Projective Fibrations in Weighted Scrolls

Geoffrey Mboya
St Peter's College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Michaelmas 2022

To my family for their support and love.

Abstract

This thesis studies classification problems of codimension 1 and codimension 2 threefold families

$$f : X \to \mathbb{P}^1,\ X \subset \mathbb{F}_A$$

fibred in low-degree K3 surfaces over $\mathbb{P}^1$ and embedded in weighted scrolls $\mathbb{F}_A$, where A is an integer matrix defining weights and twists. We classify X such as K3 surfaces with rational double points, Calabi–Yau 3-folds with isolated singularities which extend results of Mullet, K3 surfaces and Calabi–Yau 3-folds with at most Gorenstein singularities. Further, the thesis revisits degeneration of quartic K3 fibred Calabi–Yau threefolds studied some time ago by Gross and Ruan.

The thesis also sets the stage for a general study of fibrations embedded in weighted scrolls

$$(f : X \to B) \hookrightarrow (\pi : \mathbb{F}_A \to \mathbb{P}^{k-1}[c_i])$$

over any nonsingular base $B \subset \mathbb{P}^{k-1}[c_i]$ by proposing a multi-graded $\mathbb{C}$-algebra geometry method for constructing them. We initiate the study of fibrations embedded in families of key varieties such as weighted Grassmannians fibrations, leaving detailed investigations for later work.

Acknowledgements

I am deeply grateful to my supervisors, Prof Balázs Szendrői and Prof Dominic Joyce. Balazs for his guidance and tireless, generous support throughout this research. Dominic from whose research grant I got an electronics device that supported my research as well as for his guidance during the first and final few months of my research. I also express my gratitude to the Simons Foundation and University of Oxford for financing my research. My remote research during COVID-19 Pandemic was also supported through Africa Oxford Initiative Graduate Thrive Fund and St Peter's College graduate travel grant for which I am deeply grateful. I also benefited from useful suggestions and discussion from Prof Frances Kirwan and Prof Alan Lauder which helped improve this thesis.

It is not easy having a young family more than 4000 miles away while doing research. My son was 3 months old when I first held him in my hands. "Daddy, mummy *ameleta brother kutoka hospitali. Kuja umwone*", my daughter would break the news to me everyday. I would like to, in a special way, express deep gratitute to my wife, Winnie Atieno, for her endurance and for holding up the family together during the long times I was periodically physically unavailable.

I am forever grateful to my academic brothers Søren Gammelgaard and Aurelio Carlucci for many helpful discussions. I also benefited from correspondence with Alan Thompson, Stephen Coughlan, Roberto Pignatelli and Dominic Bunnett.

I would like to thank friends and family who believed in and/or prayed for me and my family during this remarkable period. To single out, I would like to thank my mother who, in her ill-health, has always been there for me during my lowest moments. *Mama, Nyasaye osetimo ng'uono!*

Contents

List of Figures

Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I declare that the work contained in this thesis has not been submitted towards any other degree or award at this institution or at any other institution. Sections 4.1-4.3 are based on joint work with Professor Balázs Szendrői.

Chapter 1

Introduction

Let (X,D) be a polarized projective variety over $\mathbb{C}$: a pair consisting of a normal projective complex variety X and an ample $\mathbb{Q}$-Cartier Weil divisor D on X, with associated divisorial sheaf $\mathscr{O}_X(D)$. We can then consider the graded ring

$$R(X,D) = \bigoplus_{n\geq 0} H^0(X, \mathscr{O}_X(nD)).$$

Ampleness of D is equivalent to an isomorphism $X \cong \operatorname{Proj} R$; and in particular, R is a finitely generated $\mathbb{C}$-algebra. A choice of algebra generator set $\{r_1,\dots,r_n\} \in R$ of positive integer weights $\{b_1,\dots,b_n\}$ gives a surjection

$$S = \mathbb{C}[x_1,\dots,x_n] \twoheadrightarrow R$$

of graded $\mathbb{C}$-algebras, and a corresponding embedding

$$(X,D) \hookrightarrow (\mathbb{P}[b_1,\dots,b_n], \mathscr{O}(1))$$

into a weighted projective space. Miles Reid's graded ring method studies classes of projective varieties of increasing complexity according to their codimension in this embedding. Well-studied classical examples include the "famous 95" families of codimension one K3 surfaces in [Fle00, CPR00] , codimension 2 complete intersection K3's in [Fle00] , codimension 3 Pffafians in [Alt98] . Further sets of examples include $\mathbb{Q}$-Fano 3-folds studied in [ABR02, CPR00]. Together with corresponding papers, the *Graded Ring Database* [BK04] gives more lists of this type with absolutely graded $\mathbb{C}$-algebras and information on their corresponding polarised varieties.

This thesis studies the models of projective fibration $f: X \to B$ polarized by a pair of divisors, one ample and the other relatively ample, which together embed the fibration into a weighted scroll $\mathbb{F}_A$. In this study, we take polarised projective varieties (X,H) and

(B,D) with H and D ample divisor line bundles on the respective varieties.

Assume that $f_*\mathscr{O}_X = \mathscr{O}_B$ so that the projective morphism $f : X \to B$ has connected fibres. The bipolarised variety (X,H,f^*D) has a corresponding bigraded ring

$$R(X,H,f^*D) := \bigoplus_{(p_1,p_2)\in\mathbb{Z}^2} H^0(X, p_1 f^*D + p_2 H)$$

whose structure encodes the geometry of the fibration.

The aim of this thesis is to construct Calabi–Yau varieties of dimension $n = 2,3$. A Calabi–Yau n-fold is an n-dimensional complex projective varieties X with at most canonical Gorenstein singularities and

$$K_X = \mathscr{O}_X \text{ with } H^i(X,\mathscr{O}_X) = 0 \text{ for all } 0 < i < n.$$

A **weak** Calabi–Yau n-fold is a variety whose singularities are at most Gorenstein, has trivial canonical bundle whose middle cohomologies vanish.

The focus of this thesis is to construct models of K3-fibred Calabi–Yau families over $\mathbb{P}^1$ and codimension 2 K3 fibred weak Calabi–Yau families over $\mathbb{P}^1$ which are nonsingular or quasismooth and have st worst isolated singular points. The thesis also identifies these isolated singularities as well as studying the degenerations of these anticanonical threefold families.

Suggested by Reid, Mullet in [Mul06] took the first steps in this relative construction with the assumptions:

(i) Fix the base $B = \mathbb{P}^1$.

(ii) the general fibre of f is one of the "famous95" list of Fletcher–Reid K3 hypersurfaces in weighted projective spaces;

(iii) the fibration $f : X \to \mathbb{P}^1$ embeds into a weighted scroll $\pi : \mathbb{F} \to \mathbb{P}^1$ as a quasi-smooth anticanonical hypersurface.

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{F} \\ & \overset{f}{\searrow} & \downarrow \pi \\ & & \mathbb{P}^1. \end{array}$$

Here, the 4-fold weighted scroll $\mathbb{F} = \mathbb{F}(a_j|b_j)$ is a $\mathbb{P}^3[b_j]$-bundle over $\mathbb{P}^1$ with a_j the integer twisting data

$$a_4 \geq a_3 \geq a_2 \geq a_1 \geq 0.$$

In this thesis, this problem is extended to allow isolated singularities on a general hypersurface along the base locus. This allows for a longer list of examples in low-degree K3 fibrations missed in Mullet's search: Calabi–Yau threefolds in $\mathbb{F}$ with isolated singularities and fibred in K3 surfaces $S_d \subset \mathbb{P}^3[b_j]$ for low degree $d = \sum b_j \leq 6$.

The thesis also relaxes conditions (ii) to allow for complete intersection fibres with hypersurface singularities. Using these methods, bi-polarized elliptic K3 surfaces with rational double point singularities are also constructed.

In the specific results proved in this thesis as listed below, "at most isolated singularities" means either nonsingular or having isolated singularities or, in cases where the fibres have non-trivial weights, having quotient singularities and/or isolated singularities.

Theorem 1.0.1. *[=Theorem (4.1.1)] Let $X \in |-K_{\mathbb{F}}|$ be a K3 surface with elliptic cubic fibres $E_3 \subset \mathbb{P}^2$ where $\mathbb{F} = \mathbb{F}(a_1, a_2, a_3)$ is a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$. There are 12 such families of K3 surfaces with at most isolated singularities along the base locus of $|-K_{\mathbb{F}}|$. These surfaces are listed in Table (4.1).*

Theorem 1.0.2. *[=Theorem (4.2.2)] Let $X \in |-K_{\mathbb{F}}|$ be a K3 surface with quartic elliptic fibres $E_4 \subset \mathbb{P}[1,1,2]$ where $\mathbb{F} = \mathbb{F}(a_1, a_2, a_3|1,1,2)$ is a $\mathbb{P}[1,1,2]$ bundle over $\mathbb{P}^1$. There are 24 such families of K3 surfaces with at most isolated singularities along the base locus of $|-K_{\mathbb{F}}|$. These surfaces are listed in Tables (4.5) and (4.6).*

Theorem 1.0.3. *[=Theorem (4.3.2)] Let $X \in |-K_{\mathbb{F}}|$ be a K3 surface with sextic fibres $E_6 \subset \mathbb{P}[1,2,3]$ where $\mathbb{F} = \mathbb{F}(a_1, a_2, a_3|1,2,3)$ is a $\mathbb{P}[1,2,3]$ bundle over $\mathbb{P}^1$. There are 31 such families of K3 surfaces with at most isolated singularities along the base locus of $|-K_{\mathbb{F}}|$. These surfaces are listed in Tables (4.9) and (4.10).*

Further, Theorems (5.1.1),(5.2.1), (5.3.3) and (5.3.4) classify Calabi–Yau threefolds X with at most isolated singularities and embedded in weighted scrolls $\mathbb{F}(a_j|b_j)$ over $\mathbb{P}^1$ and fibred by low degree K3 surfaces $S_4 \subset \mathbb{P}^3$, $S_5 \subset \mathbb{P}[1,1,1,2]$, $S_6 \subset \mathbb{P}[1,1,1,3]$ and $S_6 \subset \mathbb{P}[1,1,2,2]$. This extends the classification in [Mul06] as discussed in [GS22] and summarized below.

Theorem 1.0.4. *Families of anticanonical hypersurfaces $X \in |-K_{\mathbb{F}}|$ in weighted scrolls, fibred in quartic, quintic or sextic $K3$ surfaces, containing isolated singularities along the base locus $B = Bs|-K_{\mathbb{F}}|$ and quasi-smooth outside B are given in the table below.*

$\mathbb{F}=\mathbb{F}(a_j\|b_j)$	General $X\in\|-K_{\mathbb{F}}\|$ with mild isolated singularities
$\mathbb{F}(0,0,1,2\|1^4)$	General $X\in\|-K_{\mathbb{F}}\|$ has 3 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$
$\mathbb{F}(0,0,1,2\,\|\,1^3,2)$	General $X\in\|-K_{\mathbb{F}}\|$ has 4 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and one smooth curve of A_1 singularities
$\mathbb{F}(0,1,2,0\,\|\,1^3,2)$	General $X\in\|-K_{\mathbb{F}}\|$ has 2 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and one smooth curve of A_1 singularities
$\mathbb{F}(0,0,2,1\,\|\,1^3,2)$	General $X\in\|-K_{\mathbb{F}}\|$ has 3 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and one smooth curve of A_1 singularities
$\mathbb{F}(0,0,1,2\,\|\,1^3,3)$	General $X\in\|-K_{\mathbb{F}}\|$ has 5 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and three $\frac{1}{3}(1,1,1)$ singularities
$\mathbb{F}(0,0,2,1\,\|\,1^3,3)$	General $X\in\|-K_{\mathbb{F}}\|$ has 3 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and one $\frac{1}{3}(1,1,1)$ singularity
$\mathbb{F}(0,0,1,2\,\|\,1^2,2^2)$	General $X\in\|-K_{\mathbb{F}}\|$ has 4 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and a smooth curve of A_1 singularities
$\mathbb{F}(0,2,0,1\,\|\,1^2,2^2)$	General $X\in\|-K_{\mathbb{F}}\|$ has 2 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and two disjoint smooth curves of A_1 singularities

Table 1.1: Weighted fourfold scrolls with general anticanonical hypersurfaces fibred in quartic, quintic or sextic $K3$ surfaces with at worst isolated singularities along the base locus

It is worth noting that only ODPs arise in Table 1.1 when $\sum b_j \leq 6$. It is an interesting question whether non-canonical isolated singularities ever arise for other (b_j) in the 'Famous 95'.

The thesis also classifies codimension two surfaces and threefolds in the following results:

Theorem 1.0.5. *[=Theorem (6.1.2)] Let $X=\mathbb{V}(f_1,f_2)\subset\mathbb{F}$ be a codimension two surface in a 4-fold scroll $\mathbb{F}=\mathbb{F}(0,a_2,a_3,a_4)$ and having a trivial canonical class*

$$L_{p_1,2}+L_{2-p_1-a_2-a_3-a_4,2}+K_{\mathbb{F}}=\mathscr{O}_X \text{ with } f_1\in|L_{p_1,2}|,\ f_2\in|L_{2-p_1-a_2-a_3-a_4,2}|.$$

Suppose X is fibred by $E_{2,2}\subset\mathbb{P}^3$ complete intersection of two quadrics and embedded in $\mathbb{F}$ over $\mathbb{P}^1$. There are 18 such families of weak K3 surfaces with at most isolated singularities along either of the base loci $Bs(|L_{p_1,2}|),Bs(|L_{2-p_1-a_2-a_3-a_4,2}|)$. These surfaces are listed in Tables (6.1) and (6.2).

Theorem 1.0.6. *[=Theorem (6.2.2)] Let $X=\mathbb{V}(f_1,f_2)\subset\mathbb{F}$ be a codimension two threefold in a 5-fold scroll $\mathbb{F}=\mathbb{F}(0,a_2,a_3,a_4,a_5)$ and having a trivial canonical class*

$$L_{p_1,2}+L_{2-p_1-a_2-a_3-a_4-a_5,3}+K_{\mathbb{F}}=\mathscr{O}_X \text{ with } f_1\in|L_{p_1,2}|,\ f_2\in|L_{2-p_1-a_2-a_3-a_4-a_5,3}|.$$

Suppose X is fibred by K3 surface complete intersection of a quadric and a cubic $X_{2,3}\subset\mathbb{P}^4$ over $\mathbb{P}^1$ and embedded in $\mathbb{F}$ over $\mathbb{P}^1$. There are 12 such families of codimension two weak Calabi–Yau threefolds X with at most isolated singularities along either of the base loci $Bs(|L_{p_1,2}|),Bs(|L_{2-p_1-a_2-a_3-a_4-a_5,3}|)$. The Table (6.3) summarizes the classification.

The thesis also revisits the degeneration of quartic K3 fibred Calabi–Yau threefolds studied by Gross in [Gro97]. By considering the universal extension $\mathbf{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}(-1))$ over $\mathbb{P}^1$ and adding two copies of a trivial bundle and projectivizing, Gross got a deformation family $\mathscr{F} \to \mathbb{A}^1$ with central fibre $\mathscr{F}_{\prime} \cong \mathbb{F}(0,1,1,2)$ and all other fibres isomorphic to $\mathscr{F}_\infty \cong \mathbb{P}^1 \times \mathbb{P}^3$. He noticed, see also [GS22], that the nonsingular general members of the anticanonical families $X_1 \subset \mathscr{F}_\infty$ and $X_0 \subset \mathscr{F}_{\prime}$ cannot be smoothly deformed into each other from his construction. However, he proves that X_1 specialises in the deformation family $\mathscr{F}$ to a singular Calabi–Yau threefold $\overline{X}_0$, with a curve of canonical singularities along the base locus $B \cong \mathbb{P}^1 \subset \mathbb{F}(0,1,1,2)$ of the anticanonical system of the central fibre. The 3-fold X_0 is a smoothing of $\overline{X}_0$. The point of this construction is that there is one singular variety $\overline{X}_0$ which has two different smoothings: to X_1, and also to X_0. But these latter smooth varieties are in two different families. Ruan observes in [Ruan96][Theorem A.4.3] that though X_1, X_0 are diffeomorphic, with Hodge numbers $(2,86)$, they are not symplectic deformation equivalent. By first using a half-anticanonical embedding of $\mathscr{F}_i$ then studying the images of X_1, X_0, this thesis describes a projective version of this specialisation including a toric description that meets Ruan's description. The This example has also been studied in [CDT18, Tho00].

The thesis, in Proposition 7.1.2, introduces a set-up where Mullet's condition (i) is replaced by any nonsingular base B. This allows one to embed fibrations in weighted scrolls

$$(f : X \to B) \hookrightarrow (\pi : \mathbb{F} \to \mathbb{P}^{k-1}[c_i]).$$

It goes further to generalize this construction by finding interesting examples of fibrations where fibres of π are other key varieties such as weighted Grassmannians.

Other than Mullet's, other studies on classification of varieties fibred over a base include [Reid89, Lop89] where B is assumed to be quasiprojective or affine. The fibration $f : X \to B$ is then equipped with a relatively ample divisor, say relative canonical divisor $K = K_{X/B}$ defined as

$$K_{X/B} := K_X - f^* K_B$$

or something closely related. The object of study in this setup is the relative (sheaf of) ring(s)

$$\mathscr{R}(X/B) = \bigoplus_{n \geq 0} f_*(K^{\otimes n}).$$

The discussion is often restricted to a small open neighbourhood of some point b in B "near a 2-connected fibre". Following on from these works is [CP22]; here threefolds of general type embedded in relative Proj of $(1,1,2,5)$ weighted free symmetric algebra are constructed. These threefolds are constructed by first fixing the fibres as surfaces of general type with invariants $p_g(S) = 2, K_S^2 = 1$ (second row of Table (2.3)). In [Tho11], Thompson uses data from the base curve to construct an algebra whose relative Proj over the curve is the ambient space for threefolds fibred by $S_6 \subset \mathbb{P}[1,1,1,3]$. Also among the initial motivations for this thesis is [BCZ05] where the geography of Mori fibre spaces is studied in ways that include embedding the nonsingular total space X in scrolls of type $(n,k) = (3,2)$ and $(3,1)$ [that is, $\mathbb{P}^n$ bundles over $\mathbb{P}^k$]. Also related to the work in this thesis are [Kuh03, Kuh04] where Kähler cones of hypersurfaces in scrolls of type $(n,k) = (1,3),(3,1)$ and $(2,2)$ are studied.

Throughout this thesis, all varieties are over the field of complex numbers.

The outline of the thesis is as follows:

In **Chapter 2**, we start by a brief review of the geometry of the absolute case (X,H) associated to the finitely generated graded ring $R(X,H)$. We end by giving an introduction to Mori theory.

In **Chapter 3**, three equivalent definitions of scrolls are presented as well as a study of the space of sections of line bundles on a scroll and the related rational maps. We also discuss some properties of divisors and line bundles on scrolls and a computation of cones of surface scrolls.

In **Chapter 4**, we construct models of bipolarised elliptic fibred K3 surfaces over $\mathbb{P}^1$ embedded in weighted scrolls.

Chapter 5 extends the results in Chapter 3 to construct interesting families of mildly singular K3 fibred Calabi–Yau varieties embedded in weighted scrolls. We also discuss the deformations and degenerations of these fibrations.

In **Chapter 6**, we study families of dimensions $2,3$ Complete Intersection Calabi–Yaus [CICYs] in scrolls. There is a rich background of such CICY3 families in the pioneering works [BB96, CDLS98] constructed in Gorenstein toric Fano varieties and in products of projective spaces.

Finally, in **Chapter 7**, we introduce a set-up for probing the geometry of projective fibrations bipolarised by an ample and a relatively ample divisor. In this way, we set the stage for developing the theory of fibrations in scrolls understood in a wider context. This theory has potential of generalizing Reid's Graded Ring Methods in interesting ways. Another potential direction is to apply techniques used in chapters 4 and 5 to study weighted Grassmannian fibrations and other relative key varieties.

Chapter 2

Graded Rings and Projective Varieties

2.1 Graded Rings from Varieties

This section is a summary of material from [Reid79], [Reid02], [GW78] and [Fle00].

Let the pair (X,H) be a complex nonsingular[1] projective variety X over $\mathbb{C}$ polarised by an ample divisor H on X. For an integer $n \geq 0$, we have the finite dimensional vector space of rational functions

$$H^0(X,nH) = H^0(X,\mathcal{O}_X(nH)) = \{f \in \mathbb{C}(X) | div f + nH \geq 0\}$$

on X with divisors of poles of at most nH; it is called the Riemann–Roch (RR) space of nH. The graded vector space

$$R(X,H) = \bigoplus_{n \geq 0} H^0(X,nH)$$

of X is naturally a ring through the map

$$H^0(X,nH) \times H^0(X,mH) \xrightarrow{(f,g) \mapsto fg} H^0(X,(n+m)H).$$

When H is ample, the graded ring $R(X,H)$ is finitely generated, so can be described by a finite number of generators and relations. Further, $R(X,H)$ is a Gorenstein ring whenever $K_X = nH$ for some integer n. Details of this can be found in [GW78]. The generators of $R(X,nH)$ are weighted so that $\operatorname{Proj} R(X,nH)$ is embedded in a weighted projective space discussed in [Fle00] . By standard algebraic geometry, we also have a rational map $\varphi_{|H|} : X \dashrightarrow \mathbb{P}^N = \mathbb{P}^{h^0(H)-1}$ of X to a straight projective space.

The following toy example illustrates the graded ring methods.

[1] X can be allowed to have mild singularities. We have chosen it to be nonsingular to start our discussion.

Example 2.1.1. In this example, we compute the canonical ring $R(C,K_C)$ for a genus 3 nonsingular projective curve C and conclude that it is a plane quartic curve under additional assumptions. By the Riemann–Roch theorem for curves

$$\dim H^0(D) - \dim H^0(K_C - D) = 1 - g + \deg D, \tag{2.1}$$

where $g = g(C) = 3$ is the genus of C and $D = \sum n_i P_i \in \operatorname{Div} C$ has degree $\deg D = \sum n_i$. We get that

$$h^0(nK_C) := \dim H^0(nK_C) = \begin{cases} 1 & \text{n=0,} \\ g = 3 & \text{n=1,} \\ (2n-1)(g-1) = 4n-2 & \text{n} \geq 2. \end{cases} \tag{2.2}$$

Since $g = \dim H^0(C,K_C) = 3$, a choice of basis x_1, x_2, x_3 of $H^0(C,K_C)$ defines a rational map $\varphi_{|K_C|} : C \dashrightarrow \mathbb{P}^2$. Now, with distinct points $P, Q \in C$, we have by Riemann–Roch that

$$h^0(K_C - P) = 1 - g + \deg(K_C - P) + h^0(P) = 2$$

since $\deg K_C = 2g - 2 = 4$ and there are no functions with only a simple pole at P.

We use Equation (2.2) to make algebraic arguments about the graded ring

$$R(C,K_C) = \bigoplus_{n \geq 0} H^0(C, nK_C).$$

We have

$$\dim H^0(C, 2K_C) = \dim(\mathbf{Sym}^2(x_1,x_2,x_3)) = 6$$

quadric monomials and

$$\dim H^0(C, 3K_C) = \dim(\mathbf{Sym}^3(x_1,x_2,x_3)) = 10$$

cubic monomials. However, there are $\dim(\mathbf{Sym}^4(x_1,x_2,x_3)) = 15$ quartic monomials which do not match the prediction $\dim H^0(C,4K_C) = 14$ of Equation (2.2). We conclude with a prediction that the simplest form of the graded ring $R(C,K_C)$ is

$$R(C,K_C) = \mathbb{C}[x_1,x_2,x_3]/(f_4) = \mathbb{C}[\mathbb{P}^2]/(f_4)$$

with the plane quartic $C_4 = \mathbb{V}(f_4)$ the image of the corresponding canonical embedding

$$\varphi_{|K_C|} : C \hookrightarrow \mathbb{P}^2_{[x_i]}.$$

For the nonhyperelliptic case, with the predicted f_4, we need to show that

$$\phi : \mathbb{C}[\mathbb{P}^2]/(f_4) \to R(C, K_C)$$

is a ring isomorphism. It is injective since there is a linear relation between degree 4 monomials and that there are no new relations in higher degrees. Surjectivity follows from the fact that, in degree n,

$$\begin{aligned}
\dim_{\mathbb{C}}\left(\left(\mathbb{C}[\mathbb{P}^2_{[x_i]}]/(f_4)\right)_n\right) &= \dim_{\mathbb{C}}\left(\mathbb{C}[\mathbb{P}^2_{[x_i]}]_n\right) - \dim_{\mathbb{C}}(f_4)_n \\
&= \dim_{\mathbb{C}}\left(\mathbb{C}[\mathbb{P}^2_{[x_i]}]_n\right) - \dim_{\mathbb{C}}\left(\mathbb{C}[\mathbb{P}^2_{[x_i]}]_{n-4}\right) \\
&= \binom{n+2}{2} - \binom{n-4+2}{2} = 4n-2 \\
&= \dim H^0(C, nK_C).
\end{aligned}$$

In the hyperelliptic case, the quotient ring

$$\mathbb{C}[\mathbb{P}^3[1,1,1,2]_{[x_i,y]}]/(q_2(x_i), y^2 - f_4(x_i))$$

is the canonical ring. To compensate for the relation given by the conic defined by q_2, a new degree 2 variable y is added to the previous ring $\mathbb{C}[x_1, x_2, x_3]$. Now, with $\widetilde{f_4} = y^2 - f_4(x_i)$, we have a double cover $C \to \{q_2(x_i) = 0\}$ of the quartic over the conic with the ramification locus $\{q_2 = \widetilde{f_4} = 0\}$. The ambient space of C in this case is $\mathbb{P}[1^3, 2]$. Even better, we can consider a degenerating family

$$\mathbb{V}(\lambda y - q_2(x_i), y^2 - f_4(x_i)) \to \mathbb{A}^1_\lambda$$

which gives a hyperelliptic curve for $\lambda = 0$ and nonhyperelliptic curves for $\lambda \neq 0$.

2.2 Hilbert Series

The material on Hilbert series in this section is adapted from [Reid79, Reid97] and [Eis95]. The Hilbert series of a finitely generated $\mathbb{C}$-algebra R says a lot about the variety $\operatorname{Proj} R$ that would otherwise be, in higher dimensions, more difficult to obtain using explicit analysis such as that used in (2.1) above.

Definition 2.2.1. The Hilbert function of a graded $\mathbb{C}$-algebra $R = \oplus_{d \geq 0} R_d$ over the field of complex numbers is the function

$$P : \mathbb{Z} \xmapsto{d \mapsto \dim_{\mathbb{C}} R_d} \mathbb{Z}.$$

The formal power series

$$P_R(t) = \sum_{d\geq 0} P(d)t^d$$

is referred to as the **Hilbert series** of R.

The Hilbert series P_R is a rational function whose denominator is $\prod_{i=1}^{n}(1-t^{a_i})$ where R is finitely generated with n generators x_i of weights $wt(x_i) = a_i > 0$. The numerator is the polynomial

$$N(t) = \prod_{i=1}^{n}(1-t^{a_i})P_R(t)$$

which encodes syzygies of R of different orders.

The following are some easy examples.

Example 2.2.2.

1. Let $R = \mathbb{C}[\mathbb{P}^n] = \mathbb{C}[x_0,\ldots,x_n] = \bigoplus_{d\geq 0}\mathbb{C}[x_0,\ldots,x_n]_d$. The Hilbert series is thus

$$P_R(t) = \sum_{d\geq 0}\binom{n+d}{d}t^d = \frac{1}{(1-t)^{n+1}}$$

2. Genus g hyperelliptic curve C and H a g^1_2 on C; that is C can be represented as a double cover $C \to \mathbb{P}^1$. Now, by Riemann–Roch

$$\dim H^0(C,nH) = \begin{cases} n+1 & \text{if n} \leq g \\ 1-g+2n & \text{if n} \geq g. \end{cases}$$

The Hilbert series of the graded ring $R(X,H) = \oplus_{n\geq 0}H^0(C,nH)$ is thus

$$P_{(C,H)}(t) = 1+2t+3t^2+\ldots+gt^{g-1}+(g+1)t^g+(g+3)t^{g+1}+\ldots$$

Multiplication of $P_{(C,H)}(t)$ by increasing powers of $1-t$ yield a closed numerator in finite steps

$$(1-t)P_{(C,H)}(t) = 1+t+t^2+\ldots+t^g+2t^{g+1}+\ldots \tag{2.3}$$
$$(1-t)^2P_{(C,H)}(t) = 1+t^{g+1}$$

which is then manipulated by multiplying both sides of Equation (2.3) by $(1-t^a)^b$ for positive integers a,b to yield the Hilbert series

$$P_{(C,H)}(t) = \frac{1+t^{g+1}}{(1-t)^2} = \frac{1-t^{2g+2}}{(1-t)^2(1-t^{g+1})}$$

whose numerator is palindromic; that is, a polynomial of degree d with coefficients $a_k = -a_{d-k}$. The genus g hyperelliptic curve C can then be thought of as having $\mathbb{P}[1,1,g+1]$ as its ambient space and defined by one polynomial of degree $2g+2$.

The graded ring techniques result in various varieties depending on codimension as listed in Table (2.1). Construction of some variety (X,H) by graded ring techniques involves

Codim	Geometry
1	hypersurfaces, see [Fle00]
$2+$ conditions	complete intersections, see [Fle00]
$3+$ conditions	Pfaffians of an antisymmetric matrix, see [BE77, Alt98]
≥ 4	No structure theory yet but many interesting constructions

Table 2.1: Geometry of graded ring in different codimensions.

studying the structure of its graded ring $R(X,H)$ under the natural correspondence

$$X = \operatorname{Proj}(R,H) \leftrightarrow R(X,H).$$

From the graded pieces of the ring, the Hilbert series of (X,H) in $\mathbb{P}[a_i^{b_i}]$ takes the form

$$\begin{aligned} P_{(X,K_X)}(t) &= \sum_{n=0}^{\infty} \dim H^0(X,nH)t^n \\ &= \sum_{n=0}^{\infty} P_n(X)t^n \\ &= \frac{N(t)}{\prod_{\text{finite}}(1-t^{a_i})^{b_i}} \end{aligned}$$

where the Hilbert Numerator $N(t)$ is a palindromic polynomial ("Gorenstein symmetric") whenever $R(X,H)$ is Gorenstein [GW78]. Special cases are when H is

- K_X and ample with (X,H) a canonically polarised variety of general type,
- $-K_X$ and ample with (X,H) an anti-canonically polarised Fano variety,
- ample and $\mathscr{O}_X = K_X \neq H$ hence (X,H) is a polarised Calabi–Yau.

The Hilbert series $P_{(X,H)}(t)$ encodes a likely form of $R(X,\mathscr{O}_X(1))$ by giving a guess on the number of its generators and their syzygies, hence a prediction of a model of X as a variety embedded in a weighted projective space (w.p.s) with the generators and defining equations.

Example 2.2.3.

1. Let E be an Elliptic curve. We would like to associate a graded ring $R = R(E,H)$ to a divisor $H = P \in \mathrm{Div}(E)$ where $P \in E$. We denote by $H^0(E,P)$ in $\mathbb{C}(E)$ the Riemann–Roch space associated to the divisor $H = P$. By the R-R theorem for curves, the dimension $h^0(E,nP) = \deg(nP) = n \in \mathbb{N}$. To describe the graded ring

$$R(E,P) = \bigoplus_{n\geq 0} H^0(E,\mathscr{O}_E(nP)),$$

choose $x,y,z \in R$ so that

$$H^0(E,\mathscr{O}_E(P)) =< x >,\ H^0(E,\mathscr{O}_E(2P)) = \langle x^2,y \rangle \text{ and } H^0(E,\mathscr{O}_E(3P)) = \langle x^3,xy,z \rangle.$$

By proving that x^2,y defines a 2-to-1 map $\varphi_{|2p|} : E \to \mathbb{P}^1$ so that x,y are algebraically independent, one proceeds to prove that y has a pole at P of order 2 and z a pole of order 3. Further, we write out monomials in x,y,z of weighted degree 4 and prove that they form a basis of $H^0(E,\mathscr{O}_E(4P))$, and similarly for degree 5. Next, one shows that there must be a relation in degree 6, and that it must involve y^3 and z^2. Finally, using $z \mapsto z+\alpha_3(x,y)$ and $y \mapsto y+\beta_2(x)$, unwanted terms in the relation are cleared by completing squares and cubes to obtain $f_6 = z^2 - y^3 - ax^4y - bx^6$. We can repeat the process for $H = kP$ for $k = 2,3,4$ and 5 and tabulate the data obtained in Table (2.2) below.

k	P(t)	Degree	Ambient Space	Description of E
1	$\frac{1-t^6}{(1-t)(1-t^2)(1-t^3)}$	6	$\mathbb{P}[1,2,3]$	the classical Weierstrass eqn
2	$\frac{1-t^4}{(1-t)^2(1-t^2)}$	4	$\mathbb{P}[1,1,2]$	Double cover over $\mathbb{P}^1$ with 4 branch pts
3	$\frac{1-t^3}{(1-t)^3}$	3	$\mathbb{P}^2$	plane cubic with a flex point at infinity
4	$\frac{(1-t^2)^2}{(1-t)^4}$	2,2	$\mathbb{P}^3$	Intersection of two quadrics
5	$\frac{1-5t^2+5t^3-t^5}{(1-t)^5}$	5	$\mathbb{P}^4$	$\mathbb{P}^4$ sections of $Gr(2,5) \subset \mathbb{P}^9$

Table 2.2: Graded ring geometry for elliptic curves.

2. Table (2.3) shows some regular surfaces of general type (Y,K_Y) and the corresponding data on their canonical ring $R(Y,K_Y)$.

   We illustrate the computation of the Hilbert function of the surfaces Y from their numerical invariants, say

$$p_g = h^0(Y,K_Y) = h^1(Y,\mathscr{O}_Y) = 4,\ K_Y^2 = 6.$$

No.	$(p_g(Y),\ K_Y^2)$	Ambient Space	(# Eqns, degree)	N(t)
1	$(1,1)$	$\mathbb{P}^4[1,2^2,3^2]$	(2,6)	$1-2t^6+t^{12}$
2	$(2,1)$	$\mathbb{P}^3[1^2,2,5]$	(1,10)	$1-t^{10}$
3	$(2,2)$	$\mathbb{P}^4[1^2,2^2,3]$	(1,4),(1,6)	$1-t^4-t^6+t^{10}$
4	$(3,2)$	$\mathbb{P}^3[1^3,4]$	(1,8)	$1-t^8$
5	$(3,3)$	$\mathbb{P}^3[1^3,2]$	(1,6)	$1-t^6$
6	$(3,4)$	$\mathbb{P}^4[1^3,2^2]$	(2,4)	$1-2t^4+t^8$
7	$(3,5)$	$\mathbb{P}^5[1^3,2^3]$	(5,4)	$1-5t^4+5t^6-t^{10}$
8	$(4,5)$	$\mathbb{P}^3$	(1,5)	$1-t^5$
9	$(4,6)$	$\mathbb{P}^4[1^4,2]$	(1,3),(1,4)	$1-t^3-t^4+t^7$
10	$(4,7)$	$\mathbb{P}^5[1^4,2^2]$	(2,3),(3,4)	$1-2t^3-3t^4+3t^5+2t^6-t^9$
11	$(5,9)$	$\mathbb{P}^4$	(2,3)	$1-2t^3+t^6$
12	$(7,16)$	$\mathbb{P}^6$	(4,2)	$1-4t^2+6t^4-4t^6+t^6$

Table 2.3: Hilbert series data of the canonical ring of some regular surfaces Y of general type from the data $(p_g(Y),K_Y^2)$.

From the graded pieces of the canonical ring

$$R(Y,K_Y)=\bigoplus_{n\geq 0}H^0(Y,nK_Y),$$

the Hilbert series of $(Y,K_Y)\subset\mathbb{P}[a_i^{b_i}]$ takes the form

$$\begin{aligned}P_{(Y,K_Y)}(t)&=\sum_{n=0}^{\infty}P_n(Y)t^n\\&=\sum_{n=0}^{\infty}h^0(Y,\mathscr{O}_Y(nK_Y))t^n\\&=\frac{N(t)}{\prod_{\text{finite}}(1-t^{a_i})^{b_i}}.\end{aligned}$$

Suppose K_Y is ample, then nK_Y is ample for $n\geq 1$. Therefore, from Kodaira vanishing,

$$h^i(K_Y+D)=0 \text{ for } i\geq 0, \text{ ample } D$$

$h^1((n+1)K_Y)=0$ for $n\geq 1$ or $h^1(mK_Y)=0$ for $m\geq 2$. The canonical ring $R(Y,K_Y)$ is therefore Gorenstein [GW78]. Now from RR

$$h^0(nD))+h^0(K_Y-nD)-h^1(nD)=\chi(\mathscr{O}_Y(nD))+\frac{1}{2}(nD-K_Y).nD,$$

for $D=K_Y$, we get that for $n\geq 2$,

$$h^0(nK_Y)+\cancelto{0}{h^0(-(n-1)K_Y))}-\cancelto{0}{h^1(nK_Y)}=\chi(\mathscr{O}_Y(nK_Y))+\frac{1}{2}n(n-1)K_Y^2. \qquad (2.4)$$

Hence the plurigenera

$$P_n(Y) = \begin{cases} 1 & \text{if } n = 0 \\ p_g = 4 & \text{if } n = 1 \\ 1 + p_g + \binom{n}{2} K_Y^2 = 5 + 3n(n-1) & \text{if } n \geq 2 \end{cases}$$

Therefore,

$$P_{(Y,K_Y)}(t) = 1 + 4t + \sum_{n=2}^{\infty} [5 + 3n(n-1)]t^n.$$

We express $P_{(Y,K_Y)}(t)$ as a rational function by multiplying both sides by increasing powers of $1-t$ until we get a closed form on the Right Hand Side

$$P_{(Y,K_Y)}(t) = \frac{t^4 + t^3 + 2t^2 + t + 1}{(1-t)^3}.$$

The coefficients of closed form numerator in the equation above sum to $\deg Y = K_Y^2 - = 6$. We can further use computer algebra to manipulate the numerator into a Gorenstein Symmetric form

$$P_{(Y,K_Y)}(t) = \frac{1 - t^3 - t^4 + t^7}{(1-t)^4(1-t^2)}$$

from which the degree 7 Hilbert numerator $N(t)$ encodes the resolution structure of $R(Y, K_Y)$ hence useful information about Y; the surface Y can be thought of as a codimension 2 subvariety of $\mathbb{P}[1^4, 2]$ defined by 2 equations; a cubic and a quartic.

In more complicated cases, $X \subset \mathbb{P}^{n-1}[b_1, \ldots, b_n]$ is often contained in a weighted "key" variety, say $X \subset Z \subset \mathbb{P}^{n-1}[b_1, \ldots, b_n]$, as a complete intersection. For example, if X is codimension 3 then Z is a weighted Grassmannian [CR02] , see also section (7.2.2).

The graded ring approach is applicable in several situations, such as

(i) constructing varieties such as

- ample $H = K_X$ gives (X, K_X) a variety of general type,
- ample $H = -K_X$ gives $(X, -K_X)$ a Fano variety or
- $H \neq K_X = \mathcal{O}_X$ gives (X, H) a Calabi–Yau variety;

(ii) probing the structure theory of graded rings and resulting embeddings;

(iii) studying interesting classes of varieties such as cluster varieties [ST20] .

2.3 Cones and the basics of Mori Theory

The material in this section is from [Deb16, HM70] and [KM98]. Let $(X, \mathscr{O}_X)$ be a smooth complex projective variety and $\mathfrak{K}_X$ the sheaf of rational functions on X with $\mathscr{O}_X \subset \mathfrak{K}_X$.

A line bundle $\mathscr{O}_X(D)$ corresponding to a (Cartier) divisor $D \in \mathrm{Div}(X) := \Gamma(X, \mathfrak{K}_X^*/\mathscr{O}_X^*)$ is given by local data $\{D_i = \mathbb{V}(f_i), f_i \in \Gamma(D_i, \mathfrak{K}_X^*)\}$ glued by

$$f_i = g_{ij} f_j \text{ where } g_{ij} \in \Gamma(D_{ij}, \mathscr{O}_X^*).$$

The cycle map

$$\mathrm{Div}(X) \ni D \mapsto [D] \in H^2(X, \mathbb{Z}),$$

which associates to D its class $[D]$ on X, descends to a map

$$Cl(X) := \frac{\mathrm{Div}(X)}{\mathrm{PrDiv}(X)} = \frac{\mathrm{Div}(X)}{\sim_{lin}} \to H^2(X, \mathbb{Z})$$

on the **class group** $Cl(X)$, where the subgroup of principal divisors is

$$\mathrm{Div}(X) \supset \mathrm{PrDiv}(X) := \{(f) = (f)_0 - (f)_\infty | f \in \mathbb{C}(X)\}.$$

Thus $D_1, D_2 \in \mathrm{Div}(X)$ are linearly equivalent, denoted as $D_1 \sim_{lin} D_2$, whenever $D_1 - D_2 \in \mathrm{PrDiv}(X)$. We then have the following definitions:

- $\mathrm{Pic}(X)$:=isomorphism classes of invertible sheaves on X (line bundles on X) where

$$\frac{\mathrm{Div}(X)}{\mathrm{PrDiv}(X)} = Cl(X) \xrightarrow{\sim} \mathrm{Pic}(X) \text{ defined by } D \mapsto \mathscr{O}_X(D).$$

- Canonical divisor $K = K_X \in \mathrm{Div}(X)$ is such that

$$K_X \mapsto \mathscr{O}_X(K) \cong \Omega_X^{\dim X} := \wedge^{\dim X} \Omega_X^1$$

  under the map above.

- The evaluation of the intersection form on a Cartier divisor D and a curve C on X is given by $D.C := \deg(C|_D)$.

- $D_1, D_2 \in \mathrm{Div}(X)$ on a surface X are **numerically equivalent**, written as $D_1 \sim_{num} D_2$, if $D_1.C = D_2.C$ for all curves $C \subset X$. The **Néron-Severi group**

$$N^1(X)_{\mathbb{R}} := \frac{\mathrm{Div}(X) \otimes \mathbb{R}}{Num(X)} = \frac{\mathrm{Div}(X) \otimes \mathbb{R}}{\sim_{num}} = \mathrm{Div}(X, \mathbb{Z}) \otimes \mathbb{R}$$

  is a $\mathbb{R}-$vector space of finite dimension $\rho(X) = rank\, N^1(X)$ called the **Picard rank** of X.

Further, for a curve $C \subset X$, we can equivalently and dually define its intersection product with a divisor $D \in \mathrm{Div}(X)$ via the cup product map

$$\cup : H^2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{Z} \tag{2.5}$$

$$([D],[C]) \mapsto D.C := [D].[C]. \tag{2.6}$$

One can then infer that linear equivalence of divisors imply their numerical equivalence

$$D_1 \sim_{lin} D_2 \iff \mathscr{O}_X(D_1) \cong \mathscr{O}_X(D_2) \implies D_1 \sim_{num} D_2.$$

Further, two 1-cycles C_1 and C_2 on X are numerically equivalent if they have the same intersection number with every Cartier divisor $D \in \mathrm{Div}(X)$; this is denoted by $C_1 \equiv C_2$. Define

$$N_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes \mathbb{R} := \frac{C_1(X)}{\equiv}.$$

By extending (2.5), we get the non-degenerate (by definition) intersection pairing map

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}.$$

In particular, $N_1(X)_{\mathbb{R}}$ is a finite-dimensional real vector space.

