}C$$

   is the category of representations of a quiver Q or coherent sheaves on a smooth proper curve C. See [KS].

Moreover, in [PS, Prop. 3.10] Porta and Sala show that for $\mathcal{A} = \mathrm{Coh}(X)$ coherent sheaves on a smooth proper scheme over $\mathbf{C}$, then q has Tor amplitude in $(-\infty, \dim \mathcal{A} - 1]$, so it is quasismooth when $\mathcal{A}$ has dimension two and below.

2. $\dim \mathcal{A} \leqslant 2$. $\mathrm{H}^{\mathrm{BM}}_{\bullet}(\mathcal{M}_{\mathcal{A}})$ is a cohomological Hall algebra when

$$\mathcal{A} \;=\; \mathrm{Rep}P_Q,\ \mathrm{Higgs}C,\ \mathrm{Coh}S$$

is the category of representations of the preprojective algebra of a quiver Q, Higgs sheaves on a smooth proper curve C, or coherent sheaves on a smooth proper surface S. The first two are just the cotangent categories of $\mathrm{Rep}Q$ and $\mathrm{Coh}C$, meaning that their moduli stacks are just $\mathbf{T}^*_{\mathcal{M}_{\mathrm{Rep}Q}}$ and $\mathbf{T}^*_{\mathcal{M}_{\mathrm{Coh}C}}$. See [YZ1, YZ2, KV2].

3. The third example is related to the 3 Calabi Yau category

$$\mathcal{A} \;=\; \mathrm{Rep}(Q,W)$$

the representations of a quiver with potential. In [KS, §7], Kontsevich and Soibelman in [KS] consider the "critical cohomology", a certain dual of the compactly supported cohomology of a sheaf of vanishing cycles. Note that

$$\mathrm{H}^{\bullet}_c(X,\varphi)^{\vee} \;=\; \mathrm{H}^{\bullet}(X,\mathbf{D}\varphi),$$

so this really is analogous to Theorem 3.1.10,[3] see [KS].

3.1.12. **History.** Cohomological Hall algebras in the modern form of Theorem 3.1.10 were first introduced by Kontsevich and Soibelman in [KS], which was preceded by other attempts to define CoHAs for quivers. They defined the CoHA of a quiver with potential and showed that its Poincare polynomial is given in terms of DT invariants.

In certain cases the category $\mathrm{Coh}(Y)_0$ of zero dimensional torsion sheaves on a 3 Calabi Yau variety can be realised as representations of a certain quiver with potential. When $Y = \mathbf{C}^3$, Rapčák, Soibelman, Yang and Zhao [RSYZ1] showed that the double of the equivariant spherical CoHA is the affine Yangian of $\mathfrak{gl}(1)$, and showed that there is an action on the cohomology of the moduli space of spiked instantons in $\mathbf{P}^2$, and generalise this to 3 Calabi Yau toric threefolds in [RSYZ2]. This allowed them to make progress on the conjecture that assigns to any toric Calabi Yau n-fold X a certain vertex algebra $\mathcal{W}_{X,r_1,\dots,r_n}$.

We now turn to dimension two categories.

The construction of CoHAs is very similar to the construction by Grojnowski and Nakajima [Groj, Na] of an action of the Heisenberg algebra (W algebra of $\mathfrak{gl}(1)$) on the cohomology $\mathrm{H}^{\bullet}(\mathrm{Hilb}(S))$

[3] In [KS] they work with a variant of the category of mixed Hodge modules.

of the Hilbert scheme of points on a smooth surface S, which is an example of the AGT correspondence for $\mathfrak{gl}(1)$. The relation to three dimensional CoHAs is by "dimensional reduction", as proposed in [KS], see also work of Davison [Da], which relates the dimensional reduction of the CoHA attached to a quiver with potential with Yangians.

Moreover, Yang and Zhao [YZ1, YZ2] have related the Drinfeld double of $\mathrm{H}^{\mathrm{BM}}_{\bullet}(\mathcal{M}_{\mathrm{Rep}P_Q})$ to affine Yangians, Kapranov and Vasserot [KV2] have related the CoHA of a surface to factorisation algebras. Sala and Schiffmann [SS] have given a description of the CoHA of Higgs bundles on a curve. Working in the analytic category, Kapranov and Vasserot [KV2] have shown that the CoHA of dimension zero torsion sheaves on a smooth proper surface $\mathrm{Coh}(S)_0$ carries a (topological) factorisation algebra structure.

3.1.13. **What we prove.** How is this relevant to the current work? Some common features of CoHAs as above are:

1. the (often mysterious) relation to "affine" objects: Yangians and vertex algebras,
2. the existence of a (localised) coproduct on the CoHA.[4]

In the setting of dimension one abelian categories, we combine the two: the CoHA *is* a vertex coalgebra (not just has an action of one), and this vertex coalgebra structure is a (vertex analogue of) a coproduct on the CoHA. Moreover, we explain the singularities occuring in the localised coproducts and formulas for CoHAs: they are the Euler classes turning up when one computes the CoHA using torus localisation.

3.2 The bivariant Euler class

3.2.1. **Euler classes in topology.** Let X be a topological space and $E \to X$ a complex vector bundle over it. Associated to this a long exact sequence on cohomology called the *Thom-Gysin* sequence:

$$\cdots \;\to\; \mathrm{H}^{\bullet-2\mathrm{rk}E}(X) \;\to\; \mathrm{H}^{\bullet}(X) \;\to\; \mathrm{H}^{\bullet}(E\backslash X) \;\to\; \cdots \tag{3.7}$$

Since these are all maps of $\mathrm{H}^{\bullet}(X)$ modules, the first map is multiplication by an element $e(E) \in \mathrm{H}^{2\mathrm{rk}E}(X)$, defined to be the *Euler class* of E.

[4]We did not state it explicitly, but every time in the literature a CoHA is compared to a Yangian, one needs to Drinfeld double the CoHA, for which one needs a coproduct compatible with the algebra structure.

3.2.2. The Thom-Gysin sequence (3.7) can lifted to a distinguished triangle of sheaves on X, meaning that taking derived global sections gives (3.7).

Here and in the following, we use the language of the six functors, see section 2.7 for a review.

Write $\mathrm{Sh}(X)$ for the derived category of sheaves of k vector spaces on X. The *Gysin sequence* is the distinguished triangle in $\mathrm{Sh}(X)$

$$i^! k_E \;\to\; i^* k_E \;\to\; i^* j_* j^* k_E \;\xrightarrow{+1} \tag{3.8}$$

where $i : X \to E$ denotes the zero section, $j : E \backslash X \to E$ is its open complement and k_E is the constant sheaf with fibre k.[5] Thus $i^! k_E \to i^* k_E$ is a sheaf level description of multiplication by the Euler class.

3.2.3. **Euler classes in geometry.** We now consider *any* category of spaces X admitting a category of sheaves $\mathrm{Sh}(X)$ with the six functor formalism (see section 2.7).

If $i : X \to Y$ is any closed embedding, we still have a Gysin sequence

$$i^! k_Y \;\to\; i^* k_Y \;\to\; i^* j_* j^* k_Y \;\xrightarrow{+1} \tag{3.9}$$

taking cohomology of which gives the Thom Gysin sequence

$$\cdots \;\to\; \mathrm{H}^\bullet(X/Y) \;\to\; \mathrm{H}^\bullet(X) \;\to\; \mathrm{H}^\bullet(X, i^* j_* j^* k_Y) \;\to \cdots \tag{3.10}$$

whose first term is the bivariant homology of $X \to Y$, and the third should be thought of as the cohomology of a small neighbourhood of X in Y, with X removed. As before, j is the open complement of i.

Definition 3.2.4. Let $i : X \to Y$ be a closed embedding admitting a retraction $Y \twoheadrightarrow X$. Its *bivariant Euler class* $e(Y/X)$ is the element of the bivariant homology $\mathrm{H}^\bullet(Y/X)$,

$$e(Y/X) \;=\; i_* 1,$$

where $i_* : \mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(Y/X)$.

Here, i_* denotes the proper pushforward map on bivariant homology $\mathrm{H}^\bullet(X) = \mathrm{H}^\bullet(X/X) \to \mathrm{H}^\bullet(Y/X)$. It is easy to show that the first map in the Thom Gysin sequence (3.10) is cup product on the right with $e(Y/X)$, via

$$\mathrm{H}^\bullet(X/Y) \otimes \mathrm{H}^\bullet(Y/X) \;\to\; \mathrm{H}^\bullet(X/X) \;\simeq\; \mathrm{H}^\bullet(X).$$

This prompts the following definition.

[5] Traditionally $i^! k_E$ is called the *local cohomology sheaf* of the closed embedding i.

Definition 3.2.5. For *any* closed embedding $i : X \to Y$, (*right*) *multiplication by* its *bivariant Euler class* is

$$\cdot e(Y/X) \;:\; \mathrm{H}^\bullet(X/Y) \;\to\; \mathrm{H}^\bullet(X),$$

the first term in the Thom Gysin sequence, induced by the natural transformation $i^! \Rightarrow i^*$.

3.2.6. **Relation to ordinary Euler classes.** The bivariant Euler class of a vector bundle should give an actual cohomology class on the base, the usual notion of Euler class. This holds more generally in the *smooth* setting:

Definition 3.2.7. Let $i : X \to Y$ be a closed embedding admitting a *smooth* retraction $p : Y \to X$ of dimension d. Its *Euler class* $e(Y)$ is the element of $\mathrm{H}^{2d}(X)$ defined by

$$\begin{tikzcd} \mathrm{H}^\bullet(X/Y) \arrow[rr, "\cdot[Y/X]", "\sim"'] \arrow[dr, "\cdot e(Y/X)"'] & & \mathrm{H}^{\bullet-2d}(X) \arrow[dl, dashed, "\cdot e(Y)"] \\ & \mathrm{H}^\bullet(X) & \end{tikzcd} \tag{3.11}$$

This indeed defines an element because all maps in (3.11) are graded $\mathrm{H}^\bullet(X)$ module morphisms. Here $[Y/X]$ is the fundamental class (section C.4), which is an isomorphism by smoothness, since then by purity the fundamental class gives an isomorphism $k_Y \xrightarrow{\sim} p^! k_X[2d]$. A consequence of the definition is

$$e(Y/X) \;=\; [Y/X] \cdot e(Y) \;=\; p^* e(Y) \cdot [Y/X]. \tag{3.12}$$

3.2.8. **Functoriality.** Take a map between two closed embeddings: a pullback square

$$\begin{tikzcd} \overline{X} \arrow[r, "\bar{\imath}"] \arrow[d, "\bar{f}"] & \overline{Y} \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

It is then easy to show that

Lemma 3.2.9. *For any class* $\alpha \in \mathrm{H}^\bullet(X/Y)$,

$$f^*(\alpha \cdot e(Y/X)) \;=\; (f^*\alpha) \cdot e(\overline{Y}/\overline{X}).$$

Next take two closed embeddings admitting a retraction, and a map between them, meaning two pullback squares

$$\begin{tikzcd} \overline{X} \arrow[r, "\bar{\imath}"] \arrow[d, "\bar{f}"] & \overline{Y} \arrow[r, "\bar{\pi}"] \arrow[d, "f"] & \overline{X} \arrow[d, "\bar{f}"] \\ X \arrow[r, "i"] & Y \arrow[r, "\pi"] & X \end{tikzcd}$$

Then we have

Proposition 3.2.10. *As classes in* $\mathrm{H}^\bullet(\overline{Y}/\overline{X})$,

$$f^* e(Y/X) \;=\; e(\overline{Y}/\overline{X}).$$

Proof. Because pullback and pushforward commute,

$$f^* e(Y/X) \;=\; f^* i_* 1_X \;=\; \bar{\imath}_* f^* 1_X \;=\; \bar{\imath}_* 1_{\overline{X}} \;=\; e(\overline{Y}/\overline{X}).$$

□

3.2.11. **Whitney sum, smooth case.** The classical Whitney sum formula says that given a short exact sequence of vector bundles

$$0 \;\to\; E_1 \;\to\; E \;\to\; E_2 \;\to\; 0 \tag{3.13}$$

we have $e(E) = e(E_1) \cdot e(E_2)$, so the Euler class is multiplicative. In particular, it descends to a map on K theory. To rephrase this in terms of bivariant Euler classes, note that there is a (homotopy) pullback diagram

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\psi"] & E_2 \end{tikzcd} \tag{3.14}$$

and ψ induces an isomorphism

$$\psi^* \;:\; \mathrm{H}^\bullet(E/E_2) \;\xrightarrow{\sim}\; \mathrm{H}^\bullet(E_1/X).$$

In particular, even though $E \to E_2$ does not in general admit a section, we can define

$$e(E/E_2) \;:=\; (\psi^*)^{-1} e(E_1/X).$$

We then get

Proposition 3.2.12 (Whitney sum)**.** *Under a short exact sequence of vector bundles (3.13),*

$$e(E/X) \;=\; e(E/E_2) \cdot e(E_2/X).$$

Then applying (3.11) gives $e(E) = e(E_1)e(E_2)$ *as elements of* $\mathrm{H}\bullet(X)$*.*

Proof. Apply the projection formula (section 2.7.5) to the diagram

$$\begin{tikzcd}
E_1 \arrow[r, "\overline{\psi}"] \arrow[d] & E \arrow[d] & \\
X \arrow[r, "\psi"] & E_2 \arrow[r] & X
\end{tikzcd}$$

we get that

$$\overline{\psi}_*(\psi^* e(E/E_2)\cdot 1) \;=\; e(E/E_2)\cdot \psi_* 1$$

as elements of $\mathrm{H}^\bullet(E/X)$. The left hand side is $\overline{\psi}_* e(E_1/X) = e(E/X)$, and the right hand side is $e(E/E_2)\cdot e(E_2/X)$. □

3.2.13. **Example.** For an example we compute the universal Euler class of vector bundles. Recall that every line bundle is the pullback by a map into $\mathrm{B}\mathbf{G}_m$ of the tautological line bundle $\gamma = \mathbf{A}^1/\mathbf{G}_m$. The Thom Gysin sequence then involves multiplication by the Euler class $e(\gamma)$:

$$\cdots \;\to\; \mathrm{H}^{\bullet-2}(\mathrm{B}\mathbf{G}_m) \;\overset{\cdot e(\gamma)}{\to}\; \mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m) \;\to\; \mathrm{H}^\bullet(\mathrm{pt}) \;\to\; \cdots$$

since $(\mathbf{A}^1\backslash 0)/\mathbf{G}_m = \mathrm{pt}$. We can identify all the cohomology groups,

$$\cdots \;\to\; k[t] \;\overset{\cdot e(\gamma)}{\to}\; k[t] \;\to\; k \;\to\; \cdots$$

and so $e(\gamma) = t$, rescaling t if necessary. A more delicate analysis, e.g. using integral ℓ-adic cohomology, will show that $e(\gamma^{\otimes n}) = n\cdot e(\gamma)$.

3.2.14. We can repeat this analysis for the vector bundle

$$\gamma \;=\; V/G \;\to\; \mathrm{B}G$$

where G is any complex reductive group and V is any finite dimensional representation. Taking $V = \mathbf{C}^n$ the standard representation of $G = \mathrm{GL}_n$ gives the universal rank n vector bundle. Choosing a maximal torus T and taking the pullback

$$\begin{tikzcd}
V/T \arrow[r] \arrow[d] & \mathrm{B}T \arrow[d] \\
V/G \arrow[r] & \mathrm{B}G
\end{tikzcd}$$

we get that the image of $e(\gamma)$ under

$$\mathrm{H}^\bullet(\mathrm{B}G) \;\simeq\; \mathrm{H}^\bullet(\mathrm{B}T)^W \;\hookrightarrow\; \mathrm{H}^\bullet(\mathrm{B}T)$$

is

$$e(\gamma) \;=\; \prod_{\lambda\in\Lambda} \lambda^{\dim V_\lambda}.$$

Here we have identified $\mathrm{H}^\bullet(\mathrm{B}T) \simeq \mathbf{C}[\Lambda]$ as a polynomial algebra generated by the character lattice $\Lambda = \mathrm{Hom}_{\mathrm{Grp}}(T, \mathbf{G}_m)$ in degree two, and V_λ is the summand of V on which T acts by λ. For the standard representation of GL_n, the Euler class $e(\mathbf{C}^n/\mathrm{GL}_n)$ is the element

$$t_1 \cdots t_n \ \in \ \mathrm{H}^\bullet(\mathrm{B}T) \ \simeq \ \mathbf{C}[t_1, ..., t_n].$$

3.3 Localisation

Inverting equivariant cohomology classes is a powerful tool because two conflicting effects are often simultaneously true:

1. inverting does not lose much information (none if inverting non zero divisors), yet
2. dissimilar spaces can have the same cohomologies after inversion: abelian localisation (section 3.5) covers the case of a closed subspace, and we will see that the localised cohomology of singular spaces behaves like for smooth spaces (see section 3.4).

In this section we set up the notation.

3.3.1. Fix a base Artin stack B, and let $S \subseteq \mathrm{H}^\bullet(B)$ be a multiplicative subset. We call

$$M_{\mathrm{loc}} \ := \ M[S^{-1}]$$

the *localisation* of a $\mathrm{H}^\bullet(B)$ module M.

3.3.2. **Specialisation and concentration.**

Definition 3.3.3 (Concentration)**.** An Artin stack Y over B is (S-)*concentrated* if

$$\mathrm{H}^\bullet(Y)_{\mathrm{loc}} \ \simeq \ 0.$$

We say a closed embedding $i : X \to Y$ over B is (S-)*concentrated* if $Y \backslash X$ is concentrated.

Definition 3.3.4 (Specialisation)**.** A closed embedding $i : X \to Y$ over B is (S-)*specialised* if

$$\cdot e(Y/X) \ : \ \mathrm{H}^\bullet(X/Y)_{\mathrm{loc}} \ \to \ \mathrm{H}^\bullet(X)_{\mathrm{loc}}$$

is an isomorphism.

It may be useful to see the following commuting diagram of $\mathrm{H}^\bullet(X)$ modules, whose rows are long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^\bullet(X/Y) \arrow[r, "i_*"] \arrow[d, equal] & \mathrm{H}^\bullet(Y) \arrow[r] \arrow[d, "i^*"] & \mathrm{H}^\bullet(Y\backslash X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^\bullet(X/Y) \arrow[r, "\cdot e(Y/X)"] & \mathrm{H}^\bullet(X) \arrow[r] & \mathrm{H}^\bullet(X, i^* j_* k) \arrow[r] & \cdots
\end{tikzcd} \tag{3.15}$$

where j is the open complement of i. So concentration/specialisation just says that, upon localising, the right term in the top/bottom row of (3.15) vanishes, or equivalently the left map is an isomorphism. It also follows that

Lemma 3.3.5. *If i^* is an isomorphism then concentration and specialisation are equivalent*

For example, this is the case when the fibres of Y over X are cohomologically trivial, like when Y is a vector bundle, cone bundle or perfect complex over X.

3.3.6. **Pullbacks.**

Lemma 3.3.7. *Let $i : X \to Y$ be a concentrated and specialised closed embedding, and consider the Cartesian square*

$$\begin{tikzcd}
\Omega Y \arrow[r, "\bar{\imath}"] \arrow[d] & X \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

Then the pullback map $i^* : \mathrm{H}^\bullet(X/Y) \to \mathrm{H}^\bullet(\Omega Y/X)$ is an isomorphism after localisation.

Proof. We have a commuting diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(X/Y) \arrow[r, "i_*"] \arrow[d, "i^*"] & \mathrm{H}^\bullet(Y) \arrow[d, "i^*"] \\
\mathrm{H}^\bullet(\Omega Y/X) \arrow[r, "\bar{\imath}_*"] & \mathrm{H}^\bullet(X)
\end{tikzcd}$$

Note that $\bar{\imath}_*$ is an isomorphism on the nose since $(\Omega Y)_{cl} = X$. Both i_* and $i^* : \mathrm{H}^\bullet(Y) \to \mathrm{H}^\bullet(X)$ are isomorphisms after localisation by the hypothesis, proving the claim. $\square$

Lemma 3.3.8. *Let $i : X \to Y$ be a concentrated closed embedding, and $W \to X$ be any map. Then the cup product*

$$\mathrm{H}^\bullet(W/X) \otimes_{\mathrm{H}^\bullet(X)} \mathrm{H}^\bullet(X/Y) \;\to\; \mathrm{H}^\bullet(W/Y)$$

is an isomorphism after localisation, i.e. we have

$$\mathrm{H}^\bullet(W/X)_{\mathrm{loc}} \otimes_{\mathrm{H}^\bullet(X)_{\mathrm{loc}}} \mathrm{H}^\bullet(X/Y)_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{H}^\bullet(W/Y)_{\mathrm{loc}}$$

Proof. After stratifying Y, we get a commuting diagram of long exact sequences

$$\begin{tikzcd}
\mathrm{H}^\bullet(W_0/Y) \arrow[r] & \mathrm{H}^\bullet(W/Y) \arrow[r] & \mathrm{H}^\bullet(W_1/Y) \arrow[r, "+1"] & {} \\
\mathrm{H}^\bullet(W_0/X) \otimes_{\mathrm{H}^\bullet(X)} \mathrm{H}^\bullet(X/Y) \arrow[r] \arrow[u, "\cup"] & \mathrm{H}^\bullet(W/X) \otimes_{\mathrm{H}^\bullet(X)} \mathrm{H}^\bullet(X/Y) \arrow[r] \arrow[u, "\cup"] & \mathrm{H}^\bullet(W_1/X) \otimes_{\mathrm{H}^\bullet(X)} \mathrm{H}^\bullet(X/Y) \arrow[r, "+1"] \arrow[u, "\cup"] & {}
\end{tikzcd}$$

where $W_0 = W \times_Y X$ and $W_1 = W \times_Y (Y \backslash X)$. The right two vector spaces vanish after localisation because W_1 is concentrated. Thus it suffices to prove the Lemma after replacing W with W_0, which follows immediately since $(W_0)_{cl} = X$.

□

Proposition 3.3.9. *Let $X \to Y_k$ be maps admitting restrictions ($k = 1, 2, 3$), fitting into the Cartesian square*

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d] & X \arrow[d] \\
Y_2 \arrow[r] & Y_3
\end{tikzcd} \tag{3.16}$$

Assume that the map $i_3 : X \to Y_3$ is a concentrated and specialised closed embedding. Then the pullback map on bivariant homology

$$i_3^* \;:\; \mathrm{H}^\bullet(Y_2/Y_3)_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{H}^\bullet(Y_1/X)_{\mathrm{loc}}$$

is an isomorphism.

Proof. Noting that (3.17) is a diagram over Y_3; we now stratify Y_3 by

$$X \;\to\; Y_3 \;\leftarrow\; Y_3 \backslash X$$

and the pullbacks of (3.17) are

$$\begin{tikzcd}
Y_1 \backslash \tilde{Y}_1 \arrow[r] \arrow[d] & X \arrow[d] & \tilde{Y}_1 \arrow[r] \arrow[d] & \varnothing \arrow[d] \\
Y_2 \backslash \tilde{Y}_2 \arrow[r] & X & \tilde{Y}_2 \arrow[r] & Y_3 \backslash X
\end{tikzcd} \tag{3.17}$$

where $\tilde{Y}_i = Y_i \times_{Y_3} (Y_3 \backslash X)$, is contained in $Y_i \backslash X$. This gives a commuting diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(Y_1\backslash\tilde{Y}_1/X) \arrow[r] & \mathrm{H}^\bullet(Y_1/X) \arrow[r] & \mathrm{H}^\bullet(\tilde{Y}_1/X) \arrow[r, "+1"] & {} \\
\mathrm{H}^\bullet(Y_2\backslash\tilde{Y}_2/Y_3) \arrow[r] \arrow[u, "i_3^*"] & \mathrm{H}^\bullet(Y_2/Y_3) \arrow[r] \arrow[u, "i_3^*"] & \mathrm{H}^\bullet(\tilde{Y}_2/Y_3) \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd}$$

Both spaces on the right vanish after localisation, the top because $\tilde{Y}_1 = \varnothing$, and the bottom by concentration, as $\mathrm{H}^\bullet(Y_3\backslash X)_{\mathrm{loc}} = 0$ implies that $\mathrm{H}^\bullet(\tilde{Y}_2)_{\mathrm{loc}} = 0$.

It remains to show the left vertical arrow is an isomorphism after localisation. This follows because it fits as the bottom arrow into the following commuting diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(Y_2\backslash\tilde{Y}_2/X) \otimes_{\mathrm{H}^\bullet(X)} \mathrm{H}^\bullet(X/Y_3) \arrow[r, "\bar{\imath}_3^*\otimes i_3^*"] \arrow[d, "\cup"] & \mathrm{H}^\bullet(Y_1\backslash\tilde{Y}_1/\Omega Y_3) \otimes_{\mathrm{H}^\bullet(X)} \mathrm{H}^\bullet(\Omega Y_3/X) \arrow[d, "\cup"] \\
\mathrm{H}^\bullet(Y_2\backslash\tilde{Y}_2/Y_3) \arrow[r, "i_3^*"] & \mathrm{H}^\bullet(Y_1\backslash\tilde{Y}_1/X)
\end{tikzcd} \tag{3.18}$$

where $\Omega Y_3 = X \times_{Y_3} X$, coming from the pair of pullback squares

$$\begin{tikzcd}
Y_1\backslash\tilde{Y}_1 \arrow[r] \arrow[d] & \Omega Y_3 \arrow[r] \arrow[d] & X \arrow[d] \\
Y_2\backslash\tilde{Y}_2 \arrow[r] & X \arrow[r] & Y_3
\end{tikzcd}$$

The top rightwards arrow in (3.18) is an is an isomorphism after localisation because $\bar{\imath}_3 : \Omega Y_3 \to X$ is an isomorphism on classical parts and by Lemma 3.3.7. The vertical arrows are isomorphisms after localisation by Lemma 3.3.8 applied to the concentrated closed embedding $X \to Y_3$, finishing the proof. $\square$

Corollary 3.3.10. *In the setting of Proposition 3.3.9, if $Y_2 \to Y_3$ is quasismooth and*

$$[Y_2/Y_3]\cdot \ : \ \mathrm{H}^\bullet(Y_3) \ \to \ \mathrm{H}^\bullet(Y_2/Y_3)$$

is an isomorphism after localisation, the same is true for

$$[Y_1/X]\cdot \ : \ \mathrm{H}^\bullet(X) \ \to \ \mathrm{H}^\bullet(Y_1/X).$$

Proof. Follows from the commuting diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(Y_3) \arrow[r, "{[Y_2/Y_3]}"] & \mathrm{H}^\bullet(Y_3/Y_2) \\
\mathrm{H}^\bullet(X) \arrow[r, "{[Y_2/Y_3]}"] \arrow[u, "i_3^*"] & \mathrm{H}^\bullet(Y_1/X) \arrow[u, "i_3^*"]
\end{tikzcd}$$

□

The most basic example of this is

Corollary 3.3.11. *Let E be a strict $\mathbf{G}_m$ equivariant perfect complex in tor amplitude $\leqslant 1$ and nonzero $\mathbf{G}_m$ weights. In particular it is a bounded complex of vector bundles E_i. Then after inverting $S = (e(E_1), e(E_2), \dots)$,*

$$[E/X]\cdot \ : \ \mathrm{H}^\bullet(X)_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{H}^\bullet(E/X)_{\mathrm{loc}}$$

is an isomorphism.

Proof. Begin by writing $E_{\leqslant 0}$ and $E_{\geqslant 1}$ for the perfect complexes formed by discarding all vector bundles in the complex except in degrees $\leqslant 0$ and $\geqslant 1$, respectively. Then E is the fibre

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X \arrow[d] \\ E_{\leqslant 0} \arrow[r] & E_{\geqslant 1}[1] \end{tikzcd}$$

Note that $X \to E_{\geqslant 1}[1]$ is a concentrated and specialised closed embedding with respect to $S = (e(E_1), e(E_2), \dots)$, by Lemma 3.3.30. Moreover, since $E_{\leqslant 0}$ and $E_{\geqslant 1}[1]$ are smooth over X, it follows that multiplication by the fundamental class of $E_{\leqslant 0} \to E_{\leqslant 1}[1]$ is an isomorphism on the nose. Applying Corollary 3.3.11, we get that multiplication by $[E/X]$ is an isomorphism after localisation. □

3.3.12. **Functoriality.** We now turn to the functoriality properties of concentration and specialisation.

Lemma 3.3.13. *Let $Y \to Y'$ be a surjective map of spaces over X whose fibres have trivial cohomology. Then Y is concentrated if and only if Y' is.*

Proof. Applying the Leray sequence to this map gives that $\mathrm{H}^\bullet(Y_2) = \mathrm{H}^\bullet(Y)$ as $\mathrm{H}^\bullet(X)$-modules. □

Proposition 3.3.14. *Let $Y \to Y'$ be a map of spaces over X. Then if Y' is concentrated, so is Y.*

Proof. There is a map of algebras $\mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(Y') \to \mathrm{H}^\bullet(Y)$, and so

$$\mathrm{H}^\bullet(Y) \ = \ \mathrm{Res}^{\mathrm{H}^\bullet(Y')}_{\mathrm{H}^\bullet(X)} \mathrm{H}^\bullet(Y)$$

is the restriction of $\mathrm{H}^\bullet(Y)$ viewed as a $\mathrm{H}^\bullet(Y')$-module to a $\mathrm{H}^\bullet(X)$-module. In particular, since $\mathrm{H}^\bullet(Y')_{\mathrm{loc}} = 0$ we have

$$\mathrm{H}^\bullet(Y)_{\mathrm{loc}} \;=\; \mathrm{H}^\bullet(Y) \otimes_{\mathrm{H}^\bullet(X)} \mathrm{H}^\bullet(X)_{\mathrm{loc}} \;=\; \mathrm{H}^\bullet(Y) \otimes_{\mathrm{H}^\bullet(Y')} \mathrm{H}^\bullet(Y')_{\mathrm{loc}} \;=\; 0.$$

□

The above proof can be summarised as

Lemma 3.3.15. *Let $\varphi : A \to B$ be a map of commutative rings and $S \subseteq A$ a multiplicative subset such that $B[\varphi(S)^{-1}] = 0$. If M is an A module arising from restriction of a B module, then $M[S^{-1}] = 0$.*

We now turn to sheafifying this.

3.3.16. **Sheaves of algebras.** Let $\mathcal{A}$ be a commutative monoid in $\mathrm{Sh}(X)$. This means that it admits product and unit maps

$$e \,:\, k_X \;\to\; \mathcal{A} \qquad\qquad m \,:\, \mathcal{A} \otimes \mathcal{A} \;\to\; \mathcal{A}$$

satisfying the axioms of a commutative monoid. Call such $\mathcal{A}$ a *sheaf of algebras* over X.

3.3.17. For instance, k_X is a sheaf of algebras.

3.3.18. This structure is preserved by:

1. $*$-pullbacks. Given a map $f : Y \to X$, the functor $f^* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is monoidal, so $f^*\mathcal{A}$ is naturally a sheaf of algebras.
2. $*$-pushforwards. Given a map $g : X \to Z$, the sheaf $g_*\mathcal{A}$ is a sheaf of algebras with product

$$g_*\mathcal{A} \otimes g_*\mathcal{A} \;\to\; g_*g^*(g_*\mathcal{A} \otimes g_*\mathcal{A}) \;\simeq\; g_*(g^*g_*\mathcal{A} \otimes g^*g_*\mathcal{A}) \;\to\; g_*(\mathcal{A} \otimes \mathcal{A}) \;\overset{g_*m}{\to}\; g_*\mathcal{A},$$

and unit

$$k_Z \;\to\; g_*g^*k_Z \;\simeq\; g_*k_X \;\overset{g_*e}{\to}\; g_*\mathcal{A}.$$

These two structures are compatible as follows:

Lemma 3.3.19. *The maps $\mathcal{A} \to f_*f^*\mathcal{A}$ and $g^*g_*\mathcal{A} \to \mathcal{A}$, induced by the (co)units of the adjunctions (f^*, f_*) and (g^*, g_*), are maps of sheaves of algebras over X.*

As a basic example of the this, the cohomology $\mathrm{H}^\bullet(X, \mathcal{A})$ is an algebra and there is a map $\mathrm{H}^\bullet(X, \mathcal{A})_X \to \mathcal{A}$ from the constant sheaf of algebras with value $\mathrm{H}^\bullet(X, \mathcal{A})$.

3.3.20. **Sheaves of modules.** Let $\mathcal{M} \in \mathrm{Sh}(X)$ be an $\mathcal{A}$-module, i.e. it comes with a map

$$a \;:\; \mathcal{A} \otimes \mathcal{M} \;\to\; \mathcal{M}$$

compatible with the commutative monoid structure on $\mathcal{A}$. We sometimes call this a *sheaf of $\mathcal{A}$ modules* over X.

3.3.21. This structure is preserved by:

1. $*$-pullbacks. Given a map $f : Y \to X$, then $f^*\mathcal{M}$ is a $f^*\mathcal{A}$ module.
2. $*$-pushforwards. Given a map $g : X \to Z$, then $g_*\mathcal{M}$ is a $g_*\mathcal{A}$ module.
3. !-pullbacks. Given a map $f : Y \to X$, then $f^!\mathcal{M}$ is a $f^*\mathcal{A}$ module, via the projection formula

$$f_!(f^*\mathcal{A} \otimes f^!\mathcal{M}) \;\simeq\; \mathcal{A} \otimes f_! f^!\mathcal{M} \;\to\; \mathcal{A} \otimes \mathcal{M} \;\xrightarrow{a}\; \mathcal{M}$$

   which corresponds by adjunction to a map

$$f^*\mathcal{A} \otimes f^!\mathcal{M} \;\to\; f^!\mathcal{M}.$$

4. !-pushforwards. Given a map $g : X \to Z$, then $g_!\mathcal{M}$ is a $g_*\mathcal{A}$ module, with action given by the projection formula:

$$g_*\mathcal{A} \otimes g_!\mathcal{M} \;\simeq\; g_!(g^*g_*\mathcal{A} \otimes \mathcal{M}) \;\to\; g_!(\mathcal{A} \otimes \mathcal{M}) \;\xrightarrow{g_!a}\; g_!\mathcal{M}.$$

Moreover, the (co)units of the (f^*, f_*) and $(f_!, f^!)$ adjunctions applied to $\mathcal{M}$ are maps of modules over the appropriate sheaves of algebras.

3.3.22. For instance, considering $\mathcal{A} = k_X$, we get that the cohomology and compactly supported cohomology of any sheaf $\mathcal{M}$

$$p_!\mathcal{M} \;=\; \mathrm{H}_c^\bullet(X, \mathcal{M}) \qquad\qquad p_*\mathcal{M} \;=\; \mathrm{H}^\bullet(X, \mathcal{M})$$

is a module over $\mathrm{H}^\bullet(X)$, where $p : X \to \mathrm{pt}$. The same statement applied to a map $f : X \to Z$ gives that the cohomology of the fibres f_*k_X acts on the relative cohomology $f_*\mathcal{M}$ and compactly supported cohomology $f_!\mathcal{M}$.

3.3.23. We have the following useful proposition:

Proposition 3.3.24. *Let $\mathcal{A}$ be a sheaf of algebras over X and $\mathcal{M} \in \mathrm{Sh}(X)$. If $\mathcal{F}$ is an $\mathcal{A}$ module, then the action of k_X on $\mathcal{F}$ factors through the map*

$$k_X \;\to\; \mathcal{A}.$$

In particular, the action of $\mathrm{H}^\bullet(X)$ on $\mathrm{H}^\bullet(X, \mathcal{F})$ factors through $\mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(X, \mathcal{A})$.

Proof. By tensor Hom adjunction we have a map of sheaves of algebras in $\mathrm{Sh}(X)$

$$\mathcal{A} \;\to\; \mathcal{E}\mathrm{nd}(\mathcal{M}).$$

In particular, because k_X is initial in the category of sheaves of algebras over X, its action on $\mathcal{M}$ will factor

$$\begin{tikzcd} & k_X \arrow[dl, dashed] \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{E}\mathrm{nd}(\mathcal{M}) \end{tikzcd}$$

□

This leads easily to many corollaries. Let $\mathcal{F} \in \mathrm{Sh}(X)$.

Corollary 3.3.25. *For any map $g : X \to Z$, the action of $\mathrm{H}^\bullet(Z)$ on $\mathrm{H}^\bullet(Z, g_!\mathcal{F})$ factors through $\mathrm{H}^\bullet(Z) \to \mathrm{H}^\bullet(X)$.*

Corollary 3.3.26. *For any two maps*

$$X \;\xrightarrow{g}\; Z \;\xleftarrow{h}\; W$$

the action of $\mathrm{H}^\bullet(Z)$ on $\mathrm{H}^\bullet(W, h_a h^b g_c \mathcal{F})$, where $a, b, c \in \{*, !\}$, factors through each of the maps

$$\begin{tikzcd} \mathrm{H}^\bullet(Z) \arrow[r] \arrow[d] & \mathrm{H}^\bullet(X) \arrow[d] \\ \mathrm{H}^\bullet(W) \arrow[r] & \mathrm{H}^\bullet(W, h^* g_* k_X) \end{tikzcd}$$

Proof. There is a map from the diagram of sheaves of algebras

$$\begin{tikzcd} & g_* k_X \arrow[d] \\ h_* h^* k_Z \arrow[r] & h_* h^* g_* k_X \end{tikzcd} \tag{3.19}$$

into $\mathcal{E}\mathrm{nd}(h_a h^b g_c \mathcal{F})$. Indeed, this follows for the horizontal arrow in (3.19) by using Proposition 3.3.24 on $g_*\mathcal{F}$ and then applying h_*h^*. For the vertical arrow of (3.19), it follows by the definition of the action of g_*k_X on $h_*h^*g_*k_X$. □

When $j : X \to Z$ is an open embedding and $i : W \to Z$ is its complementary closed embedding, this says that the action of $\mathrm{H}^\bullet(Z)$ on the cohomology of $i^b j_c \mathcal{F}$ factors through the cohomology $\mathrm{H}^\bullet(W, i^* j_* k)$, which may be thought of as the cohomology of a punctured neighbourhood of the closed subspace. This in particular admits a map from the cohomology of both the open and the closed subspaces.

3.3.27. We can build more interesting example as follows, where the modules themselves are commutative monoids. Let $f_i : Y_i \to X$ be any maps and $\mathcal{A}$ a commutative monoid in $\mathrm{Sh}(X)$. Then we have a sequence of maps of commutative monoids in $\mathrm{Sh}(X)$

$$k_X \;\to\; \mathcal{A} \;\to\; f_{1*}f_1^*\mathcal{A} \;\to\; f_{2*}f_2^*f_{1*}f_1^*\mathcal{A} \;\to\; \cdots$$

and taking cohomologies gives a sequence of maps of algebras

$$\mathrm{H}^\bullet(X) \;\to\; \mathrm{H}^\bullet(X, \mathcal{A}) \;\to\; \mathrm{H}^\bullet(Y_1, f_1^*\mathcal{A}) \;\to\; \mathrm{H}^\bullet(Y_2, f_2^*f_{1*}f_1^*\mathcal{A}) \;\to\; \cdots \tag{3.20}$$

It follows from Proposition 3.3.24 that

Corollary 3.3.28. *If $S \subseteq \mathrm{H}^\bullet(X)$ is a multiplicative subset and one algebra in (3.20) localises to zero, so does every algebra to its right.*

3.3.29. **Example: cohomology sheaves.** Let $E \in \mathrm{Perf}(X)$ be a strict $\mathbf{G}_m$ equivariant perfect complex over Artin stack X with nonnegative weights, i.e. quasiisomorphic to a bounded complex of vector bundles over X

$$E \;\simeq\; (\,\cdots \to E_{-1} \xrightarrow{d_{-1}} E_0 \xrightarrow{d_0} E_1 \to \cdots\,).$$

Then

Lemma 3.3.30. *Assume in addition all the E_i are concentrated. If we set $S_i = (e(E_i))$, the total space (see [To, §3.3]) of $\mathcal{H}^i(E)$ is S_i-concentrated and S_i-specialised.*

Proof. Because $\mathcal{H}^i(E) \to X$ has contractible fibres, concentration and specialisation are equivalent. Then we apply Proposition 3.3.14 to $\ker d_i \to E_i$ to give that E_i being concentrated implies that $\ker d_i$ is concentrated, then Lemma 3.3.13 to $\ker d_i \to \mathcal{H}^i(E)$, whose fibres are contractible, to give that $\mathcal{H}^i(E)$ is concentrated too. □

For instance, let $\mathbf{G}_m$ act on Y trivially, and $E \in \mathrm{Perf}(Y/\mathbf{G}_m)$ be a strict perfect complex with nonzero $\mathrm{B}\mathbf{G}_m$ weights. Then E is a direct sum of strict perfect complexes $E(n)$ on X with weight $n \in \mathbf{Z}\backslash 0$, and so each term $E(n)_i$ in the bounded complex of vector bundles quasiisomorphic to $E(n)$ is concentrated, and so each $E_i = \bigoplus E(n)_i$ is concentrated and the conditions of the Lemma are satisfied.

3.4 Application to stacks and singular spaces

3.4.1. The reason Fulton and MacPherson [FM] invented bivariant homology was to study singular spaces. So it is not surprising that the bivariant Euler class will be well-suited to singular spaces.

3.4.2. The general picture is that the cohomology of sheaves on singular spaces is much more complicated than for smooth spaces. However, if all maps are T-equivariant for the action of a torus T, the *free part* of cohomology often behaves exactly as in the smooth case!

3.4.3. Instead of requiring that we localise $\mathrm{H}^\bullet_T(\mathrm{pt})$ modules by tensoring with the fraction field, we localise with respect to an arbitrary multiplicative subset in $\mathrm{H}^\bullet(X)$. This is necessary when dealing with Artin stacks, see Example 3.5.27.

3.4.4. **Relation to inverted Euler classes.** When the morphism is concentrated, the bivariant Euler class defines a honest localised cohomology class on X:

Definition 3.4.5. Let $i : X \to Y$ be a closed embedding admitting a *quasismooth* retraction $Y \twoheadrightarrow X$ of dimension d. If i is specialised, its (*inverse*) *localised Euler class* is the element $e(Y)^{-1}$ of $\mathrm{H}^{-2d}(X)$ defined by

$$\begin{tikzcd} \mathrm{H}^\bullet(X/Y) \arrow[rr, "\cdot{[Y/X]}"] \arrow[dr, "\cdot e(Y/X)"', "\sim"] & & \mathrm{H}^{\bullet-2d}(X) \\ & \mathrm{H}^\bullet(X) \arrow[ur, dashed, "\cdot e(Y)^{-1}"'] & \end{tikzcd} \tag{3.21}$$

If $S \subseteq \mathrm{H}^\bullet(X)$ be a multiplicative subset such that upon localising $\cdot[Y/X]$ is an isomorphism, then the *localised Euler class* is its inverse $e(Y) \in \mathrm{H}^{2d}(X)$.

Note that since (3.21) is a diagram of $\mathrm{H}^\bullet(X)_{\mathrm{loc}}$ modules, the dotted map is multiplication by an element of $\mathrm{H}^{-2d}(X)_{\mathrm{loc}}$. In the smooth setting, $e(Y)^{-1}$ is inverse to the Euler class $e(Y)$ of Definition 3.2.7, i.e. we can take $S = (1)$ in the above.

3.4.6. **Singular Whitney sum formula.** The bivariant Euler class can do more for us. Consider a homotopy fiber in the category of spaces (which we recall for us means derived Artin stacks) over X admitting a section which is a closed embedding

$$Y_1 \xrightarrow{\alpha} Y \xrightarrow{\beta} Y_2 \tag{3.22}$$

3.4.7. Note that (3.22) being a homotopy fiber product means that those maps fit into a pullback square

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "i_2"] & Y_2 \end{tikzcd} \tag{3.23}$$

Let $S \subseteq \mathrm{H}^\bullet(X)$ be a multiplicative subset.

Lemma 3.4.8. *If $X \to Y_2$ is S-concentrated, pullback by i_2 induces an isomorphism upon localisation*

$$i_2^* \;:\; \mathrm{H}^\bullet(Y/Y_2)_{\mathrm{loc}} \;\xrightarrow{\sim}\; \mathrm{H}^\bullet(Y_1/X)_{\mathrm{loc}}.$$

Proof. Expand the diagram (3.23) to a diagram of pullback squares whose rows are complementary closed and open embeddings

$$\begin{tikzcd} Y_1 \arrow[r, "\alpha"] \arrow[d, "\beta_0"] & Y \arrow[d, "\beta"] & Y\backslash Y_1 \arrow[l, "\bar{j}_2"'] \arrow[d, "\bar{\beta}"] \\ X \arrow[r, "i_2"] & Y_2 & Y_2\backslash X \arrow[l, "j_2"'] \end{tikzcd}$$

Then we have distinguished triangle

$$\beta^! j_{2!} j_2^! k \;\to\; \beta^! k \;\to \beta^! i_{2*} i_2^* k \;\xrightarrow{+1}$$

whose long exact sequence on cohomology is

$$\cdots \;\to\; \mathrm{H}^\bullet(Y, \beta^! j_{2!} k) \;\to\; \mathrm{H}^\bullet(Y/Y_2) \;\xrightarrow{i_2^*}\; \mathrm{H}^\bullet(Y_1/X) \;\to\; \cdots$$

Note that the action of $\mathrm{H}^\bullet(X)$ on $\mathrm{H}^\bullet(Y, \beta^! j_{2!} k)$ factors through the pullback $\mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(Y_2\backslash X)$. Thus since $\mathrm{H}^\bullet(Y_2\backslash X)_{\mathrm{loc}} = 0$ by concentration, the localisation of this module $\mathrm{H}^\bullet(Y, \beta^! j_{2!} k)_{\mathrm{loc}}$ also vanishes, proving the Lemma. □

It follows from Lemma 3.4.8 that we can define an element

$$e(Y/Y_2) \;\in\; \mathrm{H}^\bullet(Y/Y_2)_{\mathrm{loc}}$$

by setting $i_2^* e(Y/Y_2) = e(Y_1/X)$.

Proposition 3.4.9 (Singular Whitney sum)**.** *If $X \to Y_2$ is S-concentrated, then as elements of $\mathrm{H}^\bullet(Y/X)_{\mathrm{loc}}$,*

$$e(Y/X) \;=\; e(Y/Y_2)\cdot e(Y_2/X).$$

Proof. Exactly the same as the proof of 3.2.12, using Lemma 3.4.8. □

3.4.10. **Example: distinguished triangle of perfect complexes.** A large class of examples of homotopy fibres (3.22) come from distinguished triangles of perfect complexes in nonnegative degree (as otherwise the zero section may no longer be a closed embedding, e.g. as for vector bundle stacks)

$$E \;\to\; E' \;\to\; E'' \;\xrightarrow{+1}$$

since the map on total spaces (see [To]) is a homotopy fibre (and cofiber)

$$E \;\to\; E' \;\to\; E'' \tag{3.24}$$

If E' is S-concentrated (equivalently, S-specialised) then the singular Whitney sum gives

$$e(E'/X) \;=\; e(E'/E'')e(E''/X). \tag{3.25}$$

If all three zero sections are concentrated then we also get the singular Whitney sum for the left rotated distinguished triangle

$$E' \;\to\; E'' \;\to\; E[1] \;\xrightarrow{+1}$$

but not necessarily the right rotated unless E is concentrated in degrees $\geqslant 1$.

3.4.11. With the right quasismoothness assumptions we can use fundamental classes to turn (3.25) into a statement in the honest localised cohomology $\mathrm{H}^\bullet(X)_{\mathrm{loc}}$.

Assume that each term in (3.24) is quasismooth (so the perfect complexes have tor amplitude in $[0,1]$), and the map $E' \to E''$ is also quasismooth. Then we have the commuting diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(X)\otimes\mathrm{H}^\bullet(X) \arrow[rr, "\cdot"] \arrow[d, "\pi_{E''}^*\otimes\mathrm{id}"', "\sim"] & & \mathrm{H}^\bullet(X) \arrow[dd, "{[E'/X]\cdot}"] \\
\mathrm{H}^\bullet(E'')\otimes\mathrm{H}^\bullet(E''/X) \arrow[d, "{[E'/E'']\otimes[E''/X]\cdot}"'] & & \\
\mathrm{H}^\bullet(E'/E'')\otimes\mathrm{H}^\bullet(E''/X) \arrow[rr, "\cdot"] & & \mathrm{H}^\bullet(E'/X)
\end{tikzcd}$$

In particular, if the fundamental classes over X of each term in (3.24) give isomorphisms on localised cohomologies, we get an equality in $\mathrm{H}^\bullet(X)_{\mathrm{loc}}$

$$e(E') \;=\; e(E)\cdot e(E'')$$

where the above elements of $\mathrm{H}^\bullet(X)_{\mathrm{loc}}$ are defined as

$$[E'/X]\cdot e(E') \;=\; e(E'/X) \qquad\qquad [E''/X]\cdot e(E'') \;=\; e(E''/X),$$

as well as $[E'/E'']\cdot \pi_{E''}^* e(E) = e(E'/E'')$, which is equivalent to

$$[E/X]\cdot e(E) \;=\; i_{E''}^*([E'/E'']\cdot \pi_{E''}^* e(E)) \;=\; e(E/X).$$

3.4.12. To be extremely explicit, we consider the homotopy fibre $E = (E_0 \xrightarrow{\varphi} E_1)$ of a map of vector bundles:

$$E \;\to\; E_0 \;\to\; E_1$$

Note that each perfect complex has tor amplitude in $[0,1]$ and so its total space is quasismooth. Because E_0 and E_1 are smooth over X, the map $E_0 \to E_1$ is quasismooth and the above assumption on fundamental classes being isomorphisms holds.

We make $\mathbf{G}_m$ act trivially on X, which lifts to an action on each of the above by scaling the fibres, giving a homotopy fibre

$$E/\mathbf{G}_m \;\to\; E_0/\mathbf{G}_m \;\to\; E_1/\mathbf{G}_m$$

over $X/\mathbf{G}_m \;=\; X\times \mathrm{B}\mathbf{G}_m$. Noting that the zero section $X/\mathbf{G}_m \to E_1/\mathbf{G}_m$ is concentrated if we invert the equivariant Euler class $S = (e_{\mathbf{G}_m}(E_1))$, the singular Whitney sum formula gives

$$e_{\mathbf{G}_m}(E_0/X) \;=\; e_{\mathbf{G}_m}(E_0/E_1)e_{\mathbf{G}_m}(E_1/X) \tag{3.26}$$

as elements of $\mathrm{H}^\bullet_{\mathbf{G}_m}(E_0/X)_{\mathrm{loc}}$. Using section 3.4.11 we get the usual sorts of expressions one encounters when dealing with virtual abelian localisation (e.g. [GP])

$$e_{\mathbf{G}_m}(E) \;=\; e_{\mathbf{G}_m}(E_0)/e_{\mathbf{G}_m}(E_1).$$

3.5 Abelian localisation

3.5.1. **Background.** Take an action of a torus T on X (a manifold, scheme, stack, ...). *Abelian localisation* says that under suitable conditions the equivariant cohomology of X and its fixed locus X^T are almost equal

$$\mathrm{H}^\bullet_T(X) \text{ “}\approx\text{” } \mathrm{H}^\bullet_T(X^T). \tag{3.27}$$

This can really simplify computations with cohomology, e.g. giving *integration formulae* for equivariant classes on X.

3.5.2. What (3.27) means more precisely is their localisations agreee after localising with respect to some multiplicative subset of $\mathrm{H}^\bullet_T(X)$ (see section 3.3.1). For instance, when X is a scheme or manifold the classical Theorem 3.5.7 due to Atiyah and Bott says that as $\mathrm{H}^\bullet_T(\mathrm{pt})$ modules their free parts agree:

$$\mathrm{H}^\bullet_T(X) \otimes_{\mathrm{H}^\bullet_T(\mathrm{pt})} \mathrm{Frac}\mathrm{H}^\bullet_T(\mathrm{pt}) \;\simeq\; \mathrm{H}^\bullet_T(X^T) \otimes_{\mathrm{H}^\bullet_T(\mathrm{pt})} \mathrm{Frac}\mathrm{H}^\bullet_T(\mathrm{pt}). \tag{3.28}$$

Note that $\mathrm{H}^\bullet_T(\mathrm{pt})$, the equivariant cohomology of a point, is a polynomial algebra in $\mathrm{rk}T$ variables. So when $T = \mathbf{G}_m$ has rank one this simply says

$$\mathrm{H}^\bullet_{\mathbf{G}_m}(X)[t^{-1}] \;\simeq\; \mathrm{H}^\bullet_{\mathbf{G}_m}(X^{\mathbf{G}_m})[t^{-1}].$$

3.5.3. Abelian localisation is one of the main techniques in enumerative geometry, e.g. to compute Gromov Witten [Beh, MNOP] or Donaldson Thomas [Th, MNOP] invariants. These are defined as integrals of certain "tautological" cohomology classes on moduli stacks, and when these moduli stacks have a torus action we can use abelian localisation computations onto the fixed locus. Sometimes the fixed locus is even a disjoint union of points, reducing us to a weighted point count (i.e. combinatorics).

3.5.4. Abelian localisation was first proven by Atiyah and Bott [AB], and Berline and Vergne [BVe]. In algebraic geometry, abelian localisation for Chow homology was proven for schemes by Edidin and Graham [EG], Delgine Mumford stacks by Kresch, and generalised to the singular Deligne Mumford setting the setting by Graber and Panharipande [GP]. Aranha, Khan, Latyntsev, Park and Ravi [AKLPR] generalise this to general reductive groups instead of just tori, in arbitrary characteristic, general Borel-Moore homology theories, and for Artin stacks whose stabilisers are small enough (e.g. Deligne Mumford).

3.5.5. **Remark.** An *equivariant* map between stacks with G actions is a map f_G between their quotient stacks

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X/G \arrow[r, "f_G"] & Y/G \end{tikzcd}$$

When the context is clear we will often drop the subscript G from f_G, e.g. pullback on equivariant cohomology will be denoted f^* not f^*_G. Note that f does not determine f_G uniquely, for instance each homomorphism $G \to G$ determines a map $\mathrm{B}G \to \mathrm{B}G$ which lifts to the trivial map pt $\to$ pt.

3.5.6. **Localisation for smooth schemes.** We begin by translating Atiyah and Bott's [AB] proof of abelian localisation (for manifolds) into algebraic geometry (smooth schemes). This will also explain what we need to generalise in order to prove abelian localisation for more complicated spaces.

Theorem 3.5.7 (Abelian localisation for smooth schemes)**.** *Fix T a torus acting on smooth schemes of finite type Z, X, on Z trivially. Suppose $i : Z \to X$ is a T-equivariant closed embedding and on all points on the complement the stabiliser is a proper subgroup of T. Then the localised pullback and pushfoward maps are isomorphisms -*

$$i_* \ : \ \mathrm{H}^\bullet_T(Z)_{\mathrm{loc}} \ \tilde{\to} \ \mathrm{H}^{\bullet+2d}_T(X)_{\mathrm{loc}} \qquad\qquad i^* \ : \ \mathrm{H}^\bullet_T(X)_{\mathrm{loc}} \ \tilde{\to} \ \mathrm{H}^\bullet_T(Z)_{\mathrm{loc}}$$

where d is the codimension of i. We have written loc *for localisation of a* $\mathrm{H}^\bullet_T(\mathrm{pt})$ *module with respect to all nonzero elements:*

$$M_{\mathrm{loc}} \ = \ M \otimes_{\mathrm{H}^\bullet_T(\mathrm{pt})} \mathrm{Frac}\mathrm{H}^\bullet_T(\mathrm{pt}).$$

Why does the fixed locus X^T not appear in the statement? Consider that abelian localisation is certainly true for $Z = X$. Thus, it is not important whether Z is a fixed locus or not, only that the torus action on its complement is close to being free (has low dimensional stabilisers).

Corollary 3.5.8. *The Euler class $e(\mathbf{N}_i) \in \mathrm{H}^{2d}(Z)_{\mathrm{loc}}$ of the normal bundle is invertible, and*

$$\mathrm{id} \ = \ i_* \frac{i^*(-)}{e(\mathbf{N}_i)} \tag{3.29}$$

as endomorphisms of $\mathrm{H}^\bullet_T(X)_{\mathrm{loc}}$.

Proof. Both claims follow since the map $i^* i_*$ is multiplication by $e(\mathbf{N}_i)$. □

Corollary 3.5.9 (Integration Formula)**.** *If Z and X are proper,*

$$\int_X \ = \ \int_Z \frac{i^*(-)}{e(\mathbf{N}_i)}$$

as maps $\mathrm{H}^\bullet_T(X)_{\mathrm{loc}} \to \mathrm{H}^\bullet_T(\mathrm{pt})_{\mathrm{loc}}$.

Proof. (Equivariant) integration $\int_Z$ means proper pushforward along $Z/T \to \mathrm{B}T$. Because this map factors as $Z/T \to X/T \to \mathrm{B}T$, we have $\int_Z = \int_X i_*$, so we are done by applying $\int_Z$ to (3.29). □

3.5.10. **Proof of Theorem 3.5.7.** We proceed in two steps:

1. *Concentration.* Show $\mathrm{H}^\bullet_T(X\backslash Z)_{\mathrm{loc}} = 0$ by Thomason's generic slice Theorem.
2. *Specialisation.* Show the Euler class is unit in localised cohomology, by reducing to the case that X is a vector bundle over Z using "the exponential map".

Recall from section 3.3.2 the commuting diagram of $\mathrm{H}^\bullet_T(\mathrm{pt})$ modules whose rows are long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^{\bullet-2d}_T(Z) \arrow[r, "i_*"] \arrow[d, equal] & \mathrm{H}^\bullet_T(X) \arrow[r] \arrow[d, "i^*"] & \mathrm{H}^\bullet_T(X\backslash Z) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{\bullet-2d}_T(Z) \arrow[r, "\cdot e(X/Z)"] & \mathrm{H}^\bullet_T(Z) \arrow[r] & \mathrm{H}^\bullet_T(Z, i^*j_*k) \arrow[r] & \cdots
\end{tikzcd} \tag{3.30}$$

Thus concentration will give that i_* is an isomorphism after localisation. Specialisation gives that $\cdot e(X/Z)$ is an isomorphism after localisation, hence i^* is too.

3.5.11. Proving concentration is simple in the rank one case $T = \mathbf{G}_m$. The quotient stack $(X\backslash Z)/T$ is Deligne Mumford since all its stabilisers are étale [Ol, Cor. 8.4.2]. As it is in addition of finite type, its cohomology $\mathrm{H}^\bullet_T(X\backslash Z)$ is finite dimensional by [Ed, Prop. 4.39]. In particular, the degree two generator of $\mathrm{H}^\bullet_T(\mathrm{pt}) = k[t]$ acts nilpotently, and so its localisation is zero.

3.5.12. Note that we did not use the smoothness assumption at all. We will now prove concentration in the higher rank case for Z, X as in the Theorem but without the smoothness assumption, proceeding by induction on the dimension of $X\backslash Z$. Finally, note that if the action on $X\backslash Z$ were free or had étale stabilisers the above proof would apply.

3.5.13. To prove concentration in general we as in [AKLPR] we use Thomason's generic slice Theorem [Th'n, Thm. 4.10], which says that any scheme with a T action admits a T *invariant* nonempty affine open U and as stacks over $\mathrm{B}T$,

$$U/T \ \simeq V \times \mathrm{B}T'$$

for some subgroup $T' \subseteq T$ and some affine scheme V. In particular, the action of $\mathrm{H}^\bullet_T(\mathrm{pt})$ on any sheaf cohomology $\mathrm{H}^\bullet_T(U, \mathcal{F})$ factors through

$$\mathrm{H}^\bullet_T(\mathrm{pt}) \ \twoheadrightarrow \ \mathrm{H}^\bullet_{T'}(\mathrm{pt}).$$

Applying this to $X\backslash Z$, by assumption on the stabilisers this map has nonzero kernel because $T' \subseteq T$ is a proper subgroup and so $\mathrm{H}^\bullet_T(U, \mathcal{F})_{\mathrm{loc}} = 0$. By iterating this, we may assume that the complement of U has dimension strictly less than U. Then if $\mathcal{F}$ is any T equivariant sheaf on $X\backslash Z$ we take the Mayer Vietoris sequence

$$\cdots \to \mathrm{H}_T^\bullet((X\backslash Z)\backslash U, i^!\mathcal{F})_{\mathrm{loc}} \to \mathrm{H}_T^\bullet(X\backslash Z, \mathcal{F})_{\mathrm{loc}} \to \mathrm{H}_T^\bullet(U, j^*\mathcal{F})_{\mathrm{loc}} \to \cdots$$

We have shown that the term on the right vanishes, and the left term vanishes by induction on dimension. Thus $\mathrm{H}_T^\bullet(X\backslash Z, \mathcal{F})_{\mathrm{loc}} = 0$ for any sheaf $\mathcal{F}$, in particular $\mathrm{H}_T^\bullet(X\backslash Z)_{\mathrm{loc}} = 0$.

3.5.14. Having proven concentration, abelian localisation follows for the zero section of the normal bundle $Z \hookrightarrow \mathbf{N}_i$, and for any closed embedding whose pullback on cohomology gives an isomorphism. In the differential geometric setting we would now consider the exponential map

$$\begin{tikzcd} & \mathbf{N}_i \arrow[d, dashed, "\exp"] \\ Z \arrow[ur, hook] \arrow[r, hook] & X \end{tikzcd}$$

to relate the Euler classes of X and $\mathbf{N}_i$. This does not exist in algebraic geometry, but we have its cohomological shadow, the *exponential* map on bivariant cohomology (section C.2) which is compatible with bivariant Euler classes via

$$\begin{tikzcd} & \mathrm{H}_T^\bullet(Z/\mathbf{N}_i) \arrow[dl, "\cdot e(\mathbf{N}_i/X)"'] \\ \mathrm{H}_T^\bullet(Z) & \\ & \mathrm{H}_T^\bullet(Z/X) \arrow[ul, "\cdot e(X/Z)"] \arrow[uu, "\mathrm{H}^\bullet(\exp_i)"', "\sim"] \end{tikzcd}$$

Because these spaces are smooth we can identify the bivariant cohomologies with the cohomology of Z using fundamental classes, giving

$$\begin{tikzcd} & \mathrm{H}_T^\bullet(Z/\mathbf{N}_i) \arrow[dl, "\cdot e(\mathbf{N}_i/X)"'] & \\ \mathrm{H}_T^\bullet(Z) & & \mathrm{H}_T^{\bullet-2d}(Z) \arrow[ul, "\cdot[Z/\mathbf{N}_i]"', "\sim"] \arrow[dl, "\cdot[Z/X]", "\sim"'] \\ & \mathrm{H}_T^\bullet(Z/X) \arrow[ul, "\cdot e(X/Z)"] \arrow[uu, "\mathrm{H}^\bullet(\exp_i)"', "\sim"] & \end{tikzcd}$$

and so since $e(\mathbf{N}_i/Z)$ is a unit in localised cohomology, so is $e(X/Z)$. This proves specialisation, and so completes the proof of Theorem 3.5.7.

3.5.15. **General abelian localisation.** We summarise the above proof in a theorem, which tautological but useful to have. As usual we work with a sheaf theory admitting the six functors (see section 2.7).

Theorem 3.5.16 (Abelian localisation)**.** *Let $i : Z \to X$ be a closed embedding of spaces and $S \subseteq \mathrm{H}^\bullet(X)$ a multiplicative subset. If i is S-concentrated (3.3.3) and S-specialised (3.3.4) then*

$$i_* \ : \ \mathrm{H}^\bullet(Z/X)_{\mathrm{loc}} \ \xrightarrow{\sim} \ \mathrm{H}^\bullet(X)_{\mathrm{loc}} \qquad\qquad i^* \ : \ \mathrm{H}^\bullet(X)_{\mathrm{loc}} \ \xrightarrow{\sim} \ \mathrm{H}^\bullet(Z)_{\mathrm{loc}}$$

are isomorphisms.

Proof. Follows from the definitions of concentration and specialisation by localising the diagram (3.15) of $\mathrm{H}^\bullet(X)$ modules. □

Informally, this theorem says that for abelian localisation to be true it is enough that the Euler class is a unit and the cohomology of the open complement is torsion.

3.5.17. By Example 3.5.27, even in the T equivariant case inverting a subset $S \subseteq \mathrm{H}^\bullet_T(\mathrm{pt})$ is often not enough to ensure concentration and specialisation. This is why in the above we invert $S \subseteq \mathrm{H}^\bullet_T(X)$.

3.5.18. **Integration formulae.** Let $i : Z \to X$ be a closed embedding which is concentrated and specialised with respect to multiplicative subset $S \subseteq \mathrm{H}^\bullet(X)$, as in Theorem 3.5.16. As for any closed embedding, $\mathbf{T}_i$ is concentrated in degrees $[1, \infty)$.

In particular, if $\mathbf{T}_i$ is concentrated in degrees $\leqslant 2$ (as is the case if Z and X are quasismooth over a common base) then the normal complex $\mathbf{N}_i = \mathbf{T}_i[1]$ is concentrated in degrees $[0, 1]$ and so its total space

$$\mathbf{N}_i \ \to \ Z$$

is quasismooth. Moreover, note that because it is an isomorphism on the level of cohomology, $Z \to \mathbf{N}_i$ is S specialised if and only if it is S concentrated. As a consequence of abelian localisation, we get

Corollary 3.5.19 (Integration formula)**.** *Assume $\mathbf{N}_i$ is concentrated in degrees $\leqslant 1$ and its zero section is S-specialised, so that by section 3.3.6 we have an isomorphism*

$$\cdot[\mathbf{N}_i/Z] \ : \ \mathrm{H}^\bullet(Z/\mathbf{N}_i)_{\mathrm{loc}} \ \xrightarrow{\sim} \ \mathrm{H}^\bullet(Z)_{\mathrm{loc}}.$$

Then identifying $\mathrm{H}^\bullet(Z/X)_{\mathrm{loc}} \simeq \mathrm{H}^\bullet(Z)_{\mathrm{loc}}$ by $\alpha \mapsto (\exp \alpha) \cdot [\mathbf{N}_i/Z]$, we have

$$\mathrm{id} \ = \ i_* \frac{i^*(-)}{e(\mathbf{N}_i)} \tag{3.31}$$

as endomorphisms of $\mathrm{H}^\bullet(X)_{\mathrm{loc}}$, where $e(\mathbf{N}_i) \in \mathrm{H}^\bullet(Z)$ is the localised Euler class of $\mathbf{N}_i$.

Proof. Follows from the commuting diagram of isomorphisms

$$\begin{tikzcd}
 & \mathrm{H}^\bullet(Z/X)_{\mathrm{loc}} \arrow[r, "\mathrm{H}^\bullet(\exp)", "\sim"'] \arrow[dl, "i_*"', "\sim"] \arrow[dd, "\cdot e(X/Z)", "\sim"'] & \mathrm{H}^\bullet(Z/\mathbf{N}_i)_{\mathrm{loc}} \arrow[r, "\cdot[\mathbf{N}_i/Z]", "\sim"'] \arrow[ddl, "\cdot e(\mathbf{N}_i/Z)"', "\sim"] & \mathrm{H}^{\bullet-2d}(Z)_{\mathrm{loc}} \arrow[ddll, bend left, "e(\mathbf{N}_i)"] \\
\mathrm{H}^\bullet(X)_{\mathrm{loc}} \arrow[dr, "i^*"', "\sim"] & & & \\
 & \mathrm{H}^\bullet(Z)_{\mathrm{loc}} & &
\end{tikzcd}$$

by section C.2, where the bottom right arrow is the definition of $e(\mathbf{N}_i)$ and d is the rank of $\mathbf{N}_i$. Note that $\mathrm{H}^\bullet(\exp)$ is an isomorphism because the middle triangle commutes and its vertical edges are isomorphisms since both i and the zero section of $\mathbf{N}_i$ are S-specialised (see section C.2). $\square$

Corollary 3.5.20 (Sheaf cohomology abelian localisation). *With notation as in Theorem 3.5.16, for any $\mathcal{F} \in \mathrm{Sh}(X)$, the maps*

$$i_* \ : \ \mathrm{H}^\bullet(Z, i^!\mathcal{F})_{\mathrm{loc}} \ \xrightarrow{\sim} \ \mathrm{H}^\bullet(X, \mathcal{F})_{\mathrm{loc}} \qquad\qquad i^* \ : \ \mathrm{H}^\bullet(X, \mathcal{F})_{\mathrm{loc}} \ \xrightarrow{\sim} \ \mathrm{H}^\bullet(Z, i^*\mathcal{F})_{\mathrm{loc}}$$

are isomorphisms.

Proof. Consider the diagram of $\mathrm{H}^\bullet(X)$ modules

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^\bullet(Z, i^!\mathcal{F}) \arrow[r, "i_*"] \arrow[d, equal] & \mathrm{H}^\bullet(X, \mathcal{F}) \arrow[r] \arrow[d, "i^*"] & \mathrm{H}^\bullet(X \backslash Z, j^*\mathcal{F}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^\bullet(Z, i^!\mathcal{F}) \arrow[r, "\cdot e(X/Z)"] & \mathrm{H}^\bullet(Z, i^*\mathcal{F}) \arrow[r] & \mathrm{H}^\bullet(Z, i^*j_*\mathcal{F}) \arrow[r] & \cdots
\end{tikzcd} \tag{3.32}$$

Note that the action of $\mathrm{H}^\bullet(X)$ on the module $\mathrm{H}^\bullet(X\backslash Z, j^*\mathcal{F})$ factors though the map of algebras

$$\mathrm{H}^\bullet(X) \ \to \ \mathrm{H}^\bullet(X\backslash Z, j^*k)$$

and the action on the module $\mathrm{H}^\bullet(Z, i^*j_*\mathcal{F})$ factors through the map of algebras[6]

$$\mathrm{H}^\bullet(X) \ \to \ \mathrm{H}^\bullet(Z, i^*j_*k).$$

Thus both $\mathrm{H}^\bullet(X)$-modules localise to zero, because the action factors through rings whose localisations vanish by Theorem 3.5.16. $\square$

Applying this to the dualising sheaf $\mathcal{F} = \omega_X$, we get the statement for Borel Moore homology

[6] Recall that if $\mathcal{A}$ is a associative algebra in $\mathrm{Sh}(X)$, then so is $f_*f^*\mathcal{A}$ for any map $f : Y \to X$, and $\mathcal{A} \to f_*f^*\mathcal{A}$ is a map of algebra objects. In particular, applying this to $\mathcal{A} = k_X$ and applying this twice we get that $i_*i^*j_*j^*k_X$ is an algebra object in $\mathrm{Sh}(X)$, and hence its cohomology is an algebra.

Corollary 3.5.21 (Borel Moore abelian localisation). *With notation as in Theorem 3.5.16, the maps*

$$i_* \ : \ \mathrm{H}^\bullet_{\mathrm{BM}}(Z)_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathrm{BM}}(X)_{\mathrm{loc}} \qquad\qquad i^* \ : \ \mathrm{H}^\bullet_{\mathrm{BM}}(X)_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathrm{BM}}(Z/X)_{\mathrm{loc}}$$

are isomorphisms, where $\mathrm{H}^\bullet_{\mathrm{BM}}(Z/X)$ *is bivariant Borel Moore homology (section 2.7.5).*

3.5.22. We make a note about how the localised Euler class relates with proper pushforwards, keeping the notation of Corollary 3.5.19. Consider a commuting diagram

$$\begin{tikzcd} Z \arrow[d, "i"] \arrow[dr, bend left, "\overline{p}"] & \\ X \arrow[r, "p"'] & Y \end{tikzcd} \tag{3.33}$$

where p and $\overline{p}$ are proper quasismooth maps. Then we have the following commuting diagram

$$\begin{tikzcd} & \mathrm{H}^\bullet(Z/Y)_{\mathrm{loc}} \arrow[dl, "i_*"'] \arrow[r, equal] & \mathrm{H}^\bullet(Z/Y)_{\mathrm{loc}} & \\ \mathrm{H}^\bullet(X/Y)_{\mathrm{loc}} & & & \\ & \mathrm{H}^\bullet(Z/X)_{\mathrm{loc}} \arrow[uu, "\cdot[X/Y]"] \arrow[r, "\mathrm{H}^\bullet(\exp)", "\sim"'] \arrow[dl, "i_*"', "\sim"] \arrow[dd, "\sim"', "\cdot e(X/Z)"] & \mathrm{H}^\bullet(Z/\mathbf{N}_i)_{\mathrm{loc}} \arrow[uu, "\cdot[\mathbf{N}_i/Y]"'] \arrow[r, "\cdot[\mathbf{N}_i/Z]", "\sim"'] \arrow[ddl, "\sim"', "\cdot e(\mathbf{N}_i/Z)\ \sim"] & \mathrm{H}^\bullet(Z)_{\mathrm{loc}} \arrow[uul, "\cdot[Z/Y]"'] \arrow[ddll, bend left, "e(\mathbf{N}_i)"] \\ \mathrm{H}^\bullet(X)_{\mathrm{loc}} \arrow[uu, "\cdot[X/Y]"] \arrow[dr, "\sim", "i^*"'] & & & \\ & \mathrm{H}^\bullet(Z)_{\mathrm{loc}} & & \end{tikzcd} \tag{3.34}$$

The middle square commutes by section C.2. Then if S comes from the pullback of a multiplicative subset of $\mathrm{H}^\bullet(Y)$, we can apply proper pushforward on localised bivariant cohomology $\mathrm{H}^\bullet(X/Y)_{\mathrm{loc}} \to \mathrm{H}^\bullet(Y)_{\mathrm{loc}}$ to get that

$$p_* \ = \ \overline{p}_* \frac{i^*(-)}{e(\mathbf{N}_i)} \tag{3.35}$$

as maps $\mathrm{H}^\bullet(X)_{\mathrm{loc}} \to \mathrm{H}^\bullet(Y)_{\mathrm{loc}}$. This is classically written as

$$\int_X \ = \ \int_{\overline{X}} \frac{i^*(-)}{e(\mathbf{N}_i)}.$$

3.5.23. **Example: vector spaces.** Let us consider the simplest case where the multiplicative group $T = \mathbf{G}_m$ acts vector space $X = V$ by scaling, and taking $Z = \{0\}$ the fixed locus. The Thom Gysin sequence is

$$\cdots \ \to \ \mathrm{H}^{\bullet-2n}_{\mathbf{G}_m}(\{0\}) \ \xrightarrow{i_*} \ \mathrm{H}^\bullet_{\mathbf{G}_m}(V) \ \to \ \mathrm{H}^\bullet(\mathbf{P}V) \ \to \ \cdots$$

Here $\mathbf{P}V = (V\backslash 0)/\mathbf{G}_m$. As a sequence of $\mathrm{H}^\bullet(\mathrm{B}T) \simeq k[t]$ modules, this is

$$0 \;\to\; k[t] \;\xrightarrow{i_*}\; k[t] \;\to\; k[t]/t^n \;\to\; 0$$

where n is the rank of V. Then since $\mathrm{H}^\bullet(\mathbf{P}V)$ is finite dimensional, it is a torsion module and so $\mathrm{H}^\bullet(\mathbf{P}V)_{\mathrm{loc}} = 0$.

3.5.24. **Example: flag varieties.** Let G be a complex reductive group, T a maximal torus B a Borel subgroup. The flag variety G/B admits an action by T and a *Bruhat stratification* into affine spaces, labelled by elements of the Weyl group

$$G/B \;=\; \coprod_{w\in W} BwB/B \;\simeq\; \coprod_{w\in \mathrm{W}} \mathbf{A}^{\ell(w)}. \tag{3.36}$$

It follows from the Mayer Vietoris sequence that $\mathrm{H}^\bullet(G/B)$ as a vector space is $\mathbf{C}[W]$ (as an algebra it is W coinvariants $\mathrm{H}^\bullet_T(\mathrm{pt}) \otimes_{\mathrm{H}^\bullet_T(\mathrm{pt})^W} \otimes k$) where w has degree $2\ell(w)$.

Alternatively, the action of T has one fixed point per stratum, so by Theorem 3.5.7 the localised equivariant cohomology is free with dimension $|W|$ over the fraction field:

$$i^* \;:\; \mathrm{H}^\bullet_T(G/B)_{loc} \;\xrightarrow{\sim}\; \bigoplus_{w\in W} \mathrm{Frac}\mathrm{H}^\bullet_T(\mathrm{pt}) \;\simeq\; \mathbf{C}[W] \otimes \mathrm{Frac}\mathrm{H}^\bullet_T(\mathrm{pt})$$

as algebras.

3.5.25. Other natural examples include partial flag varieties or smooth toric varieties. More generally for any smooth projective variety with torus action we can apply the Białynicki-Birula theorem to get a stratification into generalising (3.39) using attracting sets of the fixed points. See section 3.7.

3.5.26. **Example: stacks.** We thank Hyeonjun Park for pointing the following example out to us. Take the zero section of the tautological line bundle on $\mathrm{B}\mathbf{G}_m$

$$i \;:\; \mathrm{B}\mathbf{G}_m \;\to\; \mathbf{A}^1/\mathbf{G}_m$$

with $T = \mathbf{G}_m$ acting on the fibres by scaling. Its quotient by T is

$$i_T \;:\; \mathrm{B}(\mathbf{G}_m \times T) \;\to\; \mathbf{A}^1/(\mathbf{G}_m \times T).$$

The pushforward i_{T*} fails to be an isomorphism, even after inverting any multiplicative subset of nonzero divisors in $k[t] = \mathrm{H}^\bullet_T(\mathrm{pt})$. Indeed, it fits into the Mayer Vietoris sequence

$$\cdots \;\to\; \mathrm{H}^{\bullet-2}_{\mathbf{G}_m\times T}(\mathrm{pt}) \;\xrightarrow{i_{T*}}\; \mathrm{H}^\bullet_{\mathbf{G}_m\times T}(\mathbf{A}^1) \;\to\; \mathrm{H}^\bullet(\mathbf{G}_m/(\mathbf{G}_m\times T)) \;\to\; \cdots \tag{3.37}$$

and the last term $\mathrm{H}^\bullet(\mathbf{G}_m/(\mathbf{G}_m \times T)) \simeq k[t]$ is a torsion free $\mathrm{H}^\bullet_T(\mathrm{pt})$ module. To fix this, note that (3.40) is identified with the Gysin sequence

$$\cdots \;\to\; k[x,t] \;\overset{x+t}{\to}\; k[x,t] \;\to\; k[t] \;\to\; \cdots$$

where the first term is multiplication by the Euler class of i_T. It follows that if we take

$$S \;=\; (x+t) \;\subseteq\; \mathrm{H}^\bullet_T(\mathrm{B}\mathbf{G}_m)$$

then abelian localisation as in Theorem 3.5.7 holds. Moreover, because $x+t$ is not a zero divisor, abelian localisation does not just hold for the "stupid" reason that a zero divisor was inverted so all vector spaces are zero.

Proposition 3.5.27. *Let Z be an Artin stack over a field of characteristic zero and E a vector bundle. Assume T is a torus acting on the fibres of E. Then the T equivariant zero section*

$$i_T \;:\; Z/T \;\to\; E/T$$

is S-concentrated and S-specialised for $S = (e_T(E))$.[7] *Moreover, if T acts with nonzero weights then S consists of non zero divisors.*

Proof. Specialisation and concentration are equivalent since i_T^* is an isomorphism. Specialisation follows by definition, since multiplication by the Euler class is clearly an isomorphism if we invert it. To show the second claim, writing $\mathrm{H}^\bullet_T(\mathrm{pt}) = k[t_1, ..., t_n]$, we note that

$$e_T(E) \;=\; (w_1 + x_1)\cdots(w_r + x_r) \;\in\; \mathrm{H}^\bullet_T(Z) \;\simeq\; \mathrm{H}^\bullet(Z)[t_1, ..., t_n]$$

where $w_i = \sum W_{ij} t_j$ are the weights of the action (with W_{ij} locally constant functions on Z) and x_i are the chern roots. Recall that these are defined by pulling back the map to BGL_n given by E/T:

$$\begin{tikzcd}
\mathcal{F}l_{E/T}(Z/T) \arrow[r] \arrow[d, "p"] & \mathrm{B}\mathbf{G}_m^n \arrow[d] \\
Z/T \arrow[r] & \mathrm{BGL}_n
\end{tikzcd}$$

Here $\mathcal{F}l$ denotes the flag bundle. Note that $p^*(E/T)$ splits as a direct sum of line bundles, the first chern class of the ith is denoted $w_i + x_i$. Since the weights w_i are nonzero, $w_1 \cdots w_r$ is not a zero divisor and hence $e_T(E)$ is also not a zero divisor. $\square$

[7] Here $e_T(E)$ denotes equivariant Euler class of E, defined as the Euler class of E/T.

3.5.28. **Remark.** Take a cartesian square of smooth schemes

$$\begin{tikzcd} \overline{Z} \arrow[r, "\overline{\imath}"] \arrow[d, "\overline{f}"] & \overline{X} \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

Assume this diagram is equivariant for the action of a torus T, that the horizontal maps are closed embeddings which are specialised and concentrated, and the vertical maps are proper. Then applying the integration formula (3.29) twice gives

$$\frac{i^* f_*(-)}{e(\mathbf{N}_i)} \;=\; i^* i_* \frac{i^* f_*(-)}{e(\mathbf{N}_i)} \;=\; i^* f_*(-) \;=\; \overline{f}_* \overline{\imath}^*(-) \;=\; \overline{f}_* \left(\frac{\overline{\imath}^*(-)}{e(\mathbf{N}_{\overline{\imath}})} \right) \tag{3.38}$$

as maps on localised cohomology. Equation (3.41) is sometimes called the functorial integration formula.

3.5.29. There is a folk conjecture that the Grothendieck Riemann Roch Theorem can be formulated as being a special case of abelian localisation. Let $p : X \to Y$ be a smooth and proper map of schemes, and consider the cartesian square of formal schemes

$$\begin{tikzcd} X \arrow[r, "\overline{s}"] \arrow[d, "p"] & \mathcal{L}X \arrow[d, "\mathcal{L}p"] \\ Y \arrow[r, "s"] & \mathcal{L}Y \end{tikzcd}$$

where $\mathcal{L}X$ is the free loop space of X, see [KV1, §3]. Then the idea is that the Grothendieck Riemann Roch formula

$$\mathrm{ch}(p_* \mathcal{F}) \mathrm{td}(\mathbf{T}_Y) \;=\; p_*(\mathrm{ch}(\mathcal{F}) \mathrm{td}(\mathbf{T}_X)),$$

as elements of Chow homology and for $\mathcal{F} \in \mathrm{K}_0(X)$, closely resembles the functorial integration formula (3.41), with the Todd class $\mathrm{td}(\mathbf{T}_Y)$ taking the role of the inverse of the undefined Euler class "$e(\mathbf{N}_s)$". This is of course not defined using the above framework because $\mathcal{L}Y \to Y$ is not locally of finite type, so we do not have access to the usual six functor formalism to study it. See [Liu] or [At1] for more details.

3.6 CoHA products by abelian localisation

We can use the integration formula to compute CoHA products on *non-equivariant* cohomology, like so.

3.6.1. **Linear algebra.** So as to make the proceeding clearer, we record an easy fact. Consider maps of vector spaces

$$\begin{tikzcd} V_1 \arrow[r, "\varphi_V"] \arrow[d, "\alpha_1"] & V_2 \arrow[d, "\alpha_2"] \\ W_1 \arrow[r, "\varphi_W"] & W_2 \end{tikzcd} \tag{3.39}$$

Then if α_1 has a section, the following also commutes

$$\begin{tikzcd} V_1 \arrow[r, "\varphi_V"] & V_2 \arrow[d, "\alpha_2"] \\ W_1 \arrow[u, hook] \arrow[r, "\varphi_W"] & W_2 \end{tikzcd}$$

In particular,

Lemma 3.6.2. *If α_1 has a section, φ_V uniquely determines φ_W.*

3.6.3. The examples we will care about will come from equivariant cohomology:

$$\begin{tikzcd} \mathrm{H}^\bullet_G(Y_1) \arrow[r] \arrow[d] & \mathrm{H}^\bullet_G(Y_2) \arrow[d] \\ \mathrm{H}^\bullet(Y_1) \arrow[r] & \mathrm{H}^\bullet(Y_2) \end{tikzcd}$$

Here G acts on Artin stacks Y_i. If both the G actions are trivial then the section comes from the identity $k \to \mathrm{H}^\bullet_G(\mathrm{pt})$, giving

$$\mathrm{H}^\bullet(Y_i) \;\to\; \mathrm{H}^\bullet_G(Y_i) \;\simeq\; \mathrm{H}^\bullet(Y_i) \otimes \mathrm{H}^\bullet(\mathrm{B}G).$$

Thus if we want to prove results in the nonequivariant setting about maps $\mathrm{H}^\bullet(Y_1) \to \mathrm{H}^\bullet(Y_2)$, Lemma 3.6.2 allows us to instead prove results in the equivariant setting. We will sometimes abuse notation, when given a map $\varphi : \mathrm{H}^\bullet_G(Y_1) \to \mathrm{H}^\bullet_G(Y_2)$, by also denoting by φ the induced map $\mathrm{H}^\bullet(Y_1) \to \mathrm{H}^\bullet(Y_2)$.

3.6.4. **CoHA products.** Fix base Artin stacks Y_1 and Y_2, and consider a correspondence

$$\begin{tikzcd} & X \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y_1 & & Y_2 \end{tikzcd} \tag{3.40}$$

3.6.5. Now we explain how to use abelian localisation to compute CoHA products. Let $i : Z \to X$ be a closed embedding

$$\begin{tikzcd} & Z \arrow[d, "i"] \arrow[ddl, bend right, "\overline{q}"'] \arrow[ddr, bend left, "\overline{p}"] & \\ & X \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y_1 & & Y_2 \end{tikzcd} \tag{3.41}$$

Note that if p and $\overline{p}$ (or q and $\overline{q}$) are quasismooth then the conormal complex $\mathbf{N}_i^* = \mathbf{T}_i^*[-1]$ is contained in Tor amplitude $\leqslant 1$, where $\mathbf{T}_i$ denotes the tangent complex of the morphism i, so that the total space

$$\mathbf{N}_i \ \to \ Z$$

is quasismooth over Z.

Theorem 3.6.6. *Assume $Z \to X$ is a closed embedding which is S-concentrated (3.3.3) and S-specialised (3.3.4) over Y_2 with respect to a multiplicative subset $S \subseteq \mathrm{H}^\bullet(Y_2)$ of nonzero divisors. Assume that p is proper and quasismooth, then as maps $\mathrm{H}^\bullet(Y_1) \to \mathrm{H}^\bullet(Y_2)$,*

$$p_* q^* \ = \ \overline{p}_* \frac{\overline{q}^*(-)}{e(\mathbf{N}_i)}. \tag{3.42}$$

Proof. We have the maps

$$\mathrm{H}^\bullet(Y_1) \ \xrightarrow{q^*} \ \mathrm{H}^\bullet(X) \ \hookrightarrow \ \mathrm{H}^\bullet(X)_{\mathrm{loc}} \ \xrightarrow{p_*} \ \mathrm{H}^\bullet(Y_2)_{\mathrm{loc}}$$

and the first claim follows by decomposing $\mathrm{id}_{\mathrm{H}^\bullet(X)_{\mathrm{loc}}}$ according to the integration formula (3.31).

Consider the commuting diagram

$$\begin{tikzcd} & \mathrm{H}^\bullet(X)_{\mathrm{loc}} \arrow[r, "p_*"] & \mathrm{H}^{\bullet-2d}(Y_2)_{\mathrm{loc}} \\ \mathrm{H}^\bullet(Y_1) \arrow[r, "q^*"] & \mathrm{H}^\bullet(X) \arrow[u] \arrow[r, "p_*"] & \mathrm{H}^{\bullet-2d}(Y_2) \arrow[u] \end{tikzcd} \tag{3.43}$$

By applying $(-)q^*$ to the relative integration formula (3.35), the upper row of (3.46) is

$$p_* q^* \ = \ \overline{p}_* \frac{\overline{q}^*(-)}{e(\mathbf{N}_i)},$$

viewed as maps $\mathrm{H}^\bullet(Y_1) \to \mathrm{H}^\bullet(Y_2)_{\mathrm{loc}}$ whose image lies in $\mathrm{H}^\bullet(Y_2) \subseteq \mathrm{H}^\bullet(Y_2)_{\mathrm{loc}}$.

$\square$

3.6.7. In line with our trichotomous Definition 3.1.10 of CoHAs, it is natural to expect a second Borel Moore and third sheaf theory version of the above Theorem 3.6.6. For instance, if p is quasismooth and we have $\mathcal{F}_i \in \mathrm{Sh}(Y_i)$ with a map $q^*\mathcal{F}_1 \to p^*\mathcal{F}_2$ satisfying an associativity condition, then as maps $\mathrm{H}^\bullet(Y_1, \mathcal{F}_1) \to \mathrm{H}^\bullet(Y_2, \mathcal{F}_2)$, we expect that

$$p_* q^* \;=\; \overline{p}_* \frac{\overline{q}^*(-)}{e(\mathbf{N}_i)}. \tag{3.44}$$

To verify this would require developing the exponential map for general sheaf cohomology, which we have not done but is probably not hard, after which the proof of this expectation is likely to proceed as above.

3.6.8. **CoHAs via equivariant cohomology.** In practice, we will usually consider the case where (3.44) consist of T-*equivariant* maps, where T is a torus, and the action on the Y_i are trivial. Then we apply Theorem 3.6.6 to the quotient diagram $(3.44)/T$, to get that

$$p_{T*} q_T^* \;=\; \overline{p}_{T*} \frac{\overline{q}_T^*(-)}{e(\mathbf{N}_{i_T})}.$$

In particular, composing with the pullback $\varphi_2 : \mathrm{H}_T^\bullet(Y_2) \to \mathrm{H}^\bullet(Y_2)$ and the section $\varphi_1 : \mathrm{H}^\bullet(Y_1) \to \mathrm{H}_T^\bullet(Y_1)$, we have

$$p_* q^* \;=\; \varphi_2(p_{T*} q_T^*)\varphi_1 \;=\; \varphi_2 \overline{p}_{T*} \frac{\overline{q}_T^* \varphi_1(-)}{e(\mathbf{N}_{i_T})}$$

as maps $\mathrm{H}^\bullet(Y_1) \to \mathrm{H}^\bullet(Y_2)$. In the following we will abuse notation by suppressing φ_i from the notation, e.g. writing

$$p_* q^* \;=\; p_{T*} q_T^* \;=\; \overline{p}_{T*} \frac{\overline{q}_T^*(-)}{e(\mathbf{N}_{i_T})}.$$

The reason this situation is useful to consider in the first place is that if the T action on the Y_i will be trivial, it is often easy to see when a multiplicative subset $S \subseteq \mathrm{H}_T^\bullet(Y_i) = \mathrm{H}^\bullet(Y_i)[t_1, ..., t_n]$ of a polynomial ring consists of nonzero divisors, e.g. see the proof of Proposition 3.5.28 for an example.

3.6.9. **Limit CoHA products.** When applying the above, in many examples it is hard to directly verify that $Z \to X$ is concentrated and specialised because its normal complex $\mathbf{N}_i \in \mathrm{Perf}(Z)$ is not strict (representable as a bounded complex of vector bundles). However, it is often easy to find an increasing open cover on which it *is* strict.

Thus, consider for each $n \geqslant 0$ a diagram

$$\mathfrak{J}_n \;=\; \begin{tikzcd} & Z_n \arrow[d, "i_n"] \arrow[ddl, bend right, "\overline{q}_n"'] \arrow[ddr, bend left, "\overline{p}_n"] & \\ & X_n \arrow[dl, "q_n"'] \arrow[dr, "p_n"] & \\ Y_{1,n} & & Y_{2,n} \end{tikzcd} \tag{3.45}$$

with maps of diagrams $\mathfrak{J}_0 \to \mathfrak{J}_1 \to \cdots \to \mathfrak{J}$ which are open embeddings on each object. Here we have written $\mathfrak{J}$ for the diagram (3.44). Finally, let $S_n \subseteq \mathrm{H}^\bullet(Y_{2,n})$ be a compatible system of multiplicative subset of nonzero divisors, and write $S = \lim S_n$.

Theorem 3.6.10. *Assume $Z_n \to X_n$ is a closed embedding which is S_n-concentrated and S_n-specialised over $Y_{2,n}$ with respect to a multiplicative subset $S_n \subseteq \mathrm{H}^\bullet(Y_{2,n})$ of nonzero divisors, for each $n \geqslant 0$. Assume that p_n is proper, quasismooth and*

$$\mathrm{H}^\bullet(Y_i) \;=\; \lim \mathrm{H}^\bullet(Y_{i,n}),$$

Finally, assume in the sequence $\mathfrak{J}_0 \to \mathfrak{J}_1 \to \cdots \to \mathfrak{J}$ that all squares with vertical arrows i_n, p_n or $\overline{p}_n$ (or in the Borel Moore case, i_n, q_n or $\overline{q}_n$) are cartesian. Then the first result of Theorem 3.6.6 holds, i.e.

$$p_* q^* \;=\; \overline{p}_* \frac{\overline{q}^*(-)}{e(\mathbf{N}_i)}$$

as maps $\mathrm{H}^\bullet(Y_1) \to \mathrm{H}^\bullet(Y_2)$. On the right hand side, we pass via S localised cohomology to divide by $e(\mathbf{N}_i)$.

Proof. By Theorem 3.6.6, we have for each finite $n \geqslant 0$ that

$$p_{n*} q_n^* \;=\; \overline{p}_{n,*} \frac{\overline{q}_n^*(-)}{e(\mathbf{N}_{i_n})}.$$

Thus the limit of both sides, which define maps $\mathrm{H}^\bullet(Y_1) \to \mathrm{H}^\bullet(Y_2)$, are the same. The limit of the left side is $p_* q^*$, and the limit of the right side is $\overline{p}_* \frac{\overline{q}^*(-)}{e(\mathbf{N}_i)}$. ☐

Likewise, we expect that there to be Borel Moore and sheaf versions of this Theorem.

3.6.11. In practice when computing (non Borel Moore) CoHAs we take the constant family $Y_{1,n} = Y_1$, a family of open embeddings $Y_{2,n}$ whose closed complement has increasing codimension, and form Z_n, X_n by pullback. Moreover, the multiplicative subset $(e(\mathbf{N}_{i_n}))$ is contained in a pullback of a multiplicative subset in $\mathrm{H}^\bullet(Y_1)$, see section 3.8.13.

3.6.12. **Heuristic to compute CoHAs.** Consider a space X as in section 3.1.9, for instance admitting a correspondence

$$\begin{tikzcd} & C \arrow[dl, "q"'] \arrow[dr, "\tilde{p}"] & \\ X \times X & & X \end{tikzcd} \tag{3.46}$$

and define cohomological Hall algebra by Definition 3.1.10. We will now explain how to compute explicit formulas for it using the previous sections

3.6.13. The idea is to use a *split locus*, which is any function

$$\underline{\oplus} \ : \ X^s \ \to \ X$$

whose pullback gives an injection on cohomology. The simpler the cohomology of X^s, the more useful this heuristic will be.

We then form a diagram by taking the pullback

$$\begin{tikzcd} & & C \times_X X^s \arrow[dl, "\oplus"'] \arrow[dr, "p"] & \\ & C \arrow[dl, "q"'] \arrow[dr, "\underline{p}"] & & X^s \arrow[dl, "\underline{\oplus}"'] \\ X \times X & & X & \end{tikzcd} \tag{3.47}$$

and we have

$$\underline{\oplus}^*(\mathrm{CoHA}) \ = \ \underline{\oplus}^* \underline{p}_* q^* \ = \ p_* \oplus^* q^*.$$

We then assume there is an action of a torus T on $C \times_X X^s$ such that p and $q\oplus$ are torus equivariant for the trivial actions on the target spaces (see sections 3.11, 3.13 and 3.13 for examples), and let $C^s \to C \times_X X^s$ be an equivariant closed embedding from a space with trivial T action. Then consider

$$\begin{tikzcd} & & C^s \arrow[d, "i"] \arrow[dll, bend right, "\overline{\oplus}"'] \arrow[dr, bend left, "\overline{p}"] & \\ & & C \times_X X^s \arrow[dl, "\oplus"'] \arrow[dr, "p"] & \\ & C \arrow[dl, "q"'] \arrow[dr, "\underline{p}"] & & X^s \arrow[dl, "\underline{\oplus}"'] \\ X \times X & & X & \end{tikzcd} \tag{3.48}$$

and applying Theorem 3.6.6, we get

Proposition 3.6.14. *Assume that $\underline{p}$ (and therefore p and $\overline{p}$) is proper, and that*

1. *p is quasismooth, or*
2. *q and $\underline{\oplus}$ are quasismooth, or*
3. *p is quasismooth and $\mathcal{F} \in \mathrm{Sh}(X)$ admits a map $q^*(\mathcal{F} \boxtimes \mathcal{F}) \to p^*\mathcal{F}$ satisfying an associativity condition.*

Assume that i is concentrated and specialised, as is the zero section of its normal complex, then for the three above CoHAs coming from Definition 3.1.10 we have the following formulas:

$$\underline{\oplus}^*(\mathrm{CoHA}) \;=\; \underline{\oplus}^* \underline{p}_* q^* \;=\; \overline{p}_* \frac{\overline{\oplus}^*(-)}{e(\mathbf{N}_i)}. \tag{3.49}$$

3.6.15. Note that for that for equation (3.52) to be useful in the third case, we should also require that the pullback $\underline{\oplus}^* : \mathrm{H}^\bullet(X, \mathcal{F}) \to \mathrm{H}^\bullet(X^s, \underline{\oplus}^*\mathcal{F})$ be injective.

3.6.16. It is sometimes useful to know a more explicit form of $\mathbf{N}_i$. As K-theory classes we have, by repeatedly applying the distinguished triangle the tangent complex of a composition,

$$[\mathbf{N}_i] \;=\; [i^* \oplus^* \mathbf{T}_{C/X}] - [\mathbf{T}_{C^s/X^s}]. \tag{3.50}$$

3.6.17. **Examples.** We will use this heuristic in two different ways when $X = \mathcal{M}_{\mathcal{A}}$ is the moduli space of objects in abelian category $\mathcal{A}$.

- To prove our main Theorem 3.10.1, we will use $X^s = X^2$ the space classifying pairs of objects in $\mathcal{A}$.
- To give explicit formulas for CoHAs in sections 3.12 and 3.13, we will take X^s to be something like tuples of rank one" objects in $\mathcal{A}$.

In both cases, $\underline{\oplus}$ is the direct sum map.

3.6.18. In both applications, the space $C \times_X X^s = \mathrm{Ext}_{\mathcal{A}} \times_{\mathcal{M}_{\mathcal{A}}} \mathcal{M}^s_{\mathcal{A}}$ classifies short exact sequences

$$0 \;\to\; \mathcal{E}_1 \;\to\; \mathcal{E} \;\to\; \mathcal{E}_2 \;\to\; 0 \tag{3.51}$$

with a splitting of the middle term $\varphi : \mathcal{E} \simeq \bigoplus \mathcal{E}_i$, either into two objects or multiple rank one objects, depending on the application.

The rank of the torus T will be the number of these summand, with a torus element $\tau \in T$ acting on the above by

$$0 \;\to\; \mathcal{E}_1 \;\overset{\tau\alpha}{\to}\; \bigoplus \mathcal{E}_i \;\overset{\beta\tau^{-1}}{\to}\; \mathcal{E}_2 \;\to\; 0.$$

Then (3.54) splits as a direct sum of exact sequences if and only if it is fixed under the T action. We write $C^s = \mathrm{Ext}^s$ for the space parametrising direct sums of two short exact sequences, or direct sums of many short exact sequences whose middle term has rank one, depending on the application.

Thus we have

$$\begin{tikzcd}
 & \mathrm{Ext}^s_{\mathcal{A}} \arrow[d, "i"] \arrow[ddl, bend right, "\overline{\oplus}"'] \arrow[ddr, bend left, "\overline{p}"] & \\
 & \mathrm{Ext}_{\mathcal{A}} \times_{\mathcal{M}_{\mathcal{A}}} \mathcal{M}^s_{\mathcal{A}} \arrow[dl, "q\oplus"'] \arrow[dr, "p"] & \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} & & \mathcal{M}^s_{\mathcal{A}}
\end{tikzcd}$$

If we write

$$0 \;\to\; \bigoplus \mathcal{E}_{1,i} \;\overset{\oplus\alpha_i}{\to}\; \bigoplus \mathcal{E}_i \;\overset{\oplus\beta_i}{\to}\; \bigoplus \mathcal{E}_{2,i} \;\to\; 0$$

for a point of $\mathrm{Ext}^s_{\mathcal{A}}$, the map $\overline{p}$ sends it to $\oplus\mathcal{E}_i$, and the map $\overline{\oplus}$ sends it to $(\oplus\mathcal{E}_{1,i}, \oplus\mathcal{E}_{2,i})$.

3.6.19. In particular, in these cases the map $\overline{\oplus}$ lifts to

$$\begin{tikzcd}
\mathcal{M}^s_{\mathcal{A}} \times \mathcal{M}^s_{\mathcal{A}} \arrow[dd, "\underline{\oplus}\times\underline{\oplus}"'] & \mathrm{Ext}^s_{\mathcal{A}} \arrow[l, dashed, bend right, "\widetilde{\oplus}"'] \arrow[d, "i"] \arrow[ddl, bend right, "\overline{\oplus}"'] \arrow[ddr, bend left, "\overline{p}"] & \\
 & \mathrm{Ext}_{\mathcal{A}} \times_{\mathcal{M}_{\mathcal{A}}} \mathcal{M}^s_{\mathcal{A}} \arrow[dl, "q\oplus"'] \arrow[dr, "p"] & \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} & & \mathcal{M}^s_{\mathcal{A}}
\end{tikzcd}$$

The lift however is not unique, e.g. we can postcompose $\widetilde{\oplus}$ with any automorphism of $\mathcal{M}^s_{\mathcal{A}}$ over $\mathcal{M}_{\mathcal{A}}$.

3.6.20. **Localised CoHA products.** Assume that the map $\overline{\oplus}$ can be a lifted to a map to the split locus:

$$
\begin{tikzcd}
 & & & C^s \arrow[d, "i"] \arrow[dlll, dashed, bend right, "\tilde{\oplus}"'] \arrow[dr, bend left, "\overline{p}"] & \\
 & & & C \times_X X^s \arrow[dl, "\oplus"'] \arrow[dr, "p"] & \\
X^s \times X^s \arrow[d, "\underline{\oplus}\times\underline{\oplus}"'] & & C \arrow[dll, "q"'] \arrow[dr, "\underline{p}"] & & X^s \arrow[dl, "\underline{\oplus}"'] \\
X \times X & & & X &
\end{tikzcd}
\tag{3.52}
$$

In this case by the integration formula (3.31) we can *lift* the CoHA map $\underline{p}_* q^*$ to a map

$$
\begin{tikzcd}
\mathrm{H}^\bullet(X^s) \otimes \mathrm{H}^\bullet(X^s) \arrow[r, "\overline{p}_* \frac{\tilde{\oplus}^*(-)}{e(\mathbf{N}_i)}"] & \mathrm{H}^\bullet(X^s)_{\mathrm{loc}} \\
\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \arrow[u, "(\underline{\oplus}\times\underline{\oplus})^*"] \arrow[r, "\underline{p}_* q^*"] & \mathrm{H}^\bullet(X) \arrow[u, "\underline{\oplus}^*"']
\end{tikzcd}
$$

The top horizontal map we call a *localised CoHA product*. It does not give an algebra structure on $\mathrm{H}^\bullet(X^s)_{\mathrm{loc}}$ because of the noncanonicity of the lift $\widetilde{\oplus}$. However, it restricts to the CoHA product on $\mathrm{H}^\bullet(X)$, and so is often useful in giving explicit formulas for it, e.g. [KS]. Compare the notion of localised coproduct in [Da].

3.7 Concentration for Białynicki-Birula style stratified spaces

To simplify the spaces we will be interested in, we will cut them up into strata whose behaviour is much simpler. For our purposes, a *stratification* of an Artin stack X is an increasing union of closed Artin substacks

$$\varnothing \;=\; \overline{X}_{-1} \;\subseteq\; \overline{X}_0 \;\subseteq\; \cdots \;\subseteq\; \overline{X}_{n-1} \;\subseteq\; \overline{X}_n \;=\; X$$

whose *strata* are $X_i = \overline{X}_i \backslash \overline{X}_{i-1}$. This allows us to use Mayer Vietoris and Gysin sequences to inductively prove statements about X by working stratawise.

3.7.1. For "Białynicki-Birula" type stratifications (see [JS, Bi] for a classical account), we now show that concentration and specialisation for the strata implies concentration for the whole space. Let $S \subseteq \mathrm{H}^\bullet(X)$ be a multiplicative subset and $s : Z \to X$ be a closed embedding.

3.7.2. Assume X_i is a stratification of X such that the induced closed embeddings

$$s_i \ : \ Z_i \ = \ Z \cap X_i \hookrightarrow X_i$$

are isomorphisms on cohomology, and admit retractions $X_i \twoheadrightarrow Z_i$. In particular, s_i is S-concentrated if and only if it is S-specialised.

Lemma 3.7.3. *If each s_i is S-specialised, then so is s.*

Proof. We want to show

$$\mathrm{H}^\bullet(Z, s^* t_* k)_{\mathrm{loc}} \ = \ 0.$$

Assume that X has two strata, an open stratum X_1 and its complement X_0, the general case proceeding similarly using induction. Let $t : U \to X$ be the open complement of Z, and form the pullback squares

$$\begin{tikzcd} U_0 \arrow[r, "\hat{\imath}"] \arrow[d, "t_0"'] & U \arrow[d, "t"'] & U_1 \arrow[l, "\hat{\jmath}"'] \arrow[d, "t_1"'] \\ X_0 \arrow[r, "i"] & X & X_1 \arrow[l, "j"'] \\ Z_0 \arrow[r, "\bar{\imath}"] \arrow[u, "s_0"] & Z \arrow[u, "s"] & Z_1 \arrow[l, "\bar{\jmath}"'] \arrow[u, "s_1"] \end{tikzcd} \tag{3.53}$$

We will now consider some commuting diagrams of long exact sequences, first coming from the horizontal direciton in (3.56), then secondly coming from the vertical direction.

First, we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^\bullet(Z, \bar{\jmath}_! s_1^* t_{1*} k) \longrightarrow \mathrm{H}^\bullet(Z, s^* t_* k) \xrightarrow{i^*} \mathrm{H}^\bullet(Z_0, s_0^* t_{0*} k) \longrightarrow \cdots \tag{3.54}$$

The localisation of the right term vanishes because s_0 is S-specialised. So it is enough to show that $\mathrm{H}^\bullet(Z, \bar{\jmath}_! s_1^* t_{1*} k)_{\mathrm{loc}} = 0$.

Second, we fit it into a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^\bullet(Z, \bar{\jmath}_! s_1^* t_{1*} k) \longrightarrow \mathrm{H}^\bullet(Z_1, s_1^* t_{1*} k) \longrightarrow \mathrm{H}^\bullet(Z_0, \bar{\imath}^* \bar{\jmath}_* s_1^* t_{1*} k) \longrightarrow \cdots \tag{3.55}$$

so it is enough to show that localising the middle and right terms kills them. The localisation of the middle term vanishes because s_1 is S-specialised.

The localisation of the right term actually vanishes for the same reason, using ideas in section 3.3.16. Note that $\mathcal{A} = j_* s_{1*} s_1^* t_{1*} t_1^* k$ is a commutative monoid in $\mathrm{Sh}(X)$ (see section 3.3.18). Moreover,

$$\mathrm{H}^\bullet(X, \mathcal{A})_{\mathrm{loc}} \ = \ 0$$

because s_1 is S-specialised. It follows from Corollary 3.3.28 that

$$\mathrm{H}^\bullet(X, i_*i^*j_*j^*\mathcal{A})_{\mathrm{loc}} \;=\; 0.$$

However, this is nothing but

$$\mathrm{H}^\bullet(Z_0, \bar{\imath}^*\bar{\jmath}_*s_1^*t_{1*}k) \;=\; \mathrm{H}^\bullet(Z, s_{0*}\bar{\imath}^*\bar{\jmath}_*s_1^*t_{1*}t_1^*k) \;=\; \mathrm{H}^\bullet(Z, i^*j_*s_{1*}s_1^*t_{1*}t_1^*k) \;=\; \mathrm{H}^\bullet(X, i_*i^*j_*j^*\mathcal{A}).$$

Thus the right term of (3.58) vanishes too, proving our claim and the proposition. □

Proposition 3.7.4. *If each s_i is S-concentrated, then so is s.*

Proof. We want to show

$$\mathrm{H}^\bullet(X \backslash Z)_{\mathrm{loc}} \;=\; 0.$$

Proceed as in Lemma 3.7.3, assuming that there is an open stratum X_1 and a closed stratum X_0, the general case proceeding similarly using induction, giving the diagram 3.56.