The **base locus** $Bs([D])$ of $[D] \subset Cl(X)$ is given by

$$Bs([D]) = \bigcap_{H \in [D]} H = \{p \in X : \forall s \in H^0(\mathscr{O}_X(D)), s(p) = 0\}.$$

Definition 2.3.1. Let $L \in \mathrm{Pic}(X)$ where X is projective variety. We say L is

(a) **movable/mobile** if and only if $codim(Bs(L),X) \geq 2$;

(b) numerically effective (**nef**) if and only if $L.C \geq 0$ for all effective curves $C \subset X$;

(c) **big** if and only if there is a constant $C > 0$ such that $h^0(X,mL) \geq C.m^{\dim X}$ for all sufficiently large m from the **semi-group**

$$N(X,L) = \{m \geq 0 : h^0(X,mL) \neq 0\};$$

(d) **effective** if and only if $H^0(X,L) \neq \{0\}$.

(e) **Base Point Free** (BPF) if and only if $\varphi_{|L|} : X \dashrightarrow \mathbb{P}^{h^0(L)-1}$ is a morphism;

(f) **very ample** if and only if $\varphi_{|L|}$ is an embedding with $L \sim_{lin} \varphi^*_{|L|}H$ for a general hyperplane $H \subset \mathbb{P}H^0(X,L)$;

(g) **ample** if and only if mL is very ample for some $m \in \mathbb{N}$.

Definition 2.3.2. Various cones associated to a complex projective variety X are:

- The cone $NE^1(X) :=$**cone spanned by effective divisors**.
- The **cone of effective 1-cycles** $\overline{NE}(X) :=$the closure of the convex cone of (smooth) curves of X in $N_1(X)_{\mathbb{R}}$.
- The ***nef* cone** $\overline{NE}(X)^\vee :=$the closed cone generated by *nef* divisors in $N^1(X)_{\mathbb{R}}$.
- The **Base Point Free cone** $\mathscr{B}(X) :=$the cone generated by base point free divisors in $N^1(X)_{\mathbb{R}}$.
- The **ample cone** $NA(X) \subset N^1(X)$ is the cone generated by the ample divisors; this is the interior of the *nef* cone. By Kleiman's criterion in Theorem (2.3.6) , the **closed cone of curves** is the dual to the nef cone $\overline{NE}(X)^\vee$.

Proposition 2.3.3. *In Definition* (2.3.1) *above, it is the case that*

$$(g) \impliedby (f) \implies (e) \implies (d) \impliedby (c)$$

so that

$$NA(X) \subset \mathscr{B}(X) \subset NE^1(X) \subset N^1(X)_{\mathbb{R}}.$$

*Further, with the natural topology on the Néron-Severi group $N^1(X)$, the **interior of the nef cone is the ample cone**.*

Proof. The implications are clear from the definitions. The last statement follows from Kleiman's criterion, Theorem (2.3.6) below. □

Proposition 2.3.4. *If L is base point free then it is nef ($(e) \implies (b)$).*

Proof. Let L be base point free and consider the map

$$\varphi_{|L|} : X \to Y$$

and $C \subset X$. The image ${\varphi_{|L|}}_* C$ of the curve C is either a curve or a point. We can express L as the pullback $L = {\varphi_{|L|}}^* H$ of a hyperplance section $H \subset Y$. By the projection formula and the facts that H is ample and ${\varphi_{|L|}}_* C$ is a curve or a point, we have that

$$L.C = \left({\varphi_{|L|}}^* H\right).[C] = H.[{\varphi_{|L|}}_* C] \geq 0$$

which demonstrates nefness of L. □

Proposition 2.3.5. *If L is base point free on a smooth projective variety X and big then* $\dim(\operatorname{Im}(\varphi_{|L|})) = dim(X)$.

Proof. Assume $L \in \mathscr{B}(X)$ for a smooth projective variety X then

$$\varphi_{|L|} : X \to \operatorname{Im}(\varphi_{|L|}) := Y \subset \mathbb{P}(H^0(X,L)^\vee) = \mathbb{P}^{h^0(L)-1} = \mathbb{P}^N.$$

In this case,

$$H^0(X,mL) = H^0(Y,\mathscr{O}_Y(m))$$

where $\mathscr{O}_Y(1) = \mathscr{O}_{\mathbb{P}^N}(1)|_Y$ as $\varphi_{|L|}{}^*\mathscr{O}_Y(1) = \mathscr{O}_X(L)$. Moreover, since L is big, we have that $h^0(Y,\mathscr{O}_Y(m)) = h^0(X,mL) \geq C.m^{\dim X}$ and using the Hilbert polynomial of X, we get that $\dim Y \geq \dim X$ from which we obtain

$$\dim Y = \dim X.$$

□

The following theorem give, in [HM70], a numerical characterization of ampleness.

Theorem 2.3.6. ***[Kleiman's criterion]*** *For a projective variety X we have that*

(a) A Cartier divisor $D \in \operatorname{Div}(X)$ is ample if and only if $D.[C] > 0$ for all $0 \neq [C] \in \overline{NE}(X)$.

(b) For $D' \in NA(X)$ ample and any $c \in \mathbb{Z}$, the set $\{[C] \in \overline{NE}(X) : D'.[C] \leq c\}$ is compact hence contains only finitely many classes of curves.

Theorem 2.3.7. ***[Mori]*** *Let X be a smooth projective variety and $H \in NA(X)$ ample. Assume that there is an irreducible curve $C \subset X$ such that $-K_X.C > 0$. Then there is a rational curve $E \subset X$ such that*

$$dim X + 1 \geq -K_X.E > 0.$$

Example 2.3.8. Let X be a smooth projective surface with K_X not nef. Then by the theorem above, we have a rational curve E such that $-3 \leq K_X.E < 0$ hence $K_X.E \in \{-3,-2,-1\}$. Since X is smooth and E is rational $E \cong \mathbb{P}^1$, we have by the adjuction formula that

$$deg\omega_E = (K_X + E).E = K_X.E + E^2 = -2$$

giving three classes of surfaces X

- $E^2 = -1$: this is Castelnúovo's criterion where $K_X.C = -1$,
- $E^2 = 0$: $X \cong F_a$, a ruled surface with E fibres and

- $E^2 = 1$: X birational to $\mathbb{P}^2$ with $L \in \mathbb{P}^2$ and $L^2 = 1$ and $K_{\mathbb{P}^2} = -2L$.

Definition 2.3.9. An **extremal ray** $R := \mathbb{R}_+[C]$ is , in the sense of convex geometry, half-line where for $C_1, C_2 \in \overline{NE}(X)$ and $C_1 + C_2 \in R$, we have that $C_1, C_2 \in R$ and in addition $K_X.R < 0$, where K_X is the canonical divisor of the nonsingular variety X.

We recall, from [KM98], the following Theorem about extremal rays as special generators of the Mori Cone $\overline{NE}(X)$.

Theorem 2.3.10. *[**Mori Cone Theorem**] Consider a non-singular projective variety X. We have that*

1. *There are countably many rational curves $C_i \subset X$ such that $dim X + 1 \geq -K_X.C_i > 0$, and*
$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}_+[C_i].$$
2. *For any $\varepsilon > 0$ and $H \in \overline{NA}(X)$ ample,*
$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\varepsilon H)\geq 0} + \sum_{\text{finite \# of extremal rays}} \mathbb{R}_+[C_i].$$

The following is a cohomological characterization of ampleness of divisors on a projective scheme.

Theorem 2.3.11. *Let X be a projective scheme over a field and let D be a Cartier divisor on X. The following statements are equivalent:*

(a) D is ample;

(b) for each coherent sheaf $\mathscr{F}$ on X, we have $H^k(X, \mathscr{F}(mD)) = 0$ for $m \gg 0$ and for all $k > 0$;

(c) for each coherent sheaf $\mathscr{F}$ on X, we have $H^1(X, \mathscr{F}(mD)) = 0$ for all $m \gg 0$.

`Proof.` Theorem 2.37 of [Deb16] . $\square$

Chapter 3

Introduction to Scrolls

This chapter uses materials mainly from [Reid89, Reid97] and [GS22]. We also refer to [Har77],[Mul06],[ADHL10],[Ful93],[CLS11] and [SP22].

3.1 The Product of Two Projective Spaces

Let us start with a special and very classical case of the more general construction we are going to discuss later. Consider the usual construction of

$$\mathbb{P}^{l_1-1}\times\mathbb{P}^{l_2-1} := \left(\mathbb{C}^{l_1}\setminus\{0\}\right)\times\left(\mathbb{C}^{l_2}\setminus\{0\}\right)/\mathbb{C}^*\times\mathbb{C}^*$$

where each $\mathbb{C}^*$ acts on one of the factors separately. The combined action of $\mathbb{C}^*\times\mathbb{C}^*$ is thus

$$(\lambda,\mu):(x_1,\dots,x_{l_1};y_1,\dots,y_{l_2})\mapsto(\lambda x_1,\dots,\lambda x_{l_1};\mu y_1,\dots,\mu y_{l_2}) \tag{3.1}$$

for $(\lambda,\mu)\in\mathbb{C}^*\times\mathbb{C}^*$.

A subvariety $X=\mathbb{V}(f_{d_1,e_1},\dots,f_{d_m,e_m})\subset\mathbb{P}^{l_1-1}\times\mathbb{P}^{l_2-1}$ is defined by bihomogeneous polynomials $\{f_{d_i,e_i}\}_{i=1}^m\subset\mathbb{C}[x_1,\dots,x_{l_1},y_1,\dots,y_{l_2}]$ of bidegree $(d_i,e_i)\in(\mathbb{Z}_{\geq 0})^2$.

Let

$$R=\mathbb{C}[x_1,\dots,x_{l_1}]=\bigoplus_{d\in\mathbb{N}\cup\{0\}}\mathbb{C}[x_1,\dots,x_{l_1}]_d$$

and, with $\deg x_i=(1,0),\deg y_j=(0,1)$,

$$S=R[y_1,\dots,y_{l_2}]=\bigoplus_{(d_1,d_2)\in(\mathbb{Z}_{\geq 0})^2}S_{(d_1,d_2)}.$$

This is the bigraded homogeneous (bihomogeneous) coordinate ring of the product $\mathbb{P}^{l_1-1}\times\mathbb{P}^{l_2-1}$, where

$$S_{(d_1,d_2)}:=\mathbf{Sym}^{d_1}\left\langle x_1,\dots,x_{l_1}\right\rangle\otimes\mathbf{Sym}^{d_2}\left\langle y_1,\dots,y_{l_2}\right\rangle$$

is the $d_1(1,0)+d_2(0,1)=(d_1,d_2)$-graded piece consisting of bihomogeneous polynomials of degree d_1 in x_i's and d_2 in y_j's.
The product $\mathbb{P}=\mathbb{P}^{l_1-1}\times\mathbb{P}^{l_2-1}$ has a Segre embedding

$$\varphi_{|\mathscr{O}_{\mathbb{P}}(l_1,\ l_2)|}:\mathbb{P}^{l_1-1}\times\mathbb{P}^{l_2-1}\xrightarrow{\cong}\Sigma_{l_1-1,\ l_2-1}\subset\mathbb{P}^{l_1l_2-1}=\mathbb{P}(\mathrm{Mat}(\mathbb{C},l_1\times l_2))$$

$$([x_1:\ldots:x_{l_1}],[y_1:\ldots:y_{l_2}])\mapsto\begin{bmatrix}x_1\\ \vdots\\ x_{l_1}\end{bmatrix}\bullet\begin{bmatrix}y_1 & \ldots & y_{l_2}\end{bmatrix}=(u_{ij}=x_iy_j|i=1,\ldots,l_1;j=1,\ldots,l_2).$$

The image is given by the condition that $\mathrm{rank}(u_{ij})\leq 1$ so that $\Sigma_{l_1-1,\ l_2-1}$ is the variety defined by the vanishing of all 2×2 minors of (u_{ij}).
The projections $\pi_1:\Sigma_{l_1-1,\ l_2-1}\to\mathbb{P}^{l_1-1}$ and $\pi_2:\Sigma_{l_1-1,\ l_2-1}\to\mathbb{P}^{l_2-1}$ are defined by

$$\pi_k((u_{ij}))=\begin{cases}\text{one of the non-zero columns of }(u_{ij}), & \text{k=1}\\ \text{one of the non-zero rows of }(u_{ij}), & \text{k=2.}\end{cases}$$

Furthermore, with $X_k\subset\mathbb{P}^{l_k-1}$, we have that $\sigma_{l_1-1,\ l_2-1}(X_1\times X_2)\subseteq\Sigma_{l_1-1,\ l_2-1}$ giving the Zariski topology on the product $X_1\times X_2$ as the induced topology inside $\mathbb{P}^{l_1l_2-1}$. The Picard group

$$\mathrm{Pic}(\mathbb{P}^{l_1-1}\times\mathbb{P}^{l_2-1})\cong\mathrm{Pic}(\mathbb{P}^{l_1-1})\times\mathrm{Pic}(\mathbb{P}^{l_2-1})=\mathbb{Z}[\pi_1{}^*\mathscr{O}_{\mathbb{P}^{l_1-1}}(1),\pi_2{}^*\mathscr{O}_{\mathbb{P}^{l_2-1}}(1)]\cong\mathbb{Z}^2$$

is generated by pullbacks of hyperplane sections.

3.2 Construction of Scrolls

To define a scroll $\mathbb{F}$, we generalise the group action (3.1) above to allow mixing of the two factors so that the first projection $\pi_1:\mathbb{F}\to\mathbb{P}^{l_1-1}$ can be a nontrivial $\mathbb{P}^{l_2-1}-$ bundle. We also allow weighted $\mathbb{C}^*$actions.
Before giving the definition of a scroll, let us fix some notation. Let $G=\mathbb{C}^*\times\mathbb{C}^*$, $(\lambda,\mu)=g\in G$ and the lattice of characters $\Omega=\mathrm{Hom}(G,\mathbb{C}^*)\cong\mathbb{Z}^2=\langle\chi_1,\chi_2\rangle$ of G with basis given by the projections

$$\chi_1(g)=\lambda,\ \chi_2(g)=\mu\text{ with }\chi(g)=(d_1\chi_1+d_2\chi_2)(g)=\lambda^{d_1}\mu^{d_2}.$$

Also, let $\Theta=\mathrm{Hom}(\Omega,\mathbb{Z})\cong\mathrm{Hom}(\mathbb{C}^*,G)=\langle\nu_1,\nu_2\rangle$ be the lattice of co-characters (oneparameter subgroups) of G with basis given by

$$\begin{aligned}\nu_1, \nu_2 :& \mathbb{C}^* \to G \\ & \nu_1(\lambda) = (\lambda, 1) \\ & \nu_2(\lambda) = (1, \lambda) \\ \text{with } x_1, x_2 \in \mathbb{Z},\ & \nu(\lambda) = (x_1\nu_1 + x_2\nu_2)(\lambda) = (\lambda^{x_1}, \lambda^{x_2}).\end{aligned}$$

We then have that $\Theta = \Omega^\vee$ through the pairing

$$\begin{aligned}&\Theta \times \Omega \to \mathbb{Z} \\ &(\nu, \chi) \mapsto n := \deg\left(\chi \circ \nu : \mathbb{C}^* \xrightarrow{z \mapsto z^n} \mathbb{C}^*\right).\end{aligned}$$

Definition 3.2.1. Let $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n), \mathbf{c} = (c_1, \dots, c_k)$ with the integers $a_j \geq 0, b_j, c_i > 0$ and

$$S = \mathbb{C}[t_1, \dots, t_k, x_1, \dots, x_n] = \bigoplus_{(d_1, d_2) \in L \subset \Omega} S_{(d_1, d_2)} \tag{3.2}$$

the graded ring generated by variables $t_i \in S_{(c_i, 0)}$, $x_j \in S_{(-a_j, b_j)}$ for some sublattice $L = \langle (c_i, 0), (-a_j, b_j) \subset \Omega$. The rational scroll $\mathbb{F}_A$ associated to the weight matrix

$$A := wt(t_1, \dots, t_k, x_1, \dots, x_n) = \begin{bmatrix} c_1 & \dots & c_k & -a_1 & \dots & -a_n \\ 0 & \dots & 0 & b_1 & \dots & b_n \end{bmatrix}$$

is the quotient

$$\mathbb{F}_A := \Big((\mathbb{C}^k \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\}) \Big) / G, \tag{3.3}$$

where G acts by

$$(\lambda, 1) : (t_1, \dots, t_k; x_1, \dots, x_n) \mapsto (\lambda^{c_1} t_1, \dots, \lambda^{c_k} t_k; \lambda^{-a_1} x_1, \dots, \lambda^{-a_n} x_n) \tag{3.4}$$

$$(1, \mu) : (t_1, \dots, t_k; x_1, \dots, x_n) \mapsto (t_1, \dots, t_k; \mu^{b_1} x_1, \dots, \mu^{b_n} x_n). \tag{3.5}$$

The action gives a map $\mathrm{Hom}_A : G \to (\mathbb{C}^*)^k \times (\mathbb{C}^*)^n$ which is compatible with the weighted $\mathbb{C}^*$ action on $\left(\mathbb{C}^k \setminus \{0\}\right)_{t_i}$ with weights $(c_1, \dots, c_k)$. The projection to the first factor $\pi : \mathbb{F} \to \mathbb{P}^{k-1}[c_i]$ defined by

$$(t_1, \dots, t_k; x_1, \dots, x_n) \mapsto [t_1 : \dots : t_k]$$

is a morphism with fibres $\pi^{-1}([t_1:\ldots:t_k]) \cong \mathbb{P}^{n-1}[b_1,\ldots,b_n]$.
We get the diagram

$$\begin{array}{ccc} \left(\mathbb{C}^k\setminus\{0\}\right)\times(\mathbb{C}^n\setminus\{0\}) & \xrightarrow{\ q\ } & \mathbb{F}_A \\ \big\downarrow{\scriptstyle pr_1} & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{C}^k\setminus\{0\} & \longrightarrow & \mathbb{P}^{k-1}[c_i], \end{array}$$

where q is the quotient map defined by the action (3.5).

The scroll $\mathbb{F}_A$ with $c_i = b_j = 1$ for $i = 1,\ldots k; j = 1,\ldots,n$ is called the straight case; we will often denote this scroll by $\mathbb{F}_k(a_1,\ldots,a_n)$ and drop the the subscript k whenever $k = 2$.
For example, we can consider the straight scroll with $k = 2$ over
$\mathbb{P}^1[c_1,c_2] \cong \mathbb{P}^1$,

$$wt(t_1,t_2;x_1,\ldots,x_n) = \begin{bmatrix} 1 & 1 & -a_1 & \ldots & -a_n \\ 0 & 0 & 1 & \ldots & 1 \end{bmatrix} = A.$$

The fibres $\pi^{-1}([t_1:t_2]) \cong \mathbb{P}^{n-1}_{[x_1:\cdots:x_n]}$ so that with $X_{21} = X_{11}^{-1} = \frac{t_2}{t_1}$,

$$\begin{aligned} \mathbb{F} =& (\mathbb{C}^1_{X_{11}} \times \mathbb{P}^{n-1}_{[x_1:\ldots:x_n]}) \cup (\mathbb{C}^1_{X_{21}} \times \mathbb{P}^{n-1}_{[x_1:\ldots:x_n]}), \\ =& \bigcup_{m=1}^{n} \left(\mathbb{C}^n_{<X_{11},\, Y_{1i} = \frac{t_1^{a_i-a_m} x_i}{x_m} | i\neq m>} \right) \cup \bigcup_{m=1}^{n} \left(\mathbb{C}^n_{<X_{21},\, Y_{2j} = \frac{t_2^{a_i-a_m} x_i}{x_m} | j\neq m>} \right) \\ =& \bigcup_{m=1}^{n} U_{1m} \cup \bigcup_{m=1}^{n} U_{2m} \end{aligned}$$

More generally, the rational scroll $\mathbb{F}(a_1,\ldots,a_n)$ is covered by $2n$ copies of

$$U_{ij} := \{t_i \neq 0, x_j \neq 0\} \cong \mathbb{C}^n, \quad i = 1,2; j = 1,\ldots,n.$$

For the chart $x_k \neq 0$, the nontrivial gluing data is given by

$$U_{1m} = \mathbb{C}^n_{<X_{11},\, Y_{1i} = \frac{t_1^{a_i-a_m} x_i}{x_m} | i\neq m>} \xrightarrow[Y_{2j} = X_{11}^{a_i-a_m} Y_{1i}]{X_{21} = X_{11}^{-1}} \mathbb{C}^n_{<X_{21},\, Y_{2j} = \frac{t_2^{a_i-a_m} x_i}{x_m} | j\neq m>} = U_{2m}.$$

The coordinates of the second factor transform by a matrix $\mathrm{diag}(X_{11}^{a_i-a_m})_{i\neq m}$.

Example 3.2.2.

1. The scroll $\mathbb{F}(a) = \mathbb{P}^{k-1}[c_i] \times pt \cong \mathbb{P}^{k-1}[c_i]$ for all $a \in \mathbb{Z}$.

2. The straight scroll $\mathbb{F}(\underbrace{0,\ldots,0}_{n}) = \mathbb{P}^{k-1}[c_i] \times \mathbb{P}^{n-1}$.

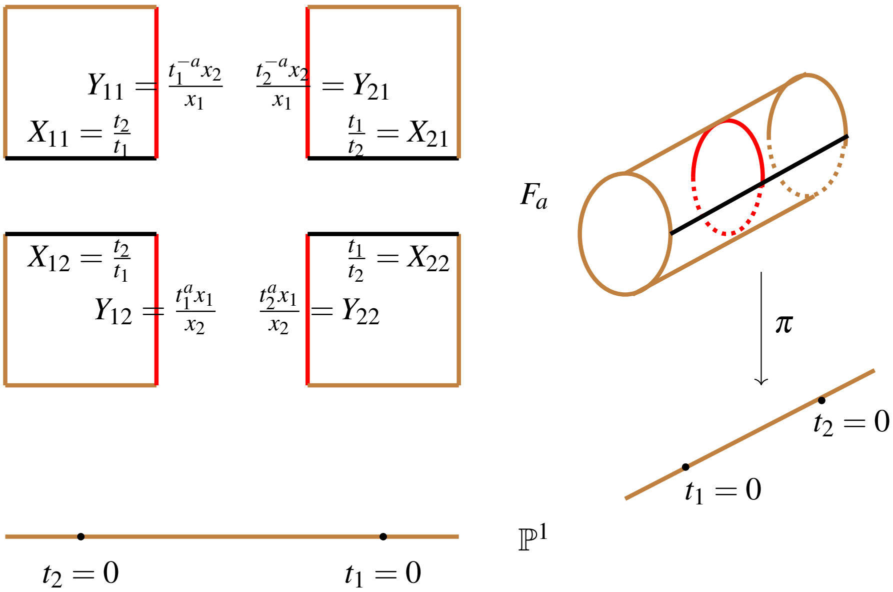


Figure 3.1: Four copies of the affine plane $\mathbb{C}^2$ covering surface scroll F_a.

3. For integer $a > 0$ the surface scroll $F_a := \mathbb{F}(a,0)$ over $\mathbb{P}^1$ is covered by 4 copies of $\mathbb{C}^2$ with the gluing data illustrated in Figure (3.1) . We study some examples further below.

4. The $3-$fold scroll $\mathbb{F}_A := \mathbb{F}(2,1,0)$ over $\mathbb{P}^1$ has $\mathbb{P}^2$ fibres and is covered by 6 copies of $\mathbb{C}^3$.

3.3 Toric Construction of Weighted Scrolls

Let $[1_m] = [1 : \ldots : 1] \in \mathbb{P}^m$. The standard toric structure

$$(\mathbb{P}^{k-1} \times \mathbb{P}^{n-1}, \mathbb{T}^{k-1} \times \mathbb{T}^{n-1}, [1_k, 1_n]) = (\mathbb{P}^{k-1} \times \mathbb{P}^{n-1}, (\mathbb{C}^{k-1})^* \times (\mathbb{C}^{n-1})^*, ([1_k],[1_n]))$$

on $\mathbb{P}^{k-1} \times \mathbb{P}^{n-1}$ comes with the standard torus action

$$(\lambda, \mu).[t_1 : t_i; x_1 : x_j] = [t_1 : \lambda_i t_i; x_1 : \mu_j x_j] \text{ where } i = 2, \ldots, k, j = 2, \ldots, n.$$

On the other hand, the homomorphism

$$\mathrm{Hom}_A : G \to (\mathbb{C}^*)^{k-1} \times (\mathbb{C}^*)^{n-1} \cong \mathbb{T}^{n+k-2}$$

described above is an open embedding with an obvious base point $([1_k],[1_n])$ on $\mathbb{F}_A$. We can then define the scroll $\mathbb{F}_A$ as a toric variety by:

$$(\mathbb{F}_A, G, ([1_k],[1_n])) := (\overline{Hom_A(G)}, \mathbb{T}^{n+k-2}, [1_k; 1_n])$$

with the torus action

$$(\lambda,\mu).[t_i;x_j] = Hom_A(\lambda,\mu)[t_i;x_j].$$

We now illustrate the toric construction of $\mathbb{F}_A$ below.

The subtorus $G = (\mathbb{C}^*)^2$ used in the definition of the scroll

$$\mathbb{F}_A = (\mathbb{C}^{n+k} \setminus Z)/G$$

gives an exact sequence of tori

$$1 \to G \to (\mathbb{C}^*)^{k+n} \to (\mathbb{C}^*)^{k+n-2} = \mathbb{T} \to 1 \tag{3.6}$$

with an open orbit of $\mathbb{F}_A$ isomorphic to $\mathbb{T}$. Now, by taking $\mathrm{Hom}(\mathbb{C}^*, \ )$ at every point of the sequence (3.6), we get an exact sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^{k+n} \xrightarrow{q} \mathrm{Hom}(\mathbb{C}^*,\mathbb{T}) = N \to 0 \tag{3.7}$$

of abelian groups. The generators $\{\sigma_i,\rho_j\} = \{q(\ell_i),q(\ell_j)\}$ of N, with $\{\ell_i,\ell_j\}$ the standard basis element of $\mathbb{Z}^{k+n}$, are the one-dimensional cones of the fan $\Sigma_{\mathbb{F}_A}$ of the projective toric variety $\mathbb{F}_A$. It suffices to define f on the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{Z}^2$ as

$$f := \begin{bmatrix} f(1,0) \\ f(0,1) \end{bmatrix} = A$$

where A is the weight matrix associated to $\mathbb{F}_A$. Associated to a one-dimensional cone σ_i is a Weil divisor $D_{\sigma_i} \in Cl(\mathbb{F}_A)$. By extension, we have the dual sequence

$$0 \to N^\vee = \mathrm{Hom}(\mathbb{T},\mathbb{C}^*) \xrightarrow{\tilde{f}} \mathbb{Z}^{k+n-2} = \bigoplus_{i=2}^{k} \mathbb{Z}[D_{\sigma_i}] \bigoplus \bigoplus_{j=2}^{n} \mathbb{Z}[D_{\rho_j}] \xrightarrow{\tilde{q}} Cl(\mathbb{F}_A) \to 0 \tag{3.8}$$

of SES (3.15) with

$$\tilde{f}(m) = \sum_{i=2}^{k} < m, v_{\sigma_i} > D_{\sigma_i} + \sum_{j=2}^{n} < m, v_{\rho_j} > D_{\rho_j}$$

where $< m, v_x >$ is a valuation at x of a character $m \in M$ with v_x a generator of the ray of 1-dimensional cone $x \in \Sigma$.

Now, with $p = (\sigma_1,\ldots,\sigma_k,\rho_1,\ldots,\rho_n) \in N$, we have that

$$N = \mathbb{Z}[\sigma_i,\rho_j]/\left\langle A(\sigma_i,\rho_j)' \right\rangle \cong \mathbb{Z}^{k+n-2} =< \sigma_2,\ldots,\sigma_k,\rho_2,\ldots,\rho_n > . \tag{3.9}$$

Proposition 3.3.1. *With N as defined by Equation* (3.9)*, define a fan $\Sigma \subset N_{\mathbb{R}}$ by declaring that $2n$ maximal cones of $\mathbb{F}(a_1, b_j|1, b_j)$ over $\mathbb{P}^1$ are*

$$\tau_{i,j} = Span(\sigma_i, \rho_1, \ldots, \widehat{\rho}_j, \ldots, \rho_n).$$

Using standard toric notation in [Ful93], we have that

$$\mathbb{F}(a_1, b_j|1, b_j) \cong X_{N,\Sigma}$$

with its Cox ring the bigraded ring $Cox(X_{N,\Sigma}) = \bigoplus_{(d_1,d_2)\in\mathbb{Z}^2} \mathbb{C}[t_i, x_j]_{(d_1,d_2)}$.

Proof. The first map of SES (3.6) is defined by

$$(\lambda, \mu) \mapsto (\lambda, \lambda, \lambda^{-a_1}\mu, \lambda^{-a_j}\mu^{b_j}).$$

This precisely describes the action of $G = (\mathbb{C}^*)^2$ on $\mathbb{C}^2_{t_i} \times \mathbb{C}^n_{x_j}$. Now the scroll definition (3.2.1) is comparable to the standard toric description of $X_{N,\Sigma}$ found in [Ful93], to check that everything agrees. In particular, the irrelevant ideal corresponding to our chosen fan Σ is the ideal

$$I_{irr} = \{t_i x_j : 1 \leq i \leq 2, 1 \leq j \leq n\} \lhd \mathbb{C}[t_i, x_j],$$

with $Z = \mathbb{V}(I_{irr}) = \mathbb{C}^2 \times \mathbb{C}^n \setminus U$ where $U = \mathbb{C}^2_{t_i} \setminus \{0\} \times \mathbb{C}^n_{x_j} \setminus \{0\}$. □

3.4 Scrolls as Projective Bundles

Let B be a nonsingular projective variety over $\mathbb{C}$ and $\mathscr{E}$ a rank n vector bundle over B with fibres $\mathscr{E}_b \cong \mathbb{C}^n$ over $b \in B$. In this section, we would like to construct the projective bundle $\mathbb{P} := \mathrm{Proj}_B(\mathscr{E})$, and its tautological line bundle $\mathscr{O}_{\mathbb{P}}(1)$. We will also show that this general construction covers the previous one $\mathbb{F}_A$ in the special case when $B = \mathbb{P}^{k-1}$ and $\mathscr{E}$ is a sum of line bundles over B with

$$A := wt(t_1, \ldots, t_k, x_1, \ldots, x_n) = \begin{bmatrix} 1 & \ldots & 1 & -a_1 & \ldots & -a_n \\ 0 & \ldots & 0 & 1 & \ldots & 1 \end{bmatrix}.$$

We first note that

$$\mathbb{P}_b := \mathrm{Proj}(\mathbf{Sym}^\bullet \mathscr{E}_b) \cong \mathbb{P}^{n-1}.$$

Further, denote $S_d := \mathbf{Sym}^d \mathscr{E} \to B$, a vector bundle over B with fibre

$$(S_d)_b \cong \mathbf{Sym}^d \mathscr{E}_b \cong \Gamma\left(\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(d)\right).$$

We then take the graded vector bundle

$$S = \mathbf{Sym}^\bullet \mathscr{E} := \bigoplus_d \mathbf{Sym}^d \mathscr{E} = \bigoplus_d S_d$$

from which we construct a projective bundle $\mathbb{P}_B(\mathscr{E}) := \mathrm{Proj}_B(S)$ over B as follows:

- Pick a cover $\{U_i | i \in I\}$ of the base B over which $\mathscr{E}$ is trivialized. For $b_i \in U_i$, the trivialization $\mathscr{E}|_{U_i} \cong U_i \times \mathscr{E}_{b_i}$ gives $S|_{U_i} \cong U_i \times \mathbf{Sym}^\bullet \mathscr{E}_{b_i}$.

- Identify the trivializations $\mathscr{E}|_{U_i}$ and $\mathscr{E}|_{U_j}$ by sections $\varphi_{ij} \in \Gamma\left(U_{ij}, \widetilde{GL(n)}\right)$ where $\widetilde{GL(n)}$ is a constant sheaf on B with fibres $GL(n)$, so $\varphi_{ij}(b) \in GL(n)$ for all $b \in U_{ij} = U_i \cap U_j$.

- Take $V_i := \mathrm{Proj}_B(S|_{U_i}) = U_i \times \mathrm{Proj}(\mathbf{Sym}^\bullet \mathscr{E}_{b_i}) \cong U_i \times \mathbb{P}^{n-1}$.

- Glue V_i and V_j by transition maps $\overline{\varphi_{ij}} \in \Gamma\left(U_{ij}, \widetilde{PGL(n)}\right)$ on B, where $\overline{\varphi_{ij}}$ is the projectivisation of φ_{ij}.

- Get $\mathbb{P}(\mathscr{E}) = \mathbb{P}_B(\mathscr{E}) := \mathrm{Proj}_B(S) = \bigcup_i V_i$.

- We shall call $\pi : \mathbb{P}_B(\mathscr{E}) \to B$ the natural projection to the base B obtained by gluing the projections π_i

$$\begin{array}{ccc} \mathbb{P}^{n-1} & & \\ \uparrow {\scriptstyle f_i} & & \\ V_i & \hookrightarrow & \mathbb{P}_B(\mathscr{E}) \\ \downarrow {\scriptstyle \pi_i} & & \downarrow {\scriptstyle \pi} \\ U_i & \hookrightarrow & B. \end{array}$$

  The fibres of π are $\pi^{-1}(b) = \mathbb{P}_b \cong \mathbb{P}^{n-1}$.

Definition 3.4.1. [Tautological Subbundle] The bundle $\pi^*\mathscr{E} \to \mathbb{P}_B(\mathscr{E})$ on B is locally given by rank n vector bundles $\pi^*\mathscr{E}|_{V_i} := V_i \times \mathscr{E}_b = V_i \times \mathbb{C}^n$ at $b_i \in U_i \subset B$ which are glued along the isomorphisms

$$V_{ij} \times \mathbb{C}^n \to V_{ij} \times \mathbb{C}^n \tag{3.10}$$

$$((b,[v]),w) \mapsto ((b, \overline{\varphi_{ij}}[v]), \varphi_{ij} w). \tag{3.11}$$

Its tautological subbundle $\mathscr{S} \subset \pi^*\mathscr{E}$ is defined as follows:

1. Define $\mathscr{S}_i$ on V_i by

$$V_i|_{U_i} \times \mathscr{E}_{b_i} \cong \mathbb{P}(\mathscr{E}_{b_i}) \times \mathscr{E}_{b_i} \supset \mathscr{S}_i := \left\{ ([v], w) : \mathrm{rnk} \begin{bmatrix} v \\ w \end{bmatrix} \leq 1 \right\}.$$

For $b_i \in U_i$, take the local coordinates $[v] = [x_1 : \ldots : x_n] = x \in \mathbb{P}(\mathscr{E}_{b_i})$ and $w = (y_1, \ldots, y_n) = y \in \mathscr{E}_{b_i}$ from which we get

$$\mathscr{S}_i := \bigcap_{i,j=1}^{n} \left\{((b_i,x),y) : x_i y_j - x_j y_i = 0\right\} \subset \mathbb{P}^n_{b_i} \times \mathbb{C}^n. \tag{3.12}$$

Indeed, for $b_i \in U_i$, we have that the fibres $(\mathscr{S}_i)_{b_i} \subset \mathbb{P}^n_{b_i} \times \mathbb{C}^n$ are tautological line bundles on $\mathbb{P}^n_{b_i}$; so $\mathscr{S}_i$ is a line bundle on V_i.

2. By equation (3.12), the definition of $\mathscr{S}_i$ is independent of the choice of basis of $\mathscr{E}_{b_i}$ so that $\cup \mathscr{S}_i = \mathscr{S} \to \mathbb{P}(\mathscr{E}) = \cup V_i$ is compatible with the gluing inside $\pi^*\mathscr{E}$. Geometrically, for $(b_i, x) \in \mathbb{P}(\mathscr{E}_{b_i})$, the line $\pi(\mathscr{S}_{(b_i,[v])}) \subset \mathscr{E}_{b_i}$ is such that

$$\mathbb{P}(\pi(\mathscr{S}_{(b_i,x)})) = x \in \mathbb{P}(\mathscr{E}_{b_i}).$$

The following definition of the Serre line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ will be restricted to the case associated to the line bundle $\mathscr{E} = \bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^{k-1}}(a_i)$ over $\mathbb{P}^{k-1}$ with $\mathbb{C}[\mathbb{P}(\mathscr{E})]$ thought of as having weights

$$wt(t_1, \ldots, t_k, x_1, \ldots, x_n) = \begin{bmatrix} 1 & \ldots & 1 & -a_1 & \ldots & -a_n \\ 0 & \ldots & 0 & 1 & \ldots & 1 \end{bmatrix} = A.$$

Definition 3.4.2. [Serre bundle] The tautological (Serre) line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is constructed by gluing line bundles $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ over the patches $V_i := U_i \times \mathbb{P}^{n-1}$ as follows:

i. Using projections $f_i : V_i \to \mathbb{P}^{n-1}$, take the line bundle $\mathscr{O}_{V_i}(1) = f_i^* \mathscr{O}_{\mathbb{P}^{n-1}}(1)$ on V_i consisting of degree one functions in fibre $\mathbb{P}^{n-1}_{[x_k]}$ coordinates.

ii. Glue $\mathscr{O}_{V_i}(1)$ and $\mathscr{O}_{V_j}(1)$, first using $\varphi_{ij} \in \Gamma\left(U_{ij}, \widetilde{GL(n)}\right)$ to identify linear polynomials

$$\mathscr{E}_{b_i} \cong \mathbb{C}[\mathbb{P}^{n-1}_{b_i}]_1 \xrightarrow{\varphi_{ij}} \mathbb{C}[\mathbb{P}^{n-1}_{b_j}]_1 \cong \mathscr{E}_{b_j}$$

in fibre coordinates where $b_i \in U_i \subset B,\ b_j \in U_j \subset B$ followed by $\overline{\varphi_{ij}} \in \Gamma(U_{ij}, \widetilde{PGL(n)})$ to identify V_i and V_j where $V_{ij} := U_{ij} \times \mathbb{P}^{n-1}$.

The degree $d \in \mathbb{Z}$ line bundle on $\mathbb{P}(\mathscr{E})$ is then defined by $\mathscr{O}_{\mathbb{P}_B(\mathscr{E})}(d) := \mathscr{O}_{\mathbb{P}_B(\mathscr{E})}(1)^{\otimes d}$.

The proposition below captures some properties of the Serre bundle on $\mathbb{P}(\mathscr{E})$.

Proposition 3.4.3. *The Serre bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ and subbundle $\mathscr{S}$ on $\mathbb{P}_B(\mathscr{E})$ have the following properties* *(a)* $\mathscr{E} \cong \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

(b) $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1) \cong \mathscr{S}$.

`Proof.` Compare sections of the respective sheaves locally. For part (b), with $p_i \in \mathbb{C}\left(\mathbb{P}^{n-1}_{[x_i]}\right)$, the morphism is given by $\varphi \mapsto (p_i = \varphi.x_i)$. □

Proposition 3.4.4. *For any line bundle $L \in \operatorname{Pic}(B)$ we have that*

$$\mathbb{P}(\mathscr{E} \otimes L) := \mathbb{P}_B(\mathscr{E} \otimes L) \cong \mathbb{P}_B(\mathscr{E}) = \mathbb{P}(\mathscr{E})$$

so that the triangle

$$\begin{array}{ccc} \mathbb{P}(\mathscr{E}) & \xrightarrow{\varphi_L} & \mathbb{P}(\mathscr{E} \otimes L) \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & B & \end{array}$$

commutes and that

$$\varphi_L{}^* \mathscr{O}_{\mathbb{P}(\mathscr{E} \otimes L)}(-1) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1) \otimes \pi'^* L.$$

`Proof.` Consequence of Chapter V Prop 2.2 of [Har77]. □

Theorem 3.4.5. *Consider the rank n vector bundle $\mathscr{E} = \mathscr{O}_{\mathbb{P}^{k-1}}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^{k-1}}(a_n)$ over $B = \mathbb{P}^{k-1}$. The rational scroll $\mathbb{F}_A$ with*

$$A := wt(t_1, \ldots, t_k, x_1, \ldots, x_n) = \begin{bmatrix} 1 & \ldots & 1 & -a_1 & \ldots & -a_n \\ 0 & \ldots & 0 & 1 & \ldots & 1 \end{bmatrix}$$

is isomorphic to the projective bundle

$$\mathbb{F}_A \cong \mathbb{P}_{\mathbb{P}^{k-1}}(\mathscr{E}).$$

`Proof.` We provide a proof for $k = 2$ but the general case is the same. Using gluing data on $\mathbb{F}$ and on $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$, we would like to show that there is an isomorphism f which fits into the following diagram

$$\begin{array}{ccccccc} V_i := \mathbb{F}|_{U_i} & \hookrightarrow & \mathbb{F} & \xrightarrow{f} & \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) & \hookleftarrow & \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}|_{U_i}) \\ \downarrow {\scriptstyle \pi|_{V_i}} & & \downarrow {\scriptstyle \pi} & & \downarrow {\scriptstyle \pi'} & & \downarrow {\scriptstyle \pi'_i} \\ U_i & \hookrightarrow & \mathbb{P}^1 & = & \mathbb{P}^1 & \hookleftarrow & U_i. \end{array}$$

From the quotient construction, we have gluing data on the open covers

$$\begin{aligned}
\mathbb{F} =& V_1 \cup V_2 \\
=& \left(U_1 \times \mathbb{P}^{n-1}_{[x_1:\dots:x_n]}\right) \cup \left(U_2 \times \mathbb{P}^{n-1}_{[x_1:\dots:x_n]}\right) \\
=& \left(\mathbb{C}^1_r \times \mathbb{P}^{n-1}_{[x_1:\dots:x_n]}\right) \cup \left(\mathbb{C}^1_s \times \mathbb{P}^{n-1}_{[x_1:\dots:x_n]}\right) \\
=& \bigcup_{i=1}^{n} \left(\mathbb{C}^n_{\left\langle r=\frac{t_1}{t_2},\, y_i:=\frac{t_1^{a_i-a_2} x_i}{x_2} \middle| i\neq 2 \right\rangle} \right) \cup \bigcup_{j=1}^{n} \left(\mathbb{C}^n_{\left\langle s=\frac{t_2}{t_1},\, y_j^1:=\frac{t_2^{a_j-a_1} x_j}{x_1} \middle| j\neq 1 \right\rangle} \right)
\end{aligned}$$

where the affine patches of the fibre $\mathbb{P}^{n-1}_{[x_i]}$ are glued by

$$\varphi_{ij} = \operatorname{diag}\left(s^{a_i-a_j}\right)_{i\neq j} \in \Gamma\left(\mathbb{C}^*, \widetilde{PGL(n)}\right) \tag{3.13}$$

on the base $\mathbb{P}^1 = U_1 \cup U_2$. Explicitly, with $U_{12} = U_1 \cap U_2 \cong \mathbb{C}^*$, the gluing is given by

$$\begin{aligned}
& U_{12} \times \mathbb{P}^{n-1}_{[x_i]} = V_{12} = V_1|_{U_2} \to V_2|_{U_1} = V_{21} = U_{21} \times \mathbb{P}^{n-1}_{[x_i]} \\
& \left(r,\, y_i^2 := \frac{t_1^{a_i-a_2} x_i}{x_2} \middle| i \neq 2 \right) \mapsto \left(s,\, y_j^1 := \frac{t_2^{a_j-a_1} x_j}{x_1} \middle| j \neq 1 \right) \\
& (r, y_1^2, \widehat{y_2^2}, y_3^2, \dots, y_n^2) \mapsto (s, \widehat{y_1^1}, y_2^1, y_3^1, \dots, y_n^1) = \\
& \left(r^{-1}, (y_1^2)^{-1} r^{a_1-a_2}, y_3^2 (y_1^2)^{-1} r^{a_1-a_3}, \dots, y_j^2 (y_1^2)^{-1} r^{a_1-a_j}, \dots, y_n^2 (y_1^2)^{-1} r^{a_1-a_n} \middle| j \neq 2 \right).
\end{aligned}$$

From the Proj construction, we glue V_1 to V_2 through the gluing information on $\mathscr{E} = \bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^1}(a_i)$ where, locally, we get a map $\mathscr{E}|_{U_{12}} \cong \mathscr{O}_{U_1}^{\oplus n}|_{U_2} \to \mathscr{O}_{U_2}^{\oplus n}|_{U_1} \cong \mathscr{E}|_{U_{21}}$ defined through gluing of $\mathscr{E}|_{U_i} = U_i \times \mathbb{C}_{t_i \neq 0}$ using global sections

$$\sigma_i \in \Gamma(U_i = \{t_i \neq 0\}, \mathbb{P}^1) = \mathbb{C}\left[t_1, t_2\right]_{t_i}$$

for $i = 1,2$ and transition matrices $M_{ij} = \left(\frac{t_i}{t_j}\right)^{a_i-a_j}$ to yield

$$M_{ij}\sigma_i = \sigma_j$$

on U_{ij}. Then by equation (3.13) we see that the transition matrices $M_{ij} = \operatorname{diag}(s^{a_i-a_j})_{i\neq j}$ are used in gluing the same affine pieces on the scroll. Hence

$$\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) = \operatorname{Proj}_{\mathbb{P}^1}\left(\bigoplus_{d\geq 0} \mathbf{Sym}^d \bigoplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^1}(a_i) \right) = \left(\mathbb{C}^1_{\frac{t_2}{t_1}} \times \mathbb{P}^{n-1}_{<x_1,\dots,x_n>} \right) \cup \left(\mathbb{C}^1_{\frac{t_1}{t_2}} \times \mathbb{P}^{n-1}_{<x_1,\dots,x_n>} \right)$$

glued by $\overline{\varphi_{12}} \in \Gamma\left(\mathbb{C}^1_{t_1,t_2\neq 0}, \widetilde{PGL(n)}\right)$ on $\mathbb{P}^1$. We therefore have the morphism

$$f : \{t_i \neq 0\} \times \mathbb{C}^{n-1} \to \{t_i \neq 0\} \times \mathbb{C}^{n-1}, i = 1,2$$

defined by

$$f\left(r,\, y^k_j := \frac{t_1^{a_j - a_k} x_j}{x_k}\middle| k \text{ fixed } j \in \{1,\ldots,\widehat{k},\ldots,n\}\right) = (r, y^k_j | k \text{ fixed } j \in \{1,\ldots,\widehat{k},\ldots,n\}).$$

With the gluing data on both sides, f extends to a morphism on $\mathbb{F}$ over $\mathbb{P}^1$.

$$\begin{array}{ccc} \mathbb{F} := \mathbb{F}(a_1,\ldots,a_n) & \xrightarrow{\quad f \quad} & \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) := \operatorname{Proj}\left(\mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^1}(a_n)\right) \\ & \searrow^{\pi} \quad \swarrow_{\pi'} & \\ & \mathbb{P}^1_{[t_1:t_2]} & \end{array}$$

□

Corollary 3.4.6. *For $L = \mathscr{O}_{\mathbb{P}^1}(-c)$, we get an isomorphism*

$$\mathbb{F}(a_1+c,\ldots,a_n+c) = \operatorname{Proj}_{\mathbb{P}^1}\left(\bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^1}(a_i+c)\right) \xrightarrow{\varphi_L} \operatorname{Proj}_{\mathbb{P}^1}\left(\bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^1}(a_i)\right) = \mathbb{F}(a_1,\ldots,a_n).$$

`Proof.` Combine Proposition (3.4.4) with Theorem (3.4.5). □

Example 3.4.7. For the surface scroll F_2, we have that

$$\ldots \cong \mathbb{F}(4,2) \cong \mathbb{F}(3,1) \cong \mathbb{F}(2,0) = F_2.$$

For a more general construction, we have the following definition involving weighted locally free sheaves.

Definition 3.4.8. [Weighted projective bundle] Let B be any base. Consider a locally free sheaf $\mathscr{E} = \bigoplus_{i=1}^m \mathscr{E}_i$, with $\mathscr{E}_i$ of rank n_i thought of as having weights $b_i \geq 0$. Consider the sheaf of graded algebras

$$S = \mathbf{Sym}^\bullet \mathscr{E} = \mathbf{Sym}^\bullet\left(\bigoplus_{i=1}^m \mathscr{E}_i\right) = \bigoplus_{d \geq 0} S_d.$$

This inherits the grading of the individual pieces $\mathscr{E}_i$. We then have the weighted projective bundle

$$\mathbb{P}_B(\mathscr{E}) := \operatorname{Proj}(S) \xrightarrow{\pi} B$$

with fibres $\pi^{-1}(b) = \mathbb{P}(\mathbf{Sym}^\bullet \mathscr{E}_b) \cong \mathbb{P}^{n-1}[b_1^{n_1},\ldots,b_m^{n_m}]$ weighted projective spaces.

Example 3.4.9. Let $B = \mathbb{P}^1_{[t_1:t_2]}$ and $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2 \oplus \mathscr{E}_3$ be a vector bundle over B where $\mathscr{E}_1 = \mathscr{E}(a_1) \oplus \mathscr{E}(a_2)$ is of rank 2 and weight 1 and $\mathscr{E}_2 = \mathscr{E}(a_3)$, $\mathscr{E}_3 = \mathscr{E}(a_4)$ both of rank 1 and weight $2,3$ respectively. We then have the graded sheaf

$$\mathfrak{S} := \mathbf{Sym}^\bullet(\mathscr{E}) = \bigoplus_{d \geq 0} \mathbf{Sym}^d(\mathscr{E})$$

over B with the resulting weighted scroll

$$\mathrm{Proj}(\mathbf{Sym}^\bullet \mathscr{E}) = \left(\mathbb{C}_{\frac{t_1}{t_2}} \times \mathbb{P}^3[1^2,2,3]\right) \cup \left(\mathbb{C}_{\frac{t_2}{t_1}} \times \mathbb{P}^3[1^2,2,3]\right)$$

whose projection $\pi : \mathrm{Proj}(\mathbf{Sym}^\bullet \mathscr{E}) \to \mathbb{P}^1$ has $\mathbb{P}^3[1^2,2,3]_{<x_1,x_2,y,z>}$ as its fibres with x_1, x_2, y, z the local sections of $\mathfrak{S}$. The choice of the vector $(a_1,a_2,a_3,a_4) \in \mathbb{Z}^4$ determines the gluing information and hence the geometry of the weighted scroll.

3.5 Class Groups of Weighted scrolls

This section describes the group $Cl(\mathbb{F}_A)$ of Weil divisors on a weighted scroll $\mathbb{F}_A = X_{N,\Sigma}$, see Proposition (5.4), associated to the weight matrix

$$wt(t_1,\ldots,t_k,x_1,\ldots,x_n) = \begin{bmatrix} c_1 & \ldots & c_k & -a_1 & \ldots & -a_n \\ 0 & \ldots & 0 & b_1 & \ldots & b_n \end{bmatrix} = A.$$

The one-dimensional cones $\sigma_1,\ldots,\sigma_k,\rho_1,\ldots,\rho_n$ in the fan Σ corresponds to Weil divisors $D_{\sigma_i} = \mathbb{V}(t_i), D_{\rho_j} = \mathbb{V}(x_j)$. The following proposition summarizes the property of $Cl(\mathbb{F}_A)$ for $c_1 = \ldots = c_k = 1$.

Proposition 3.5.1. *Let $L = [D_{\sigma_1}], M = [D_{\rho_1}] \subset Cl(\mathbb{F}_A)$ and suppose $b_1 = 0$ which guarantees that $a_1 = 0$ with other weights in the weight matrix A not fixed. That is, assume that $wt(x_1) = (-a_1,b_1) = (0,1), wt(x_j) = (-a_j,b_j)$ and $wt(t_i) = (c_i,0)$. We have that*

(i) $Cl(\mathbb{F}_A) = \mathbb{Z}L \oplus \mathbb{Z}M$.

(ii) The canonical class is given by

$$K_{\mathbb{F}_A} = \left(-k + \sum_{j=1}^{n} a_j\right) L - \left(\sum_{j=1}^{n} b_j\right) M$$

(iii) For $d_1, d_2 \in \mathbb{Z}$, we have that

$$H^0(d_1L + d_2M) = S_{d_1,d_2},$$

the bidegree (d_1,d_2) piece of the graded ring $S = \mathbb{C}[\mathbb{F}_A] = \mathbb{C}[t_i,x_j]$.

(iv) Every divisor on $\mathbb{F}_A$ is $\mathbb{Q}$-Cartier.

`Proof.` For (i), we remember the dual sequence (3.8) below

$$0 \to N^\vee = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \xrightarrow{\tilde{f}} \mathbb{Z}^{k+n-2} = \bigoplus_{i=2}^{k} \mathbb{Z}[D_{\sigma_i}] \bigoplus \bigoplus_{j=2}^{n} \mathbb{Z}[D_{\rho_j}] \xrightarrow{\tilde{q}} Cl(\mathbb{F}_A) \to 0.$$

Here,

$$\tilde{f}(m) = \sum_{i=2}^{k} < m, \sigma_i > D_{\sigma_i} + \sum_{j=2}^{n} < m, \rho_j > D_{\rho_j}$$

with $m \in N^\vee$ and the angle brackets $<,>$ denote the canonical pairing between $N^\vee$ and N; a rational function associated to a character of $\mathbb{T}$.

Note that the elements of $N^\vee$ give the following relations between the classes of various torus-invariant divisors in $Cl(\mathbb{F}_A)$

$$\rho_1 = -\sum_{j=2}^{n} b_j \rho_j$$

and

$$\sigma_1 = -\sum_{i=2}^{k} \sigma_i + a_1\rho_1 + \sum_{j=2}^{n} a_j \rho_j = -\sigma_2 + \sum_{j=2}^{n} (a_j - b_j a_1)\rho_j;$$

the former being valid since $wt(x_1) = (0,1)$. From these relations, we have a $\mathbb{Z}$-basis of $Cl(\mathbb{F}_A)$. Therefore with $p = (\sigma_1, \ldots, \sigma_k, \rho_1, \ldots, \rho_n) \in N$, we have that

$$N = \mathbb{Z}[\sigma_i, \rho_j] / \langle A(\sigma_i, \rho_j)' \rangle \cong \mathbb{Z}^{k+n-2} = < \sigma_2, \ldots, \sigma_k, \rho_2, \ldots, \rho_n >$$

whereas the class group $Cl(\mathbb{F}_A)$ is free of rank 2 defined by

$$Cl(\mathbb{F}_A) = \mathbb{Z}[D_{\sigma_i}, D_{\rho_j}] / \langle \tilde{f}(\sigma_2^\vee), \ldots, \tilde{f}(\sigma_k^\vee), \tilde{f}(\rho_2^\vee), \ldots, \tilde{f}(\rho_n^\vee) \rangle = \mathbb{Z}[D_{\sigma_1}, D_{\rho_1}].$$

For (ii), with the basis $\{f_2 = \sigma_2^\vee, \ldots, f_k = \sigma_k^\vee, g_2 = \rho_2^\vee, \ldots, g_n = \rho_n^\vee\}$ of $N^\vee$, we have that

$$\tilde{f}(f_i) = -D_{\sigma_1} + D_{\sigma_i} \implies [D_{\sigma_1}] = [D_{\sigma_i}] = L \text{ for all } i = 2, \ldots, k$$

and

$$\begin{aligned}\tilde{f}(g_j) &= < \rho_j^\vee, \sigma_1 > D_{\sigma_1} + < \rho_j^\vee, \rho_1 > D_{\rho_1} + < \rho_j^\vee, \sigma_2 > D_{\rho_1} \\ &= (a_j - b_j a_1) D_{\sigma_1} - b_j D_{\sigma_1} + D_{\sigma_j} \\ &\implies [D_{\rho_j}] = b_j [D_{\rho_1}] + (a_1 b_j - a_j) L \text{ for for all } j = 2, \ldots, n.\end{aligned}$$

Hence

$$\begin{aligned}
K_{\mathbb{F}_A} &= -\sum_{i=1}^{k}[D_{\sigma_i}] - \sum_{j=1}^{n}[D_{\rho_j}] \\
&= -kL - [D_{\sigma_1}] - \sum_{j=2}^{n}(b_j[D_{\rho_1}] + (a_1 b_j - a_j)L) \\
&= \left(-k + \sum_{j=2}^{n} a_j + a_1 - a_1 b_1 - a_1 \sum_{j=2}^{n} b_j\right) L - \left(1 + \sum_{j=2}^{n} b_j\right)[D_{\sigma_1}] \\
&= \left(-k + \sum_{j=1}^{n} a_j - a_1 \sum_{j=1}^{n} b_j\right) L - \left(\sum_{j=1}^{n} b_j\right) M \\
&= \left(-k + \sum_{j=2}^{n} a_j\right) L - \left(1 + \sum_{j=2}^{n} b_j\right) M \text{ since } wt(x_1) = (0,1).
\end{aligned}$$

For (iii), we know that the $(\mathbb{C}^*)^2$-action induces a bigrading on the ring

$$S = \mathbb{C}[\mathbb{C}^2_{t_i} \times \mathbb{C}^n_{x_j}]$$

with $deg(t_i) = (1,0), \deg(x_j) = (-a_j, b_j)$. On $\mathbb{F}_A$, the coordinates $t_i, x_j \in \mathbb{C}[\mathbb{F}_A]$ are treated as weighted homogeneous just like in weighted projective spaces. In particular, the sections of $L := L_{1,0}$ and $M = L_{0,1}$ corresponding to $wt(t_1) = (1,0)$ and $wt(x_1) = (0,1)$ respectively are

$$H^0(\mathbb{F}(a_1,\ldots,a_n), L) = \mathbf{Sym}^{\bullet}(t_1,\ldots,t_k) = S_{(0,1)} \subset S = \mathbb{C}[t_i, x_j]$$

and, if $a_i \geq 0$ for all i,

$$H^0(\mathbb{F}(a_1,\ldots,a_n), M) = \mathbf{Sym}^{a_1}(t_1,\ldots,t_k)x_1 \oplus \ldots \oplus \mathbf{Sym}^{a_n}(t_1,\ldots,t_k)x_n.$$

Generally, the global sections of $L_{d_1,d_2} = d_1 L + d_2 M$ are given by

$$H^0(\mathbb{F}_A, L_{d_1,d_2}) := \bigoplus_{(q_i)_w \vdash d_2,\ w=(b_i),\ wt(t_i)=c_i} \mathbf{Sym}^{d_1 + \sum_{j=1}^{n} a_j q_j}(t_1,\ldots,t_k) x_1^{q_1} \ldots x_n^{q_n} \cong S_{(d_1,d_2)}. \tag{3.14}$$

where we use the convention that $\mathbf{Sym}^m$ is empty for $m < 0$. Here $(q_i)_w \vdash d_2$ denotes a weighted partition of d_2 with fixed non zero integral weights $w = (b_i)$ such that $\sum b_i q_i = d_2$.

Finally, for (iv), consider the SES

$$0 \to \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^{k+n} \xrightarrow{q} \operatorname{Hom}(\mathbb{C}^*, \mathbb{T}) = N \to 0 \tag{3.15}$$

of abelian groups. The generators $\{\sigma_i, \rho_j\} = \{q(\ell_i), q(\ell_j)\}$ of N, with $\{\ell_i, \ell_j\}$ the standard basis element of $\mathbb{Z}^{k+n}$, are the one-dimensional cones of the fan $\Sigma = \Sigma_{\mathbb{F}_A} \subset N_{\mathbb{R}}$. For any collection of integer labels on the generators $\{\sigma_i, \rho_j\}$ there is a linear function on the cone that takes these values at the generators and rational values at other integral points. A multiple of this function is integral-valued on the lattice $\mathbb{N}_{\mathbb{R}}$, hence corresponds to a $\mathbb{Q}$-Cartier divisor. $\square$

Since $G = (\mathbb{C}^*)^2$ acts with finite stabilizers on $Z = (\mathbb{C}^k \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$, the scroll $\mathbb{F}_A = Z//G$ is a geometric quotient. Along the locus where there are non-trivial finite stabilizers, we obtain finite quotient singularities on the quotient $\mathbb{F}_A$. Just as the weighted projective space $\mathbb{P}^n[b_i]$ has singularities while $\mathbb{P}^n$ is nonsingular, weighted scrolls $\mathbb{F}_A$ have quotient singularities whereas straight scrolls $\mathbb{F}(a_1, \ldots, a_n)$ are nonsingular.

Remark 3.5.2. *Consider the general weight matrix*

$$A = \begin{bmatrix} c_1 & \ldots & c_k & -a_1 & \ldots & -a_n \\ 0 & \ldots & 0 & b_1 & \ldots & b_n \end{bmatrix}.$$

In this case, the sections of Weil divisors L_{d_1,d_2} define a rational map

$$\varphi_{|d_1L+d_2M|} : \mathbb{F}_A \dashrightarrow \mathbb{P}(S_{(d_1,d_2)}) := \mathbb{P}^N_{[y_i]}$$

under the assumption that

$$N = \sum_{(q_i)_w \vdash d_2} (d_1 + \sum_{i=1}^{n} a_i q_i) - 1 > 0.$$

Let us return to the case of a straight scroll $\mathbb{F}(a_1, \ldots, a_n)$, with non-negative integral weights.

Theorem 3.5.3. *Assume integers $a_i \geq 0$ for all i. Then we have that*

$$H^0(\mathbb{F}(a_1, \ldots, a_n), M) = \mathbf{Sym}^{a_1}(t_1, \ldots, t_k)x_1 \oplus \ldots \oplus \mathbf{Sym}^{a_n}(t_1, \ldots, t_k)x_n = S_{(0,1)}$$

gives a morphism

$$\varphi_{|M|} : \mathbb{F}(a_1, \ldots, a_n) \to \mathbb{P}(S_{(0,1)}) = \mathbb{P}^{(a_1+\ldots+a_n)+n-1}_{[y_i]}$$

defined by

$$(t_1, \ldots, t_k; x_1, \ldots, x_n) \overset{\varphi}{\mapsto} [\mathbf{Sym}^{a_1}(t_1, \ldots, t_k)x_1 : \ldots : \mathbf{Sym}^{a_n}(t_1, \ldots, t_k)x_n]$$

with the image $\varphi(\mathbb{F}(a_1, \ldots, a_n)) = \mathbb{V}(I) \subset \mathbb{P}^{(a_1+\ldots+a_n)+n-1}_{[y_i]}$ where

$$I = \bigwedge^2 \left[\begin{array}{ccc|ccc|c|ccc} y_1 & \cdots & y_{a_1} & y_{a_1+2} & \cdots & y_{a_1+a_2+1} & \cdots & y_{a_1+\ldots+a_{n-1}+n} & \cdots & y_N \\ y_2 & \cdots & y_{a_1+1} & y_{a_1+3} & \cdots & y_{a_1+a_2+2} & \cdots & y_{a_1+\ldots+a_{n-1}+n+1} & \cdots & y_{N+1} \end{array} \right].$$

Here I is a homogeneous ideal generated by all 2×2 minors within and between coordinates of the projective subspaces $\mathbb{P}(\mathbf{Sym}^{a_j}(t_i)x_j)$ for all $j=1,\ldots,n$. Moreover, M is very ample if $a_i>0$ for all i.

`Proof.` Theorem 2.5 of [Reid97] proves the $k=2$ case which generalizes easily to any case $k>2$. □

The following proposition characterizing base point free divisor classes on $\mathbb{F}_A$ will be useful in the later chapters.

Proposition 3.5.4. *Let $\Sigma\subset N_{\mathbb{R}}$ be the simplicial fan associated to the scroll $\mathbb{F}_A=X_{\Sigma,N}$. Any linear system which contains the vertices of facets $F^n_{\tau_{i,j}}$ of Σ is base point free.*

`Proof.` Proposition 3.2 of [CDT18]. Using our notations, the idea of proof uses the implication of Proposition (3.5.1) (iv) that the normal fan Σ of N is simplicial. Its facets are given by

$$F^n_{\tau_{i,j}}=\text{ConvexHull}(v_x|x\in\sigma_i,\rho_1,\ldots,\widehat{\rho}_j,\ldots,\rho_n)$$

where $\tau_{i,j}=\text{Span}(\sigma_i,\rho_1,\ldots,\widehat{\rho}_j,\ldots,\rho_n)$ is a maximal cone, v_x is a ray through a point x in a one dimensional cone. For all $f\in F^n_{\tau_{i,j}}$, a lattice point

$$m\in<\sigma_2^\vee,\ldots,\sigma_k^\vee,\rho_2^\vee,\ldots,\rho_n^\vee>=N^\vee,$$

is such that $<m,f>=-1$. Therefore $<m,v_x>=-1$ for all $x\in F^1_{\tau_{i,j}}$.

□

Example 3.5.5.

1. We revisit $\mathbb{F}(0,0)\cong\mathbb{F}(1,1)\cong\mathbb{P}^1\times\mathbb{P}^1$ by noting that

   $$\mathbb{F}(1,1)\overset{\varphi}{\hookrightarrow}\mathbb{P}^3_{[y_0:y_1:y_2:y_3]}$$

   where

   $$\varphi(t_1,t_2;x_1,x_2)=[t_1x_1:t_2x_1:t_1x_2:t_2x_2].$$

   This is nothing but the standard Segre embedding

   $$\mathbb{P}^1_{[t_1:t_2]}\times\mathbb{P}^1_{[x_1:x_2]}\overset{\sigma_{1,1}}{\hookrightarrow}\Sigma_{1,1}:=\mathbb{V}(y_0y_2-y_1y_3)=\varphi(\mathbb{F}(1,1))\subset\mathbb{P}^3_{[y_i]}.$$

2. $F_1 := \mathbb{F}(1,0) \xrightarrow{\varphi} \mathbb{P}^2_{[y_j]}$ defined by $\varphi(t_1,t_2;x_1,x_2) = [t_1x_1 : t_2x_1 : x_2]$ is a surjective morphism. If $y_1 = t_2x_1 \neq 0$, we can set $t_2 = 1$ then

$$(t_1,t_2;x_1,x_2) = (\frac{y_0}{y_1}, 1 : y_1 : y_2)$$

hence

$$\varphi^{-1}(\{y_1 \neq 0\}) = \{t_2 \neq 0\} \times \mathbb{P}^1$$

which implies that φ is an isomorphism over $\{y_1 \neq 0\} \subset \mathbb{P}^2$.
On the other hand, the image of the projective line $\mathbb{V}(x_1)$ is the cone point

$$\varphi((t_1,t_2;0,x_2)) = [0:0:1] = p \in \mathbb{P}^2_{[y_i]}.$$

To understand the morphism in the neighbourhood of this point, recall the standard blowup

$$\mathbb{P}^2_{[y_j]} \times \mathbb{P}^1_{[X:Y]} \supset \mathbb{C}^2_{\left\langle x=\frac{y_0}{y_2}, y=\frac{y_1}{y_2}\right\rangle} \times \mathbb{P}^1_{[X:Y]} \supset Bl_{(0,0)}\mathbb{C}^2 = \mathbb{V}(xY - yX) \xrightarrow{\sigma} \mathbb{C}^2_{\left\langle x,y \right\rangle}$$

with the exceptional curve $\sigma^{-1}((0,0)) \cong \mathbb{P}^1_{[X:Y]}$. We then have the following diagram

$$\begin{array}{ccccc} F_1 & \xrightarrow{\quad\varphi\quad} & \mathbb{P}^2 & \xhookleftarrow{\quad j\quad} & \mathbb{C}^2 \\ \Big\downarrow \pi & & \Big\uparrow pr1 & & \Big\uparrow \sigma \\ \sigma^{-1}((0,0)) & \xleftarrow{\quad pr2\quad} & \mathbb{P}^2_{[y_j]} \times \mathbb{P}^1_{[X:Y]} & \xhookleftarrow{\quad i\quad} & Bl_{(0,0)}\mathbb{C}^2. \end{array}$$

The fibres

$$\overline{pr1 \circ i \circ \sigma^{-1}(\mathbb{C}^2 \setminus \{(0,0)\})} \cong \mathbb{P}^1_{[t_1:t_2]}$$

of the scroll map π are the strict transforms of the lines L_p through $p \in \mathbb{P}^2_{[y_j]}$. We conclude, therefore, that

$$\mathbb{F}(1,0) \cong Bl_p\mathbb{P}^2.$$

3. $F_2 := \mathbb{F}(0,2) \xrightarrow{\varphi} \mathbb{P}^3_{[y_0:y_1:y_2:y_3]}$ where

$$\varphi(t_1,t_2;x_1,x_2) = [x_1 : t_1^2x_2 : t_1t_2x_2 : t_2^2x_2].$$

We have that

$$\varphi(\mathbb{F}(2,0)) = \mathbb{V}\left(\bigwedge^2 \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix}\right) = \mathbb{V}(y_1y_3 - y_2^2) = Q_3 \subset \mathbb{P}^3_{[y_i]}.$$

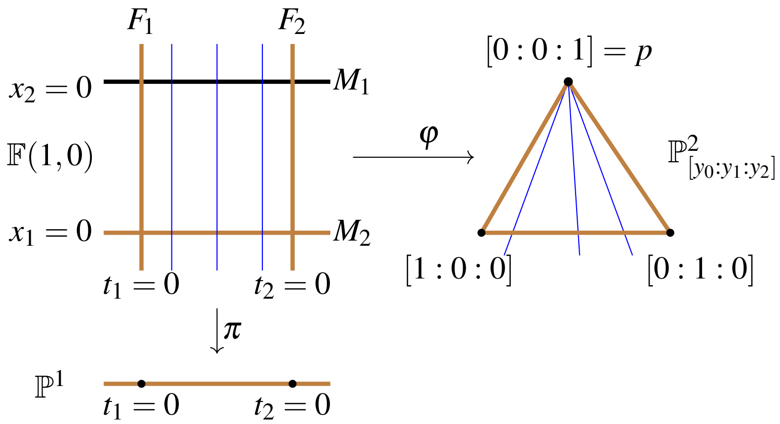


Figure 3.2: The surface scroll $\mathbb{F}(1,0) \cong Bl_{[0:0:1]}\mathbb{P}^2$ is the blowup of a point in $\mathbb{P}^2$.

The map $\varphi : \mathbb{F}(2,0) \to Q_3$ is a contraction of $M_1 = \mathbb{V}(x_1) \subset \mathbb{F}(0,2)$ to a singular cone point $[1:0:0:0]$ of the quadric Q_3.

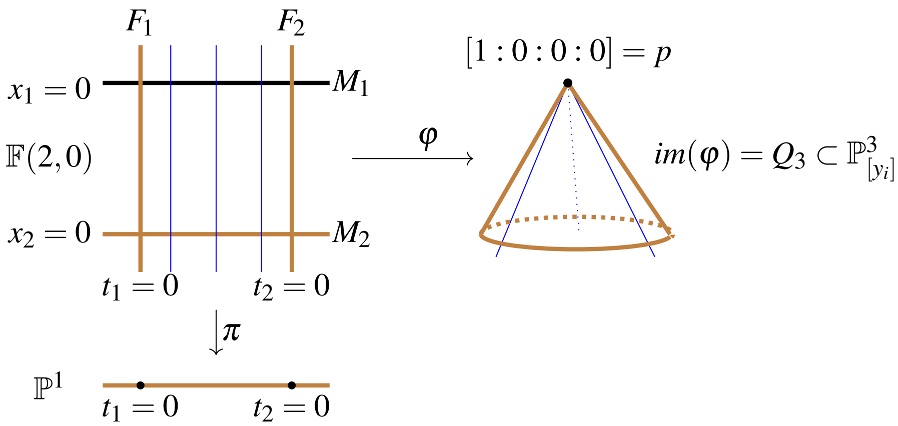


Figure 3.3: Contraction of $\{x_1 = 0\} \subset \mathbb{F}(0,2)$ to a singular cone point $p \in Q_3$.

4. We can infer that the blowup of $\mathbb{P}^2$ embeds in $\mathbb{P}^4$

$$Bl_p\mathbb{P}^2 \cong \mathbb{F}(1,0) \cong \mathbb{F}(2,1) \hookrightarrow \mathbb{P}^4.$$

The following example explicitly computes line bundles on nonsingular scrolls

Example 3.5.6. The divisor class group of the straight scroll $\mathbb{F} := \mathbb{F}(a_1,\dots,a_n)$ fibred over $\mathbb{P}^{k-1}$ is the free abelian group

$$\mathrm{Pic}(\mathbb{F}(a_1,\dots,a_n)) = \mathbb{Z}.L \oplus \mathbb{Z}.M$$

generated by $L = [\{t_i = 0\}]$, the class of fibres, and $M = [\{x_1 = 0\}]$, the class of sections of

$$\pi : \mathbb{F} \to \mathbb{P}^{k-1}.$$

Assume the integers $a_i \geq 0$ for all i, we have the expressions $[M_j] = -a_jL + M$ where

$$M_j = \mathbb{F}(a_1, \ldots, \widehat{a_j}, \ldots, a_n) = \{x_j = 0\}$$

is the j-th subscroll of $\mathbb{F}$.

Indeed, the first assertion follows from

$$(a_iL + M_i) - (a_jL + M_j) = \left(\frac{g_{a_i}(t_s)x_i}{g_{a_j(t_s)}x_j} \right)$$

with $\frac{g_{a_i}(t_s)x_i}{g_{a_j}(t_s)x_j} \in \mathbb{C}(\mathbb{F})$ where $g_{a_i}(t_s) \in \mathbf{Sym}^{a_i}(t_1, \ldots, t_k)$; we conclude that $a_iL + M_i$ and $a_jL + M_j$ are linearly equivalent. Moreover, from the weight matrix

$$wt(t_1, \ldots, t_k, x_1, \ldots, x_n) = \begin{bmatrix} 1 & \ldots & 1 & -a_1 & \ldots & -a_n \\ 0 & \ldots & 0 & 1 & \ldots & 1 \end{bmatrix},$$

we have that $H^0(\mathbb{F}, -a_jL + M) = < x_j >$ hence $[M_j] = -a_jL + M$ for all j.

To demonstrate linear independence of L and M, suppose $aL + bM \sim_{lin} 0$ then we have , by restriction in any $\mathbb{P}^{n-1}$ fibre of $\pi : \mathbb{F} \to \mathbb{P}^{k-1}$, that for $n - k \geq 0$

$$0 = (aL + bM)^{n-k+1}|_{L^{k-1}} = b,$$

hence $0 = (aL)^{n-k+1}|_{M^{k-1}} = a$.

Finally, every irreducible codimension one subvariety $D \subset \mathbb{F}$ is a hypersurface. So D corresponds to a height 1 ideal in $S = \mathbb{C}[t_i, x_j]$ which must be principal by Krull's theorem. We can find a monomial $x^\alpha \in S$ with multi-index

$$\alpha = (e_1, \ldots, e_k, q_1, \ldots, q_n) = (d_1 + a_1d_2, 0, \ldots, 0, d_2, 0, \ldots, 0),$$

that is

$$f,\ t_1^{d_1+ad_2}x_1^{d_2} \in S_{(d_1,d_2)}$$

hence $D - (d_1L + d_2M) = \left(\frac{f}{t_1^{d_1+ad_2}x_1^{d_2}} \right)$ with $\frac{t_1^{-(d_1+a_1d_2)}f}{x_1^{d_2}} \in \mathbb{C}(\mathbb{F})$.

Example 3.5.7. We have already proved that

$$\text{Pic}(\mathbb{F}(a, 0)) = \mathbb{Z}.L \oplus \mathbb{Z}.M,$$

with $L = [F_i] = [\{t_i = 0\}], M = M_2 = [\{x_2 = 0\}]$ and $M_1 = -aL + M$. Further, we have that

$$\begin{bmatrix} L^2 & LM_1 & LM \\ LM_1 & M_1^2 & M_1M \\ LM_2 & M_1M & M^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -a & 0 \\ 1 & 0 & a \end{bmatrix}.$$

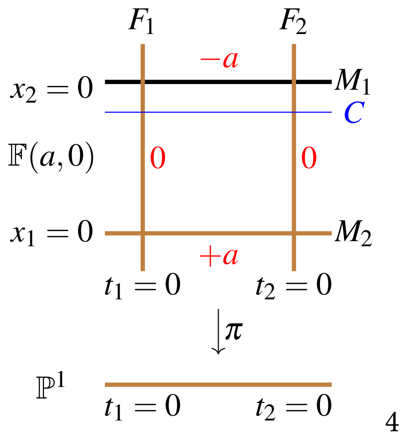


Figure 3.4: Intersection numbers on surface scroll $\mathbb{F}(a,0)$

Indeed, by definition, $M = aL + M_1 = M_2$ so that $M_1 = -aL + M$. Since

$$M_1 \cap M_2 = \emptyset, F_1 \cap F_2 = \emptyset \text{ with } L = [F_i],$$

we have that $M_1.M = 0$ and $M.L = 1$ and $M_1.L = 1$ since L is a fibre of

$$\pi : \mathbb{F}(a,0) \to \mathbb{P}^1$$

whereas M_i are sections. Now, by intersecting $M_1 = -aL + M$ with M_1, we get $M_1.M_1 = -a$; also with M to get $0 = -a + M.M$ which gives the last self-intersection $M^2 = a$.

Example 3.5.8. In this example, we use alternative methods to compute (hence confirm) that the canonical divisor of a scroll $\mathbb{F}_A$ where

$$A := wt(t_1, \dots, t_k, x_1, \dots, x_n) = \begin{bmatrix} 1 & \dots & 1 & -a_1 & \dots & -a_n \\ 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}$$

is given by

$$K_{\mathbb{F}_A} = (-k + a_1 + \dots + a_n)L - nM.$$

In particular, for the surface scroll $F_a = \mathbb{F}(0,a)$ we have that

$$K_{F_a} = (-2 + a)L - 2M.$$

Consider the following two $(n+k-2)-$affine charts of $\mathbb{F}_A$;

$$U_{11} = \{t_1 \neq 0, x_1 \neq 0\} \cong \mathbb{C}^{n+k-2}_{\left\langle r_{i1} = \frac{t_i}{t_1},\ y_j := \frac{t_1^{a_j - a_1} x_j}{x_1} \middle| i \neq 1, j \neq 1 \right\rangle}$$

and

$$U_{22} = \{t_2 \neq 0, x_2 \neq 0\} \cong \mathbb{C}^{n+k-2}\left\langle s_{i2}=\frac{t_i}{t_2},\ z_j:=\frac{t_2^{a_j-a_2}x_j}{x_2}\middle| i\neq 2, j\neq 2\right\rangle$$

with the transition functions

$$(s_{12}, s_{i2}, z_1, z_j)_{i=3,\ j=3}^{i=k,\ j=n} \mapsto (r_{21}^{-1}, r_{21}^{-1}r_{i1},\ r_{21}^{a_1-a_2}y_2^{-1},\ r_{21}^{a_j-a_2}y_2^{-1}y_j)_{i=3,\ j=3}^{i=k,\ j=n}.$$

One can compute the $(n+k-2)-$form

$$\begin{aligned}\eta &= ds_{12}\wedge\ldots\wedge ds_{k1}\wedge dz_1\wedge\ldots\wedge dz_n\\ &= -r_{21}^{-2}dr_{21}\wedge r_{21}^{-1}dr_{31}\wedge\ldots\wedge r_{21}^{-1}dr_{k1}\wedge -r_{21}^{a_1-a_2}y_2^{-2}dy_2\wedge -r_{21}^{a_3-a_2}y_2^{-1}dy_3\wedge\ldots\wedge -r_{21}^{a_n-a_2}y_2^{-1}dy_n\\ &= (-1)^{n-1}r_{21}^{(-k+\sum(a_j-a_2))}y_2^{-n}dr_{21}\wedge\ldots\wedge dr_{k1}\wedge dy_2\wedge\ldots\wedge dy_n\end{aligned}$$

Hence

$$K_{\mathbb{F}_A} = \operatorname{div}(\eta) = (-k+\sum(a_j-a_2)\operatorname{div}(r_{21}) - n\operatorname{div}(y_2).$$

Obviously, $\operatorname{div}(r_{21})$ and $\operatorname{div}(y_2)$ in U_{11} are

$$\operatorname{div}(t_2) = L \text{ and } \operatorname{div}(x_2) = M_2 = -a_2L + M$$

on $\mathbb{F}_A$ respectively. Hence,

$$\begin{aligned}K_{\mathbb{F}_A} &= (-k+\sum(a_j-a_2))L - n(-a_2L+M)\\ &= (-k+\sum a_j)L - nM.\end{aligned}$$

Other than through differential forms, we could use intersection numbers to confirm that $K_{F_a} = (-2+a)L - 2M$ for the surface scroll $F_a = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1}(a)\oplus\mathscr{O}_{\mathbb{P}^1})$.

Let $K_{F_a} = k_1L + k_2M$ where $k_1, k_2 \in \mathbb{Z}$ are to be determined. Now, we use the adjuction formula in the form

$$K_D = (K_X + D)|_D,\ \text{ where } D \in \operatorname{Div}(X)$$

is a nonsingular hypersurface and X nonsingular complex variety. Taking degrees on the surface X gives

$$2g_a(D) - 2 = (K_X + D).D\ .$$

So with $g_a(L) = g_a(M_1) = g_a(M_2) = 0$, we have that

$$2g_a(L) - 2 = ((k_1+1)L + k_2M).L,$$

and

$$2g_a(M) - 2 = (k_1L - M).M$$

from which we get $k_2 = -2,\ k_1 = -2 + a$ so that

$$K_{F_a} = (-2+a)L - 2M.$$

The following Lemma from [Reid97] shows a general computaton of intersection numbers in an n-fold straight scroll $\mathbb{F} := \mathbb{F}(a_1, \dots, a_n)$ fibred over $\mathbb{P}^1$.

Lemma 3.5.9. *$M^{n-1}L = 1$ and $M^n = \sum a_j$.*

Proof. Let $M_j = \mathbb{F}(a_1, \dots, \widehat{a_j}, \dots, a_n) = \{x_j = 0\}$ be the j-th subscroll so that $M_j \sim_{lin} -a_jL + M$. On each $\mathbb{P}^{n-1}$ fibre, the M_j for $j = 1, \dots, n$ are coordinate hyperplanes $x_j = 0$ which are disjoint so that $\cap_{j=0}^n M_j = \emptyset$. We therefore have that $M_1M_2 \dots M_n = 0$. However, any $n-1$ hyperplanes $M_1, \dots, \widehat{M_j}, \dots, M_n$ intersect transversally in a curve section of $\pi : \mathbb{F} \to \mathbb{P}^1$ (this is the point $e_j \in \mathbb{P}^{n-1}_{[x_k]}$ with 1 in the j-th place and zeros in the remaining $n-1$ places).
We therefore have, say for j=1, that

$$1 = LM_2 \dots M_n = L(-a_2L + M) \dots (-a_nL + M) = LM^{n-1}$$

because $L^2 = 0$, and

$$\begin{aligned} 0 = M_1M_2 \dots M_n &= (-a_1L + M)(-a_2L + M) \dots (-a_nL + M) \\ &= -\sum_{j=1}^{n} a_jLM^{n-1} + M^n. \end{aligned}$$

□

Example 3.5.10.

1. Consider the fibrations

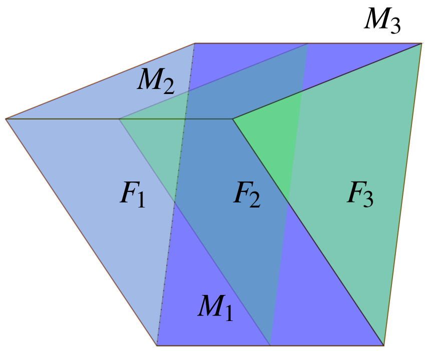
$$\begin{array}{ccc} \mathbb{F} = \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}(0,0) & \xrightarrow{\varphi_{|L_{0,1}|}} & \mathbb{P}^1_{[x_1:x_2]} \\ \Big\downarrow \varphi_{|L_{1,0}|} & & \\ \mathbb{P}^1_{[t_1:t_2]} = B. & & \end{array}$$

   Here, $M := L_{0,1}$ is base point free but **not** big since $\dim \mathrm{Im}(\varphi_M) = 1 < 2 = \dim \mathbb{F}$.

2. The morphism $F_2 := \mathbb{F}(2,0) \xrightarrow{\varphi_{|M|}} Q_3 \subset \mathbb{P}^3$ is such that $dim\, \mathrm{Im}(\varphi_{|M|}) = 2$. We have shown in example 3 of (3.5.5) that φ is not an isomorphism but a contraction of a proper subvariety $\{x_2 = 0\} = \mathbb{F}(2) \cong \mathbb{P}^1 \subset F_2$. We therefore have that $\varphi_{|M|}$ is base point free and big.

3. Consider a rank 3 vector bundle $\mathscr{E} = \mathscr{O}(2) \oplus \mathscr{O}(1) \oplus \mathscr{O} \to \mathbb{P}^2$ and $\mathbb{F}_A = \mathbb{F}(2,1,0) = \mathbb{P}(\mathscr{E})$ where $A = wt(t_1,t_2,t_3;x_1,x_2,x_3) = \begin{bmatrix} 1 & 1 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

$$\begin{array}{ccc} \mathbb{F}(2,1,0) = \mathbb{P}(\mathscr{E}) & \xhookrightarrow{\varphi_{|L_{0,1}|}} & \mathbb{P}\left(\bigoplus_j \mathbf{Sym}^{a_j}(t_i)x_j\right) = \mathbb{P}^{6+3+1-1}_{[y_k]} \\ \Big\downarrow \pi & & \\ \mathbb{P}^2_{[t_i]} & & \end{array}$$



$M_j = \mathbb{V}(x_j)$
M_3 Plane at the back
$F_i = \mathbb{V}(t_i)$ Vertical planes
$L = [F_i]$ Class of vertical planes

Figure 3.5: Toric representation of 3-fold scroll $\mathbb{F}(2,1,0)$.