First, we have a commuting diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^\bullet(X, j_!k) \arrow[r] \arrow[d] & \mathrm{H}^\bullet(X) \arrow[r, "i^*"] \arrow[d, "s^*"] & \mathrm{H}^\bullet(X_0) \arrow[r] \arrow[d, "\sim"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^\bullet(Z, \bar{\jmath}_!k) \arrow[r] & \mathrm{H}^\bullet(Z) \arrow[r, "\bar{\imath}^*"] & \mathrm{H}^\bullet(Z_0) \arrow[r] & \cdots
\end{tikzcd} \tag{3.56}$$

The right vertical map is an isomorphism because pullback by s_0 gives an isomorphism on cohomology. We then consider another commuting diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^\bullet(X, j_!k) \arrow[r] \arrow[d] & \mathrm{H}^\bullet(X_1) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^\bullet(X_0, i^*j_*k) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^\bullet(Z, \bar{\jmath}_!k) \arrow[r] & \mathrm{H}^\bullet(Z_1) \arrow[r] & \mathrm{H}^\bullet(Z_0, \bar{\imath}^*\bar{\jmath}_*k) \arrow[r] & \cdots
\end{tikzcd} \tag{3.57}$$

The middle vertical map is an isomorphism because pullback by s_1 gives an isomorphism. The cone of the right vertical map is

$$\mathrm{H}^\bullet(X_0, i^*j_*t_{1!}k).$$

To show that its localisation vanishes, we again use the using ideas in section 3.3.16. Writing $\mathcal{A} = j_*t_{1*}t_1^*k$, we have that

$$\mathrm{H}^\bullet(X, \mathcal{A})_{\mathrm{loc}} \;=\; \mathrm{H}^\bullet(U_1)_{\mathrm{loc}} \;=\; 0$$

because s_1 is concentrated. It follows from Corollary 3.3.28 that the localised cohomology of

$$\mathcal{A}' \;=\; i_*i^*j_*j^*\mathcal{A}$$

also vanishes. Note that by section 3.3.21, the cohomology of $\mathcal{A}'$ acts on $\mathrm{H}^\bullet(X_0, i^* j_* t_{1!} k) = \mathrm{H}^\bullet(X, i_* i^* j_* j^* j_* t_{1!} t_1^* k)$, and the action of $\mathrm{H}^\bullet(X)$ factors through $\mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(X, \mathcal{A}')$. Thus it follows that

$$\mathrm{H}^\bullet(X_0, i^* j_* t_{1!} k)_{\mathrm{loc}} \;=\; 0.$$

We have shown that all maps in (3.60), and hence (3.59), are isomorphisms after localisation. In particular, $s^* : \mathrm{H}^\bullet(X)_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{H}^\bullet(Z)_{\mathrm{loc}}$.

To finish, we consider the commuting diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^\bullet(X, t_! k) \arrow[r] \arrow[d, equal] & \mathrm{H}^\bullet(X) \arrow[r, "s^*"] \arrow[d, "t^*"] & \mathrm{H}^\bullet(Z) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^\bullet(X, t_! k) \arrow[r] & \mathrm{H}^\bullet(X \backslash Z) \arrow[r] & \mathrm{H}^\bullet(Z, s^* t_* k) \arrow[r] & \cdots
\end{tikzcd} \tag{3.58}$$

so we have

$$\mathrm{H}^\bullet(X \backslash Z)_{\mathrm{loc}} \;\simeq\; \mathrm{H}^\bullet(Z, s^* t_* k)_{\mathrm{loc}}.$$

The right side fits into a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^\bullet(Z_0, s_0^* t_{0*} \hat{\imath}^! k) \longrightarrow \mathrm{H}^\bullet(Z, s^* t_* k) \xrightarrow{i^*} \mathrm{H}^\bullet(Z, s^* j_* t_{1*} t_1^* k) \longrightarrow \cdots \tag{3.59}$$

Now, the action of $\mathrm{H}^\bullet(X)$ on the left term factors through $\mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(X_0, t_{0*} t_0^* k) = \mathrm{H}^\bullet(X_0 \backslash Z_0)$ (see section 3.3.23), whose localisation is zero because s_0 is concentrated. Similarly, the action on the right term factors through $\mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(X_1, t_{1*} t_1^* k) = \mathrm{H}^\bullet(X_1 \backslash Z_1)$, whose localisation vanishes for the same reason. Thus localisation kills all terms in (3.62), and so $\mathrm{H}^\bullet(X \backslash Z)_{\mathrm{loc}} = 0$. □

3.7.5. **Fundamental classes.** Note that we also have the following result

Proposition 3.7.6. *Given a quasismooth map $\pi : X \to Z$ over a base B, the fundamental class*

$$[X/Z] \;:\; \mathrm{H}^\bullet(Z/B)_{\mathrm{loc}} \;\to\; \mathrm{H}^\bullet(X/B)_{\mathrm{loc}}$$

is an isomorphism if any only if they are stratawise, i.e. writing $X_i = X \times_{Z_i} Z$,

$$[X_i/Z_i] \;:\; \mathrm{H}^\bullet(Z_i/B)_{\mathrm{loc}} \;\to\; \mathrm{H}^\bullet(X_i/B)_{\mathrm{loc}}$$

are isomorphisms.

Proof. For simplicity assume that $Z = Z_0 \cup Z_1$ has two strata, one closed and one open. Dropping loc subscripts, in this case the Proposition follows from the five lemma applied to the following commuting diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(X_0/B) \arrow[r] & \mathrm{H}^\bullet(X/B) \arrow[r] & \mathrm{H}^\bullet(X_1/B) \arrow[r, "+1"] & {} \\
\mathrm{H}^\bullet(Z_0/B) \arrow[r] \arrow[u, "{[X_0/Z_0]}", "\sim"'] & \mathrm{H}^\bullet(Z/B) \arrow[r] \arrow[u, "{[X/Z]}"] & \mathrm{H}^\bullet(Z_1/B) \arrow[r, "+1"] \arrow[u, "{[X_1/Z_1]}", "\sim"'] & {}
\end{tikzcd}$$

The general case proceeds by induction on the number of strata. □

Proposition 3.7.7. *Given a quasismooth map $\pi : X \to Z$, the fundamental class*

$$[X/Z] \ : \ \mathrm{H}^\bullet(Z)_{\mathrm{loc}} \ \to \ \mathrm{H}^\bullet(X/Z)_{\mathrm{loc}}$$

is an isomorphism if any only if they are stratawise, i.e.

$$[X_i/Z_i] \ : \ \mathrm{H}^\bullet(Z_i)_{\mathrm{loc}} \ \to \ \mathrm{H}^\bullet(X_i/Z_i)_{\mathrm{loc}}$$

are isomorphisms.

Proof. By the previous Proposition applied to $B = Z$ we just need to show that

$$[X_i/Z_i] \ : \ \mathrm{H}^\bullet(Z_i/Z)_{\mathrm{loc}} \ \to \ \mathrm{H}^\bullet(X_i/Z)_{\mathrm{loc}}$$

are isomorphisms. Dropping the loc subscripts, consider

$$\begin{tikzcd}
\mathrm{H}^\bullet(Z_i) \otimes_{\mathrm{H}^\bullet(Z_i)} \mathrm{H}^\bullet(Z_i/Z) \arrow[r, "{[X_i/Z_i]\otimes \mathrm{id}}", "\sim"'] \arrow[d, "\sim"', "\cup"] & \mathrm{H}^\bullet(X_i/Z_i) \otimes_{\mathrm{H}^\bullet(Z_i)} \mathrm{H}^\bullet(Z_i/Z) \arrow[d, "\sim"', "\cup"] \\
\mathrm{H}^\bullet(Z_i/Z) \arrow[r, "{[X_i/Z_i]}"] & \mathrm{H}^\bullet(X_i/Z)
\end{tikzcd}$$

The right vertical arrow is an isomorphism because the following is an isomorphism the diagram is an isomorphism

$$\begin{tikzcd}
\mathrm{H}^\bullet(Z_i) \otimes_{\mathrm{H}^\bullet(Z_i)} \mathrm{H}^\bullet(Z_i) \arrow[r, "{[X_i/Z_i]\cdot\otimes \mathrm{id}}", "\sim"'] \arrow[d, "\sim"', "\cup"] & \mathrm{H}^\bullet(X_i/Z_i) \otimes_{\mathrm{H}^\bullet(Z_i)} \mathrm{H}^\bullet(Z_i) \arrow[d, "\cup"] \\
\mathrm{H}^\bullet(Z_i) \arrow[r, "{[X_i/Z_i]\cdot}", "\sim"'] & \mathrm{H}^\bullet(X_i/Z_i)
\end{tikzcd}$$

finishing the proof. □

3.8 Application: moduli stacks

In this section we apply the results of section 3.7 to the moduli stacks we are interested in.

3.8.1. Let X be a moduli stack parametrising objects in an abelian category, i.e. either a stack as in section 3.11.2, or just the moduli of vector bundles, representations of a quiver or coherent sheaves on a curve. Also consider

$$\mathcal{N} \;=\; \mathrm{Ext}, \qquad\qquad \mathcal{M} \;=\; \mathrm{Ext}\times_X X^2$$

the moduli stacks parametrising an object with a subobject, and two objects with a subobject of their direct sum, respectively.

3.8.2. Write $\gamma_1 \subseteq \gamma_2$ and $\gamma_1 \subseteq \gamma_2^- \oplus \gamma_2^+$ for the tautological vector bundles or quiver representation bundles on $\mathcal{N}\times\mathcal{N}$, or coherent sheaves on $C\times\mathcal{N}\times\mathcal{N}$.

3.8.3. There is a map (which in the three main examples is a closed embedding)

$$i \;:\; \mathcal{N}\times\mathcal{N} \;\to\; \mathcal{M} \qquad\qquad (\mathcal{F}^+\subseteq\mathcal{E}^+,\mathcal{F}^-\subseteq\mathcal{E}^-) \;\mapsto\; (\mathcal{F}^+\oplus\mathcal{F}^-\subseteq\mathcal{E}^+\oplus\mathcal{E}^-).$$

Now, this map usually does not admit a retraction. However, using the obvious $\mathbf{G}_m$ action

$$\mathbf{G}_m\times\mathcal{M} \;\to\; \mathcal{M} \qquad\qquad t\cdot(\mathcal{F}\subseteq\mathcal{E}^+\oplus\mathcal{E}^-) \;=\; (\left(\begin{smallmatrix} t & \\ & 1\end{smallmatrix}\right)\cdot\mathcal{F}\subseteq\mathcal{E}^+\oplus\mathcal{E}^-),$$

one might have expected that taking the $t\to 0$ limit could give a retraction to i:

$$\mathcal{M} \text{ ``}\to\text{'' } \mathcal{N}\times\mathcal{N} \qquad\qquad (\mathcal{F}\subseteq\mathcal{E}^+\oplus\mathcal{E}^-) \;\mapsto\; (\ker\mathcal{F}\subseteq\mathcal{E}^+,\mathrm{im}\mathcal{F}\subseteq\mathcal{E}^-)$$

given by taking the kernel and image of the map $\mathcal{F}\hookrightarrow\mathcal{E}^+\oplus\mathcal{E}^-\twoheadrightarrow\mathcal{E}^-$. The reason this is does not define a map is that whilst one can take image and kernels in abelian categories, this is not true *in families*. For instance, when X is the moduli stack of vector spaces, a map to $\mathcal{N}\times\mathcal{N}$ is uniquely determined by the pullbacks of the tautological vector bundles $\gamma_1^\pm\subseteq\gamma_2^\pm$ on $\mathcal{N}\times\mathcal{N}$, however we cannot set the pullbacks of γ_1^+ and γ_1^- to be $\ker\gamma_1$ and $\mathrm{im}\gamma_1$, since kernels and images of vector bundles are not themselves vector bundles.

3.8.4. However, we *can* define a *stratified* retraction to the map i. Write $\check{\mathcal{M}}$ for the moduli stack parametrising two objects with a subobject

$$\mathcal{F} \;\subseteq\; \mathcal{E}^+\oplus\mathcal{E}^-$$

such that the image and kernel of $\mathcal{F}$ under the map $\mathcal{F}\to\mathcal{E}^+\oplus\mathcal{E}^-\twoheadrightarrow\mathcal{E}^-$ are also objects of the category familywise. We get a commuting diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathrm{im}\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^+ & \longrightarrow & \mathcal{E}^+\oplus\mathcal{E}^- & \longrightarrow & \mathcal{E}^- & \longrightarrow & 0 \end{array} \tag{3.60}$$

which thus defines a map

$$\lim \;:\; \check{\mathcal{M}} \;\to\; \mathcal{N}\times\mathcal{N} \qquad\qquad (\mathcal{F}\subseteq \mathcal{E}^+\oplus\mathcal{E}^-) \;\mapsto\; (\ker\mathcal{F}\subseteq\mathcal{E}^+, \mathrm{im}\mathcal{F}\subseteq\mathcal{E}^-)$$

which is a retraction of $i:\mathcal{N}\times\mathcal{N}\to\check{\mathcal{M}}$. One can show that $\check{\mathcal{M}}$ is a disjoint union of the strata in a stratification of $\mathcal{M}$.

3.8.5. We spell out how to define i and lim very explicitly. As noted, $\mathcal{N}\times\mathcal{N}$ carries four tautological vector bundles, $\gamma_1^+\subseteq\gamma_2^+$ and $\gamma_1^-\subseteq\gamma_2^-$, and the space $\check{\mathcal{M}}$ carries $\gamma_1\subseteq\gamma_2^+\oplus\gamma_2^-$ as well as $\ker\gamma_1$ and $\mathrm{im}\gamma_1$. We thus define $\lim:\check{\mathcal{M}}\to\mathcal{N}\times\mathcal{N}$ by $\lim^*\gamma_2^\pm=\gamma_2^\pm$ and

$$\lim{}^*\gamma_1^+ \;=\; \ker\gamma_1, \qquad\qquad \lim{}^*\gamma_1^- \;=\; \mathrm{im}\gamma_1.$$

Similarly, $i:\mathcal{N}\times\mathcal{N}\to\check{\mathcal{M}}$ is defined by $i^*\gamma_2^\pm=\gamma_2^\pm$ and $i^*\gamma_1=\gamma_1^+\oplus\gamma_1^-$.

3.8.6. The fibres of lim above $(\mathcal{F}^+\subseteq\mathcal{E}^+,\mathcal{F}^-\subseteq\mathcal{E}^-)$ consist of maps of extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^+ \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, "\alpha"] & \mathcal{F}^- \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}^+ \arrow[r] & \mathcal{E}^+\oplus\mathcal{E}^- \arrow[r] & \mathcal{E}^- \arrow[r] & 0
\end{tikzcd} \tag{3.61}$$

We can now compute the fibre of lim using

Lemma 3.8.7. *Let $\mathcal{A}$ be an abelian category with objects $\mathcal{F}^\pm\subseteq\mathcal{E}^\pm$. Then the choices of diagram of extensions (3.64) biject with* $\mathrm{Hom}(\mathcal{F}^-,\mathcal{E}^+/\mathcal{F}^+)$.

Proof. Given a map of extensions (3.64), we get a map

$$\mathcal{F}^- \;=\; \mathcal{F}/\mathcal{F}^+ \;\xrightarrow{\alpha}\; (\mathcal{E}^+\oplus\mathcal{E}^-)/\mathcal{F}^+ \;\twoheadrightarrow\; \mathcal{E}^+/\mathcal{F}^+.$$

Conversely, given a map $A:\mathcal{F}^-\to\mathcal{E}^+/\mathcal{F}^+$, we can form

$$\begin{tikzcd}
\mathcal{F}^+ \arrow[r] & \mathcal{E}^+\times_{\mathcal{E}^+/\mathcal{F}^+}\mathcal{F}^- \arrow[r] \arrow[d] & \mathcal{F}^- \arrow[d] \\
 & \mathcal{E}^+ \arrow[r] & \mathcal{E}^+/\mathcal{F}^+
\end{tikzcd}$$

It is easy to check that the top row is an exact sequence, that the induced diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^+ \arrow[r] \arrow[d] & \mathcal{E}^+\times_{\mathcal{E}^+/\mathcal{F}^+}\mathcal{F}^- \arrow[r] \arrow[d] & \mathcal{F}^- \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}^+ \arrow[r] & \mathcal{E}^+\oplus\mathcal{E}^- \arrow[r] & \mathcal{E}^+ \arrow[r] & 0
\end{tikzcd}$$

commutes, and that this gives a bijection. $\square$

When $\mathcal{A}$ is semisimple, we can choose a splitting $\mathcal{F} \simeq \mathcal{F}^+ \oplus \mathcal{F}^-$ and having made this choice the diagrams (3.64) biject with $\mathrm{Hom}(\mathcal{F}^-, \mathcal{E}^+)$. Since the choices of splitting are parametrised by $\mathrm{Hom}(\mathcal{F}^-, \mathcal{F}^+)$, and $\mathrm{Hom}(\mathcal{F}^-, \mathcal{E}^+)/\mathrm{Hom}(\mathcal{F}^-, \mathcal{F}^+) = \mathrm{Hom}(\mathcal{F}^-, \mathcal{E}^+/\mathcal{F}^+)$, we again recover the above result in the semisimple case.

It follows from Lemma 3.8.7 that

Corollary 3.8.8. *$\check{\mathcal{M}}$ is the total space of $\mathrm{Hom}(\gamma_1^-, \gamma_2^+/\gamma_1^+)$, where:*

1. *When $\mathcal{A}$ is the category of representations of a finite quiver Q, $\gamma_i^\pm$ means the tautological Q representation bundles on $\mathcal{N} \times \mathcal{N}$, and $\mathrm{Hom}(\gamma_1^-, \gamma_2^+/\gamma_1^+)$ means the coherent sheaf on $\mathcal{N} \times \mathcal{N}$ given by Q representation coherent sheaf morphisms (as in section 3.12.3).*
2. *In particular, when $\mathcal{A}$ is the category of finite dimensional vector spaces, $\gamma_i^\pm$ are the tautological vector bundles on $\mathcal{N} \times \mathcal{N}$ and $\mathrm{Hom}(\gamma_1^-, \gamma_2^+/\gamma_1^+) = \mathcal{H}\mathrm{om}(\gamma_1^-, \gamma_2^+/\gamma_1^+)$ is the ordinary internal vector bundle Hom.*
3. *When $\mathcal{A}$ is the category of coherent sheaves on a curve C, $\gamma_i^\pm$ are the tautological coherent sheaves on $C \times \mathcal{N} \times \mathcal{N}$ and $\mathrm{Hom}(\gamma_1^-, \gamma_2^+/\gamma_1^+) = \mathcal{H}\mathrm{om}_C(\gamma_1^-, \gamma_2^+/\gamma_1^+)$ denotes the relative sections $\mathcal{H}\mathrm{om}(\gamma_1^-, \gamma_2^+/\gamma_1^+)$ along C.*

3.8.9. The $\mathbf{G}_m$ action on the fibres of lim sends

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}^+ \arrow[r, "\alpha"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, "\beta"] & \mathcal{F}^- \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{E}^+ \arrow[r] & \mathcal{E}^+ \oplus \mathcal{E}^- \arrow[r] & \mathcal{E}^- \arrow[r] & 0 \end{tikzcd}$$

to

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}^+ \arrow[r, "\begin{pmatrix} t & \\ & 1 \end{pmatrix}^{-1}\alpha"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, "\beta\begin{pmatrix} t & \\ & 1 \end{pmatrix}"] & \mathcal{F}^- \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{E}^+ \arrow[r] & \mathcal{E}^+ \oplus \mathcal{E}^- \arrow[r] & \mathcal{E}^- \arrow[r] & 0 \end{tikzcd}$$

and all other maps in the diagram are unchanged. In particular, the induced $\mathbf{G}_m$ action on the vector space $\mathrm{Hom}(\mathcal{F}^-, \mathcal{E}^+/\mathcal{F}^+)$ has weight one. This all upgrades to a $\mathbf{G}_m$ action on $\check{\mathcal{M}}$, by acting on $\mathcal{H}\mathrm{om}(\gamma_1^-, \gamma_2^+/\gamma_1^+)$ with weight one.

3.8.10. **Specialisation.** In this section, we show that the total space of the normal complex to $i : (\mathcal{N} \times \mathcal{N})/\mathbf{G}_m \to \check{\mathcal{M}}/\mathbf{G}_m$ is specialised (or equivalently, concentrated) over $\mathcal{N} \times \mathcal{N}/\mathbf{G}_m$. More than this, we have a $\mathbf{G}_m$ equivariant diagram

$$\begin{tikzcd} \mathcal{N} \times \mathcal{N} \arrow[d, "i"] \arrow[ddr, bend left, "\overline{p}"] & \\ \check{\mathcal{M}} \arrow[dr, "p"] & \\ & X \times X \end{tikzcd} \tag{3.62}$$

and we will show that $\mathbf{N}_i/\mathbf{G}_m$ is specialised with respect to a multiplicative subset for S of $\mathrm{H}^\bullet_{\mathbf{G}_m}(X \times X)$.

3.8.11. **Moduli of vector spaces.** In this case $\check{\mathcal{M}}$ is a vector bundle of $\mathbf{G}_m$ weight one over $\mathcal{N}\times\mathcal{N}$, hence $\check{\mathcal{M}} = \mathbf{N}_i$. By Proposition (3.5.28), $\mathbf{N}_i/\mathbf{G}_m$ is specialised after inverting the equivariant Euler class

$$e(\mathbf{N}_i) \;=\; e_{\mathbf{G}_m}(\mathcal{H}\mathrm{om}(\gamma_1^-, \gamma_2^+/\gamma_1^+)) \;=\; \prod (x_i - t) \;\in\; \mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{N}\times\mathcal{N}),$$

where x_i are the chern roots of $\mathbf{N}_i$. Next, because $\mathcal{H}\mathrm{om}(\gamma_1^-, \gamma_2^+/\gamma_1^+)$ is the subbundle of a quotient of $\mathcal{H}\mathrm{om}(\gamma_2^-, \gamma_2^+)$,

$$e_{\mathbf{G}_m}(\mathcal{H}\mathrm{om}(\gamma_1^-, \gamma_2^+/\gamma_1^+)) \mid e_{\mathbf{G}_m}(\mathcal{H}\mathrm{om}(\gamma_2^-, \gamma_2^+))$$

and so i is $S = \big(e_{\mathbf{G}_m}(\mathcal{H}\mathrm{om}(\gamma_2^-, \gamma_2^+)\big)$ concentrated.

3.8.12. **Moduli of quiver representations.** For notation on quiver representation and their moduli, see section 3.12. As before,

$$\check{\mathcal{M}} \;=\; \mathcal{H}\mathrm{om}_Q(\gamma_1^-, \gamma_2^+/\gamma_1^+)$$

and $\mathcal{H}\mathrm{om}_Q(\gamma_1^-, \gamma_2^+/\gamma_1^+)$ is a sub of a quotient of the vector bundle $\mathcal{H}\mathrm{om}_Q(\gamma_2^-, \gamma_2^+)$ in the category of coherent sheaves on $\mathcal{N}\times\mathcal{N}$. In particular, applying Lemma 3.3.30 we get that i is specialised with respect to $S = \big(e_{\mathbf{G}_m}(\mathcal{H}\mathrm{om}_Q(\gamma_2^-, \gamma_2^+))\big)$.

3.8.13. **Moduli of coherent sheaves on a curve.** As in the above two cases, i is specialised with respect to any multiplicative subset for which the total space of $\mathcal{H}\mathrm{om}_C(\gamma_2^-, \gamma_2^+)$. However, we need to be careful because in this case $\mathcal{H}\mathrm{om}_C(\gamma_2^-, \gamma_2^+)$ is not a vector bundle, nor does it have a global resolution by vector bundles.

Note that $\mathcal{H}\mathrm{om}_C(\gamma_2^-, \gamma_2^+)$ is the zeroeth cohomology sheaf of the perfect complex $\mathcal{E}\mathrm{xt}_C(\gamma_2^-, \gamma_2^+)$, which as for any perfect complex is strict (is quasiisomorphic to a bounded complex of vector bundles) when restricted to any quasi-compact open.

Pick an ample line bundle on our curve C. The moduli stack X of coherent sheaves is an increasing union of the moduli stack X_m of m regular sheaves for $m \in \mathbf{Z}$, which are quasicompact, see [Hu, §1.7]. We thus get a $\mathbf{G}_m$ equivariant commuting diagram

$$\begin{tikzcd} \mathcal{N}_m \times \mathcal{N}_m \arrow[d, "i_m"] \arrow[ddr, bend left, "\overline{p}_m"] & \\ \check{\mathcal{M}}_m \arrow[dr, "p_m"] & \\ & X_m \times X_m \end{tikzcd} \tag{3.63}$$

the pullback via the open embedding $X_m \times X_m \to X \times X$ of (3.65). Note that since $\overline{p}, \overline{p}_m$ are proper they are in particular quasicompact, hence $\mathcal{N}_m \times \mathcal{N}_m$ is quasicompact. Thus the Ext complex is quasiisomorphic to a bounded complex of vector bundles

$$\mathcal{E}\mathrm{xt}_C(\gamma_2^+, \gamma_2^-)|_{\mathcal{N}_m \times \mathcal{N}_m} \;\simeq\; (\;\cdots \to E_m^{-1} \xrightarrow{d_{-1}} E_m^0 \xrightarrow{d_0} E_m^1 \to \cdots\;).$$

Thus, by Lemma 3.3.30 the total space of $\mathcal{H}\mathrm{om}_C(\gamma_2^+, \gamma_2^-)|_{\mathcal{N}_m \times \mathcal{N}_m}$ is specialised with respect to the multiplciative subset $S_m = (e_{\mathbf{G}_m}(E_m^0))$ of $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{N}_m \times \mathcal{N}_m)$. Now, since X is smooth and on each connected component the codimension of $X_m \times X_m$ in $X \times X$ tends to infinity as $m \to \infty$,[8] we have that $\mathrm{H}^\bullet(X) = \lim \mathrm{H}^\bullet(X_m)$.

Thus, enlarging S_m if necessary, we have a compatible system $(S_m)_{m \in \mathbf{Z}}$ of multiplicative subsets of $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{N}_m \times \mathcal{N}_m)$ with respect to which $(\mathbf{N}_i/\mathbf{G}_m)|_{\mathcal{N}_m \times \mathcal{N}_m}$ is specialised. Then since $\mathrm{H}^\bullet_{\mathbf{G}_m}(X \times X) = \lim \mathrm{H}^\bullet_{\mathbf{G}_m}(X_m \times X_m)$, we may use the process in 3.6.9, applying Theorem 3.6.10 to compute the CoHA products of the moduli of coherent sheaves on a curve.

3.9 Cup product compatibility

3.9.1. Let $(\mathcal{M}, 0)$ be a commutative monoid in pointed spaces as in section 2.6.3. Then by (2.6.4) its cohomology $H = \mathrm{H}^\bullet(\mathcal{M})$ is a cocommutative Hopf algebra, under cup product and coproduct the pullback by the monoid structure map $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$.

3.9.2. **CoHA compatibility.** Assume that moreover that $\mathcal{M} \in \mathrm{Alg}(\mathrm{Art}^{corr})$, i.e. it is an algebra in the category of correspondences, giving

$$\begin{tikzcd} & \mathrm{Ext} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{M}^2 & & \mathcal{M} \end{tikzcd} \tag{3.64}$$

[8]Indeed, note that $X = \cup_{m \in \mathbf{Z}} X_m$ and $X_m \backslash X_m$ is closed.

and moreover that p is proper and quasismooth, so that by Definition 3.1.10 its cohomology is given the cohomological algebra structure.

Proposition 3.9.3. *If* $p^* = q^* \oplus^*$ *as maps* $\mathrm{H}^\bullet(\mathcal{M}) \to \mathrm{H}^\bullet(\mathrm{Ext})$*, then the cohomological Hall algebra makes* $\mathrm{H}^\bullet(\mathcal{M})$ *into an algebra internal to* $\mathrm{H}^\bullet(\mathcal{M})$-Mod.

Proof. We need to show that CoHA multiplication is a map in $\mathrm{Rep}_{\mathcal{M}}$, i.e. that for every $\alpha \in \mathrm{H}^\bullet(\mathcal{M})$,

$$\begin{tikzcd} \mathrm{H}^\bullet(\mathcal{M}) \otimes \mathrm{H}^\bullet(\mathcal{M}) \arrow[r, "p_* q^*"] \arrow[d, "\oplus^* \alpha \cdot"] & \mathrm{H}^\bullet(\mathcal{M}) \arrow[d, "\alpha \cdot"] \\ \mathrm{H}^\bullet(\mathcal{M}) \otimes \mathrm{H}^\bullet(\mathcal{M}) \arrow[r, "p_* q^*"] & \mathrm{H}^\bullet(\mathcal{M}) \end{tikzcd} \tag{3.65}$$

Since the cup product is induced by pullback along the diagonal, (3.68) commutes because the following diagram of spaces commutes and its right square is a pullback

$$\begin{tikzcd} & \mathrm{Ext} \arrow[dl, "q"'] \arrow[d, "p \times \mathrm{id}"] \arrow[dr, "p"] & \\ \mathcal{M}^2 \arrow[d, "(\oplus \times \mathrm{id}) \cdot \Delta"'] & \mathcal{M} \times \mathrm{Ext} \arrow[dl, "\mathrm{id} \times q"'] \arrow[dr, "\mathrm{id} \times p"] & \mathcal{M} \arrow[d, "\Delta"] \\ \mathcal{M} \times \mathcal{M}^2 & & \mathcal{M} \times \mathcal{M} \end{tikzcd} \tag{3.66}$$

$\square$

It is unclear whether this generalises to cohomological Hall algebras on Borel Moore homology or sheaf cohomology (CoHAs 2. and 3. in Definition 3.1.10).

3.9.4. **Vertex algebra compatibility.** The compatibility with the (nonlocal) vertex algebra structure is more subtle. It is *not* true that $\mathrm{H}^\bullet(\mathcal{M})$ is a (nonlocal) vertex coalgebra in the symmetric monoidal category H-Mod: it interacts nontrivially with the derivation on H.

Let $\mathcal{M}$ be a space as in section 2.6. Then

Proposition 3.9.5. *The nonlocal Joyce vertex coalgebra structure (see Theorem 2.6.18) makes* $\mathrm{H}^\bullet(\mathcal{M})$ *into a nonlocal vertex coalgebra in the spectral symmetric monoidal category* H-Mod_∂ *(see Definitions A.4.2 and A.4.12) of the category of* H *modules with a compatible derivation.*

Proof. We want to show that for every $\alpha \in \mathrm{H}^\bullet(\mathcal{M})$, the cofield map is compatible with the action of H:

$$\begin{array}{ccc} \mathrm{H}^\bullet(\mathcal{M}) & \xrightarrow{\Delta(z)} & \mathrm{H}^\bullet(\mathcal{M})\otimes \mathrm{H}^\bullet(\mathcal{M})((z^{-1})) \\ \downarrow{\scriptstyle \alpha\cdot} & & \downarrow{\scriptstyle \mathrm{act}_1^*\oplus^*\alpha\cdot} \\ \mathrm{H}^\bullet(\mathcal{M}) & \xrightarrow{\Delta(z)} & \mathrm{H}^\bullet(\mathcal{M})\otimes \mathrm{H}^\bullet(\mathcal{M})((z^{-1})) \end{array} \tag{3.67}$$

This follows directly from the definition $\Delta(\alpha, z) = \Psi(\theta)\mathrm{act}_1^*(\oplus^*\alpha)$. □

3.9.6. The same is true if we consider the (nonlocal) Joyce vertex coalgebra structure with derivation attached to an orientation ε, of Theorem 2.6.21.

3.10 Main result

In this section we state the main Theorem 3.10.1 and give the proof (section 3.10.4).

Let $\mathcal{M}_\mathcal{A}$ be the moduli stack of representations of a quiver Q, or coherent sheaves on a smooth proper curve C. Its cohomology $\mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})$ has the following structures:

1. A cocommutative Hopf algebra structure with derivation, which we denote by $H = \mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})$.
2. A cohomological Hall algebra (Definition 3.1.10).
3. The Joyce nonlocal vertex coalgebra structure (Theorem 2.6.18).

Moreover, as we have seen in section (3.9), the CoHA and vertex coalgebra structures are compatible with the Hopf algebra structure, i.e. are internal to the category $H\text{-Mod}_\partial$ of H modules with compatible derivation.

Theorem 3.10.1. *The cohomology $\mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})$ forms a vertex bialgebra in the vertex symmetric monoidal category $H\text{-Mod}_\partial$ induced by Yang Baxter matrix (see section A.5)*

$$S(z) \;=\; \Psi(\theta, z)/\Psi(\sigma^*\theta^\vee, z).$$

To be very explicit, this means that the following diagram of vector spaces commutes

$$\begin{array}{ccc} \mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})^{\otimes 2} & \xrightarrow{\Delta(z)\otimes\Delta(z)} & \mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})^{\otimes 4}((z)) \\ & & \downarrow{\scriptstyle \mathrm{id}\otimes\sigma S(z)\otimes\mathrm{id}} \\ \downarrow{\scriptstyle m} & & \mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})^{\otimes 4}((z)) \\ & & \downarrow{\scriptstyle m\otimes m} \\ \mathrm{H}^\bullet(\mathcal{M}_\mathcal{A}) & \xrightarrow{\Delta(z)} & \mathrm{H}^\bullet(\mathcal{M}_\mathcal{A})^{\otimes 2}((z)) \end{array} \tag{3.68}$$

where $\Delta(z)$ is Joyce's vertex coalgebra structure, m is the cohomological Hall algebra structure, and σ is the spectral symmetric monoidal structure (see section A.4). Equivalently, σ is the ordinary symmetric monoidal structure swapping the factors, and $S(z) = \Psi(\theta, z)/\Psi(\sigma^*\theta, z)$; we will use this notation for the rest of the proof.

The proof rests on a method which computes CoHA-style products using abelian localisation. An application of them will later allow us to give explicit formulas for CoHA products (sections 2.3, 3.12 and 3.13).

3.10.2. **Example.** We will first demonstrate this method in the zero dimensional case, where we use it to compute the CoHA product for $\mathcal{A} = \mathrm{Vect}_K^{f.d.}$ the category of finite dimensional vector spaces.

We give a detailed description of the moduli stacks in section 3.11, but recall that $\mathcal{M}_{\mathcal{A}}$ is the union of BGL_n over nonnegative n, and the CoHA correspondence is a disjoint union of

$$\begin{tikzcd} & \mathrm{B}P_{n,m} \arrow[dl] \arrow[dr] & \\ \mathrm{BGL}_n \times \mathrm{BGL}_m & & \mathrm{BGL}_{n+m} \end{tikzcd}$$

where $P_{n,m} \subseteq \mathrm{GL}_{n+m}$ is the stabiliser of a fixed dimension n subspace.

The first step in the method is to consider a *split locus* of the target: any map $\oplus$ pullback along which gives an injection on cohomology. To compute the CoHA product we will choose

$$\begin{tikzcd} & \mathrm{B}P_{n,m} \arrow[dl] \arrow[dr] & & \mathrm{B}T_{n+m} \arrow[dl, "\oplus"'] \\ \mathrm{BGL}_n \times \mathrm{BGL}_m & & \mathrm{BGL}_{n+m} & \end{tikzcd}$$

where $T_{n+m} \subseteq \mathrm{GL}_{n+m}$ are the diagonal matrices. Thus $\mathrm{B}T_{n+m}$ classifies $n+m$ tuples of line bundles, and the map $\oplus$ sends

$$\oplus \ : \ (\mathcal{L}_1, ..., \mathcal{L}_{n+m}) \ \mapsto \ \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{n+m}$$

and identifies $\mathrm{H}^\bullet(\mathrm{BGL}_{n+m})$ with $\mathfrak{S}_{n+m}$ invariants inside $\mathrm{H}^\bullet(\mathrm{B}T_{n+m}) \simeq k[t_1, ..., t_{n+m}]$.

Second, we take the cartesian product

$$\begin{tikzcd}[column sep=small]
 & & P_{n,m}\backslash \mathrm{GL}_{n+m}/T_{n+m} \arrow[dl] \arrow[dr] & \\
 & \mathrm{B}P_{n,m} \arrow[dl] \arrow[dr] & & \mathrm{B}T_{n+m} \arrow[dl, "\oplus"'] \\
\mathrm{BGL}_n \times \mathrm{BGL}_m & & \mathrm{BGL}_{n+m} &
\end{tikzcd} \tag{3.69}$$

The cartesian product classifies short exact sequence of vector bundles with a splitting of the middle term

$$0 \ \to \ \mathcal{E}_n \ \overset{\alpha}{\to} \ \mathcal{E}_{n+m} \ \overset{\beta}{\to} \ \mathcal{E}_m \ \to \ 0 \qquad\qquad \varphi \ : \ \mathcal{E}_{n+m} \ \simeq \ \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{n+m}. \tag{3.70}$$

Third, note that this stack admits an action of a torus T_{n+m}, coming from its action on $\mathrm{B}T_{n+m}$. Explicitly, this sends

$$t \ : \ (\alpha, \beta, \varphi) \ \mapsto \ (\varphi^{-1} t \varphi \alpha, \beta \varphi^{-1} t^{-1} \varphi, t\varphi)$$

As the closed substack playing the role of the fixed locus, we consider the stack classifying

$$0 \ \to \ \mathcal{L}_{i_1} \oplus \cdots \oplus \mathcal{L}_{i_n} \ \to \ \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{n+m} \ \to \ \mathcal{L}_{j_1} \oplus \cdots \oplus \mathcal{L}_{j_m} \ \to \ 0 \tag{3.71}$$

where the $\mathcal{L}_i, \mathcal{L}_j$ are line bundles, which is labelled by the partitions of $\{1, ..., n+m\}$ into two sets of sizes n and m. Write Σ for the set of these.

$$\begin{tikzcd}
 & \coprod_{\sigma \in \Sigma} \mathrm{B}T_{n+m} \arrow[d, "i"] \arrow[ddl, bend right, "\overline{q}"'] \arrow[dr, bend left, "\overline{p}"] & \\
 & \mathrm{P}_{n,m}\backslash \mathrm{GL}_{n+m}/T_{n+m} \arrow[dl, "q"] \arrow[r, "p"] & \mathrm{B}T_{n+m} \\
\mathrm{BGL}_n \times \mathrm{BGL}_m & &
\end{tikzcd}$$

We can now compute the CoHA product. The point is that the map $\overline{p}$ is extremely simple, it is just the identity on each component. Thus the hardest part of the CoHA product (integration) is replaced by a triviality. Moreover, the map $\overline{q}$ on each component is the composite

$$\overline{q}_\sigma \ : \ \mathrm{B}T_{n+m} \ \overset{\pi_\sigma}{\to} \ \mathrm{B}T_n \times \mathrm{B}T_m \ \to \ \mathrm{BGL}_n \times \mathrm{BGL}_m$$

where π_σ is the projection corresponding to the partition σ.

Thus by the integration formula (3.29), we get that

$$p_* q^* \;=\; \overline{p}_{T*}\frac{\overline{q}_T^*(-)}{e(\mathbf{N}_i)} \;=\; \sum_{\sigma\in\Sigma}\frac{\overline{q}_{\sigma,T}^*(-)}{e(\mathbf{N}_{\sigma,i})}.$$

To be extremely explicit, we have

$$e(\mathbf{N}_{\sigma,i}) \;=\; \prod_{i=1}^{n}\prod_{j=1}^{m}(t_{\sigma i}-t_{\sigma j})$$

where we have made a choice of lift the partition σ to a pair of jointly surjective maps σ : $\{1,...,n\},\{1,...,m\}\to\{1,...,n+m\}$. Thus the CoHA map sends

$$f(t_1,...,t_n)\cdot g(t_1,...,t_m) \;=\; \sum_{\sigma\in\Sigma}\frac{1}{e(\mathbf{N}_{\sigma,i})}f(t_{\sigma 1},...,t_{\sigma n})g(t_{\sigma 1},...,t_{\sigma m}). \tag{3.72}$$

This recovers the formula in [KS, Thm. 2].

3.10.3. Notice that the right side of (3.75) is a priori valued in the localised cohomology $\mathrm{H}^\bullet(\mathrm{B}T_{n+m})_{\mathrm{loc}}$, i.e. the Euler class in the denominator may not be cancelled. However, when the inputs are symmetric group invariant, the denominator cancels and the right side is also symmetric group invariant: we have a commuting diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(\mathrm{B}T_n)_{\mathrm{loc}}\otimes\mathrm{H}^\bullet(\mathrm{B}T_m)_{\mathrm{loc}} \arrow[r] & \mathrm{H}^\bullet(\mathrm{B}T_{n+m})_{\mathrm{loc}} \\
\mathrm{H}^\bullet(\mathrm{BGL}_n)\otimes\mathrm{H}^\bullet(\mathrm{BGL}_m) \arrow[r] \arrow[u] & \mathrm{H}^\bullet(\mathrm{BGL}_{n+m}) \arrow[u]
\end{tikzcd} \tag{3.73}$$

We stress that to define the top arrow in (3.76), we need to choose of lift of every partition $\sigma\in\Sigma$ to a jointly surjective pair of functions $\tilde{\sigma}$: $\{1,...,n\},\{1,...,m\}\to\{1,...,n+m\}$. In particular, there is no reason to expect it to define an algebra structure.

3.10.4. **Proof of Theorem 3.10.1.** We use the split locus consisting of objects which are a direct sum of two subobjects:

$$\underline{\oplus}\;:\;\mathcal{M}_{\mathcal{A}}^2\;\to\;\mathcal{M}_{\mathcal{A}},$$

and apply the method of section 3.10.2. Note that since $\underline{\oplus}$ admits a section it is injective on cohomology. We have a commuting diagram

$$\begin{tikzcd}
 & & \mathrm{Ext}^2_{\mathcal{A}} \arrow[d, "i"] \arrow[dddll, bend right, "\overline{\oplus}"'] \arrow[ddr, bend left, "\overline{p}"] & \\
 & & \mathrm{Ext}_{\mathcal{A}} \times_{\mathcal{M}_{\mathcal{A}}} \mathcal{M}^2_{\mathcal{A}} \arrow[dl, "\oplus"'] \arrow[dr, "p"] & \\
 & \mathrm{Ext}_{\mathcal{A}} \arrow[dl, "q"'] \arrow[dr, "\underline{p}"] & & \mathcal{M}^2_{\mathcal{A}} \arrow[dl, "\underline{\oplus}"'] \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} & & \mathcal{M}_{\mathcal{A}} &
\end{tikzcd} \tag{3.74}$$

The following subdiagram of (3.77) is equivariant for both the action T

$$\begin{tikzcd}
 & \mathrm{Ext}^2_{\mathcal{A}} \arrow[d, "i"] \arrow[ddl, bend right, "\overline{\oplus}"'] \arrow[ddr, bend left, "\overline{p}"] & \\
 & \mathrm{Ext}_{\mathcal{A}} \times_{\mathcal{M}_{\mathcal{A}}} \mathcal{M}^2_{\mathcal{A}} \arrow[dl, "q\oplus"'] \arrow[dr, "p"] & \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} & & \mathcal{M}^2_{\mathcal{A}}
\end{tikzcd} \tag{3.75}$$

and we can use abelian localisation (Theorem 3.5.16) to compute[9]

$$\underline{\oplus}^*(\text{CoHA product}) \;=\; \underline{\oplus}^*\underline{p}_*q^* \;=\; \overline{p}_{T*}\frac{\overline{\oplus}^*_T(-)}{e(\mathbf{N}_i)}. \tag{3.76}$$

See section 3.6.8 for the relation to the T equivariant structure. Note also that $\overline{\oplus}$ and $\overline{p}$ are equivariant for the $\mathrm{B}\mathbf{G}_m$ action on the spaces scaling the left factors; we write act for the corresponding pullback map on cohomology after identifying $\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m) \simeq k[z]$. Finally, $\overline{\oplus}$ factors T- and $\mathrm{B}\mathbf{G}_m$-equivariantly as

$$\begin{tikzcd}
 & \mathrm{Ext}^2_{\mathcal{A}} \arrow[d, "i"] \arrow[dl, bend right, "\tilde{\oplus}"'] \arrow[ddr, bend left, "\overline{p}"] & \\
\mathcal{M}^2_{\mathcal{A}} \times \mathcal{M}^2_{\mathcal{A}} \arrow[d, "\underline{\oplus}\times\underline{\oplus}"'] & \mathrm{Ext}_{\mathcal{A}} \times_{\mathcal{M}_{\mathcal{A}}} \mathcal{M}^2_{\mathcal{A}} \arrow[dl, "q\oplus"'] \arrow[dr, "p"] & \\
\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} & & \mathcal{M}^2_{\mathcal{A}}
\end{tikzcd} \tag{3.77}$$

[9]Note that i is of finite presentation and its normal complex is perfect because the same is true for $\overline{p}$ and p, so the conditions of Theorem 3.6.6 are met.

We now turn to proving the theorem. To do this, we first write out in detail the diagram (3.71) relating the vertex algebra and cohomological Hall algebra structures:

$$
\begin{tikzcd}
\mathrm{H}^\bullet(\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}}) \arrow[r, "(\underline{\oplus}\times\underline{\oplus})^*"] \arrow[dd, "\underline{p}_* q^*"] & \mathrm{H}^\bullet(\mathcal{M}^2_{\mathcal{A}} \times \mathcal{M}^2_{\mathcal{A}}) \arrow[r, "\mathrm{act}^*(-)"] \arrow[dd, dashed, "\alpha"] & \mathrm{H}^\bullet(\mathcal{M}^2_{\mathcal{A}} \times \mathcal{M}^2_{\mathcal{A}})((z)) \arrow[r, "\Psi(\theta\boxplus\theta)\cdot"] \arrow[dd, dashed, "\beta"] & \mathrm{H}^\bullet(\mathcal{M}^2_{\mathcal{A}} \times \mathcal{M}^2_{\mathcal{A}})((z)) \arrow[d, "\sigma_{23}\cdot S_{23}(z)"] \\
 & & & \mathrm{H}^\bullet(\mathcal{M}^2_{\mathcal{A}} \times \mathcal{M}^2_{\mathcal{A}})((z)) \arrow[d, "(\underline{p}\times\underline{p})_*(q\times q)^*"] \\
\mathrm{H}^\bullet(\mathcal{M}_{\mathcal{A}}) \arrow[r, "\underline{\oplus}^*"] & \mathrm{H}^\bullet(\mathcal{M}^2_{\mathcal{A}}) \arrow[r, "\mathrm{act}^*(-)"] & \mathrm{H}^\bullet(\mathcal{M}^2_{\mathcal{A}})((z)) \arrow[r, "\Psi(\theta)\cdot"] & \mathrm{H}^\bullet(\mathcal{M}^2_{\mathcal{A}})((z))
\end{tikzcd}
\tag{3.78}
$$

Firstly, by the integration formula (3.79), the left square commutes if we pick

$$\alpha \;=\; \overline{p}_{T*}\frac{\widetilde{\oplus}^*_T(-)}{e(\mathbf{N}_i)}.$$

Second, by $\mathrm{B}\mathbf{G}_m$ equivariance of the maps, the middle square commutes if we take

$$\beta \;=\; \overline{p}_{T*}\frac{\widetilde{\oplus}^*_T(-)}{\Psi(\mathbf{N}_i)}.$$

Thirdly and finally, we need the right square to commute, that is

$$\Psi(\theta)\overline{p}_{T*}\frac{\overline{\oplus}^*_T(-)}{\Psi(\mathbf{N}_i)} \;=\; (\underline{p}_T \times \underline{p}_T)_*(q_T \times q_T)^*\sigma^*_{23}\left(S_{23}(z)\Psi(\theta\boxplus\theta)(-)\right). \tag{3.79}$$

The rest of the proof then consists of showing (3.82).

To begin with, since $\overline{p} = \underline{p} \times \underline{p}$ equation (3.82) would be implied by the equality in $\mathrm{H}^\bullet(\mathrm{Ext}^2_{\mathcal{A}})((z))$

$$\Psi(\overline{p}^*_T\theta)\frac{1}{\Psi(\mathbf{N}_i)} \;=\; (q_T \times q_T)^*\sigma^*_{23}\left(S_{23}(z)\Psi(\theta\boxplus\theta)\right).$$

We can further reduce since since $\underline{p}^* = q^*\underline{\oplus}^*$ and q^* is an isomorphism, to the equality in $\mathrm{H}^\bullet(\mathcal{M}^2_{\mathcal{A}} \times \mathcal{M}^2_{\mathcal{A}})((z))$

$$\Psi((\underline{\oplus}_T \times \underline{\oplus}_T)^*\theta)\frac{1}{((q_T \times q_T)^*)^{-1}\Psi(\mathbf{N}_i)} \;=\; \sigma^*_{23}S_{23}(z)\sigma^*_{23}\Psi(\theta\boxplus\theta). \tag{3.80}$$

To understand the pieces of (3.82), we first compute the $\Psi(\mathbf{N}_i)$ term. Note that as $\Psi(-)$ defines a map on K theory, so we will proceed by simplifying $[\mathbf{N}_i]$.

Lemma 3.10.5. *We have* $((q\times q)^*)^{-1}[\mathbf{N}_i] = [\sigma^*_{23}(\underline{\oplus}\times\underline{\oplus})^*\theta] - [\theta\boxplus\theta]$.

Proof. By repeatedly applying the distinguished triangle for the tangent complex of a composition we get that

$$[\mathbf{N}_i] \;=\; [i^*\oplus^*\mathbf{T}_{\mathrm{Ext}_{\mathcal{A}}/\mathcal{M}_{\mathcal{A}}}] - [\mathbf{T}_{\mathrm{Ext}^2_{\mathcal{A}}/\mathcal{M}^2_{\mathcal{A}}}] \;=\; [\overline{\oplus}^*\theta] - [q^*\theta\boxplus q^*\theta].$$

Then since $\widetilde{\oplus} = \sigma_{23}(q \times q)$, this is

$$[\mathbf{N}_i] \;=\; [(q\times q)^*\sigma_{23}^*(\underline{\oplus}\times\underline{\oplus})^*\theta] - [(q\times q)^*(\theta \boxplus \theta)].$$

$\square$

We next simplify all the terms in (3.83). Label the connected components of $\mathcal{M}_{\mathcal{A}}^2 \times \mathcal{M}_{\mathcal{A}}^2$ by quadruples of connected components of $\mathcal{M} = \mathcal{M}_{\mathcal{A}}$:

$$\mathcal{M}_{\mathcal{A}}^2 \times \mathcal{M}_{\mathcal{A}}^2 \;=\; \coprod(\mathcal{M}_{\alpha_1}\times\mathcal{M}_{\beta_1})\times(\mathcal{M}_{\alpha_2}\times\mathcal{M}_{\beta_2}).$$

As usual, denote by $\theta_{\alpha_1,\beta_2}$ for the pullback of θ under the projection

$$(\mathcal{M}_{\alpha_1}\times\mathcal{M}_{\beta_1})\times(\mathcal{M}_{\alpha_2}\times\mathcal{M}_{\beta_2}) \;\to\; \mathcal{M}_{\alpha_1}\times\mathcal{M}_{\beta_2},$$

and similarly for other indices. Then writing s for the section of $q\times q$ given by taking trivial extensions, and so $s^* = ((q\times q)^*)^{-1}$, we have the following T equivariantly

1. $\overline{p}s$ is the direct sum map

   $$\overline{p}s \;:\; (\mathcal{M}_{\alpha_1}\times\mathcal{M}_{\beta_1})\times(\mathcal{M}_{\alpha_2}\times\mathcal{M}_{\beta_2}) \;\to\; \mathcal{M}_{\alpha_1+\beta_1}\times\mathcal{M}_{\alpha_2+\beta_2}$$

   and so we have

   $$(\underline{\oplus}\times\underline{\oplus})^*\theta \;=\; s^*\overline{p}^*\theta \;=\; \theta_{\alpha_1,\alpha_2}\oplus\theta_{\alpha_1,\beta_2}\oplus\theta_{\beta_1,\alpha_2}\oplus\theta_{\beta_1,\beta_2}.$$

2. Likewise, $\theta\boxplus\theta = \theta_{\alpha_1,\beta_1}\oplus\theta_{\alpha_2,\beta_2}$.

3. Together with Lemma 3.10.5 this implies that $((q\times q)^*)^{-1}[\mathbf{N}_i] = [\theta_{\alpha_1,\beta_2}] + [\theta_{\alpha_2,\beta_1}]$.

Thus equation (3.83) is equivalent to

$$\frac{\Psi(\theta_{\alpha_1,\alpha_2})\Psi(\theta_{\alpha_1,\beta_2})\Psi(\theta_{\beta_1,\alpha_2})\Psi(\theta_{\beta_1,\beta_2})}{\Psi(\theta_{\alpha_1,\beta_2})\Psi(\theta_{\alpha_2,\beta_1})} \;=\; \sigma_{23}^* S_{23}(z)\Psi(\theta_{\alpha_1,\alpha_2})\Psi(\theta_{\beta_1,\beta_2}).$$

Simplifying further, this is

$$S_{23}(z) \;=\; \frac{\Psi(\theta_{\alpha_2,\beta_1})}{\Psi(\theta_{\beta_1,\alpha_2})} \tag{3.81}$$

which thus completes the proof of Theorem 3.10.1.

3.11 Example: Vector spaces

3.11.1. The most basic example is when

$$\mathcal{A} \;=\; \mathrm{Vect}_K^{f.d.}$$

is finite dimensional vector spaces over a field K. It is a zero dimensional Calabi Yau category. Its moduli stack of objects is

$$\mathcal{M}_{\mathcal{A}} \;=\; \coprod_{n\geqslant 0} \mathrm{BGL}_n.$$

Giving a map into $\mathcal{M}_{\mathcal{A}}$ is equivalent to giving a vector bundle on the source. Writing $T_n \subseteq \mathrm{GL}_n$ for the diagonal matrices, we have an identification

$$\mathrm{H}^\bullet(\mathrm{BGL}_n) \;=\; \mathrm{H}^\bullet(\mathrm{B}T_n)^{\mathfrak{S}_n} \;\simeq\; k[t_1,...,t_n]^{\mathfrak{S}_n}.$$

Writing γ_n for the universal rank n vector bundle over BGL_n, this is freely generated by the chern classes $c_i(\gamma_n)$, which are identified with the elementary symmetric polynomials in the t_i. It follows that the cohomology of the moduli stack is

$$\mathrm{H}^\bullet(\mathcal{M}_{\mathcal{A}}) \;=\; \bigoplus_{n\geqslant 0} k[t_{1,n},...,t_{n,n}]^{\mathfrak{S}_n}.$$

3.11.2. **Cohomological Hall algebra.** The moduli stack of extensions is

$$\mathrm{Ext}_{\mathcal{A}} \;=\; \coprod_{n,m\geqslant 0} \mathrm{B}P_{n,m}$$

where $P_{n,m} \subseteq \mathrm{GL}_{n+m}$ is the stabiliser of a fixed dimension n subspace. Thus to map into $\mathrm{B}P_{n,m}$ is to give a rank $n+m$ vector bundle with a rank n subbundle. Note that $P_{n,m}$ is a parabolic subgroup, so $\mathrm{GL}_{n+m}/P_{n,m}$ is smooth and proper, and the quotient by its unipotent radical gives a short exact sequence

$$1 \;\to\; U_{n,m} \;\to\; P_{n,m} \;\to\; \mathrm{GL}_n\times\mathrm{GL}_m \;\to\; 1.$$

Thus the connected components of the CoHA extension correspondence (3.3) are

$$\begin{tikzcd} & \mathrm{BP}_{n,m} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{BGL}_n\times\mathrm{BGL}_m & & \mathrm{BGL}_{n+m} \end{tikzcd}$$

The map p, whose fibres are $\mathrm{GL}_{n+m}/P_{n,m}$, is representable, smooth and proper. Pullback by the map q, whose fibres $\mathrm{B}U_{n,m}$ are cohomologically trivial, gives an isomorphism on cohomology. In any case, both p and q must be quasismooth because all involved stacks are smooth.

3.11.3. **Computation.** In section 3.10.2, we used abelian localisation to compute the CoHA product for $\mathcal{A}$, recovering the formula of [KS, Thm. 2]. So the reader can appreciate that method more, we will now demonstrate how one might compute the CoHA product in a "brute force" way. It is much harder to repeat for higher dimensional categories, and the answer it gives is less explicit.

3.11.4. Note that $\mathrm{B}P_{n,m}$ carries a tautological short exact sequence of vector bundles

$$0 \;\to\; \gamma_n \;\to\; \gamma_{n+m} \;\to\; \gamma_m \;\to\; 0.$$

This is a slight abuse of notation: these are the pullbacks by q and p of γ_n, γ_m and γ_{n+m}. The CoHA product on cohomology is then

$$\mathrm{H}^\bullet(\mathrm{BGL}_n) \otimes \mathrm{H}^\bullet(\mathrm{BGL}_m) \;\overset{q^*}{\to}\; \mathrm{H}^\bullet(\mathrm{B}P_{n,m}) \;\overset{p_*}{\to}\; \mathrm{H}^{\bullet-2d}(\mathrm{BGL}_{n+m})$$

where $d = \dim \mathrm{GL}_{n+m}/P_{n,m}$. There is a general formula for the cohomology of fibre bundles whose fibres are partial flag varieties (see e.g. [And, Prop. 5.1]), giving us that $\mathrm{H}^\bullet(\mathrm{B}P_{n,m})$ is generated by the chern classes of γ_n and γ_m subject to the single relation

$$c(\gamma_{n+m}) \;=\; c(\gamma_n)c(\gamma_m),$$

which can be rewritten as

$$\mathrm{H}^\bullet(\mathrm{B}P_{n,m}) \;\simeq\; \mathrm{H}^\bullet(\mathrm{BGL}_{n+m})\,[c_i(\gamma_n)]_i/\left(c(\gamma_{n+m})/c(\gamma_n)_{(k)} : k > m\right)$$

and has a basis over $\mathrm{H}^\bullet(\mathrm{BGL}_{n+m})$ given by

$$\prod c_i(\gamma_n)^{k_i} \quad \text{for} \quad \textstyle\sum_i k_i \leqslant m.$$

It follows that for some constant $\kappa \in k$,

$$p_* \;=\; \kappa \cdot \mathrm{coeff}_{c_n(\gamma_n)^m}(-). \tag{3.82}$$

3.11.5. Over a point, $\mathrm{GL}_{n+m}/P_{n,m}$ carries a tautological rank n vector bundle $\mathcal{E}_n \subseteq \mathcal{O}^{n+m}$, and by the above

$$\mathrm{H}^\bullet(\mathrm{GL}_{n+m}/P_{n,m}) \;=\; k\,[c_i(\mathcal{E}_n)]_i/\left(c(\gamma_n)^{-1}_{(1+m)}, ..., c(\gamma_n)^{-1}_{(n+m)}\right).$$

It follows that the top dimensional cohomology is generated by

$$c_n(\mathcal{E}_n)^m \;\in\; \mathrm{H}^{\mathrm{top}}(\mathrm{GL}_{n+m}/P_{n,m}).$$

We can thus compute the coefficient in (3.85) as

$$\kappa \;=\; \int_{\mathrm{GL}_{n+m}/P_{n,m}} c_n(\mathcal{E}_n)^m.$$

3.12 Example: Representations of a quiver

3.12.1. We now consider the example where

$$\mathcal{A} \;=\; \mathrm{Rep}_k Q$$

is the category of representations of a finite quiver Q (meaning finitely many vertices).

3.12.2. A quiver is a set $|Q|$ of *vertices* and a set of arrows $e : p \to q$ between pairs of vertices. A *representation* of Q is a vector space attached to each vertex and a linear map between the relevant vector spaces attached to each arrow. The *dimension* of a representation is the element $\gamma \in \mathbf{N}^{|Q|}$ representing the dimension of these vector spaces.

3.12.3. Let Q be a quiver. A Q *representation bundle* V is a vector bundle V_q attached to each vertex $q \in |Q|$ and a map of vector bundles $\rho_e : V_p \to V_q$ for every edge $e : p \to q$ of Q. When $Q = \bullet$ is the one vertex no loops quiver this is just a vector bundle. If V, W are Q representation bundles, then their tensor product $V \otimes W$ another Q representation bundle, defined by

$$(V \otimes W)_q \;=\; V_q \otimes W_q, \qquad\qquad \rho_{V\otimes W,q} \;=\; \rho_{V,q} \otimes \rho_{W,q}.$$

Likewise, the hom space $\mathcal{H}\mathrm{om}_Q(V, W)$ is the vector subbundle

$$\mathcal{H}\mathrm{om}_Q(V, W) \;\subseteq\; \textstyle\prod_q \mathcal{H}\mathrm{om}(V_q, W_q)$$

of maps $\varphi_q : V_q \to W_q$ intertwining the $\{\rho_{V,p}\}$ and $\{\rho_{W,p}\}$. See also equation (3.86).

3.12.4. Similarly, we can define Q representation coherent sheaves, and likewise for any other notion of sheaf. We can define $\otimes$ and $\mathcal{H}\mathrm{om}_Q$ exactly as above.

3.12.5. **Moduli spaces.** The moduli stack of quiver representations

$$\mathcal{M}_{\mathcal{A}} \;=\; \coprod_{\gamma \in \mathbf{N}^{|Q|}} \mathcal{M}_{\mathcal{A},\gamma}$$

has connected components labelled by the dimension of the representation

$$\mathcal{M}_{\mathcal{A},\gamma} \;=\; \textstyle\prod_{e:p\to q} \mathrm{Hom}(k^{\gamma_p}, k^{\gamma_q}) / \prod_{q \in |Q|} \mathrm{GL}(k^{\gamma_q}).$$

Thus a map into $\mathcal{M}_{\mathcal{A}}$ is precisely a Q representation bundle. In particular, the cohomology of $\mathcal{M}_{\mathcal{A},\gamma}$ is symmetric group invariants inside a polynomial algebra:

$$\mathrm{H}^\bullet(\mathcal{M}_{\mathcal{A},\gamma}) \;\simeq\; \textstyle\bigotimes_{q \in |Q|} k[x_{q,1}, \ldots, x_{q,\gamma_q}]^{\mathfrak{S}_{\gamma_q}}.$$

3.12.6. Another way to view the construction is this. We have the vector bundles $\mathcal{E}_q$, the pullbacks of the tautological vector bundle via

$$\prod_{q\in|Q|}\mathrm{BGL}(k^{\gamma_q}) \;\to\; \mathrm{BGL}(k^{\gamma_q}),$$

then the moduli space is the total space of a hom space:

$$\mathcal{M}_{\mathcal{A},\gamma} \;=\; \prod_{e:p\to q}\mathrm{Hom}(\mathcal{E}_p,\mathcal{E}_q).$$

Note that $\mathcal{M}_{\mathcal{A}}$ carries a tautological vector bundle $\check{\mathcal{E}}_q$ for each vertex q and a map of vector bundles $\check{e}:\check{\mathcal{E}}_q\to\check{\mathcal{E}}_p$ for every edge e in the quiver.[10] Use the subscript i to denote pullback of a vector bundle, map etc. by the ith projection $\mathcal{M}_{\mathcal{A}}\times\mathcal{M}_{\mathcal{A}}\to\mathcal{M}_{\mathcal{A}}$.

It is now also easy to describe the Ext complex $\mathrm{Ext}\in\mathrm{Perf}(\mathcal{M}_{\mathcal{A}}\times\mathcal{M}_{\mathcal{A}})$: it has a global resolution by a two term complex of vector bundles in degrees $[0,1]$ corresponding to the usual complex used to compute Ext groups of quiver representations:

$$\mathrm{Ext} \;=\; \left(\prod_q\mathrm{Hom}(\check{\mathcal{E}}_{q,1},\check{\mathcal{E}}_{q,2}) \;\to\; \prod_{e:p\to q}\mathrm{Hom}(\check{\mathcal{E}}_{p,1},\check{\mathcal{E}}_{q,2})\right) \tag{3.83}$$

sending

$$(\varphi_q)_q \;\mapsto\; \left(\check{e}_2\varphi_{p,1}-\varphi_{q,2}\check{e}_1\right)_{e:p\to q}.$$

3.12.7. In particular, there is a clear analogue for the *derived* category of representations of Q. We set

$$\widehat{\mathcal{M}}_{\mathcal{A}} \;=\; \coprod_{\gamma\in\mathbf{Z}^{|Q|}}\widehat{\mathcal{M}}_{\mathcal{A},\gamma}$$

whose components are now labelled by the entire Grothendieck group $\mathbf{Z}^{|Q|}$ and not just the positive cone $\mathbf{N}^{|Q|}$, where

$$\widehat{\mathcal{M}}_{\mathcal{A},\gamma} \;=\; \prod_{e:p\to q}\mathrm{Hom}(\mathcal{E}_p,\mathcal{E}_q)$$

where $\mathcal{E}_q$ is the (total space of) the perfect complex induced by the pullback of the tautological perfect complex via

$$\prod_{q\in|Q|}\mathrm{Perf}_{\gamma_q} \;\to\; \mathrm{Perf}_{\gamma_q}.$$

Thus points of $\widehat{\mathcal{M}}_{\mathcal{A}}$ correspond to a perfect complex for each vertex of Q, and a map (not just a map up to quasi-isomorphism) between appropriate perfect complexes for each edge. This is not quite the same thing as an object

[10] $\check{\mathcal{E}}_q$ is the pullback of $\mathcal{E}_q$ under the projection $\mathcal{M}_{\mathcal{A}}\twoheadrightarrow\coprod_{\gamma\in\mathbf{N}^{|Q|}}\prod_{q\in|Q|}\mathrm{BGL}(k^{\gamma_q})$.

3.12.8. **Vertex algebra structure.** See section 4.3 and in particular Proposition 4.3.8, which says that the homology of the moduli stack of *derived* category of quiver representations equipped with the symmetrised Ext complex is a lattice vertex algebra. In particular the homology of the moduli stack of quiver representations is a vertex subalgebra - note that the inclusion $\mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{D^b(\mathcal{A})}$ pulls back tautological perfect complexes to tautological vector bundles and so gives a surjection on cohomology and injection on homology.

3.12.9. **Cohomological Hall algebra.** Drop all $\mathcal{A}$'s from the subscript from now on. To demonstrate our method (section 3.6.12) we will compute the formula for the quiver CoHA in [KS, Thm. 2].

Take as the split locus the moduli space parametrising tuples of rank one representations

$$\mathcal{M}^s \ = \ \coprod_J \left(\textstyle\prod_{j\in J} \mathcal{M}_1 \right)$$

which admits a direct sum map $\underline{\oplus} : \mathcal{M}^s \to \mathcal{M}$. The connected components are labelled by finite sets J.

Given a short exact sequence

$$0 \ \to \ \mathcal{E}_1 \ \to \ \mathcal{E} \ \to \ \mathcal{E}_2 \ \to \ 0$$

and a splitting of the middle term as a sum of rank one representations (say summands labelled by J), if the whole exact sequence splits as a direct sum over J we can write it as a sum using rank one representations

$$0 \ \to \ \mathcal{L}_{j_1} \ \to \ \mathcal{L}_{j_1} \ \to \ 0 \ \to \ 0$$
$$0 \ \to \ 0 \ \to \ \mathcal{L}_{j_2} \ \to \ \mathcal{L}_{j_2} \ \to \ 0$$

as j_1 and j_2 vary over disjoint subsets J_1 and J_2 with $J = J_1 \amalg J_2$.

3.12.10. Thus, the connected components of

$$\mathrm{Ext}^s \ = \ \coprod_{J=J_1\cup J_2} \left(\textstyle\prod_{j_1\in J_1} \mathrm{Ext}_{1,0} \times \prod_{j_2\in J_2} \mathrm{Ext}_{0,1} \right)$$

are labelled by pairs of finite sets J_1, J_2, where $\mathrm{Ext}_{1,0}$ parametrises extensions of a rank zero object by a rank one object and vice-versa for $\mathrm{Ext}_{0,1}$. Since there are no nontrivial such extensions, we have

$$\mathrm{Ext}_{1,0} \ \simeq \ \mathrm{Ext}_{0,1} \ \simeq \ \mathcal{M}_1$$

and we have

$$\mathrm{Ext}^s \ = \ \coprod_{J=J_1\cup J_2} \left(\textstyle\prod_{j_1\in J_1} \mathcal{M}_1 \times \prod_{j_2\in J_2} \mathcal{M}_1 \right).$$

3.12.11. The localised CoHA correspondence (see section 3.6.20) is

$$
\begin{tikzcd}[column sep=small]
& \coprod_{J=J_1\amalg J_2}\left(\prod_{j_1\in J_1}\mathcal{M}_1 \times \prod_{j_2\in J_2}\mathcal{M}_1\right) \arrow[dl, "\tilde{\oplus}"'] \arrow[dr, "\overline{p}"] & \\
\coprod_{J_1}\left(\prod_{j_1\in J_1}\mathcal{M}_1\right) \times \coprod_{J_2}\left(\prod_{j_2\in J_2}\mathcal{M}_1\right) & & \coprod_{J}\left(\prod_{j\in J}\mathcal{M}_1\right)
\end{tikzcd}
\tag{3.84}
$$

where $\tilde{\oplus}$ is the identity, and $\overline{p}$ is the identity on each connected component. Moreover, by equation (3.53) we have

$$[\mathbf{N}_i] \;=\; [i^* \oplus^* \mathbf{T}_{\mathrm{Ext}/\mathcal{M}}] - [\mathbf{T}_{\mathrm{Ext}^s/\mathcal{M}^s}] \;=\; [\tilde{\oplus}^*(\underline{\oplus}\times\underline{\oplus})^*\mathrm{Ext}] - 0$$

where Ext is the Ext complex in (3.86).

3.12.12. It is possible to give an explicit formula for $e(\mathbf{N}_i) \in \mathrm{H}^\bullet(\mathrm{Ext}^s)_{\mathrm{loc}}$. Writing $\mathcal{L}_j$ for the pullback of the tautological line bundle under

$$\mathcal{M}^s \times \mathcal{M}^s \;=\; \coprod_{J_1}\left(\prod_{j_1\in J_1}\mathcal{M}_1\right) \times \coprod_{J_2}\left(\prod_{j_2\in J_2}\mathcal{M}_1\right) \;\to\; \mathcal{M}_j$$

we have that $\underline{\oplus}^*\check{\mathcal{E}}_q = \prod_{j\in J}\mathcal{L}_j$, and so

$$(\underline{\oplus}\times\underline{\oplus})^*\mathrm{Ext} \;=\; \prod_{j_1,j_2}\left(\prod_q \mathrm{Hom}(\mathcal{L}_{q,j_1},\mathcal{L}_{q,j_2}) \;\to\; \prod_{e:p\to q}\mathrm{Hom}(\mathcal{L}_{p,j_1},\mathcal{L}_{q,j_2})\right)$$

as perfect complexes on the J_1, J_2th component of $\mathcal{M}^s\times\mathcal{M}^s$. It follows that

$$e(\mathbf{N}_i) \;=\; \prod_{p,q}\prod_{j_1,j_2}(x_{p,j_1} - x_{q,j_2})^{\chi(p,q)} \tag{3.85}$$

where $\chi(p,q) = \delta_{p,q} - a_{p,q}$ is the Euler form, and $a_{p,q}$ is the number of edges from p to q.

3.12.13. Putting all this together, the integration formula 3.29 recovers the explicit formula for the (localised) CoHA product [KS, Thm. 2]: the localised CoHA map

$$\mathrm{H}^\bullet(\mathcal{M}^s)\otimes\mathrm{H}^\bullet(\mathcal{M}^s) \;\to\; \mathrm{H}^\bullet(\mathcal{M}^s)_{\mathrm{loc}}$$

on the connected component of $\mathcal{M}^s\times\mathcal{M}^s$ labelled by finite sets J_1 and J_2, is given by

$$f_1(x_{q,j_1})\cdot f_2(x_{q,j_2}) \;=\; \sum_{J\simeq J_1\amalg J_2} \frac{f_1(x_{q,j_1})f_2(x_{q,j_2})}{e(\mathbf{N}_i)} \tag{3.86}$$

with $e(\mathbf{N}_i)$ as in (3.88). Here f_i is a polynomial in variables x_{q,j_i} labelled by elements of $|Q|$ and J_i, and the sum is taken over all the ways

$$J_1 \amalg J_2 \;\xrightarrow{\sim}\; J$$

of writing a fixed finite set J as a disjoint of J_1 and J_2. Given any such partition, we identity elements of J_1 and J_2 with the corresponding element of J. Now, in particular, restricting to symmetric group invariant polynomials, the same formula gives the genuine CoHA product

$$\mathrm{H}^\bullet(\mathcal{M}) \otimes \mathrm{H}^\bullet(\mathcal{M}) \ \to\ \mathrm{H}^\bullet(\mathcal{M}).$$

3.13 Example: Coherent sheaves on a curve

Let C be a smooth projective curve over an algebraically closed field, and

$$\mathcal{A} \ = \ \mathrm{Coh}(C)$$

the category of coherent sheaves over it. Applying our heuristic is trickier than in the quiver case because there are nontrivial rank zero objects.

3.13.1. The moduli stack of objects in $\mathcal{A}$ is defined by functor of points

$$\mathrm{Maps}(S, \mathcal{M}_{\mathcal{A}}) \ = \ \{\mathcal{F} \in \mathrm{Coh}(S \times C),\ \mathcal{F} \text{ is flat over } C\}.$$

One can show that this is an Artin stack locally of finite type, which we will denote by $\mathrm{Coh} = \mathcal{M}_{\mathcal{A}}$. Its connected components are labelled by the rank r and degree d of the coherent sheaf:

$$\mathrm{Coh} \ = \ \coprod_{r \in \mathbf{N}, d \in \mathbf{Z}} \mathrm{Coh}_r^d.$$

3.13.2. **Cohomology.** The cohomology of these moduli stacks is fairly simple, just polynomial algebra on a super vector space. However, there is a slight subtlety coming from the fact that there are nontrivial rank zero objects. To begin, consider the tautological coherent sheaf $\mathcal{E}$ on $C \times \mathcal{M}$. If we pick a basis of the cohomology of C

$$\mathrm{H}^0(C) \ = \ k\{1\}, \quad \mathrm{H}^1(C) \ = \ k\{b_1, ..., b_{2g}\}, \quad \mathrm{H}^2(C) \ = \ k\{\sigma\}$$

we may decompose the chern classes of $\mathcal{E}$ in $\mathrm{H}^\bullet(C \times \mathcal{M})$ as

$$c_i(\mathcal{E}) \ = \ 1 \otimes \alpha_i \ + \ \sum b_{k,i} \otimes \beta_{k,i} \ + \ \sigma \otimes \gamma_i.$$

Fixing the following graded vector space

$$W_C \ = \ k\{a_{i-1}, b_{1,i}, ..., b_{2g,i}, c_i\}_{i \geqslant 1} \tag{3.87}$$

where $|a_i| = 2i$, $|b_{k,i}| = 2i - 1$ and $|c_i| = 2i - 2$, we have

Proposition 3.13.3 ([He]). *For positive rank $r > 0$, there is an isomorphism of graded supercommutative algebras*

$$\mathrm{Sym}(W_C) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathcal{M}_r^d)$$

sending $(a_i, b_{k,i}, c_i) \mapsto (\alpha_i, \beta_{k,i}, \gamma_i)$.

The rank zero case is different because the *support* of the tautological coherent sheaf gives more cohomology classes. To begin,

Lemma 3.13.4. *The degree one rank zero moduli space is $\mathcal{M}_0^1 \simeq C \times \mathrm{B}\mathbf{G}_m$.*

Proof. Given a coherent sheaf $\mathcal{F}$ on $C \times S$ which is flat over C, we may take its support to give an S-valued point of C, $\mathrm{Supp}\mathcal{F} : S \to C$. Moreover, we get a line bundle $p_*\mathcal{F}$ on S by projection along $S \times C \to S$. This defines a map $\mathcal{M}_0^1 \to C \times \mathrm{B}\mathbf{G}_m$. The inverse map is by taking an S-valued point $c : S \to C$ and a line bundle $\mathcal{L}$ on S, and sending it to $p^*\mathcal{L} \otimes \mathcal{O}(c)$, which defines a degree one rank zero coherent sheaf on S. $\square$

We consider the graded vector space

$$V_C \;=\; \mathrm{H}^\bullet(C \times \mathrm{B}\mathbf{G}_m). \tag{3.88}$$

Heinloth then shows

Proposition 3.13.5 ([He]). *The cohomology of the rank zero degree d moduli stack is*

$$\mathrm{H}^\bullet(\mathcal{M}_d^0) \;\simeq\; \mathrm{Sym}^d(V_C).$$

Note that $\mathcal{M}_d^0 = \varnothing$ if $d < 0$.

3.13.6. **Vertex algebra structure.** By work of Gross, the vertex algebra structure on the moduli stack of the derived category of coherent sheaves is a lattice vertex superalgebra attached to the superlattice $(\mathrm{K}^\bullet_{top}(\mathrm{Coh}C), \overline{\chi})$, where $\mathrm{K}^\bullet_{top}$ is the higher topological K-theory and $\overline{\chi}$ is the symmetrised Euler form (see [Gro2]). In particular, for the same reason as in section 3.12.8 the moduli stack of coherent sheaves will be a vertex subalgebra.

3.13.7. **Extensions.** The moduli space of extensions has connected components

$$\mathrm{Ext}_{r,r'}^{d,d'}$$

labelled by the rank and degree $(r,d),(r',d')$ of the subobject and quotient: these are the ranks and degrees of the terms of the tautological short exact sequence of coherent sheaves on $C\times \mathrm{Ext}^{d,d'}_{r,r'}$:

$$0 \;\to\; \mathcal{E}^d_r \;\to\; \mathcal{E}^{d+d'}_{r+r'} \;\to\; \mathcal{E}^{d'}_{r'} \;\to\; 0.$$

3.13.8. **Cohomological Hall algebra.** Take as split locus the moduli space parametrising finite direct sums of rank zero and one coherent sheaves. Its connected components are

$$(\mathrm{Coh}^{e_1}_0\times\cdots\times\mathrm{Coh}^{e_n}_0)\times(\mathrm{Coh}^{d_1}_1\times\cdots\times\mathrm{Coh}^{d_m}_1)$$

for integers d_i and positive integers e_i. In other words,

$$\mathrm{Coh}^s \;=\; \coprod_{I\to\mathbf{N}}\left(\textstyle\prod_{i\in I}\mathrm{Coh}^{e_i}_0\right)\times\coprod_{J\to\mathbf{Z}}\left(\textstyle\prod_{j\in J}\mathrm{Coh}^{d_j}_1\right)$$

where the union is over all finite sets I,J and functions $e:I\to\mathbf{N}$ and $d:J\to\mathbf{Z}$.

3.13.9. The split locus map $\underline{\oplus}:\mathrm{Coh}^s\to\mathrm{Coh}$ takes the direct sum. It gives an injection on cohomology. In fact, just the map

$$\underline{\oplus}\;:\;\mathrm{Coh}^{d_1}_1\times\cdots\times\mathrm{Coh}^{d_r}_1\;\to\;\mathrm{Coh}^d_r$$

gives an injection on cohomology, where $r\geqslant 1$ and d_i are any integers with $d=\sum d_i$. Indeed, the pullback of the tautological coherent sheaf $\mathcal{E}$ on $C\times\mathrm{Coh}^d_r$ is $\bigoplus_j\mathcal{E}_j$, where $\mathcal{E}_j$ is the tautological coherent sheaf on $C\times\mathrm{Coh}^{d_j}_1$. Then for any homology class $\alpha\in\mathrm{H}_\bullet(C)$, we have

$$(\mathrm{id}\times\underline{\oplus})^*\left(\alpha\cdot\mathrm{ch}_k(\mathcal{E})\right)\;=\;\alpha\cdot(\mathrm{id}\times\underline{\oplus})^*\mathrm{ch}_k(\mathcal{E})\;=\;\sum\alpha\cdot\mathrm{ch}_k(\mathcal{E}_j).$$

Thus, the composition $\mathrm{H}^\bullet(\mathrm{Coh}^d_r)\xrightarrow{\underline{\oplus}^*}\mathrm{H}^\bullet(\prod\mathrm{Coh}^{d_j}_1)\to\mathrm{H}^\bullet(\mathrm{Coh}^{d_j}_1)$ sends is an isomorphism and so $\underline{\oplus}^*$ is an injection on cohomology.

3.13.10. Likewise, for Ext^s take the stack classifying tuples of short exact sequences of coherent sheaves whose middle term has rank zero or one. Its connected components are products of three types of extension moduli spaces

$$\mathrm{Coh}^{e,e'}_{0,0},\quad\mathrm{Coh}^{d,e}_{1,0},\quad\mathrm{Coh}^{e,d}_{0,1}.$$

In formulae this reads

$$\mathrm{Ext}^s\;=\;\coprod_{K\rightrightarrows\mathbf{N}}\left(\textstyle\prod_{k\in K}\mathrm{Coh}^{e_{1k},e_{2k}}_{0,0}\right)\times\coprod_{J=J_1\amalg J_2\to\mathbf{Z}\times\mathbf{N}}\left(\textstyle\prod_{j_1\in J_1}\mathrm{Coh}^{d_{j_1},e_{j_1}}_{1,0}\times\prod_{j_2\in J_2}\mathrm{Coh}^{e_{j_2},d_{j_2}}_{0,1}\right).$$

The first union is over all finite sets K and pairs of functions $e_1,e_2:K\to\mathbf{N}$. The second is over all finite sets J with a partition into two subsets $J=J_1\amalg J_2$, and functions $d:J\to\mathbf{Z}$ and $e:J\to\mathbf{N}$.

3.13.11. We proceed with computing the localised CoHA product. The top row

$$\begin{tikzcd} & \mathrm{Ext}^s \arrow[dl, bend right, "\tilde{\oplus}"'] \arrow[dr, bend left, "\overline{p}"] & \\ \mathrm{Coh}^s \times \mathrm{Coh}^s & & \mathrm{Coh}^s \end{tikzcd}$$

of the diagram (3.55) can be understood in terms of the following three simple correspondences

$$\begin{tikzcd}[column sep=small] & \mathrm{Coh}_{0,0}^{e_1,e_2} \arrow[dl, "q_\alpha"'] \arrow[dr, "\alpha"] & & & \mathrm{Coh}_{0,1}^{e,d} \arrow[dl, "q_\beta"'] \arrow[dr, "\beta"] & & & \mathrm{Coh}_{1,0}^{d,e} \arrow[dl, "q_\gamma"'] \arrow[dr, "\gamma"] & \\ \mathrm{Coh}_0^{e_1} \times \mathrm{Coh}_0^{e_2} & & \mathrm{Coh}_0^{e_1+e_2} & \mathrm{Coh}_0^{e} \times \mathrm{Coh}_1^{d} & & \mathrm{Coh}_1^{e+d} & \mathrm{Coh}_1^{d} \times \mathrm{Coh}_0^{e} & & \mathrm{Coh}_1^{d+e} \end{tikzcd} \tag{3.89}$$

More precisely, on each connected component of Ext^s, it is

$$\begin{tikzcd} & (\prod_{k\in K}\mathrm{Coh}_{0,0}^{e_{1k},e_{2k}}) \times (\prod_{j_1\in J_1}\mathrm{Coh}_{1,0}^{d_{j_1},e_{j_1}} \times \prod_{j_2\in J_2}\mathrm{Coh}_{0,1}^{e_{j_2},d_{j_2}}) \arrow[dl, bend right, "\tilde{\oplus}"'] \arrow[d, "\overline{p}"] \\ \begin{matrix} (\prod_{k\in K}\mathrm{Coh}_0^{e_{1k}}) \times (\prod_{j_1\in J_1}\mathrm{Coh}_1^{d_{j_1}} \times \prod_{j_2\in J_2}\mathrm{Coh}_0^{e_{j_2}}) \\ \times(\prod_{k\in K}\mathrm{Coh}_0^{e_{2k}}) \times (\prod_{j_1\in J_1}\mathrm{Coh}_0^{e_{j_1}} \times \prod_{j_2\in J_2}\mathrm{Coh}_1^{d_{j_2}}) \end{matrix} & (\prod_{k\in K}\mathrm{Coh}_0^{e_{1k}+e_{2k}}) \times (\prod_{j_1\in J_1}\mathrm{Coh}_1^{d_{j_1}+e_{j_1}} \times \prod_{j_2\in J_2}\mathrm{Coh}_1^{d_{j_2}+e_{j_2}}) \end{tikzcd} \tag{3.90}$$

which is simply a product of the correspondences (3.92). The associated maps on connected components send

$$\begin{tikzcd} & (K, J_1, J_2) \arrow[dl, mapsto, bend right, "\tilde{\oplus}"'] \arrow[dr, mapsto, bend left, "\overline{p}"] & \\ ((K \amalg J_2, J_1), (K \amalg J_1, J_2)) & & (K, J_1 \amalg J_2) \end{tikzcd}$$

In particular, given a connected component of $\mathrm{Coh}^s \times \mathrm{Coh}^s$ labelled by finite sets $((I_1, J_1), (I_2, J_2))$, to define a lift $\tilde{\oplus}$ requires a (non-canonical) choice of partitions

$$K \amalg J_2 \xrightarrow{\sim} I_1 \qquad\qquad K \amalg J_1 \xrightarrow{\sim} I_2. \tag{3.91}$$

3.13.12. We fix some notation. Fix a connected component c of Ext^s, i.e. fix background finite sets K, J_1, J_2, functions $e_1, e_2 : K \to \mathbf{N}$ on K, and $d : J \to \mathbf{Z}$ and $e : J \to \mathbf{N}$ on $J = J_1 \amalg J_2$.

- For any $k \in K$, write $\mathbf{T}_{\alpha,k}$ for the pullback of $\mathbf{T}_\alpha$ via the kth projection

$$\mathrm{Ext}_c^s \to \mathrm{Coh}_{0,0}^{e_{1k},e_{2k}}.$$

- Similarly for $\mathbf{T}_{\beta,j_1}$ and $\mathbf{T}_{\gamma,j_2}$.

Similarly, fix a connected component c' of $\mathrm{Coh}^s \times \mathrm{Coh}^s$, i.e. fix background finite sets $(I_1, J_1), (I_2, J_2)$ and functions $e : I_1 \to \mathbf{N}$, $d : J_1 \to \mathbf{Z}$ and $e' : I_2 \to \mathbf{N}$, $d' : J_2 \to \mathbf{Z}$.

- For any $a_1, a_2 \in K$, write θ_{a_1,a_2} for the pullback of the Ext complex by the a, bth projetion

$$(\mathrm{Coh}^s \times \mathrm{Coh}^s)_{c'} \;\to\; \mathrm{Coh}_0^{e_{a_1}} \times \mathrm{Coh}_0^{e'_{a_2}}.$$

- Similarly define θ_{a_1,a_2} for any $a_1, a_2 \in K \amalg J_1 \amalg J_2$.

Having chosen a partition (3.94), we can define $\widetilde{\oplus}$ and the above perfect complexes are related by

$$\widetilde{\oplus}^* \;:\; (\theta_{k,k}, \theta_{j_1,j_1}, \theta_{j_2,j_2}) \;\mapsto\; (\mathbf{T}_{\alpha,k}, \mathbf{T}_{\beta,j_1}, \mathbf{T}_{\gamma,j_2}).$$

3.13.13. We now compute the Euler class of the normal complex of the closed embedding $i : \mathrm{Ext}^s \to \mathrm{Ext} \times_{\mathrm{Coh}} \mathrm{Coh}^s$, using (3.53), which says that

$$[\mathbf{N}_i] \;=\; [i^* \oplus^* \mathbf{T}_{\mathrm{Ext}/\mathrm{Coh}}] - [\mathbf{T}_{\mathrm{Ext}^s/\mathrm{Coh}^s}] \;=\; [\widetilde{\oplus}^* (\underline{\oplus} \times \underline{\oplus})^* \theta] - [\mathbf{T}_{\overline{p}}]$$

where $\theta = \mathrm{Ext}$ is the Ext complex. The second summand is easy to compute because $\overline{p}$ is just a product of the maps α, β and γ, so the tangent complex is just a direct sum

$$\mathbf{T}_{\overline{p}} \;=\; \textstyle\prod_{k \in K} \mathbf{T}_{\alpha,k} \oplus \prod_{j_1 \in J_1} \mathbf{T}_{\beta,j_1} \oplus \prod_{j_2 \in J_2} \mathbf{T}_{\gamma,j_2}.$$

The first summand is

$$(\underline{\oplus} \times \underline{\oplus})^* \theta \;=\; \textstyle\bigoplus_{a_i \in I_i \amalg J_i} \theta_{a_1,a_2}.$$

Thus given a choice of identifications (3.94), we can write this as

$$(\underline{\oplus} \times \underline{\oplus})^* \theta \;=\; \textstyle\bigoplus_{a_i \in K \amalg J_1 \amalg J_2} \theta_{a_1,a_2}.$$

It follows that

Lemma 3.13.14. $[\mathbf{N}_i]$ *is the sum of* $[\widetilde{\oplus}^* \theta_{a_1,a_2}]$ *over* $a_i \in K \amalg J_1 \amalg J_2$ *with* $a_1 \neq a_2$.

3.13.15. We can now use the integration formula to compute the localised CoHA product for the moduli stack of coherent sheaves on C. It is defined on the vector space

$$\mathrm{H}^\bullet(\mathrm{Coh}^s) \;=\; \bigoplus_{\substack{I \to \mathbf{N} \\ J \to \mathbf{Z}}} \bigotimes_{i \in I} V_C^i \otimes \bigotimes_{j \in J} W_C^j$$

where V_C^i and W_C^j are just copies of the vector spaces (3.91) for $d = d_i$ and (3.90). Applying the integration formula then gives that

Theorem 3.13.16. *Fix two connected components of* Coh^s*, labelled by finite sets* $(I_1, J_1), (I_2, J_2)$ *and functions on them as in section 3.13.8. The localised CoHA product*

$$\mathrm{H}^\bullet(\mathrm{Coh}^s) \otimes \mathrm{H}^\bullet(\mathrm{Coh}^s) \ \to \ \mathrm{H}^\bullet(\mathrm{Coh}^s)_{\mathrm{loc}}$$

on each summand

$$\left(\bigotimes\nolimits_{i_1 \in I_1} V_C^{i_1} \otimes \bigotimes\nolimits_{j_1 \in J_1} W_C^{j_1}\right) \otimes \left(\bigotimes\nolimits_{i_2 \in I_2} V_C^{i_2} \otimes \bigotimes\nolimits_{j_2 \in J_2} W_C^{j_2}\right) \ \to \ \left(\bigotimes\nolimits_{k \in K} V_C^{k} \otimes \bigotimes\nolimits_{j \in J_1 \amalg J_2} W_C^{j}\right)_{\mathrm{loc}}$$

is given by

$$f_1 \otimes f_2 \ \mapsto \ \sum_{\substack{I_1 \simeq K \amalg J_2 \\ I_2 \simeq K \amalg J_1}} \left(\bigotimes_{k, j_1, j_2} \alpha_*^k \otimes \beta_*^{j_1} \otimes \gamma_*^{j_2} \right) \left(\frac{f_1 \otimes f_2}{e(\mathbf{N}_i)} \right). \tag{3.92}$$

Here K *is a finite set of size* $|I_1| - |J_2| = |I_2| - |J_1|$*, the product being zero if these two quantities are not equal, and the sum is over identifications of* I_1, I_2 *with the marked sets as in (3.94). The Euler class* $e(\mathbf{N}_i)$ *is given as in (3.13.14).*

In the above we have used the pushforwards by α, β, γ to define maps

$$\alpha_*^k \ : \ V_C^k \otimes V_C^k \ \to \ V_C^k \qquad \beta_*^{j_1} \ : \ V_C^{j_1} \otimes W_C^{j_1} \ \to \ W_C^{j_1} \qquad \gamma_*^{j_2} \ : \ W_C^{j_2} \otimes V_C^{j_2} \ \to \ W_C^{j_2}.$$

Thus their tensor product in (3.95) along with the identifications $I_1 \simeq K \amalg J_2$ and $I_2 \simeq K \amalg J_1$ as in (3.94) define a map

$$\begin{aligned} &\left(\bigotimes\nolimits_{i_1 \in I_1} V_C^{i_1} \otimes \bigotimes\nolimits_{j_1 \in J_1} W_C^{j_1}\right) \otimes \left(\bigotimes\nolimits_{i_2 \in I_2} V_C^{i_2} \otimes \bigotimes\nolimits_{j_2 \in J_2} W_C^{j_2}\right) \\ &\qquad \xrightarrow{\sim} \ \left(\bigotimes\nolimits_{k \in k} V_C^{k} \otimes \bigotimes\nolimits_{j_2 \in J_2} V_C^{j_2} \otimes \bigotimes\nolimits_{j_1 \in J_1} W_C^{j_1}\right) \otimes \left(\bigotimes\nolimits_{k \in k} V_C^{k} \otimes \bigotimes\nolimits_{j_1 \in J_1} V_C^{j_1} \otimes \bigotimes\nolimits_{j_2 \in J_2} W_C^{j_2}\right) \\ &\qquad \to \ \left(\bigotimes\nolimits_{k \in K} V_C^{k} \otimes \bigotimes\nolimits_{j \in J_1 \amalg J_2} W_C^{j}\right)_{\mathrm{loc}}. \end{aligned}$$

This gives a fairly explicit description of the CoHA product, modulo computing the pushforwards $\alpha_*, \beta_*, \gamma_*$.

3.13.17. **The CoHA α.** First consider the rank zero correspondence

$$\begin{tikzcd} & \mathrm{Coh}_{0,0}^{e_1, e_2} \arrow[dl, "q_\alpha"'] \arrow[dr, "\alpha"] & \\ \mathrm{Coh}_0^{e_1} \times \mathrm{Coh}_0^{e_2} & & \mathrm{Coh}_0^{e_1+e_2} \end{tikzcd}$$

The work [He] of Heinloth can be easily adapted to show that the CoHA product $\alpha_* q^*$ is thus the usual algebra structure on

$$\mathrm{Sym}(V_C) \ = \ \bigoplus_{e \geqslant 0} \mathrm{H}^\bullet(\mathrm{Coh}_0^e).$$

Since α is generically finite, $\mathbf{T}_\alpha$ generically vanishes.

3.13.18. **Stratification.** To continue, it is useful to consider the stratification on Coh given by the length ℓ of torsion subsheaf:

$$\mathrm{Coh}_r^d \;=\; \coprod_{\ell\geqslant 0} \mathrm{Coh}_r^{d,\ell}.$$

The closure relations of this stratification is $\overline{\mathrm{Coh}_r^{d,\ell}} \;=\; \coprod_{\ell'\geqslant\ell}\mathrm{Coh}_r^{d,\ell'}$. All strata are smooth. Sending a coherent sheaf to its torsion subsheaf and torsion-free quotient gives

$$\mathrm{Coh}_r^{d,\ell} \;\to\; \mathrm{Coh}_0^{\ell}\times\mathrm{Bun}_r^{d-\ell}.$$

This is a vector bundle of rank $r\ell$, the zero section being the direct sum map. It follows that the ℓth stratum has codimension $r\ell$ inside Coh_r^d.

3.13.19. **The CoHA β.** Turn secondly to

$$\begin{tikzcd} & \mathrm{Coh}_{0,1}^{e,d} \arrow[dl, "q"'] \arrow[dr, "\beta"] & \\ \mathrm{Coh}_0^e\times\mathrm{Coh}_1^d & & \mathrm{Coh}_1^{e+d} \end{tikzcd}$$

Whilst β_* is complicated to work out, its stratified pieces with respect to the stratification $\mathrm{Coh}_1^{d+e,\ell}\subseteq\mathrm{Coh}_1^{d+e}$ by length ℓ of torsion subsheaf are easy to compute. Setting $f=d+e-\ell$, we have

$$\begin{tikzcd} \mathrm{Coh}_0^e\times(\mathrm{Coh}_0^{\ell-e}\times\mathrm{Pic}^f) \arrow[d] & \mathrm{Coh}_{0,0}^{e,\ell-e}\times\mathrm{Pic}^f \arrow[l] \arrow[r] \arrow[d] & \mathrm{Coh}_0^{\ell}\times\mathrm{Pic}^f \arrow[d] \\ \mathrm{Coh}_0^e\times\mathrm{Coh}_1^{d,\ell-e} \arrow[d] & \mathrm{Coh}_{0,1}^{e,d,\ell} \arrow[l, "q_\ell"'] \arrow[r, "\beta_\ell"] \arrow[d] & \mathrm{Coh}_1^{d+e,\ell} \arrow[d] \\ \mathrm{Coh}_0^e\times\mathrm{Coh}_1^d & \mathrm{Coh}_{0,1}^{e,d} \arrow[l, "q"'] \arrow[r, "\beta"] & \mathrm{Coh}_1^{d+e} \end{tikzcd}$$

We have defined $\mathrm{Coh}_{0,1}^{d,e,\ell}$ so the lower right square is Cartesian. The upper right square is Cartesian because there are no nonzero maps from a torsion sheaf into a line bundle. The top row of vertical arrows are all vector bundles, so that

$$\beta_{\ell,*}q_\ell^* \;=\; \text{rank zero CoHA product} \;\otimes\; \mathrm{id}_{\mathrm{H}^\bullet(\mathrm{Pic}^f)}.$$

Now, one can show that q^* and β_* are uniquely determined by their restriction to the strata, and so the above uniquely determines the first postive rank CoHA product β_*q^*. Moreover, by the above $\mathbf{T}_{\beta_\ell}$ are given in terms of $\mathbf{T}_\alpha$.

3.13.20. **The CoHA γ.** Finally we consider the last positive rank case

$$\begin{tikzcd} & \mathrm{Coh}_{1,0}^{d,e} \arrow[dl, "q"'] \arrow[dr, "\gamma"] & \\ \mathrm{Coh}_1^d \times \mathrm{Coh}_0^e & & \mathrm{Coh}_1^{d+e} \end{tikzcd}$$

Partial information about the CoHA product can be computed by stratifying the base of γ:

$$\begin{tikzcd} & \mathrm{Coh}_{1,0}^{d,e,\ell} \arrow[r, "\gamma_\ell"] \arrow[d] & \mathrm{Coh}_1^{d+e,\ell} \arrow[d] \\ \mathrm{Coh}_1^d \times \mathrm{Coh}_0^e & \mathrm{Coh}_{1,0}^{d,e} \arrow[l, "q"'] \arrow[r, "\gamma"] & \mathrm{Coh}_1^{d+e} \end{tikzcd}$$

However, in contrast to last section, $\mathrm{Coh}_{1,0}^{d,e,\ell}$ is more complicated because there exist nonzero maps from line bundles into torsion sheaves. To proceed in computing $\gamma_{\ell *}$ we apply an argument suggested to us by Kevin Lin. First fix some notation:

1) $\mathrm{Coh}_1^{d+e,\ell}$ classifies rank one degree $d+e$ coherent sheaves $\mathcal{E}$ whose torsion part $\mathcal{T}$ has length ℓ. That is to say, it classifies short exact sequences

$$\mathcal{T} \to \mathcal{E} \to \mathcal{Q}$$

of a degree ℓ torsion sheaf $\mathcal{T}$ by a degree $f = d+e-\ell$ line bundle $\mathcal{Q}$.

2) $\mathrm{Coh}_{1,0}^{d,e}$ classifies extensions

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

where $\mathcal{E}'$ has rank one and degree d, and $\mathcal{E}''$ has rank zero and degree e.

3) The pullback $\mathrm{Coh}_{1,0}^{d,e,\ell}$ classifies

$$\begin{tikzcd} \mathcal{T}' \arrow[r] \arrow[d] & \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{Q}' \arrow[d] & & ? & d & ? \\ \mathcal{T} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[d] & & \ell & d+e & f \\ \mathcal{T}'' \arrow[r] & \mathcal{E}'' \arrow[r] & \mathcal{Q}'' & & ? & e & ? \end{tikzcd}$$

where all rows and columns are exact sequences, the left horizontal arrows are maximal torsion subsheaves, and the degrees are indicated on the right.

Note that fixing one of the ?'s determines the rest. Thus there is a stratification $\mathrm{Coh}_{1,0}^{d,e,\ell,\ell'} \subseteq \mathrm{Coh}_{1,0}^{d,e,\ell}$ given by bounding the length of $\mathcal{T}'$, classifying the same data as above, except that the degrees are fixed:

$$
\begin{tikzcd}
\mathcal{T}' \arrow[r] \arrow[d] & \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{Q}' \arrow[d] & & \leqslant \ell' & d & \geqslant f' \\
\mathcal{T} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[d] & & \ell & d+e & f \\
\mathcal{T}'' \arrow[r] & \mathcal{E}'' \arrow[r] & \mathcal{Q}'' & & \geqslant \ell'' & e & \leqslant f''
\end{tikzcd}
$$

Notice that the strata are labelled by $0 \leqslant \ell' \leqslant \ell$, so in particular there are finitely many strata.

3) Write $\mathcal{M}$ for the space classifying

$$
\begin{tikzcd}
\mathcal{T}' \arrow[d] & & \mathcal{Q}' \arrow[d] & & \ell' & & f' \\
\mathcal{T} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] & \mathcal{Q} \arrow[d] & & \ell & d+e & f \\
\mathcal{T}'' & & \mathcal{Q}'' & & \ell'' & & f''
\end{tikzcd}
$$

with notation as in (3)) above. Similarly, write $\widetilde{\mathcal{M}}$ for the space classifying

$$
\begin{tikzcd}
\mathcal{T}' \arrow[r] \arrow[d] & \mathcal{T}' \oplus \mathcal{Q}' \arrow[r] \arrow[d] & \mathcal{Q}' \arrow[d] & & \ell' & d & f' \\
\mathcal{T} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[d] & & \ell & d+e & f \\
\mathcal{T}'' \arrow[r] & \mathcal{T}'' \oplus \mathcal{Q}'' \arrow[r] & \mathcal{Q}'' & & \ell'' & e & f''
\end{tikzcd}
$$

The point of considering $\mathcal{M}$ is that we have the pullback

$$
\begin{tikzcd}
\mathrm{Coh}_{0,0}^{\ell'',\ell''} \times \mathrm{Pic}_{1,0}^{f',f''} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\pi"] \\
\mathrm{Coh}_0^{\ell} \times \mathrm{Pic}^{f} \arrow[r] & \mathrm{Coh}_1^{d+e,\ell}
\end{tikzcd}
$$

and the horizontal arrows give isomorphisms on cohomology, so π_* is easy to compute. This can be used to gain information about $\gamma_{\ell *}$, using the diagram

$$\begin{tikzcd}
& \widetilde{\mathcal{M}} \arrow[dl, bend right] \arrow[dr, bend left] \arrow[ddd, "\widetilde{\pi}"] & \\
\mathcal{M} \arrow[ddr, bend right, "\pi"'] & & \mathrm{Coh}_{1,0}^{d,e,\ell,\ell'} \arrow[d, "j_{\ell'}"] \\
& & \mathrm{Coh}_{1,0}^{d,e,\ell} \arrow[dl, "\gamma_\ell"] \\
& \mathrm{Coh}_1^{d+e,\ell} &
\end{tikzcd}$$

The bottom two maps are proper, and the top two are affine space fibrations. Applying cohomology to $\widetilde{\pi}_!\widetilde{\pi}^!k \to k$ thus gives

$$\begin{tikzcd}
& \mathrm{H}^\bullet(\widetilde{\pi}_!k) \arrow[dl, bend right, "\sim"'] \arrow[dr, bend left, "\sim"] & \\
\mathrm{H}^\bullet(\mathcal{M}) \arrow[ddr, bend right, "\pi_*"'] & & \mathrm{H}^\bullet(\mathrm{Coh}_{1,0}^{d,e,\ell,\ell'}, j_{\ell'!}k) \arrow[d] \\
& & \mathrm{H}^\bullet(\mathrm{Coh}_{1,0}^{d,e,\ell,\ell'}) \arrow[dl, "\gamma_{\ell*}"] \\
& \mathrm{H}^\bullet(\mathrm{Coh}_1^{d+e,\ell}) &
\end{tikzcd}$$

where we have omitted grading shifts from the notation. This determines what $\gamma_{\ell*}$ is on the image of $\mathrm{H}^\bullet(\mathrm{Coh}_{1,0}^{d,e,\ell,\ell'}, j_{\ell'!}k)$.

3.13.21. Finally, we make the obvious comment that although the above does give a partial description of γ_*, one hopes that there exists a more explicit one.

Chapter 4

Quantum groups and vertex algebras

4.1 Drinfeld Jimbo quantum groups

4.1.1. **Quantisation of algebras.** Given a k algebra A, one can ask what deformations it admits, i.e. a flat algebra $\widehat{A}$ over a base augmented algebra B, whose fibre over k is A: there is a pushout in the category of k algebras

$$\begin{tikzcd} A & \widehat{A} \arrow[l] \\ k \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd} \tag{4.1}$$

For instance, the algebra $A = \mathbf{C}[x, p]$ from classical mechanics admits a quantisation

$$\widehat{A} \;=\; \mathbf{C}[[h]]\langle x, p\rangle \qquad \text{where} \quad [x, p] \;=\; \hbar$$

over the base $B = \mathbf{C}[[\hbar]]$. There is a large body of work about quantising (functions on) spaces with Poisson structure, for instance see [Kon]. However, we will be interested in deforming noncommutative algebras. For more on deformation theory, see [Ge, Ha].

4.1.2. One of the most interesting algebras to consider is the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra. This may be thought of as a sort of noncommutative space of functions on the Poisson space $\mathfrak{g}^*$, so one might expect it to have interesting deformation theoretic properties. However,

Proposition 4.1.3 (e.g. [BMP])**.** *Every deformation of the algebra $U(\mathfrak{g})$ over $k[[\hbar]]$ is equivalent to the trivial deformation.*

Proof sketch. Let $\widehat{A}$ be any such deformation. As a vector space, $\widehat{A} = U(\mathfrak{g})[[\hbar]]$, with the multiplication map given by

$$m_{\widehat{A}}(\ ,\ ) \ = \ m_{U(\mathfrak{g})}(\ ,\ ) \ + \ \hbar\mu_1(\ ,\ ) \ + \ \hbar^2\mu_2(\ ,\ ) \ + \ \cdots$$

where $\mu_n : U(\mathfrak{g})^{\otimes 2} \to U(\mathfrak{g})$. However, one can show that the first nonzero μ_n defines an element of $\mathrm{HH}^2(U(\mathfrak{g}), U(\mathfrak{g}))$. One can show that this vanishes, hence so too do all of the μ_n. $\square$

So at first glance it might seem like the deformation theory of $U(\mathfrak{g})$ is uninteresting. The fundamental insight of Drinfeld [Dr] was that to properly study the deformations of $U(\mathfrak{g})$, one needs to remember its coalgebra structure.

4.1.4. By a *deformation* of an algebra A with extra structure (bialgebra, Hopf algebra,...) over a base augmented algebra B, we mean an algebra $\widehat{A}$, flat over B, with the same structure and a map $A \to \widehat{A}$ preserving that structure, fitting into a pushout of algebras (4.1). A deformation over $k[[\hbar]]$ is called a *one parameter deformation.*

4.1.5. **Drinfeld Jimbo quantum groups.** The universal enveloping algebra of any Lie algebra $U(\mathfrak{g})$ carries a cocommutative coalgebra structure, given by

$$\Delta x \ = \ 1 \otimes x \ + \ x \otimes 1$$

which makes $U(\mathfrak{g})$ into Hopf algebra with antipode $S = (-1)^{\deg}$ given by the degree function on $U(\mathfrak{g})$.

Drinfeld discovered that

Proposition 4.1.6. [Dr, Ex. 6.2] *Let $\mathfrak{g}$ be any simple Kac Moody Lie algebra (see Appendix B) over $\mathbf{C}$, e.g. a finite dimensional semisimple Lie algebra. There is a nontrivial one parameter deformation $U_\hbar(\mathfrak{g})$ of the Hopf algebra $U(\mathfrak{g})$, such that*

1) It admits an involution θ which is an (co)algebra (anti)automorphism, such that $\theta \mod \hbar$ is the Cartan involution.

2) The is a cocommutative Hopf subalgebra C (here $C = U_\hbar(\mathfrak{t})$) stable under θ such that the map $C/\hbar \to U(\mathfrak{g})$ is injective with image $U(\mathfrak{t})$.

Moreover, any other such one parameter deformation is isomorphic to $U_\hbar(\mathfrak{g})$ as algebras over $\mathbf{C}[[\hbar]]/\hbar^2$.

We will now give an explicit description of this deformation.