With the sections $M_3 = M,\ M_2 = -L + M,\ M_1 = -2L + M$ we have that

$$K_{\mathbb{F}} = (-3 + \sum a_j)L - nM = -3M \text{ and } \mathscr{O}_{\mathbb{F}}(1) = M$$

so that $-K_{\mathbb{F}} = \mathscr{O}_{\mathbb{F}}(3)$. The anticanonical map is

$$\varphi_{|-K_{\mathbb{F}}|} = \varphi_{|3M|} : \mathbb{F} \to \mathbb{P}\left(S_{(0,3)}\right) = \mathbb{P}^{108=28+21+2\times15+2\times10+6+3+1-1}_{[y_k]}$$

where

$$\begin{aligned} S_{(0,3)} = & \bigoplus_{q_1+q_2+q_3=3} \mathbf{Sym}^{2q_1+q_2}(t_1,t_2,t_3)x_1^{q_1}x_2^{q_2}x_3^{q_3} \\ & = \mathbf{Sym}^6(t_i)x_1^3 \oplus \mathbf{Sym}^5(t_i)x_1^2x_2 \oplus \mathbf{Sym}^4(t_i)\left(x_1^2x_3 + x_1x_2^2\right) \\ & \oplus \mathbf{Sym}^3(t_i)\left(x_1x_2x_3 + x_2^3\right) \oplus \mathbf{Sym}^2(t_i)x_2^2x_3 \oplus \mathbf{Sym}^1(t_i)x_2x_3^2 \oplus x_3^3. \end{aligned}$$

The anticanonical divisor $-K_{\mathbb{F}}$ is therefore base point free hence nef by proposition (2.3.4). However, the map $\varphi_{|-K_{\mathbb{F}}|}$ is not an embedding as it contracts the subscroll

$$\{x_3 = 0\} = \mathbb{F}(2,1) \cong \mathbb{F}(1,0)$$

of $\mathbb{F}$ (we know that $\operatorname{Im}\mathbb{F}(1,0) \cong \mathbb{P}^2 \cong \mathbb{P}(\mathscr{O})$) to a point in $\operatorname{Im}(\varphi_{|-K_{\mathbb{F}}|})$. We then conclude that since $\dim \operatorname{Im}(\varphi_{|-K_{\mathbb{F}}|}) = 4$, it follows from proposition (2.3.5) that the anticanonical divisor $-K_{\mathbb{F}}$ is big .

3.6 Base locus of a Divisor Class in Scrolls

The following results about the base locus of a divisor class $|D|$ for a divisor $D \in Cl(\mathbb{F}_A)$ will be useful in later chapters.

Proposition 3.6.1. *Let the quotient torus $\mathbb{T} = \mathbb{T}^{k+n-4} = \mathbb{T}^{k+n-2}/G$ act on the scroll $\mathbb{F}_A = X_{\Sigma,N}$ as induced by the SES* (3.6). *For a Cartier divisor $D \in \operatorname{Div}\mathbb{F}_A$, the base locus $B(|D|)$ is $\mathbb{T}$-invariant.*

Proof. We note that $Cl(\mathbb{F}_A) = \mathbb{Z}[L,M]$ is discrete and that $\mathbb{T}$ is continuous on $\mathbb{F}_A = X_{\Sigma,N}$ so that for every $D = L_{d_1,d_2} \in Cl(\mathbb{F}_A)$, the torus $\mathbb{T}$ fixes $|D|$. So $\mathbb{T}$ acts on the set of sections of $\mathscr{O}_X(D)$ so the intersection of all $E \in |D|$ is $\mathbb{T}$-invariant. □

Lemma 3.6.2. *Let* $0 \le a_1 \le \ldots \le a_m \le b < a_{m+1} \le \ldots \le a_n$ *and*

$$L_{-b-1,1} \in \operatorname{Pic}(\mathbb{F}(a_1,\ldots,a_n)) = \mathbb{Z}[L,M].$$

Then

(1) the base locus

$$Bs(|L_{-b-1,1}|) = \mathbb{F}(a_1,\ldots,a_m) = \{x_j = 0| \text{ for all } j > m\} = B_b.$$

(2) with $b = a_m$, $B_b = \mathbb{F}(a_1,\ldots,a_m)$ is contained in $Bs(|L_{d_1,d_2}|)$ with multiplicity $< \mu$ iff

$$d_1 + a_m d_2 + (\mu - 1)(a_n - a_m) \ge 0. \tag{3.16}$$

Proof. This is proved in Secton 2.8 of [Reid97]. The idea of proof lies in the observation that:

(1) for $X = \mathbb{V}(f) \in |L_{-b-1,1}|$ with $a_i \le b < (b+1)$, we must have that

$$f = \oplus_{i=1}^{m} \mathbf{Sym}^{a_i+b+1}(t_i)x_i$$

with $x_i = 0$ for all $i \ge m+1$.

(2) Inequality (3.16) says that there is a monomial of bidegree (d_1,d_2) whose degree in $x_{m+1},\ldots,x_n$ is less than μ. That is $X = \mathbb{V}(g) \in |L_{d_1,d_2}|$ is such that $\alpha(t_i)x_m^{d_1-\mu+1}x_n^{\mu-1}$ is a monomial of g precisely when

$$\deg(\alpha(t_i)) = d_1 + a_m d_2 + (\mu - 1)(a_n - a_m).$$

□

3.7 Hypersurfaces in the Surface Scroll F_a

Choose a character $(m,n) \in \mathbb{Z}^2$ of $G = \mathbb{C}^* \times \mathbb{C}^*$ with the corresponding line bundle $\mathscr{O}_{F_a}(mL+nM) \in \overline{NA}^1(F_a)$ ample on F_a. We shall, in Lemma (3.8.1) , discuss how to make this choice. Meanwhile, once such a choice is made, we study the curves

$$X := H \cap \varphi_{|L_{m,n}|}(F_a)$$

in the linear system $|mL+nM)|$ where

$$H \cong \mathbb{P}^{\left(m+1+\frac{1}{2}an\right)(n+1)1-2} \subset \mathbb{P}((S_{(m,n)}) \xleftarrow{\varphi_{|mL+nM|}} F_a$$

is a general hyperplane.
We start with an example of the linear system of a surface scroll F_2. We study a general $X \in |mL+nM|$ and the fibration $\pi|_X : X \to \mathbb{P}^1$.

$$\begin{array}{ccc} X \subset \mathbb{F}(2,0) & \xhookrightarrow{\varphi_{|L_{m,n}|}} & \mathbb{P}_{[y_i]}{}^{(m+n+1)(n+1)-1} := \mathbb{P}(S_{(m,n)}) \\ \downarrow \pi & & \\ \mathbb{P}^1_{[t_1:t_2]} & & \end{array}$$

where

$$S_{(m,n)} = H^0(F_2, nL+mM) = \left\langle \mathbf{Sym}^{m+2(n-d)}(t_1,t_2).x_1^{n-d}x_2^d : 0 \leq d \leq n \right\rangle.$$

Example 3.7.1. In this example $(m,n) = (1,1)$. From

$$F_2 := \mathbb{F}(2,0) \xrightarrow{\varphi_{11}} Y \subset \mathbb{P}^5_{[y_i]}$$

defined by $\varphi_{11}(t_1,t_2;x_1,x_2) = [t_1^3x_1 : t_1^2t_2x_1 : t_1t_2^2x_1 : t_2^3x_1 : t_1x_2 : t_2x_2]$, we get

$$X := H \cap Y = \mathbb{V}(c_1t_1^3x_1 + c_2t_1^2t_2x_1 + c_3t_1t_2^2x_1 + c_4t_2^3x_1 + c_5t_1x_2 + c_6t_2x_2).$$

That is

$$f := \pi|_X : X \to \mathbb{P}^1_{[t_1:t_2]}$$

with fibres X_t,

$$[b(t_i) : -a(t_i)] = pt \in X_t := f^{-1}(t) = \mathbb{V}(a(t_i)x_1 + b(t_ix_2) \cong \mathbb{P}^1_{[x_1:x_2]}.$$

So for general X, f is an isomorphism $X \cong \mathbb{P}^1$ by the Riemann-Hurwitz theorem for curves.

We now return to the general case $X := \mathbb{V}(f_n(x_1,x_2)) \in |mL+nM|$ on a surface scroll $F_a = \mathbb{F}(a,0)$ over $\mathbb{P}^1$. Here,

$$f_{m,n}(x_1,x_2) := f_n = \alpha_0 x_1^n + \alpha_1 x_1^{n-1} x_2 + \ldots + \alpha_n x_2^n$$

where $\alpha_i \in Sym^{m+a(n-i)}(t_1,t_2)$ have arbitrary coefficients. As in the example above, the fibres $X_t := (\pi|_X)^{-1}(t)$ are obtained by evaluating $f_{m,n}(x_j)$ at general points in the base $\mathbb{P}^1$. We note that X, chosen generally in the linear system, is nonsingular by Bertini's theorem. The Riemann-Hurwitz theorem applied to the n-to-1 map $\pi|_X : X \to \mathbb{P}^1$ gives

$$g(X) = 1 - n + \frac{1}{2}\delta(m,n) \geq 0$$

where $n = deg f_{m,n}$ on the fibres and the finite sum

$$\delta(f_n)(m) = \delta(m,n) := \sum_{P \in X} (e_P - 1)$$

where e_P is the multiplicity of $f_{m,n}$ at the point P.

Note that, $\delta(f_n)$ is a function of m since $\pi|_X$ is an $n-$cover of $\mathbb{P}^1$ branched over as many points in $\mathbb{P}^1$ as the number of zeros of the discriminant $\Delta(\widetilde{f_n})(x) \in \mathbb{C}[x]$. The dehomogenised $f_{m,n}$ is given by

$$\widetilde{f_n} = \widetilde{\alpha_0}(x) + \widetilde{\alpha_1}(x)y + \ldots + \widetilde{\alpha_n}(x)y^n$$

in the affine chart, say $\mathbb{C}^2_{<x,y>} \subset F_2$ where $x = \frac{t_2}{t_1}$ is the base coordinate and $y = \frac{t_1^{-a} x_2}{x_1}$ coordinate in $\mathbb{A}^1$ fibre of π. Further, the roots of $\widetilde{f_n}$ form a fibre of $\pi|_X$.

Example 3.7.2. Consider a curve $X = \mathbb{V}(f_2) \in |L_{m,2}|$ on a surface scroll F_a with

$$f_2 = g_{m+2a}(t_1,t_2)x_1^2 + g_{m+a}(t_1,t_2)x_1x_2 + g_m x_2^2.$$

Since ramification is a local property, it suffices to consider affine patches of X which, without loss of generality, are defined by a monic quadric $\widetilde{f_2}$ in y

$$X_{aff} := X \cap \{t_1 \neq 0, x_1 \neq 0\} = \mathbb{V}(f_2) \cap \mathbb{A}^2_{\left\langle x=\frac{t_2}{t_1}, y=\frac{t_1^{-a}x_2}{x_1}\right\rangle}$$

therefore

$$X_{aff} = \mathbb{V}(\widetilde{f_2}) = \mathbb{V}(\widetilde{g_{m+2a}}(x) + \widetilde{g_{m+a}}(x)y + y^2) \xrightarrow{\pi|_{X_{aff}}} \mathbb{A}^1_x$$

where $\pi|_{X_{aff}}$ is the composition

$$X_{aff} := \mathbb{V}(\widetilde{f_2}) \to \mathbb{A}^2_{<x,y>} \to \mathbb{A}^1_{<x>}.$$

This map corresponds to the $\mathbb{C}-$algebra map $(\pi|_{X_{aff}})^* : \mathbb{C}[s] \xrightarrow{s \mapsto x} \mathbb{C}[x,y]/(\widetilde{f_2})$ which locally at a point $p = (x_0, y_0) \in X_{aff}$ is

$$\mathbb{C}[s]_{(s-x_0)} \xrightarrow{(s-x_0)\mapsto(x-x_0)} \mathbb{C}[x,y]/(\widetilde{f_2})_{\mathfrak{m}_p}$$

where $\mathfrak{m}_P = (x-x_0, y-y_0)$. We also note that

$$e_{(x_0,y_0)} = mult_{(x_0,y_0)}\pi := Max_{k\in\mathbb{N}}\{k : (y-y_0)^k | \widetilde{f_2}(x_0,y)\}.$$

Since the discriminant $\Delta(\widetilde{f_2}) \in \mathbb{C}[x]$ vanishes at repeated zeros of $\widetilde{f_2}$, every point in $\mathbb{V}(\Delta(\widetilde{f_2}))$ adds one to $\delta(f_2)$. By the Fundamental Theorem of Algebra and the fact that ramification indices sum to the degree $n = 2$, we have that

$$\begin{aligned} \delta(f_n)(m) : &= \sum_{P\in X}(e_P - 1) \\ &= \text{ number of roots of } \Delta(\widetilde{f_2}) \\ &= \deg \Delta(\widetilde{f_2}) \end{aligned}$$

hence

$$\begin{aligned} g(X) &= 1 - 2 + \frac{1}{2}(\deg \Delta(\widetilde{f_2})) \\ &= -1 + \frac{1}{2}(2m + 2a) \\ &= a + m - 1. \end{aligned}$$

For instance, a general $X \in |L_{m,2}|$ in a surface scroll F_2 is of genus $g(X) = m+1$.

The general case is more easily approached using intersection numbers. Take $K_{F_a} = L_{-2+a,-2}$ and $L_{m,n}$ in the ample cone $\overline{NA}^1(F_a)$ and a general hyperplane H

$$H \cong \mathbb{P}^{\left(m+1+\frac{1}{2}an\right)(n+1)-2} \subset \mathbb{P}((S_{(m,n)}) \xleftarrow{\varphi_{|mL+nM|}} F_a.$$

We have that the curve $X := H \cap \varphi(F_a) \in |L_{m,n}|$ of degree

$$\deg X = \#H \cap X = \text{Hyperplane class. } L_{m,n}$$

is such that

$$\begin{aligned} 2g(X) - 2 = deg(K_X) = (K_{F_a} + X).X &= ((-2+a)L - 2M + mL + nM).(mL + nM) \\ &= ((m+a-2)L + (n-2)M)(mL+nM) \\ &= n(m+a-2) + m(n-2) + na(n-2). \end{aligned}$$

Hence ,

$$g(X) = \frac{1}{2}na(n-1) + (m-1)(n-1) = \left(m + \frac{1}{2}an - 1\right)(n-1)$$

or

$$g(X) = a\binom{n}{2} + (m-1)(n-1).$$

For $a = 2$, the genus $g(X) = (m+n-1)(n-1)$. Now, for a general hyperplane H, all curves $X := H \cap \operatorname{Im} \varphi_{|L_{m,1}|}$ are projective lines, $Y := H \cap \operatorname{Im} \varphi_{|L_{m,2}|}$ are curves of genus $g(X) = m+1$. These were the cases discussed in Examples (3.7.1) and (3.7.2) respectively.

3.8 Cones of Surface Scrolls

We begin with a lemma characterizing the effective cone $\overline{NE}(F_a)^{\vee}$ and the ample cone $NA(F_a)$ of a surface scroll $F_a = \mathbb{F}(a,0)$.

Lemma 3.8.1. *The divisor* $L_{d_1,d_2} \in \operatorname{Pic}(F_a) = \mathbb{Z}[L,M]$ *is*

- *effective if* $d_2 \geq 0$ *and* $d_1 + ad_2 \geq 0$ *and*
- *ample if and only if* $d_1, d_2 \geq 1$.

`Proof.` By definition, L_{d_1,d_2} is effective if $|L_{d_1,d_2}| \neq \emptyset$. Hence, the sections are such that

$$H^0(F_a, L_{d_1,d_2}) := \left\langle \mathbf{Sym}^{d_1+aq_1}(t_1,t_2)x_1^{q_1}x_2^{d_2-q_1} \middle| d_1 + aq_1 \geq 0 \right\rangle$$

with nonnegative powers of x_j's making d_2 to be at least zero. Applying the extreme partition $(q_1, d_2 - q_1) = (d_2, 0)$ of d_2 to $d_1 + aq_1 \geq 0$ gives the second condition of effectiveness. For ampleness of $L_{d_1,d_2} = d_1L + d_2M$ on

$$\begin{aligned} \mathbb{F}(a,0) &= U_{11} \cup U_{12} \cup U_{21} \cup U_{22} \\ &\cong \mathbb{C}^2_{\left\langle X_{11}=\frac{t_2}{t_1}, Y_{11}=\frac{t_1^{-a}x_2}{x_1} \right\rangle} \cup \mathbb{C}^2_{\left\langle X_{12}=\frac{t_2}{t_1}, Y_{12}=\frac{t_1^{a}x_1}{x_2} \right\rangle} \cup \mathbb{C}^2_{\left\langle X_{21}=\frac{t_1}{t_2}, Y_{21}=\frac{t_2^{-a}x_2}{x_1} \right\rangle} \cup \mathbb{C}^2_{\left\langle X_{22}=\frac{t_1}{t_2}, Y_{22}=\frac{t_2^{a}x_1}{x_2} \right\rangle}, \end{aligned} \tag{3.17}$$

we demonstrate that the restriction maps $\phi_{ijk} : U_{ij} \to V_k$ of

$$\phi_{|L_{d_1,d_2}|} : \mathbb{F}(a,0) \to \mathbb{P}\left(\left\langle \mathbf{Sym}^{d_1+aq_1}(t_1,t_2)x_1^{q_1}x_2^{d_2-q_1} \middle| d_1 + aq_1 \geq 0 \right\rangle\right) = \mathbb{P}^N_{z_k}$$

to affine covers of the respective spaces are isomorphisms (defined by polynomial maps in affine coordinates).

It suffices, in the diagram below, to show that ϕ_{111} is an isomorphism. This follows from the fact that

$$\widetilde{\phi_{111}} := g_1 \circ \phi_{111} \circ f_{11}^{-1}$$

is an isomorphism

$$\begin{array}{ccc}
U_{11} & \xrightarrow{\quad \phi_{111} \quad} & V_1 \\
\Big\downarrow f_{11} & & \Big\downarrow g_1 \\
\mathbb{C}^2_{\left\langle X_{11}=\frac{t_2}{t_1}, Y_{11}=\frac{t_1^{-a}x_2}{x_1} \right\rangle} & \xrightarrow{\quad \widetilde{\phi_{111}} \quad} & \mathbb{C}^N_{\left\langle \widehat{\frac{z_1}{z_1}}, \frac{z_2}{z_1}, \ldots, \frac{z_{N+1}}{z_1} \right\rangle}
\end{array}$$

defined by

$$\begin{aligned}
\widetilde{\phi_{111}}(X_{11},Y_{11}) &= (h_1,\ldots,h_N) \\
&= \left(X_{11}, X_{11}^2, \ldots, X_{11}^{d_1}, Y_{11}^{-1}, X_{11}Y_{11}^{-1}, \ldots, X_{11}^{a-1}Y_{11}^{-1}, \ldots, Y_{11}^{-d_2}, X_{11}Y_{11}^{-d_2}, \ldots, X_{11}^{d_2}Y_{11}^{-d_2}\right).
\end{aligned} \tag{3.18}$$

This shows that the line bundle d_1L+d_2M is ample if all bidegrees $deg(h_t) > 0$ in X_{11}, Y_{11}. This imply that $a \geq 0, d_1, d_2 > 0$.
Conversely, suppose $d_2 \leq 0$ and $d_1 \in [-ad_2, -1] \subset \mathbb{R}$, a line bundle L_{d_1,d_2} is such that $L.L_{d_1,d_2} = d_2 \leq 0$, $M.L_{d_1,d_2} = d_1 + ad_2 \leq 0$. Hence by (a) of Theorem (2.3.6) , we have that L_{d_1,d_2} is not ample. □

The result above is a version of the Linear Embedding Theorem; L and M are not ample as L contracts to $\mathbb{P}^1$ and M contracts a rational curve.
Similar arguments work for other divisors of scrolls of higher dimension.

3.9 Hilbert Series of Scrolls

We start with a direct computation of the Hilbert series of the bigraded ring corresponding to the surface scroll F_a.

Proposition 3.9.1. *Let $\mathbb{F}_A$ be an $n+k-2$-fold scroll with*

$$A := wt(t_1,\ldots,t_k,x_1,\ldots,x_n) = \begin{bmatrix} c_1 & \ldots & c_k & -a_1 & \ldots & -a_n \\ 0 & \ldots & 0 & b_1 & \ldots & b_n \end{bmatrix}.$$

The Hilbert series associated to the bigraded ring $S := \mathbb{C}[\mathbb{F}_A] = \mathbb{C}[t_i, x_j]$ is given by

$$P_{\mathbb{F}}(s_1,s_2) = \sum_{(m,n)\in L} h^0(\mathbb{F}, L_{m,n}) s_1^m s_2^n = \prod_{i=1}^{k} \frac{1}{(1-s_1^{c_i})} \prod_{j=1}^{n} \frac{1}{(1-s_1^{-a_j} s_2^{b_j})}.$$

Proof. Consider the bigraded ring

$$R = \mathbb{C}[t_1, \ldots, t_k, x_1, \ldots, x_n] = \mathbb{C}[t_i] \otimes \mathbb{C}[x_j].$$

We then have that

$$P_R(s_1, s_2) = P_{\mathbb{C}[t_i]}(s_1, s_2) \times P_{\mathbb{C}[x_j]}(s_1, s_2).$$

□

Theorem 3.9.2. *Suppose the character $\chi = (m,n) \in \Omega = \mathrm{Hom}(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}^2$ is such that L_χ is in the ample cone $NA(F_a)$ of a surface scroll F_a. We have, for $a = 2$, that $h^0(F_2, L_{m,n})$ is give by*

$$\begin{cases} \sum_{q_1=k}^{n}(m+2q_1+1) = \frac{1}{2}(m+2n+1)(m+2n+2) , & m = -2k < 0, k \in \mathbb{N} \\ \sum_{q_1=k}^{n}(m+2q_1+1) = \frac{1}{2}(m+2n+1)(m+2n+2) - 1 , & m = -2k+1 < 0, k \in \mathbb{N} \\ \sum_{q_1=0}^{n}(m+2q_1+1) = (m+n+1)(n+1) , & m \geq 0. \end{cases}$$

Proof. This proof finds the closed formula for $h^0(F_2, L_{m,n})$ and the corresponding Hilbert series $P_{F_2}(s_1, s_2)$. Note that $t_1, t_2 \in S_{(1,0)}, x_1 \in S_{(-a,1)}$ and $x_2 \in S_{(0,1)}$ or equivalently, the line bundle $L_{m,n} \in NE^1(F_a)$ corresponds to (m,n) in the sub-lattice $L = \langle (1,0), (-a,1) \rangle$ of Ω satisfying, from lemma (3.8.1) ,

$$m + aq_1 \geq 0 \text{ and } n \geq q_1 \geq \max\left\{ \left\lceil \frac{-m}{a} \right\rceil, 0 \right\}.$$

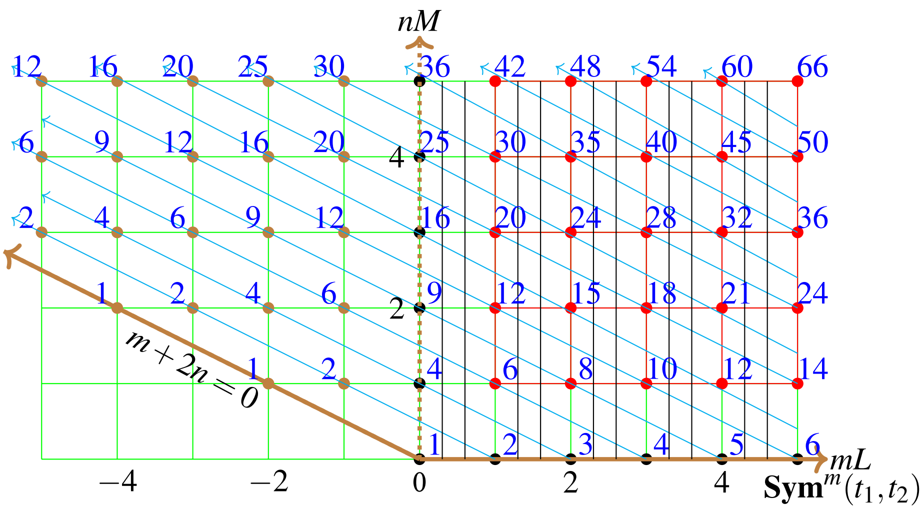


Figure 3.6: The effective cone $\overline{NE}(F_2)^\vee \supset$ **ample cone** $NA(F_2) \supset$ very ample cone in $N^1(F_2)_\mathbb{R}$. The dimensions of the spaces $H^0(F_2, L_{m,n})$ of section are shown in blue.

We then have, from equation (3.14) that

$$H^0(F_a,L_{m,n}) := \left\langle \mathbf{Sym}^{m+aq_1}(t_1,t_2)x_1^{q_1}x_2^{n-q_1} : n \geq q_1 \geq \max\left\{\left\lceil \frac{-m}{a}\right\rceil, 0\right\}\right\rangle = S^a_{(m,n)}$$

Hence for $a=2$, we get that $h^0(F_2,L_{m,n})$ is given by

$$\begin{cases} \sum_{q_1=k}^{n}(m+2q_1+1) = \frac{1}{2}(m+2n+1)(m+2n+2)\ , & m=-2k<0, k\in\mathbb{N} \\ \sum_{q_1=k}^{n}(m+2q_1+1) = \frac{1}{2}(m+2n+1)(m+2n+2)-1\ , & m=-2k+1<0, k\in\mathbb{N} \\ \sum_{q_1=0}^{n}(m+2q_1+1) = (m+n+1)(n+1)\ , & m\geq 0. \end{cases}$$

Indeed,

$$\begin{aligned} P_{F_2}(s_1,s_2) = \sum_{(m,n)\in L} h^0(F_2,L_{m,n})s_1^m s_2^n = \sum_{m=-2n}^{-1}\sum_{n=1}^{\infty}\frac{1}{2}(m+2n+1)(m+2n+2)s_1^m s_2^n + \\ \sum_{m=\lceil\frac{-2n+1}{2}\rceil}^{-1}\sum_{n=1}^{\infty}\left(\frac{1}{2}(m+2n+1)(m+2n+2)-1\right)s_1^m s_2^n + \\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+n+1)(n+1)s_1^m s_2^n \\ = \frac{1}{(1-s_1)^2(1-s_1^{-2}s_2)(1-s_2)}. \end{aligned}$$

□

Remark 3.9.3. *With $K=K_{F_a}=L_{-2+a,-2}$, the results above are consistent with the RR theorem for F_a*

$$\begin{aligned} h^0(F_a,\mathcal{O}_{F_a}(mL+nM)) &= 1+\frac{1}{2}((mL+nM)^2-(mL+nM).K) \\ &= \frac{1}{2}(2m+an+2)(n+1),\ \textit{for } m\geq 0. \end{aligned}$$

Define $c(m)=\max\left\{\left\lceil\frac{-m}{a}\right\rceil,0\right\}$ whenever $m<0$. In this case, a correction term is involved

$$\begin{aligned} &h^0(F_a,\mathcal{O}_{F_a}(mL+nM)) = \#\{(e_1,e_2,q_1,q_2)|q_1+q_2=n\geq 0, e_1+e_2-aq_1=m\} \\ &= \sum_{q_1=0}^{n}\#\{(e_1,e_2)\in\mathbb{Z}^2_{\geq 0}|e_1+e_2=m+aq_1\} \\ &= \sum_{q_1=0}^{n}\begin{cases} 0 & \textit{if } m+aq_1<0 \\ m+aq_1+1 & \textit{if } m+aq_1\geq 0 \end{cases} \\ &= \sum_{q_1=c(m)=\max\left\{\left\lceil\frac{-m}{a}\right\rceil,0\right\}}^{n}(m+aq_1+1) = (m+1)(n-c(m)+1)+a\sum_{q_1=c(m)}^{n}q_1 \\ &= \frac{1}{2}(2m+an+2)(n+1)-\frac{1}{2}(2m+a(c(m)-1)+2)c(m). \end{aligned}$$

3.10 Wellformedness of hypersurfaces in Weighted Scrolls

A codimension m variety $X \subset \mathbb{P}[b_1,\dots,b_n] = \mathbb{P}$ is well-formed, if the codimension of $X \cap Sing(\mathbb{P}) \subset X$ denoted by $\mathrm{Codim}_X(X \cap Sing(\mathbb{P}))$, is at least $m+1$. For a hypersurface, $\mathrm{Codim}_X(X \cap Sing(\mathbb{P})) \geq 2$ is the condition for X to be wellformed.

The following Lemma characterizes wellformedness of subvarieties of $\mathbb{P} = \mathbb{P}[b_1,\dots,b_n]$.

Lemma 3.10.1. *[6.10, 6.11 [Fle00]]*

1. *$\mathbb{P} = \mathbb{P}[b_1,\dots,b_n]$ is wellformed if any $n-1$ of the weights are coprime. Otherwise, if $h = hcf(b_1,\dots,\widehat{b_j},\dots,b_n) \neq 1$ for some j, then we have an isomorphism*

$$\mathbb{P}[b_1,\dots,b_n] \cong \mathbb{P}[\beta_1,\dots,\beta_{j-1},b_j,\beta_{j+1},\dots,\beta_n],\ \beta_k = \frac{b_k}{h}, k \neq j.$$

   This can be repeated until we get a wellformed weighted projective space.

2. *A hypersurface $X_d \in \mathbb{P}[b_1,\dots,b_n]$ is well-formed if any $n-1$ of the weights are coprime and d is a multiple of the hcf of any $n-2$ of the weights.*

3. *A codimension two subvariety $X_{d_1,d_2} \subset \mathbb{P}$ is wellformed if any $n-1$ of the weights are coprime, both d_1 and d_2 are multiples of the hcf of any $n-2$ of the weights and either of d_1 or d_2 is a multiple of the hcf of any $n-3$ of the weights.*

We say that a $\mathbb{P}^{n-1}[b_j]$-bundle over $\mathbb{P}^{k-1}[c_i]$ (that is, the weighted scroll $\mathbb{F}_A = \mathbb{F}(a_j|b_j)$ over $\mathbb{P}^{k-1}[c_i]$ is well-formed) if both the fibre and the base are well-formed. In this thesis, all weighted scrolls are well-formed from construction.

Definition 3.10.2. [Wellformedness in Scrolls] A codimension m variety $X \subset \mathbb{F}_A = \mathbb{F}(a_j|b_j)$ is well-formed, if $\mathrm{Codim}_X(X \cap Sing(\mathbb{F}_A)) \geq m+1$.

Proposition 3.10.3. *An anticanonical Gorenstein wellformed threefold hypersurface $X \subset \mathbb{F}_A$ with singularities $Z = Sing\mathbb{F}_A \cap X$ is a Calabi–Yau threefold.*

`Proof.` Since $X_0 = X \setminus Z$ is smooth and X is Gorenstein [=sheaf of 3-forms Ω^3_X is well defined on Z], we have that

$$\omega_X|_{X_0} = \Omega^3_{X_0}$$

where ω_X is the canonical bundle on X. Now, consider the codimension two (≥ 2) embedding $i : X_0 \hookrightarrow X$. That X is Gorenstein [in a neighbourhood of $d_i \in Z$ we may choose a generating global section $\eta \in H^0(\Omega^3_{X_0})$ that ensures that ω_X is rank one] implies that ω_X is locally free so that, by adjunction, we have the canonical isomorphism

$$\omega_X \xrightarrow{\cong} i_* i^* \omega_X = i_*(\omega_X|_{X_0}) = i_* \Omega^3_{X_0}.$$

Hence $i_* \mathscr{O}_{X_0} \cong \mathscr{O}_X$. □

3.11 Embedded Deformations of Hypersurfaces in Scrolls

The concept of embedded deformations becomes clearer if one starts with nonsingular genus $g = g(C) = \binom{d-1}{2}$ plane curves $C_d \subset \mathbb{P}^2$ of degree d. Consider the open subspace

$$U \subset \mathbb{P}^{(\binom{d+2}{2}-1)}(H^0(\mathscr{O}_{\mathbb{P}^2}(d)))$$

of nonsingular curves in $\mathbb{P}^2$. We declare (as we should) that two nonsingular plane curves are the same if they differ by automorphisms of $\mathbb{P}^2$.

We then have the moduli space

$$\mathscr{M}_{C_d/\mathbb{P}^2} = U/PGL(3) \tag{3.19}$$

of degree d nonsingular curves in $\mathbb{P}^2$.

As one would expect, we have that

$$\begin{aligned} \dim(\mathscr{M}_{C_d/\mathbb{P}^2}) &= \dim(U) - \dim(PGL(3)) \\ &= \left(\binom{d+2}{2} - 1\right) - (3^2 - 1) \\ &= 3\binom{d-1}{2} - \left(2\binom{d-1}{2} - 3(d-3)\right) \\ &\leq 3\binom{d-1}{2} - 3 = \dim(\mathscr{M}_{C_d}) = \dim(\mathscr{M}_g); \end{aligned}$$

the embedded deformations are asymptotically much fewer than the abstract (total) deformations of the degree d curves for $d > 4$. These embedded deformations are more accessible.

The rest of the deformations can be found by considering the canonical embeddings

$$C_d \hookrightarrow \mathbb{P}^{(\binom{d-1}{2}-1)}(H^0(\omega_{C_d})^\vee)$$

over all distinct $[C_d] = \{hC_d : h \in PGL(3)\}$.

Now, let $X_d \subset \mathbb{P}[b_0, \ldots, b_n] = \mathbb{P}$ be a quasismooth, well-formed degree d hypersurface and $U \subset \mathbb{P}(H^0(\mathscr{O}_{\mathbb{P}}(d)))$ an open set of such hypersurfaces. We then have the moduli space

$$\mathscr{M}_{X_d/\mathbb{P}} = U/\operatorname{Aut}(\mathbb{P}[b_j]). \tag{3.20}$$

the construction of the Non-Reductive GIT quotient in Equation (3.20) is the central question in [Bun19].

The following definition of embedded deformation in scrolls makes no assumptions of quasismooth and is therefore both relative and general.

Definition 3.11.1. Let $\mathbb{F}_A = \mathbb{F}(a_j|b_j)$ be a $\mathbb{P}^{n-1}[b_j]$-bundle over $\mathbb{P}^{k-1}[c_i]$. Assume that $L_{d_1,d_2} \in Cl(\mathbb{F}_A)$ is such that a general well-formed hypersurface $X \in |L_{d_1,d_2}|$ has codimension ≥ 2 singularities along the base locus of $|L_{d_1,d_2}|$ and $\mathbb{P}(S_{d_1,d_2})$ is the space of all hypersurfaces $X \in |L_{d_1,d_2}|$. Consider the open subspace

$U := \{\text{hypersurfaces } X \text{ with codimension} \geq 2 \text{ singularities along } Bs(|L_{d_1,d_2}|)\} \subset \mathbb{P}(S_{d_1,d_2})$.

Now, define by

$$\mathscr{M}_{L_{d_1,d_2}} := U/\operatorname{Aut}(\mathbb{F}_A) \tag{3.21}$$

the moduli space of sections of $|L_{d_1,d_1}|$ with codimension ≥ 2 singularities along the base locus $Bs(|L_{d_1,d_1}|)$.

The dimension of the moduli space $\mathscr{M}_{L_{d_1,d_2}}$ is

$$\dim(\mathscr{M}_{L_{d_1,d_2}}) = \dim(\mathbb{P}(S_{d_1,d_2})) - \dim(\operatorname{Aut}(\mathbb{F}_A)).$$

It is only sensible, therefore, to define a classifying map as follows.

$$\begin{array}{ccccc} X & \stackrel{i}{\hookrightarrow} & \mathbb{F}_A & \stackrel{\varphi_{|L_{d_1,d_2}|}}{\dashrightarrow} & \mathbb{P}(S_{(d_1,d_2)}) \\ \downarrow f & & \downarrow \pi & & \\ B & \stackrel{j}{\hookrightarrow} & \mathbb{P}^{k-1}[c_i]. & & \end{array}$$

Call X_0 and B_0, the smooth loci of the total space X and base B of the fibration f. For $b \in B_0 \subset B \subset \mathbb{P}^{k-1}[c_i]$, we have that $f^{-1}(b) = X_{d_2} \subset \mathbb{P}^{n-1}[b_j] = \pi^{-1}(j(b))$ which is identified with the point

$$b \mapsto [f^{-1}(b)] \in \mathscr{M}_{X_{d_2}/\mathbb{P}^{n-1}[b_j]}$$

defining a "classifying map"

$$\sigma : B_0 \to \mathscr{M}_{X_{d_2}/\mathbb{P}^{n-1}[b_j]}.$$

This hints at how rich the study of projective fibrations in weighted scrolls is.

Chapter 4

Elliptic K3 Hypersurfaces in Weighted Scrolls

We recall, from Table (2.2), that there are five projective models constructed from their graded rings. In this section, we would like to construct $K3$ surfaces X with at worst perhaps rational double point singularities and which are fibred over $\mathbb{P}^1$ by the first three of the five projective models. We do this by classifying ambient scrolls $\mathbb{F}$ such that the fibration $\varphi : X \to \mathbb{P}^1$ is embedded in the scroll map $\pi : \mathbb{F} \to \mathbb{P}^1$.

4.1 Elliptic fibrations with plane cubic fibres

Theorem 4.1.1. *Let $\mathbb{F} = \mathbb{F}(0, a_2, a_3)$ be a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$ with elliptic cubic fibres $E_3 \subset \mathbb{P}^2$. There are* 12 *families of mildly singular elliptically fibred K3 surfaces embedded in* 3*-fold straight scrolls $\mathbb{F}$. The Table* (4.1) *below gives reasons why each family is mildly singular; that is, either nonsingular or has isolated singularities.*

`Proof.` In this case $\mathbb{F} = \mathbb{F}(a_1, a_2, a_3)$ is a 3-fold scroll over $\mathbb{P}^1$ with integral ordered weights

$$a_1 \leq a_2 \leq a_3.$$

We will assume $a_1 = 0$ using the standard isomorphism

$$\mathbb{F}(a_1, a_2, a_3) \cong \mathbb{F}(a_1 + k, a_2 + k, a_3 + k)$$

where $k \in \mathbb{Z}$. The anticanonical divisor class of $\mathbb{F}(0, a_2, a_3)$ is given by

$$-K_{\mathbb{F}(0,a_2,a_3)} = (2 - a_2 - a_3)L + 3M = L_{2-a_2-a_3,3}$$

Assume that $|L_{2-a_2-a_3,3}| \neq \emptyset$ and take a general surface

$$X = \mathbb{V}(f_3) = \mathbb{V}\left(\sum_{(q_1,q_2,q_3)\vdash 3} \alpha_{(q_j)}(t_1, t_2) x_1^{q_1} x_2^{q_2} x_3^{q_3}\right) \in |L_{2-a_2-a_3,3}|$$

No.	$\mathbb{F} = \mathbb{F}(0,a_2,a_3)$	General $X \in \lvert -K_{\mathbb{F}} \rvert$
1	$\mathbb{F}(0,0,0)$	$-K_{\mathbb{F}}$ is very ample, so a general X is nonsingular $K3$ surface
2	$\mathbb{F}(0,0,1)$	$-K_{\mathbb{F}}$ is very ample, so a general X is nonsingular $K3$ surface
3	$\mathbb{F}(0,0,2)$	$-K_{\mathbb{F}}$ is base-point-free but not very ample. General X is nonsingular $K3$ surface.
4	$\mathbb{F}(0,1,1)$	$-K_{\mathbb{F}}$ is base-point-free but not very ample. General X is nonsingular $K3$ surface.
5	$\mathbb{F}(0,1,2)$	$-K_F$ has a base point but a general X is still nonsingular $K3$ surface
6	$\mathbb{F}(0,2,2)$	$-K_F$ has a base point but a general X is still nonsingular $K3$ surface
7	$\mathbb{F}(0,2,3)$	$-K_F$ has a base point but a general X is still nonsingular $K3$ surface
8	$\mathbb{F}(0,2,4)$	$-K_F$ has a base point but a general X is still nonsingular $K3$ surface
9	$\mathbb{F}(0,2,5)$	$-K_F$ has a base point but a general X is still nonsingular $K3$ surface
10	$\mathbb{F}(0,2,6)$	$-K_F$ has a base point but a general X is still nonsingular $K3$ surface
11	$\mathbb{F}(0,1,3)$	General $X \in \lvert -K_{\mathbb{F}(0,1,3)} \rvert$ has a $\frac{1}{2}(1,1)$ singularity
12	$\mathbb{F}(0,1,4)$	General $X \in \lvert -K_{\mathbb{F}(0,1,4)} \rvert$ has a $\frac{1}{3}(1,2)$ singularity

Table 4.1: $\mathbb{F} = \mathbb{F}(0,a_2,a_3)$ for which a general $X \in |-K_{\mathbb{F}}|$ is smooth genus-1-fibred K3 surface or can be resolved to give such a surface.

where $\alpha_{(q_j)}(t_1,t_2) \in \mathbb{C}[\mathbb{P}_{t_i}]$ is a polynomial in t_1,t_2 of degree

$$\deg \alpha_{(q_j)}(t_i) = 2 + (q_2-1)a_2 + (q_3-1)a_3,$$

the coefficient of $x_1^{q_1}x_2^{q_2}x_3^{q_3}$. The surface X fits in the diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{F}(0,a_2,a_3) \\ & \overset{\varphi}{\searrow} & \downarrow \pi \\ & & \mathbb{P}^1_{[t_i]}. \end{array}$$

The fibration π induces the fibration

$$\varphi : X \to \mathbb{P}^1$$

whose fibres are cubic curves; elliptic if nonsingular

$$E_3 = \mathbb{V}(f(t,x_j)) \subset \mathbb{P}^2_{[x_j]},\ t \in \mathbb{P}^1_{[t_i]}$$

with $f(t_i,x_j)$ being the sum of monomials $\alpha_{(q_1,q_2,q_3)}(t_i)x_1^{q_1}x_2^{q_2}x_3^{q_3}$ for fixed $t = [t_1 : t_2]$ over partitions (q_1,q_2,q_3) of 3. We can represent these partitions in a triangle giving a Newton triangle of monomials spanning $\mathbb{C}[\mathbb{P}^2_{[x_j]}]_3$

			(3,0,0)			
		(2,1,0)		(2,0,1)		
	(1,2,0)		(1,1,1)		(1,0,2)	
(0,3,0)		(0,2,1)		(0,1,2)		(0,0,3)

Table 4.2: Unweighted partitions of 3

			x_1^3			
		$x_1^2x_2$		$x_1^2x_3$		
	$x_1x_2^2$		$x_1x_2x_3$		$x_1x_3^2$	
x_2^3		$x_2^2x_3$		$x_2x_3^2$		x_3^3.

Table 4.3: Newton triangle basis of $\mathbb{C}[\mathbb{P}^2_{[x_j]}]_3$

The Newton triangle of corresponding degrees $2+(q_2-1)a_2+(q_3-1)a_3$ of coefficients of the monomials $x_1^{q_1}x_2^{q_2}x_3^{q_3}$ is
These degrees increase from left to right along horizontal rows of the triangle by a_3-a_2, down the left side of the triangle by a_2 and down the right side of the triangle by a_3.

If X is nonsingular, then by adjunction it is a K3 surface, and the map $\varphi : X \to \mathbb{P}^1$ is a fibration by plane cubic curves $E_3 \subset \mathbb{P}^2$.