Definition 4.1.7. Let $\mathfrak{g}$ be a simple Kac Moody Lie algebra over $\mathbf{C}$. The *Drinfeld Jimbo quantum group* $U_\hbar(\mathfrak{g})$ is the Hopf algebra over $\mathbf{C}[[\hbar]]$ defined as follows. It is generated as a topological algebra by $\{x_i^-, h_i, x_i^+\}$ where i varies over the simple roots of $\mathfrak{g}$, with algebra relations

$$[h_i, h_j] \; = \; 0 \qquad\qquad [h_i, x_j^\pm] \; = \; \pm A_{ij} x_j^\pm \tag{4.2}$$

$$[x_i^+, x_j^-] \; = \; \delta_{i,j} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}} \tag{4.3}$$

and the quantum Serre relations, labelled by pairs of different simple roots $i \neq j$:

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \binom{1-A_{ij}}{k}_{q_i} (x_i^\mp)^k x_j^\pm (x_j^\pm)^{1-A_{ij}-k} \; = \; 0, \tag{4.4}$$

where A is the Cartan matrix of $\mathfrak{g}$, $q_i = \exp(d_i \hbar)$ for $d_i \in \mathbf{N}$ the exponents,[1] and $\binom{}{}_q$ are the quantum binomial coefficients. Its coproduct is defined by

$$\Delta h_i \; = \; 1 \otimes h_i + h_i \otimes 1 \tag{4.5}$$

$$\Delta x_i^+ \; = \; 1 \otimes x_i^+ + x_i^+ \otimes q_i \qquad\qquad \Delta x_i^- \; = \; q_i^{-1} \otimes x_i^- + x_i^- \otimes 1 \tag{4.6}$$

its counit by $\varepsilon(h_i) = \varepsilon(x_i^\pm) = 0$, and its antipode by

$$S(h_i) \; = \; -h_i \qquad\qquad S(x_i^\pm) \; = \; -q^{\mp h_i} x_i^\pm. \tag{4.7}$$

One can check by hand that this defines a Hopf algebra structure on $U_\hbar(\mathfrak{g})$.

4.1.8. **Drinfeld double.** We now explain in a more conceptual way why $U_\hbar(\mathfrak{g})$ is a quasitriangular Hopf algebra, and where its definition 4.1.7 came from.

4.1.9. Recall that the *opposite* of a coalgebra A with extra structure (bialgebra, Hopf algebra,...) is the same vector space A^{op} with the opposite coproduct $\Delta^{op} = \sigma\Delta$, and the other structure unchanged.

[1]These are defined as the unique set of coprime positive integers such that the matrix $(d_i A_{ij})$ is symmetric, see [CP, § A.1]. In the ADE cases we will be considering, all $d_i = 1$.

4.1.10. Any Hopf algebra A is the "positive part" of a larger Hopf algebra $D(A)$ called its *Drinfeld double*:

Proposition 4.1.11 ([Dr]). *Let A be a Hopf algebra. There is a unique quasitriangular Hopf algebra structure on*

$$D(A) \;=\; A \otimes A^{\vee,op}$$

such that the natural inclusions of A and $A^{\vee,op}$ are Hopf algebra maps, with R matrix the image of the canonical element under the embedding $A \otimes (A^{\vee})^{op} \hookrightarrow D(A) \otimes D(A)$.

Also see [Maj, EGNO]. The Drinfeld double can be viewed as taking the E_3 centre, see [Lur2].

4.1.12. To apply this to give another construction of Drinfeld Jimbo quantum groups, pick complementary Borel subalgebras $\mathfrak{b}_{\pm} \subseteq \mathfrak{g}$ and endow the algebra $U_\hbar(\mathfrak{b}_{\pm})$ with a *cocommutative* Hopf algebra structure

$$\Delta h_i \;=\; 1 \otimes h_i + h_i \otimes 1 \qquad\qquad \Delta x_i^{\pm} \;=\; 1 \otimes x_i^{\pm} + x_i^{\pm} \otimes 1. \tag{4.8}$$

4.1.13. To relate its Drinfeld double to quantum groups, we then consider the *Drinfeld pairing* (see [ES, §12.3])

$$\langle\ ,\ \rangle \;:\; U_\hbar(\mathfrak{b}_+)\widehat{\otimes} U_\hbar(\mathfrak{b}_-)^{op} \;\to\; \mathbf{C}((\hbar)).$$

This is perfect pairing of bialgebras, meaning that the induced maps

$$U_\hbar(\mathfrak{b}_+) \;\to\; U_\hbar(\mathfrak{b}_-)^{\vee,op} \qquad\qquad U_\hbar(\mathfrak{b}_-)^{op} \;\to\; U_\hbar(\mathfrak{b}_+)^{\vee} \tag{4.9}$$

are algebra isomorphisms. The Drinfeld pairing is defined by its values on the generators $1, h_i, x_i^{\pm}$, which are all zero except

$$\langle 1,1\rangle \;=\; 1 \qquad \langle h_i, h_j\rangle \;=\; \frac{1}{\hbar}\kappa(h_i,h_j) \qquad \langle x_i^+, x_j^-\rangle \;=\; \frac{\delta_{i,j}}{q_i - q_i^{-1}}. \tag{4.10}$$

where $\kappa : \mathfrak{t}\otimes\mathfrak{t} \to \mathbf{C}$ is the Killing form.

4.1.14. Because the Drinfeld pairing is perfect, the Drinfeld double is identified with $U_\hbar(\mathfrak{b}_+)\widehat{\otimes} U_\hbar(\mathfrak{b}_-)$. We recover the Drinfeld Jimbo quantum groups after quotienting by the diagonal copy of $U_\hbar(\mathfrak{t})$

$$0 \;\to\; U_\hbar(\mathfrak{t}) \;\to\; D(U_\hbar(\mathfrak{b}_+)) \;\to\; U_\hbar(\mathfrak{g}) \;\to\; 0.$$

Because these are maps of Hopf algebras, the R matrix of $U_\hbar(\mathfrak{g})$ can be recovered by taking the image of the R matrix of the Drinfeld double. Letting a_α be a basis of $U_\hbar(\mathfrak{n}_+)$ and a^α the dual basis of $U_\hbar(\mathfrak{n}_-)$ with respect to the Drinfeld pairing, it follows that the R matrix is

$$R \;=\; e^{\hbar\sum h_i\otimes h_i}\sum a_\alpha \otimes a^\alpha,$$

see [ES, §12.13]. For instance, when $\mathfrak{g} = \mathfrak{sl}_2$,

$$R \;=\; q^{\frac{1}{2}h\otimes h} \sum_{n\geqslant 0} q^{\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]_q!} e^n \otimes f^n.$$

4.1.15. **Rational and integral forms.** A *rational form* of the Drinfeld Jimbo quantum group is a $\mathbf{Q}(q)$ algebra whose base change to $\mathbf{C}[[\hbar]]$ is $U_\hbar(\mathfrak{g})$.

Definition 4.1.16. [CP, §9.1] Let $\mathfrak{g}$ be a simple Kac Moody Lie algebra over $\mathbf{C}$. The (*adjoint*) *rational form* $U_q(\mathfrak{g})$ of the Drinfeld Jimbo quantum group is the Hopf algebra over $\mathbf{Q}(q)$ generated as a $\mathbf{Q}(q)$-subalgebra of $U_\hbar(\mathfrak{g})$ by $\{x_i^-, k_i, x_i^+\}$ where $k_i = q_i^{h_i}$ and i varies over simple roots of $\mathfrak{g}$.

An *integral form* of the Drinfeld Jimbo quantum group is a $\mathbf{Z}[q,q^{-1}]$ algebra whose base change to $\mathbf{Q}(q)$ is $U_q(\mathfrak{g})$, see [CP, §9.2]. Thus, given any integral form we may specialise it to any nonzero value of q. There are three main integral forms, whose specialisations agree unless q is a root of unity. As a $\mathbf{Z}[q,q^{-1}]$ subalgebra of the rational form, they are

1. The *Lusztig quantum group* $U^{Lus}(\mathfrak{g})$ introduced by Lusztig [Lus, Thm. 6.7], is generated by the divided powers $x_i^{\pm}/[n]_{q_i}!$ and
$$\binom{k_i;0}{n} \;=\; \prod_{m=1}^{n} \frac{k_i q_i^{1-m} - k_i^{-1} q_i^{m-1}}{q_i - q_i^{-1}}$$
for $n \geqslant 0$.
2. The *Kac-DeConcini quantum group* $U_q^{KC}(\mathfrak{g})$ introduced by Kac and De Concini [DK], is generated by $x_i^{\pm}, k_i$ and $\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$.
3. The *small quantum group* $u_q(\mathfrak{g})$ is defined as the image of the map $U_q^{KC}(\mathfrak{g}) \to U^{Lus}(\mathfrak{g})$, which is not injective.

4.1.17. An interesting question following this chapter is the relation between integral forms of quantum groups and integral (e.g. integral singular or ℓ-adic) cohomology of moduli stacks.

4.1.18. **Kazhdan Lusztig equivalence.** We very briefly note that quantum groups are intimately related with the other material in this thesis, i.e. two dimensional conformal field theories. On the physics side this comes from the work of Reshetikhin and Turaev [RT] relating $3d$ TQFTs and $2d$ CFTs. The main mathematical incarnation is the Kazhdan Lusztig equivalence [KL] between something close to $U_q(\mathfrak{g})$-Mod where $\mathfrak{g}$ is a simple finite dimensional complex Lie algebra and q is a root of unity, and the category $(\widehat{\mathfrak{g}}\text{-Mod})_k^{G(\mathcal{O})}$ of integrable modules over $\widehat{\mathfrak{g}}$ at a certain level. See work of Chen and Fu [CF] for a conceptual explanation involving factorisation machinery.

4.2 Quantum affine algebras

There are two different realisations of the universal enveloping algebra $U(\widehat{\mathfrak{g}})$ of an affine Lie algebra, which are best understood in terms of the *double* loop space, or the associated toroidal algebra $\hat{\widehat{\mathfrak{g}}}$. Both notions admit q deformations, which are called quantum affine and quantum toroidal algebras.

4.2.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra of rank r, with simple roots $\alpha_1, ..., \alpha_r$. We can form the affine Lie algebra $\widehat{\mathfrak{g}}$, but there are *two* different ways of realising it:

1. View $\widehat{\mathfrak{g}}$ as a central extension of the loop algebra of $\mathfrak{g}$. Thus $U(\widehat{\mathfrak{g}})$ is generated by

$$x^-_{\alpha,n}, h_{\alpha,n}, x^+_{\alpha,n}, \qquad \alpha \;=\; \alpha_1, ..., \alpha_r, \quad n \in \mathbf{Z},$$

subject to the relations

$$[h_{\alpha_i}(z), x^\pm_{\alpha_j}(w)] \;=\; A_{ij} x^\pm_{\alpha_j}(z)\delta(z-w), \qquad [h_{\alpha_i}(z), h_{\alpha_j}(w)] \;=\; 0,$$

$$[x^-_{\alpha_i}(z), x^+_{\alpha_j}(w)] \;=\; \delta_{i,j} h_{\alpha_i}(z)\delta(z-w) \;+\; \kappa(\alpha_i,\alpha_j)\partial_z\delta(z-w),$$

and the power series analogues of the Serre relations, where for an element $a \in \mathfrak{g}$ we have set

$$a(z) \;=\; \sum_{n\in\mathbf{Z}} a_n z^{-n-1}.$$

2. View $\widehat{\mathfrak{g}}$ as a Kac Moody Lie algebra in its own right, so that $U(\widehat{\mathfrak{g}})$ is generated by $r+1$ many $\mathfrak{sl}_2$ triples of Chevalley generators

$$x^-_\alpha, h_\alpha, x^+_\alpha, \qquad \alpha \;=\; \alpha_0, \alpha_1, ..., \alpha_r,$$

satisfying the relations set out by the affine Cartan matrix $\widehat{A}$,

$$[h_{\alpha_i}, x^\pm_{\alpha_j}] \;=\; \widehat{A}_{ij} x^\pm_{\alpha_j}, \qquad [h_{\alpha_i}, h_{\alpha_j}] \;=\; 0,$$

$$[x^-_{\alpha_i}, x^+_{\alpha_j}] \;=\; \delta_{i,j} h_{\alpha_i},$$

and the Serre relations, where we emphasise that here i, j vary among $0, 1, ..., r$.

4.2.2. **Toroidal algebras.** Both of these realisations naturally live inside the *toroidal* (*Lie*) *algebra* $\hat{\widehat{\mathfrak{g}}}$, see e.g. [MRY].

4.2.3. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with its normalised invariant form κ and A any commutative algebra, over $\mathbf{C}$. Then Kassel [Kas] showed that

$$\mathfrak{u} \;=\; A \otimes \mathfrak{g} \;\oplus\; \Omega^1_A / d\Omega^0_A$$

is the universal central extension of the Lie algebra $A \otimes \mathfrak{g}$ which is perfect (Lie bracket is surjective). The $\Omega^1_A/d\Omega^0_A$ is central and the rest of the Lie bracket is given by

$$[a \otimes x, b \otimes y] \;=\; ab \otimes [x, y] \;-\; (a \cdot db)\kappa(x, y).$$

4.2.4. In particular, if we apply this to the algebra of functions on the n torus

$$A \;=\; \mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

then the universal central extension we get is denoted $\mathfrak{u} = \mathfrak{g}_{[n]}$, or for small values of n, $\mathfrak{g}, \widehat{\mathfrak{g}}, \hat{\hat{\mathfrak{g}}}, \ldots$. It should be thought of as (a central extension of) the Lie algebra to the higher loop space

$$\mathrm{Maps}((\mathbf{C}^\times)^n, G).$$

We will call $\mathfrak{g}_{[2]} = \hat{\hat{\mathfrak{g}}}$ the *toroidal Lie algebra.*

4.2.5. A map of commutative algebras $A_1 \to A_2$ induces a map on the associated central extensions $\mathfrak{u}_1 \to \mathfrak{u}_2$. In particular, the two different maps

$$\mathbf{C}[t^{\pm 1}] \;\to\; \mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}] \qquad\qquad t \;\mapsto\; t_i$$

induces two different maps

$$\widehat{\mathfrak{g}} \;\hookrightarrow\; \hat{\hat{\mathfrak{g}}}$$

which are exchanged by the involution on $\hat{\hat{\mathfrak{g}}}$ induced by swapping t_1 and t_2. The point is then that these correspond to the two realisations of $\widehat{\mathfrak{g}}$. Indeed, the algebra $\hat{\hat{\mathfrak{g}}}$ is generated by

$$x^-_{\alpha,n}, h_{\alpha,n}, x^+_{\alpha,n}, \qquad\qquad \alpha \;=\; \alpha_0, \alpha_1, ..., \alpha_r, \quad n \;\in\; \mathbf{Z}$$

with relations similar to the above (see [MRY, §3]). Ignoring α_0, or ignoring all nonzero n, gives the two copies of $\widehat{\mathfrak{g}}$ inside $\hat{\hat{\mathfrak{g}}}$, whose intersection in $\mathfrak{g}$.

4.2.6. **Quantum analogues.** The whole above story can be q-deformed. There are two subalgebras whose intersection is the Drinfeld Jimbo quantum group

$$\begin{tikzcd} U_q(\mathfrak{g}) \arrow[r] \arrow[d] & U_q(\widehat{\mathfrak{g}}) \arrow[d] \\ U_q(\widehat{\mathfrak{g}}) \arrow[r] & U_q(\hat{\hat{\mathfrak{g}}}) \end{tikzcd} \tag{4.11}$$

See [FJW2, GKV, He].

4.2.7. We will instead describe the formal case, where we have an intersection of topological algebras over $\mathbf{C}[[\hbar]]$

$$\begin{array}{ccc} U_\hbar(\mathfrak{g}) & \longrightarrow & U_\hbar(\hat{\mathfrak{g}}) \\ \downarrow & & \downarrow \\ U_\hbar(\hat{\mathfrak{g}}) & \longrightarrow & U_\hbar(\hat{\hat{\mathfrak{g}}}) \end{array}$$

which contains (4.11). Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. The *quantum toroidal algebra* $U_\hbar(\hat{\hat{\mathfrak{g}}})$ is the topological algebra over $\mathbf{C}[[\hbar]]$ generated by

$$c^{\pm 1}, h_{i,n}, x^{\pm}_{i,n} \qquad\qquad i \;=\; 0,1,...,r, \quad n \in \mathbf{Z},$$

with $c^{\pm 1}$ central, subject to

$$\begin{aligned} [h_{i,n}, h_{j,m}] &= \delta_{n,-m}\frac{1}{n}[nA_{ij}]_{q_i}\frac{c^{2n}-c^{-2n}}{q_j - q_j^{-1}} \\ [h_{i,0}, x^{\pm}_{j,m}] &= \pm A_{ij} x^{\pm}_{j,m} \\ [h_{i,n}, x^{\pm}_{j,m}] &= \frac{1}{n}[nA_{ij}]_{q_i} c^{\mp|n|} x^{\pm}_{j,n+m} \\ x^{\pm}_{i,n+1}x^{\pm}_{j,m} - q_i^{\pm A_{ij}} x^{\pm}_{j,m}x^{\pm}_{i,n+1} &= q^{\pm A_{ij}} x^{\pm}_{i,n}x^{\pm}_{j,m+1} - x^{\pm}_{j,m+1}x^{\pm}_{i,n} \\ [x^{+}_{i,n}, x^{-}_{j,m}] &= \delta_{i,j}\frac{c^{(n-m)}\phi^{+}_{i,n+m} - c^{-(n-m)}\phi^{-}_{i,n+m}}{q_i - q_i^{-1}} \end{aligned}$$

along with the quantum Serre relations

$$\sum_{\pi\in\Sigma_m}\sum_{k=0}^{m}(-1)^k \binom{m}{k}_{q_i} x^{\pm}_{i,r_{\pi(i)}} \cdots x^{\pm}_{k,r_{\pi(k)}} x^{\mp}_{j,m} x^{\pm}_{k,r_{\pi(k+1)}} \cdots x^{\pm}_{m,r_{\pi(m)}}$$

where $m = 1 - A_{ij}$ and $r_1, ..., r_m$ is any sequence of integers. Here,

$$\phi^{\pm}_i(z) \;=\; \sum_{k\geqslant 0}\phi^{\pm}_{i,\pm k}z^{\pm k} \;=\; \exp\left(\hbar h_i \pm (q_i - q_i^{-1})\sum_{\ell\geqslant 0} h_{i,\pm\ell}z^{\pm\ell}\right).$$

As before, $q_i = \exp(d_i\hbar)$. In the notation of [CP], we have replaced c by $2c$. See also [He] for a discussion of the rational form.

4.3 Moduli stack of derived quiver representations

4.3.1. Recall from section 3.12 that for any finite quiver Q, the moduli stack of finite dimensional representations of Q can be written as a total space

$$\mathcal{M}^{\heartsuit}_Q \;=\; \coprod_{d\in\mathbf{N}^{|Q|}} \prod\nolimits_{e:p\to q}\mathcal{H}\mathrm{om}(\gamma_p^{\heartsuit,\otimes d_p}, \gamma_q^{\heartsuit,\otimes d_q})^{a_{pq}} \;\twoheadrightarrow\; \prod\nolimits_q \mathrm{BGL}_{d_q} \tag{4.12}$$

where $\gamma_q^\heartsuit$ is the tautological rank d_q vector bundle over BGL_{d_q}, which we have identified with its pullback along $\prod_q \mathrm{BGL}_{d_q} \to \mathrm{BGL}_{d_q}$. Here a_{pq} is the number of edges $e : p \to q$ in Q.

4.3.2. We now turn to what should be the moduli stack of the *derived* category $D^b(\mathrm{Rep}Q)$ of quiver representations. To begin with, attached to each element of the category is a *virtual* multidimension vector $d \in \mathbf{Z}^{|Q|}$. To form the moduli stack, the correct analogue of BGL_{d_q} is Perf_{d_q}, the higher stack parametrising perfect complexes, see [To]. It carries a universal perfect complex γ_{d_q} of rank d_q, and so we define the moduli stack of objects in the derived category $D^b(\mathrm{Rep}Q)$ to be the total space

$$\mathcal{M}_Q \;=\; \coprod_{d\in\mathbf{Z}^{|Q|}} \prod_{e:p\to q} \mathcal{H}\mathrm{om}(\gamma_p^{\otimes d_p}, \gamma_q^{\otimes d_q})^{a_{pq}} \;\twoheadrightarrow\; \prod_q \mathrm{Perf}_{d_q}. \tag{4.13}$$

4.3.3. One can show that for each $n \in \mathbf{Z}$ the cohomology[2]

$$\mathrm{H}^\bullet(\mathrm{Perf}_n) \;=\; k[c_1, c_2, \dots\,]$$

is generated by the chern classes $c_i = c_i(\gamma)$ of the tautological Perfect complex γ on the higher Artin stack Perf_n, see e.g. [To]. Moreover, because the fibres of the map (4.13) are contractible, we have that

$$\mathrm{H}^\bullet(\mathcal{M}_Q) \;=\; \bigoplus_{d\in\mathbf{Z}^{|Q|}} \otimes_{q\in|Q|} k[c_{1,q}, c_{2,q}, \dots\,],$$

is freely generated by chern classes of pullbacks of tautological perfect complexes on $\prod_q \mathrm{Perf}_{d_q}$.

4.3.4. Note that these constructions are functorial in the quiver: if $Q \to Q'$ is a map of quivers, we can restrict (derived) representations of Q' to Q and so getting maps of moduli stacks

$$\mathcal{M}_{Q'}^\heartsuit \;\to\; \mathcal{M}_Q^\heartsuit \qquad\qquad \mathcal{M}_{Q'} \;\to\; \mathcal{M}_Q.$$

The maps (4.12) and (4.13) come from taking the inclusion $|Q| \to Q$ of the quiver with the same vertices as Q and no edges.

4.3.5. Maps into $\mathcal{M}_Q$ classify a perfect complex for each vertex of Q, and a map between the associated perfect complexes for each edge of Q. This is a perfect complex analogue of Q representation bundle. Thus $\mathcal{M}_Q$ carries tautological perfect complexes, also denoted γ_q, and maps between them $\varphi_{p,q} : \gamma_p \to \gamma_q$ for each edge.

[2]Remember that cohomology $\mathrm{H}^\bullet(-)$ is defined for any higher Artin stack, see section 2.7.

4.3.6. **Vertex algebra structure.** Both the spaces $\mathcal{M}_Q^\heartsuit$ and $\mathcal{M}_Q$ carry the structures in section 2.6, so making their homologies into vertex algebras. We spell this out for $\mathcal{M}_Q$. The point

$$0 \;:\; \mathrm{pt} \;\to\; \mathcal{M}_Q$$

is given by the zero perfect complex, i.e. defined by the pullbacks of all tautological perfect complexes being zero. The commutative monoid structure

$$\oplus \;:\; \mathcal{M}_Q \times \mathcal{M}_Q \;\to\; \mathcal{M}_Q$$

is defined by $\oplus^*\gamma_q \;=\; \gamma_q \boxplus \gamma_q$ and $\oplus^*\varphi_{p,q} = \varphi_{p,q} \boxplus \varphi_{p,q}$. The $\mathrm{B}\mathbf{G}_m$ action

$$\mathrm{act} \;:\; \mathrm{B}\mathbf{G}_m \times \mathcal{M}_Q \;\to \mathcal{M}_Q$$

is defined by $\mathrm{act}^*\gamma_q = \gamma \boxtimes \gamma_q$ and $\mathrm{act}^*\varphi_{p,q} = \mathrm{id} \boxtimes \varphi_{p,q}$, where γ is the tautological line bundle over $\mathrm{B}\mathbf{G}_m$. Thus the tautological perfect complexes on $\mathcal{M}_Q$ all have weight one with respect to the $\mathrm{B}\mathbf{G}_m$ action. The perfect complex

$$\theta \;\in\; \mathrm{Perf}(\mathcal{M}_Q \times \mathcal{M}_Q)$$

is defined as in section 3.8.12 as the symmetrisation $\theta = \overline{\theta} \oplus \sigma^*\overline{\theta}^\vee$ of the Ext complex $\overline{\theta}$, which is a cone

$$\overline{\theta} \;\to\; \textstyle\prod_q \mathcal{H}\mathrm{om}(\gamma_{q,1}, \gamma_{q,2}) \;\to\; \textstyle\prod_{e:p\to q} \mathcal{H}\mathrm{om}(\gamma_{p,1}, \gamma_{q,2}) \;\xrightarrow{+1}$$

of the map sending $(f_q)_q \mapsto \big(\rho_{\gamma_q,q} f_p - f_q \rho_{V,e}\big)_{e:p\to q}$. Here the subscript i refers to pullback with respect to the ith projection $\mathcal{M}_Q \times \mathcal{M}_Q \twoheadrightarrow \mathcal{M}_Q$. In particular, as a K theory class

$$[\overline{\theta}] \;=\; \bigoplus_{p,q\in|Q|} (\delta_{p,q} - a_{p,q})[\gamma_{p,1}^\vee \otimes \gamma_{q,2}]$$

as an element of $\mathrm{K}(\mathcal{M}_Q \times \mathcal{M}_Q)$, as well as

$$[\theta] \;=\; \bigoplus_{p,q\in|Q|} (2\delta_{p,q} - a_{p,q} - a_{q,p})[\gamma_{p,1}^\vee \otimes \gamma_{q,2}].$$

4.3.7. The connected components of $\mathcal{M}_Q$ are labelled by the lattice $\check{\Delta} \simeq \mathbf{Z}^{|Q|}$, and the symmetrised Euler form defines on it a bilinear form

$$\kappa \;:\; \check{\Delta} \times \check{\Delta} \;\to\; \mathbf{C} \qquad\qquad \kappa(\alpha, \beta) \;=\; \mathrm{rk}(\theta|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta}).$$

When Q is a Dynkin quiver these are the coroot lattice and normalised Killing form, and its values on simple roots give the Cartan matrix: $\kappa(\alpha_i, \alpha_j) = A_{ij}$. Finally, as noted in [FK] there is a unique two cocycle coming from the central extension

$$0 \;\to\; \{\pm 1\} \;\to\; \tilde{\check{\Delta}} \;\to\; \check{\Delta} \;\to\; 0$$

such that the commutator of $\alpha, \beta \in \check{\Delta}$ is

$$\alpha\beta\alpha^{-1}\beta^{-1} \;=\; (-1)^{\kappa(\alpha,\beta)}.$$

Note that this defines an orientation as in section 2.6.20 because $\kappa(\alpha, \alpha)$ is even for all $\alpha \in \check{\Delta}$. Thus, we fix the choice of this orientation for the rest of the chapter.

Proposition 4.3.8. *Joyce's vertex algebra structure on* $\mathrm{H}_\bullet(\mathcal{M}_Q)$ *attached to the above data is the lattice vertex algebra attached to the lattice* $(\check{\Delta}, \kappa)$*.*

Proof. Arguing just as in section 2.6.25 we can compute the vertex algebra structure on the homology of the zero connected component to be the vertex algebra

$$\mathrm{H}_\bullet(\mathcal{M}_{Q,0}) \;\simeq\; V_1(\widehat{\mathfrak{t}}) \qquad\qquad \mathfrak{t} \;=\; \check{\Delta} \otimes_{\mathbf{Z}} \mathbf{C}.$$

The same computation shows that the $\widehat{\mathfrak{t}}$ action on $\mathrm{H}_\bullet(\mathcal{M}_{Q,\lambda})$ gives it the structure of a level one weight λ representation of $\widehat{\mathfrak{t}}$. In the rank one case this follows because in the notation of section 2.6.25, $Y(\mathrm{ch}_1^\vee, z)1_{\mathcal{M}_{Q,n}} = z^{\langle\lambda,\lambda\rangle}\mathrm{ch}_1(\gamma|_{\mathcal{M}_{Q,n}})^\vee + \cdots$, and the higher rank cases follow similarly. As discussed in section 2.3.10 uniquely determines $\mathrm{H}_\bullet(\mathcal{M}_Q)$ to be the lattice vertex algebra attached to $(\check{\Delta}, \kappa)$. $\square$

4.4 Free field realisations

4.4.1. One useful way to work with (vertex) algebras is using generators and relations, realising them as a quotient of something simpler. Free field realisations are in a sense dual to this: to work with a (vertex) algebra, realise it as a subalgebra of something simpler. This simpler vertex algebra is often a lattice vertex algebra, which are sometimes referred to in the physics literature as free fields.

4.4.2. The prototypical "geometric" example of free field realisations is when one has a sheaf of algebras $\mathcal{A}$ over X, then the restriction to an open

$$\mathcal{A}(X) \;\to\; \mathcal{A}(U)$$

is often often injective, and if the geometry of U much simpler than X then $\mathcal{A}(U)$ itself tends to also be simpler.

For instance, let G be a complex algebraic group with Borel subgroup B, and $\mathfrak{g}$ its Lie algebra. Then $U(\mathfrak{g})$ acts as differential operators on the flag variety G/B, and Beilinson and Bernstein localisation [BB] says that restricting to the big cell $U \simeq \mathbf{A}^n$ gives an injection

$$U(\mathfrak{g})_0 \;=\; \mathcal{D}(G/B) \;\hookrightarrow\; \mathcal{D}(\mathbf{A}^n) \;\simeq\; \mathbf{C}\langle x_1, ..., x_n, \partial_{x_1}, ..., \partial_{x_n}\rangle$$

from (a central quotient of) the universal envloping algebra, realised as differential operators on the flag variety, into the Weyl algebra.

4.4.3. This can be generalised to the *Wakimoto* free field realisation due to Wakimoto, B. Feigin and E. Frenkel [Wa, FF]. Replacing differential operators with chiral differential operators, they obtain at critical level maps of vertex algebras

$$V_{\kappa_{crit}}(\mathfrak{g}) \;\twoheadrightarrow\; \mathcal{D}_{ch}(G/B) \;\hookrightarrow\; \mathcal{D}_{ch}(U) \;\simeq\; \mathcal{D}_{ch}(\mathbf{A}^n)$$

and the kernel of the first map is also given by a central character, see [AMa], giving as image $L_{\kappa_{crit}}(\mathfrak{g})$. They also deform this map to non-critical level.

4.4.4. **FKS isomorphism.** The *Frenkel-Kac-Segal* [FK, Se1] free field realisations of affine vertex algebras we will be considering in this section are different. Firstly, they are defined at level *one* rather than at critical level, second, they are only defined for ADE type Lie algebras, and third, the free field vertex algebra in question is a lattice vertex algebra rather than the Wakimoto module. The simplest version gives an isomorphism

$$L_1(\mathfrak{sl}_2) \;\xrightarrow{\sim}\; V_{\sqrt{2}\mathbf{Z}}$$

expressing the lattice vertex algebra $V_{\sqrt{2}\mathbf{Z}}$ as the simple affine vertex algebra $L_1(\mathfrak{sl}_2)$.

4.4.5. Let $\mathfrak{g}$ denote a finite dimensional Lie algebra of ADE type, $\check{\Delta}$ its root lattice, $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra and $\{\alpha_i\} \subseteq \check{\Delta}$ a basis of simple roots. Let κ be the normalised invariant bilinear form on $\mathfrak{g}$, giving the basis $\{h_i\} \subseteq \mathfrak{h}$ of the coroot lattice Δ dual to the simple roots.

We can make the above choices so that this forms a part of a Chevalley basis $\{x_i^{\pm}, h_i\} \subseteq \mathfrak{g}$ of the algebra $U(\mathfrak{g})$, which we assume from now on.

4.4.6. Denote by $V_{\check{\Delta},\kappa}$ the resulting lattice vertex algebra. As a $\widehat{\mathfrak{t}}$ module,

$$V_{\check{\Delta},\kappa} \;\simeq\; V_{\mathfrak{t},\kappa} \otimes \mathbf{C}[\check{\Delta}]$$

where the $\widehat{\mathfrak{t}}$ action on the second component is trivial. There are three families of endomorphisms acting on this vector space:

- h_n for $h \in \mathfrak{t}$ and $n \in \mathbf{Z}$, defined by the $\widehat{\mathfrak{t}}$ action.
- e^α for $\alpha \in \check{\Delta}$, defined by the group algebra structure on $\mathbf{C}[\check{\Delta}]$.
- ∂_α for for $\alpha \in \check{\Delta}$, defined on $\mathbf{C}[\check{\Delta}]$ by $\partial_\alpha e^\beta = \kappa(\alpha, \beta) e^\beta$ and extended trivially to $V_{\check{\Delta},\kappa}$.

4.4.7. The main result of [FJW1] was an explicit description of this vertex algebra.

Theorem 4.4.8. [FJW1, 7.3] *There is a surjection of vertex algebras*

$$\pi \ : \ V_1(\mathfrak{g}) \ \to \ V_{\check{\Delta},\kappa}.$$

inducing an isomorphism $L_1(\mathfrak{g}) \xrightarrow{\sim} V_{\check{\Delta},\kappa}$.

Proof. For any element of $\alpha \in \check{\Delta} \subseteq \mathfrak{t}$, we get an endomorphism valued power series by

$$Y_\alpha^{\pm}(z) \ := \ \check{\varepsilon}_{\alpha,-} \exp\left(\sum_{n\geqslant 1} \frac{1}{n}\alpha_{-n} z^n\right) \exp\left(-\sum_{n\geqslant 1}\frac{1}{n}\alpha_n z^{-n}\right) e^\alpha z^{\partial_\alpha}. \tag{4.14}$$

To define the map π, it is enough to dictate where the generating fields attached to the Chevalley basis are sent to:

$$x_i^{\pm}(z) \ \mapsto \ Y_{\alpha_i}^{\pm}(z), \quad h_i(z) \ \mapsto \ h_i(z),$$

where $h(z) = \sum_{n \in \mathbf{Z}} h_n z^{-n-1}$. To show that this is a map of vertex algebras, it is enough to show that these fields also generate $V_{\check{\Delta},\kappa}$ and that π preserves the operator product expansions: this is done in [FJW1] where they compute the operator product expansions of these fields explicitly. It is then clear that π is surjective, and comparing characters (e.g. using the Weyl-Kac character formula) gives the isomorphism to $L_1(\mathfrak{g})$. □

Thus the action of $U(\widehat{\mathfrak{t}})$ on $V_{(\check{\Delta},\kappa)}$ extends to an action of $U(\widehat{\mathfrak{g}})$.

4.4.9. Note that in [FJW1] the signs $\check{\varepsilon}$ are incorporated into their definition [FJW1, (7.1)] of the toroidal Lie algebra $\hat{\hat{\mathfrak{g}}}$, whereas we incorporate them into the fields (4.14). Our above definition of $\hat{\hat{\mathfrak{g}}}$ has $\mathfrak{g}[t_1^{\pm 1}, t_2^{\pm 1}]$ as a quotient, which would not be the case for the definition in [FJW1].

4.4.10. **Geometric interpretation.** The geometric origin of the FKS isomorphism was discovered by Zhu in [X.Zh]. Let G be a simply connected algebraic group of ADE type and T a maximal torus. This gives a closed embedding on the associated Beilinson Drinfeld Grassmannians

$$\begin{tikzcd} \mathrm{Gr}_{T,X} \arrow[rr, "i"] \arrow[dr, "p_T"'] & & \mathrm{Gr}_{G,X} \arrow[dl, "p_G"] \\ & \mathrm{Ran}X & \end{tikzcd}$$

where as usual X is a smooth curve over a field of characteristic zero. By [X.Zh, 3.3.1] their determinant bundles are related by

$$i^*\mathcal{L}_G \;\simeq\; \mathcal{L}_T, \qquad\qquad i_*\mathcal{L}_T \;\simeq\; \mathcal{L}_G \otimes \mathcal{O}_{\mathrm{Gr}_{T,X}}.$$

The FKS isomorphism will then be induced by the unit of the adjunction

$$\mathcal{L}_G \;\to\; i_*i^*\mathcal{L}_G = \mathcal{L}_G \otimes \mathcal{O}_{\mathrm{Gr}_{T,X}}. \tag{4.15}$$

Indeed, proceeding similarly to in section 2.5, it is noted in [X.Zh] that the simple affine vertex algebra attached of level k attached to G and the lattice vertex algebra attached to $\check{\Delta}$ is

$$\overline{\mathcal{A}}_G \;=\; (p_{G*}\mathcal{L}_G^{\otimes k})^\vee, \qquad\qquad \mathcal{A}_{\check{\Delta}} \;=\; (p_{T*}\mathcal{L}_T)^\vee \;=\; (p_{G*}(\mathcal{L}_G \otimes \mathcal{O}_{\mathrm{Gr}_{T,X}}))^\vee.$$

Thus the unit adjunction (4.15) gives a map of factorisation algebras

$$\mathcal{A}_{\check{\Delta}} \;\to\; \overline{\mathcal{A}}_G, \tag{4.16}$$

and so a map of vertex algebras $V_{\check{\Delta},\kappa} \to L_1(\mathfrak{g})$. Notice that this is in the other direction that might be expected. This map is the FKS isomorphism:

Theorem 4.4.11. [X.Zh, 3.3.2] *The map (4.16) is an isomorphism of factorisation algebras.*

4.5 Moduli interpretation

In this section we begin by giving a moduli space interpretation of the FKS isomorphism (Theorem 4.4.8), as a warm up to the q-deformed case in what follows. Let Q be an ADE quiver attached to Lie algebra $\mathfrak{g}$, with Cartan matrix A_{ij}. Reserve the letter d for a virtual multidimension vector $d \in \mathbf{Z}^{|Q|}$.

4.5.1. As an algebra under cup product, the cohomology of $\mathcal{M}_d = \mathcal{M}_{Q,d}$ is freely generated by chern characters

$$\mathrm{H}^\bullet(\mathcal{M}_d) \;=\; k[\{\mathrm{ch}_\ell(\gamma_i|_{\mathcal{M}_d})\}]$$

where ℓ ranges over positive integers and i the simple roots.

4.5.2. Turning to homology, consider the dual classes $\sigma^d_{\ell,i} \in \mathrm{H}_\bullet(\mathcal{M}_d) = \mathrm{H}^\bullet(\mathcal{M}_d)^\vee$ defined by sending $\mathrm{ch}_\ell(\gamma_i|_{\mathcal{M}_d}) \mapsto 1$ and all other monomials in chern characters to zero. The direct sum map $\oplus : \mathcal{M}^2 \to \mathcal{M}$ includes an algebra structure on homology $\mathrm{H}_\bullet(\mathcal{M})$, which we denote by $\cdot$.

Lemma 4.5.3. *The product* $\prod_{\ell,i}(\sigma^0_{\ell,i})^{n_{\ell,i}}$ *sends*

$$\prod \mathrm{ch}_\ell(\gamma_i|_{\mathcal{M}_0})^{n_{\ell,i}} \;\mapsto\; \prod n_{\ell,i}!$$

and kills all other monomials in chern characters.

Proof. The product of $\sigma^0_{\ell_1,i_1},...,\sigma^0_{\ell_n,i_n}$ is the element of $\mathrm{H}^\bullet(\mathcal{M}_0)^\vee$ given by

$$\mathrm{H}^\bullet(\mathcal{M}_0) \stackrel{\oplus^*}{\to} \mathrm{H}^\bullet(\mathcal{M}_0)^{\otimes n} \stackrel{\otimes\sigma^0_{\ell_k,i_k}}{\to} k^{\otimes n} \simeq k.$$

Thus the lemma follows by additivity of chern characters under direct sum. □

Corollary 4.5.4. *The subalgebra* $\mathrm{H}_\bullet(\mathcal{M}_0) \subseteq \mathrm{H}_\bullet(\mathcal{M})$ *is freely generated by the* $\sigma^0_{\ell,i}$*, where* ℓ *ranges over positive integers and* i *the simple roots.*

4.5.5. The cup product and direct sum map combine to give a bialgebra structure on $\mathrm{H}^\bullet(\mathcal{M})$, as in section 2.6. Together with the antipode given by taking monomial degree, we get that $\mathrm{H}^\bullet(\mathcal{M})$ is a Hopf algebra, and if we write

$$\widehat{\mathfrak{t}}_+ \hookrightarrow \mathrm{H}^\bullet(\mathcal{M})$$

for the vector subspace generated by chern characters $\mathrm{ch}_\ell(\gamma_i)$, the above results then imply

Proposition 4.5.6. $\mathrm{H}^\bullet(\mathcal{M}_0) \simeq U(\widehat{\mathfrak{t}}_+)$ *as graded Hopf algebras.*

Dually, writing

$$\widehat{\mathfrak{t}}_- \hookrightarrow \mathrm{H}_\bullet(\mathcal{M}_0)$$

for the vector subspace generated by the dual chern characters $\sigma^0_{\ell,i}$, we have that that $\mathrm{H}_\bullet(\mathcal{M}_0) \simeq U(\widehat{\mathfrak{t}}_-)$ as Hopf algebras. It follows from this, or from an explicit computation, that

Corollary 4.5.7. *The subspace* $\widehat{\mathfrak{t}}_+ \subseteq \mathrm{H}^\bullet(\mathcal{M}_0)$ *(likewise* $\widehat{\mathfrak{t}}_- \subseteq \mathrm{H}_\bullet(\mathcal{M}_0)$*) consists of primitive elements.*

4.5.8. **Heisenberg algebra action.** As for any Hopf algebra, any element $\rho \in \mathrm{H}_\bullet(\mathcal{M}_0)$ of its dual defines an endomorphism

$$\rho^\vee : \mathrm{H}^\bullet(\mathcal{M}_0) \stackrel{\oplus^*}{\to} \mathrm{H}^\bullet(\mathcal{M}_0) \otimes \mathrm{H}^\bullet(\mathcal{M}_0) \stackrel{\rho\otimes\mathrm{id}}{\to} k \otimes \mathrm{H}^\bullet(\mathcal{M}_0) \simeq \mathrm{H}^\bullet(\mathcal{M}_0).$$

This makes $\mathrm{H}^\bullet(\mathcal{M}_0)$, actually all of $\mathrm{H}^\bullet(\mathcal{M})$, into a module for $\mathrm{H}_\bullet(\mathcal{M}_0)$. Since the Hopf algebra is cocommutative it does not matter whether we acted by ρ on the right or left. Moreover, from Lemma 2.6.10, if $\rho \in \mathrm{H}_\bullet(\mathcal{M}_0)$ is a primitive element $\rho^\vee$ is a derivation.

4.5.9. Similarly, ρ defines an endomorphism of any Hopf module, for instance $\mathrm{H}^\bullet(\mathcal{M})$, and we also denote this by $\rho^\vee$. If the Hopf module is a Hopf algebra as in the case $\mathrm{H}^\bullet(\mathcal{M})$, $\rho^\vee$ is again a derivation.

4.5.10. We can use this to get an action of the Heisenberg algebra on $\mathrm{H}^\bullet(\mathcal{M}_0)$, identifying it with the Heisenberg Lie algebra by

$$\mathbf{C}[t,t^{-1}] \;=\; \mathbf{C}[t]\oplus t^{-1}\mathbf{C}[t^{-1}] \;\simeq\; \hat{\mathfrak{t}}_+\oplus\hat{\mathfrak{t}}_- \;=\; \hat{\mathfrak{t}}.$$

Note that $\hat{\mathfrak{t}}_\pm$ are abelian Lie subalgebras of $\hat{\mathfrak{t}}$. Now,

Proposition 4.5.11. $\mathrm{H}^\bullet(\mathcal{M})$ *is a representation of the Heisenberg algebra* $\hat{\mathfrak{t}}$ *of level one, under the identifications*

$$h_{i,-n-1} \;=\; \frac{1}{n!}{\sigma^0_{n+1,i}}^\vee, \quad h_{i,n} \;=\; \sum_j A_{ij}n!\mathrm{ch}_n(\gamma_j) \tag{4.17}$$

for $n\geqslant 0$.

Proof. We need to show that the generators (4.17) satisfy the commutation relations of the Heisenberg algebra

$$[h_{i,n},h_{j,m}] \;=\; n\delta_{n,-m}A_{ij}\mathrm{id}$$

where $n,m\in\mathbf{Z}$. This is implied by the Weyl relations $[\sigma^{0,\vee}_{k,i},\mathrm{ch}_\ell(\gamma_j)] \;=\; \delta_{i,j}\delta_{k,\ell}\mathrm{id}$. The Weyl relations themselves follow because

$$\sigma^0_{k,i}(\mathrm{ch}_\ell(\gamma_j)) \;=\; \delta_{i,j}\delta_{k,\ell}$$

and because as the $\sigma^0_{k,i}$ are primitive, they define derivations on $\mathrm{H}^\bullet(\mathcal{M})$. $\square$

Moreover, $\mathrm{H}^\bullet(\mathcal{M}_d)\simeq V_\lambda$ is the Verma module of weight $\lambda = 2\sum d_i\alpha_i$.

4.5.12. We make a note of what structures the $\mathrm{B}\mathbf{G}_m$ action gives rise to. Begin by noting that as a Hopf algebra $\mathrm{H}_\bullet(\mathrm{B}\mathbf{G}_m)\simeq U(\mathbf{C}\tau)$ is the symmetric algebra on one generator. Moreover, the Hopf algebra $\mathrm{H}^\bullet(\mathcal{M})$ is a Hopf module for $\mathrm{H}_\bullet(\mathrm{B}\mathbf{G}_m)$, so in particular $\tau^\vee:\mathrm{H}^\bullet(\mathcal{M})\to\mathrm{H}^\bullet(\mathcal{M})$ is a derivation. Moreover,

Lemma 4.5.13. $\tau^\vee\mathrm{ch}_\ell(\gamma_i)=\mathrm{ch}_{\ell-1}(\gamma_i)$ *and* $\tau\sigma^0_{k,i}=\sigma^0_{\ell+1,i}$ *when* $\ell>0$.

Proof. The definition of $\tau^\vee$ is

$$\mathrm{H}^\bullet(\mathcal{M}) \;\overset{\mathrm{act}^*}{\to}\; \mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)\otimes\mathrm{H}^\bullet(\mathcal{M}) \;\overset{[c_1(\gamma_1)]\otimes\mathrm{id}}{\to}\; k\otimes\mathrm{H}^\bullet(\mathcal{M}) \;\simeq\; \mathrm{H}^\bullet(\mathcal{M})$$

where γ_1 is the tautological line bundle on $\mathrm{B}\mathbf{G}_m$ and $[c_1(\gamma_1)] \in \mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)^\vee$ takes the $c_1(\gamma_1)$ coefficient of a cohomology class. Thus the first claim holds since $\mathrm{act}^*\gamma_i = \gamma_1 \boxtimes \gamma_i$, which then implies the second by duality. □

4.5.14. **Free field realisation.** There is an isomorphism

$$\mathfrak{t} \ \xrightarrow{\sim}\ \mathrm{H}_2(\mathcal{M}_0)$$

sending h_i to $\sigma^0_{1,i}$.

Lemma 4.5.15. *For any $\beta \in \mathrm{H}_\bullet(\mathcal{M})$*

$$\mathrm{ch}_k(\theta)\cdot(\sigma^0_{1,i}\otimes\beta) \ = \ \sum_j A_{ij}\cdot(1_0\otimes \mathrm{ch}_{k-1}(\gamma_j)\beta)$$

whenever $k > 0$.

Proof. Recall that as a K theory class on $\mathcal{M}^2$,

$$[\theta] \ = \ \sum_{i,j} A_{ij}[\mathrm{Hom}(\gamma_i,\gamma_j)].$$

The result now follows from additivity of chern characters under direct sums, after noting that $\mathrm{ch}_0(\gamma_i|_{\mathcal{M}_0}) = 0$ implies

$$\mathrm{ch}_k(\mathrm{Hom}(\gamma_i|_{\mathcal{M}_0},\gamma_j))\sigma^0_{1,i}\otimes\beta \ = \mathrm{ch}_k(\gamma_i^\vee|_{\mathcal{M}_0}\boxtimes\gamma_j)\sigma^0_{1,i}\otimes\beta \ = \ 1_0\otimes\mathrm{ch}_{k-1}(\gamma_j)\beta.$$

□

We can now compute the field in Joyce's vertex algebra structure on $\mathrm{H}_\bullet(\mathcal{M})$

$$\begin{aligned}
\mathrm{Y}(\sigma^0_{1,i},z)\beta \ &= \ \oplus_* \sum_{k\geqslant 0}\frac{z^k t^k}{k!}\exp\left(-\sum_{k\geqslant 1}\frac{(-z)^{-k}}{k}k!\mathrm{ch}_k\theta\right)\sigma^0_{1,i}\otimes\beta \\
&= \ \oplus_* \sum_{k\geqslant 0}\frac{z^k t^k}{k!}\left(\sigma^0_{1,i}\otimes\beta + 1_0\otimes\left(\sum_{k\geqslant 1,j} A_{ij}\frac{(-z)^{-k}}{k}k!\mathrm{ch}_{k-1}(\gamma_j|_{\mathcal{M}_e})\right)\beta\right) \\
&= \ \sum_{k\geqslant 0}\frac{z^k\sigma^0_{k+1,i}}{k!}\cdot\beta + \sum_{k\geqslant 1,j} A_{ij}\frac{(-z)^{-k}}{k}k!\mathrm{ch}_{k-1}(\gamma_j|_{\mathcal{M}_e})\beta.
\end{aligned}$$

4.5.16. Next, writing $\delta_i \in \mathbf{Z}^{Q_0}$ for the dimension vector $(\delta_i)_j := \delta_{i,j}$, we identify

$$\mathrm{H}_0(\mathcal{M}_{\pm\delta_i}) \ \simeq \ \mathfrak{g}_{\pm\alpha_i} \ \hookrightarrow \ \mathfrak{g}$$

under which the Chevalley generators $x_i^\pm \in \mathfrak{g}_{\pm\alpha_i}$ are identified with $1_{\pm i}$, the homology class dual to the identity cohomology class.

4.5.17. There is an isomorphism

$$e^{\alpha} \ := \ 1_{\alpha} \cdot \ : \ \mathrm{H}_{\bullet}(\mathcal{M}_d) \ \xrightarrow{\sim} \ \mathrm{H}_{\bullet}(\mathcal{M}_{d+\alpha})$$

and its dual map on cohomology, which is also denoted e^{α}.

Lemma 4.5.18. *For any finite set of vertices i_n and positive integers $\ell_n > 0$,*

$$e^{\alpha} \prod \sigma_{\ell_n, i_n}^{d_n} \ = \ \prod \sigma_{\ell_n, i_n}^{d_n+\alpha}.$$

Proof. By additivity of chern characters under direct sum, the dual map on cohomology acts as $e^{\alpha} \prod \mathrm{ch}_{\ell_n}(\gamma_{i_n}|_{\mathcal{M}_{d_n+\alpha}}) = \prod \mathrm{ch}_{\ell_n}(\gamma_{i_n}|_{\mathcal{M}_{d_n}})$, from which the lemma follows. □

It follows from this that

$$[e^{\alpha}, \mathrm{ch}_{\ell}(\gamma_i)] \ = \ \begin{cases} \langle \delta_i, \alpha \rangle & \text{if } \ell = 0 \\ 0 & \text{if } \ell > 0 \end{cases}$$

where $\langle -, - \rangle$ is the bilinear form on $\mathbf{Z}^{|Q|}$ with orthonormal basis δ_i. Moreover, by the associativity of $\oplus$, $(e^{\alpha} A) \cdot B = e^{\alpha}(A \cdot B) = A \cdot (e^{\alpha} B)$. Finally, we have

Proposition 4.5.19. *As* $\mathrm{H}_{\bullet}(\mathcal{M}_0)$ *valued power series,*

$$\exp\left(\sum_{k>0} \frac{\pm \sigma_{k,i}^0}{k!} z^k \right) \ = \ e^{\mp \alpha_i} \exp(zt)\, e^{\pm \alpha_i} 1_0.$$

Proof. We first expand the left hand side. Its z^nth coefficient is the sum of

$$\frac{1}{m_1! \cdots m_r!} \left(\frac{\sigma_{k_1,i}^0}{k_1!} \right)^{m_1} \cdots \left(\frac{\sigma_{k_r,i}^0}{k_r!} \right)^{m_r}$$

summed over all finite sets of positive integers m_j, and pairwise Fdistinct positive integers k_j with $\sum m_j k_j = n$. Thus combining this with lemma 4.5.3, we have

$$\exp\left(\sum_{k>0} \frac{\sigma_{k,i}^0}{k!} z^k \right) \prod \mathrm{ch}_{k_j}(\gamma_i|_{\mathcal{M}_0})^{m_j} \ = \ \left(\tfrac{1}{k_1!}\right)^{m_1} \cdots \left(\tfrac{1}{k_r!}\right)^{m_r} z^{k_1 m_1 + \cdots + k_r m_r}.$$

For the right side, we prepare by noticing that $e^{-\alpha} t^{\vee} e^{\alpha}$ is a derivation of $\mathrm{H}^{\bullet}(\mathcal{M})$ sending

$$e^{-\alpha} t^{\vee} e^{\alpha} \mathrm{ch}_{\ell}(\gamma_i|_{\mathcal{M}_d}) \ = \ \begin{cases} \mathrm{ch}_0(\gamma_i|_{\mathcal{M}_{d+\alpha}}) & \text{if } \ell = 0 \\ \mathrm{ch}_{\ell-1}(\gamma_i|_{\mathcal{M}_d}) & \text{if } \ell > 0 \end{cases}$$

Thus writing $n = \sum m_j k_j$, this observation allows us to compute

$$\begin{aligned} e^{-\alpha_i} \exp(zt)\, e^{\alpha_i} 1_0 \cdot \mathrm{ch}_{k_j}(\gamma_i|_{\mathcal{M}_0})^{m_j} \ &= \ 1_0 \cdot e^{+\alpha_i} \exp(zt^{\vee})\, e^{-\alpha_i} \mathrm{ch}_{k_j}(\gamma_i|_{\mathcal{M}_0})^{m_j} \\ &= \ 1_0 \cdot \frac{1}{n!} (e^{+\alpha_i} z t^{\vee} e^{-\alpha_i})^n \mathrm{ch}_{k_j}(\gamma_i|_{\mathcal{M}_0})^{m_j}; \end{aligned}$$

since $\mathrm{ch}_0(\gamma_i|_{\mathcal{M}_{\delta_i}}) = 1$, it follows that this is also equal to $\left(\frac{1}{k_1!}\right)^{m_1} \cdots \left(\frac{1}{k_r!}\right)^{m_r} z^{k_1 m_1 + \cdots + k_r m_r}$. $\square$

We can now compute the field of $1_{\pm i}$ as

$$\begin{aligned}
\left(\check{\varepsilon}_{\pm\alpha_i, b} z^{\pm\kappa(\alpha_i, b)}\right)^{-1} \cdot \mathrm{Y}(1_{\pm i}, z)\beta &= \oplus_* \sum_{k \geqslant 0} \frac{z^k t^k}{k!} \exp\left(\sum_{k \geqslant 1} \frac{z^{-k}}{k} k! \mathrm{ch}_k \theta\right)(1_{\pm i} \otimes \beta) \\
&= 1_{\pm i} \cdot \exp\left(\sum_{k \geqslant 1, j} A_{ij} \frac{z^{-k}}{k} k! \mathrm{ch}_k \gamma_j\right)\beta \\
&= e^{\pm\alpha_i} e^{\mp\alpha_i} e^{zt} e^{\pm\alpha_i} 1_0 \cdot \exp\left(\sum_{k \geqslant 1, j} A_{ij} \frac{z^{-k}}{k} k! \mathrm{ch}_k \gamma_j\right)\beta \\
&= \exp\left(e^{\mp\alpha_i} z t e^{\pm\alpha_i}\right) 1_0 \cdot \exp\left(\sum_{k \geqslant 1, j} A_{ij} \frac{z^{-k}}{k} k! \mathrm{ch}_k \gamma_j\right) e^{\pm\alpha_i}\beta \\
&= \exp\left(\sum_{k > 0} \frac{\sigma^0_{i,k}}{k!} z^k\right) \exp\left(\sum_{k > 1, j} A_{ij} \frac{z^{-k}}{k} k! \mathrm{ch}_k \gamma_j\right) e^{\pm\alpha_i}\beta
\end{aligned}$$

where we have used the fact that $(e^\alpha A) \cdot B = A \cdot (e^\alpha B)$. Here $\check{\varepsilon} \in \{\pm 1\}$ are the orientations, as in section 2.6.20, and $\beta \in \mathrm{H}_\bullet(\mathcal{M}_b)$.