If $L_{2-a_2-a_3,3}$ being very ample on the scroll $\mathbb{F}(0,a_2,a_3)$, then X a is nonsingular surface by the classical form of Bertini's theorem. The divisor class of $\mathbb{F}(0,a_2,a_3)$ is very ample if the extremal vector spaces

$$\mathbf{Sym}^{2-a_2-a_3+3a_3}(t_1,t_2)x_3^3, \mathbf{Sym}^{2-a_2-a_3+3a_2}(t_1,t_2)x_2^3 \text{ and } \mathbf{Sym}^{2-a_2-a_3}(t_1,t_2)x_1^3$$

are of dimension greater than 1. In particular, $-K_{\mathbb{F}(0,a_2,a_3)} = L_{2-a_2-a_3,3}$ is very ample on $\mathbb{F}(0,a_2,a_3)$ is equivalent to the vertices of the Newton triangle of degrees of coefficients of $\prod_{(q_1,q_2,q_3)\vdash 3} x_j^{q_j}$ being positive

$$2-a_2+2a_3>0,\ 2+2a_2-a_3>0,\ 2-a_2-a_3>0. \tag{4.1}$$

The Inequality (4.1) together with the asssumption that $a_3 \geq a_2 \geq 0$ gives the three-fold scrolls

$$\mathbb{F}(0,a_2,a_3) = \mathbb{F}(0,0,0) \text{ and } \mathbb{F}(0,0,1)$$

in which the respective K3 surface families $X \in |-K_{\mathbb{F}(0,a_2,a_3)}|$ are nonsingular.

Nonsingularity of X also holds if $L_{2-a_2-a_3,3}$ is base points free but not very ample on

$$
\begin{array}{ccccccc}
 & & & 2-a_2-a_3 & & & \\
 & & 2-a_3 & & 2-a_2 & & \\
 & 2+a_2-a_3 & & 2 & & 2-a_2+a_3 & \\
2+2a_2-a_3 & & 2+a_2 & & 2+a_3 & & 2-a_2+2a_3.
\end{array}
$$

Table 4.4: The degrees $2+(q_2-1)a_2+(q_3-1)a_3$ of the coefficients of $x_1^{q_1}x_2^{q_2}x_3^{q_3}$

$\mathbb{F}(0,a_2,a_3)$ which is equivalent to the Inequalities (4.1) being non strict

$$2-a_2+2a_3 \geq 0,\ 2+2a_2-a_3 \geq 0,\ 2-a_2-a_3 \geq 0. \tag{4.2}$$

In this case we get the further 3-fold scrolls

$$\mathbb{F}(0,a_2,a_3) = \mathbb{F}(0,0,2) \text{ and } \mathbb{F}(0,1,1)$$

in which the nonsingular elliptic-fibred K3 surfaces X are embedded.

We would like to find a weaker condition than very ampleness or base point freeness of $L_{2-a_2-a_3,3}$ that still implies nonsingularity of X. Our argument will be as follows: assuming the base locus $|L_{2-a_2-a_3,3}|$ is nonempty, we find explicit sections of the linear system $|L_{2-a_2-a_3,3}|$ that are nonsingular at the base locus. We want X to be nonsingular in a neighbourhood of the base locus of $|L_{2-a_2-a_3,3}|$.

The base locus $Bs(|L_{2-a_2-a_3,3}|)$ is given by setting some of the variables t_1,t_2,x_1,x_2,x_3 to zero with $0 \leq \dim Bs(|L_{2-a_2-a_3,3}|) \leq 2$. This is a direct consequence of Proposition (3.6.1).

We start with $\dim(Bs(|L_{2-a_2-a_3,3}|)) = 2$. For a generic choice of the coefficients

$$\alpha_{(d_1,d_2,d_3)\vdash 3}(t_i) \in \mathbf{Sym}^{2+a_2(d_2-1)+a_3(d_3-1)}(t_1,t_2) \text{ with } d_1+d_2+d_3=3, d_i \geq 0;$$

we have that $t_i \not| f(t_i,x_j)$. In this dimension, the worst case would give a singular

$$X = \mathbb{V}(f_3) = X' \cup Bs(|L_{2-a_2-a_3,3}|)$$

where $Bs(|L_{2-a_2-a_3,3}|)$ is the surface scroll $\{x_i = 0\} \subset \mathbb{F}(0,a_2,a_3)$. This would happen when $f_3 = x_i q$ is reducible with $q(t_i,x_j)$ a quadratic in x_j whereas X' is the quadric surface $X' = \mathbb{V}(q) \subset \mathbb{F}(0,a_2,a_3)$; we want to exclude this case, so for nonsingularity of X we must have $x_3 \not| f(t_i,x_j)$, $x_2 \not| f(t_i,x_j)$ and $x_1 \not| f(t_i,x_j)$. The condition for this is that at least one of the monomials x_1^3, $x_1^2x_2$, $x_1x_2^2$ and x_2^3 on the left edge of the Newton triangle must occur in $f(t_i,x_j)$. Therefore, it is enough to see a nonzero $\alpha_{(030)}(t_i)x_2^3$ in $f(t_i,x_j)$ which gives us the condition that the corresponding degree of coefficients

$$2+2a_2-a_3 \geq 0 \tag{4.3}$$

should be nonnegative. Equivalently, as a consequence of Lemma (3.6.2), we have that nonsingularity of X is implied by $B_{a_2} = \mathbb{F}(0,a_2), \mu B_{a_1} = 2\mathbb{P}^1 \not\subset X$; that is $\alpha_{(030)}(t_i)x_2^3, \alpha_{(201)}(t_i)x_1^2x_3$ are respectively in $f(t_i,x_j)$.

Suppose $\dim(Bs(|L_{2-a_2-a_3,3}|)) = 1$. The inequality (4.3) implies that the equation of a general section of $L_{2-a_2-a_3,3}$ has both a nonzero term involving x_2^3 term and a nonzero term involving x_3^3 term hence

$$f(t_i,x_j)|_{\{x_1=x_2=0\}} \not\equiv 0 \text{ and } f(t_i,x_j)|_{\{x_1=x_3=0\}} \not\equiv 0.$$

Therefore, the curve $\{x_2 = x_3 = 0\} = C_{2,3} = Bs(|L_{2-a_2-a_3,3}|)$ is the only base locus of $|L_{2-a_2-a_3,3}|$ which corresponds to there being no nonzero x_1^3 term in $f(t_i,x_j)$. Then we need to investigate what happens along $C_{2,3}$. In this case, we have in an affine chart $U_{11} = \{t_1 = x_1 = 1\} \subset \mathbb{F}(0,a_2,a_3)$ that

$$\begin{aligned} Sing(\widetilde{X}) \cap \widetilde{C_{2,3}} \cap \widetilde{U_{11}} &= \mathbb{V}\left(f(t_i,x_j), \frac{\partial f(t_i,x_j)}{\partial t_2}, \frac{\partial f(t_i,x_j)}{\partial x_2}, \frac{\partial f(t_i,x_j)}{\partial x_3}, x_2, x_3\right) \\ &\cong \mathbb{V}(\alpha_{(210)})(t_2), \alpha_{(201)}(t_2)) \subset \widetilde{\mathbb{A}^3_{t_2,x_2,x_3}} \subset U \end{aligned} \tag{4.4}$$

where

$$\widetilde{X} = q^{-1}(X) \subset \mathbb{C}^2_{t_i} \setminus \{(0,0)\} \times \mathbb{C}^3_{x_j} \setminus \{(0,0,0)\} = U \xrightarrow{q} U/(\mathbb{C}^*)^2 = \mathbb{F}(0,a_2,a_3).$$

We get 3 cases

1. If both $\alpha_{(201)}(t_2)$ and $\alpha_{(210)}(t_2)$ are zero then $Sing(\widetilde{X}) \cap \widetilde{C_{2,3}} = \widetilde{C_{2,3}}$. In this case, the surface X is singular along the whole of the curve $C_{2,3}$. We eliminate such X from our search of mildly singular $K3$ surfaces.

2. If exactly one of $\alpha_{(2,1,0)})(t_2)$ and $\alpha_{(201)}(t_2)$ is nonzero, then we have that $\alpha_{(201)}(t_2) = 0$ and $\alpha_{(210)}(t_2) \neq 0$. This is because the former has a larger degree. We equivalently have that
$$2 - a_2 \geq 0 \text{ and } 2 - a_3 < 0. \tag{4.5}$$
Two situations arise in this case, namely $\alpha_{(201)}(t_2)$ is either a nonconstant or a nonzero constant

   (a) If $\alpha_{(201)}(t_2)$ is noncostant, that is $2 - a_2 > 0$, we have that

$$\begin{array}{c} Sing(\widetilde{X}) = Sing(\widetilde{X}) \cap \widetilde{C_{2,3}} \cap \widetilde{U_{11}} = \{(T_k,0,0) \in \widetilde{\mathbb{A}^3_{t_2,x_2,x_3}} : 1 \leq k \leq 2 - a_2\} \subset U \\ \Big\downarrow q \\ Sing(X) = \{[1 : T_k; 1 : 0 : 0] \in \mathbb{A}^3_{t_2,x_2,x_3} : 1 \leq k \leq 2 - a_2\} \subset \mathbb{F}(0,a_2,a_3) \end{array}$$

corresponding to the roots of $\alpha_{(201)}(t_2)$. These singular anticanonical hypersurfaces are subvarieties of 3-fold scrolls $\mathbb{F}(0,a_2,a_3)$ where (a_2,a_3) are lattice points of the polytope defined by $0 \le a_2 \le a_3$, Inequality (4.3) and $0 \le a_2 < 2 < a_3$. Such hypersurfaces X are subvarieties of either of the 3-fold scrolls

$$\mathbb{F}(0,a_2,a_3) = \mathbb{F}(0,1,3) \text{ and } \mathbb{F}(0,1,4).$$

For the scroll $\mathbb{F}(0,1,3)$ above, the degrees of coefficients of the monomials $x_1^{q_1}x_2^{q_2}x_3^{q_3}$ are

$$\begin{array}{ccccccc} & & & \emptyset & & & \\ & & \emptyset & & 1 & & \\ & 0 & & 2 & & 4 & \\ 1 & & 3 & & 5 & & 7. \end{array}$$

The corresponding hypersurface is

$$X = \mathbb{V}(f(t_i,x_j)) = \mathbb{V}\left(x_1x_2^2 + r(t_i)x_1^2x_3 + \text{higher order terms}\right) \subset \mathbb{F}(0,1,3)$$

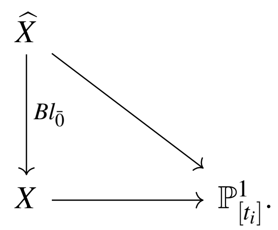
with r linear in t_i.

Locally in the chart $U_{11} = \{t_1 = x_1 = 1\} \subset \mathbb{F}(0,a_2,a_3)$ with $r = \mu t_2 + 1$, we get, up to higher order terms,

$$X_{aff} = \mathbb{V}(f(t_2,x_2,x_3)) = \mathbb{V}(x_2^2 + rx_3) \subset \mathbb{A}^3_{<r,x_2,x_3>}$$

which is a $\frac{1}{2}(1,1)$ or A_1 surface singularity which can be resolved by a standard blowup $\widehat{X_{aff}} := Bl_{(0,0,0)}X_{aff}$.

Performing this blowup on the projective variety X, we get an elliptically fibred K3 surface $\widehat{X}$.



Similarly for the scroll $\mathbb{F}(0,1,4)$ above, the degrees of coefficients of the monomials $x_1^{q_1}x_2^{q_2}x_3^{q_3}$ are

$$\begin{array}{ccccccc} & & & \emptyset & & & \\ & & \emptyset & & 1 & & \\ & \emptyset & & 2 & & 5 & \\ 0 & & 3 & & 6 & & 9. \end{array}$$

The corresponding hypersurface is

$$Y = \mathbb{V}(f(t_i, x_j)') = \mathbb{V}\left(x_2^3 + s(t_i)x_1^2x_3 + \text{higher order terms}\right) \subset \mathbb{F}(0,1,4)$$

with s linear in t_i.

Locally in the chart $U_{11} \subset \mathbb{F}(0, a_2, a_3)$ with $s = \nu t_2 + 1$; we get, upto higher order terms,

$$Y_{aff} = \mathbb{V}(f(t_2, x_2, x_3)') = \mathbb{V}(x_2^3 + sx_3) \subset \mathbb{A}^3_{<s,x_2,x_3>}$$

which is a $\frac{1}{3}(1,2)$ or A_2 surface singularity which can be resolved by a standard blowup $\widehat{Y_{aff}} := Bl_{(0,0,0)}Y_{aff}$.

Performing this blowup on the projective variety Y, we have a cubic fibred K3 surfaces $\widehat{Y}$

$$\begin{array}{ccc} \widehat{Y} & & \\ \big\downarrow {\scriptstyle Bl_{\bar{0}}} & \searrow & \\ Y & \longrightarrow & \mathbb{P}^1_{[t_i]}. \end{array}$$

(b) If $\alpha_{(2,1,0)}(t_2)$ is nonzero constant, that is $2 - a_2 = 0$ or $a_2 = 2$ then $Sing(X) = \emptyset$. Together with the condition that $a_3 \geq a_2 \geq 0$, Inequality (4.3) and Inequality (4.5), this case results in the lattice points (a_2, a_3) corresponding to 3-fold scrolls

$$\mathbb{F}(0, a_2, a_3) = \mathbb{F}(0,2,3),\ \mathbb{F}(0,2,4),\ \mathbb{F}(0,2,5) \text{ and } \mathbb{F}(0,2,6)$$

in which nonsingular genus-1-fibred K3 surfaces X are embedded.

3. If both $\alpha_{(2,0,1)})(t_2)$ and $\alpha_{(2,1,0)}(t_2)$ are nonzero or equivalently

$$2 - a_3 \geq 0 \implies 2 - a_2 \geq 0, \tag{4.6}$$

then geometrically the roots of $\alpha_{(2,0,1)})(t_2)$ are different from the roots of $\alpha_{(2,1,0)}(t_2)$ in which case $Sing(X) = \emptyset$. Together with the condition that $a_3 \geq a_2 \geq 0$ and Inequality (4.3), this case result in the lattice points (a_2, a_3) corresponding to other 3-fold scrolls

$$\mathbb{F}(0, a_2, a_3) = \mathbb{F}(0,1,2) \text{ and } \mathbb{F}(0,2,2)$$

in which the nonsingular genus-1-fibred K3 surfaces X are embedded.

We then obtain the Table (4.1) summarising with reason the models of 3-fold straight scrolls $\mathbb{F}(0,a_2,a_3)$ in which the anticanonical elliptically fibred K3 surfaces are embedded. $\square$

Example 4.1.2. For the special case $\mathbb{F}(0,2,6)$,

$$X = \mathbb{V}(x_2^3 + x_1^2 x_3 + \text{ other terms }) \subset \mathbb{F}(0,2,6)$$

is the nonsingular K3 surface X whose equation involves constant coefficients of x_2^3 and $x_1^2 x_3$ so that every fibre of $\varphi : X \to \mathbb{P}^1$ is an elliptic curve in Weierstrass normal form.

4.2 Elliptic fibrations with weighted quartic fibres

Lemma 4.2.1. *The unique normal form of the weighted 3-fold scroll $\mathbb{F}(a,b,c|1,1,2)$ is $\mathbb{F}(0,a_2,a_3|1,1,2)$ with $a_2 \geq 0,\ a_3 \in \mathbb{Z}$.*

Proof. Note that for $k \in \mathbb{Z}$

$$\mathbb{F}(a,b,c|1,1,2) = \mathbb{F}\begin{bmatrix} 1 & 1 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \cong \mathbb{F}\begin{bmatrix} 1 & 1 & -a+k & -b+k & -c+2k \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

We could swap the weights $wt(x_1) = (a,1), wt(x_2) = (b,1)$ and hence conveniently conclude that $\mathbb{F}(a,b,c|1,1,2) = \mathbb{F}(0,a_2,a_3)$ with $a_2 \geq 0,\ a_3 \in \mathbb{Z}$. $\square$

Theorem 4.2.2. *Let $\mathbb{F} = \mathbb{F}(a_1,a_2,a_3|1,1,2)$ be a $\mathbb{P}[1,1,2]$ bundle over $\mathbb{P}^1$ with quartic elliptic fibres. There are* 24 *mildly singular quartic fibred K3 surfaces with the properties in the Tables* (4.5), (4.6).

No.	$\mathbb{F}=\mathbb{F}(0,a_2,a_3\|1^2,2)$	General $X\in\|-K_{\mathbb{F}}\|$
1	$\mathbb{F}(0,2,0\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ is very ample, so a general X is nonsingular $K3$ surface
2	$\mathbb{F}(0,1,1\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{2}(1,1)$ singularities.
3	$\mathbb{F}(0,1,0\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1)$ singularity
4	$\mathbb{F}(0,1,-1\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ is very ample, so a general X is nonsingular $K3$ surface
5	$\mathbb{F}(0,0,2\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $4\times\frac{1}{2}(1,1)$ singularities.
6	$\mathbb{F}(0,0,1\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{2}(1,1)$ singularities
7	$\mathbb{F}(0,0,0\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{2}(1,1)$ singularities
8	$\mathbb{F}(0,0,-1\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1)$ singularity.
9	$\mathbb{F}(0,0,-2\|1,1,2)$	$\|-K_{\mathbb{F}}\|$ is very ample, so a general X is nonsingular $K3$ surface
10	$\mathbb{F}(0,1,2\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|)=\{x_2=y=0\}$. General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{2}(1,1)$ singularities along$\{x_1=x_2=0\}$
11	$\mathbb{F}(0,2,1\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|)=\{x_2=y=0\}$. General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1)$ singularities along$\{x_1=x_2=0\}$
12	$\mathbb{F}(0,2,2\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|)=\{x_2=y=0\}$. General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{2}(1,1)$ singularities along$\{x_1=x_2=0\}$
13	$\mathbb{F}(0,2,3\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|)=\{x_2=y=0\}$. General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{2}(1,1)$ singularities along$\{x_1=x_2=0\}$
14	$\mathbb{F}(0,2,4\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|)=\{x_2=y=0\}$. General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $4\times\frac{1}{2}(1,1)$ singularities along$\{x_1=x_2=0\}$
15	$\mathbb{F}(0,2,5\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|)=\{x_2=y=0\}$. General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $5\times\frac{1}{2}(1,1)$ singularities along$\{x_1=x_2=0\}$

Table 4.5: $\mathbb{F}=\mathbb{F}(0,a_2,a_3|1,1,2)$ for which a general $X\in|-K_{\mathbb{F}}|$ is weighted quartic fibred K3 surface or with quotient singularities.

No.	$\mathbb{F} = \mathbb{F}(0, a_2, a_3 \| 1^2, 2)$	General $X \in \|-K_{\mathbb{F}}\|$
16	$\mathbb{F}(0,2,6\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|) = \{x_2 = y = 0\}$. General $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $6 \times \frac{1}{2}(1,1)$ singularities along $\{x_1 = x_2 = 0\}$
17	$\mathbb{F}(0,2,7\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|) = \{x_2 = y = 0\}$. General $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $7 \times \frac{1}{2}(1,1)$ singularities along $\{x_1 = x_2 = 0\}$
18	$\mathbb{F}(0,2,8\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|) = \{x_2 = y = 0\}$. General $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $8 \times \frac{1}{2}(1,1)$ singularities along $\{x_1 = x_2 = 0\}$
19	$\mathbb{F}(0,3,2\|1,1,2)$	$Bs(\|-K_{\mathbb{F}}\|) = \{x_2 = y = 0\}$. General $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1)$ singularities along $\{x_1 = x_2 = 0\}$
20	$\mathbb{F}(0,4,2\|1,1,2)$	$-K_{\mathbb{F}}$ is very ample, so a general X is nonsingular $K3$ surface
21	$\mathbb{F}(0,3,1\|1,1,2)$	$-K_{\mathbb{F}}$ is very ample, so a general X is nonsingular $K3$ surface
22	$\mathbb{F}(0,1,3\|1,1,2)$	General $X \in \|-K_{\mathbb{F}}\|$ is singular with $4 \times \frac{1}{2}(1,1)$ singularities along $\{x_1 = x_2 = 0\}$ and $\frac{1}{2}(1,1)$ along $Bs(\|-K_{\mathbb{F}}\|) = \{x_2 = y = 0\}$.
23	$\mathbb{F}(0,1,4\|1,1,2)$	General $X \in \|-K_{\mathbb{F}}\|$ is singular with $5 \times \frac{1}{2}(1,1)$ singularities along $\{x_1 = x_2 = 0\}$ and $\frac{1}{3}(1,2)$ along $Bs(\|-K_{\mathbb{F}}\|) = \{x_2 = y = 0\}$.
24	$\mathbb{F}(0,1,5\|1,1,2)$	General $X \in \|-K_{\mathbb{F}}\|$ is singular with $6 \times \frac{1}{2}(1,1)$ singularities along $\{x_1 = x_2 = 0\}$ and $\frac{1}{4}(1,3)$ along $Bs(\|-K_{\mathbb{F}}\|) = \{x_2 = y = 0\}$.

Table 4.6: $\mathbb{F} = \mathbb{F}(0, a_2, a_3|1,1,2)$ for which a general $X \in |-K_{\mathbb{F}}|$ is a weighted quartic fibred K3 surface with quotient singularities.

Proof. By Lemma (4.2.1), we have that

$$\mathbb{F} = \mathbb{F}(a_1, a_2, a_3|1,1,2) \cong \mathbb{F}(0, a_2, a_3|1,1,2), \text{ with } a_2 \geq 0, a_3 \in \mathbb{Z}.$$

Assume that $|-K_{\mathbb{F}}| = |L_{2-a_2-a_3,4}| \neq \emptyset$ and take a general surface

$$X = \mathbb{V}(f(t_i, x_j, y)) = \mathbb{V}\left(\sum_{(q_1,q_2,q_3)_{(1,1,2)} \vdash 4} \alpha_{(q_j)}(t_1,t_2) x_1^{q_1} x_2^{q_2} y^{q_3}\right) \in |L_{2-a_2-a_3,4}|.$$

The fibration $\pi : \mathbb{F}(0, a_2, a_3|1,1,2) \to \mathbb{P}^1$ induces the quartic fibration

$$\varphi : X \to \mathbb{P}^1$$

whose fibres are quartic curves

$$E_4 = \mathbb{V}(f(t, x_j, y)) = \{f_4(t, x_2) + y f_2(t, x_1, x_2) + \alpha_{(002)}(t) y^2 = 0\} \subset \mathbb{P}^2_{[x,y,z]}[1,1,2]$$

for fixed $t \in \mathbb{P}^1_{[t_i]}$.

$$
\begin{array}{ccccccc}
 & & & & x_1^4 & & \\
 & & & x_1^3x_2 & & & \\
 & & & & & x_1^2y & \\
 & & x_1^2x_2^2 & & & & \\
 & x_1x_2^3 & & & x_1x_2y & & \\
x_2^4 & & & x_2^2y & & & y^2.
\end{array}
$$

Table 4.7: Newton triangle for the basis of $\mathbb{C}[\mathbb{P}^2_{[x_1:x_2:y]}[1,1,2]]_4$

The degrees of coefficients

$$\deg \alpha_{(q_j)_{(1,1,2)}\vdash 4}(t_i) = 2 + (q_2-1)a_2 + (q_3-1)a_3$$

of the monomials $x_1^{q_1}x_2^{q_2}y^{q_3}$ increase down the $x_1 - x_2$ edge of the triangle by a_2 and increase, decrease or stay constant on the other two edges depending on the value of $a_3 \in \mathbb{Z}$.

The scroll $\mathbb{F}(0,a_2,a_3|1,1,2) = U/(\mathbb{C}^*)^2$ has finite $\frac{1}{2}(1,1,0)$ quotient singularities where

$$Sing(\mathbb{F}(0,a_2,a_3|1,1,2)) = \{x_1 = x_2 = 0\} = \{(t_1 : t_2; 0:0:1)\} \cong \mathbb{P}_{[t_i]}.$$

Therefore, for a quasismooth quartic K3 surface

$$
\begin{aligned}
X &= \mathbb{V}(f(t_i,x_j)) \\
&= \{\alpha_{(002)}(t_i)y^2 + y(\alpha_{(201)}(t_i)x_1^2 + \alpha_{(111)}(t_i)x_1x_2 + \alpha_{(021)}(t_i)x_2^2) + \\
&\sum_{k=1}^{4} \alpha_{((4-k)k0)}(t_i)x_1^{4-k}x_2^k = 0\},
\end{aligned}
$$

we have that singularities on X are inherited from the ambient space

$$
\begin{aligned}
Sing\ X =& X \cap Sing(\mathbb{F}(0,a_2,a_3|1,1,2)) = \mathbb{V}(\alpha_{(002)}(t_2)) \times \{[0:0:1]\} \\
=& \{p_k = [\beta_{1k} : \beta_{2k}; 0:0:1] | 1 \leq k \leq \deg \alpha_{(002)}\}.
\end{aligned}
$$

So, near p_k, we have that

$$X \cap \{y = t_2 = 1\} = \mathbb{V}(f(t_1,x_j)) = \{\alpha_{(002)}(t_1) + \text{Other Terms} = 0\}.$$

With β_{1k} a simple root of $f(t_1,x_j)$, we have that $\frac{\partial f(t_1,x_j)}{\partial t_1} \neq 0$ by the implicit function theorem. Therefore, up to the stabilizer $Stab_{(\mathbb{C}^*)^2}(\mathbb{F}(0,a_2,a_3|1,1,2))$, we have that (x_1,x_2) is a set of local coordinates on X satisfying

$$(x_1,x_2) \mapsto (-x_1,-x_2) \text{ hence } \frac{1}{2}(1,1) \text{ singularity}$$

at p_k for every k.

A general section of $-K_{\mathbb{F}} = L_{2-a_2-a_3,4}$ having x_1^4, x_2^4 and y^2 terms in f_4 implies base point freeness (bpf) of $-K_{\mathbb{F}}$ which in turn implies quasismoothness of a general $X \in |-K_{\mathbb{F}}|$. Equivalently, bpf is implied by the degrees of the coefficients $\alpha_{(q_j)_{(1,1,2)}\vdash 4}(t_i)$ of monomials $x_1^{q_1}x_1^{q_2}y^{q_3}$ at the vertices of Newton triangle (4.7) being non-negative.

$$2-a_2-a_3 \geq 0,\ 2+3a_2-a_3 \geq 0,\ 2-a_2+a_3 \geq 0. \tag{4.7}$$

The Inequality (4.7) together with Lemma (4.2.1) corresponds to the list $1-9$ of threefold scrolls in which the respective quasismooth families of quartic -fibred K3 surfaces $X \in |-K_{\mathbb{F}(0,a_2,a_3|1,1,2)}|$ are embedded. The $(2-a_2+a_3) \times \frac{1}{2}(0,1,1)$ singularities on each X can then be resolved by blow up.

Further, and weaker than bpf, we check for quasismooth sections across the base locus of $|-K_{\mathbb{F}(0,a_2,a_3|1,1,2)}|$. The base loci are of dimension at most two by setting all or some of the variables x_1, x_2 and y to zero with $0 \leq \dim Bs(|-K_{\mathbb{F}}|) \leq 2$. This is a direct consequence of Proposition (3.6.1).

If $x_1 | f(t_i, x_j, y)$ then no term involving monomials on the opposite $x_2 - y$ edge of the Newton triangle (4.7) would be in $f(t_i, x_j, y)$; this is also true for x_2 and y with monomials on the corresponding opposite sides of the triangle. Therefore $\dim(Bs(L_{2-a_1-a_2-a_3,4})) \leq 1$ for a general $X \in |L_{2-a_2-a_3,4}|$ to be irreducible. That is, $x_1, x_2, y \not| f(t_i, x_j, y)$ which precisely means that $f(t_i, x_j, y)$ must have one term from each edge of the Newton triangle (4.7). Equivalently, it is enough to have nonzero $\alpha_{(040)}(t_i)x_2^4$ and the larger (in terms of degree of coefficient α) of $\alpha_{(400)}(t_i)x_1^4$ or $\alpha_{(002)}(t_i)y^2$ nonzero

$$2+3a_2-a_3 \geq 0,\ \mathrm{Max}(2-a_2-a_3, 2-a_2+a_3) \geq 0, a_3 \neq 0. \tag{4.8}$$

If $\dim(Bs(|L_{2-a_2-a_3,4}|)) = 1$ then we have, in addition to Inequality (4.8), that

$$\mathrm{Min}(2-a_2-a_3, 2-a_2+a_3) < 0. \tag{4.9}$$

The Inequalities

$$\begin{aligned} a_3 &> 0, \\ a_2 &\geq 0, \\ 2+3a_2-a_3 &\geq 0, \\ 2-a_2+a_3 &\geq 0 \text{ and} \\ 2-a_2-a_3 &< 0 \end{aligned}$$

imply absence of x_1^4 in $f(t_i,x_j)$ hence

$$Bs(|-K_{\mathbb{F}}|) = \{x_2 = y = 0\} = C_{2y}.$$

The Inequalities

$$\begin{aligned} a_3 &< 0, \\ a_2 &\geq 0, \\ 2+3a_2-a_3 &\geq 0, \\ 2-a_2+a_3 &< 0 \text{ and} \\ 2-a_2-a_3 &\geq 0 \end{aligned}$$

imply absence of y^2 in $f(t_i,x_j)$ hence

$$Bs(|-K_{\mathbb{F}}|) = \{x_1 = x_2 = 0\} = C_{12}.$$

This results in two cases

(i) Where $Bs(|-K_{\mathbb{F}}|) = C_{2y}$, we get that the singularities on the cone $\widetilde{X} = q^{-1}(X) \subset U$ over X are

$$Sing(\widetilde{X}) \cap \widetilde{C_{2y}} = \mathbb{V}(\alpha_{(310)})(t_i), \alpha_{(201)}(t_i)) \subset U \tag{4.10}$$

where $\widetilde{C_{2y}} = q^{-1}(C_{2y}) \subset U$ with

$$q : \mathbb{C}^2 \setminus \{(0,0)\} \times \mathbb{C}^3 \setminus \{(0,0,0)\} = U \to U/(\mathbb{C}^*)^2 = \mathbb{F}(0,a_2,a_3|1,1,2)).$$

To get isolated singularities, we must have that $x_1 \not| \frac{\partial f(t_i,x_j)}{\partial x_2}, \frac{\partial f(t_i,x_j)}{\partial y}$; equivalently, we get a further constraint

$$\text{Max}(\deg \alpha_{(310)}, \deg \alpha_{(201)}) = \text{Max}(2-a_3, 2-a_2) \geq 0. \tag{4.11}$$

The Inequalities corresponding to $Bs(-K_{\mathbb{F}(0,a_2,a_3|1,1,2)}) = C_{2y}$ with $a_2 = 2$ and $a_3 \geq 2$ (or $a_2 \geq 2$ and $a_3 = 2$) are such that

$$Sing(\widetilde{X}) \cap \widetilde{C_{2y}} = \mathbb{V}(1) = \emptyset.$$

This results in the list of ambient threefold scrolls $\mathbb{F}(a_1,a_2,a_3|1,1,2)$

$$\begin{aligned} &\mathbb{F}(0,1,2|1,1,2), \mathbb{F}(0,2,1|1,1,2), \mathbb{F}(0,2,2|1,1,2), \mathbb{F}(0,2,3|1,1,2), \\ &\mathbb{F}(0,2,4|1,1,2), \mathbb{F}(0,2,5|1,1,2), \mathbb{F}(0,2,6|1,1,2), \mathbb{F}(0,2,7|1,1,2), \\ &\mathbb{F}(0,2,8|1,1,2), \mathbb{F}(0,3,2|1,1,2) \text{ and } \mathbb{F}(0,4,2|1,1,2) \end{aligned}$$

in which the respective general K3 surfaces $X \in |-K_{\mathbb{F}(0,a_2,a_3|1,1,2)}|$ are quasismooth and quartic fibred. Their $(2-a_2+a_3)\times\frac{1}{2}(1,1)$ singularities along $Sing\mathbb{F}(0,a_2,a_3|1,1,2) = C_{12}$ are resolved by blowing them up. Otherwise (without the constraints on a_2 and a_3), other than their respective $(2-a_2+a_3)\times\frac{1}{2}(1,1)$ singularities along C_{12}, the K3 surfaces X embedded in the scrolls

$$\mathbb{F}(a_1,a_2,a_3|1,1,2) = \mathbb{F}(0,1,3|1,1,2), \mathbb{F}(0,1,4|1,1,2), \mathbb{F}(0,1,5|1,1,2)$$
$$\text{and } \mathbb{F}(0,3,1|1,1,2)$$

have additional isolated singularities along $Bs(-K_{\mathbb{F}(0,a_2,a_3|1,1,2)}) = C_{2y}$. From the degrees of the coefficient of $x_1^{q_1}x_2^{q_2}y^{q_3}$ in $f(t_i,x_j)$, we have the following equations up

					$2-a_2-a_3$		
			$2-a_3$				
						$2-a_2$	
		$2+a_2-a_3$					
	$2+2a_2-a_3$			2			
$2+3a_2-a_3$			$2+a_2$				$2-a_2+a_3$

Table 4.8: Degrees $\deg\left(\alpha_{(q_j)_{(1,1,2)}\vdash 4}(t_i)\right)$

to higher order terms:

(i) $X_1 = \mathbb{V}\left(x_1^2x_2^2 + r_1(t_i)x_1^2y + s_1(t_i)x_1x_2^3 + \text{H.O.T}\right) \subset \mathbb{F}(0,1,3|1,1,2)$;

(ii) $X_2 = \mathbb{V}\left(x_1x_2^3 + r_2(t_i)x_1^2y + s_2(t_i)x_2^4 + \text{H.O.T}\right) \subset \mathbb{F}(0,1,4|1,1,2)$;

(iii) $X_3 = \mathbb{V}\left(x_2^4 + r_3(t_i)x_1^2y + \text{H.O.T}\right) \subset \mathbb{F}(0,1,5|1,1,2)$ and

(iv) $X_4 = \mathbb{V}\left(y^2 + r_4(t_i)x_1^3x_2 + \text{H.O.T}\right) \subset \mathbb{F}(0,3,1|1,1,2)$

where r_k, s_k are linear in t_i for $1 \le k \le 4$.

Now, in the chart $\{t_1 = x_1 = 1\} \cong \mathbb{A}^3_{<t_2,x_2,y>}$ with $X_k' = X_k \cap \mathbb{A}^3_{<t_2,x_2,y>}$, we have

(i) $X_1' = \mathbb{V}\left(x_2^2 + r_1(t_2)y + s_1(t_2)x_2^3 + \text{H.O.T}\right) \implies \frac{1}{2}(1,1)$;

(ii) $X_2' = \mathbb{V}\left(x_2^3 + r_2(t_2)y + s_2(t_2)x_2^4 + \text{H.O.T}\right) \implies \frac{1}{3}(1,2)$;

(iii) $X_3' = \mathbb{V}\left(x_2^4 + r_3(t_2)y + \text{H.O.T}\right) \implies \frac{1}{4}(1,3)$ and

(iv) $X_4' = \mathbb{V}\left(y^2 + r_4(t_2)x_2 + \text{H.O.T}\right) \implies \frac{1}{2}(1,1)$

where r_k, s_k are linear in t_2 for $1 \leq k \leq 4$.

(ii) Along the curve

$$Bs(-K_{\mathbb{F}(0,a_2,a_3|1,1,2)}) = C_{12} = Sing\mathbb{F}(0,a_2,a_3|1,1,2) \cong \mathbb{P}^1_{[t_i]},$$

the singularities

$$Sing(\widetilde{X}) \cap \widetilde{C_{12}} = \mathbb{V}\left(f(t_i,x_j), \frac{\partial f(t_i,x_j)}{\partial x_1}, \frac{\partial f(t_i,x_j)}{\partial x_2}, x_1, x_2\right) = U$$

coincide with those of the ambient space $\mathbb{F}(0,a_2,a_3|1,1,2)$ which are not isolated. In conclusion, sections of $-K_{\mathbb{F}}$ are not considered.

□

4.3 Elliptic fibrations with weighted sextic fibres

Lemma 4.3.1. *The unique normal form of the weighted 3-fold scroll* $\mathbb{F}(a,b,c|1,2,3)$ *is* $\mathbb{F}(0,a_2,a_3|1,2,3)$ *with* $a_2, a_3 \in \mathbb{Z}$.

Proof. Note that for $k \in \mathbb{Z}$

$$\mathbb{F}(a,b,c|1,2,3) = \mathbb{F}\begin{bmatrix} 1 & 1 & -a & -b & -c \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \cong \mathbb{F}\begin{bmatrix} 1 & 1 & -a+k & -b+2k & -c+3k \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Hence, with $k = a$, we can conclude that, uniquely, $\mathbb{F}(a,b,c|1,2,3) \cong \mathbb{F}(0,a_2,a_3)$ with $a_2, a_3 \in \mathbb{Z}$. □

Theorem 4.3.2. *Let* $\mathbb{F} = \mathbb{F}(a_1,a_2,a_3|1,2,3)$ *be a* $\mathbb{P}[1,2,3]$ *bundle over* $\mathbb{P}^1$ *with sextic fibres. There are* 31 *mildly singular sextic fibred K3 surfaces with the properties on the Tables* (4.9) *and* (4.10).

No.	$\mathbb{F}=\mathbb{F}(0,a_2,a_3\|1,2,3)$	General $X \in \|-K_{\mathbb{F}}\|$
1	$\mathbb{F}(0,2,0\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $6\times\frac{1}{3}(1,1)$ singularity
2	$\mathbb{F}(0,1,1\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{2}(1,1), 3\times\frac{1}{3}(1,1)$ singularity.
3	$\mathbb{F}(0,1,0\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1), 4\times\frac{1}{3}(1,1)$ singularity
4	$\mathbb{F}(0,1,-1\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{2}(1,1), 3\times\frac{1}{3}(1,1)$ singularity.
5	$\mathbb{F}(0,0,2\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $4\times\frac{1}{2}(1,1)$ singularity.
6	$\mathbb{F}(0,0,1\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{2}(1,1), \frac{1}{3}(1,1)$ singularities
7	$\mathbb{F}(0,0,0\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{2}(1,1), 2\times\frac{1}{3}(1,1)$ singularities
8	$\mathbb{F}(0,0,-1\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1), 3\times\frac{1}{3}(1,1)$ singularities.
9	$\mathbb{F}(0,0,-2\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $4\times\frac{1}{3}(1,1)$ singularity.
10	$\mathbb{F}(0,-1,-1\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{2}(1,1), \frac{1}{3}(1,1)$ singularities.
11	$\mathbb{F}(0,-1,-2\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1), 2\times\frac{1}{3}(1,1)$ singularity.
12	$\mathbb{F}(0,-2,-3\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1), \frac{1}{3}(1,1)$ singularity.
13	$\mathbb{F}(0,-1,0\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{2}(1,1)$ singularity.
14	$\mathbb{F}(0,-1,-3\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{3}(1,1)$ singularity.
15	$\mathbb{F}(0,-2,-2\|1,2,3)$	$-K_{\mathbb{F}}$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{2}(1,1)$ singularity.
16	$\mathbb{F}(0,-2,-4\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{3}(1,1)$ singularity.
17	$\mathbb{F}(0,-3,-4\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1)$ singularity.
18	$\mathbb{F}(0,-3,-5\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X \in \|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{3}(1,1)$ singularity.
19	$\mathbb{F}(0,-4,-6\|1,2,3)$	General $X \in \|-K_{\mathbb{F}}\|$ is quasismooth. No isolated singularities

Table 4.9: $\mathbb{F}=\mathbb{F}(0,a_2,a_3|1,2,3)$ for which a general $X \in |-K_{\mathbb{F}}|$ is weighted sextic fibred K3 surface or with quotient singularities along $Bs(|-K_{\mathbb{F}}|)$

No.	$\mathbb{F}=\mathbb{F}(0,a_2,a_3\|1,2,3)$	General $X\in\|-K_{\mathbb{F}}\|$
20	$\mathbb{F}(0,4,2\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $8\times\frac{1}{3}(1,1)$ singularity.
21	$\mathbb{F}(0,3,2\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1),6\times\frac{1}{3}(1,1)$ singularity.
22	$\mathbb{F}(0,3,1\|1,2,3)$	General $X\in\|-K_{\mathbb{F}}\|$ is singular with $7\times\frac{1}{3}(1,1)$ singularities along $Sing(\mathbb{F})$ and $\frac{1}{2}(1,1)$ singularity along $Bs(\|-K_{\mathbb{F}}\|)=\{y=z=0\}$.
23	$\mathbb{F}(0,2,6\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $6\times\frac{1}{2}(1,1)$ singularity.
24	$\mathbb{F}(0,2,5\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $5\times\frac{1}{2}(1,1),\frac{1}{3}(1,1)$ singularity.
25	$\mathbb{F}(0,2,4\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $4\times\frac{1}{2}(1,1),2\times\frac{1}{3}(1,1)$ singularity.
26	$\mathbb{F}(0,2,3\|1,2,3)$	$\|-K_{\mathbb{F}}\|$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{2}(1,1),3\times\frac{1}{3}(1,1)$ singularity.
27	$\mathbb{F}(0,2,2\|1,2,3)$	$Bs(\|-K_{\mathbb{F}}\|)=\{y=z=0\}$. General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $4\times\frac{1}{2}(1,1)$ singularities along $\{x_1=x_2=0\}$
28	$\mathbb{F}(0,2,1\|1,2,3)$	General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{2}(1,1),5\times\frac{1}{3}(1,1)$ singularities along $\{x_1=x_2=0\}$
29	$\mathbb{F}(0,1,4\|1,2,3)$	General $X\in\|-K_{\mathbb{F}}\|$ is singular with $5\times\frac{1}{2}(1,1)$ singularities along $Sing(\mathbb{F})$ and $\frac{1}{3}(1,2)$ singularity along $Bs(\|-K_{\mathbb{F}}\|)=\{y=z=0\}$.
30	$\mathbb{F}(0,1,3\|1,2,3)$	General $X\in\|-K_{\mathbb{F}}\|$ is singular with $4\times\frac{1}{2}(1,1),\frac{1}{3}(1,1)$, singularities along $Sing(\mathbb{F})$ and $\frac{1}{2}(1,1)$ singularity along $Bs(\|-K_{\mathbb{F}}\|)=\{y=z=0\}$.
31	$\mathbb{F}(0,1,2\|1,2,3)$	General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{2}(1,1),2\times\frac{1}{3}(1,1)$ singularities along $\{x_1=x_2=0\}$

Table 4.10: $\mathbb{F}=\mathbb{F}(0,a_2,a_3|1,2,3)$ for which a general $X\in|-K_{\mathbb{F}}|$ is weighted sextic fibred K3 surface or with quotient singularities along $Bs(|-K_{\mathbb{F}}|)$

`Proof.` Let $\mathbb{F}=\mathbb{F}(a_1,a_2,a_3|1,2,3)$ be as in the Lemma (4.3.1) and assume that the anticanonical divisor class

$$-K_{\mathbb{F}}=(2-a_2-a_3)L+(1+2+3)M=L_{2-a_2-a_3,6}$$

of $\mathbb{F}$ is such that $|L_{2-a_2-a_3,6}|\neq\emptyset$. Take a general surface

$$X=\mathbb{V}(f_6)=\mathbb{V}\left(\sum_{(q_1,q_2,q_3)_{(1,2,3)}\vdash 6}\alpha_{(q_j)}(t_1,t_2)x^{q_1}y^{q_2}z^{q_3}\right)\in|L_{2-a_1-a_2-a_3,6}|$$

where the coefficient $\alpha_{(q_j)}(t_1,t_2)\in\mathbb{C}[\mathbb{P}_{t_i}]$ of $x^{q_1}y^{q_2}z^{q_3}$ is of degree

$$\deg\alpha_{(q_j)}(t_i)=2+(q_2-1)a_2+(q_3-1)a_3.$$

As before, the fibration $\pi : \mathbb{F}(0,a_2,a_3|1,2,3) \to \mathbb{P}^1$ induces the elliptic fibration

$$\varphi : X \to \mathbb{P}^1$$

whose fibres are sextic curves

$$E_6 = \mathbb{V}(f(t,x,y,z)) \subset \mathbb{P}^2_{[x,y,z]}[1,2,3], \text{ for fixed } t \in \mathbb{P}^1_{[t_i]}.$$

We can represent these partitions $(q_1,q_2,q_3)_{(1,2,3)} \vdash 6$, hence the monomials $x^{q_1}y^{q_2}z^{q_3}$, in a sparser Newton triangle.

$$\begin{array}{ccccccc}
 & & & & x^6 & & \\
 & & & x^4y & & & \\
 & & & & & x^3z & \\
 & & x^2y^2 & & & & \\
 & & & & xyz & & \\
y^3 & & & & & & z^2.
\end{array}$$

Table 4.11: Newton triangle for the basis of $\mathbb{C}[\mathbb{P}^2_{[x:y:z]}[1,2,3]]_6$

The degrees of coefficients of the monomials $x^{q_1}y^{q_2}z^{q_3}$ increases, decreases or stays constant on the three edges depending on the value of $a_2, a_3 \in \mathbb{Z}$.