4.5.20. We summarise what we have shown in a Theorem. Take the identification $V_{\check{\Delta},\kappa} \xrightarrow{\sim} \mathrm{H}_\bullet(\mathcal{M}_Q)$ sending

$$|\alpha\rangle \mapsto 1_\alpha, \qquad \alpha_{i,-1}|0\rangle \mapsto \sigma^0_{1,i} \tag{4.18}$$

where i varies over the simple roots of $\mathfrak{g}$. As before,

Theorem 4.5.21. *Compose the identification (4.18) with the FKS isomorphism. The resulting isomorphism of vertex algebras $L_1(\mathfrak{g}) \xrightarrow{\sim} \mathrm{H}_\bullet(\mathcal{M}_Q)$ has an explicit description on the level of fields as*

$$\begin{aligned}
x_i^\pm(z) &\mapsto \check{\varepsilon}_{\pm\alpha_i,-} \exp\left(\sum_{k > 0} \frac{\sigma^0_{i,k}}{k!} z^k\right) \exp\left(\sum_{k > 1, j} A_{ij} \frac{z^{-k}}{k} k! \mathrm{ch}_k \gamma_j\right) e^{\pm\alpha_i} z^{\kappa(\pm\alpha_i,-)}, \\
h_i(z) &\mapsto \sum_{k \geqslant 0} \frac{z^k \sigma^0_{k+1,i}}{k!} + \sum_{k \geqslant 1, j} A_{ij} \frac{(-z)^{-k}}{k} k! \mathrm{ch}_{k-1}(\gamma_j).
\end{aligned}$$

4.5.22. **The Cartan involution.** Any automorphism of a lattice Λ induces an automorphism of the vertex algebra V_Λ. In particular, the involution $-\mathrm{id} : \check{\Delta} \to \check{\Delta}$ induces the *Cartan involution* τ on $V_{\check{\Delta}}$, sending

$$\tau \ : \ Y(e^\alpha, z) \mapsto Y(e^{-\alpha}, z), \qquad\qquad \tau \ : \ \alpha(z) \mapsto -\alpha(z)$$

for $\alpha \in \check{\Delta} \subseteq \mathfrak{t}$.

4.5.23. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the Cartan involution $\tau : \mathfrak{g} \to \mathfrak{g}$ induces a vertex algebra involution $\tau : V_k(\mathfrak{g}) \to V_k(\mathfrak{g})$ of the affine vertex algebra, and of its simple quotient $L_k(\mathfrak{g})$. It sends

$$\tau \ : \ x_i^{\pm}(z) \ \mapsto \ x_i^{\mp}(z), \qquad\qquad \tau \ : \ h_i(z) \ \mapsto \ -h_i(z).$$

Then if $\mathfrak{g}$ is of ADE type the maps in the FKS theorem 4.4.8

$$V_k(\mathfrak{g}) \ \to \ L_k(\mathfrak{g}) \ \xrightarrow{\sim} \ V_{\check{\Delta}}$$

are equivariant for the actions of these involutions.

4.5.24. The moduli stack interpretation of this is the following. The shift operator $[1] : D^b(\mathrm{Rep}Q) \to D^b(\mathrm{Rep}Q)$ induces a map $\tau : \mathcal{M} \to \mathcal{M}$ preserving the commutative monoid and $\mathrm{B}\mathbf{G}_m$ action structures and θ, since $\mathrm{Ext}^\bullet(\mathcal{E}[1], \mathcal{F}[1]) = \mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{F})$. Note that τ is *not* an involution, but τ^2 is $\mathbf{A}^1$ homotopic to the identity, so τ induces an involution on (co)homology, see [Jo2]. Thus it induces an involution of the vertex algebra $\mathrm{H}_\bullet(\mathcal{M})$, and upon identifying $\mathrm{H}_\bullet(\mathcal{M}) \simeq V_{\check{\Delta}}$ it corresponds to the Cartan involution.

4.6 Quantum FKS isomorphism

4.6.1. We now review the matter of q-deforming the FKS isomorphism.

The first problem we encounter is what sort of objects a "quantum FKS isomorphism" should be a map between. One would hope that it should be a map between some notion of quantum vertex algebras. In the first work on the subject by Frenkel, Jing and Wang in [FJW2], they essentially only considered it as a map of $U_q(\widehat{\mathfrak{g}})$ modules given by certain power series which they called q vertex operators.

In future work, we plan to interpret these power series as fields in a quantum vertex algebra, using the moduli stack interpretation of the quantum FKS isomorphism which we will explore below.

4.6.2. To begin with, one deforms the $U(\widehat{\mathfrak{t}})$ module structure to a $U_q(\widehat{\mathfrak{t}})$ module structure

$$V_{\check{\Delta},\kappa,q} \ \simeq \ V_{\mathfrak{t},\kappa,q} \otimes \mathbf{C}[\check{\Delta}]$$

where $V_{\mathfrak{t},\kappa,q}$ is the Verma representation of $U_q(\widehat{\mathfrak{t}})$ of level one. Define the endomorphisms h_n, e^α and ∂_α as before in 4.4.6.

Theorem 4.6.3. [FJW2, 8.7] *There is a map of $U_q(\widehat{\mathfrak{g}})$ modules*

$$\pi_q \;:\; V_{1,q}(\mathfrak{g}) \;\to\; V_{\check{\Delta},\kappa,q} \tag{4.19}$$

inducing an isomorphism $L_{1,q}(\mathfrak{g}) \xrightarrow{\sim} V_{\check{\Delta},\kappa,q}$ and specialising at $q = 1$ to the FKS map $\pi_1 = \pi$.

Proof. For any element of $\alpha \in \check{\Delta} \subseteq \mathfrak{t}$ and integer $k \in \mathbf{Z}$ we define an endomorphism valued power series by

$$Y^{\pm}_{\alpha,k}(z) \;:=\; \check{\varepsilon}_{\alpha,-}\exp\left(\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{-n}(q^{(k\mp k)/2}z)^{n}\right)\exp\left(-\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{n}(q^{(k\pm k)/2}z)^{-n}\right)e^{\alpha}z^{\partial_\alpha}.$$

To define the map π_q, it is enough to say where the generating fields attached to the Chevalley basis are sent to:

$$\pi_q \;:\; x^{\pm}_i(z) \;\mapsto\; Y^{\pm}_{\alpha_i,-1}(z), \qquad\qquad \pi_q \;:\; h_{i,n} \;\mapsto\; h_{i,n}.$$

The proof proceeds as in the classical case, and is outlined in [FJW2]. □

4.6.4. To be explicit, we send

$$x^{+}_i(z) \;\overset{\pi_q}{\mapsto}\; \check{\varepsilon}_{\alpha_i,-}\exp\left(\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,-n}z^{n}\right)\exp\left(-\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,n}(q^{-1}z)^{-n}\right)e^{\alpha_i}z^{\partial_{\alpha_i}},$$

$$x^{-}_i(z) \;\overset{\pi_q}{\mapsto}\; \check{\varepsilon}_{-\alpha_i,-}\exp\left(-\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,-n}(q^{-1}z)^{n}\right)\exp\left(\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,n}z^{-n}\right)e^{-\alpha_i}z^{\partial_{-\alpha_i}}.$$

Or to be more symmetric (and removing q factors from the first exponential), we have

$$q^{-\frac{1}{2}\partial_{\alpha_i}}x^{+}_i(z) \;\overset{\pi_q}{\mapsto}\; \check{\varepsilon}_{\alpha_i,-}\exp\left(\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,-n}z^{n}\right)\exp\left(-\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,n}(q^{-1}z)^{-n}\right)e^{\alpha_i}z^{\partial_{\alpha_i}}q^{-\frac{1}{2}\partial_{\alpha_i}}, \tag{4.20}$$

$$q^{-\frac{1}{2}\partial_{-\alpha_i}}x^{-}_i(qz) \;\overset{\pi_q}{\mapsto}\; \check{\varepsilon}_{-\alpha_i,-}\exp\left(-\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,-n}z^{n}\right)\exp\left(\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,n}(qz)^{-n}\right)e^{-\alpha_i}z^{\partial_{-\alpha_i}}q^{-\frac{1}{2}\partial_{-\alpha_i}}. \tag{4.21}$$

4.6.5. **Variant.** We make a note that may be safely skipped. As remarked in [FJW2], note that there is another map which works, defined by

$$x^{\pm}_i(z) \;\overset{\omega\pi_q}{\mapsto}\; Y^{\mp}_{-\alpha_i,1}(z), \quad h_{i,n} \;\overset{\omega\pi_q}{\mapsto}\; h_{i,n}.$$

To be explicit, this sends

$$x^{+}_i(z) \;\overset{\omega\pi_q}{\mapsto}\; \check{\varepsilon}_{-\alpha_i,-}\exp\left(-\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,-n}(qz)^{n}\right)\exp\left(\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,n}z^{-n}\right)e^{-\alpha_i}z^{\partial_{-\alpha_i}},$$

$$x^{-}_i(z) \;\overset{\omega\pi_q}{\mapsto}\; \check{\varepsilon}_{\alpha_i,-}\exp\left(\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,-n}z^{n}\right)\exp\left(-\sum_{n\geqslant 1}\frac{1}{[n]}\alpha_{i,n}(qz)^{-n}\right)e^{\alpha_i}z^{\partial_{\alpha_i}}.$$

Written more symmetrically,

$$q^{-\frac{1}{2}\partial_{-\alpha_i}} x_i^+(q^{-1}z) \overset{\omega\pi_q}{\mapsto} \check{\varepsilon}_{-\alpha_i,-} \exp\left(-\sum_{n\geqslant 1} \frac{1}{[n]} \alpha_{i,-n} z^n\right) \exp\left(\sum_{n\geqslant 1} \frac{1}{[n]} \alpha_{i,n} (q^{-1}z)^{-n}\right) e^{-\alpha_i} z^{\partial_{-\alpha_i}} q^{-\frac{1}{2}\partial_{-\alpha_i}}, \tag{4.22}$$

$$q^{-\frac{1}{2}\partial_{\alpha_i}} x_i^-(z) \overset{\omega\pi_q}{\mapsto} \check{\varepsilon}_{\alpha_i,-} \exp\left(\sum_{n\geqslant 1} \frac{1}{[n]} \alpha_{i,-n} z^n\right) \exp\left(-\sum_{n\geqslant 1} \frac{1}{[n]} \alpha_{i,n} (qz)^{-n}\right) e^{\alpha_i} z^{\partial_{\alpha_i}} q^{-\frac{1}{2}\partial_{\alpha_i}}. \tag{4.23}$$

Note that $\omega\pi_q$ and π_q are *not* dual by the quantum Cartan involution (see below). Rather, they differ by the deformation of the identity $\omega = \mathrm{id} \otimes \check{q}$, where $\check{q}$ is the endomorphism of $\mathbf{C}[q, q^{-1}]$ induced by sending q^n to q^{-n}. We will not consider this variant in the following.

4.6.6. **Quantum Cartan involution.** We expect the quantum Cartan involution should exchange

$$(4.20) \leftrightarrow (4.21).$$

This is achieved sending $\alpha_{i,n} \mapsto -\alpha_{i,n}$, $e^{\alpha_i} \mapsto e^{-\alpha_i}$ and $q \mapsto q^{-1}$. Let us make this more precise.

4.6.7. For a lattice Λ we will define the *quantum Cartan involution* $\tau_q = \tau \otimes \check{q}$ on the vector space $V_{\Lambda,q} = V_\Lambda \otimes_{\mathbf{C}} \mathbf{C}[q, q^{-1}]$, where τ is the usual Cartan involution of section 4.5.22 and $\check{q}$ is the involution of $\mathbf{C}[q, q^{-1}]$ sending $q^n \mapsto q^{-n}$. Thus it sends

$$\tau_q : \lambda(z) \mapsto -\lambda(z), \qquad \tau_q : e^{\lambda(z)} \mapsto e^{-\lambda(z)}, \qquad \tau_q : q^n \mapsto q^n$$

where $\lambda \in \Lambda \subseteq \Lambda \otimes_{\mathbf{Z}} \mathbf{C}$. Note that τ_q fixes the quantum integers $[n]$, and does indeed swap $(4.20), (4.21)$ when applied to the lattice $\Lambda = \check{\Delta}$.

4.6.8. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. So as for the quantum FKS map of Theorem 4.6.3 to be involution equivariant when $\mathfrak{g}$ is of ADE type, the *quantum Cartan involution* τ_q on the vector space $V_{k,q}(\mathfrak{g})$ we *define* so that it swaps the right sides of $(4.20), (4.21)$ if we work in $V_{k,q^{1/2}}(\mathfrak{g})$. We send

$$\tau_q : x_i^+(z) \mapsto q^{\partial_{\alpha_i}} x_i^-(qz), \qquad \tau_q : x_i^-(z) \mapsto q^{\partial_{-\alpha_i}} x_i^+(q^{-1}z),$$

$$\tau_q : h_i(z) \mapsto -h_i(z), \qquad \tau_q : q^n \mapsto q^{-n}.$$

Here for α a root we write ∂_α for the endomorphism of $V_k(\mathfrak{g})$ (and $V_{k,q}(\mathfrak{g})$ when extended q linearly) which multiplies an eigenvector of $[h_{\alpha,0}, -]$ by its eigenvalue. By considering a PBW basis, under the quantum FKS isomorphism of Theorem 4.6.3 this corresponds to ∂_α as in the usual notation for lattice vertex algebras (see section 4.4.6). We expect that the above induces an involution on $L_{k,q}(\mathfrak{g})$. One also expects that it should define a map of q-deformed vertex algebras.

4.7 Moduli interpretation, quantum case

4.7.1. To begin with, recall that we have an isomorphism of $U(\hat{\mathfrak{t}})$ modules

$$V_{\check{\Delta}} \;\simeq\; \mathrm{H}_\bullet(\mathcal{M})$$

where the $U(\hat{\mathfrak{t}})$ module structure on the right is defined using the commutative monoid structure on $\mathcal{M}$ and the cap product action of cohomology on homology.

We want to deform this to an identification of $\mathbf{C}[q,q^{-1}]$ modules

$$V_{\check{\Delta},q} \;\simeq\; \mathrm{H}_\bullet(\mathcal{M})_q \tag{4.24}$$

restricting to isomorphisms $V^\alpha_{\check{\Delta},q} \simeq \mathrm{H}_\bullet(\mathcal{M}_\alpha)_q$ for each $\alpha \in \check{\Delta}$. Such an identification is *equivalent* to the choice of a deformation of the $U(\hat{\mathfrak{t}})$ action on $\mathrm{H}_\bullet(\mathcal{M})$ to a $U_q(\hat{\mathfrak{t}})$ action on $\mathrm{H}_\bullet(\mathcal{M})_q$, such that each $\mathrm{H}_\bullet(\mathcal{M}_\alpha)$ is a level one highest weight Verma module of the appropriate weight.

4.7.2. Thus, to construct an identification (4.24) we make the following construction

Proposition 4.7.3. $\mathrm{H}^\bullet(\mathcal{M})_q$ *is a representation of the quantum Heisenberg algebra* $U_q(\hat{\mathfrak{t}})$ *of level one, under the identifications*

$$\alpha_{i,-n-1} \;=\; \frac{[n+1]}{n+1}\frac{1}{n!}\sigma^{0}_{n+1,i}{}^{\vee}, \quad \alpha_{i,n} \;=\; \frac{[n]}{n}\sum_j \frac{[A_{ij}n]}{[n]}n!\mathrm{ch}_n(\gamma_j) \;=\; \sum_j [A_{ij}n](n-1)!\mathrm{ch}_n(\gamma_j) \tag{4.25}$$

for $n \geqslant 0$.

Proof. We need to show that the generators (4.25) satisfy the commutation relations of the Heisenberg algebra

$$[h_{i,n}, h_{j,m}] \;=\; \delta_{n,-m}\frac{[A_{ij}n]}{n}[n]\mathrm{id}$$

where $n,m \in \mathbf{Z}$. As in the classical case, this is implied by the Weyl relations. □

To be explicit, the induced identification (4.24) sends

$$e^\alpha \;\mapsto\; 1_\alpha$$

and acts on the other modes as (4.25).

4.7.4. We note again the fact that there is an identification $\check{\Delta}\otimes_{\mathbf{Z}}\mathbf{C} \simeq \mathfrak{t}$, so we freely view elements of $\check{\Delta}$ like α_i as elements of $\mathfrak{t}$.

4.7.5. **Quantum Cartan involution.** Having fixed an identification (4.24), what is the induced involution on $\mathrm{H}_\bullet(\mathcal{M})_q = \mathrm{H}_\bullet(\mathcal{M}) \otimes_{\mathbf{C}} \mathbf{C}[q, q^{-1}]$? It follows from equations (4.25) and that e^α and 1_α are interchanged that it is simply $\tau_q = \tau \otimes \check{q}$, where τ is as in section 4.5.22 induced by $[1] : D^b(\mathrm{Rep}Q) \to D^b(\mathrm{Rep}Q)$.

4.7.6. To summarise the last two sections: if $\mathfrak{g}$ is of ADE type, then we get a map of $U_q(\widehat{\mathfrak{t}})$ modules

$$V_{1,q}(\mathfrak{g}) \ \twoheadrightarrow\ V_{\check{\Delta},q} \ \xrightarrow{\sim}\ \mathrm{H}_\bullet(\mathcal{M})_q$$

which is equivariant for the quantum Cartan involution defined on all three vector spaces.

4.8 q-deformed Joyce vertex algebra

4.8.1. In this section we are going to define a q-deformation of the Joyce vertex algebra structure of Theorem 2.6.21. It will be defined on the vector space $\mathrm{H}_\bullet(\mathcal{M})_q = \mathrm{H}_\bullet(\mathcal{M}) \otimes_{\mathbf{C}} \mathbf{C}[q, q^{-1}]$.

4.8.2. The main question is in this q deformed Joyce vertex algebra structure what should be the fields $Y(1_{\pm\alpha_i}, z)$. It follows from Proposition 4.7.3 and Theorem 4.6.3 that

Proposition 4.8.3. *The composite map*

$$V_{1,q}(\mathfrak{g}) \ \twoheadrightarrow\ V_{\check{\Delta},q} \ \xrightarrow{\sim}\ \mathrm{H}_\bullet(\mathcal{M})_q$$

acts on the level of fields as

$$x_i^{\pm}(z) \ \mapsto\ \check{\varepsilon}_{\pm\alpha_i,-}(q^{\pm 1}z)^{\kappa(\alpha_i,-)} \exp\left(\sum_{k>0} \frac{\sigma^0_{i,k}}{k!} z^k\right) \exp\left(\sum_{k\geqslant 1,j} \frac{[A_{ij}k]}{[k]} \frac{z^{-k}}{k} k!\mathrm{ch}_k\gamma_j\right) e^{\pm\alpha_i}, \tag{4.26}$$

$$h_i(z) \ \mapsto\ \sum_{k\geqslant 0} \frac{z^k \sigma^0_{k+1,i}}{k!} \ +\ \sum_{k\geqslant 1,j} \frac{[A_{ij}k]}{[k]} \frac{(-z)^{-k}}{k} k!\mathrm{ch}_{k-1}(\gamma_j). \tag{4.27}$$

Moreover, since the quantum Cartan involution τ_q on $\mathrm{H}_\bullet(\mathcal{M})_q$ exchanges 1_α and $1_{-\alpha}$, we require that it interchange $Y(1_\alpha, z)$ and $Y(1_{-\alpha}, z)$.

In particular, we should not set $Y(1_{\pm\alpha_i}, z)$ to be $x_i^{\pm}(z)$ since these are not swapped under the quantum Cartan involution, but rather

$$Y(1_{\pm\alpha_i}, z) \ =\ q^{-\frac{1}{2}\partial_{\pm\alpha_i}} x_i^{+}(q^{\mp 1/2} z) \tag{4.28}$$

since it is (4.20) and (4.21) that are swapped by the quantum Cartan involution.[3]

4.8.4. **Deformed bicharacter.** If we want to make (4.28) true, we need to interpret the $q^{\pm 1}$ factors. We do so by choosing a linear functional

$$\psi \;:\; \pi_0(\mathcal{M}) \;=\; \check{\Delta} \;\to\; \mathbf{C}$$

defined uniquely by $\psi(\alpha_i) = 1$. Then we set

$$\Psi_q(\theta) \;=\; (q^{\frac{1}{2}\psi_1}z)^{\mathrm{rk}\theta} \exp\left(\sum_{k\geqslant 1} \frac{(-q^{\psi_1}z)^{-k}}{k} k!\mathrm{ch}_k(\oplus_{i,j}\frac{[A_{ij}k]}{[k]}\gamma_i^\vee \boxtimes \gamma_j)\right) \tag{4.29}$$

where ψ_1 is the locally constant function taking value $\psi(a)$ on $\mathcal{M}_a \times \mathcal{M}$. Here, we have extended the chern character $\mathbf{C}[q^{1/2}, q^{-1/2}]$ linearly to $\mathrm{K}(\mathcal{M}\times\mathcal{M})_{q^{1/2}}$, so that the above defines an element of $\mathrm{H}^\bullet(\mathcal{M}\times\mathcal{M})_{q^{1/2}}((z^{-1}))$. Its specialisation

$$\Psi_1(\theta) \;=\; \Psi(\theta)$$

recovers the usual bicharacter (2.19) inducing Joyce's vertex algebra structure on $\mathrm{H}_\bullet(\mathcal{M})$.

4.8.5. By q linearity we extend the maps on homology arising from the geometry of $\mathcal{M}$ (see section 2.6) to

$$\oplus_* \;:\; \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}} \otimes \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}} \;\to\; \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}}, \qquad \mathrm{act}_* \;:\; \mathrm{H}_\bullet(\mathrm{B}\mathbf{G}_m) \otimes \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}} \;\to\; \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}}.$$

Definition 4.8.6. In the above setting, the *q-deformed Joyce vertex algebra* structure on $\mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}}$ is the map

$$Y \;:\; \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}} \otimes \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}} \;\to\; \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}}((z))$$

sending

$$Y(\alpha, z)\beta \;=\; \check{\varepsilon} \oplus_* \left(e^{zt} \otimes \mathrm{id} \cdot \Psi_q(\theta)\alpha \otimes \beta\right).$$

[3]To make the formula more symmetric we have made the variable change $z \mapsto q^{-1/2}z$.

4.8.7. **Fields.** We now compare Proposition 4.7.3 to the fields in the q-deformed Joyce vertex algebra structure. We compute

$$\begin{aligned}
\mathrm{Y}(\sigma^0_{1,i}, z)\beta &= \oplus_* \sum_{k\geqslant 0} \frac{z^k t^k}{k!} \Psi_q(\theta)\sigma^0_{1,i} \otimes \beta \\
&= \oplus_* \sum_{k\geqslant 0} \frac{z^k t^k}{k!} \exp\left(\sum_{k\geqslant 1} \frac{(-z)^{-k}}{k} k!\mathrm{ch}_k(\oplus_{i,j} \frac{[A_{ij}k]}{[k]} \gamma_i^\vee \boxtimes \gamma_j) \right) \sigma^0_{1,i} \otimes \beta \\
&= \oplus_* \sum_{k\geqslant 0} \frac{z^k t^k}{k!} \left(\sigma^0_{1,i} \otimes \beta + 1_0 \otimes \left(\sum_{k\geqslant 1, j} \frac{[A_{ij}k]}{[k]} \frac{(-z)^{-k}}{k} k!\mathrm{ch}_{k-1}(\gamma_j) \right) \beta \right) \\
&= \oplus_* \sum_{k\geqslant 0} \frac{z^k t^k}{k!} \left(\sigma^0_{1,i} \otimes \beta + 1_0 \otimes \left(\sum_{k\geqslant 1, j} \frac{[A_{ij}k]}{[k]} \frac{(-z)^{-k}}{k} k!\mathrm{ch}_{k-1}(\gamma_j) \right) \beta \right) \\
&= \sum_{k\geqslant 0} \frac{z^k \sigma^0_{k+1,i}}{k!} \cdot \beta + \sum_{k\geqslant 1, j} \frac{[A_{ij}k]}{[k]} \frac{(-z)^{-k}}{k} k!\mathrm{ch}_{k-1}(\gamma_j)\beta.
\end{aligned}$$

We have used that $\psi(0) = 0$. Under the identifications (4.25) in Proposition 4.7.3,

$$Y(\sigma^0_{1,i}, z) = \sum_{k\geqslant 0} \frac{n}{[n]} h_{i,n} z^{-n-1}.$$

4.8.8. Similarly, using Lemma 4.5.19 we can compute

$$\begin{aligned}
&\left(\check{\varepsilon}_{\pm\alpha_i,b}(q^{\pm 1}z)^{\kappa(\alpha_i,b)}\right)^{-1} \cdot Y(1_{\pm i}, z)\beta \\
&\quad = \oplus_* \sum_{k\geqslant 0} \frac{z^k t^k}{k!} \Psi_q(\theta)(1_{\pm i} \otimes \beta) \\
&\quad = \oplus_* \sum_{k\geqslant 0} \frac{z^k t^k}{k!} \left(\sum_{k\geqslant 1} \frac{(-z)^{-k}}{k} k!\mathrm{ch}_k(\oplus_{i,j} \frac{[A_{ij}k]}{[k]} \gamma_i^\vee \boxtimes \gamma_j) \right) (1_{\pm i} \otimes \beta) \\
&\quad = e^{zt} 1_{\pm i} \cdot \exp\left(\sum_{k\geqslant 1, j} \frac{[A_{ij}k]}{[k]} \frac{z^{-k}}{k} k!\mathrm{ch}_k \gamma_j \right) \beta \\
&\quad = e^{\pm\alpha_i} e^{\mp\alpha_i} e^{zt} e^{\pm\alpha_i} 1_0 \cdot \exp\left(\sum_{k\geqslant 1, j} \frac{[A_{ij}k]}{[k]} \frac{z^{-k}}{k} k!\mathrm{ch}_k \gamma_j \right) \beta \\
&\quad = \exp\left(e^{\mp\alpha_i} z t e^{\pm\alpha_i} \right) 1_0 \cdot \exp\left(\sum_{k\geqslant 1, j} \frac{[A_{ij}k]}{[k]} \frac{z^{-k}}{k} k!\mathrm{ch}_k \gamma_j \right) e^{\pm\alpha_i}\beta \\
&\quad = \exp\left(\sum_{k>0} \frac{\sigma^0_{i,k}}{k!} z^k \right) \exp\left(\sum_{k\geqslant 1, j} \frac{[A_{ij}k]}{[k]} \frac{z^{-k}}{k} k!\mathrm{ch}_k \gamma_j \right) e^{\pm\alpha_i}\beta
\end{aligned}$$

where $\beta \in \mathrm{H}^\bullet(\mathcal{M}_b)$.

4.8.9. We summarise the above in a Theorem.

Theorem 4.8.10. *The quantum FKS isomorphism and the q deformed Joyce structure are compatible in the sense that the FKS map $V_{1,q^{1/2}}(\mathfrak{g}) \to V_{\check{\Delta},\kappa,q^{1/2}} \xrightarrow{\sim} \mathrm{H}_\bullet(\mathcal{M})_{q^{1/2}}$ sends*

$$q^{-\frac{1}{2}\partial_{\pm\alpha_i}} x_i^{\pm}(q^{\mp 1/2}z) \;\mapsto\; Y(1_{\pm\alpha_i}, z), \qquad\qquad h_i(z) \;\mapsto\; Y(\sigma^0_{1,i}, z).$$

4.9 Axiomatics

4.9.1. In this section we ask the open question of how to interpret Definition 4.8.6 as an example of a sort of q-deformed vertex algebra. We also recall the notion of Joyce of deformed vertex algebra, which should be closely related into any answer that is given to this question.

4.9.2. **History.** Over the years there have been many attempts to find a q-analogue of vertex algebras, e.g. due to Borcherds, Etingof and Kac, Frenkel and Reshitikin and Li [Bo2, EK, FR, Li2]. However, many of the basic questions about them have yet to be answered, for instance whether there is a q-analogue notion of factorisation algebra. Many structures appearing in algebra like quantum affine algebras, quantum Yangians and quantum W algebras may have attached q-vertex algebras, which would be a powerful new (geometric) lens to understand these structures.

4.9.3. **Open question.** We are left with the question: is there a definition of q-deformed vertex algebra which includes the structure on $\mathrm{H}_\bullet(\mathcal{M})_q$ introduced in Definition 4.8.6? Moreover, is there a definition of q-deformed affine vertex algebra such that the quantum FKS map in Theorem 4.6.3 comes from a map of q-deformed vertex algebras?

4.9.4. **q-deformed vertex algebras.** We recall what [Jo2] calls q-deformed vertex algebra. The definition is inspired by an equivalent definition of vertex algebras, which in terms of factorisation algebras on a curve X takes account of the sheaves living over X^n for all $n \geqslant 0$ and not just $n = 0, 1, 2$. One considers the maps

$$V^{\otimes n} \;\to\; V[[z_1, ..., z_n]][(z_i - z_j)^{-1}]^n_{i,j=1}$$

which in terms of the usual vertex algebra definition are just $\alpha_1 \otimes \cdots \otimes \alpha_n \mapsto Y(\alpha_1, z_1) \cdots Y(\alpha_n, z_n)|0\rangle$. Thus the vertex algebra structure is encoded in maps as above, together with compatibility conditions between them.

Definition 4.9.5. [Jo2] Let $q \in \mathbf{C}^\times$. A *q-deformed nonlocal vertex algebra* is a vector space V and maps

$$Y_n \;=\; Y_n(z_1,...,z_n) \;:\; V^{\otimes n} \;\to\; V[[z_1,...,z_n]][(z_i - q^k z_j)^{-1} : k \in \mathbf{Z}]_{i,j=1}^n$$

for all $n = 0, 1 \ldots$ satisfying:

1. Identity. $Y_0 = \mathrm{id}_k$.
2. Exponential. $Y_1(z_1)Y_1(w_1) = Y_1(z_1 + w_1)$ and $Y_1(0) = \mathrm{id}_V$.
3. Associativity. Given elements $\alpha_1, ..., \alpha_n \in V$ (though of as being at points $z_1, ..., z_n$) and $\beta_1, ..., \beta_m \in V$ (at $w_1, ..., w_m$), we have

   $$Y_n(z_1,...,z_{i-1},0,z_{i+1},...,z_n)\,(\alpha_1 \otimes \cdots \otimes \alpha_{i-1} \otimes [Y_m(w_1,...,w_m)\beta_1 \otimes \cdots \otimes \beta_m] \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_n)$$

   for any i is equal to

   $$Y_{n+m-1}(z_1,...,z_{i-1},w_1,...,w_m,z_{i+1},...,z_n)(\alpha_1 \otimes \cdots \otimes \alpha_{i-1} \otimes \beta_1 \otimes \cdots \otimes \beta_m \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_n).$$

If in addition the Y_n are $\mathfrak{S}_n$ invariant, it is called a *q-deformed vertex algebra.*

4.9.6. By the associativity axiom, the Y_n for $n \geqslant 3$ may be defined in terms of Y_2, in which case the associativity axiom becomes a set of compatibility relations between the Y_1 and Y_2. By the exponential axiom, we may write $Y_1(z) = e^{zT}$ for some operator T.

4.9.7. Definition 4.9.5 should be viewed as a working definition, to be updated as the theory is worked out more, e.g. the notion of q-factorisation algebra discovered. It is also pleasant that it is very close to the definitions of [EK, Li2], and admits certain vertex algebraic properties lie the Zhu algebra.

4.9.8. How is thus structure related to the structure in Definition 4.8.6?

To begin with, let $\mathcal{M}, \theta, (\varepsilon)$ be as in 2.6.17 (or 2.6.21), so that the homology $\mathrm{H}_\bullet(\mathcal{M})$ is canonically a (nonlocal) vertex algebra.

In addition to this, we then let $\mathbf{G}_m$ act on the fibres of θ. As for all actions of tori on Artin stacks, we have

$$D_{qc}(\mathcal{M} \times \mathcal{M} \times \mathrm{B}\mathbf{G}_m) \;\simeq\; \prod_{k \in \mathbf{Z}} D_{qc}(\mathcal{M} \times \mathcal{M}),$$

see [AKLPR]. In particular, any perfect complex θ on $\mathcal{M}\times\mathcal{M}\times \mathrm{B}\mathbf{G}_m$ splits as a direct sum

$$\theta \;=\; \textstyle\bigoplus_{k\in\mathbf{Z}}\theta(k)\otimes\gamma^{\otimes k}.$$

Now, we can extend the chern character to a map

$$\mathrm{K}(\mathcal{M}\times\mathcal{M}\times \mathrm{B}\mathbf{G}_m) \;\to\; \mathrm{H}^\bullet(\mathcal{M}\times\mathcal{M})[q,q^{-1}]$$

sending $\sum_{k\in\mathbf{Z}}\mathcal{V}(k)\otimes\gamma^{\otimes k}\mapsto\sum_{k\in\mathbf{Z}}\mathrm{ch}(\mathcal{V}(k))q^k$ where $\mathcal{V}(k)\in\mathrm{K}(\mathcal{M}\times\mathcal{M})$. The proof of the following proposition is essentially the same as the proof of Theorems 2.6.17 or 2.6.21.

Proposition 4.9.9. *Keeping the the above notation,*

$$Y_n \;=\; \oplus_*\left(e^{z_1t}\otimes\cdots\otimes e^{z_nt}\prod_{i\neq j}\breve{\varepsilon}_{i,j}\Psi_q(\theta_{i,j};z_i,z_j)(-)\right)$$

is a q-deformed nonlocal vertex algebra as in Definition 4.9.5. Here θ_{ij} is the pullback of θ via the ijth projection $\mathcal{M}^n\to\mathcal{M}^2$ and for $\mathcal{V}\in\mathrm{Perf}(\mathcal{M}\times\mathcal{M}\times\mathrm{B}\mathbf{G}_m)$ we set

$$\Psi_q(\mathcal{V};z,w) \;=\; \sum_{k\in\mathbf{Z}}\sum_{r,s\geqslant 0}(z-q^{-n}w)^{\frac{1}{2}\mathrm{rk}\mathcal{V}(k)-r}(q^nz-w)^{\frac{1}{2}\mathrm{rk}\mathcal{V}(k)-s}c_{r,s}(\mathcal{V}) \;\in\; \mathrm{H}^\bullet(\mathcal{M}\times\mathcal{M})[q,q^{-1}]$$

where $c_{r,s}(\mathcal{V})$ is the $\mathrm{H}^{2r}(\mathcal{M})\otimes\mathrm{H}^{2s}(\mathcal{M})$ summand of $c_{r+s}(\mathcal{V})\in\mathrm{H}^{2r+2s}(\mathcal{M}\times\mathcal{M})$. Finally, $\breve{\varepsilon}_{i,j}$ is $\breve{\varepsilon}$ as in section 2.6.20 acting on the ijth component of $\mathrm{H}^\bullet(\mathcal{M})^{\otimes n}$.

4.9.10. In particular, setting $n=2$ and $w=0$,

$$Y_2(\alpha,\beta;z,0)$$

gives back the Definition 4.8.6, but with the ψ term absent. Thus, the problem becomes how to alter Definition 4.9.5 so as to incorporate the ψ shift.

4.9.11. **Affine q-deformed vertex algebras.** Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. Given the (so far unmade) definition of q deformed vertex algebra, we would like to endow the Verma module $V_{k,q}(\mathfrak{g})$ with such a structure, in a way that the FKS map 4.19 is a map of q deformed vertex algebras.

4.9.12. Attached to each generator $h_i,x_i^\pm$ in a Chevalley basis, it is reasonable to guess that the following might define a q deformed vertex algebra:

$$h_i(z) \;:=\; h_{-1}z^{-1}+\sum_{n\in\mathbf{Z}\backslash 0}\frac{n}{[n]}h_{i,n}z^{-n-1}, \qquad\qquad x_i^\pm(z) \;:=\; \sum_{n\in\mathbf{Z}}x_{i,n}^\pm z^{-n-1}.$$

For instance, we have the following Proposition (and notice that the power series $\sum h_{i,n}z^{-n-1}$ do not admit similar operator product expansions):

Proposition 4.9.13. *The fields $h_i(z)$ have operator product expansions*

$$h_i(z)h_j(w) = \frac{k\mathrm{id}}{(z-q^{r-1}w)^2} + \frac{k\mathrm{id}}{(z-q^{r-3}w)^2} + \cdots + \frac{k\mathrm{id}}{(z-q^{-(r-1)}w)^2} \mod V[[z,w]]$$

where $r = \kappa(h_i, h_j)$ and $q = q_i$.

Proof. We compute

$$\begin{aligned}
[h_i(z), h_j(w)] &= \sum_{n,m} \frac{nm}{[n][m]}[h_{i,n}, h_{j,m}]z^{-n-1}w^{-m-1} \\
&= \sum_{n,m} \frac{n^2}{[n]^2}\delta_{n+m,0}\frac{[\kappa(h_i,h_j)n]}{n}[n]cz^{-n-1}w^{-m-1} \\
&= \sum_{n} \frac{[\kappa(h_i,h_j)n]}{[n]}cz^{-n-1}nw^{n-1} \\
&= \partial_w \sum_n \frac{q^{n\kappa(h_i,h_j)} - q^{-n\kappa(h_i,h_j)}}{q^n - q^{-n}}cz^{-n-1}w^n \\
&= \partial_w \sum_n (q^{n(\kappa(h_i,h_j)-1)} + q^{n(\kappa(h_i,h_j)-3)} + \cdots + q^{-n(\kappa(h_i,h_j)-1)})cz^{-n-1}w^n \\
&= \partial_w\delta(z-q^{r-1}w)c + \cdots + \partial_w\delta(z-q^{-(r-1)}w)c.
\end{aligned}$$

Finally, note that the central element c acts as multiplication by k on $V_k(\mathfrak{g})$. □

Similarly, we have

Proposition 4.9.14. *The fields $h_i(z), x_i^{\pm}(z)$ have operator product expansions*

$$x_i^{\pm}(z)h_j(w) = \frac{x_j^{\pm}(w)}{z-q^{r-1}w} + \frac{x_j^{\pm}(w)}{z-q^{r-3}w} + \cdots + \frac{x_j^{\pm}(w)}{z-q^{-(r-1)}w} \mod V[[z,w]]$$

where $r = \kappa(h_i, h_j)$ and $q = q_i$.

Proof. We compute

$$\begin{aligned}
[h_i(z), x_j^{\pm}(w)] &= \sum_{n,m} \frac{n}{[n]}[h_{i,n}, x_{j,m}^{\pm}]z^{-n-1}w^{-m-1} \\
&= \sum_{n,m} \frac{[nA_{ij}]}{[n]}x_{j,n+m}^{\pm}z^{-n-1}w^{-m-1} \\
&= \sum_{n,m} (q^{n(r-1)} + q^{n(r-3)} + \cdots + q^{-n(r-1)})x_{j,n+m}^{\pm}z^{-n-1}w^{-m-1} \\
&= \sum_{n,m} (q^{n(r-1)} + q^{n(r-3)} + \cdots + q^{-n(r-1)})z^n w^{-n-1} \cdot \sum_{k\in\mathbf{Z}} x_{j,k}^{\pm}w^{-k-1} \\
&= \left(\delta(z-q^{r-1}w) + \cdots + \delta(z-q^{-(r-1)}w)\right)x_i^{\pm}(w).
\end{aligned}$$

□

Arguing similarly, we get

Proposition 4.9.15. *The fields $x_i^{\pm}(z)$ have operator product expansions*

$$x_i^+(z)x_j^-(w) \;=\; \delta_{i,j}\frac{1}{(z-w)}\cdot\frac{\phi_i^+(w)-\phi^-(w)}{q-q^{-1}} \qquad \mathrm{mod}\ V[[z,w]]$$

where $r=\kappa(h_i,h_j)$ and $q=q_i$.

Since $[x_{i,n}^{\pm}, x_{j,m}^{\pm}]$ does not admit a simple closed form expression in quantum toroidal or affine algebras, it is less clear what the operator product expansion of $x_i^{\pm}(z)$ with itself will be. We leave this to future work.

Appendix A

Variant notions

A.1 Varying the background category

A.1.1. Ordinarily one defines vertex algebras inside the category Vect $=$ Vect$_k$ of vector spaces over a field k. We recall how to replace Vect with a more general background category (Definition A.1.14). Applying this to the categories Vect$_{\mathbf{Z}/2}$ and Vect$_{\mathbf{Z}}$ of vector superspaces and graded vector spaces will recover the notion of vertex superalgebra and graded vertex algebra. Taking Vect$_A$, vector spaces graded by an abelian group A, recovers the definition of Dong and Lepowsky [DL].

A.1.2. For completeness we first write down some category theory definitions, see e.g. [Ric, §8] or [EGNO].

A.1.3. **Structures on monoidal categories.**

Definition A.1.4. A *monoidal category* is a category $\mathcal{C}$ with a functor

$$\otimes \;:\; \mathcal{C}\times\mathcal{C} \;\to\; \mathcal{C}$$

an object e and natural isomorphisms

$$\alpha_{c_1,c_2,c_3} \;:\; (c_1\otimes c_2)\otimes c_3 \;\tilde{\to}\; c_1\otimes(c_2\otimes c_3) \qquad \lambda_c \;:\; e\otimes c \;\tilde{\to}\; c \qquad \rho_c \;:\; c\otimes e \;\tilde{\to}\; c$$

for all c, c_1, c_2, c_3, satisfying the *pentagon identity*

$$
\begin{tikzcd}
 & (c_1 \otimes c_2) \otimes (c_3 \otimes c_4) \arrow[rr, "\alpha_{c_1 \otimes c_2, c_3, c_4}"] & & ((c_1 \otimes c_2) \otimes c_3) \otimes c_4) \arrow[dr, "\alpha_{c_1,c_2,c_3} \otimes \mathrm{id}"] & \\
c_1 \otimes (c_2 \otimes (c_3 \otimes c_4))) \arrow[ur, "\alpha_{c_1,c_2,c_3 \otimes c_4}"] \arrow[drr, "\mathrm{id} \otimes \alpha_{c_2,c_3,c_4}"'] & & & & (c_1 \otimes (c_2 \otimes c_3)) \otimes c_4 \\
 & & c_1 \otimes ((c_2 \otimes c_3) \otimes c_4) \arrow[urr, "\alpha_{c_1, c_2 \otimes c_3, c_4}"'] & &
\end{tikzcd}
$$

and the *triangle identity*

$$
\begin{tikzcd}
c_1 \otimes (e \otimes c_2) \arrow[rr, "\alpha_{c_1,e,c_2}"] \arrow[dr, "\mathrm{id} \otimes \lambda_{c_2}"'] & & (c_1 \otimes e) \otimes c_2 \arrow[dl, "\rho_{c_1} \otimes \mathrm{id}"] \\
 & c_1 \otimes c_2 &
\end{tikzcd}
$$

That is, these diagrams commute for all c_1, c_2, c_3, c_4.

Definition A.1.5. Monoidal category $\mathcal{C}$ is called *symmetric monoidal* if in addition there are binatural isomorphisms

$$\tau_{c_1,c_2} \ : \ c_1 \otimes c_2 \ \xrightarrow{\sim} \ c_2 \otimes c_1$$

with *symmetry condition* $\tau_{c_1,c_2}\tau_{c_2,c_1} = \mathrm{id}$, and satisfying the *hexagon identity*

$$
\begin{tikzcd}
 & (c_1 \otimes c_2) \otimes c_3 \arrow[r, "\tau_{c_1 \otimes c_2, c_3}"] & c_3 \otimes (c_1 \otimes c_2) \arrow[dr, "\alpha_{c_3,c_1,c_2}"] & \\
c_1 \otimes (c_2 \otimes c_3) \arrow[ur, "\alpha_{c_1,c_2,c_3}"] \arrow[dr, "\mathrm{id} \otimes \tau_{c_2,c_3}"'] & & & (c_3 \otimes c_1) \otimes c_2 \\
 & c_1 \otimes (c_3 \otimes c_2) \arrow[r, "\alpha_{c_1,c_3,c_2}"] & (c_1 \otimes c_3) \otimes c_2 \arrow[ur, "\tau_{c_1,c_3} \otimes \mathrm{id}"'] &
\end{tikzcd}
\tag{A.1}
$$

and $\rho_c = \lambda_c \tau_{c,e}$.

Often the symbol σ instead of τ is used to denote the symmetric braiding in Definition A.1.5. We now weaken this notion by discarding the symmetry condition:

Definition A.1.6. Monoidal category $\mathcal{C}$ is called *braided monoidal* if in addition there are binatural isomorphisms

$$\beta_{c_1,c_2} \ : \ c_1 \otimes c_2 \ \xrightarrow{\sim} \ c_2 \otimes c_1$$

satisfying the two *hexagon identities*, (A.1) and

$$\begin{tikzcd}[column sep=small]
& c_1 \otimes (c_2 \otimes c_3) \arrow[r, "\beta_{c_1,c_2\otimes c_3}"] & (c_2 \otimes c_3) \otimes c_1 \arrow[dr, "\alpha^{-1}_{c_2,c_3,c_1}"] & \\
(c_1 \otimes c_2) \otimes c_3 \arrow[ur, "\alpha^{-1}_{c_1,c_2,c_3}"] \arrow[dr, "\beta_{c_1,c_2}\otimes\mathrm{id}"'] & & & c_2 \otimes (c_3 \otimes c_1) \\
& (c_2 \otimes c_1) \otimes c_3 \arrow[r, "\alpha^{-1}_{c_2,c_1,c_3}"] & c_2 \otimes (c_1 \otimes c_3) \arrow[ur, "\mathrm{id}\otimes\beta_{c_1,c_3}"'] &
\end{tikzcd} \tag{A.2}$$

and $\rho_c = \lambda_c \beta_{c,e}$ and $\beta_{e,c}\rho_c = \lambda_c$.

These structures A.1.4, A.1.5 and A.1.6 are are called *strict* if all the α's are identities.

A.1.7. These definitions are more natural in the language of ∞-categories, see [Lur2, Ex. 1.2.4]. One can define monoidal, braided monoidal and symmetric monoidal structures on ∞-categories, and show they the same thing as E_1, E_2 and E_∞ monoidal structures. Thus monoidal categories are associative algebras, and symmetric monoidal categories are commutative algebras, in the category of ∞-categories.

A.1.8. **Tensor categories.** Let $\mathcal{C}$ be a k linear abelian monoidal category, such that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is cocontinuous (in particular additive), and bilinear on morphisms, and the structure maps $\alpha_{c_1,c_2,c_3}, \rho_c, \lambda_c$ are additive in each variable and multilinear on morphisms. We call such a category *k linear monoidal.*

A *k linear symmetric* (*braided*) *monoidal* structure on $\mathcal{C}$ is a symmetric (braided) monoidal structure on the underlying category such that τ_{c_1,c_2} (β_{c_1,c_2}) are additive in each variable and multilinear on morphisms.

Again, these structures are called *strict* if the α's are identities.

A.1.9. **Power series.** Let $\mathcal{C}$ be a k linear symmetric monoidal category. We define

$$\mathcal{C}[[z]] \;=\; \mathcal{C} \otimes_{\mathrm{Vect}_k} k[[z]]\text{-Mod} \;:=\; k[[z]]\text{-Mod}(\mathcal{C}).$$

Thus, an object of $\mathcal{C}[[z]]$ is an object c of $\mathcal{C}$ together with a map $m_c : k[[z]] \otimes_{\mathcal{C}} c \to c$ respecting the algebra structure on $k[[z]] \in \mathrm{Ob}\mathcal{C}$. Since $k[[z]]$ is a cocommutative bialgebra, $\mathcal{C}[[z]]$ inherits a symmetric monoidal structure from $\mathcal{C}$ and $k[[z]]$-Mod, i.e. $c \otimes_{\mathbf{C}[[z]]} c' = c \otimes_{\mathcal{C}} c'$ as objects of $\mathcal{C}$, with the $k[[z]]$ action induced by the coproduct $k[[z]] \to k[[z]] \otimes_{\mathcal{C}} k[[z]]$.

A.1.10. For instance, writing $k \in \mathcal{C}$ for the unit object, $k[[z]]$ is an object in $\mathcal{C}[[z]]$. It is a commutative monoid in fact, and $\mathcal{C}[[z]]$ is its category of algebras in $\mathcal{C}$.

A.1.11. As with all module categories, the "tensor by $k[[z]]$" functor is left adjoint to the forgetful functor,

$$\otimes k[[z]] \ : \ \mathcal{C} \ \to \ \mathcal{C}[[z]], \qquad \text{For} \ : \ \mathcal{C}[[z]] \ \to \mathcal{C}.$$

Note that For is conservative. There is also the trivial module functor

$$\text{triv} \ : \ \mathcal{C} \ \to \ \mathcal{C}[[z]]$$

which together with For allows us identify with of $\mathcal{C}$ as elements of $\mathcal{C}[[z]]$ and vice versa.

A.1.12. Similarly, we can define modifications of $\mathcal{C}$ for other power series rings, e.g. $\mathcal{C}((z,w))$, $\mathcal{C}((z)), \ldots$ and the above remarks in section A.1.11 will also apply. However, these categories will not necessarily be symmetric monoidal so much of the material in section A.4 will not apply if we replace $\mathcal{C}[[z]]$ with them.

A.1.13. **Vertex algebras in a general symmetric monoidal category.** Let $\mathcal{C}$ be a symmetric monoidal k linear category, with unit denoted k.

Definition A.1.14. A *vertex algebra* in $\mathcal{C}$ is an object c with a map

$$Y(z) \ : \ c \otimes c \ \to \ c((z))$$

and maps $|0\rangle : k \to c$, $T : c \to c$, satisfying the axioms in Definition 2.1.9.

A.1.15. For instance, let us show how to define the notion of weak commutativity in this setting. Take the map

$$c \otimes c \otimes c \ \overset{\text{id}\otimes Y(w)}{\to} \ (c \otimes c)((w)) \ \overset{Y(z)\otimes\text{id}}{\to} \ c((w))((z)) \ \to \ c[[z^{\pm 1}, w^{\pm 1}]]$$

and the map

$$c \otimes c \otimes c \ \overset{\tau\otimes\text{id}}{\to} \ c \otimes c \otimes c \ \overset{\text{id}\otimes Y(z)}{\to} \ (c \otimes c)((z)) \ \overset{Y(w)\otimes\text{id}}{\to} \ c((z))((w)) \ \to \ c[[z^{\pm 1}, w^{\pm 1}]].$$

Thus for any elements $\alpha, \beta : k \to c$ we get two maps

$$c \ \to \ c[[z^{\pm 1}, w^{\pm 1}]].$$

which are the analogues of $Y(\alpha, z)Y(\beta, w)$ and $Y(\beta, w)Y(\alpha, z)$. Weak commutativity says that their difference composed with $\text{id} \otimes (z-w)^n$ vanishes when n is high enough.

A.2 Varying the algebraic structure

There are some variants of the notion of vertex algebra, which we briefly discuss in this section. Vertex algebras are like commutative algebras in two ways: a commutative algebra with derivation is a vertex algebra by section 2.3.2, and factorisation algebras (which are commutative algebras in some category) give rise to vertex algebras.

The variants we will discuss, and their unaffinised analogues, are

(Co)commutative (co)algebra	$\rightsquigarrow$	Vertex (co)algebra
(Co)algebra	$\rightsquigarrow$	Nonlocal vertex (co)algebra
Bialgebra	$\rightsquigarrow$	Vertex bialgebra
Quasitriangular bialgebra	$\rightsquigarrow$	Quasitriangular vertex bialgebra

Given a structure on the left, if we equip it with a compatible derivation, we get the structure on the right, see e.g. [Gro2]. We leave the question of interpreting these structures in terms of the chiral and $*$ symmetric monoidal structures on $\mathrm{Sh}(\mathrm{Ran}X)$.

A.2.1. **Nonlocal vertex algebras.** First we recall the vertex analogue of associative algebra due to Bakalov, Kac and Li in [BK, Li1, Li2].

Definition A.2.2. A *nonlocal vertex algebra* is a vertex algebra as in Definition A.1.14, but with the weak commutativity condition replaced with weak associativity: for all $\alpha, \beta \in V$,

$$(z-w)^n Y(Y(\alpha,z)\beta,-w)\gamma \;=\; (z-w)^n Y(\alpha,z-w)Y(\beta,-w)\gamma \tag{A.3}$$

for $n \gg 0$.

A.2.3. The reader is warned that in the literature the word *(non)commutative* vertex algebra means (non)holomorphic, not (non)local.

A.2.4. **Vertex coalgebras.** We now give the dual notion of a vertex algebra.

Definition A.2.5. A *vertex coalgebra* is a vector space V with a linear functional $\langle 0| : V \to k$, an endomorphism T satisfying $\langle 0|T = 0$, and a map

$$\Delta(z) \;:\; V \;\to\; V \otimes V((z^{-1}))$$

which weakly cocommutes, $(T \otimes 1)\Delta(\alpha, z) = \partial_z \Delta(\alpha, z)$ and

$$(\langle 0| \otimes \mathrm{id})\Delta \;=\; \mathrm{id}, \qquad\qquad (\mathrm{id} \otimes \langle 0|)\Delta = \mathrm{id} \mod zV \otimes V[z].$$

A slightly modified version of this definition appears in [Hu]. In the case where V is a graded vertex coalgebra with finite dimensional weight spaces however, both definitions are equivalent, see section A.2.7 below.