The scroll $\mathbb{F}(0,a_2,a_3|1,2,3) = U/(\mathbb{C}^*)^2$

$$\begin{array}{ccccc}
X & \hookrightarrow & \mathbb{F}(0,a_2,a_3|1,2,3) & \xleftarrow{\ q\ } & U \\
 & \searrow^{f=\pi|_X} & \downarrow \pi & & \downarrow pr1 \\
 & & \mathbb{P}^1_{[t_i]} & \longleftarrow & \mathbb{C}^2_{t_i,t_2} \setminus \{(0,0)\},
\end{array}$$

where

$$U = \mathbb{C}^2_{t_i,t_2} \setminus \{(0,0)\} \times \mathbb{C}^3_{x,y,z} \setminus \{(0,0,0)\},$$

has finite quotient singularities. These are

$$\frac{1}{3}(1,1,0) \text{ along } \{x=y=0\}$$

and

$$\frac{1}{2}(1,0,1) \text{ along } \{x=z=0\}$$

where

$$Sing(\mathbb{F}(0,a_2,a_3|1,2,3)) = \{x = y = 0\} \cup \{x = z = 0\}.$$

A quasismooth sextic K3 surface $X \in |-K_{\mathbb{F}(0,a_2,a_3|1,2,3)}|$ inherits its singularities

$$\begin{aligned} Sing\ X =& \{p_m = [\beta_{1m} : \beta_{2m}; 0 : 0 : 1] : 1 \leq m \leq \deg \alpha_{(002)}\} \cup \\ & \{q_n = [\beta_{1n} : \beta_{2n}; 0 : 1 : 0] : 1 \leq n \leq \deg \alpha_{(030)}\} \end{aligned}$$

from the ambient space.

Locally around p_m, up to $(\mathbb{C}^*)^2$ stabilizers on the ambient space, we have that

$$(x,y) \mapsto (\varepsilon x, \varepsilon y) \text{ with } \varepsilon = \exp\left(\frac{2\pi i}{\mathrm{wt}(z)}\right) \neq 1 \text{ where } \mathrm{wt}(z) = 3 \text{ on } \mathbb{P}[1,2,3].$$

These p_m are indeed, therefore, $\frac{1}{3}(1,1)$ isolated singularities on X. We deduce similarly, that there are $\frac{1}{2}(1,1)$ isolated singularities at q_n .

Base point freeness of $-K_{\mathbb{F}} = L_{2-a_2-a_3,6}$ on $\mathbb{F} = \mathbb{F}(0,a_2,a_3|1,2,3)$ is implied by its general section having terms involving monomials at the vertices of the triangle in Table (4.11). Equivalently,

$$2 - a_2 - a_3 \geq 0,\ 2 + 2a_2 - a_3 \geq 0,\ 2 - a_2 + a_3 \geq 0. \tag{4.12}$$

That is, we have nonnegative degrees at the vertices of the triangle below.

			$2-a_2-a_3$		
		$2-a_3$			
				$2-a_2$	
	$2+a_2-a_3$				
			2		
$2+2a_2-a_3$					$2-a_2+a_3.$

Table 4.12: Degrees $\deg\left(\alpha_{(q_j)_{(1,2,3)}\vdash 6}(t_i)\right)$

The Inequality (4.12) together with Lemma (4.3.1) results in the list of ambient threefold scrolls $\mathbb{F}(0,a_2,a_3|1,2,3)$

$$\begin{aligned}&\mathbb{F}(0,0,0|1,2,3),\mathbb{F}(0,0,1|1,2,3),\mathbb{F}(0,1,0|1,2,3),\mathbb{F}(0,0,-1|1,2,3),\\&\mathbb{F}(0,-1,-1|1,2,3),\mathbb{F}(0,-1,-2|1,2,3),\mathbb{F}(0,-2,-3|1,2,3),\\&\mathbb{F}(0,0,2|1,2,3),\mathbb{F}(0,1,1|1,2,3),\mathbb{F}(0,2,0|1,2,3),\mathbb{F}(0,1,-1|1,2,3),\\&\mathbb{F}(0,0,-2|1,2,3),\mathbb{F}(0,-1,0|1,2,3),\mathbb{F}(0,-1,-3|1,2,3),\\&\mathbb{F}(0,-2,-2|1,2,3),\mathbb{F}(0,-2,-4|1,2,3),\mathbb{F}(0,-3,-4|1,2,3),\\&\mathbb{F}(0,-3,-5|1,2,3)\text{ and }\mathbb{F}(0,-4,-6|1,2,3)\end{aligned}$$

in which a general $X \in |L_{2-a_2-a_3,6}|$ is quasismooth sextic fibred K3 surfaces; they are numbered $1-19$ in the Table (4.9).

Further, we want X to be quasismooth in a neighbourhood of the base locus of $|L_{2-a_2-a_3,6}|$ with weaker condition than base point freeness. As a consequence of Proposition (3.6.1), base loci is of dimension at most two by setting all or some of the variable x,y and z to zero.

The dimension $\dim(Bs(|L_{2-a_2-a_3,6}|)) \neq 2$ for if otherwise, a general $X \in |L_{2-a_2-a_3,6}|$ would be reducible. We must therefore have that $x,y,z \not| f(t_i,x,y,z)$ or equivalently

$$\begin{aligned}&m = \text{Max}\{2-a_2-a_3, 2+2a_2-a_3, 2-a_2+a_3\} \geq 0,\\&\text{Max}\{2-a_2-a_3, 2+2a_2-a_3, 2-a_2+a_3\} \setminus \{m\} \geq 0.\end{aligned} \tag{4.13}$$

Suppose the dimension $\dim(Bs(|L_{2-a_2-a_3,6}|)) = 1$. The Inequalities (4.13) imply the following three cases of base locus $Bs(|L_{2-a_2-a_3,6}|)$ (six Inequalities if we order the two nonnegative vertices in Table (4.12)) Inequalities (4.14) below

$$\begin{cases}\{y=z=0\}=C_{yz} & \text{if } 2-a_2+a_3 \geq 0, 3a_2-2a_3 \geq 0, 2-a_2-a_3 < 0 \text{ or}\\ & \text{if } 2+2a_2-a_3 \geq 0, -3a_2+2a_3 \geq 0, 2-a_2-a_3 < 0\\ & \implies \text{ no } x^6 \text{ in } f(t_i,x_j);\\ \{x=z=0\}=C_{xz} & \text{if 2-a}_2+a_3 \geq 0, -2a_3 \geq 0, 2+2a_2-a_3 < 0 \text{ or}\\ & \text{if } 2-a_2-a_3 \geq 0, 2a_3 \geq 0, 2+2a_2-a_3 < 0\\ & \implies \text{ no } y^3 \text{ in } f(t_i,x_j);\\ \{x=y=0\}=C_{xy} & \text{if 2+2a}_2-a_3 \geq 0, -3a_2 \geq 0, 2-a_2+a_3 < 0 \text{ or}\\ & \text{if } 2-a_2-a_3 \geq 0, 3a_2 \geq 0, 2-a_2+a_3 < 0\\ & \implies \text{ no } z^2 \text{ in } f(t_i,x_j).\end{cases} \tag{4.14}$$

This result in three cases

(i) Along the curve $Bs(-K_{\mathbb{F}(0,a_2,a_3|1,2,3)}) = C_{yz}$, we get that the singularities on the cone $\widetilde{X} = q^{-1}(X) \subset U$ over X are

$$Sing(\widetilde{X}) \cap \widetilde{C_{yz}} = \mathbb{V}(\alpha_{(410)})(t_i), \alpha_{(301)}(t_i)) \subset U. \tag{4.15}$$

where $\widetilde{C_{yz}} = q^{-1}(C_{yz}) \subset U$. To get isolated singularities, we get a further constraint

$$\text{Max}(\deg \alpha_{(410)}, \deg \alpha_{(301)}) = \text{Max}(2-a_3, 2-a_2) \geq 0. \tag{4.16}$$

The Inequalities (4.14) corresponding to $Bs(-K_{\mathbb{F}(0,a_2,a_3|1,2,3)}) = C_{yz}$ with $a_2 = 2$ and $a_3 \geq 2$ (or $a_2 \geq 2$ and $a_3 = 2$) are such that

$$Sing(\widetilde{X}) \cap \widetilde{C_{yz}} = \mathbb{V}(1) = \emptyset.$$

These are threefold scrolls $\mathbb{F}(0,a_2,a_3|1,2,3)$

$$\mathbb{F}(0,1,2|1,2,3), \mathbb{F}(0,2,1|1,2,3), \mathbb{F}(0,2,2|1,2,3), \mathbb{F}(0,2,3|1,2,3),$$
$$\mathbb{F}(0,2,4|1,2,3), \mathbb{F}(0,2,5|1,2,3), \mathbb{F}(0,2,6|1,2,3), \mathbb{F}(0,3,2|1,2,3)$$
$$\text{and } \mathbb{F}(0,4,2|1,2,3)$$

in which the sextic fibred respective general K3 surfaces $X \in |-K_{\mathbb{F}(0,a_2,a_3|1,2,3)}|$ have $(2-a_2+a_3) \times \frac{1}{2}(1,1)$ and $(2+2a_2-a_3) \times \frac{1}{3}(1,1)$ quotient singularities. Without the constraints on a_2 and a_3, we expect singularities on X other than the $(2-a_2+a_3) \times \frac{1}{2}(1,1)$ and the $(2+2a_2-a_3) \times \frac{1}{3}(1,1)$ quotient singularities from $Sing(\mathbb{F}(0,a_2,a_3|1,2,3) \cong \mathbb{P}^1 \cup \mathbb{P}^1$. These K3 surfaces X are embedded in the scrolls

$$\mathbb{F}(0,a_2,a_3|1,2,3) = \mathbb{F}(0,1,3|1,2,3), \mathbb{F}(0,1,4|1,2,3) \text{ and } \mathbb{F}(0,3,1|1,2,3)$$

with additional isolated singularities along $Bs(-K_{\mathbb{F}(0,a_2,a_3|1,2,3)}) = C_{yz}$. From the triangle (4.12) of the coefficient of $x^{q_1}y^{q_2}z^{q_3}$ in $f(t_i,x_j)$, we have the following up to higher order terms:

(i) $X_1 = \mathbb{V}\left(x^2y^2 + r_1(t_i)x^3z + s_1(t_i)y^3 + \text{H.O.T}\right) \subset \mathbb{F}(0,1,3|1,2,3)$;

(ii) $X_2 = \mathbb{V}\left(y^3 + r_2(t_i)x^3z + \text{H.O.T}\right) \subset \mathbb{F}(0,1,4|1,2,3)$ and

(iii) $X_3 = \mathbb{V}\left(z^2 + r_3(t_i)x^4y + \text{H.O.T}\right) \subset \mathbb{F}(0,3,1|1,2,3)$

where r_k, s_k is linear in t_i for $1 \leq k \leq 3$.

Locally in the chart $\{x = t_1 = 1\} \cong \mathbb{A}_{<t_2,y,z>}$, we have the following singularities on $X_k' = X_k \cap \mathbb{A}^3_{<t_2,y,z>}$ up to higher order terms:

(i) $X_1' = \mathbb{V}\left(y^2 + r_1(t_2)z + s_1(t_2)y^3 + \text{H.O.T}\right) \implies \frac{1}{2}(1,1)$;

(ii) $X_2' = \mathbb{V}\left(y^3 + r_2(t_2)z + \text{H.O.T}\right) \implies \frac{1}{3}(1,2)$ and

(iii) $X_3' = \mathbb{V}\left(z^2 + r_3(t_2)y + \text{H.O.T}\right) \implies \frac{1}{2}(1,1)$

where r_k, s_k is linear in t_i for $1 \leq k \leq 3$.

(ii) Along each of the other curves $Bs(-K_{\mathbb{F}(0,a_2,a_3|1,2,3)}) = C_{xz}, C_{xy}$ we have that $Sing(\widetilde{X}) \cap \widetilde{C_*} = U$ which is not interesting to us; these singularities are not isolated hence ignored.

□

Chapter 5

K3 fibred Calabi–Yau Hypersurfaces in Weighted scrolls

In this chapter, we construct models of Calabi–Yau threefolds fibred by K3 surfaces $S_4 \in \mathbb{P}^3$ and $S_6 \subset \mathbb{P}[1,1,1,3]$ from Miles Reid's "Famous 95" list in [Fle00]. In particular, we extend lists in appendix A of [Mul06] by relaxing the stronger quasismoothness condition on the Calabi–Yau threefold families; we would like a general K3 fibred Calabi–Yau threefold $X \in |-K_{\mathbb{F}(a_j|b_j)}|$ to be well-formed and have cannonical singularities along the base locus of $|-K_{\mathbb{F}(a_j|b_j)}|$. Starting with known fibre data (b_j), finding the twisting data (a_j) is equivalent to classifying K3 fibred Calabi–Yau threefolds with orbifold singularities and a finite number of isolated singularities along the base locus of $|-K_{\mathbb{F}(a_j|b_j)}|$.

5.1 Calabi–Yau threefolds with quartic K3 fibres

5.1.1 Construction

By the standard isomorphism theorem and the assumption that $a_1 = 0$, let

$$\mathbb{F}(a_1,a_2,a_3,a_4) \cong \mathbb{F}(0,a_2,a_3,a_4)$$

be a 4-fold scroll over $\mathbb{P}^1$ with $a_4 \geq a_3 \geq a_2 \geq 0$. Assuming that the anticanonical linear system

$$|-K_{\mathbb{F}(0,a_2,a_3,a_4)}| = |L_{2-a_2-a_3-a_4,4}| \neq \emptyset$$

is nonempty, we take a general 3-fold

$$X = \mathbb{V}\left(\sum_{(q_1,q_2,q_3,q_4)\vdash 4} \alpha_{(q_j)}(t_1,t_2)x_1^{q_1}x_2^{q_2}x_3^{q_3}x_4^{q_4}\right) \subset \mathbb{F}(0,a_2,a_3,a_4)$$

in $|L_{2-a_2-a_3-a_4,4}|$ with $x_1^{q_1}x_2^{q_2}x_3^{q_3}x_4^{q_4}$ from the Newton tetrahedron in Figure (5.1).

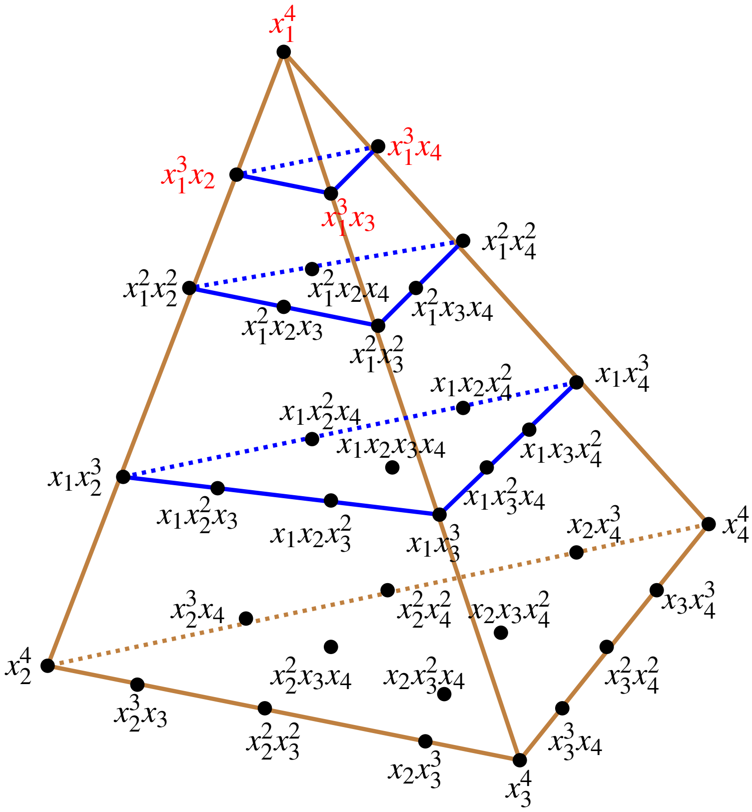


Figure 5.1: Newton tetrahedron of the 35 monomial basis of $\mathbb{C}[\mathbb{P}^3_{x_j}]_4$.

The corresponding degrees

$$\deg \alpha_{(q_j)} = 2 + (q_2 - 1)a_2 + (q_3 - 1)a_3 + (q_4 - 1)a_4$$

of coefficients $\alpha_{(q_j)}$ of the monomials $x_1^{q_1}x_2^{q_2}x_3^{q_3}x_4^{q_4}$ increase

1. by a_2 down the $x_1 - x_2$-edge, by a_3 down the $x_1 - x_3$-edge and by a_4 down the $x_1 - x_4$-edge,
2. by $a_3 - a_2$ from the $x_1 - x_2$-edge to the $x_1 - x_3$-edge,
3. by $a_4 - a_3$ from the $x_1 - x_3$-edge to the $x_1 - x_4$-edge,
4. by $a_4 - a_2$ from the $x_1 - x_2$-edge to the $x_1 - x_4$-edge.

The 3-fold X fits in the diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{F}(0,a_2,a_3,a_4) \\ & \searrow^{\varphi} & \downarrow \pi \\ & & \mathbb{P}^1_{[t_i]}. \end{array}$$

The fibres of the fibration

$$\varphi : X \to \mathbb{P}^1$$

induced by the fibration π are quartic surfaces

$$C_{4,t} = \mathbb{V}(f(t_i, x_4)) \subset \mathbb{P}^3_{[x_j]},\ t \in \mathbb{P}^1_{[t_i]}$$

with

$$f(t_i, x_j) = \sum_{(q_1,q_2,q_3,q_4)\vdash 4} \alpha_{(q_1,q_2,q_3,q_4)}([t_1 : t_2]) x_1^{q_1} x_2^{q_2} x_3^{q_3} x_4^{q_4} \in \mathbb{C}[\mathbb{P}^3_{[x_j]}]_4 \text{ for fixed } t = [t_1 : t_2].$$

The following theorem characterizes quartic fibred Calabi–Yau threefolds with at most orbifold singularities and a finite number of isolated singularities along the base locus of $|-K_{\mathbb{F}(a_j)}|$..

Theorem 5.1.1. *There are exactly* 10 *families of quartic-fibred CY3 embedded in* 4*-fold straight scrolls* $\mathbb{F}(0, a_2, a_3, a_4)$ *as anticanonical hypersurfaces and whose general member is either nonsingular or has only isolated singularities. The Table* (5.1) *below gives a reason why each family is either nonsingular or has only isolated singularities. The table also shows the dimension of the moduli space of embedded deformations.*

No.	$\mathbb{F} = \mathbb{F}(0, a_2, a_3, a_4)$	General $X \in \lvert -K_{\mathbb{F}}\rvert$ is nonsingular or with mild isolated singularities	$\dim(\mathscr{M}_{-K_{\mathbb{F}}})$
1	$\mathbb{F}(0,0,0,0)$	$-K_{\mathbb{F}}$ is base-point-free with a general X nonsingular	86
2	$\mathbb{F}(0,0,0,1)$	$-K_{\mathbb{F}}$ is base-point-free with a general X nonsingular	118
3	$\mathbb{F}(0,0,0,2)$	$-K_{\mathbb{F}}$ is base-point-free with a general X nonsingular	83
4	$\mathbb{F}(0,0,1,1)$	$-K_{\mathbb{F}}$ is base-point-free with a general X nonsingular	86
5	$\mathbb{F}(0,0,1,2)$	General $X \in \lvert -K_{\mathbb{F}}\rvert$ has 3 isolated ODP singularities on $Bs(\lvert -K_{\mathbb{F}}\rvert)$	86
6	$\mathbb{F}(0,0,2,2)$	General $X \in \lvert -K_{\mathbb{F}}\rvert$ is nonsingular	91
7	$\mathbb{F}(0,1,1,1)$	General $X \in \lvert -K_{\mathbb{F}}\rvert$ is nonsingular	73
8	$\mathbb{F}(0,1,1,2)$	General $X \in \lvert -K_{\mathbb{F}}\rvert$ is nonsingular	86
9	$\mathbb{F}(0,1,1,3)$	General $X \in \lvert -K_{\mathbb{F}}\rvert$ is nonsingular	89
10	$\mathbb{F}(0,1,1,4)$	General $X \in \lvert -K_{\mathbb{F}}\rvert$ is nonsingular	95

Table 5.1: $\mathbb{F} = \mathbb{F}(0, a_2, a_3, a_4)$ for which a general $X \in |-K_{\mathbb{F}}|$ has at most threefold Ordinary Double Point singularities along $Bs(|-K_{\mathbb{F}}|)$.

Proof. If X is nonsingular, its canonical divisor class K_X is trivial by adjunction and the map $\varphi : X \to \mathbb{P}^1$ is a fibration by quartic surfaces $C_{4,t} \subset \mathbb{P}^3$.

By Proposition (3.5.4), base point freeness of $-K_{\mathbb{F}} = L_{2-a_2-a_3-a_4,4}$ is equivalent to the Inequalities

$$\begin{aligned} 2 - a_2 - a_3 + 3a_4 &\geq 0, \\ 2 - a_2 + 3a_3 - a_4 &\geq 0, \\ 2 + 3a_2 - a_3 - a_4 &\geq 0, \\ 2 - a_2 - a_3 - a_4 &\geq 0. \end{aligned} \tag{5.1}$$

The Inequalities (5.1) then correspond to smooth threefolds families $X \in |-K_{\mathbb{F}(a_1,a_2,a_3,a_4)}|$ with the four-fold scrolls

$$\mathbb{F}(a_1,a_2,a_3,a_4) = \mathbb{F}(0,0,0,0),\ \mathbb{F}(0,0,0,1), \mathbb{F}(0,0,0,2) \text{ and } \mathbb{F}(0,0,1,1).$$

These are Calabi–Yau threefolds by adjunction.

We would, as in the previous discussions, like to find a weaker condition than base point freeness of $-K_{\mathbb{F}(a_1,a_2,a_3,a_4)} = L_{2-a_2-a_3-a_4,4}$ that would result in a not-too-singular X. That is, assuming $Bs(|L_{2-a_2-a_3-a_4,4}|) \neq \emptyset$ we would like to find explicit sections of $|L_{2-a_2-a_3-a_4,4}|$ that have isolated singularities along the base locus; resulting in an X that is not too singular in a neighbourhood of the base locus of $|L_{2-a_2-a_3-a_4,4}|$.

The base locus $Bs(|L_{2-a_2-a_3-a_4,4}|)$ is of dimension at most three by Proposition (3.6.1) and is defined by setting all or some of the variables to zero. The worst case, $\dim(Bs(L_{2-a_2-a_3-a_4,3})) = 3$, would give a singular

$$X = \mathbb{V}(f_4) = X' \cup Bs(|L_{2-a_2-a_3-a_4,4}|)$$

where $Bs(|L_{2-a_2-a_3-a_4,4}|) = \{x_i = 0\} \subset \mathbb{F}(0,a_2,a_3,a_4)$. This would happen when $f_4 = x_i h$ is reducible with $h(t_i,x_j)$ a cubic in x_j where X' is the cubic $X' = \mathbb{V}(h)$. For X to be not too singular, we must therefore have $x_4 \not| f_4$, $x_3 \not| f_4$, $x_2 \not| f_4$ and $x_1 \not| f_4$. For a generic choice of the coefficients $\alpha_{(q_1,q_2,q_3,q_4)\vdash 4}(t_i)$, we also have that $t_i \not| f(t_i,x_j)$. The condition for this is that at least one of the monomials on the $x_1 - x_2 - x_3$ face of the Newton tetrahedron in Figure (5.1) occurs in $f(t_i,x_j)$. It therefore suffices to have $\alpha_{(0,0,4,0)}(t_i)x_3^4$ in $f(t_i,x_j)$. The equivalent condition is that the largest corresponding degree of coefficients of the 15 monomials $x_1^{q_1}x_2^{q_2}x_3^{q_3}x_4^{q_4}$ in the $x_1 - x_2 - x_3$-face should be nonnegative

$$2 - a_2 + 3a_3 - a_4 \geq 0. \tag{5.2}$$

Suppose $\dim(Bs(|L_{2-a_2-a_3-a_4,4}|)) = 2$. The inequality (5.2) implies that the equation of a general section of $L_{2-a_2-a_3-a_4,4}$ simultaneously has nonzero terms involving x_3^4 and x_4^3 hence

$$f(t_i,x_j)|_{\{x_i=x_j=0\}} \not\equiv 0$$

for all $i,j < 3$. Therefore of the six surfaces $\{x_i = x_j = 0 : 1 \leq i < j \leq 4\}$, the base locus is

$$Bs(|L_{2-a_2-a_3-a_4,4}|) = \{x_3 = x_4 = 0\} = D_{34}.$$

This is equivalent to there being no terms on the $x_1 - x_2$-edge of the Newton tetrahedron (5.1) in $f(t_i,x_j)$; that is, there are no $x_1^k x_2^{4-k}$ terms in $f(t_i,x_j)$ for all $k = 0,1,2,3,4$ for if otherwise $f(t_i,x_j)|_{x_3=x_4=0} \not\equiv 0$ which contradicts our interest for X_{34} to be a base locus of $|L_{2-a_2-a_3-a_4,4}|$. Since x_2^4 has the coefficient of highest degree of the five terms, we equivalently have that there is no x_2^4 term in $f(t_i,x_j)$ and that

$$2+3a_2-a_3-a_4<0 \tag{5.3}$$

Further, with $f(t_i,x_j)$ expressed as a linear combination of monomials in the Newton tetrahedron in Figure (5.1) with corresponding coefficients $\alpha_{(q_j)}(t_1,t_2)$, it is then observed from

$$\left.\frac{\partial f(t_i,x_j)}{\partial x_4}\right|_{x_3=x_4=0} = \alpha_{3001}(t_1,t_2)x_1^3+\alpha_{2101}(t_1,t_2)x_1^2x_2+\alpha_{1201}(t_1,t_2)x_1x_2^2+\alpha_{0301}(t_1,t_2)x_2^3 \text{ and} \tag{5.4}$$

$$\left.\frac{\partial f(t_i,x_j)}{\partial x_3}\right|_{x_3=x_4=0} = \alpha_{3010}(t_1,t_2)x_1^3+\alpha_{2110}(t_1,t_2)x_1^2x_2+\alpha_{1210}(t_1,t_2)x_1x_2^2+\alpha_{0310}(t_1,t_2)x_2^3 \tag{5.5}$$

that

$$\begin{aligned} Sing(X)\cap X_{34} &= \mathbb{V}\left(f,\frac{\partial f}{\partial t_1},\frac{\partial f}{\partial t_2},\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3},\frac{\partial f}{\partial x_4},x_3,x_4\right) \\ &= \mathbb{V}\left(\frac{\partial f}{\partial x_3},\frac{\partial f}{\partial x_4},x_3,x_4\right),\ \text{since } f,\frac{\partial f}{\partial t_1},\frac{\partial f}{\partial t_2},\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}\in<x_3,x_4> \end{aligned} \tag{5.6}$$

and where $f=f(t_i,x_j)$ under the assumption of Inequality (5.3).

We also note that, for X to have isolated singularities, then the first term of Equation (5.4) and the last term of Equation (5.5) must be nonzero so that both $x_1,x_2 \not| \frac{\partial f(t_i,x_j)}{\partial x_4}$ and $x_1,x_2 \not| \frac{\partial f(t_i,x_j)}{\partial x_3}$. Equivalently,

$$2-a_2-a_3\geq 0 \text{ and } 2+2a_2-a_4\geq 0. \tag{5.7}$$

Therefore, with the condition that $a_4\geq a_3\geq a_2\geq 0$, Inequalities (5.2), (5.3) and (5.7) result in the lattice point (a_2,a_3,a_4) corresponding to a 4-fold scroll

$$\mathbb{F}(0,a_2,a_3,a_4)=\mathbb{F}(0,0,1,2) \text{ and } \mathbb{F}(0,0,2,2)$$

in which Calabi–Yau threefolds with isolated singularities are embedded.

For these scroll $\mathbb{F}(0,0,1,2)$ and $\mathbb{F}(0,0,2,2)$ the degrees of coefficients of the monomials $x_1^{q_1}x_2^{q_2}x_3^{q_3}x_4^{q_4}$ are given in Figures 5.2 and 5.3 .

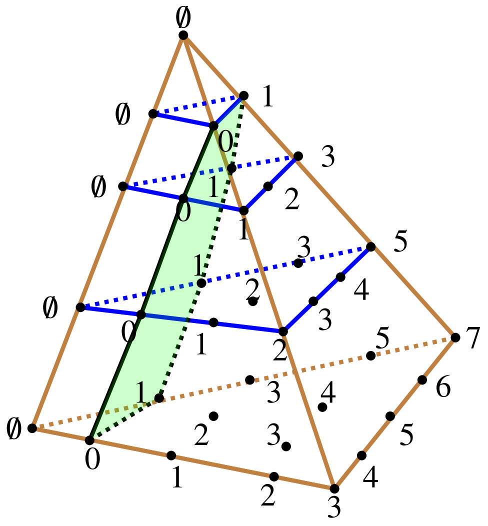


Figure 5.2: Degrees of $\alpha_{(q_j)}(t_i)$ of $f(t_i,x_j)$ for $X = \mathbb{V}(f(t_i,x_j)) \subset \mathbb{F}(0,0,1,2)$

and

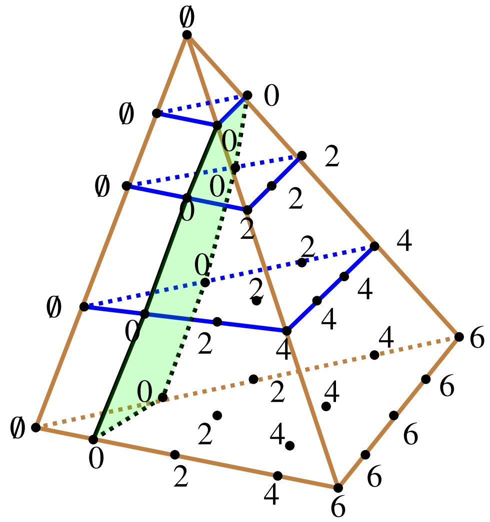


Figure 5.3: Degrees of $\alpha_{(q_j)}(t_i)$ of g_4 for $Y = \mathbb{V}(g_4) \subset \mathbb{F}(0,0,2,2)$

For $\mathbb{F}(0,0,1,2)$, the corresponding threefold is

$$X = \mathbb{V}(f(t_i,x_j)) = \mathbb{V}(c_1x^{(3010)} + c_2x^{(2110)} + c_3x^{(1210)} + c_4x^{(0310)} + \tag{5.8}$$
$$\alpha_1(t_i)x^{(3001)} + \alpha_1(t_i)x^{(2101)} + \alpha_1(t_i)x^{(1201)} + \alpha_1(t_i)x^{(0301)} + \alpha_1(t_i)x^{(2020)} + \alpha_1(t_i)x^{(1120)} +$$
$$\alpha_1(t_i)x^{(0220)} + \alpha_2(t_i)x^{(2011)} + \alpha_2(t_i)x^{(1111)} + \alpha_2(t_i)x^{(1030)} + \alpha_2(t_i)x^{(0130)} +$$
$$\alpha_2(t_i)x^{(0211)} + \alpha_3(t_i)x^{(1021)} + \alpha_3(t_i)x^{(1102)} + \alpha_3(s)x^{(1201)}\alpha_3(t_i)x^{(0202)} + \alpha_3(t_i)x^{(0121)} +$$
$$\alpha_4(t_i)x^{(1012)} + \alpha_4(t_i)x^{(0112)} + \alpha_4(t_i)x^{(0031)} + \alpha_5(t_i)x^{(2002)} + \alpha_5(t_i)x^{(1003)} + \alpha_5(t_i)x^{(0022)} +$$
$$\alpha_6(t_i)x^{(0013)} + \alpha_7(t_i)x^{(0103)}), \quad \text{with general } c_j \in \mathbb{C}.$$

The threefold isolated singularities are

$$Sing(X) \cap D_{34} = \mathbb{V}\left(\frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, x_3, x_4\right)$$
$$= \mathbb{V}(c_1x^{(3000)} + c_2x^{(2100)} + c_3x^{(1200)} + c_4x^{(0300)},$$
$$\alpha_1(t_i)x^{(3000)} + \alpha_1(t_i)x^{(2100)} + \alpha_1(t_i)x^{(1200)} + \alpha_1(t_i)x^{(0300)})$$
$$= \{d_i = [\gamma_{1i} : \gamma_{2i}; \beta_{1i} : \beta_{2i} : 0 : 0] : i = 1,2,3 = (0,3).(1,3)\} \subset D \subset X \subset \mathbb{F}(0,0,1,2)$$

where $([\gamma_{1i} : \gamma_{2i}], [\beta_{1i} : \beta_{2i}]) \in D_{34}$ satisfying

$$c_1\beta_{1i}^3 + c_2\beta_{1i}^2\beta_{2i} + c_3\beta_{1i}\beta_{2i}^2 + c_4\beta_{2i}^3 = 0;$$

that is $[\beta_{1i}:\beta_{2i}]$ is one of the 3 roots of a general homogeneous cubic in x_1,x_2 and

$$\alpha_1(\gamma_{1i},\gamma_{2i})\beta_{1i}^3+\alpha_1(\gamma_{1i},\gamma_{2i})\beta_{1i}^2\beta_{2i}+\alpha_1(\gamma_{1i},\gamma_{2i})\beta_{1i}\beta_{2i}^2+\alpha_1(\gamma_{1i},\gamma_{2i})\beta_{2i}^3=0,$$

a linear equation in t_1,t_2 for a fixed $[\beta_{1i}:\beta_{2i}]$.

Now, locally on the chart $U_{12}=\{t_1=x_2=1\}\cong\mathbb{A}^4_{t_2,x_1,x_3,x_4}$, a local change of coordinates $t_2=t_2'+\gamma_i,\ x_1=x_1'+\beta_{1i}$ results in

$$\begin{aligned}X\cap U_{12}=&\mathbb{V}(x_3(c_1(x_1'+\beta_{1i})^3+c_2(x_1'+\beta_{1i})^2+c_3(x_1'+\beta_{1i})+c_4)+x_4(\alpha_1(t_2'+\gamma_i)(x_1'+\beta_{1i})^3+\\&\alpha_1(t_2'+\gamma_i)(x_1'+\beta_{1i})^2+\alpha_1(t_2'+\gamma_i)(x_1'+\beta_{1i})+\alpha_1(t_2'+\gamma_i))+\text{ higher order terms})\\=&\mathbb{V}(\delta_1x_1'x_3+x_4(\delta_2t_2'+\delta_3x_1')+\text{ higher order terms}),\ \text{let } t_2'':=\delta_2t_2'+\delta_3x_1'\\=&\mathbb{V}(x_1'x_3+x_4t_2''+\text{ higher order terms})\subset\mathbb{A}^4_{(t_2'',x_1',x_3,x_4)}\subset\mathbb{F}(0,0,1,2),\end{aligned}$$

with a full rank quadratic part $f_2=x_1'x_3+x_4t_2''$. Therefore, up to higher order terms, the Calabi–Yau threefold X has 3 isolated threefold Ordinary Double Point (ODP) singularities $\{d_i\}$.

On the other hand, for $\mathbb{F}(0,0,2,2)$, the corresponding threefold is

$$\begin{aligned}Y=\mathbb{V}(g_4)=&\mathbb{V}(d_1x^{(3010)}+d_2x^{(2110)}+d_3x^{(1210)}+d_4x^{(0310)}+d_5x^{(3001)}+\\&d_6x^{(2101)}+d_7x^{(1201)}+d_8x^{(0301)}+\alpha_2(t_i)x^{(2020)}+\alpha_2(t_i)x^{(2011)}+\alpha_2(t_i)x^{(2002)}+\alpha_2(t_i)x^{(1120)}+\\&\alpha_2(t_i)x^{(1111)}+\alpha_2(t_i)x^{(1102)}+\alpha_2(t_i)x^{(0220)}+\alpha_2(t_i)x^{(0211)}+\alpha_2(t_i)x^{(0202)}+\alpha_4(t_i)x^{(1030)}+\\&\alpha_4(t_i)x^{(1021)}+\alpha_4(t_i)x^{(1012)}+\alpha_4(t_i)x^{(1003)}+\alpha_4(t_i)x^{(0130)}+\alpha_4(t_i)x^{(0121)}+\alpha_4(t_i)x^{(0112)}+\alpha_4(t_i)x^{(0103)}+\\&\alpha_6(t_i)x^{(0040)}+\alpha_6(t_i)x^{(0031)}+\alpha_6(t_i)x^{(0022)}+\alpha_6(t_i)x^{(0013)}+\alpha_6(t_i)x^{(0004)}),\quad\text{with general } d_k\in\mathbb{C}.\end{aligned}\tag{5.9}$$

In this case, we have that

$$\begin{aligned}&Sing(Y)\cap Y_{34}\cap U_{12}=\mathbb{V}\left(\frac{\partial f}{\partial x_3},\frac{\partial f}{\partial x_4},x_3,x_4\right)\\&=\{d_1x^{(3000)}+d_2x^{(2100)}+d_3x^{(1200)}+d_4x^{(0300)}=0,\\&d_5x^{(3000)}+d_6x^{(2100)}+d_7x^{(1200)}+d_8x^{(0300)}=0\}=\emptyset\subset\mathbb{F}(0,0,2,2).\end{aligned}$$

This gives a family $Y\subset\mathbb{F}(0,0,2,2)$ of nonsingular Calabi–Yau 3-folds.

Suppose $\dim(Bs(|L_{2-a_2-a_3-a_4,4}|))=1$. Since $g_4|_{x_1=1}\not\equiv0$, a general section of $L_{2-a_2-a_3-a_4,4}$ must **have no** $\alpha_{(4000)}(t_i)x_1^4$ term, **there is** $\alpha_{(0400)}(t_i)x_2^4$ and at least one of

$$\alpha_{(3100)}(t_2)x_1^3x_2,\alpha_{(3010)}(t_2)x_1^3x_3,\ \text{or}\ \alpha_{(3001)}(t_2)x_1^3x_4$$

appears in $f(t_i, x_j)$ or equivalently

$$2 - a_2 - a_3 - a_4 < 0, \; 2 + 3a_2 - a_3 - a_4 \geq 0 \; \text{ and } 2 - a_2 - a_3 \geq 0. \tag{5.10}$$

This would mean

$$f(t_i, x_j)|_{\{x_2 = x_3 = x_4 = 0\}} \equiv 0$$

so that the one-dimensional base locus is

$$X_{234} = \{x_2 = x_3 = x_4 = 0\} \subset X = \mathbb{V}(f(t_i, x_j)) \subset \mathbb{F}(0, a_2, a_3, a_4).$$

Along the curve X_{234} and on the chart $U_{11} = \{t_1 = x_1 = 1\}$ we have that

$$\begin{aligned} Sing(X) \cap X_{234} \cap U_{11} &= \mathbb{V}\left(\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, x_2, x_3, x_4\right) \\ &= \mathbb{V}(\alpha_{(3100)}(t_2), \alpha_{(3010)}(t_2), \alpha_{(3001)}(t_2)) \subset \mathbb{A}^1_{t_2}. \end{aligned}$$

By assuming non-negativity of one, two, or all of the polynomials $\alpha_{(q_j)}(t_2)$ defining $Sing(X) \cap X_{234} \cap U_{11}$, it is the case when the generic choice of the polynomial $\alpha_{(3001)}(t_2)$ is a nonzero constant and in particular when

$$\deg \alpha_{(3001)}(t_2) = 2 - a_2 - a_3 = 0$$

which results in $Sing(X) \cap X_{234} \cap U_{11}$ being empty. This results in 4 families of nonsingular Calabi–Yau threefolds embedded in fourfold scrolls

$$\mathbb{F}(0, a_2, a_3, a_4) = \mathbb{F}(0,1,1,1), \; \mathbb{F}(0,1,1,2), \; \mathbb{F}(0,1,1,3) \text{ and } \mathbb{F}(0,1,1,4).$$

From definition (3.11.1), we have that $\dim(\mathscr{M}_{-K_{\mathbb{F}}})$ is given by

$$\begin{aligned} &= -1 + \sum_{(q_j) \vdash 4, \; \deg \alpha_{(q_j)} \geq 0} (1 + \deg \alpha_{(q_j)}(t_i)) - \dim \mathrm{Aut}(\mathbb{F}) \\ &= -1 + \sum_{(q_j)_{(b_j)} \vdash d_2} (d_1 + \sum_{i=1}^{n} a_i q_i) - \dim \mathrm{Aut}(\mathbb{P}^1) - \dim \mathrm{Aut}(\mathbb{F}/\mathbb{P}^1). \end{aligned}$$

We then have, for instance, that:

1. For the family of $X_{2,4} \subset \mathbb{F}(0,0,0,0) = \mathbb{P}^1 \times \mathbb{P}^3$, we have that $\deg \alpha_{(q_j)}(t_i) = 2$ for all the 35 monomials of Newton tetrahedron (5.1) . We then compute that

$$\begin{aligned} \dim(\mathscr{M}_{-K_{\mathbb{F}}}) &= -1 + (2 \times 35 + 35) - \dim \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^3) \\ &= 104 - \dim PGL(2) - \dim PGL(4) \\ &= 104 - (4-1) - (16-1) = 86 \end{aligned}$$

2. Consider the family of $X \subset \mathbb{F}(0,0,0,1)$. We know that on the Newton tetrahedron (5.1), the degrees satisfy

   (i) $\deg \alpha_{(q_j)}(t_i) = 2$ on the $x_1 - x_2 - x_3$ face.

   (ii) $\deg \alpha_{(q_j)}(t_i)$ increase by 1 : along $x_1 - x_3$-edge, from the $x_1 - x_3$-edge to the $x_1 - x_4$-edge and from the $x_1 - x_2$-edge to the $x_1 - x_4$-edge.