A.2.6. The definition of weak cocommutativity is the dual notion of weak commutativity. $\Delta(z)$ is said to weakly cocommute if for every $\varphi \in (V^{\otimes 3})^{\vee}$,

$$(z-w)^n \varphi\left((\mathrm{id}\otimes\Delta(z))\Delta(w) - (\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\Delta(w))\Delta(z)\right) \;=\; 0$$

for $n \gg 0$.

A.2.7. If V is a vertex algebra, then its dual V^* is a vertex coalgebra, and vice versa. Indeed, note that weak commutativity can be written as: for all $v \in V^{\otimes 3}$,

$$(z-w)^n (Y(z)(Y(w)\otimes\mathrm{id}) - Y(w)(\mathrm{id}\otimes Y(z))(\sigma\otimes\mathrm{id}))v \;=\; 0$$

for $n \gg 0$, where we have written $Y(z) : V\otimes V \to V((z))$ for the vertex algebra field map. Likewise, if V is a graded vertex algebra then its contragredient dual $V^{\vee}$ is a graded vertex coalgebra, and vice versa. Since $V^{\vee\vee} \simeq V$ canonically if the weight spaces are finite dimensional, this sets up an equivalence of categories between graded vertex algebras and coalgebras whose graded pieces are finite dimensional.

Likewise, the dual analogue of nonlocal vertex algebra is

Definition A.2.8. A *nonlocal vertex coalgebra* is a vertex coalgebra as in A.2.5, but with weak cocommutativity replaced with weak coassociativity: for all $\gamma \in V$ and $\varphi \in (V^{\otimes 3})^{\vee}$,

$$(z-w)^n \varphi\left((\Delta(z)\otimes\mathrm{id})\Delta(-w)\gamma - (\mathrm{id}\otimes\Delta(z))\Delta(z-w)\gamma)\right) \;=\; 0$$

for $n \gg 0$.

As before, there is an equivalence between graded nonlocal vertex algebras and graded nonlocal vertex coalgebras whose graded pieces are finite dimensional.

A.2.9. **Skew symmetry.** Let A be a coalgebra in a symmetric monoidal category $\mathcal{C}$. Its *opposite* is the coalgebra with the same counit, and coproduct

$$\Delta^{op} \;=\; \sigma\Delta.$$

It is cocommuative if and only if $\Delta^{op} = \Delta$.

A.2.10. Let V be a nonlocal vertex coalgebra. Its *opposite* is the nonlocal vertex coalgebra with the same covacuum, and cofield map

$$\Delta^{op}(z) \;=\; \sigma\Delta(-z)e^{zT}.$$

It is local (i.e. defines a vertex coalgebra) if and only if $\Delta^{op}(z) = \Delta(z)$, see [Hu, Prop. 2.3].

A.2.11. Dually, one can define the opposite algebra product $a \cdot_{op} b = b \cdot a$, which is equal to the original if and only if it is commutative, and the opposite nonlocal vertex algebra $Y^{op}(\alpha, z)\beta = e^{zT}Y(\beta, -z)\alpha$, with $Y^{op} = Y$ if and only if the vertex algebra is local, see [BK, Rem. 4.8]

A.2.12. For instance, consider Joyce's nonlocal vertex algebra (see Theorem 2.6.18). We have

$$\Delta(\alpha, z) \;=\; \Psi(\theta, z)\mathrm{act}^*_{1,z} \oplus^* \alpha$$

and so, since $e^{zT} = \mathrm{act}^*_z$ by Lemma 2.6.12,

$$\Delta^{op}(\alpha, z) \;=\; \Psi(\sigma^*\theta, -z)\mathrm{act}^*_{2,-z} \oplus^* \mathrm{act}^*_z\alpha \;=\; \Psi(\sigma^*\theta^\vee, z)\mathrm{act}^*_{1,z} \oplus^* \alpha.$$

That is, the opposite of the Joyce nonlocal vertex (co)algebra attached to θ is the Joyce nonlocal vertex (co)algebra attached to $\sigma^*\theta^\vee$.

Similarly, if we want to include orientations, the opposite of the Joyce nonlocal vertex (co)algebra of Theorem 2.6.21 attached to $\theta, \check{\varepsilon}$ is the Joyce nonlocal vertex (co)algebra attached to $\sigma^*\theta^\vee, \sigma^*\check{\varepsilon}$.

A.2.13. **(Quasitriangular) vertex bialgebras.** We first recall some standard definitions from algebra, see e.g. [Dr, ES, EGNO].

Definition A.2.14. A *bialgebra* is an associative algebra A with a compatible coalgebra structure (A, Δ, ε), meaning

$$\Delta(a \cdot b) \;=\; \Delta(a) \cdot \Delta(b), \tag{A.4}$$

for all $a, b \in A$, and $\varepsilon \otimes \varepsilon(a \otimes b) = \varepsilon(a \cdot b)$, $\varepsilon(1) = 1$ and $\Delta(1) = 1 \otimes 1$.

A bialgebra A is *cocommutative* if $\sigma\Delta = \Delta$ where $\sigma : A \otimes A \to A \otimes A$ is the map swapping the factors, and *almost cocommutative* if there is an invertible element $R \in A \otimes A$ with

$$\sigma\Delta(a) = \; R\Delta(a)R^{-1} \tag{A.5}$$

if all $a \in A$. It is called a *quasitriangular bialgebra* if in addition the *hexagon identities* hold:

$$(\Delta \otimes \mathrm{id})(R) \;=\; R_{13}R_{23}, \qquad\qquad (\mathrm{id} \otimes \Delta)(R) \;=\; R_{13}R_{12}. \tag{A.6}$$

Here e.g. R_{23} denotes the element $1 \otimes R \in A^{\otimes 3}$.

Lemma A.2.15. *If A is a quasitriangular bialgebra, R sastisfies the Yang Baxter equation*

$$R_{12}R_{13}R_{23} \;=\; R_{23}R_{13}R_{12}.$$

Proof. We have $R_{12}R_{13}R_{23} = R_{12}(\Delta\otimes\mathrm{id})(R) = (\sigma\Delta\otimes\mathrm{id})(R)R_{12} = R_{23}R_{13}R_{12}$. $\square$

A.2.16. To organise these definitions, recall from [Lur2] the sequence of operads

$$E_0 \;\to\; E_1 \;\to\; E_2 \to\; \cdots \to\; E_\infty.$$

Their algebras inside the category Cat of categories are (see [Lur2, §1.2])

Pointed categories $\leftarrow$ Monoidal categories $\leftarrow$ Braided monoidal categories $\leftarrow\;\cdots$

$\cdots\;\leftarrow$ Symmetric monoidal categories.

Now let A be an associative algebra in Vect, and consider its category A-Mod of left modules.

A priori, it carries no extra structure other than having a distinguished object A, i.e. it is an E_0 category. If we want it to be a monoidal category, i.e. the tensor product $M\otimes_k N$ of every module to carry a A module structure, then the action of A on $A\otimes_k A$ would give a map

$$A \;\to\; \mathrm{End}_{A\text{-Mod}}(A\otimes_k A) \;\simeq\; A\otimes_k A,$$

which defines a coproduct, and makes A into a bialgebra. This is symmetric monoidal if A is cocommutative, and braided monoidal if A is quasitriangular, with with braiding given by

$$M\otimes N \;\xrightarrow{R\cdot}\; M\otimes N \;\xrightarrow{\sigma}\; N\otimes M,$$

which is A linear by almost commutativity, see [ES, Prop. 14.2]. Thus the corresponding algebraic structures to the above are

Algebra $\leftarrow$ Bialgebra $\leftarrow$ Quasitriangular bialgebra $\leftarrow\;\cdots\;\leftarrow$ Cocommutative biaglebra.

More precisely, there is an equivalence of categories between lifts of the associative algebra structure on A to a bialgebra, quasitriangular bialgebra and cocommutative bialgebra structure, and lifts of the E_1, E_2 and E_∞ structures along A-Mod $\to$ Vect, respectively.

A.2.17. We now turn to the vertex analogue of the above, loosely following [Jo2].

Definition A.2.18. A *vertex bialgebra* is a nonlocal vertex coalgebra V with a compatible associative algebra structure, meaning

$$\Delta(\alpha \cdot \beta, z) \;=\; \Delta(\alpha, z) \cdot \Delta(\beta, z) \tag{A.7}$$

for all $\alpha, \beta \in V$, and $\langle 0| \otimes \langle 0|(a \otimes b) = \langle 0|(ab)$, $\langle 0|(1) = 1$ and $\Delta(1, z) = 1 \otimes 1$.

A vertex bialgebra V is *almost local* if there is an invertible $R(z) \in V \otimes V((z^{-1}))$ such that

$$\Delta^{op}(\alpha, z) \;=\; R(z)\Delta(\alpha, z)R(z)^{-1} \tag{A.8}$$

for all $\alpha \in V$, as well as compatibility with T

$$(e^{wT} \otimes \mathrm{id})R(z) \;=\; R(z+w)(e^{wT} \otimes \mathrm{id}), \qquad (\mathrm{id} \otimes e^{wT})R(z) \;=\; R(z-w)(\mathrm{id} \otimes e^{wT}). \tag{A.9}$$

It is a *quasitriangular vertex bialgebra* if in addition the *hexagon identities* hold

$$(\Delta(z)\otimes\mathrm{id})(R(w)) \;=\; R_{13}(z+w)R_{23}(w), \qquad (\mathrm{id}\otimes\Delta(z))(R(w)) \;=\; R_{13}(w)R_{12}(w-z). \tag{A.10}$$

A.2.19. **Almost locality.** There is another possible generalisation that does not require the algebra structure on V. A vertex coalgebra V is called *weakly almost local* if there is an invertible $k((z^{-1}))$ linear endomorphism $S(z) \in \mathrm{End} V \otimes V((z^{-1}))$ such that

$$\Delta^{op}(\alpha, z) \;=\; S(z) \cdot \Delta(\alpha, z)$$

for all $\alpha \in V$, as well as as well as compatibility with T

$$(e^{wT} \otimes \mathrm{id})S(z) \;=\; S(z+w)(e^{wT} \otimes \mathrm{id}), \qquad (\mathrm{id} \otimes e^{wT})S(z) \;=\; S(z-w)(\mathrm{id} \otimes e^{wT}). \tag{A.11}$$

It is a *weak quasitriangular vertex coalgebra* if in addition the *hexagon identities* hold

$$(\Delta(z) \otimes \mathrm{id})S(w) \;=\; S_{13}(z+w)S_{23}(w), \qquad (\mathrm{id} \otimes \Delta(z))S(w) \;=\; S_{13}(w)S_{12}(w-z). \tag{A.12}$$

Our computations in section A.2.12 show that

Proposition A.2.20. *Consider Joyce's nonlocal vertex coalgebra structure on* $\mathrm{H}^\bullet(\mathcal{M})$ *of Theorem 2.6.18. It is weakly almost local, for*

$$S(z) \;=\; \Psi(\sigma^*\theta^\vee, z)/\Psi(\theta, z).$$

Proof. Compatibility with T and the hexagon identities follow from the Commutation Lemma 2.6.19. That $\Delta^{op}(\alpha, z) = S(z) \cdot \Delta(\alpha, z)$ holds follows from section A.2.12. □

A.3 Representations of a holomorphic vertex bialgebra

Before discussing spectral monoidal categories in the next section, we write down the motivating case, with the main example being the Hopf algebra structure on $\mathrm{H}^\bullet(\mathcal{M})$ given by $\oplus^*$ and cup product (see section 2.6).

A.3.1. Let H be a holomorphic vertex bialgebra (in any background k linear symmetric monoidal category $\mathcal{V}$), or equivalently a cocommutative bialgebra H along with a derivation ∂. Write $\mathcal{C} = H\text{-Mod}_\partial$ for the category of modules with derivation over the underlying associative algebra. This is symmetric monoidal.

The motivating question is: what extra structure does the holomorphic vertex coalgebra structure endow to $\mathcal{C}$?

A.3.2. **The product $\otimes_z$.** To begin with, for any H modules M, N, the vertex coalgebra map

$$\Delta(z^{-1}) \;:\; H \;\to\; H \otimes H[[z]]$$

gives a map

$$H \otimes (M \otimes N) \;\to\; (M \otimes N)[[z]]$$

or equivalently, a $k[[z]]$ linear map

$$H \otimes (M \otimes N)[[z]] \;\to\; (M \otimes N)[[z]]$$

compatible with the algebra structure of H. Thus, we get

$$(M \otimes N)[[z]] \;\in\; H\text{-Mod}(k[[z]]\text{-Mod})$$

where H is endowed with a trivial $k[[z]]$ module action. In particular, endowing $k[[z]]$ with a trivial H module structure,

$$H\text{-Mod}(k[[z]]\text{-Mod}) \;=\; H \otimes k[[z]]\text{-Mod} \;=\; k[[z]]\text{-Mod}(H\text{-Mod}) \;=\; k[[z]]\text{-Mod}(\mathcal{C}),$$

which we have previously denoted $\mathcal{C}[[z]]$. Thus, we get a binatural functor

Definition A.3.3. If H is a holomorphic vertex bialgebra and $\mathcal{C}$ the category of modules over the underlying associative algebra, we set

$$\otimes_z \;=\; ((-) \otimes (-))\,[[z]] \;:\; \mathcal{C} \times \mathcal{C} \;\to\; \mathcal{C}[[z]].$$

We then see that

Lemma A.3.4. $\otimes_z$ *is a lax monoidal functor.*

Proof. Given modules M_1, M_2 and M_1', M_2', we have a natural map

$$(M_1 \otimes M_2)[[z]] \otimes (M_1' \otimes M_2')[[z]] \;\to\; ((M_1 \otimes M_1') \otimes (M_2 \otimes M_2'))[[z]]$$

and $1_{\mathcal{C}[[z]]} = k \to 1_{\mathcal{C}} \otimes_z 1_{\mathcal{C}} = k[[z]]$. It is easy to check associativity and unitality. $\square$

In particular, this allows us to compose the functors $\otimes_z$ and $\otimes_w$:

$$\begin{aligned} \mathcal{C} \times \mathcal{C} \times \mathcal{C} &\overset{\otimes_z}{\to} \mathcal{C}[[z]] \times \mathcal{C} \\ &= (k[[z]], k)\text{-Mod}(\mathcal{C} \times \mathcal{C}) \\ &\overset{\otimes_w}{\to} (k[[z]] \otimes_w k)\text{-Mod}(\mathcal{C}[[w]]) \\ &= k[[z, w]]\text{-Mod}(\mathcal{C}[[w]]) \\ &= k[[z]]\text{-Mod}(\mathcal{C}[[w]]) \\ &= \mathcal{C}[[z, w]]. \end{aligned}$$

We have used that $\otimes_w$ is lax monoidal hence induces a map on module categories. The result of the above will be denoted $\otimes_w(\otimes_z, \mathrm{id})$, and similarly for other compositions. To be very explicit, this functor is induced by the map

$$H \overset{\Delta(w^{-1})}{\to} (H \otimes H)[[w]] \overset{\Delta(z^{-1}) \otimes \mathrm{id}}{\to} ((H \otimes H)[[z]] \otimes H)[[w]] \hookrightarrow (H \otimes H \otimes H)[[z, w]]. \quad \text{(A.13)}$$

A.3.5. Note that strictly speaking $\otimes_w(\otimes_z, \mathrm{id})$ is an abuse of notation, since the functors $\otimes_w$ and $(\otimes_z, \mathrm{id})$ are not composable: instead of $\otimes_w$ we more precisely mean the map induced by $\otimes_w$ on module categories. Note that we have

$$\begin{tikzcd} & \mathcal{C}[[z]] \times \mathcal{C} \arrow[rr, "\otimes_w"] & & \mathcal{C}[[z, w]] \arrow[dr, "\mathrm{oblv}"] & \\ \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[ur, "(\otimes_z, \mathrm{id})"] \arrow[dr, "(\otimes_z, \mathrm{id})"'] & & & & \mathcal{C} \\ & \mathcal{C}[[z]] \times \mathcal{C} \arrow[r, "(\mathrm{oblv}, \mathrm{id})"] & \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes_w"] & \mathcal{C}[[w]] \arrow[ur, "\mathrm{oblv}"'] & \end{tikzcd}$$

so that the upper composition (which we are referring to as oblv composed with $\otimes_w(\otimes_z, \mathrm{id})$) is computed by applying $(\otimes_z, \mathrm{id})$ then forgetting the $k[[z]]$ module structure, then applying $\otimes_w$ and forgetting the $k[[w]]$ module structure.

A.3.6. **Associativity.** We may consider $\otimes_w(\mathrm{id}, \otimes_z)$ and $\otimes_z(\otimes_w, \mathrm{id})$. These give two different H module structures on $(L \otimes M \otimes N)[[z,w]] \in k[[z,w]]$-Mod, one induced by (A.13) and the other by

$$H \overset{\Delta(w^{-1})}{\to} (H \otimes H)[[w]] \overset{\mathrm{id}\otimes\Delta(z^{-1})}{\to} (H \otimes (H \otimes H)[[z]])[[w]] \hookrightarrow (H \otimes H \otimes H)[[z,w]]. \quad \text{(A.14)}$$

where oblv is the forgetful functor. In our notation above, these module structures are denoted $(L \otimes_z M) \otimes_w N$ and $L \otimes_w (M \otimes_z N)$. Notice that the two variables have swapped. Thus, we get two different functors which we denote

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \;\to\; \mathcal{C}[[z,w]]. \quad \text{(A.15)}$$

A.3.7. To understand (A.15), it says that there are two actions of $h \in H$ on the vector space $L \otimes M \otimes N$. Firstly, the H module structure on $((L \otimes M) \otimes N)[[z,w]]$ is as multiplication by

$$(e^{w\partial} \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(e^{z\partial} \otimes \mathrm{id})\Delta h \;=\; (e^{(z+w)\partial} \otimes e^{z\partial} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta h, \quad \text{(A.16)}$$

where we have used $\Delta e^{z\partial} = (e^{z\partial} \otimes e^{z\partial})\Delta$, and secondly the H module $L \otimes (M \otimes N)[[z,w]]$ is as multiplication by

$$(\mathrm{id} \otimes e^{w\partial} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(e^{z\partial} \otimes \mathrm{id})\Delta h \;=\; (e^{z\partial} \otimes e^{w\partial} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta h. \quad \text{(A.17)}$$

These differ by an (invertible) factor of $(e^{w\partial} \otimes e^{(z-w)\partial} \otimes \mathrm{id})$.

A.3.8. Before we define the analogue of an associator α, we first need a commutation Lemma

Lemma A.3.9. *Let (A, ∂) be an associative algebra with derivation and (M, ∂_M) an A module with derivation. Then as elements of* $\mathrm{End}(M[[z]])$*, for each $a \in A$ we have*

$$(e^{z\partial}a) \;=\; e^{z\partial_M} \cdot a \cdot e^{-z\partial_M}.$$

Proof. Follows from the definition $\partial_M(h \cdot m) = \partial(h) \cdot m + h \cdot \partial_M(m)$ of compatible derivation. □

Comparing equations (A.16) and (A.17) and applying Lemma A.3.9, it then follows that

Corollary A.3.10. *If L, M, N are H modules with compatible derivations, then there is an $k[[z,w]]$ linear isomorphism*

$$\alpha_{L,M,N}(z,w) \;:\; ((L \otimes M) \otimes N)[[z,w]] \;\tilde{\to}\; (L \otimes (M \otimes N))[[z,w]]$$

sending

$$(l \otimes m \otimes n) \;\mapsto\; (e^{w\partial_L} \otimes e^{(z-w)\partial_M} \otimes \mathrm{id})(l \otimes m \otimes n),$$

moreover it is an isomorphism of H modules

$$\alpha_{L,M,N}(z,w) \;:\; (L \otimes_z M) \otimes_w N \;\xrightarrow{\sim}\; L \otimes_w (M \otimes_z N).$$

Again, notice that the variables z, w have swapped. Here we have suppressed the identifications $(L \otimes M) \otimes N \simeq L \otimes (M \otimes N)$ coming from the background symmetric monoidal category from the notation.

Note that the above does *not* mean that H-Mod is a symmetric monoidal category, since we had to adjoin the formal variables z, w. In that case, how do we think about this structure? Answering this question is the point of the next section A.4.

A.3.11. **Failure of the naive pentagon identity.** For any objects c_1, c_2, c_3, c_4 of $\mathcal{C}$, we would like the following to commute

$$\begin{tikzcd}
 & (c_1 \otimes_w c_2) \otimes_z (c_3 \otimes_u c_4) \arrow[rr, "\alpha_{c_1 \otimes_w c_2, c_3, c_4}(z,u)"] & & ((c_1 \otimes_w c_2) \otimes_u c_3) \otimes_z c_4) \arrow[dr, "\alpha_{c_1,c_2,c_3}(w,u) \otimes_z \mathrm{id}"] & \\
c_1 \otimes_z (c_2 \otimes_w (c_3 \otimes_u c_4)) \arrow[ur, "\alpha_{c_1,c_2,c_3 \otimes_u c_4}(z,w)"] \arrow[drr, "\mathrm{id} \otimes_z \alpha_{c_2,c_3,c_4}(w,u)"'] & & & & (c_1 \otimes_u (c_2 \otimes_w c_3)) \otimes_z c_4 \\
 & & c_1 \otimes_z ((c_2 \otimes_u c_3) \otimes_w c_4) \arrow[urr, dashed, "?", "\alpha_{c_1,c_2 \otimes_u c_3, c_4}(z,w)"'] & &
\end{tikzcd}$$

However, the dotted arrow is not defined, since its codomain should be $(c_1 \otimes_w (c_2 \otimes_u c_3)) \otimes_z c_4$. To resolve this, we thus need a way by which we can switch the variable order.

A.3.12. **Variable commutativity.** Let L, M, N be H modules with a compatible derivation. Then $h \in H$ acts on $L \otimes_z (M \otimes_w N)$ as multiplication by

$$(\mathrm{id} \otimes e^{w\partial} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(e^{z\partial} \otimes \mathrm{id})\Delta h \;=\; (e^{z\partial} \otimes e^{w\partial} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta h.$$

and acts on $L \otimes_w (M \otimes_z N)$ as multiplication by

$$(\mathrm{id} \otimes e^{z\partial} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(e^{w\partial} \otimes \mathrm{id})\Delta h \;=\; (e^{w\partial} \otimes e^{z\partial} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta h.$$

Thus we have another Corollary of Lemma A.3.9

Corollary A.3.13. *If L, M, N are H modules with compatible derivations, then there is an $k[[z,w]]$ linear isomorphism*

$$\gamma_{L,M,N}(z,w) \;:\; (L \otimes (M \otimes N))[[z,w]] \;\xrightarrow{\sim}\; (L \otimes (M \otimes N))[[z,w]]$$

sending

$$(l \otimes m \otimes n) \;\mapsto\; (e^{(w-z)\partial_L} \otimes e^{(z-w)\partial_M} \otimes \mathrm{id})(l \otimes m \otimes n),$$

moreover it is an isomorphism of H modules

$$\gamma_{L,M,N}(z,w) \;:\; L \otimes_z (M \otimes_w N) \;\xrightarrow{\sim}\; L \otimes_w (M \otimes_z N).$$

It is then straightforward to check

Proposition A.3.14. *For any $c_1, c_2, c_3, c_4 \in \mathcal{C}$, the modified pentagon identity*

$$\begin{tikzcd}
& (c_1 \otimes_w c_2) \otimes_z (c_3 \otimes_u c_4) \arrow[rr, "\alpha_{c_1\otimes_w c_2, c_3, c_4}(z,u)"] & & ((c_1 \otimes_w c_2)\otimes_u c_3)\otimes_z c_4 \arrow[dr, "\alpha_{c_1,c_2,c_3}(w,u)\otimes_z \mathrm{id}"] & \\
c_1 \otimes_z (c_2 \otimes_w (c_3 \otimes_u c_4)) \arrow[ur, "\alpha_{c_1,c_2,c_3\otimes_u c_4}(z,w)"] \arrow[ddrr, "\mathrm{id}\otimes_z \alpha_{c_2,c_3,c_4}(w,u)"'] & & & & (c_1 \otimes_u (c_2 \otimes_w c_3))\otimes_z c_4 \\
& & & & (c_1 \otimes_w (c_2\otimes_u c_3))\otimes_z c_4 \arrow[u, "\gamma_{c_1,c_2,c_3}(u,w)\otimes_z \mathrm{id}"'] \\
& & c_1 \otimes_z ((c_2\otimes_u c_3)\otimes_w c_4) \arrow[urr, "\alpha_{c_1,c_2\otimes_u c_3,c_4}(z,w)"'] & &
\end{tikzcd} \tag{A.18}$$

commutes.

Note that if we ignore the explicit forms for γ and α, the content of this Proposition is that the modification needed to make the pentagon identity commute only depends on the first three factors and on u, w. We do not preclude the fact that there may be further compatibilities between γ and α.

A.3.15. Similarly, conjugating γ by α gives an isomorphism of H modules

$$\overline{\gamma}_{L,M,N}(z,w) \;:\; (L\otimes_z)M \otimes_w N \;\xrightarrow{\sim}\; (L \otimes_w M) \otimes_z N.$$

A.3.16. **Units.** Note that there is no obvious analogue of a right unit. However, there is a quotient map

$$M \otimes_z 1_{\mathcal{C}} \;\simeq\; M[[z]] \;\to\; M$$

where $h \in H$ acts on $M[[z]]$ as multiplication by $(e^{z\partial}h)$ and $k[[z]]$ as multiplication by z, and $k[[z]]$ acts trivially on M. Similarly, there is no left unit but an isomorphism

$$k \otimes_z M \;\simeq\; M[[t]]$$

where the right hand side is $\mathrm{triv}(\mathrm{oblv} M[[z]])$, i.e. equal to $M[[z]]$ as an element of $\mathcal{C}$ but with the trivial $k[[z]]$ action.

A.3.17. **Symmetric structure.** There is also an analogue of a symmetric monoidal structure.

Proposition A.3.18. *If M, N are H modules with compatible derivations, then there is an isomorphism of $k[[z]]$ modules*

$$\tau_{M,N}(z) \;:\; (M \otimes N)[[z]] \;\tilde{\to}\; (N \otimes M)[[z]]$$

sending

$$(m \otimes n) \;\mapsto\; (e^{-z\partial_N} \otimes e^{z\partial_M})(n \otimes m),$$

moreover it is an isomorphism of H modules

$$\tau_{M,N}(z) \;:\; M \otimes_z N \;\tilde{\to}\; N \otimes_z M.$$

It is easy to check that

Proposition A.3.19. *For any $c_1, c_2, c_3 \in \mathcal{C}$, the modified hexagon identity*

$$\begin{tikzcd}
 & (c_1 \otimes_w c_2) \otimes_z c_3 \arrow[r, "\tau_{c_1 \otimes_z c_2, c_3}(w)"] & c_3 \otimes_w (c_1 \otimes_z c_2) \arrow[dr, "\alpha_{c_3,c_1,c_2}(w,z)"] & \\
 & & & (c_3 \otimes_z c_1) \otimes_w c_2 \\
c_1 \otimes_z (c_2 \otimes_w c_3) \arrow[uur, "\alpha_{c_1,c_2,c_3}(z,w)"] \arrow[ddr, "\mathrm{id} \otimes_z \tau_{c_2,c_3}(w)"'] & & & \\
 & & & (c_3 \otimes_z c_1) \otimes_w c_2 \arrow[uu, "\gamma_{c_3,c_1,c_2}(z,w)"'] \\
 & c_1 \otimes_z (c_3 \otimes_w c_2) \arrow[r, "\alpha_{c_1,c_3,c_2}(z,w)"] & (c_1 \otimes_w c_3) \otimes_z c_2 \arrow[ur, "\tau_{c_1,c_3}(z) \otimes_z \mathrm{id}"'] &
\end{tikzcd} \tag{A.19}$$

commutes.

A.3.20. We remark that it is clearly an interesting question to ask what analogous structure the category of modules of a *non* holomorphic vertex bialgebra has, with the expectation that it is likely to be easier to work over the Ran space instead of piecemeal definitions involving power series. Also note that the above does not trivially generalise upon replacing $\mathcal{C}[[z]]$ with $\mathcal{C}((z))$, since $\mathcal{C}((z))$ does not inherit a symmetric monoidal structure from $\mathcal{C}$ as $k((z))$ does not have a natural coproduct and hence is not naturally a cocommutative bialgebra.

A.4 Spectral symmetric monoidal categories

Our definition of spectral symmetric monoidal categories is set up to describe the naturally arising examples we consider in section 2.6. One hopes that it is related to a (unmade as far the author knows) definition of a symmetric or braided monoidal category "over the Ran space of a curve, factorisably".

Spectral R matrices and spectral Yang Baxter equations were first discovered by Cherednik [Che]. Our definition is potentially related is Soibelman's [So] definition of a meromorphic tensor category.

We recommend reading section A.3 first, since that section contains the main example; this section is just an axiomatisation of that one.

A.4.1. **Spectral monoidal category.**

Definition A.4.2. A *spectral monoidal category* is a k linear symmetric monoidal category $\mathcal{C}$ with a lax monoidal functor

$$\otimes_z \;:\; \mathcal{C}\times\mathcal{C} \;\to\; \mathcal{C}[[z]]$$

together with natural isomorphisms[1]

$$\alpha_{c_1,c_2,c_3}(z,w) \;:\; c_1\otimes_z(c_2\otimes_w c_3) \;\tilde{\to}\; (c_1\otimes_w c_2)\otimes_z c_3$$

$$\gamma_{c_1,c_2,c_3}(z,w) \;:\; c_1\otimes_z(c_2\otimes_w c_3) \;\tilde{\to}\; c_1\otimes_w(c_2\otimes_z c_3)$$

for all $c, c_1, c_2, c_3 \in \mathcal{C}$, where α, γ are isomorphisms in the category $\mathcal{C}[[z,w]]$, satisfying the *spectral pentagon identity* (A.18) for all $c_1, c_2, c_3, c_4 \in \mathcal{C}$. It is called *unital* if there are natural maps

$$\lambda_c(z) \;:\; 1_{\mathcal{C}}\otimes_z c \;\to c \qquad\qquad \rho_c(z) \;:\; c\otimes_z 1_{\mathcal{C}} \;\to\; \mathrm{triv}c$$

satisfying the *spectral triangle identity*

$$\begin{tikzcd}[column sep=small]
c_1\otimes_z(1_{\mathcal{C}}\otimes_w c_2) \arrow[rr, "\alpha_{c_1,1_{\mathcal{C}},c_2}(z,w)"] \arrow[dr, "\mathrm{id}\otimes\lambda_{c_2}(w)"'] & & (c_1\otimes_w 1_{\mathcal{C}})\otimes_z c_2 \arrow[dl, "\rho_{c_1}(w)\otimes\mathrm{id}"] \\
& c_1\otimes_z c_2 &
\end{tikzcd}$$

for all $c_1, c_2 \in \mathcal{C}$.

A.4.3. Via the functor $\mathrm{triv} : \mathcal{C} \to \mathcal{C}[[z]]$, we may identify monoidal categories as (fairly trivial) examples of spectral monoidal categories, whose tensor product does not depend on z.

A.4.4. Let H be a holomorphic vertex bialgebra. Then the purpose of section A.3 was to show that the category H-Mod_∂ of H modules with compatible derivation is a unital spectral monoidal category, with

$$M\otimes_z N \;=\; (M\otimes N)[[z]] \qquad\qquad h\cdot(m\otimes n) \;:=\; \big((e^{z\partial}\otimes\mathrm{id})\Delta h\big)\cdot(m\otimes n).$$

[1]As before we have written $c[[t]] = \mathrm{triv}(\mathrm{oblv}c[[z]])$.

A.4.5. However, we note that $H\text{-Mod}_\partial$ is also a (nonunital) spectral monoidal category for the product

$$M \otimes_z N \;=\; (M\otimes N)((z)) \qquad\qquad h\cdot(m\otimes n) \;:=\; \big((e^{z\partial}\otimes \mathrm{id})\Delta h\big)\cdot(m\otimes n).$$

A.4.6. More generally, if A is any $k[[z]]$ algebra then

$$M \otimes_z N \;=\; (M\otimes N)\otimes A \qquad\qquad h\cdot(m\otimes n) \;:=\; \big((e^{z\partial}\otimes \mathrm{id})(\Delta h\otimes 1_A)\big)\cdot(m\otimes n)\otimes 1_A$$

defines a spectral monoidal structure, where $z\partial = z\otimes\partial$ is an endomorphism of $H\otimes A$. It is unital if there is a map of $k[[z]]$ algebras $A\to k$.

A.4.7.

Definition A.4.8. Let $\mathcal{C}$ be a spectral k linear monoidal category. A *spectral symmetric monoidal* structure on $\mathcal{C}$ is a $k((z))$ linear binatural isomorphism

$$\tau_{c_1,c_2}(z) \;:\; c_1\otimes_z c_2 \;\xrightarrow{\sim}\; c_2\otimes_z c_1$$

with *symmetry condition* $\tau_{c_1,c_2}(z)\tau_{c_2,c_1}(-z) = \mathrm{id}$, satisfying the *spectral hexagon identity* (A.19) and the following commutes

$$\begin{tikzcd} 1_{\mathcal{C}}\otimes_z c \arrow[rr, "\tau_{1_{\mathcal{C}},c}(z)"] \arrow[dr, "\lambda_c(z)"'] & & 1_{\mathcal{C}}\otimes_z c \arrow[dl, "\rho_c(z)"] \\ & c & \end{tikzcd}$$

for all $c\in\mathcal{C}$, which we write as $\rho_c(z)\tau_{1_{\mathcal{C}},c}(z) = \lambda_c(z)$.

A.4.9.

Definition A.4.10. Let $\mathcal{C}$ be a k linear spectral monoidal category. A *spectral braided monoidal* structure on $\mathcal{C}$ is a binatural isomorphism

$$\beta_{c_1,c_2}(z) \;:\; c_1\otimes_z c_2 \;\to\; c_2\otimes_z c_1$$

and satisfying the *spectral hexagon identities*: (A.19) and

$$
\begin{tikzcd}
 & c_1 \otimes_w (c_2 \otimes_z c_3) \arrow[r, "\beta_{c_1,c_2\otimes_z c_3}(w)"] & (c_2 \otimes_z c_3) \otimes_w c_1 \arrow[dr, "\alpha_{c_2,c_3,c_1}(w,z)^{-1}"] & \\
(c_1 \otimes_z c_2) \otimes_w c_3 \arrow[ur, "\alpha_{c_1,c_2,c_3}(w,z)^{-1}"] \arrow[ddr, "\beta_{c_1,c_2}(z)\otimes_w \mathrm{id}"'] & & & c_2 \otimes_w (c_1 \otimes_z c_3) \\
 & & & c_2 \otimes_z (c_3 \otimes_w c_1) \arrow[u, "\gamma_{c_3,c_1,c_2}(z,w)"'] \\
 & (c_2 \otimes_z c_1) \otimes_w c_3 \arrow[r, "\alpha_{c_2,c_1,c_3}(w,z)^{-1}"] & c_2 \otimes_w (c_1 \otimes_w c_3) \arrow[ur, "\mathrm{id}\otimes_z \beta_{c_1,c_3}(w)"'] &
\end{tikzcd}
\tag{A.20}
$$

and finally $\rho_c(z) = \lambda_c(z)\beta_{c,e}(z)$ and $\beta_{e,c}(-z)\rho_c(z) = \lambda_c(z)$.

A.4.11. **Vertex coalgebras in a general spectral symmetric monoidal category.** Let $\mathcal{C}$ be a spectral symmetric monoidal k linear category, and denote $1_{\mathcal{C}} = k$.

Definition A.4.12. A (*nonlocal*) *vertex coalgebra* in $\mathcal{C}$ is an object c with a map in $\mathcal{C}$

$$\Delta(z) \ : \ c \ \to \ c \otimes_z c$$

as well as maps $\langle 0| : c \to k$ and $T : c \to c$, satisfying the analogous axioms to those in Definition A.2.5 (A.2.8). Here we have written $c \otimes_z c$ for $\mathrm{oblv}(c \otimes_z c)$.

A.4.13. Taking $\mathcal{C} = \mathrm{Vect}_k$ recovers the usual definition of coalgebra, holomorphic vertex coalgebra and vertex coalgebra, by taking

$$\otimes_z \ = \ \otimes, \ ((-) \otimes (-))[[z]], \ ((-) \otimes (-))((z)).$$

A.4.14. Let $\mathcal{C} = H\text{-}\mathrm{Mod}_\partial$ be as in section A.3 and τ is the spectral symmetric braiding as in section A.3.17. If we are in Joyce's example where $H = (\mathrm{H}^\bullet(\mathcal{M}), \cdot, \oplus^*)$ and $V = \mathrm{H}^\bullet(\mathcal{M})$ is as in Theorem 2.6.18 then the commutation relation

$$(e^{-zt} \otimes e^{zt})\Psi(\theta, z) \ = \ \Psi(\theta, -z)(e^{-zt} \otimes e^{zt})$$

implies that

$$\sigma\Delta(-z)e^{zT} \ = \ \tau_{V,V}(z)\Delta(z).$$

In particular, this suggests that the opposite of a vertex coalgebra as in Definition A.4.12 should be defined as $\tau_{V,V}(z)\Delta(z)$.

A.5 Yang Baxter matrices

A.5.1. Let $\mathcal{V}$ be a symmetric monoidal category, with symmetric braiding σ.

A.5.2. Let $H = (H, \cdot, \oplus^*)$ be a cocommutative bialgebra in $\mathcal{V}$, and $R \in H \otimes H$ an invertible element with:

1. $\sigma \oplus^* (h) = R \oplus^* (h) R^{-1}$, e.g. R is central (since $\oplus^*$ cocommutative).
2. R satisfies the hexagon identities

$$(\oplus^* \otimes \mathrm{id})(R) \;=\; R_{13}R_{23} \qquad\qquad (\mathrm{id} \otimes \oplus^*)(R) \;=\; R_{13}R_{12}, \tag{A.21}$$

   where for instance R_{12} means $R \otimes 1$.

It follows that

Lemma A.5.3. *R satisfies the Yang Baxter equation*

$$R_{12}R_{13}R_{23} \;=\; R_{23}R_{13}R_{12}. \tag{A.22}$$

Proof. We have

$$\begin{aligned} R_{12}R_{13}R_{23} \;&=\; R_{12}(\oplus^* \otimes \mathrm{id})(S) \;=\; (\sigma \oplus^* \otimes \mathrm{id})(S)R_{12} \\ &=\; R_{23}R_{13}R_{12}. \end{aligned}$$

□

Definition A.5.4. Such an $R \in H \otimes H$ is called a *Yang Baxter matrix.*

Lemma A.5.5. *There is an equivalence of categories between symmetric monoidal structures on* $\mathcal{C} = H$-Mod *and invertible elements R satisfying* 1. *and* 2.

Proof sketch. Any such R induces the braiding, denoted β_R,

$$\beta_{R,M,N} \;:\; M \otimes N \xrightarrow{R\cdot} M \otimes N \xrightarrow{\sigma} N \otimes M.$$

Conversely, if β is a braiding, we can take $R = \sigma\beta_{H,H}(1)$. □

A.5.6. Let A be a cocommutative coalgebra in $(H\text{-Mod}, \beta_R)$. Equivalently, this means it is a coalgebra with

1. $\beta_R \Delta(a) = \Delta(a)$ for all $a \in A$.
2. $\sigma\Delta(a) = R^{-1}\Delta(a)$ for all $a \in A$.

Lemma A.5.7. *If A is in addition a cocommutative bialgebra, R satisfies the Δ hexagon identities*

$$(\Delta \otimes \mathrm{id}) \cdot S \;=\; R_{13}R_{23} \cdot (\Delta \otimes \mathrm{id}) \qquad\qquad (\mathrm{id} \otimes \Delta) \cdot S \;=\; R_{13}R_{12} \cdot (\Delta \otimes \mathrm{id})$$

as morphisms $A^{\otimes 2} \to A^{\otimes 3}$.

Proof. By H linearity we have $\Delta(ah) = \oplus^*(a) \cdot \Delta(a)$, then the Lemma follows by the $\oplus^*$ hexagon identities (A.21). $\square$

A.5.8. **Spectral Yang Baxter matrices.** Let $\mathcal{V}$ be a background k linear symmetric monoidal category with symmetric braiding σ.

A.5.9. Let $H = (H, \cdot, \oplus^*, T)$ be a cocommutative bialgebra with derivation in $\mathcal{V}$. Equivalently, H is a holomorphic vertex bialgebra with

$$\oplus^*(z) \;=\; (e^{zT} \otimes \mathrm{id}) \cdot \oplus^* \;:\; H \;\to\; H \otimes H[[z]].$$

A.5.10. We endow the category $\mathcal{C} = H\text{-Mod}_\partial$ of H modules with a compatible derivation with the spectral monoidal structure $M \otimes_z N = (M \otimes N)[[z^{\pm 1}]]$. Thus $\oplus^*(z)$ can be viewed as a map

$$\oplus^*(z) \;:\; H \;\to\; H \otimes_z H.$$

Since $\otimes_z$ is lax monoidal, there is an algebra structure on $H \otimes_z H$, and for any modules $M, N \in H\text{-Mod}_\partial$, $M \otimes_z N$ is a module for this algebra.

Definition A.5.11. A *spectral Yang Baxter matrix* is an invertible element $R(z) \in H \otimes_z H$ or invertible map $R(z) : H \otimes_z H \to H \otimes_z H$ in $\mathcal{C}[[z]]$, with:

1. $\oplus^{*,op}(h, z) = R(z) \oplus^* (h, z) R(z)^{-1}$, e.g. $R(z)$ is central.[2]
2. $R(z)$ satisfies the spectral hexagon identities

$$(\oplus^*(w) \otimes_z \mathrm{id})(R(z)) \;=\; R_{13}(z + w)R_{23}(z) \qquad (\mathrm{id} \otimes_z \oplus^*(w))(R(z)) \;=\; R_{13}(z)R_{12}(z - w). \tag{A.23}$$

[2]Since $\oplus^*(z)$ is a (local) vertex coalgebra and so $\oplus^*(z) = \oplus^{*,op}(z)$.

Here we have denoted $R_{13}(z+w)$ and $R_{23}(z)$ for the images of $R(w)\otimes_z \mathrm{id}$ and $\mathrm{id}\otimes R(w)$ under

$$(H\otimes_w H)\otimes_z H \ \xrightarrow{\sim}\ H\otimes_z (H\otimes_w H) \ \xrightarrow{\mathrm{id}\otimes\tau_{H,H}(w)}\ H\otimes_z (H\otimes_w H) \ \xrightarrow{\sim}\ (H\otimes_w H)\otimes_z H$$

$$H\otimes_z (H\otimes_w H) \ \xrightarrow{\sim}\ (H\otimes_w H)\otimes_z H$$

respectively, and likewise for $R_{13}(z)$ and $R_{12}(z-w)$.[3]

A.5.12. In our running moduli stack example $H = (\mathrm{H}^\bullet(\mathcal{M}), \oplus^*, \cdot)$, the hexagon relations are satisfied by $R(z) = \Psi(\mathcal{V}, z)$ or $R(z) = \Psi(\mathcal{W}, z)^{-1}$ for any $\mathcal{V}, \mathcal{W} \in \mathrm{Perf}(\mathcal{M}\times\mathcal{M})$ with $\mathrm{B}\mathbf{G}_m$ weights $(-1,1)$ and $(1,-1)$ respectively. In this case, we can understand the above notation in the hexagon identities more easily: as an element of $H\otimes H\otimes H$ we have

$$R_{ij}(t) \ = \ \pi_{ij}^* R(t)$$

for any variable t, where $\pi_{ij} : \mathcal{M}^3 \to \mathcal{M}^2$ is the ijth projection.

A.5.13. When $R(z)$ is an endomorphism that is not simply multiplication by an element in the algebra structure of $H\otimes_z H$, we sometimes also denote it by $S(z)$.

Lemma A.5.14. *$R(z)$ satisfies the spectral Yang Baxter equation*

$$R_{12}(z)R_{13}(z+w)R_{23}(w) \ = \ R_{23}(w)R_{13}(z+w)R_{12}(z). \tag{A.24}$$

Lemma A.5.15. *Invertible elements (or endomorhisms) $R(z)$ satisfying* 1. *and* 2. *induce a spectral symmetric monoidal structure on the category H-Mod_∂ of H modules with compatible derivation.*

Proof sketch. Any such $R(z)$ induces the braiding, denoted $\beta_R(z)$,

$$\beta_{R,M,N}(z) \ : \ M\otimes_z N \ \xrightarrow{R(z)\cdot}\ M\otimes_z N \ \xrightarrow{\tau_{M,N}(z)}\ N\otimes_z M.$$

□

We expect that the converse to this Lemma is also true.

[3] These definitions are made precisely so that if we take the spectral hexagon identities (A.19), (A.20) and replace $\tau(z)$ with $R(z)\tau(z)$, then the diagrams still commute.

A.5.16. Let V be a local vertex coalgebra in $(H\text{-Mod}, \beta_R(z))$, see Appendix A. Equivalently, this means it is a nonlocal vertex coalgebra with

1. $\beta_R(z)\Delta(e^{zT}\alpha, -z) = \Delta(\alpha, z)$ for all $\alpha \in V$.
2. $\Delta^{op}(\alpha, z) = \sigma\Delta(e^{zT}\alpha, -z) = R(z)^{-1}\Delta(\alpha, z)$ for all $\alpha \in V$.

Lemma A.5.17. *If V is in addition a local vertex bialgebra, $R(z)$ satisfies the $\Delta(z)$ hexagon identities*

$$(\Delta(z) \otimes \mathrm{id}) \cdot R(w) = R_{13}(z + w)R_{23}(w) \cdot (\Delta(z) \otimes \mathrm{id}), \tag{A.25}$$

$$(\mathrm{id} \otimes \Delta(z)) \cdot R(w) = R_{13}(w)R_{12}(w - z) \cdot (\mathrm{id} \otimes \Delta(z)). \tag{A.26}$$

A.5.18. We note that there is a discrepancy with section A.2.19. Indeed, one would hope that Joyce's nonlocal vertex coalgebra structure on $\mathrm{H}^\bullet(\mathcal{M})$ is almost local, for the same

$$S(z) = \Psi(\theta, z)/\Psi(\sigma^*\theta, z)$$

as appearing in Theorem 3.10.1, which defines a spectral symmetric monoidal structure on $\mathcal{C}$. However, Joyce's nonlocal vertex coalgebra structure is actually almost nonlocal in the sense of section A.2.19 for

$$S(z) = \Psi(\theta, z)/\Psi(\sigma^*\theta^\vee, z).$$

Note that $\theta, \sigma^*\theta^\vee$ both have weights $(-1, 1)$, so this second expression for $S(z)$ cannot be expected to satisfy the hexagon relations. This suggests that the definition of almost locality in a spectral monoidal category should be possible to make, but is different from that definition appearing in section A.2.19.

Appendix B

Kac Moody algebras

Studying Kac Moody Lie algebras gives a common framework under which to study both the Lie algebras of finite dimensional algebraic groups G and of their loop spaces LG, centrally extended. Despite sometimes being infinite dimensional, they are almost as well behaved as finite dimensional Lie algebras, e.g. in their representation theory.

In this section we give a brief review of Kac Moody algebras and their representations. See e.g. [Ca, Kac, CP] for more details.

B.1 Finite dimensional Lie algebras

For motivation we begin by reviewing some facts about finite dimensional Lie algebras. We will work over an algebraically closed ground field of characteristic 0.

B.1.1. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Pick a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ (sometimes denoted $\mathfrak{b}^+$), and an opposite Borel subalgebra $\mathfrak{b}^-$. Their intersection $\mathfrak{b}^+ \cap \mathfrak{b}^-$, which is a Cartan subalgebra, denoted $\mathfrak{t}$. Write $\mathfrak{n}$ and $\mathfrak{n}^-$ for the unipotent radicals of $\mathfrak{b}$ and $\mathfrak{b}^-$; we have $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, and

$$\mathfrak{g} \;=\; \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+.$$

The action of $\mathfrak{t}$ splits $\mathfrak{g}$ into eigenspaces

$$\mathfrak{g} \;=\; \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

and the nonzero eigenvalues $\Phi \subseteq \mathfrak{t}^*$ are called *roots*. They consist of *positive* and *negative roots* $\Phi^\pm \subseteq \Phi$, defined as the eigenvalues of the action of $\mathfrak{t}$ on $\mathfrak{n}^\pm$. A positive root which is not the sum of any two other positive roots is called a *simple root*, write $\{\alpha_i\} \subseteq \Phi^+$ for these.

Their $\mathbf{Z}$ span of the roots forms a lattice called the *root lattice* $\Delta \subseteq \mathfrak{t}^*$, and the cones associated to $\Phi^\pm$ are denoted $\Delta^\pm$. The simple roots form a basis for Δ.

B.1.2. For example, $\mathfrak{g} = \mathfrak{sl}_2$ splits into three one dimensional eigenspaces

$$\begin{array}{ccc} e & h & f \\ \bullet & \bullet & \bullet \end{array}$$

and the there is one irreducible representation of $\mathfrak{sl}_2$ for each dimension $n \geqslant 0$, whose one dimensional h-eigenspaces are

$$\begin{tikzcd} \overset{-n}{\bullet} \arrow[r, bend left] & \overset{-n+2}{\bullet} \arrow[l, bend left] \arrow[r, bend left] & \cdots \arrow[l, bend left] \arrow[r, bend left] & \overset{n-2}{\bullet} \arrow[l, bend left] \arrow[r, bend left] & \overset{n}{\bullet} \arrow[l, bend left] \end{tikzcd}$$

where we have labelled the eigenvalue (or *weight*) and represented the action of e and f on the eigenspaces by left and right moving arrows. The roots are $\Phi = \{\pm 2\}$ and the root lattice is $\Delta = 2\mathbf{Z}$.

B.1.3. Returning to the general case, $\dim \mathfrak{g}_\alpha = 1$ for any root α. By the Jacobson Morozov theorem there is a unique copy of $\mathfrak{sl}_2$

$$\mathfrak{sl}_{2,\alpha} \;=\; k\{f_\alpha, h_\alpha, e_\alpha\} \;\subseteq\; \mathfrak{g}$$

where f_α and e_α lie in the eigenspaces $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_\alpha$. Requiring that $\alpha(h_\alpha) = 2$ uniquely determines these three generators, and implies

$$\alpha(h_\beta) \;=\; \text{weight of } \mathfrak{g}_\beta \text{ under the action of } \mathfrak{sl}_{2,\alpha} \text{ on } \mathfrak{g}.$$

The set of the h_α are called *coroots* and are denoted $\Phi^\vee \subseteq \mathfrak{t}$, and their $\mathbf{Z}$ span is the *coroot lattice* $\Delta^\vee \subseteq \mathfrak{t}$.

B.1.4. The *Cartan* bilinear form on Δ is defined on positive roots by

$$A \;:\; \Delta \times \Delta \;\to\; \mathbf{Z} \qquad\qquad (\alpha, \beta) \;\mapsto\; \alpha(h_\beta)$$

and extended it to all of Δ by linearity. It is not symmetric in general. With respect to the basis of simple roots α_i, it is represented by the matrix $A_{ij} = A(\alpha_i, \alpha_j)$, which one can show is an example of

Definition B.1.5. A *generalised Cartan matrix* is an integral matrix A_{ij} with $A_{ii} = 2$, $A_{ij} \leqslant 0$ for $i \neq j$, and $A_{ji} = 0$ whenever $A_{ij} = 0$.

Indecomposable generalised Cartan matrices split into three classes:

1. *Finite* if all principal minors are positive.

2. *Affine* if all of its *proper* principal minors are positive and $\det A = 0$.

3. *Indefinite* otherwise.

If $\mathfrak{g}$ is a simple finite dimensional Lie algebra, its Cartan matrix has finite type, and conversely

Theorem B.1.6. *There is a bijection between simple finite dimensional Lie algebras and indecomposable generalised Cartan matrices which have finite type.*

B.2 Kac Moody algebras

The inverse construction producing a Lie algebra from Cartan matrix works for *any* $n \times n$ matrix A. The resulting Lie algebra $\mathfrak{g}(A)$ is called the *Kac-Moody Lie algebra* attached to A.

B.2.1. First, we take the Cartan $\mathfrak{t}(A)$, which is a vector space of dimension

$$\dim \mathfrak{t}(A) \;=\; n + \mathrm{corank} A,$$

along with subsets $\{h_1, ..., h_n\} \subseteq \mathfrak{t}(A)$ and $\{\alpha_1, ..., \alpha_n\} \subseteq \mathfrak{t}(A)^*$ with $\alpha_i(h_j) \;=\; A_{ij}$. There is a unique choice of this data up to isomorphism. Then we form the Lie algebra $\mathfrak{g}_0(A)$ with so-called "Chevalley" generators e_i, h_i, f_i and relations

$$\begin{aligned} [h_i, h_j] &= 0, & \qquad [h_i, e_j] &= \alpha_j(h_i) e_j, \\ [e_i, f_j] &= \delta_{i,j} h_i, & \qquad [h_i, f_j] &= -\alpha_j(h_i) f_j. \end{aligned}$$

Writing $\mathfrak{r}$ for its maximal ideal, the Kac Moody algebra is $\mathfrak{g}(A) := \mathfrak{g}_0(A)/\mathfrak{r}$. When A is indecomposable of finite type, $\mathfrak{r}$ is generated by the *Serre relations* for $i \neq j$:

$$(\mathrm{ad} e_i)^{1-A_{ij}} e_j \;=\; (\mathrm{ad} f_i)^{1-A_{ij}} f_j \;=\; 0.$$

B.2.2. Write $\mathfrak{n}$ (or $\mathfrak{n}^+$) and $\mathfrak{n}^-$ for the subalgebras generated by the e_i and the f_i, respectively. For all Kac Moody Lie algebras $\mathfrak{g} = \mathfrak{g}(A)$ we have

$$\mathfrak{g} \;=\; \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+. \tag{B.1}$$

Similarly, we can write $\mathfrak{b}$ and $\mathfrak{b}^-$ for the direct sum of $\mathfrak{n}$ and $\mathfrak{n}^-$ with $\mathfrak{t}$.

B.2.3. A well behaved category of representations is *category* $\mathcal{O}$. Its objects are representations V on which $\mathfrak{t}$ acts diagonalisably with finite dimensional weight spaces, and so that the set of weights with nonzero eigenspaces is contained in the downward closure of a finite set $\lambda_1, ..., \lambda_s \in \mathfrak{t}^*$.

The *Verma module* of weight $\lambda \in \mathfrak{t}^*$ is

$$M_\lambda \;:=\; \mathrm{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})} k_\lambda$$

where we have composed $\mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \simeq \mathfrak{t}$ to get a one dimensional representation of $\mathfrak{b}$. Its simple quotient is written L_λ. It is easy to check that both live in category $\mathcal{O}$.

Theorem B.2.4. [Kac, 9.3] *The L_λ are the irreducible representations in $\mathcal{O}$.*

B.2.5. The Cartan matrix is related to the the *Killing form*. If $\mathfrak{g}$ is finite dimensional it is the invariant bilinear form defined by

$$\kappa \;:\; \mathfrak{g} \times \mathfrak{g} \;\to\; k \qquad\qquad (x, y) \;\mapsto\; \mathrm{tr}_{\mathfrak{g}}(xy)$$

and if $\mathfrak{g}$ is simple it is the unique such form up to scaling. For any matrix A which is symmetrisable (the product DS of an invertible diagonal matrix D and a permutation matrix S), the Kac Moody algebra $\mathfrak{g}(A)$ also carries an invariant bilinear form by [Kac, 2.2]. Its restriction to the Cartan determines the Cartan matrix

$$A(\alpha_i, \alpha_j) \;=\; 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

When $\mathfrak{g} = \mathfrak{sl}_2$ we have $(h, h) = 8$, $(e, f) = 4$ and other products are zero. Its Cartan matrix is $A = (2)$.

B.3 Affine Lie algebras

The Lie algebras of loop spaces LG give rise to Kac Moody algebras of affine type. In physics one studies projective representations of LG, or equivalently representations of a central extension of LG.

B.3.1. The *loop algebra* of a finite dimensional Lie algebra $\mathfrak{g}$ is

$$L\mathfrak{g} \;:=\; \mathfrak{g} \otimes_k k[t, t^{-1}].$$

Writing $x_n = x \otimes t^n$, the Lie algebra structure is given by $[x_n, y_m] = [x, y]_{n+m}$. Fix an invariant bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$. The *affine Lie algebra* is the central extension

$$0 \;\to\; kc \;\to\; \widehat{\mathfrak{g}} \;\to\; L\mathfrak{g} \;\to\; 0,$$

and having choosing a splitting as a vector space, its Lie bracket is

$$[x_n, y_m] \ := \ [x,y]_{n+m} + n\delta_{n+m,0}\kappa(x,y)c.$$

B.3.2. Central extensions are classified by $\mathrm{H}^2(L\mathfrak{g}, k)$. Attached to κ we get the residue two cocycle

$$f(t) \wedge g(t) \ \mapsto \ \mathrm{Res}_{t=0}\kappa(f'(t), g(t))dt$$

and $\widehat{\mathfrak{g}}$ is the resulting central extension.

B.3.3. When $\mathfrak{g}$ is simple we can (non-centrally) extend once more to get a Kac Moody vertex algebra

$$0 \ \to \ \widehat{\mathfrak{g}} \ \to \ \widehat{\mathfrak{g}}' \ \to \ kd \ \to \ 0$$

where $[d, -]$ acts on $\widehat{\mathfrak{g}}$ as $t\partial_t$, so in particular x_n is an eigenvector with eigenvalue n.

Proposition B.3.4. [Kac, §7.4] *$\widehat{\mathfrak{g}}' = \mathfrak{g}(\widehat{A})$ is a Kac Moody algebra with Cartan matrix*

$$\widehat{A} \ = \ \begin{pmatrix} 2 & -\theta(h_j) \\ -\alpha_i(h_\theta) & A_{ij} \end{pmatrix}.$$

Its Cartan subalgebra is

$$\mathfrak{t}(\widehat{A}) \ = \ \mathfrak{t} \oplus kc \oplus kd,$$

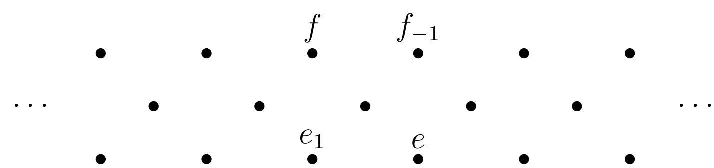
within which the simple coroots are $h_0 = c - h_\theta$ along with the simple coroots h_i of $\mathfrak{t}$. Likewise, using the the dual basis $\mathfrak{t}(\widehat{A})^* = \mathfrak{t}^* \oplus kc^* \oplus kd^*$ we have simple root $\alpha_0 = d^* - \theta$, from which we can deduce the form of the Cartan matrix above. Writing e_i, f_i for the Chevalley generators of $\mathfrak{g}$, the Chevalley generators of $\widehat{\mathfrak{g}}'$ are

$$E_i \ = \ e_i, \ F_i \ = \ f_i, \qquad E_0 \ = \ e_\theta \otimes t, \ F_0 \ = \ f_\theta \otimes t^{-1}$$

where $e_\theta \in \mathfrak{g}_{-\theta}$ and $f_\theta \in \mathfrak{g}_\theta$ form part of an $\mathfrak{sl}_2$ triple where θ is the highest root of $\mathfrak{g}$.

One usually denotes $\Lambda = c^*$.

B.3.5. For instance, in the $\widehat{\mathfrak{sl}}_2$ case the Chevalley generators are $e, f_{-1} = f \otimes t^{-1}$ and $e_1 = e \otimes t, f$. Thus, we can draw the weight space decomposition of $\widehat{\mathfrak{sl}}_2$ as



This picture is a subset of the two dimensional space $\mathfrak{t}^* \oplus kd^* \subseteq \mathfrak{t}(\widehat{A}_1)$ with horizontal and vertical axes given by the α_0 and α_1 coefficients.

B.3.6. The triangular decomposition (B.1) for affine vertex algebras given by the Chevalley generators in equation

$$\widehat{\mathfrak{g}} \;=\; (t^{-1}\mathfrak{g}[t^{-1}] \oplus \mathfrak{n}_-) \oplus (\mathfrak{t} \oplus kc) \oplus (\mathfrak{n}_+ \oplus t\mathfrak{g}[t]).$$

In the $\widehat{\mathfrak{sl}}_2$ case, this corresponds in the root space picture to negative, zero and positive values of the horizontal coordinate.

B.3.7. **Representations.** Since $c \in \widehat{\mathfrak{g}}$ is central, it acts on any irreducible representation of $\widehat{\mathfrak{g}}$ by a scalar, which is called the *level* of that representation. Thus irreducible representations are parametrised by levels $\ell \in k$ and a weight $\lambda \in \mathfrak{t}^*$ of the finite dimensional Lie algebra.

Appendix C

Fundamental classes and the exponential map

This appendix collects a lot of the technical machinery we will need. If $s : Z \hookrightarrow X$ is closed embedding of Artin stacks, in the first two sections we construct the cohomological exponential map

$$\mathrm{H}^\bullet(\exp_s) \; : \; \mathrm{H}^\bullet(Z/X) \; \to \; \mathrm{H}^\bullet(Z/\mathbf{N}_{Z/X})$$

and explain how it interlaces to the bivariant Euler classes (Definition 3.2.4), and, when it is defined, fundamental classes (Definition C.4.1)

$$\cdot e(Z/X) \; : \; \mathrm{H}^\bullet(Z/X) \; \to \; \mathrm{H}^\bullet(X), \qquad\qquad \cdot[Z/X] \; : \; \mathrm{H}^\bullet(Z) \; \to \; \mathrm{H}^\bullet(Z/X),$$

with $e(Z/\mathbf{N}_s)$ and $[Z/\mathbf{N}_s]$.

We also give relative statements of these results. In practice, often the closed embedding s is not quasismooth but the individual spaces Z and X are (over some common base), and so although the fundamental class $[Z/X]$ is not defined, $[Z]$ and $[X]$ are. In this case, we say how the cohomological exponential map interlaces $[Z]$ and $[X]$.

In the last two sections, we record the construction of fundamental classes and umkehr maps for the reader's convenience.

C.1 Toy model: topological case

If $i : Z \to X$ is a closed embedding of manifolds, we can define the exponential map on an open neighbourhood of Z inside the normal bundle $\mathbf{N}_i$

$$\begin{tikzcd} & \mathbf{N}_i \arrow[d, dashed, "\exp"] \\ Z \arrow[ur, hook] \arrow[r, hook] & X \end{tikzcd}$$

By the tubular neighbourhood theorem, exp gives is an open immersion of manifolds. If exp were to extend to the whole normal bundle, we would get a fundamental class

$$[\mathbf{N}_i/X] \in \mathrm{H}^0(\mathbf{N}_i/X),$$

multiplication by which is compatible with Euler and fundamental classes

$$\begin{tikzcd} \mathrm{H}^\bullet(Z) \arrow[r, equal] \arrow[d, "\cdot e(Z/X)"'] & \mathrm{H}^\bullet(Z) \arrow[d, "\cdot e(Z/\mathbf{N}_i)"] \\ \mathrm{H}^\bullet(Z/X) & \mathrm{H}^\bullet(Z/\mathbf{N}_i) \arrow[l, "\cdot[\mathbf{N}_i/X]"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{H}^\bullet(Z) \arrow[r, equal] \arrow[d, "{[Z/X]}"'] & \mathrm{H}^\bullet(Z) \arrow[d, "{[Z/\mathbf{N}_i]}"] \\ \mathrm{H}^{\bullet-c}(Z/X) & \mathrm{H}^{\bullet-c}(Z/\mathbf{N}_i) \arrow[l, "\cdot[\mathbf{N}_i/X]"', "\sim"] \end{tikzcd} \tag{C.1}$$

Here c is the real codimension of the closed embedding.

C.2 Cohomological exponential map

In algebraic geometry we cannot define an exponential map. Fortunately, we *can* in algebraic geometry define analogues of the horizontal arrows in (C.1).

Let $s : Z \to X$ be any map for which deformation to the normal complex is defined, e.g. closed embeddings of finite presentation or quasismooth maps. Define the *cohomological exponential map*

$$\exp \ : \ \mathrm{H}^\bullet(Z/\mathbf{N}_s) \ \to \ \mathrm{H}^\bullet(Z/X)$$

by composing the two dotted arrows

$$\begin{tikzcd} \mathrm{H}^\bullet(Z/X) \arrow[dr, "{[X/X\times\mathbf{A}^1]}", "\sim"'] & & & \mathrm{H}^\bullet(Z\times\mathbf{G}_m/X\times\mathbf{G}_m) \arrow[lll, dashed] \\ & \mathrm{H}^\bullet(Z/X\times\mathbf{A}^1) & \mathrm{H}^\bullet(Z\times\mathbf{A}^1/X\times\mathbf{A}^1) \arrow[l, "{[Z/Z\times\mathbf{A}^1]}"', "\sim"] \arrow[ur, "j^*_{X\times\mathbf{A}^1}"] & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{H}^\bullet(Z\times\mathbf{G}_m/X\times\mathbf{G}_m) & & & \mathrm{H}^\bullet(Z/\mathbf{N}_s) \arrow[lll, dashed] \arrow[dl, "{[\mathbf{N}_s/D_s]}"'] \\ & \mathrm{H}^\bullet(Z\times\mathbf{A}^1/D_s) \arrow[ul, "j^*_{D_s}"] \arrow[r, "{[Z/Z\times\mathbf{A}^1]}", "\sim"'] & \mathrm{H}^\bullet(Z/D_s) & \end{tikzcd}$$

The first dotted map is defined by picking a retraction of $j^*_{X\times\mathbf{A}^1}$, so the the map is independent of this choice. Note that $\mathbf{N}_s \to D_s$ is the pullback of $0 \to \mathbf{A}^1$, so is quasismooth and has a fundamental class.

It is easy to show that

Proposition C.2.1. *If $i : Z \to X$ is a closed embedding of finite presentation (resp. and is quasismooth), we have the left (resp. and the right) commuting diagrams*

$$\begin{tikzcd} \mathrm{H}^\bullet(Z) \arrow[r, equal] \arrow[d, "\cdot e(Z/X)"'] & \mathrm{H}^\bullet(Z) \arrow[d, "\cdot e(Z/\mathbf{N}_i)"] \\ \mathrm{H}^\bullet(Z/X) & \mathrm{H}^\bullet(Z/\mathbf{N}_i) \arrow[l, "\exp"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{H}^\bullet(Z) \arrow[r, equal] \arrow[d, "\cdot[Z/X]"'] & \mathrm{H}^\bullet(Z) \arrow[d, "\cdot[Z/\mathbf{N}_i]"] \\ \mathrm{H}^{\bullet-c}(Z/X) & \mathrm{H}^{\bullet-c}(Z/\mathbf{N}_i) \arrow[l, "\exp"'] \end{tikzcd} \tag{C.2}$$

Proof. Follows from the compatibility of Euler and fundamental classes, and for the right diagram, the composition rule for fundamental classes. □

In particular,

Corollary C.2.2. (Cohomological tubular neighbourhood theorem) *If Z and X are smooth,* exp *is an isomorphism. More generally, if $Z \to X$ and $Z \to \mathbf{N}_i$ are specialised, then* exp *is an isomorphism modulo torsion.*

Moreover,

Proposition C.2.3. *If $i : Z \to X$ is a closed embedding of finite presentation over a base B. Then if Z, X are quasismooth over B, we have a commuting diagram*

$$\begin{tikzcd} \mathrm{H}^\bullet(Z/B) \arrow[r, equal] & \mathrm{H}^\bullet(Z/B) \\ \mathrm{H}^{\bullet-c}(Z/X) \arrow[u, "\cdot[X/B]"] & \mathrm{H}^{\bullet-c}(Z/\mathbf{N}_i) \arrow[l, "\exp"'] \arrow[u, "\cdot[\mathbf{N}_i/B]"'] \end{tikzcd} \tag{C.3}$$

Note that the assumptions of the Proposition imply that $\mathbf{N}_i \to Z$ (and so $\mathbf{N}_i \to B$) is quasismooth.

Finally, we note a functoriality statement

Proposition C.2.4. *If s is any map as in (C.2), then any pullback*

$$\begin{tikzcd} \overline{Z} \arrow[r, "\overline{s}"] \arrow[d, "\overline{f}"] & \overline{X} \arrow[d, "f"] \\ Z \arrow[r, "s"] & X \end{tikzcd} \quad \text{\textit{gives commuting diagram}} \quad \begin{tikzcd} \mathrm{H}^\bullet(\overline{Z}/\mathbf{N}_{\overline{s}}) \arrow[r, "\exp_{\overline{s}}"] & \mathrm{H}^\bullet(\overline{Z}/\overline{X}) \\ \mathrm{H}^\bullet(Z/\mathbf{N}_s) \arrow[r, "\exp_s"] \arrow[u, "\underline{f}^*"] & \mathrm{H}^\bullet(Z/X) \arrow[u, "f^*"] \end{tikzcd}$$

where we have denoted $\underline{f} : \mathbf{N}_{\overline{s}} = \mathbf{N}_s \times_Z \overline{Z} \to \mathbf{N}_s$.

Proof. Follows since the construction of D_s is stable under arbitrary base change of X. □

C.3 Borel Moore variant

Let $s : Z \to X$ be any map defined over base space B for which deformation to the normal complex is defined, e.g. closed embeddings of finite presentation or quasismooth maps.

Assume that the localised pushforward along $i : \mathbf{N}_s \to D_s$ vanishes:

$$i_* \ : \ \mathrm{H}^\bullet(\mathbf{N}_s/B)_{\mathrm{loc}} \ \to \ \mathrm{H}^\bullet(D_s/B)_{\mathrm{loc}}.$$

Define the (*Borel Moore*) *cohomological exponential map*

$$\exp \ : \ \mathrm{H}^\bullet(X/B)_{\mathrm{loc}} \ \to \ \mathrm{H}^\bullet(\mathbf{N}_s/B)_{\mathrm{loc}}$$

by composing the two dotted arrows

$$\begin{tikzcd}
\mathrm{H}^\bullet(X/B)_{\mathrm{loc}} \arrow[rrr, dashed] \arrow[dr, "{[B/B\times\mathbf{A}^1]}", "\sim"'] & & & \mathrm{H}^\bullet(X\times\mathbf{G}_m/B\times\mathbf{G}_m)_{\mathrm{loc}} \\
& \mathrm{H}^\bullet(X/B\times\mathbf{A}^1)_{\mathrm{loc}} & \mathrm{H}^\bullet(X\times\mathbf{A}^1/B\times\mathbf{A}^1)_{\mathrm{loc}} \arrow[l, "{[X/X\times\mathbf{A}^1]}"', "\sim"] \arrow[ur, "j^*_{B\times\mathbf{A}^1}"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{H}^\bullet(X\times\mathbf{G}_m/B\times\mathbf{G}_m)_{\mathrm{loc}} \arrow[rrr, dashed] & & & \mathrm{H}^\bullet(\mathbf{N}_s/B)_{\mathrm{loc}} \\
& \mathrm{H}^\bullet(D_s/B\times\mathbf{A}^1)_{\mathrm{loc}} \arrow[ul, "j^*_{B\times\mathbf{A}^1}"'] \arrow[r, "{[B/B\times\mathbf{A}^1]}", "\sim"'] & \mathrm{H}^\bullet(D_s/B)_{\mathrm{loc}} \arrow[ur, "{[\mathbf{N}_s/D_s]}"] &
\end{tikzcd}$$

Our assumption allows us to take a section of $j^*_{B\times\mathbf{A}^1}$ as before.

C.3.1. Assume that Z, X are quasismooth over a common base. Then $D_s \to X\times\mathbf{A}^1$ is quasismooth and we have a commuting diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(Z/B)_{\mathrm{loc}} \arrow[r, "0"] \arrow[d, "{[\mathbf{N}_s/Z]\cdot}"] & \mathrm{H}^\bullet(X\times\mathbf{A}^1/B)_{\mathrm{loc}} \arrow[r, hook] \arrow[d, hook, "{[D_s/X\times\mathbf{A}^1]\cdot}"] & \mathrm{H}^\bullet(X\times\mathbf{G}_m/B)_{\mathrm{loc}} \arrow[r, "+1"] \arrow[d, equal] & {} \\
\mathrm{H}^\bullet(\mathbf{N}_s/B)_{\mathrm{loc}} \arrow[r, "i_*"] & \mathrm{H}^\bullet(D_s/B)_{\mathrm{loc}} \arrow[r, "j^*"] & \mathrm{H}^\bullet(X\times\mathbf{G}_m/B)_{\mathrm{loc}} \arrow[r, "+1"] & {}
\end{tikzcd}$$

and the following are equivalent: $i_* = 0$, j^* is an injection, or either vertical map is an isomorphism.

C.4 Virtual fundamental classes

We recap the construction of the (virtual) fundamental class from [Kh]. Let $f : X \to Y$ be a quasismooth map, which gives the deformation to the normal complex sitting in a diagram of stacks over Y:

$$Y \times \mathbf{G}_m \xrightarrow{j} D_{X/Y} \xleftarrow{i} \mathbf{N}_{X/Y}$$

Consider the natural transformation $\psi = p_{D_{X/Y}*}(-)p^!_{D_{X/Y}}$. If we replace $D_{X/Y}$ with the trivial family $Y \times \mathbf{A}^1$,

$$Y \times \mathbf{G}_m \xrightarrow{j} Y \times \mathbf{A}^1 \xleftarrow{i} Y$$

we likewise get a natural transformation ψ_{triv}. Then as endofunctors of $\mathrm{Sh}(Y)$ we have

$$\psi_{\mathrm{triv}}(i_*i^!) \;=\; \mathrm{id} \qquad\qquad \psi(i_*i^!) \;=\; p_{\mathbf{N}_{X/Y}*}p^!_{\mathbf{N}_{X/Y}} \;\simeq\; f_*f^!\langle 2d\rangle$$

where we have used that $\mathbf{N}_{X/Y} \to X$ is smooth of dimension d, and

$$\psi_{\mathrm{triv}}(j_*j^!) \;=\; p_{Y\times\mathbf{G}_m*}p^!_{Y\times\mathbf{G}_m} \;=\; \psi(j_*j^!).$$

We now argue as for the construction of the cohomological exponential map, noting that for the trivial family, in the Borel Moore distinguished triangle

$$\mathrm{id} \;\Rightarrow\; j_*j^! \;\overset{\partial_{Y\times\mathbf{A}^1}}{\Rightarrow}\; i_*i^![1]$$

the boundary map admits a canonical section $\gamma_{Y\times\mathbf{A}^1}$, see [DJK, 3.2.2]. Thus we have

$$\psi_{\mathrm{triv}}(i_*i^!) \;\overset{\psi_{\mathrm{triv}}(\gamma_{Y\times\mathbf{A}^1})}{\Rightarrow}\; \psi_{\mathrm{triv}}(j_*j^![-1]) \;=\; \psi(j_*j^![-1]) \;\overset{\psi(\partial_{D_{X/Y}})}{\Rightarrow}\; \psi(i_*i^!). \tag{C.4}$$

Definition C.4.1. Let $f : X \to Y$ be a quasismooth map of relative dimension d. The *fundamental class map* is the natural transformation induced by (C.4) and (f^*, f_*) adjunction:

$$f^* \;\Rightarrow\; f^!\langle 2d\rangle.$$

Taking cohomology, the *fundamental class* $[X/Y] \in \mathrm{H}^{-2d}(X/Y)$ is the the image of $1 \in \mathrm{H}^0(Y)$.

Theorem C.4.2 ([Kh])**.** *Let $f : X \to Y$ be a quasismooth map of derived Artin stacks. Then*

1) Under any pullback of derived stacks

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & W \arrow[d, "g"] \\ X \arrow[r] & Y \end{tikzcd}$$

we have $[Z/W] = g^*[X/Y]$.

2) Given two quasismooth maps $X \to Y \to Z$, we have $[X/Z] = [X/Y] \cdot [Y/Z]$.

3) (Purity) *If f is smooth, the fundamental class $f^* \Rightarrow f^!\langle 2d\rangle$ is an equivalence.*

Lemma C.4.3. *If $f : X \to Y$ is smooth of relative dimension d, then the cup product*

$$\mathrm{H}^\bullet(W/X) \xrightarrow{\sim} \mathrm{H}^{\bullet-2d}(W/Y) \qquad \alpha \mapsto \alpha \cdot [X/Y]$$

is an isomorphism for any map $g : W \to X$.

Proof. It follows from purity that

$$\mathrm{H}^\bullet(W, g^! k_X) = \mathrm{H}^\bullet(W, g^! f^* k_Y) \simeq \mathrm{H}^{\bullet-2d}(W, g^! f^! k_Y).$$

□

C.4.4. The fundamental class induces a map $f^* k_Y \to f^! k_Y\langle 2d\rangle$, and one can show that the fundamental class map is given in terms of it using the projection formula:

$$f^*(-) = f^*(-) \otimes f^* k_Y \to f^*(-) \otimes f^! k_Y\langle 2d\rangle \to f^!((-) \otimes k_Y)\langle 2d\rangle = f^!(-)\langle 2d\rangle.$$

C.5 Umkehr maps

C.5.1. **Cohomology version.** For any map $f : X \to Y$ and any $\mathcal{F} \in \mathrm{Sh}(Y)$, recall that we can always define the pullback map

$$f^* : \mathrm{H}^\bullet(Y, \mathcal{F}) \to \mathrm{H}^\bullet(Y, f^* \mathcal{F})$$

coming from the natural transformation $\mathrm{id} \Rightarrow f_* f^*$. However, if f is quasismooth and proper of dimension d we can use the fundamental class to define the wrong-way pushforward (or *umkehr*) maps

$$f_* : \mathrm{H}^\bullet(X, f^* \mathcal{F}) \to \mathrm{H}^{\bullet-2d}(Y, \mathcal{F})$$

coming from the natural transformation

$$f_* f^* \overset{[X/Y]}{\Rightarrow} f_* f^! \langle 2d\rangle = f_! f^! \langle 2d\rangle \Rightarrow \mathrm{id}\langle 2d\rangle.$$

It is easy to see that both pushforward and pullback maps are functorial with respect to composition of maps. Next, because of the compatibility of fundamental classes with pullback squares, we have

Lemma C.5.2. *For any pullback square*

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \\ \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

such that f (and hence $\overline{f}$) is proper and quasismooth, $g^* f_* = \overline{f}_* \overline{g}^*$ as maps $\mathrm{H}^\bullet(Z) \to \mathrm{H}^\bullet(Y)$.

C.5.3. **Borel Moore version.** Likewise, for any proper map $f : X \to Y$ and any $\mathcal{F} \in \mathrm{Sh}(Y)$ we have a map

$$f_* \ : \ \mathrm{H}^\bullet(X, f^! \mathcal{F}) \ \to \ \mathrm{H}^\bullet(Y, \mathcal{F})$$

coming from the natural tranformation $f_! f^! \Rightarrow \mathrm{id}$. If f is also quasismooth (and not necessarily proper) we get the *umkehr* pullback map

$$f^* \ : \ \mathrm{H}^\bullet(Y, \mathcal{F}) \ \to \ \mathrm{H}^{\bullet - 2d}(X, f^! \mathcal{F})$$

coming from the natural transformation

$$\mathrm{id} \ \Rightarrow \ f_* f^* \ \overset{[X/Y]}{\Rightarrow} f_* f^! \langle 2d \rangle.$$

C.5.4. This gives the usual pushforward and pullback maps for cohomology and Borel Moore homology, e.g. [BM].

Bibliography

[ACJ] Anguelova, I.I., Cox, B. and Jurisich, E., 2014. *N-point locality for vertex operators: normal ordered products, operator product expansions, twisted vertex algebras.* Journal of Pure and Applied Algebra, 218(12), pp.2165-2203. https://arxiv.org/abs/1307.4830

[AKLPR] Aranha, D., Khan, A.A., Latyntsev, A., Park, H. and Ravi, C., 2022. *Localization theorems for algebraic stacks.* arXiv preprint arXiv:2207.01652. https://arxiv.org/abs/2207.01652

[AMa] Arakawa, T. and Malikov, F., 2012. *A chiral Borel–Weil–Bott theorem.* Advances in Mathematics, 229(5), pp.2908-2949. https://arxiv.org/abs/0903.1281

[AMi] Adamović, D. and Milas, A., 2008. *On the triplet vertex algebra $W(p)$.* Advances in Mathematics, 217(6), pp.2664-2699. https://arxiv.org/abs/0707.1857

[AMo] Arakawa, T. and Moreau, A., 2021. *Arc spaces and vertex algebras.*https://www.imo.universite-paris-saclay.fr/~moreau/CEMPI-arc_space-vertex_algebras.pdf

[And] Anderson, D., 2012. *Introduction to equivariant cohomology in algebraic geometry.* Contributions to Algebraic Geometry: Impanga Lecture Notes, 6, p.71.

[Ang] Anguelova, I.I., 2006. *Bicharacter Construction of Quantum Vertex Algebras* (Doctoral dissertation, University of Illinois at Urbana-Champaign).

[Ar1] Arakawa, T., 2007. *Representation theory of $\mathcal{W}$-algebras.* Inventiones mathematicae, 169(2), pp.219-320. https://arxiv.org/abs/math/0506056

[Ar2] Arakawa, T., 2017. *Introduction to W-algebras and their representation theory.* In Perspectives in Lie theory (pp. 179-250). Springer, Cham. https://arxiv.org/abs/1605.00138

[At1] Atiyah, M.F., 1985. *Circular symmetry and stationary-phase approximation.* Astérisque, 131, pp.43-59. http://www.numdam.org/item/?id=AST_1985__131__43_0

[At2] Atiyah, M.F., 1988. Topological quantum field theory. Publications Mathématiques de l'IHÉS, 68, pp.175-186. http://www.numdam.org/item/PMIHES_1988__68__175_0/

[Ay] Ayoub, J., 2007. *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique*. Montrouge: Société mathématique de France. http://www.numdam.org/item/AST_2007__315__1_0/

[AB] Atiyah, M.F. and Bott, R., 1984. *The moment map and equivariant cohomology*. Topology, 23(1), pp.1-28. https://www.math.stonybrook.edu/~mmovshev/MAT570Spring2008/BOOKS/atiyahbott_moment.pdf

[AG] Arinkin, D. and Gaitsgory, D., 2015. *Singular support of coherent sheaves and the geometric Langlands conjecture*. Selecta Mathematica, 21(1), pp.1-199. https://arxiv.org/abs/1201.6343

[Bi] Bialynicki-Birula, A., 1973. *Some theorems on actions of algebraic groups*. Annals of mathematics, 98(3), pp.480-497.

[BB] A. Beilinson, J. Bernstein, *Localisation de g-modules,* C.R. Acad. Sci. Paris, 292 (1981), 15-18, MR82k:14015, Zbl 0476.14019

[Beh] Behrend, K., 1996. *Gromov-Witten invariants in algebraic geometry.* Invent. Math. 127 (1997), no. 3, 601–617. https://arxiv.org/abs/alg-geom/9601011

[Ber] Bernstein, J., 1983. *Algebraic theory of D-modules*, lecture notes. https://gauss.math.yale.edu/~il282/Bernstein_D_mod.pdf

[BD1] Beilinson, A. and Drinfeld, V., 1991. *Quantization of Hitchin's integrable system and Hecke eigensheaves*. https://math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf

[BD2] Beilinson, A., Drinfeld, V. and Drinfeld, V.G., 2004. *Chiral algebras* (Vol. 51). American Mathematical Soc..

[BGT] Blumberg, A.J., Gepner, D. and Tabuada, G., 2013. *A universal characterization of higher algebraic K-theory.* Geometry & Topology, 17(2), pp.733-838. https://arxiv.org/abs/1001.2282

[BK] Bakalov, B. and Kac, V.G., 2002. *Field algebras.* Int. Math. Res. Not. 2003, no. 3, 123–159. https://arxiv.org/abs/math/0204282

[BLL] Beem, C., Lemos, M., Liendo, P., Peelaers, W., Rastelli, L. and Van Rees, B.C., 2015. *Infinite chiral symmetry in four dimensions*. Communications in Mathematical Physics, 336(3), pp.1359-1433. https://arxiv.org/abs/1312.5344

[BM] Borel, A. and Moore, J.C., 1960. *Homology theory for locally compact spaces.* Michigan Mathematical Journal, 7(2), pp.137-159.

[BMP] Bordemann, M., Makhlouf, A. and Petit, T., 2005. *Déformation par quantification et rigidité des algebres enveloppantes*. Journal of Algebra, 285(2), pp.623-648. https://arxiv.org/abs/math/0211416v1

[Bo1] Borcherds, R.E., 1986. *Vertex algebras, Kac-Moody algebras, and the Monster*. Proceedings of the National Academy of Sciences, 83(10), pp.3068-3071.

[Bo2] Borcherds, R.E., 1999. *Quantum vertex algebras*. https://arxiv.org/abs/math/9903038

[BPV] Belavin, A.A., Polyakov, A.M. and Zamolodchikov, A.B., 1984. *Infinite conformal symmetry in two-dimensional quantum field theory*. Nuclear Physics B, 241(2), pp.333-380.

[Bu] Butson, D.W., 2021. *Equivariant Localization in Factorization Homology and Vertex Algebras from Supersymmetric Gauge Theory* (Doctoral dissertation, University of Toronto (Canada)).

[BVe] Berline, N. and Vergne, M., 1982. *Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante.* CR Acad. Sci. Paris, 295(2), pp.539-541.

[BVi] Bakalov, B. and Villarreal, J.J., 2021. *Logarithmic vertex algebras*. arXiv preprint arXiv:2107.10206. https://arxiv.org/abs/2107.10206

[Ca] Carter, R. and Carter, R.W., 2005. *Lie algebras of finite and affine type* (No. 96). Cambridge University Press.

[CD] Cisinski, D.C. and Déglise, F., 2019. *Triangulated categories of mixed motives* (Vol. 6). Berlin: Springer. https://arxiv.org/abs/0912.2110

[Che] Cherednik, I.V., 1984. *Factorizing particles on a half-line and root systems.* Teoreticheskaya i Matematicheskaya Fizika, 61(1), pp.35-44.

[CF] Chen, L. and Fu, Y., 2021. *An Extension of the Kazhdan-Lusztig Equivalence.* arXiv preprint arXiv:2111.14606. https://arxiv.org/abs/2111.14606

[CG1] Costello, K. and Gwilliam, O., *Factorization Algebras in Quantum Field Theory Volume 1* (8 May 2016).

[CG2] Costello, K. and Gwilliam, O., 2021. *Factorization algebras in quantum field theory* (Vol. 2). Cambridge University Press.

[CP] Chari, V. and Pressley, A., 1994. *A guide to quantum groups*, Cambridge, UK: Univ.

[CS] Cisinski, D.C. and Déglise, F., 2019. *Triangulated categories of mixed motives* (Vol. 6). Berlin: Springer. https://arxiv.org/abs/0912.2110

[Da] Davison, B., 2017. *The critical CoHA of a quiver with potential*. Quarterly Journal of Mathematics, 68(2), pp.635-703. https://arxiv.org/abs/1311.7172

[Dé] Déglise, F., 2017. *Bivariant theories in motivic stable homotopy*. https://arxiv.org/abs/1705.01528

[De] Deligne, P., 1968. *Théoreme de Lefschetz et criteres de dégénérescence de suites spectrales.* Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 35(1), pp.107-126. http://www.numdam.org/item/PMIHES_1968__35__107_0/

[DGai] Drinfeld, V. and Gaitsgory, D., 2013. *On some finiteness questions for algebraic stacks.* Geometric and Functional Analysis, 23(1), pp.149-294. https://arxiv.org/abs/1108.5351

[DGal] Drew, B. and Gallauer, M., 2020. *The universal six-functor formalism.* https://arxiv.org/abs/2009.13610

[DJK] Déglise, F., Jin, F. and Khan, A.A., 2021. *Fundamental classes in motivic homotopy theory.* Journal of the European Mathematical Society, 23(12), pp.3935-3993. https://arxiv.org/abs/1805.05920

[DK] De Concini, C. and Kac, V., 1990. *Representations of quantum groups at roots of 1, Operator algebras, unitary representations, enveloping algebras, and invariant theory* (Paris, 1989), 471–506. Progr. Math, 92.

[DL] Dong, C. and Lepowsky, J., 2012. *Generalized vertex algebras and relative vertex operators (Vol. 112).* Springer Science & Business Media.

[DM] Davison, B. and Sven, M., 2020. *Cohomological Donaldson–Thomas theory of a quiver with potential and quantum enveloping algebras.* Inventiones mathematicae, 221(3), pp.777-871. https://arxiv.org/abs/1601.02479

[Dr] Drinfeld, V.G., 1986. *Quantum groups.* Zapiski Nauchnykh Seminarov POMI, 155, pp.18-49.

[DR] Dong, C. and Ren, L., 2016. *Vertex operator algebras associated to the Virasoro algebra over an arbitrary field.* Transactions of the American Mathematical Society, 368(7), pp.5177-5196. https://arxiv.org/abs/1308.0087

[Ed] Edidin, D., 2010. *Equivariant geometry and the cohomology of the moduli space of curves.* Handbook of moduli. Vol. I, 259–292, Adv. Lect. Math. (ALM), 24, Int. Press, Somerville, MA, 2013. https://arxiv.org/abs/1006.2364

[EFK] Etingof, P.I., Frenkel, I. and Kirillov, A.A., 1998. *Lectures on representation theory and Knizhnik-Zamolodchikov equations* (No. 58). American Mathematical Soc..

[EG] Edidin, D. and Graham, W., 1998. *Localization in equivariant intersection theory and the Bott residue formula.* American Journal of Mathematics, 120(3), pp.619-636. https://arxiv.org/abs/alg-geom/9508001

[EGNO] Etingof, P., Gelaki, S., Nikshych, D. and Ostrik, V., 2016. *Tensor categories* (Vol. 205). American Mathematical Soc..

[EK] Etingof, P. and Kazhdan, D., 1996. *Quantization of Lie bialgebras, I.* Selecta Mathematica, 2(1), pp.1-41. https://arxiv.org/abs/q-alg/9506005

[ES] Etingof, P.I. and Schiffmann, O., 1998. *Lectures on quantum groups.* Boston, MA: International press.

[FMS] Di Francesco, P., Mathieu, P. and Sénéchal, D., 1997. *Conformal Field Theory,* Graduate Texts in Contemprorary Physics.

[Fu] Fulton, W., 2016. *Intersection theory* (pp. 96-130). Princeton University Press.

[FBZ] Frenkel, E. and Ben-Zvi, D., 2004. *Vertex algebras and algebraic curves* (No. 88). American Mathematical Soc..

[FF] Feigin, B.L. and Frenkel, E.V., 1990. *Representations of affine Kac-Moody algebras and bosonization.* In Physics And Mathematics Of Strings: Memorial Volume for Vadim Knizhnik (pp. 271-316).

[FG] Francis, J. and Gaitsgory, D., 2012. *Chiral Koszul duality.* Selecta Mathematica, 18(1), pp.27-87. https://arxiv.org/abs/1103.5803

[Fi] Fialowski, A., 1990. *Deformations of some infinite-dimensional Lie algebras.* Journal of mathematical physics, 31(6), pp.1340-1343.

[FJW1] Frenkel, I.B., Jing, N. and Wang, W., 2000. *Vertex representations via finite groups and the McKay correspondence.* International Mathematics Research Notices, 2000(4), pp.195-222. https://arxiv.org/abs/math/9907166

[FJW2] Frenkel, I.B., Jing, N. and Wang, W., 2000. *Quantum Vertex Representations via Finite Groups and the McKay Correspondence.* Communications in Mathematical Physics, 211(2), pp.365-393. https://arxiv.org/abs/math/9907175

[FK] Frenkel, I. B.; Kac, V. G. *Basic representations of affine Lie algebras and dual resonance models.* Invent. Math. 62 (1980/81), no. 1, 23–66.

[FM] Fulton, W. and MacPherson, R., 1981. *Categorical framework for the study of singular spaces* (Vol. 243). American Mathematical Soc..

[FR] Frenkel, E. and Reshetikhin, N., 1997. *Towards deformed chiral algebras.* arXiv preprint. https://arxiv.org/abs/q-alg/9706023

[FZ] Frenkel, I.B. and Zhu, Y., 1992. *Vertex operator algebras associated to representations of affine and Virasoro algebras.* Duke Mathematical Journal, 66(1), pp.123-168.

[Ga1] Gaitsgory, D., *Geometric Representation Theory*, Fall 2005. Online notes. www.people.math.harvard.edu/~gaitsgde/267y/catO.pdf

[Ga2] Gaitsgory, D., 2010. *Notes on geometric Langlands: categories over the Ran space.* Online notes. www.people.math.harvard.edu/~gaitsgde/GL/Ran.pdf

[Ga3] Gaitsgory, D., 2021. *On factorization algebras arising in the quantum geometric Langlands theory.* Adv. Math. 391 (2021), Paper No. 107962, 120 pp. `https://arxiv.org/abs/1909.09775`

[Ga4] Gaitsgory, D., 2015. *The Atiyah-Bott formula for the cohomology of the moduli space of bundles on a curve.* arXiv preprint arXiv:1505.02331. `https://arxiv.org/abs/1505.02331`

[Ge] Gerstenhaber, M., 1964. *On the deformation of rings and algebras.* Ann. of Math. (2) 79 (1964), 59–103.

[GJT] Gross, J., Joyce, D. and Tanaka, Y., 2020. *Universal structures in $\mathbb{C}$-linear enumerative invariant theories.* I. arXiv preprint arXiv:2005.05637. `https://arxiv.org/abs/2005.05637`

[GKV] Ginzburg, V., Kapranov, M. and Vasserot, É., 1995. *Langlands reciprocity for algebraic surfaces.* Math. Res. Lett. 2 (1995), no. 2, 147–160. `https://arxiv.org/abs/q-alg/9502013`

[GL] Gaitsgory, D. and Lurie, J., 2019. *Weil's conjecture for function fields.* Princeton University Press. `https://people.math.harvard.edu/~lurie/papers/tamagawa.pdf`

[GP] Graber, T. and Pandharipande, R., 1997. *Localization of virtual classes.* Invent. Math. 135 (1999), no. 2, 487–518. `https://arxiv.org/abs/alg-geom/9708001`

[GR] Gaitsgory, D. and Rozenblyum, N., 2017. *A Study in Derived Algebraic Geometry: Volume II: Deformations, Lie Theory and Formal Geometry.* Mathematical surveys and monographs, 221.

[Gre] Green, J.A., 1995. *Hall algebras, hereditary algebras and quantum groups.* Inventiones mathematicae, 120(1), pp.361-377.

[Groj] Grojnowski, I., 1995. *Instantons and affine algebras I: the Hilbert scheme and vertex operators.* Math. Res. Lett. 3 (1996), no. 2, 275–291. `https://arxiv.org/abs/alg-geom/9506020`

[Gro1] Gross, J., 2019. *The homology of moduli stacks of complexes.* arXiv preprint. `https://arxiv.org/abs/1907.03269`

[Gro2] Gross, J., 2020. *Moduli spaces of complexes of coherent sheaves* (Doctoral dissertation, University of Oxford).

[GaR] Gaitsgory, D. and Rozenblyum, N., 2011. *Crystals and D-modules.* Pure Appl. Math. Q. 10 (2014), no. 1, 57–154. `https://arxiv.org/abs/1111.2087`

[GrR] Grinberg, D. and Reiner, V., 2020. *Hopf algebras in combinatorics.* Mathematisches Forschungsinstitut Oberwolfach gGmbH.

[Ha] Hartshorne, R., 2010. *Deformation theory (Vol. 257).* New York: Springer.

[He] Heinloth, J., 2012. *Cohomology of the moduli stack of coherent sheaves on a curve*. Geometry and arithmetic, 165–171, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012. https://www.uni-due.de/~hm0002/Artikel/CohNote_v1a.pdf

[HKR] Hekking, J., Khan, A. and Rydh, D., 2021. *Deformation to the normal cone and blow-ups via derived Weil restrictions*. In preparation.

[HL] Huybrechts, D. and Lehn, M., 2010. *The geometry of moduli spaces of sheaves*. Cambridge University Press.

[Hl] Halpern-Leistner, D., 2020. *Derived Θ-stratifications and the D-equivalence conjecture*. https://arxiv.org/abs/2010.01127

[HM] Harvey, J.A. and Moore, G., 1998. *On the algebras of BPS states*. Communications in Mathematical Physics, 197(3), pp.489-519. https://arxiv.org/abs/hep-th/9609017

[HTT] Hotta, R. and Tanisaki, T., 2007. *D-modules, perverse sheaves, and representation theory* (Vol. 236). Springer Science and Business Media.

[Hu] Hubbard, K., 2009. *Vertex coalgebras, comodules, cocommutativity and coassociativity.* Journal of Pure and Applied Algebra, 213(1), pp.109-126. http://faculty.sfasu.edu/hubbardke/vc.pdf

[Iv] Iversen, B., 2012. *Cohomology of sheaves.* Springer Science and Business Media.

[Jo1] Joyce, D., 2007. *Configurations in abelian categories. II. Ringel–Hall algebras.* Advances in Mathematics, 210(2), pp.635-706. https://arxiv.org/abs/math/0503029

[Jo2] Joyce, D., 2018. *Ringel–Hall style vertex algebra and Lie algebra structures on the homology of moduli spaces*. In preparation. https://people.maths.ox.ac.uk/joyce/hall.pdf

[Jo3] Joyce, D., 2021. *Enumerative invariants and wall-crossing formulae in abelian categories.* arXiv preprint arXiv:2111.04694. https://arxiv.org/abs/2111.04694

[JS] Jelisiejew, J. and Sienkiewicz, Ł., 2019. *Białynicki-Birula decomposition for reductive groups*. Journal de Mathématiques Pures et Appliquées, 131, pp.290-325. https://arxiv.org/abs/1805.11558

[Kac] Kac, V.G., 1990. *Infinite-dimensional Lie algebras.* Cambridge university press.

[Kas] Kassel, C., 1984. *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra.* Journal of Pure and Applied Algebra, 34(2-3), pp.265-275.

[Kh] Khan, A.A., 2019. *Virtual fundamental classes of derived stacks I.* https://arxiv.org/abs/1909.01332

[KL] Kazhdan, D. and Lusztig, G., 1993. *Tensor structures arising from affine Lie algebras.* I. Journal of the American Mathematical Society, 6(4), pp.905-947.

[Koc] Kock, J., 2004. *Frobenius algebras and 2-d topological quantum field theories (No. 59).* Cambridge University Press.

[Kon] Kontsevich, M., 2003. *Deformation quantization of Poisson manifolds.* Letters in Mathematical Physics, 66(3), pp.157-216. https://arxiv.org/abs/q-alg/9709040

[KR] Khan, A.A. and Rydh, D., 2018. *Virtual Cartier divisors and blow-ups.* arXiv preprint arXiv:1802.05702. https://arxiv.org/abs/1802.05702

[KS] Kontsevich, M. and Soibelman, Y., 2010. *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants.* https://arxiv.org/abs/1006.2706

[KV1] Kapranov, M. and Vasserot, E., 2006. *Formal loops IV: Chiral differential operators.* arXiv preprint math/0612371. https://arxiv.org/abs/math/0612371

[KV2] Kapranov, M. and Vasserot, E., 2019. *The cohomological Hall algebra of a surface and factorization cohomology.* https://arxiv.org/abs/1901.07641

[KW] Kac, V.G. and Wakimoto, M., 2017. *A remark on boundary level admissible representations.* Comptes Rendus Mathematique, 355(2), pp.128-132. https://arxiv.org/abs/1612.07423

[La] Latyntsev, A., 2021. *Cohomological Hall algebras and vertex algebras.* arXiv preprint arXiv:2110.14356. https://arxiv.org/abs/2110.14356

[Li1] Li, H., 2004. *Vertex algebras and vertex Poisson algebras.* Communications in Contemporary Mathematics, 6(01), pp.61-110. https://arxiv.org/abs/math/0209310

[Li2] Li, H., 2006. *Constructing quantum vertex algebras.* International Journal of Mathematics, 17(04), pp.441-476. https://arxiv.org/abs/math/0505293

[LO1] Laszlo, Y. and Olsson, M., 2008. *The six operations for sheaves on Artin stacks I: Finite Coefficients.* Publications mathématiques, 107(1), pp.109-168. https://arxiv.org/abs/math/0512097

[LO2] Laszlo, Y. and Olsson, M., 2008. *The six operations for sheaves on Artin stacks II: adic coefficients.* Publications Mathématiques de l'IHÉS, 107, pp.169-210. https://arxiv.org/abs/math/0603680

[Li1] Li, H., 2003. *Axiomatic G_1-vertex algebras.* Communications in Contemporary Mathematics, 5(02), pp.281-327. https://arxiv.org/abs/math/0204308

[Li2] Li, H., 2006. *Nonlocal vertex algebras generated by formal vertex operators.* Selecta Mathematica, 11(3), pp.349-397.

[Liu] Liu, K., 2006. *Localization and conjectures from string duality.* In Differential Geometry And Physics (pp. 63-105). https://arxiv.org/abs/math-ph/0701057

[Lur1] Lurie, J., 2009. *On the classification of topological field theories*. In Current developments in mathematics, 2008 (pp. 129-280). International Press of Boston. https://arxiv.org/abs/0905.0465

[Lur2] Lurie, J., 2009. *Derived algebraic geometry VI:* $\mathbf{E}[k]$ *algebras.* arXiv preprint arXiv:0911.0018. https://arxiv.org/abs/0911.0018

[Lus] Lusztig, G., 1990. *Quantum groups at roots of 1.* Geometriae Dedicata, 35(1), pp.89-113.

[LZ1] Liu, Y. and Zheng, W., 2012. *Enhanced six operations and base change theorem for higher Artin stacks*. https://arxiv.org/abs/1211.5948

[LZ2] Liu, Y. and Zheng, W., 2014. *Enhanced adic formalism and perverse t-structures for higher Artin stacks*. https://arxiv.org/abs/1404.1128

[Mac] Mac Lane, S., 2013. *Categories for the working mathematician* (Vol. 5). Springer Science & Business Media.

[Maj] Majid, S., 1991. *Doubles of quasitriangular Hopf algebras.* Communications in algebra, 19(11), pp.3061-3073.

[MNOP] Maulik, D., Nekrasov, N., Okounkov, A. and Pandharipande, R., 2006. *Gromov–Witten theory and Donaldson–Thomas theory,* i. Compositio Mathematica, 142(5), pp.1263-1285. https://arxiv.org/abs/math/0312059

[MRY] Moody, R.V., Rao, S.E. and Yokonuma, T., 1990. *Toroidal Lie algebras and vertex representations.* Geometriae dedicata, 35(1), pp.283-307.

[MV] Mirković, I. and Vilonen, K., 1999. *Perverse sheaves on affine Grassmannians and Langlands duality*. Math. Res. Lett. 7 (2000), no. 1, 13–24. https://arxiv.org/abs/math/9911050

[Na] Nakajima, H., 1994. *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras.* Duke Mathematical Journal, 76(2), pp.365-416.

[Ne] Neeman, A., 2001. *Triangulated categories (No. 148).* Princeton University Press.

[Ol] Olsson, M., 2016. *Algebraic spaces and stacks* (Vol. 62). American Mathematical Soc..

[Po] Polchinski, J.G., 1998. *String theory, volume I: An introduction to the bosonic string* (p. 2). Cambridge: Cambridge university press.

[PS] Porta, M. and Sala, F., 2019. *Two-dimensional categorified Hall algebras.* arXiv preprint arXiv:1903.07253. https://arxiv.org/abs/1903.07253

[Rap] Rapcák, M., 2021. *Branes, Quivers and BPS algebras.* arXiv preprint arXiv:2112.13878. https://arxiv.org/abs/2112.13878

[Ras] Raskin, S., 2015. *Chiral categories*. preprint available at http://math. mit. edu/∼sraskin.

[Ric] Richter, B., 2020. *From categories to homotopy theory* (Vol. 188). Cambridge University Press.

[Rin] Ringel, C.M., 1990. *Hall algebras.* Banach Center Publications, 26(1), pp.433-447.

[RSYZ1] Rapčák, M., Soibelman, Y., Yang, Y. and Zhao, G., 2020. *Cohomological Hall algebras, vertex algebras and instantons.* Communications in Mathematical Physics, 376(3), pp.1803-1873. `https://arxiv.org/abs/1810.10402`

[RSYZ2] Rapcak, M., Soibelman, Y., Yang, Y. and Zhao, G., 2020. *Cohomological Hall algebras and perverse coherent sheaves on toric Calabi-Yau 3-folds.* arXiv preprint arXiv:2007.13365. `https://arxiv.org/abs/2007.13365`

[RT] Reshetikhin, N. and Turaev, V.G., 1991. *Invariants of 3-manifolds via link polynomials and quantum groups.* Inventiones mathematicae, 103(1), pp.547-597.

[Sa] Saito, M., 1989. *Introduction to mixed Hodge modules.* Astérisque, 179(180), pp.145-162. `http://www.numdam.org/item/?id=AST_1989__179-180__145_0`

[Sc] Schiffmann, O., 2006. *Lectures on Hall algebras.* Geometric methods in representation theory. II, 1–141, Sémin. Congr., 24-II, Soc. Math. France, Paris, 2012. `https://arxiv.org/abs/math/0611617`

[Se1] Segal, Graeme *Unitary representations of some infinite-dimensional groups.* Comm. Math. Phys. 80 (1981), no. 3, 301–342.

[Se2] Segal, G.B., 1988. *The definition of conformal field theory.* In Differential geometrical methods in theoretical physics (pp. 165-171). Springer, Dordrecht.

[Se3] Segal, G., 2004. *Quantum field theory.* London Mathematical Society Lecture Note Series, 308, pp.421-577.

[SGA5] Bucur, I., Houzel, C., Jouanolou, J.P. and Serre, J.P., 2006. *Cohomologie l-adique et Fonctions L*: Seminaire de Geometrie Algebrique du Bois-Marie 1965-66, SGA 5 (Vol. 589). Springer.

[So] Soibelman, Y., 1997. *Meromorphic tensor categories.* Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 437–451, Contemp. Math., 248, Amer. Math. Soc., Providence, RI, 1999. `https://arxiv.org/abs/q-alg/9709030`

[SS] Sala, F. and Schiffmann, O., 2018. *Cohomological Hall algebra of Higgs sheaves on a curve.* Algebr. Geom. 7 (2020), no. 3, 346–376. `https://arxiv.org/abs/1801.03482`

[Sz] Szendroi, B., 2014. *Cohomological Donaldson-Thomas theory.* Proceedings of String-Math, 2015. `https://arxiv.org/abs/1503.07349`

[Ta] Tachikawa, Y., 2018. *On 'categories' of quantum field theories.* In Proceedings of the International Congress of Mathematicians (ICM 2018) (In 4 Volumes) Proceedings of the International Congress of Mathematicians 2018 (pp. 2709-2731). https://arxiv.org/abs/1712.09456

[Th] Thomas, R.P., 2000. *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations.* Journal of Differential Geometry, 54(2), pp.367-438. https://arxiv.org/abs/math/9806111

[Th'n] Thomason, R.W., 1986. *Comparison of equivariant algebraic and topological K-theory.* Duke Mathematical Journal, 53(3), pp.795-825.

[To] Toën, B., 2014. *Derived algebraic geometry.* EMS Surveys in Mathematical Sciences, 1(2), pp.153-240. https://arxiv.org/abs/1401.1044

[TV] Toën, B. and Vaquié, M., 2007. *Moduli of objects in dg-categories.* In Annales scientifiques de l'Ecole normale supérieure (Vol. 40, No. 3, pp. 387-444). https://arxiv.org/abs/math/0503269

[Wa] Wakimoto, M., 1986. *Fock representations of the affine Lie algebra $A_1^{(1)}$.* Communications in mathematical physics, 104(4), pp.605-609.

[Wil] Williams, B., 2017. *The Virasoro vertex algebra and factorization algebras on Riemann surfaces.* Letters in Mathematical Physics, 107(12), pp.2189-2237. https://arxiv.org/abs/1603.02349

[Wit] Witten, E., 1990. *Gauge theories, vertex models, and quantum groups.* Nuclear Physics B, 330(2-3), pp.285-346.

[YZ1] Yang, Y. and Zhao, G., 2018. *The cohomological Hall algebra of a preprojective algebra.* Proceedings of the London Mathematical Society, 116(5), pp.1029-1074. https://arxiv.org/abs/1407.7994

[YZ2] Yang, Y. and Zhao, G., 2018. *Cohomological Hall algebras and affine quantum groups.* Selecta Mathematica, 24(2), pp.1093-1119. https://arxiv.org/abs/1604.01865

[Y.Zh] Zhu, Y., 1996. *Modular invariance of characters of vertex operator algebras.* Journal of the American Mathematical Society, 9(1), pp.237-302.

[X.Zh] Zhu, X., 2009. *Affine Demazure modules and T-fixed point subschemes in the affine Grassmannian.* Advances in Mathematics, 221(2), pp.570-600. https://arxiv.org/abs/0710.5247