   We evaluate $\dim \mathrm{Aut}(\mathbb{F}(0,0,0,1)/\mathbb{P}^1)$ by packaging the parameters on the fibres of $\pi : \mathbb{F}(0,0,0,1) \to \mathbb{P}^1$ in a 4×4 matrix counting the $\mathbb{C}^*$ action on the projective bundle
$$\mathbb{F}(0,0,0,1) \cong \mathbb{F}(-1,-1,-1,0) \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \oplus \mathscr{O}_{\mathbb{P}^1}).$$
   Hence, $\dim(\mathscr{M}_{-K_{\mathbb{F}}})$
$$\begin{aligned}
&= -1 + (2 \times 15 + 3 \times 10 + 4 \times 6 + 5 \times 3 + 6 + 35) - \dim \mathrm{Aut}(\mathbb{F}(0,0,0,1)) \\
&= 139 - \dim \mathrm{Aut}(\mathbb{P}^1) - \dim \mathrm{Aut}(\mathbb{F}(0,0,0,1)/\mathbb{P}^1) \\
&= 139 - \dim \mathrm{Aut}(\mathbb{P}^1) - \text{sum of entries of } \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 1 \\
&= 139 - (4-1) - (19-1) = 118
\end{aligned}$$

3. Consider the family of $X \subset \mathbb{F}(0,1,1,2) \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1})$, we have that $\dim(\mathscr{M}_{-K_{\mathbb{F}}})$
$$\begin{aligned}
&= -1 + (6 + 2 \times 9 + 3 \times 6 + 4 \times 4 + 5 \times 2 + 6 + 32) - \dim \mathrm{Aut}(\mathbb{F}(0,1,1,2)) \\
&= 105 - \dim \mathrm{Aut}(\mathbb{P}^1) - \dim \mathrm{Aut}(\mathbb{F}(0,1,1,2)/\mathbb{P}^1) \\
&= 105 - \dim \mathrm{Aut}(\mathbb{P}^1) - \text{sum of entries of } \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 1 \\
&= 105 - (4-1) - (17-1) = 86.
\end{aligned}$$

□

In Theorem (5.1.1), we have found a new singular family in $|-K_{\mathbb{F}(0,0,1,2)}|$ as well as recovered the first list in the Appendix A of [Mul06]; that is the 9 smooth quartic fibred Calabi–Yau threefolds and a new interesting $X \in |-K_{\mathbb{F}(0,0,1,2)}|$ with three isolated Ordinary Double Points $\{d_i\}$. This result is achieved by allowing isolated singularities along the base locus of $Bs(|-K_{\mathbb{F}(0,0,1,2)}|) = D_{34}$.

It is also worth noting that the Weil divisor $D_{34} \subset X \subset \mathbb{F}(0,0,1,2)$ passes through all the three isolated ODP singularities $\{d_i\}$. We then have a projective small resolution

$$f : Bl_{D_{34}}X = \widehat{X} \to X$$

with $K_{\widehat{X}} = \mathscr{O}_{\widehat{X}}$. Indeed, we have seen from the proof above that $X \subset \mathbb{F}(0,0,1,2)$ is a quadric 3-fold cone up to higher order terms; one can then check locally at $\{d_i\}$ that such a threefold cone admits a small (crepant) resolution [Alt98].

On the other hand, we now re-visit Gross's [Gro97] example on Calabi–Yau threefold family number 8 in the table above. That is

$$X \subset \mathbb{F}(0,1,1,2) \cong \mathbb{F}(-1,0,0,1) = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)).$$

This family has also been studied in [CDT18, Tho00] and Appendix of [Ruan96] . The following treatment uses explicit realization of the Calabi–Yau threefold as a hypersurface $X \subset \mathbb{F}(0,1,1,2)$ to study its intrinsic geometry including deformations and degenerations.

5.1.2 Deformation

We start with an informative example of degeneration of curves in surface scroll fibrations.

Example 5.1.2. Consider the family

$$\mathscr{F} = \mathbb{V}(y_1y_3 - y_2^2 + t^2y_0^2) \subset \mathbb{A}_t^1 \times \mathbb{P}^3_{[y_k]}$$

of quadric surfaces over $\mathbb{A}_t^1$. These are deformations of the image $\overline{F_2} = \mathbb{V}(y_1y_3 - y_2^2)$ of

$$\begin{aligned} F_2 := \mathbb{F}(0,2) &\xrightarrow{\varphi} \mathbb{P}^3_{[y_0:y_1:y_2:y_3]} \\ [t_i;x_j] &\mapsto [x_1 : t_1^2x_2 : t_1t_2x_2 : t_2^2x_2]. \end{aligned}$$

We know that φ is the blowup map of the singular cone point $p = [1:0:0:0]$ of the quadric $\mathscr{F}_0$. Now, by deforming $\mathscr{F}_0$ to $\mathscr{F}_{t\neq 0} = \mathbb{V}(y_1y_3 - y_2^2 + t^2y_0^2)$ followed by a change of variables

$$(y_1, y_2 + ty_0, y_2 - ty_0, y_3) = (z_{11}, z_{12}, z_{21}, z_{22}) \text{ for } t \in \mathbb{C}^*,$$

the deformed variety becomes $\mathrm{Im}(\Sigma_{1,1}) = \Sigma_{1,1}(F_0) = \mathbb{V}(z_{11}z_{22} - z_{12}z_{21})$ where

$$\Sigma_{1,1} : F_0 = \mathbb{P}^1_{[u_i]} \times \mathbb{P}^1_{[v_j]} \to \mathbb{P}^3_{[z_{ij}]}$$

is the Segre embedding.

Blowing up the ordinary double point singularity $(0,p)$ in the threefold total space $\mathscr{F}$, we obtain the deformation $\widehat{\mathscr{F}} \to \mathbb{A}^1_t$ of surface scroll F_2 to F_0. This deformation is formed from the family $\mathscr{F} \to \mathbb{A}^1_t$ with the central fibre $\widehat{\mathscr{F}_0} = F_2 = \varphi^{-1}(\mathscr{F}_0)$.

Consider a nonsingular anticanonical curve

$$\mathbb{V}(g_{2,2}(u_i;v_j)) \subset |-K_{F_0}| = |\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)|.$$

We degenerate to $t=0$, the elliptic curve $\mathbb{V}(g_{2,2}(u_i;v_j))$ using the change of variables

$$(z_{11},z_{12},z_{21},z_{22}) = (y_1, y_2+ty_0, y_2-ty_0, y_3) \text{ for } t\in\mathbb{C}^*$$

to get

$$\begin{aligned}\mathbb{V}(g_{2,2}(u_i;v_j) =& \mathbb{V}(z_{11}^2+c_1z_{11}z_{12}+c_2z_{12}^2+c_3z_{11}z_{21}+c_4z_{11}z_{22}+c_5z_{12}z_{22}+c_6z_{21}^2+c_7z_{21}z_{22}+\\ & c_8z_{22}^2)\cap \mathrm{Im}(\Sigma_{1,1})\\ =& \mathbb{V}(y_1^2+c_1y_1(y_2+ty_0)+c_2(y_2+ty_0)^2+c_3y_1(y_2-ty_0)+c_4y_1y_3+c_5(y_2+ty_0)y_3+\\ & c_6(y_2-ty_0)^2+c_7(y_2-ty_0)y_3+c_8y_3^2)\\ \overset{t\mapsto 0}{\rightsquigarrow}& \mathbb{V}(y_1^2+(c_1+c_3)y_1y_2+(c_2+c_4+c_6)y_2^2+(c_5+c_7)y_2y_3+c_8y_3^2)\\ =& \mathbb{V}(y_1^2+(c_1+c_3)y_1y_2+(c_2+c_4+c_6)y_2^2+(c_5+c_7)y_2y_3+c_8y_3^2)\\ =& \mathbb{V}(\varphi(t_1^4x_2^2+(c_1+c_3)t_1^3t_2x_2^2+(c_2+c_4+c_6)t_1^2t_2^2x_2^2+(c_5+c_7)t_1t_2^3x_2^2+c_8t_2^4x_2^2))\\ =& \varphi(\mathbb{V}(f_{0,2}(t_i;x_j)))\subset\mathscr{F}_0.\end{aligned}$$

The anticanonical curve $\mathbb{V}(f_{0,2}(t_i;x_j)) \subset \mathscr{F}_0$ is singular at $p=[1:0:0:0]$.
We note that the moduli space $\mathscr{M}_{-K_{\mathscr{F}_{t\neq 0}}}$ of nonsingular anticanonical curves in $\mathscr{F}_{t\neq 0} = \mathbb{P}^1\times\mathbb{P}^1$ is, as expected, of dimension

$$\dim(\mathscr{M}_{-K_{\mathscr{F}_{t\neq 0}}}) = 4-(4-1)-(1-1) = 1.$$

So in summary, we have a smooth specialisation of F_0 to F_2, but in this specialisation, the general anticanonical curve in F_0 specializes to a singular anticanonical curve in F_2. This can be smoothed inside F_2 to get a nonsingular anticanonical section. The threefold families to be studied next exhibit a similar phenomenon as shown below.

We would like to study the deformation $\mathscr{F}\to B$ of nonsingular quartic fibred Calabi–Yau threefold $X\in|-K_{\mathbb{P}(\mathscr{E}|_\zeta)}|$ with $\zeta\in B$,

$$\mathscr{E}|_\zeta = \mathscr{O}_{\mathbb{P}^1}\oplus\mathscr{O}_{\mathbb{P}^1}(a_2)\oplus\mathscr{O}_{\mathbb{P}^1}(a_3)\oplus\mathscr{O}_{\mathbb{P}^1}(a_4)$$

and $\mathbb{P}(\mathscr{E}|_\zeta) = \mathbb{F}(0,a_2,a_3,a_4)$.

Now, recall that the $\mathscr{O}_{\mathbb{P}^1}(1)$ twisted $\mathbb{P}^1$ Euler sequence is a nontrivial extension

$$0 \to \mathscr{O}_{\mathbb{P}^1}(-1) \to \mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^1}(1) \to 0.$$

Consider the universal extension

$$\mathscr{E} \to \mathbf{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}(-1)) \times \mathbb{P}^1 = \mathbb{A}^1_\zeta \times \mathbb{P}^1$$

then define a family with 5-dimensional total space $\mathscr{F}$

$$\begin{array}{ccc} \mathscr{F} := \mathbb{P}(\mathscr{E} \oplus \mathscr{O}_{\mathbb{A}^1_\zeta \times \mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{A}^1_\zeta \times \mathbb{P}^1}) & \longrightarrow & \mathbb{A}^1_\zeta \times \mathbb{P}^1 \\ & \searrow^{\pi} & \downarrow \\ & & \mathbb{A}^1_\zeta. \end{array}$$

The 4-dimensional fibres of π are $\mathscr{F}_\zeta = \mathbb{P}(\mathscr{E}|_\zeta)$. The fibres are

$$\mathscr{F}_0 = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)) = \mathbb{F}(-1,0,0,1) \cong \mathbb{F}(0,1,1,2)$$

and

$$\mathscr{F}_{\zeta \neq 0} \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}^{\oplus 4}) = \mathbb{F}(0,0,0,0) = \mathbb{P}^1_{[u_i]} \times \mathbb{P}^3_{[v_j]}.$$

Starting with a general, non-singular anticanonical section

$$X = X_{2,4} = \mathbb{V}\left(\sum_{(q_j) \vdash 4} \alpha_2(u_i) v_1^{q_1} v_2^{q_2} v_3^{q_3} v_4^{q_4} \right) \subset \mathscr{F}_\zeta = \mathbb{P}^1_{[u_i]} \times \mathbb{P}^3_{[v_j]},$$

we show that it becomes a general quadric $Q \subset \mathbb{P}^{19}_{[z_{f_1 f_2 jk}]}$ where $1 \leq j \leq k \leq 4$ and f_1, f_2 either 0 or 1. We then proceed to find a degeneration $\widetilde{Q} \subset \mathbb{P}^{19}_{[y_{e_1 e_2 mn}]}$ of Q along the degeneration $\mathscr{F} \to \mathbb{A}^1_\zeta$ of the scrolls $\mathscr{F}_\zeta = \mathbb{F}(0,0,0,0) \mapsto \mathbb{F}(0,1,1,2) = \mathscr{F}_0$. In this way, we would like to understand the specialization of $X_{2,4}$ as $\zeta \rightsquigarrow 0$.

Let's make the first steps by noting that both $\mathscr{F}_\zeta$ and $\mathscr{F}_0$ are mapped to $\mathbb{P}^{19}$ by sections of $|-\frac{1}{2}K_{\mathscr{F}_\zeta}| = |\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,2)|$ and $|-\frac{1}{2}K_{\mathscr{F}_0}| = |L_{-1,2}|$ respectively. We start by understanding the images of the two scrolls $\mathscr{F}_\zeta = \mathbb{P}^1 \times \mathbb{P}^3$ and $\mathscr{F}_0 = \mathbb{F}(0,1,1,2)$ under their half-canonical maps. Thereafter, we shall study the birational geometry of $\operatorname{Im} \varphi_{|-\frac{1}{2}K_{\mathscr{F}_0}|}$.

The map $\varphi_{|\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,2)|}$ in the Proposition below factors through the degree two Veronese map on the $\mathbb{P}^3_{[v_j]}$ factor of $\mathscr{F}_\zeta$. That is, the map $\mathrm{Id} \times v_2$ has the image

$$\mathbb{P}^1_{[u_i]} \times \left\{ [z_{jk}] : \operatorname{rank} \begin{vmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{12} & z_{22} & z_{23} & z_{24} \\ z_{13} & z_{23} & z_{33} & z_{34} \\ z_{14} & z_{24} & z_{34} & z_{44} \end{vmatrix} \leq 1 \right\}.$$

This image is then mapped through the Segre embedding on the $\mathbb{P}^1_{[u_i]}$ factor of $\mathscr{F}_\zeta$ to $\mathrm{Im}((\Sigma_{1,9} \times \mathrm{Id}) \circ (\mathrm{Id} \times \nu_2))$ which is given in the statement below.

Proposition 5.1.3. *The image of* $\varphi_{|\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,2)|}$

$$\begin{array}{ccc} \mathbb{P}^1_{[u_i]} \times \mathbb{P}^9_{[z_{jk}]} & & \\ \uparrow {\scriptstyle \mathrm{Id} \times \nu_2} & \searrow^{\Sigma_{1,9}} & \\ \mathscr{F}_\zeta = \mathbb{P}^1_{[u_i]} \times \mathbb{P}^3_{[v_j]} & \xrightarrow[{[u_i;v_j] \mapsto [\mathbf{Sym}^1(u_i) \otimes \mathbf{Sym}^2(v_j)]}]{\varphi_{|\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,2)|}} & \mathbb{P}^{19}_{[z_{f_1 f_2 jk}]} . \end{array}$$

is given by

$$\varphi_{|\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,2)|}(\mathscr{F}_\zeta) = \left\{ [M_1 \quad M_2] = [z_{f_1 f_2 jk}] : \mathit{rank} \left| M_1 \quad M_2 \right| \leq 1 \right\}.$$

Here, $1 \leq j \leq k \leq 4$ *and* f_1, f_2 *either* 0 *or* 1, *while*

$$M_1 = \begin{bmatrix} z_{1011} & z_{1012} & z_{1013} & z_{1014} \\ z_{1012} & z_{1022} & z_{1023} & z_{1024} \\ z_{1013} & z_{1023} & z_{1033} & z_{1034} \\ z_{1014} & z_{1024} & z_{1034} & z_{1044} \end{bmatrix}$$

and

$$M_2 = \begin{bmatrix} z_{0111} & z_{0112} & z_{0113} & z_{0114} \\ z_{0112} & z_{0122} & z_{0123} & z_{0124} \\ z_{0113} & z_{0123} & z_{0133} & z_{0134} \\ z_{0114} & z_{0124} & z_{0134} & z_{0144} \end{bmatrix},$$

so that $[N] = [M_1 \quad M_2]$ *can be identified with an element of* $\mathbb{P}^{19}$ *using coordinates* $z_{f_1 f_2 jk}$.

Proof. We first note that

$$\mathrm{Im}(\varphi_{|\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,2)|}) \subseteq (\Sigma_{1,9} \times \mathrm{Id}) \left(\mathbb{P}^1_{[u_i]} \times \left\{ [z_{jk}] : \mathrm{rank} \begin{vmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{12} & z_{22} & z_{23} & z_{24} \\ z_{13} & z_{23} & z_{33} & z_{34} \\ z_{14} & z_{24} & z_{34} & z_{44} \end{vmatrix} \leq 1 \right\} \right).$$

Multiplying each of the 24 two-by-two minors of z_{jk} by basis of $\mathbf{Sym}^2(u_1, u_2)$ gives the quadrics

$$z_{10jk} z_{10lm} = z_{10pq} z_{10rs} \text{ from rank } |M_1| \leq 1,$$

$$z_{01jk} z_{01lm} = z_{01pq} z_{01rs} \text{ from rank } |M_2| \leq 1$$

and

$$z_{10jk} z_{01lm} = z_{10pq} z_{01rs}, \; z_{01jk} z_{10lm} = z_{01pq} z_{10rs} \text{ from rank} |N| = \text{rank} \left| M_1 \quad M_2 \right| \leq 1$$

all of which are satisfied on the image. We have shown that the equations

$$\operatorname{rank}|N| = \operatorname{rank}\begin{vmatrix} M_1 & M_2 \end{vmatrix} \leq 1$$

are satisfied on the image of $\varphi_{|\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,2)|}$. These equations are equivalent to $\operatorname{rank}\begin{vmatrix} M_1 & M_2 \end{vmatrix} = 1$ since $[M_1 \quad M_2] \in \mathbb{P}^{19}$ so M_1 and M_2 are both 0. Now, using Macaulay2 computer algebra, we can check that

$$\left\{[N] = [M_1 \quad M_2] : \operatorname{rank}\begin{vmatrix} M_1 & M_2 \end{vmatrix} \leq 1\right\} \subset \mathbb{P}^{19}_{[z_{f_1 f_2 jk}]}$$

is an irreducible projective variety of dimension 4 and of degree 32. This degree was expected since, by direct computation with $LM^3 = 1$ and $L^2 = M^4 = 0$, we have that

$$(L+2M)^4 = L^4 + 8L^3M + 24L^2M^2 + 32LM^3 + 16M^4 = 32.$$

In conclusion, we have that

$$\operatorname{Im}(\varphi_{|\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,2)|}) = \left\{[M_1 \quad M_2] : \operatorname{rank}\begin{vmatrix} M_1 & M_2 \end{vmatrix} \leq 1\right\} \subset \mathbb{P}^{19}_{[z_{f_1 f_2 jk}]}.$$

□

On the other hand, the image of the map $\varphi_{|-\frac{1}{2}K_{\mathscr{F}_0}|}$ is given in the following statement.

Proposition 5.1.4. *Let* $[y_{e_1e_2mn}] = [t_1^{e_1}t_2^{e_2}x_mx_n]$ *be coordinates on* $\mathbb{P}^{19}$ *where* $1 \leq m \leq n \leq 4$ *and* e_1, e_2 *either* 0 *or* 1. *The map*

$$\begin{aligned} \varphi_{|L_{-1,2}|} : \mathscr{F}_0 &\dashrightarrow \mathbb{P}^{19}_{[y_{e_1e_2mn}]} \\ [t_i; x_j] &\mapsto [x_1x_2 : x_1x_3 : \mathbf{Sym}^1(t_i)x_1x_4 : \mathbf{Sym}^1(t_i) \otimes \mathbf{Sym}^2(x_2, x_3) : \mathbf{Sym}^2(t_i)x_2x_4 : \\ &\qquad \mathbf{Sym}^2(t_i)x_3x_4 : \mathbf{Sym}^3(t_i)x_4^2] \end{aligned}$$

is an isomorphism away from the curve $Bs(-K_{\mathscr{F}_0})$. *Here* $\mathscr{F}_0 = \mathbb{F}(0,1,1,2)$ *with coordinates* t_i, x_j *and the curve* $Bs(-K_{\mathscr{F}_0}) = \mathbb{V}(x_2, x_3, x_4) \cong \mathbb{P}^1$. *The closure of its image is*

$$\overline{\operatorname{Im}(\varphi_{|L_{-1,2}|})} = \mathbb{V}(I_B)$$

where

$$B = \begin{bmatrix} y_{0012} & y_{1022} & y_{1023} & y_{0122} & y_{0123} & y_{2024} & y_{1124} & y_{0224} \\ y_{0013} & y_{1023} & y_{1033} & y_{0123} & y_{0133} & y_{2034} & y_{1134} & y_{0234} \\ y_{1014} & y_{2024} & y_{2034} & y_{1124} & y_{1134} & y_{3044} & y_{2144} & y_{1244} \\ y_{0114} & y_{1124} & y_{1134} & y_{0224} & y_{0234} & y_{2144} & y_{1244} & y_{0234} \end{bmatrix}$$

and $I_B = \{\operatorname{rank}|B| = 1\}$

Proof. We look for all relations within and between blocks $\mathbf{Sym}^{-1+q_2+q_3+2q_4}(t_i)x_1^{q_1}x_2^{q_2}x_3^{q_3}x_4^{q_4}$ for every $(q_j) \vdash 2$ for which $-1+q_2+q_3+2q_4 \geq 0$. First note that, since $\mathscr{F}_0 = \mathbb{F}(0,1,1,2)_{t_i,x_j}$ with the weights

$$wt(t_1,t_2,x_1,x_2,x_3,x_4) = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\begin{aligned} S_{-2,2} &= \left\langle x_1x_4, x_2^2, x_2x_3, x_3^2, t_1x_2x_4, t_2x_2x_4, t_1x_3x_4, t_2x_3x_4, t_1^2x_4^2, t_1t_2x_4^2, t_2^2x_4^2 \right\rangle, \\ S_{-1,1} &= \langle x_2, x_3, t_1x_4, t_2x_4 \rangle \text{ and} \\ S_{0,1} &= \left\langle x_1, t_1x_2, t_1x_3, t_2x_2, t_2x_3, t_1^2x_4, t_1t_2x_4, t_2^2x_4 \right\rangle. \end{aligned}$$

From these bases, we can obtain the line bundle $L_{-1,2}$ in two ways by forming the matrices

$$\begin{aligned} A &= \begin{bmatrix} t_1S_{-2,2} \\ t_2S_{-2,2} \end{bmatrix} \\ &= \begin{bmatrix} y_{1014} & y_{1022} & y_{1023} & y_{1033} & y_{2024} & y_{1124} & y_{2034} & y_{1134} & y_{3044} & y_{2144} & y_{1244} \\ y_{0114} & y_{0122} & y_{0123} & y_{0133} & y_{1124} & y_{0224} & y_{1134} & y_{0234} & y_{2144} & y_{1244} & y_{0344} \end{bmatrix} \text{ and} \\ B &= \left[S_{-1,1} \otimes S_{0,1} \right] \\ &= \begin{bmatrix} y_{0012} & y_{1022} & y_{1023} & y_{0122} & y_{0123} & y_{2024} & y_{1124} & y_{0224} \\ y_{0013} & y_{1023} & y_{1033} & y_{0123} & y_{0133} & y_{2034} & y_{1134} & y_{0234} \\ y_{1014} & y_{2024} & y_{2034} & y_{1124} & y_{1134} & y_{3044} & y_{2144} & y_{1244} \\ y_{0114} & y_{1124} & y_{1134} & y_{0224} & y_{0234} & y_{2144} & y_{1244} & y_{0344} \end{bmatrix} = [c_0c_1 \ldots c_7] \end{aligned}$$

whose ranks are both 1 on the image. Therefore, the relations are

$$I_B = \{\text{rank}|B| = 1\} \text{ and } I_A = \{\text{rank}|A| = 1\}.$$

Since $I_A \subset I_B$, we have that

$$\overline{\text{Im}(\varphi_{|L_{-1,2}|})} \subseteq \mathbb{V}(I_B) = X'.$$

Finally, using Macaulay2 computer algebra, we find that in fact X' is a 4-dimensional irreducible projective variety of degree 32 so that

$$\overline{\text{Im}(\varphi_{|L_{-1,2}|})} = \mathbb{V}(I_B).$$

□

Now, to find a deformation of $\text{Im}(\varphi_{|L_{-1,2}|})$, we first note from Proposition (5.1.4) that, from $B = [b_0b_1 \ldots b_7]$, the matrices $[b_1b_2b_5b_6]$ and $[b_3b_4b_6b_7]$ are symmetric which is reminiscent of the matrices $M_1 = [m_1m_2m_3m_4]$ and $M_2 = [m_5m_6m_7m_8]$ in Proposition

(5.1.3) except for overlapping on the column b_6. We define the columns $n_4(t) = b_6 + tb_0$ and $n_7(t) = b_6 - tb_0$ for $t \in \mathbb{C}$ so that

$$N_1(t) = [n_1n_2n_3n_4(t)] = \begin{bmatrix} y_{1022} & y_{1023} & y_{2024} & y_{1124}+ty_{0012} \\ y_{1023} & y_{1033} & y_{2034} & y_{1134}+ty_{0013} \\ y_{2024} & y_{2034} & y_{3044} & y_{2144}+ty_{1014} \\ y_{1124}+ty_{0012} & y_{1134}+ty_{0013} & y_{2144}+ty_{1014} & y_{1244}+ty_{0114} \end{bmatrix}$$

and

$$N_2(t) = [n_5n_6n_7(t)n_8] = \begin{bmatrix} y_{0122} & y_{0123} & y_{1124}-ty_{0012} & y_{0224} \\ y_{0123} & y_{0133} & y_{1134}-ty_{0013} & z_{0234} \\ y_{1124}-ty_{0012} & y_{1134}-ty_{0013} & y_{2144}-ty_{1014} & y_{1244}-ty_{0114} \\ y_{0224} & z_{0234} & y_{1244}-ty_{1014} & y_{0344} \end{bmatrix}$$

are both symmetric.

In the long discussion below, we now use the two natural maps defined on scrolls in propositions above to describe a projective degeneration of quartic K3 families using bi-homogeneous coordinates. By first considering the half-anticanonical embedding of $\mathscr{F}_\zeta = \mathbb{P}^1 \times \mathbb{P}^3$

$$\phi_1 = \phi_{|-\frac{1}{2}K_{\mathbb{P}^1\times\mathbb{P}^3}|} : \mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^{19}$$

as defined above. Consider the matrix

$$M = \begin{bmatrix} z_{1011} & z_{1012} & z_{1013} & z_{1014} & z_{0111} & z_{0112} & z_{0113} & z_{0114} \\ z_{1012} & z_{1022} & z_{1023} & z_{1024} & z_{0112} & z_{0122} & z_{0123} & z_{0124} \\ z_{1013} & z_{1023} & z_{1033} & z_{1034} & z_{0112} & z_{0122} & z_{0123} & z_{0124} \\ z_{1014} & z_{1024} & z_{1034} & z_{1044} & z_{0114} & z_{0124} & z_{0134} & z_{0144} \end{bmatrix},$$

joined from two 4×4 symmetric matrices M_1 and M_2 with independent entries. Also consider the half-anticanonical map

$$\phi_0 = \phi_{|-\frac{1}{2}K_{\mathbb{F}}|} : \mathbb{F}(0,1,1,2) \dashrightarrow \mathbb{P}^{19}$$

on the scroll $\mathscr{F}_0 \cong \mathbb{F}(0,1,1,2)$ also defined before with the indeterminacy locus $Bs(|-K_{\mathscr{F}_0}|) = \mathbb{V}(x_2,x_3,x_4) \cong \mathbb{P}^1$. Consider the matrix

$$M_0 = \begin{bmatrix} y_{0012} & y_{1022} & y_{1023} & y_{0122} & y_{0123} & y_{2024} & y_{1124} & y_{0224} \\ y_{0013} & y_{1023} & y_{1033} & y_{0123} & y_{0133} & y_{2034} & y_{1134} & y_{0234} \\ y_{1014} & y_{2024} & y_{2034} & y_{1124} & y_{1134} & y_{3044} & y_{2144} & y_{1244} \\ y_{0114} & y_{1124} & y_{1134} & y_{0224} & y_{0234} & y_{2144} & y_{1244} & y_{0234} \end{bmatrix}.$$

We have proved the first 2 results in the proposition below. In the discussion following the proposition below, the rest of the results are proved.

Proposition 5.1.5.

1. *The image of ϕ_1 inside $\mathbb{P}^{19}$ is described (scheme-theoretically) by the ideal generated by 2×2 minors of the matrix M:*

$$\operatorname{Im} \phi_1 \cong \mathbb{V}\left(\wedge^2 M\right) \subset \mathbb{P}^{19}.$$

2. *The closure $Q = \overline{\operatorname{Im} \phi_0} \subset \mathbb{P}^{19}$ is described (scheme-theoretically) by the ideal generated by 2×2 minors of the matrix M_0:*

$$Q = \overline{\operatorname{Im} \phi_0} \cong \mathbb{V}\left(\wedge^2 M_0\right) \subset \mathbb{P}^{19}.$$

3. *There is a distinguished divisor $\mathbb{P}^3 \cong D \subset Q \subset \mathbb{P}^{19}$ embedded in $\mathbb{P}^{19}$ as a projective linear subspace, defined by the condition that all variables except those in the first column of M_0 vanish.*

4. *The Hilbert polynomials of $Q \subset \mathbb{P}^{19}$ and $\mathbb{F}(0,1,1,2) \subset \mathbb{P}^{19}$ agree.*

5. *Define the variety $\mathcal{Q} \subset \mathbb{P}^{19} \times \mathbb{A}^1$ over $\mathbb{A}^1$ by the equations*

$$\mathcal{Q} = \mathbb{V}\left(\wedge^2 N\right) \subset \mathbb{P}^{19} \times \mathbb{A}^1.$$

Then the natural map $\mathcal{Q} \to \mathbb{A}^1$ is a flat family of projective varieties, with central fibre $\mathcal{Q}_0 \cong Q \subset \mathbb{P}^{19}$, and all other fibres isomorphic to $\mathcal{Q}_1 \cong \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^{19}$.

6. *An anticanonical hypersurface $X_1 \subset \mathcal{Q}_1 \cong \mathbb{P}^1 \times \mathbb{P}^3$ specialises in the family $\mathcal{Q}$ to a* reducible *threefold $\underline{X}_0 \subset Q$, which contains a double copy of the distinguished toric divisor $\mathbb{P}^3 \subset Q$.*

7. *[Ruan96, App.A] describes a fan Σ with 8 maximal dimensional cones and one-dimensional rays $\sigma_1, \sigma_2, \rho_1, \ldots, \rho_4 \in N$ in a rank-4 lattice N for which $X_{N,\Sigma} \cong \mathbb{F}(0,1,1,2)$. There is a refined fan Σ' generated by 10 cones with the same rank 4-lattice N.*

We begin by defining the deformation of

$$\varphi_{|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,2)|}(\mathscr{F}_\zeta) = \left\{[z_{f_1 f_2 jk}] : \operatorname{rank}\left|M_1 \quad M_2\right| \leq 1\right\}$$

as

$$(\widetilde{\operatorname{Im} \varphi_{|L_{-1,2}|}})_t = \left\{([z_{f_1 f_2 jk}], t) : \operatorname{rank}|c_0 \quad N_1(t) \quad N_2(t)| \leq 1\right\} \subset \mathbb{A}^1_t \times \mathbb{P}^{19}.$$

Using Macaulay2, we find that $(\widetilde{\operatorname{Im} \varphi_{|L_{-1,2}|}})_t$ is irreducible of dimension 4 and of degree 32, hence it is a sensible deformation of $\operatorname{Im} \varphi_{|L_{-1,2}|}$.

For $t \neq 0$, making a change of variables

$$[M_1 \quad M_2] = [N_1(t) \quad N_2(t)]$$

yields

$$\begin{aligned}(\widetilde{\operatorname{Im}\varphi_{|L_{-1,2}|}})_t &= \left\{([y_{e_1e_2mn}],t) : \operatorname{rank}|N_1(t) \quad N_2(t)| \leq 1\right\} \\ &\cong \left\{[z_{f_1f_2jk}] : \operatorname{rank}\left|M_1 \quad M_2\right| \leq 1\right\} \\ &= \varphi_{|\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,2)|}(\mathbb{F}(0,0,0,0) \subset \mathbb{P}^{19}_{[z_{f_1f_2jk}]} = \mathbb{P}\left(H^0\left(-\frac{1}{2}K_{\mathscr{F}_\zeta}\right)\right).\end{aligned}$$

Further, for $t = 0$, we get

$$\begin{aligned}(\widetilde{\operatorname{Im}\varphi_{|L_{-1,2}|}})_0 &= \left\{([y_{e_1e_2mn}],0) : \operatorname{rank}|c_0 \quad N_1(0) \quad N_2(0)| \leq 1\right\} \\ &\cong \overline{\varphi_{|L_{-1,2}|}(\mathbb{F}(0,1,1,2))} \subset \mathbb{P}^{19}_{[y_{e_1e_2mn}]} = \mathbb{P}\left(H^0\left(-\frac{1}{2}K_{\mathscr{F}_0}\right)\right).\end{aligned}$$

We now find the degeneration $X = X_{2,4} \rightsquigarrow X_0$ as follows. Take

$$\begin{aligned}\varphi_{|\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,2)|}(X) &= \varphi_{|\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,2)|}\left(\mathbb{V}\left(\sum_{(q_j)\vdash 4}\alpha_2(u_i)v_1^{q_1}v_2^{q_2}v_3^{q_3}v_4^{q_4}\right)\right) \\ &= \left\{[z_{f_1f_2jk}] \in \mathbb{P}^{19} : q_2(z_{10jk}, z_{01jk}), 1 \leq j,k \leq 4\right\} = Q\end{aligned}$$

which degenerates, under the change of variable, to the special quadric

$$Q_t = \mathbb{V}(q_2(y_{e_1e_2mn},t)).$$

Now, we would like to know the singularity of the special surface

$$X_0 = (\operatorname{Im}\varphi_{|-\frac{1}{2}K_{\mathscr{F}_\zeta}|})_0 \cap Q_0 = \mathbb{V}\left(q_2(y_{e_1e_2mn},0), I_{(\widetilde{\operatorname{Im}\varphi_{|L_{-1,2}|}})_0}\right) \cong \mathbb{V}(q_2, I_B)$$

from the degeneration

$$X_t = (\operatorname{Im}\varphi_{|-\frac{1}{2}K_{\mathscr{F}_\zeta}|})_t \cap Q_t = \mathbb{V}\left(q_2(y_{e_1e_2mn},t), I_{(\widetilde{\operatorname{Im}\varphi_{|L_{-1,2}|}})_t}\right)$$

of $X = X_{2,4}$.

Using Macaulay2 codes below, we deduce the following.

```
R=QQ[ z1011 , z0111 , z1012 , z0112 , z1013 , z0113 , z1014 , z0114 ,
 z1022 , z0122 , z1023 , z0123 , z1024 , z0124 , z1033 , z0133 , z1034 ,
 z0134 , z1044 , z0144 ]
```

```
T=QQ[y0012,y0013,y1014,y0114,y1022,y0122,y1023,y0123,
y1033,y0133,y2024,y1124,y0224,y2034,y1134,y0234,y3044,
y2144,y1244,y0344,t]

M=matrix{{z1011,z1012,z1013,z1014,z0111,z0112,z0113,
z0114},{z1012,z1022,z1023,z1024,z0112,z0122,z0123,z0124},
{z1013,z1023,z1033,z1034,z0113,z0123,z0133,z0134},
{z1014,z1024,z1034,z1044,z0114,z0124,z0134,z0144}}

N=matrix{{y0012, y1022,y1023,y2024,y1124+t*y0012,y0122,
y0123,y1124-t*y0012,y0224},{y0013, y1023,y1033,y2034,
y1134+t*y0013,y0123,y0133,y1134-t*y0013,y0234},{y1014,
y2024,y2034,y3044,y2144+t*y1014,y1124-t*y0012,y1134-t*y0013,
y2144-t*y1014,y1244-t*y0114},{y0114, y1124+t*y0012,
y1134+t*y0013,y2144+t*y1014,y1244+t*y0114,y0224,y0234,
y1244-t*y0114,y0344}}

K=minors(2,M)
N0=sub(N,{t=>0})
I0=ideal(minors(2,N0),t)
N1=sub(N,{t=>1})
I1=ideal(minors(2,N1),t-1)

g=map(T,R,matrix{{y1022,y0122,y1023,y0123,y2024,
-t*y0012+y1124,t*y0012+y1124,y0224,y1033,y0133,y2034,
-t*y0013+y1134,t*y0013+y1134,y0234,y3044,-t*y1014+y2144,
t*y1014+y2144,-t*y0114+y1244,t*y0114+y1244,y0344}})
```

Here, K is affine dimension 5, degree 32 prime ideal of R; the image of $\mathscr{F}_\zeta = \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^{19}_{f_1 f_2 jk}$. Whereas I0 and I1 are the ideals of (closures of) images under $\varphi_{|-\frac{1}{2}K_{\mathscr{F}_0}|}$ of $\mathscr{F}_0 = \mathbb{F}(0,1,1,2)$ and, respectively, its deformation to $t=1$; both of degree 32 and affine dimension 5 prime ideals of the ring T.

Since, from Macaulay2, it is the case that

```
i:  ideal(g(K),t-1)==I1
o:  true
```

the ring map $g : R \to T$ is an isomorphism.

With $\mathscr{F}_0 = \mathbb{F}(0,1,1,2)$, the map $\varphi_{|-\frac{1}{2}K_{\mathscr{F}_0}|}$ is not defined on the curve $C = \mathbb{V}(x_2, x_3, x_4) \cong \mathbb{P}^1$. What happens is that there is a flip of the curve C, so that we have a different birational model

$$\mathscr{F}_0^+ := \mathrm{flop}_C(\mathscr{F}_0)$$

of $\mathscr{F}_0$. The corresponding map is well defined everywhere; in fact

$$\mathscr{F}_0^+ = \overline{\operatorname{Im}\varphi_{|-\frac{1}{2}K_{\mathscr{F}_0}|}} \subset \mathbb{P}^{19}_{e_1e_2mn}.$$

The birational map

$$\mathscr{F}_0 \dashrightarrow \mathscr{F}_0^+$$

is a morphism

$$\mathscr{F}_0 \setminus C \to \mathscr{F}_0^+ \setminus S \tag{5.11}$$

for some locus $S \subset \mathscr{F}_0^+$. The locus S is a surface in P^{19} with the following interesting identity in $V(I0) = \mathscr{F}_0^+$: From Macaulay2, start with the following Macaulay2 codes.

```
L=ideal(I0 ,y0012 ,y0013)
A=minimalPrimes L
S=A#1
dim S
degree S
```

With a change of variables from

$$\mathbb{P}^5_{[y1014,y0114,y3044,y2144,y1244,y0344]}$$

(where the 6 variables are those remaining in A#1) to

$$P = \mathbb{P}^5_{[t_1x_1,t_2x_1,t_1^3x_2,t_1^2t_2x_2,t_1t_2^2x_2,t_2^3x_2]},$$

we identify the degree 4 surface $S = \mathbb{F}(1,3) \cong \mathbb{F}(0,2) = F_2$ by the 6 quadrics defining the image of the embedding

$$\varphi_{|L_{0,1}|} : \mathbb{F}(1,3) \hookrightarrow P.$$

It is worth noting that the 3 quadrics in A#1 are reducible with $\{x_4 = 0\}$ component; indeed, it was logical to set $x_1x_2 = x_1x_3 = 0$ in L because in doing so we get the base locus $C = \{x_2 = x_3 = x_4 = 0\}$ away from which we have the isomorphism (5.11) .

The (degenerate variety) special surface X_0 at $t = 0$ is defined by the following Macaulay2 codes.

```
J=g(random(2,R))+I0+S
B=minimalPrimes J
V=ideal(I0, g(random(2,R)))
C=minimalPrimes V
```

Here, V is degree 64 affine dimension 4 ideal of the degeneration X_t of a generic threefold $X \subset \mathscr{F}_0 = \mathbb{F}(0,1,1,2)$. At this level of threefolds, we have two components, one a $\mathbb{V}(C\#0) = \mathbb{P}^3$ of degree 1, doubled; and a residual irreducible threefold $\mathbb{V}(C\#1)$ of degree

62 in $\mathbb{P}^{20}_{e_1e_2mn,t}$. The threefold X_t is therefore more degenerate at $t=0$.

At the level of surfaces, we note that $\mathbb{V}(B\#0)$ is a rational curve; a section of $S=F_2$. Also the degree 6 curve $\mathbb{V}(B\#1)$ meets $\mathbb{V}(B\#0)$ in six points. We then conclude that the degenerations X_t always intersect with the fixed surface on a fixed curve; a very special property. In contrast, for example, you get a generic, irreducible, degree $8=4\times 2$ curve if you intersect X_t with a generic quadric in $\mathbb{P}^{20}_{e_1e_2mn,t}$.

Instead of finding and resolving $Sing(X_0)$, we replace the family $\mathscr{F}\to\mathbb{A}^1_\zeta$ by a new 5-dimensional family $\mathscr{F}^+\to\mathbb{A}^1_\zeta$ whose fibres are $\mathscr{F}^+_\zeta=\mathscr{F}_\zeta$ and $\mathscr{F}^+_0=\text{flop}_C(\mathscr{F}_0)$ as constructed above.

Alternatively, from the toric construction of scrolls in Subsection (3.3), we have that

$$\mathbb{F}(0,1,1,2)=X_{N,\Sigma}$$

with one-dimensional cones $\sigma_1,\sigma_2,\rho_1,\rho_2,\rho_3,\rho_4\in\Sigma(1)\subset\Sigma\subset N_{\mathbb{R}}=\mathbb{R}^4$ corresponding to Weil divisors $D_{\sigma_i}=\mathbb{V}(t_i),D_{\rho_j}=\mathbb{V}(x_j)$ with relations

$$\sigma_1=-\sigma_2+\rho_2+\rho_3+2\rho_4 \text{ and } \rho_1=-\rho_2-\rho_3-\rho_4.$$

We fix

$$N=<u_2=(1,0,0,0),v_2=(0,1,0,0),v_3=(0,0,1,0),v_4=(0,0,0,1)>\cong\mathbb{Z}^4$$

with one dimensional cones

$$\begin{aligned}
\sigma_1=<u_1=-u_2+v_2+v_3+2v_4>&=<(-1,1,1,2)>;\\
\sigma_2=<u_2>&=<(1,0,0,0)>;\\
\rho_1=<v_1=-v_2-v_3-v_4>&=<(0,-1,-1,-1)>;\\
\rho_2=<v_2>&=<(0,1,0,0)>;\\
\rho_3=<v_3>&=<(0,0,1,0)>;\\
\rho_4=<v_4>&=<(0,0,0,1)>\,.
\end{aligned}$$

The 8 maximal cones $\Sigma(4)$ of its normal fan Σ are

$$\tau_{i,j}=\mathbb{Z}<u_i,v_1,\ldots,\widehat{v_j},\ldots,v_4>\,.$$

The class group and the $\mathbb{Z}^2$-graded Cox ring of $X_{N,\Sigma}$ are, respectively,

$$Cl(X_\Sigma)\cong\mathbb{Z}[L,M]:=\mathbb{Z}\langle[\mathbb{V}(t_1)],[\mathbb{V}(x_1)]\rangle$$

and

$$S = \mathbb{C}[t_i, x_j] = \bigoplus_{(d_1,d_2)\in\mathbb{Z}^2} \mathbb{C}[t_i, x_j]_{(d_1,d_2)}$$

with weight $wt(t_1,t_2,x_1,x_2,x_2,x_4) = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$.

We have that

$$D = -D_{\sigma_1} + 2D_{\rho_1},$$

a $\mathbb{T}$-Weil divisor in the class $[D] = -\frac{1}{2}K_{X_\Sigma}$. The polytope $P_D \subset \mathbb{R}^4_{x_k}$ of sections of $[D]$ is bounded by

$$\begin{aligned} -x_1 + x_2 + x_3 + 2x_4 - 1 &\geq 0, \\ x_1 &\geq 0, \\ -x_2 - x_3 - x_4 + 2 &\geq 0, \\ x_2 &\geq 0, \\ x_3 &\geq 0 \text{ and} \\ x_4 &\geq 0. \end{aligned}$$

Using polymake with the commands

```
$P=new Polytope(INEQUALITIES=>[[-1,-1,1,1,2],[0,1,0,0,0],
[2,0,-1,-1,-1],[0,0,1,0,0],[0,0,0,1,0],[0,0,0,0,1]]);
$P->properties();
```

we get the required Polytope with $20 = \dim H^0\left(\mathbb{F}, -\frac{1}{2}K_{\mathbb{F}}\right)$ lattice points, 6 facets and 9 vertices $w_k = w_k(u_i, v_j)$.

It is important to note that the above polytope $P = P_D$ is not an integral polytope; $w_6 = \frac{1}{2}v_4$. Now, since the section (bi-graded) algebra

$$x^b \in \mathbb{C}[H^0(\mathbb{F}(0,1,1,2),[D])] = \mathbb{C}[t_1,t_2,x_1,x_2,x_3,x_4]_{-1,2}$$

of $[D] = -\frac{1}{2}K_{X_\Sigma}$ only sees the (integral) lattice points $b \in P_D$, we want to focus on the properties of the convex hull Q of the lattice points in P_D and the corresponding normal fan FQ defined by

```
$Q=new Polytope(POINTS=>$P->LATTICE_POINTS);
print $Q->VERTICES;
print $Q->N_FACETS;
print $Q->N_LATTICE_POINTS;
application "fan";
FQ = normal_fan($Q);
print primitive($FQ->RAYS);
print $FQ->MAXIMAL_CONES;
```

This lattice polytope Q has the same 20 lattice points (as those of $P = P_D$), 10 vertices and 7 facets. We then have that FQ is the normal fan of the 4-dimensional projective variety $\mathrm{Im}\,\varphi_{|-\frac{1}{2}K_{\mathscr{F}_0}|} \subset \mathbb{P}^{19}_{e_1e_2mn}$.

For purposes of comparison with the above computation, we consider the polytope $R = P_D$ where $[D] = -K_{\mathbb{F}}$ the anticanonical divisor class of the scroll $\mathscr{F}_0 = \mathbb{F}(0,1,1,2)$. The polytope R has a much larger number ($106 = \dim H^0(\mathscr{F}_0, -K_{\mathscr{F}_0})$) of lattice points than the $20 = \dim H^0\left(\mathscr{F}_0, -\frac{1}{2}K_{\mathscr{F}_0}\right)$ lattice points of P.

```
$R=new Polytope(INEQUALITIES=>[[-2,-1,1,1,2],[0,1,0,0,0],
[4,0,-1,-1,-1],[0,0,1,0,0],[0,0,0,1,0],[0,0,0,0,1]]);
$S=new Polytope(POINTS=>$R->LATTICE_POINTS);
print $S->VERTICES;
print $S->N_FACETS;
print $S->N_LATTICE_POINTS;
application "fan";
$FS = normal_fan($S);
print primitive($FS->RAYS);
print $FS->MAXIMAL_CONES;
```

The normal fans FQ and FS of the respective convex hulls Q of the polytope P and convex hull S of the polytope R compare as in the Table (5.4) .

Name	Rays of FS	9 Cones of FS
0	$-1,1,1,2$	$\{0,2,3,4\},\{0,2,4,5\},\{0,2,3,5\},\{1,2,3,5\},\{0,1,3,5\}$
1	$1,0,0,0$	$\{0,1,3,4\},\{1,2,3,4\},\{0,1,4,5\},\{1,2,4,5\}$
2	$0,-1,-1,-1$	
3	$0,1,0,0$	
4	$0,0,1,0$	
5	$0,0,0,1$	
Name	**Rays of FQ**	**10 Cones of FQ**
0	$-1,1,1,2$	$\{0,2,3,4\},\{0,2,4,5\},\{0,2,3,5\},\{1,2,3,5\},\{0,1,3,5,6\},$
1	$1,0,0,0$	$\{\not0,1,3,4,6\},\{0,\not1,3,4,6\},\{1,2,3,4\},\{0,1,4,5,6\},\{1,2,4,5\}$
2	$0,-1,-1,-1$	
3	$0,1,0,0$	
4	$0,0,1,0$	
5	$0,0,0,1$	
6	$0,1,1,1$	

Figure 5.4: Comparison between normal fan FS of $\mathrm{Im}\,\varphi_{|-K_{\mathscr{F}_0}|}$ and normal fan FQ of $\mathrm{Im}\,\varphi_{|-\frac{1}{2}K_{\mathscr{F}_0}|}$

Up to permutation of the rays, the normal fan FS of $\mathrm{Im}\,\varphi_{|-K_{\mathscr{F}_0}|}$ is the fan studied in [Ruan96] .

Related to the deformation demonstrated above, consider the family

$$\mathbb{P}(\mathscr{E} \oplus \mathscr{O}_{\mathbb{A}^1_\zeta \times \mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{A}^1_\zeta \times \mathbb{P}^1}(1)) = \mathscr{F} \to \mathbb{A}^1_t$$

whose fibres $\mathscr{F}_\zeta = \mathbb{P}(\mathscr{E}|_\zeta)$ are

$$\mathscr{F}_0 = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}) = \mathbb{F}(-1,-1,1,1) \cong \mathbb{F}(0,0,2,2) \text{ and}$$
$$\mathscr{F}_{\zeta \neq 0} \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)) = \mathbb{F}(-1,0,0,1) \cong \mathbb{F}(0,1,1,2).$$

We also have that $\mathscr{F}_0 = \mathbb{F}(0,0,2,2)$ is mapped to $\mathbb{P}^{19}$ by sections of $|-\frac{1}{2}K_{\mathscr{F}_0}|$ with the image given in the Proposition below.

Proposition 5.1.6. *The map*

$$\varphi_{|-\frac{1}{2}K_{\mathscr{F}_0}|} = \varphi_{|L_{-1,2}|} : \mathbb{F}(0,0,2,2) \dashrightarrow \mathbb{P}^{19}_{[w_{e_1 e_2 mn}]}$$
$$[t_i; x_j] \mapsto [\mathbf{Sym}^1(t_i) \otimes \mathbf{Sym}^1(x_1,x_2) \otimes \mathbf{Sym}^1(x_3,x_4) : \mathbf{Sym}^3(t_i) \otimes \mathbf{Sym}^2(x_3,x_4)]$$

is an isomorphism away from the surface $X_{34} \cong \mathbb{P}^1 \times \mathbb{P}^1$. The closure of its image is

$$\overline{\mathrm{Im}(\varphi_{|L_{-1,2}|})} = \mathbb{V}(rank|C| \leq 1, rank|C'| \leq 1)$$

where

$$C = \begin{bmatrix} w_{1013} & w_{1023} & w_{3033} & w_{2133} & w_{1233} & w_{3034} & w_{2134} & w_{1234} \\ w_{0113} & w_{0123} & w_{2133} & w_{1233} & w_{0333} & w_{2134} & w_{1234} & w_{0334} \\ w_{1014} & w_{1024} & w_{3034} & w_{2134} & w_{1234} & w_{3044} & w_{2144} & w_{1244} \\ w_{0114} & w_{0124} & w_{2134} & w_{1234} & w_{0334} & w_{2144} & w_{1244} & w_{0344} \end{bmatrix} = [c_0 c_1 \dots c_7]$$

and

$$C' = \begin{bmatrix} w_{1013} & w_{1023} & w_{1014} & w_{1024} & w_{3033} & w_{2133} & w_{1233} & w_{3034} & w_{2134} & w_{1234} & w_{3044} & w_{2144} & w_{1244} \\ w_{0113} & w_{0123} & w_{0114} & w_{0124} & w_{2133} & w_{1233} & w_{0333} & w_{2134} & w_{1234} & w_{0334} & w_{2144} & w_{1244} & w_{0344} \end{bmatrix}$$

Proof. From the weight matrix

$$wt(t_1,t_2,x_1,x_2,x_3,x_4) = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

we have that

$$S_{-2,2} = \langle x_1x_3, x_2x_3, x_1x_4, x_2x_4, t_1^2x_3^2, t_1t_2x_3^2, t_2^2x_3^2, t_1^2x_3x_4, t_1t_2x_3x_4, t_2^2x_3x_4, t_1^2x_4^2, t_1t_2x_4^2, t_2^2x_4^2 \rangle,$$
$$S_{-1,1} = \langle t_1x_3, t_2x_3, t_1x_4, t_2x_4 \rangle \text{ and}$$
$$S_{0,1} = \langle x_1, x_2, t_1^2x_3, t_1t_2x_3, t_2^2x_3, t_1^2x_4t_1t_2x_4, t_2^2x_4 \rangle.$$

The line bundle $L_{-1,2}$ is obtained from these bases in two ways by forming the matrices

$$\begin{aligned}C' &= \begin{bmatrix} t_1 S_{-2,2} \\ t_2 S_{-2,2} \end{bmatrix} \\ &= \left[\begin{matrix} w_{1013} & w_{1023} & w_{1014} & w_{1024} & w_{3033} & w_{2133} & w_{1233} & w_{3034} & w_{2134} & w_{1234} \\ w_{0113} & w_{0123} & w_{0114} & w_{0124} & w_{2133} & w_{1233} & w_{0333} & w_{2134} & w_{1234} & w_{0334} \end{matrix}\right. \\ &\qquad \left.\begin{matrix} w_{3044} & w_{2144} & w_{1244} \\ w_{2144} & w_{1244} & w_{0344} \end{matrix}\right] \\ C &= \left[S_{-1,1} \otimes S_{0,1} \right] \\ &= \begin{bmatrix} w_{1013} & w_{1023} & w_{3033} & w_{2133} & w_{1233} & w_{3034} & w_{2134} & w_{1234} \\ w_{0113} & w_{0123} & w_{2133} & w_{1233} & w_{0333} & w_{2134} & w_{1234} & w_{0334} \\ w_{1014} & w_{1024} & w_{3034} & w_{2134} & w_{1234} & w_{3044} & w_{2144} & w_{1244} \\ w_{0114} & w_{0124} & w_{2134} & w_{1234} & w_{0334} & w_{2144} & w_{1244} & w_{0344} \end{bmatrix}\end{aligned}$$

whose ranks are both 1 on the image. Therefore, all the relations within and between blocks $\mathbf{Sym}^{-1+2q_3+2q_4}(t_i)x_1^{q_1}x_2^{q_2}x_3^{q_3}x_4^{q_4}$ for every $(q_j) \vdash 2$ for which $-1+2q_3+2q_4 \geq 0$ are

$$I_C = \{\text{rank}|C| \leq 1\} \text{ and } I_{C'} = \{\text{rank}|C'| \leq 1\}.$$

Since $I_{C'} \not\subset I_C$, we have that

$$\overline{\text{Im}(\varphi_{|L_{-1,2}|})} \subseteq \mathbb{V}(I_C + I_{C'}) = Y'.$$

Using Macaulay2 computer algebra, we find that Y' is a 4-dimensional irreducible projective variety of degree 28 so that

$$\overline{\text{Im}(\varphi_{|L_{-1,2}|})} = \mathbb{V}(I_C + I_{C'}).$$

□

Since a flat family preserves degree, we know that Y' is not a deformation of $\varphi_{|-1,2|}(\mathbb{F}(0,1,1,2))$ over $\mathbb{A}^1_t$. However, using Macaulay2, we can check that

$$\mathbb{V}(I_C) = Y' \cup \varphi_{|\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,1)|}(\mathbb{P}^1 \times \mathbb{P}^3) \tag{5.12}$$

is reducible of dimension 4 and degree 32. We can therefore conclude that $\varphi_{|-1,2|}(\mathbb{F}(0,1,1,2))$ deforms into a reducible projective variety $\mathbb{V}(I_C)$.

To understand this deformation, we note from Proposition (5.1.6) that, from $C = [c_0 c_1 \ldots c_7]$, the matrices $[c_2 c_3 c_5 c_6]$ and $[c_3 c_4 c_6 c_7]$ are symmetric and overlap on the column c_3 and c_6. We define the columns

$$p_2(t) = c_3 + tc_0, \; p_5(t) = c_6 - tc_1$$

and

$$p_4(t) = c_3 + tc_1,\ p_7(t) = c_6 - tc_1$$

for $t \in \mathbb{C}$ so that

$$P_1(t) = [p_1 p_2(t) p_3 p_4(t)] =$$
$$\begin{bmatrix} w_{3033} & w_{2133}+tw_{1013} & w_{3034} & w_{2134}-tw_{1023} \\ w_{2133}+tw_{1013} & w_{1233}+tw_{0113} & w_{2134}+tw_{1014} & w_{1234}-tw_{0123}+tw_{0124} \\ w_{3034} & w_{2134}+tw_{1014} & w_{3044} & w_{2144}-tw_{1024} \\ w_{2134}-tw_{1023} & w_{1234}-tw_{0123}+tw_{0124} & w_{2144}-tw_{1024} & w_{1244}-tw_{0124} \end{bmatrix}$$

and

$$P_2(t) = [p_5(t) p_6 p_7(t) p_8] =$$
$$\begin{bmatrix} w_{2133}+tw_{1013} & w_{1233}+tw_{0113} & w_{2134}+tw_{1014}-tw_{1023} & w_{1234}+tw_{0124} \\ w_{1233}+tw_{0113} & w_{0333} & w_{1234}-tw_{0123} & w_{0334} \\ w_{2134}+tw_{1014}-tw_{1023} & w_{1234}-tw_{0123} & w_{2144}-tw_{1024} & w_{1244}-tw_{0114} \\ w_{1234}+tw_{0124} & w_{0334} & w_{1244}-tw_{0124} & w_{0344} \end{bmatrix}$$

are both symmetric.

We then define the deformation of $\operatorname{Im}\varphi_{|-\frac{1}{2}K_{\mathbb{F}(0,0,2,2)}|}$ as

$$(\widetilde{\operatorname{Im}\varphi_{|L_{-1,2}|}})_t = \left\{([w_{e_1e_2mn}],t) : \operatorname{rank}|c_0 \quad c_1 \quad P_1(t) \quad P_2(t)| \leq 1\right\} \subset \mathbb{A}^1_t \times \mathbb{P}^{19}.$$

Using Macaulay2 once again, we find that

$$(\widetilde{\operatorname{Im}\varphi_{|L_{-1,2}|}})_t = Y' \cup \varphi_{|\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,1)|}(\mathbb{P}^1 \times \mathbb{P}^3) \cup \mathbb{A}^1_t \times \mathbb{P}^1_{[w_{0333}:w_{3044}]} \tag{5.13}$$

is reducible of dimension 4 and of degree 32 hence, it is a sensible deformation of $\operatorname{Im}\varphi_{|L_{-1,2}|}$.

For $t \neq 0$, making a change of variables with coordiante entries of $[P_1(t) \quad P_2(t)]$ mapped to coordnate entries of $[N_1(t) \quad N_2(t)]$ yields

$$\begin{aligned}(\widetilde{\operatorname{Im}\varphi_{|L_{-1,2}|}})_t &= \left\{([y_{e_1e_2mn}],t) : \operatorname{rank}|N_1(t) \quad N_2(t)| \leq 1\right\} \\ &= \varphi_{|L_{-1,2}|}(\mathbb{F}(0,1,1,2)) \subset \mathbb{P}^{19}_{[y_{e_1e_2mn}]} = \mathbb{P}\left(H^0\left(-\frac{1}{2}K_{\mathscr{F}_\zeta}\right)\right).\end{aligned}$$

Further, for $t = 0$, we get

$$\begin{aligned}(\widetilde{\operatorname{Im}\varphi_{|L_{-1,2}|}})_0 &= \left\{([w_{e_1e_2mn}],0) : \operatorname{rank}|C| \leq 1\right\} \\ &= \varphi_{|L_{-1,2}|}(\mathbb{F}(0,0,2,2)) \subset \mathbb{P}^{19}_{[w_{e_1e_2mn}]} = \mathbb{P}\left(H^0\left(-\frac{1}{2}K_{\mathscr{F}_0}\right)\right).\end{aligned}$$

It is worth noting that the family number 3, that is $X \subset \mathbb{F}(0,0,1,1)$, in Table (5.1) also maps to $\mathbb{P}^{19}$ by sections of its half anticanonical divisor class $|L_{0,2}|$. Just like families numbers 1, 6 and 8 discussed above; the family number 3 is also such that $\dim(\mathscr{M}_{-K_{\mathbb{F}}}) = 86$.

$$\mathbb{P}(\mathscr{E} \oplus \mathscr{O}_{\mathbb{A}^1_\zeta \times \mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{A}^1_\zeta \times \mathbb{P}^1}) = \mathscr{F} \to \mathbb{A}^1_t$$

with fibres

$$\mathscr{F}_0 \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)) = \mathbb{F}(-1,-1,0,1) \cong \mathbb{F}(0,0,1,2) \text{ and}$$
$$\mathscr{F}_{\zeta \neq 0} = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 3}) = \mathbb{F}(-1,0,0,0) \cong \mathbb{F}(0,1,1,1).$$

An attempt at considering an "almost-half" anticanonical divisor class using sections of $|L_{0,2}|$ or $|L_{-1,2}|$ did not embed the fibres into projective spaces of same dimension.

5.2 Calabi–Yau threefolds with Quintic fibres

Consider the 4-fold weighted scroll

$$\mathbb{F}_A = \mathbb{F}\begin{pmatrix} 1 & 1 & -a_1 & -a_2 & -a_3 & -a_4 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

or use the shorthand $\mathbb{F}_A = \mathbb{F}(a_1, a_2, a_3, a_4 | 1^3, 2)$. By the standard isomorphism

$$\mathbb{F}(a_1, a_2, a_3, a_4 | 1^3, 2) \cong \mathbb{F}(a_1 + k, a_2 + k, a_3 + k, a_4 + 2k | 1^3, 2) \text{ for } k \in \mathbb{Z},$$

we can set $a_1 = 0$ so that $\mathbb{F}_A = \mathbb{F}(0, a_2, a_3, a_4 | 1^3, 2)$ with $a_3 \geq a_2 \geq a_1 = 0$ with a general $a_4 = a$.

The anticanonical divisor of $\mathbb{F}_A$ is given by

$$-K_{\mathbb{F}_A} = \left(2 - \sum_j a_j\right) L + \left(\sum_j b_j\right) M = L_{2-(a_2+a_3+a_4),5} = L_{d,5}.$$

The sections of $L_{d,5}$ are

$$x_1^{q_1} x_2^{q_2} x_1^{q_3} y^{q_4} \,\mathbf{Sym}^{d+a_2q_2+a_3q_3+a_4q_4}(t_i) \subset H^0(\mathbb{F}_A, L_{d,5}),\ q_1 + q_2 + q_3 + 2q_4 = 5.$$

We therefore have that the general 3-fold section $X \in |L_{2-(a_2+a_3+a_4),5}|$ is defined by a sum

$$X = \mathbb{V}(f(t_i, x_j, y)) = \mathbb{V}\left(\sum_{(q_j)_{(1^3,2)} \vdash 5} x_1^{q_1} x_2^{q_2} x_3^{q_3} y^{q_4} Sym^{d+a_2q_2+a_3q_3+a_4q_4}(t_1, t_2)\right)$$

of monomials $x_1^{q_1} x_2^{q_2} x_3^{q_3} y^{q_4} f_{(q_j)}(t_i)$. The degree 5 monomials $x_1^{q_1} x_2^{q_2} x_3^{q_3} y^{q_4}$ are displayed in the $(1,1,1,2)$ weighted Newton tetrahedron in Figure (5.5) . The degree of coefficients $f_{(q_j)}(t_i)$ is given by

$$\deg f_{(q_j)}(t_i) = 2 + a_2(q_2 - 1) + a_3(q_3 - 1) + a_4(q_4 - 1).$$

Theorem 5.2.1. *Let $\mathbb{F} = \mathbb{F}(a_1, a_2, a_3, a_4 | 1, 1, 1, 2)$ be a $\mathbb{P}[1^3, 2]$ bundle over $\mathbb{P}^1$ with quintic fibres. There are* 2 *quintic fibred Calabi–Yau* 3*-fold families with at worst orbifold singularities and finite number of isolated singular points along $Bs(|-K_{\mathbb{F}}|)$ embedded in scrolls as listed in Table* (5.2).

Proof. We first note that

$$Bs(\mathscr{O}_{\mathbb{P}^3[1^3,2]}(5)) = \{p = [0:0:0:1]\}$$

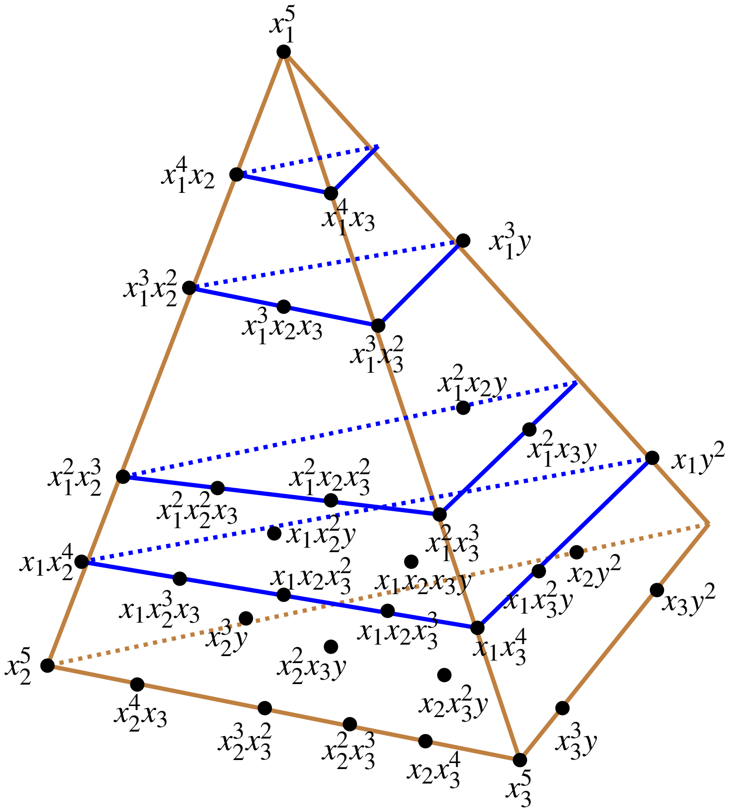


Figure 5.5: The 34 monomial generators of $\mathbb{C}[\mathbb{P}^3_{x_j,y}[1^3,2]]_5$.

No.	$\mathbb{F} = \mathbb{F}(a_j\|1^3,2)$	General $X \in \|-K_{\mathbb{F}}\|$
1	$\mathbb{F}(0,0,1,2\|1^3,2)$	X has 4 ODP singularities along $\mathbb{V}(x_3,y)$ X is quasismooth along $Sing(\mathbb{F}) = \mathbb{V}(x_1,x_2,x_3)$ with $Bs(\|-K_{\mathbb{F}}\|) = \mathbb{V}(x_1,x_2,x_3) \sqcup \mathbb{V}(x_3,y)$
2	$\mathbb{F}(0,1,2,0\|1^3,2)$	X has 2 ODP singularities along $Bs(\|-K_{\mathbb{F}}\|) = \mathbb{V}(x_2,x_3)$ X is quasismooth along $Sing(\mathbb{F}) = \mathbb{V}(x_1,x_2,x_3)$
3	$\mathbb{F}(0,0,2,1\|1^3,2)$	X has 3 ODP singularities along $\mathbb{V}(x_3,y)$ X is quasismooth along $Sing(\mathbb{F}) = \mathbb{V}(x_1,x_2,x_3)$ with $Bs(\|-K_{\mathbb{F}}\|) = \mathbb{V}(x_1,x_2,x_3) \sqcup \mathbb{V}(x_3,y)$

Table 5.2: $\mathbb{F} = \mathbb{F}(a_j|1^3,2)$ for which a general $X \in |-K_{\mathbb{F}}|$ has at worst orbifold singularities and finite number of isolated singular points along $Bs(|-K_{\mathbb{F}}|)$

and that some points of

$$Sing(\mathbb{F}(0,a_2,a_3,a_4|1^3,2)) = \mathbb{P}_{[t_i]} \times Bs(\mathcal{O}_{\mathbb{P}^3[1^3,2]})(5)$$

will be in the one dimensional base loci of $|L_{2-a_2-a_3-a_4,5}|$. These are $\frac{1}{2}(1,1,1,0)$ quotient singularities along $\mathbb{P}^1_{[t_i]} \times \{p\}$. Existence of the one-dimensional base locus $\mathbb{P}^1 \times \{p\}$ on every general quintic fibred Calabi–Yau threefold X means that $Bs(|L_{2-a_2-a_3-a_4,5}|) \neq \emptyset$.

We must also have an irreducible threefold X so that $\dim(Bs(|L_{2-a_2-a_3-a_4,5}|)) \neq 3$. This is guaranteed if a general section $f(t_i,x_j,y)$ of $|L_{2-a_2-a_3-a_4,5|}$ is not a multiple of x_j and y. That is, the largest corresponding degree of coefficients of the monomials on each face of Newton tetrahedron (5.5) is nonnegative. The "fourth vertex" is represented by the smallest integral face on which (degrees of coefficients of monomials satisfy) conventionally

$$x_1y^2 \leq x_2y^2 \leq x_3y^2$$

since $a_3 \geq a_2 \geq 0$ and $a_4 \in \mathbb{Z}$. Further, since $x_1^5 \leq x_2^5 \leq x_3^5$, we have that

$$\mathrm{Max}\{x_1^5, x_1y^2\} \leq \mathrm{Max}\{x_2^5, x_2y^2\} \leq \mathrm{Max}\{x_3^5, x_3y^2\}.$$

Now, depending on the directions of arrows of magnitude of degree of coefficients within the three edges, ireducibility of X is then implied, respectively, by the Inequalities

$$\begin{cases} 2+4a_2-a_3-a_4 \geq 0, a_4 \geq 0, -a_2+a_4 \geq 0, -a_3+a_4 \geq 0 & \text{when } x_1^5 \leq x_1y^2 \leq x_2^5 \\ 2-a_2+4a_3-a_4 \geq 0, a_4 \geq 0, -a_2+a_4 \geq 0, -a_3+a_4 < 0 & \text{when } x_1y^2 \leq x_2y^2 \leq x_3^5 \\ 2-a_3+a_4 \geq 0, a_4 \geq 0, -a_2+a_4 < 0, -a_3+a_4 < 0 & \text{when } x_1^5 \leq x_1y^2 \leq x_2y^2 \\ 2+4a_2-a_3-a_4 \geq 0, a_4 \geq 0, -a_2+a_4 < 0, -a_3+a_4 < 0 & \text{when } x_1y^2 \leq x_2y^2 \leq x_2^5 \\ 2-a_2+a_4 \geq 0, a_4 < 0, -a_2+a_4 \geq 0, -a_3+a_4 \geq 0 & \text{when } x_2^5 \leq x_2y^2 \leq x_3y^2 \\ 2-a_2+4a_3-a_4 \geq 0, a_4 < 0, -a_2+a_4 < 0, -a_3+a_4 < 0 & \text{when } x_1y^2 \leq x_2^5 \leq x_3y^2 \\ 2-a_2+a_4 \geq 0, a_4 < 0, -a_2+a_4 < 0, -a_3+a_4 < 0 & \text{when } x_1y^2 \leq x_2^5 \leq x_3y^2 \\ 2+4a_2-a_3+a_4 \geq 0, a_4 < 0, -a_2+a_4 < 0, -a_3+a_4 < 0 & \text{when } x_1y^2 \leq x_2y^2 \leq x_2^5. \end{cases} \tag{5.14}$$

This would mean that the one-dimensional base locus is

$$B = \begin{cases} \mathbb{V}(x_2,x_3,y) & \text{for (5.14)(1)-(5.14)(4)} \\ \mathbb{V}(x_1,x_2,x_3) & \text{for (5.14)(5)-(5.14)(8).} \end{cases}$$

Along this curve $B \cong \mathbb{P}^1$, we get that

$$\begin{aligned} Sing(\widetilde{X}) \cap \widetilde{B} &= \begin{cases} \mathbb{V}\left(\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial y}, x_2, x_3, y\right) & \text{for (5.14)(1)-(5.14)(4)} \\ \mathbb{V}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, x_1, x_2, x_3\right) & \text{for (5.14)(5)-(5.14)(8)} \end{cases} \\ &= \begin{cases} \mathbb{V}(\alpha_{(4100)}(t_i), \alpha_{(4010)}(t_i), \alpha_{(3001)}(t_i)) & \text{for (5.14)(1)-(5.14)(4)} \\ \mathbb{V}(\alpha_{(1002)}(t_i), \alpha_{(0102)}(t_1), \alpha_{(0012)}(t_i)) & \text{for (5.14)(5)-(5.14)(8).} \end{cases} \end{aligned}$$

We want isolated singularities along the base locus so we require, for a general choice of the polynomial, the coefficints $\alpha_{q_j}(t_i)$ of maximum degree in the equation of $Sing(\widetilde{X}) \cap \widetilde{B}$ to be nonnegative; in particular

$$\begin{cases} 2-a_2-a_3 \geq 0 & \text{for (5.14)(1)-(5.14)(4)} \\ 2-a_2+a_4 \geq 0 & \text{for (5.14)(5)-(5.14)(8).} \end{cases} \tag{5.15}$$

This results in families of Calabi–Yau threefolds embedded in fourfold scrolls $\mathbb{F}(a_1,a_2,a_3,a_4|1^3,2)$

$$\begin{cases} \mathbb{F}(0,0,0,0|1^3,2),\ \mathbb{F}(0,0,0,1|1^3,2),\ \mathbb{F}(0,0,0,2|1^3,2),\ \mathbb{F}(0,0,1,1|1^3,2), \\ \mathbb{F}(0,1,1,1|1^3,2),\ \mathbb{F}(0,1,1,2|1^3,2),\ \mathbb{F}(0,1,1,3|1^3,2),\ \mathbb{F}(0,1,1,4|1^3,2), \\ \mathbb{F}(0,1,1,5|1^3,2),\ \mathbb{F}(0,0,1,0|1^3,2),\ \mathbb{F}(0,0,2,0|1^3,2),\ \text{and}\ \mathbb{F}(0,1,1,0|1^3,2) \\ \text{for } (5.14)(1)+(5.15)(1),\dots,(5.14)(4)+(5.15)(1); \\ \mathbb{F}(0,0,0,-2|1^3,2),\ \mathbb{F}(0,0,0,-1|1^3,2),\ \mathbb{F}(0,0,1,-1|1^3,2), \\ \mathbb{F}(0,1,1,-1|1^3,2),\ \mathbb{F}(0,1,2,-1|1^3,2),\ \mathbb{F}(0,1,3,-1|1^3,2), \\ \mathbb{F}(0,1,4,-1|1^3,2)\ \text{and}\ \mathbb{F}(0,1,5,-1|1^3,2) \\ \text{for } (5.14)(4)+(5.15)(2),\dots,(5.14)(8)+(5.15)(2) \end{cases}$$

with either $\frac{1}{2}(1,1,0)$ curve singularities along $Bs(|-K_{\mathbb{F}}|)=\mathbb{V}(x_2,x_3,y)$ or

$$(2-a_2+a_4)\times\frac{1}{2}(1,1,1)$$

quotient singularities along $Bs(|-K_{\mathbb{F}}|)=\mathbb{V}(x_1,x_2,x_3)$. The new singular families are colour coded blue whereas the rest of the families are quasismooth and already classified in [Mul06].

If $\dim(Bs(|L_{2-a_2-a_3-a_4,5}|))=2$ then the following Inequalities follow line-by-line from Inequalities (5.14)

$$\begin{cases} 2-a_2-a_3+a_4\geq 0, 2-a_2-a_3-a_4<0, a_4\geq 0, -a_2+a_4\geq 0, & \\ -a_3+a_4\geq 0; & x_1^5\leq x_1y^2\leq x_2^5 \\ 2-a_2-a_3+a_4<0, 2-a_3+a_4\geq 0, a_4\geq 0, -a_2+a_4\geq 0; & x_1y^2\leq x_2y^2\leq x_3^5 \\ 2-a_2-a_3+a_4\geq 0, 2-a_2-a_3-a_4<0, a_4\geq 0, -a_2+a_4\geq 0, & \\ -a_3+a_4<0; & x_1^5\leq x_1y^2\leq x_2y^2 \\ 2-a_2-a_3+a_4<0, 2-a_3+a_4\geq 0, a_4\geq 0, -a_2+a_4<0; & x_1y^2\leq x_2y^2\leq x_2^5 \\ 2-a_3+a_4\geq 0, 2+4a_2-a_3-a_4<0, a_4<0, -a_2+a_4\geq 0, & \\ -a_3+a_4\geq 0; & x_2^5\leq x_2y^2\leq x_3y^2 \\ 2-a_3+a_4\geq 0, 2-a_2-a_3+a_4<0, a_4<0, -a_2+a_4<0, & \\ -a_3+a_4<0; & x_1y^2\leq x_2^5\leq x_3y^2 \\ 2+4a_2-a_3-a_4\geq 0, 2-a_2-a_3+a_4<0, a_4<0, -a_2+a_4<0, & \\ -a_3+a_4<0; & x_1y^2\leq x_2^5\leq x_3y^2 \\ 2-a_3+a_4\geq 0, 2-a_2-a_3+a_4<0, a_4<0, -a_2+a_4<0, & \\ -a_3+a_4<0; & x_1y^2\leq x_2y^2\leq x_2^5. \end{cases} \tag{5.16}$$

We then have that the base locus is

$$B = Bs(|L_{2-a_2-a_3-a_4,5}|) = \begin{cases} \mathbb{V}(x_3, y) \cong \mathbb{F}(0, a_2|1,1) & \text{for } (5.16)(1) - (5.16)(4), \\ \mathbb{V}(x_2, x_3) \cong \mathbb{F}(0, a_4|1,2) & \text{for } (5.16)(5) - (5.16)(8). \end{cases}$$

Further consider a general section f of $|L_{2-a_2-a_3-a_4,5}|$ as an affine equation of the cone

$$\widetilde{X} = q^{-1}(X) \subset \mathbb{A}^6_{t_i, x_j, y} \setminus Z \xrightarrow{q} \mathbb{F}(0, a_2, a_3, a_4|1^3, 2)$$

over X with $Z = \{t_1 = t_2 = 0\} \sqcup B$.

Along the base locus we have that

$$Sing(\widetilde{X}) \cap \widetilde{B} = \begin{cases} \mathbb{V}\left(\frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial y}, x_3, y\right) & \text{for } (5.16)(1) - (5.16)(4), \\ \mathbb{V}\left(\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, x_2, x_3\right) & \text{for } (5.16)(5) - (5.16)(8) \end{cases}$$

where

$$\begin{aligned}
\left.\frac{\partial f}{\partial y}\right|_{x_3=y=0} =& \alpha_{3001}(t_1,t_2)x_1^3 + \alpha_{2101}(t_1,t_2)x_1^2x_2 + \alpha_{1201}(t_1,t_2)x_1x_2^2 + \alpha_{0301}(t_1,t_2)x_2^3; \\
\left.\frac{\partial f}{\partial x_3}\right|_{x_3=y=0} =& \alpha_{4010}(t_1,t_2)x_1^4 + \alpha_{3110}(t_1,t_2)x_1^3x_2 + \alpha_{2210}(t_1,t_2)x_1^2x_2^2 + \alpha_{1310}(t_1,t_2)x_1x_2^3 + \\
& \alpha_{0410}(t_1,t_2)x_2^4; \\
\left.\frac{\partial f}{\partial x_2}\right|_{x_2=x_3=0} =& \alpha_{4100}(t_1,t_2)x_1^4 + \alpha_{2101}(t_1,t_2)x_1^2y + \alpha_{0102}(t_1,t_2)y^2, \\
\left.\frac{\partial f}{\partial x_3}\right|_{x_2=x_3=0} =& \alpha_{4010}(t_1,t_2)x_1^4 + \alpha_{2011}(t_1,t_2)x_1^2y + \alpha_{0012}(t_1,t_2)y^2.
\end{aligned}$$

Therefore for X to have isolated singularities we must have that $\frac{\partial f}{\partial x_3}$ and $\frac{\partial f}{\partial y}$ $\left(\text{ or that } \frac{\partial f}{\partial x_2} \text{ and } \frac{\partial f}{\partial x_3}\right)$ are not identically zero on B and have no common factor. Equivalently,

$$\begin{cases} \begin{cases} 2-a_2-a_4\geq 0, 2+2a_2-a_3\geq 0; \\ \begin{cases} a_4-a_3\geq 0,\ 2-a_2-a_4\geq 0, 2+2a_2-a_3\geq 0, \\ a_4-a_3<0,\ 2-a_2-a_3\geq 0, 2+3a_2-a_4\geq 0; \end{cases} \\ 2-a_2-a_4\geq 0, 2+2a_2-a_3\geq 0; \\ \begin{cases} a_4-a_3\geq 0,\ 2-a_2-a_4\geq 0, 2+2a_2-a_3\geq 0, \\ a_4-a_3<0,\ 2-a_2-a_3\geq 0, 2+3a_2-a_4\geq 0; \end{cases} \end{cases} & \text{for } (5.16)(1)-(5.16)(4) \\ \begin{cases} 2-a_3-a_4\geq 0, 2-a_2+a_4\geq 0; \\ 2-a_3-a_4\geq 0, 2-a_2+a_4\geq 0; \\ 2-a_3-a_4\geq 0, 2-a_2+a_4\geq 0; \\ 2-a_3-a_4\geq 0, 2-a_2+a_4\geq 0; \\ 2-a_3-a_4\geq 0, 2-a_2+a_4\geq 0 \end{cases} & \text{for } (5.16)(5)-(5.16)(8). \end{cases} \tag{5.17}$$

Therefore, together with the condition that $a_3 \geq a_2 \geq 0$, Inequalities (5.16) and (5.17) result in the following families of Calabi–Yau threefolds embedded in fourfold scrolls $\mathbb{F}(a_j|1^3,2)$

$$\begin{cases} \mathbb{F}(0,0,1,2|1^3,2),\ \mathbb{F}(0,0,2,2|1^3,2),\ \mathbb{F}(0,1,3,1|1^3,2),\ \mathbb{F}(0,1,2,1|1^3,2), \\ \mathbb{F}(0,1,2,0|1^3,2) \text{ and } \mathbb{F}(0,2,2,0|1^3,2) \text{ for } (5.16)(1)+(5.17)(1),\dots,(5.16)(4)+ \\ (5.17)(4a,4b); \\ \mathbb{F}(0,0,2,1|1^3,2),\ \mathbb{F}(0,0,1,-2|1^3,2),\ \mathbb{F}(0,0,2,-2|1^3,2),\ \mathbb{F}(0,0,3,-2|1^3,2), \\ \mathbb{F}(0,0,3,-1|1^3,2) \text{ and } \mathbb{F}(0,0,4,-2|1^3,2) \text{ for } (5.16)(5)+(5.17)(5),\dots,(5.16)(8)+ \\ (5.17)(8). \end{cases}$$

having either $\mathbb{P}^1$ and or a curve of $\frac{1}{2}(1,1,0)$ singularities along the base locus as the respective base locus. The new singular families are colour coded blue whereas the rest of the families are quasismooth and already classified in [Mul06].

We now analyze the singularities in the new families. We are only interested in the families with at worst isolated singularities along the base locus.

1. For $\mathbb{F} = \mathbb{F}(0,0,1,2|1^3,2)$ we have that $X \in |-K_{\mathbb{F}}| = |-L+5M|$. Also

$$Sing(\mathbb{F}) = \{x_1 = x_2 = x_3 = 0\} \cong \mathbb{F}(2) \cong \mathbb{P}^1.$$

   A general section $f(t_1,t_2,x_1,x_2,x_3,y)$ of $|-L+5M|$ has coefficients

$$\deg(\alpha_{q_j}(t_i)) = -1+q_3+2q_4$$

   as listed in Figure (5.6). There is no pure y-term so we always have that $f(t_1,t_2;0,0,0,y) =$

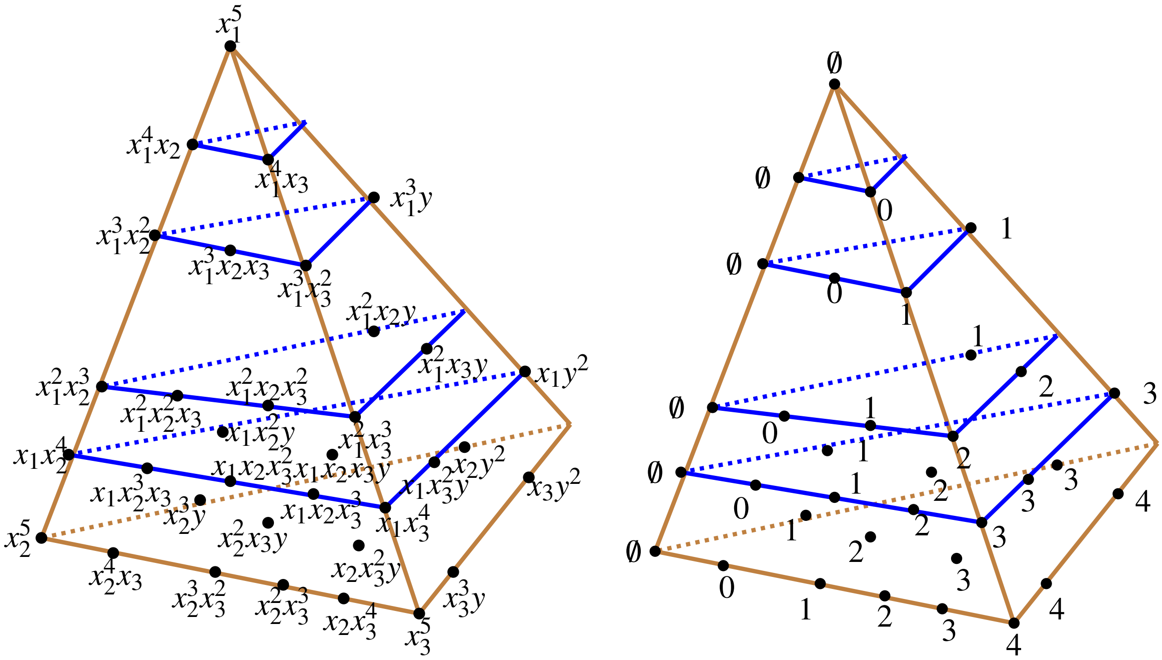


Figure 5.6: Degrees of coefficients of a section of $|-K_{\mathbb{F}(0,0,1,2|1^3,2)}|$

0 hence

$$Sing(\mathbb{F}) = \{x_1 = x_2 = x_3 = 0\} \subset Bs(|-K_{\mathbb{F}}|).$$

Further, since there is no term involving x_1^5 and x_2^5 we have a reducible two dimensional base locus

$$\begin{aligned} B = Bs(|-K_{\mathbb{F}}|) =& Sing(\mathbb{F}) \cup \{x_3 = y = 0\} \\ =& \{x_1 = x_2 = x_3 = 0\} \sqcup \mathbb{F}(0,0) \end{aligned}$$

with disjoint components.

Consider the equations

$$\begin{aligned} \left.\frac{\partial f}{\partial y}\right|_{x_3=y=0} =& \alpha_{3001}(t_1,t_2)x_1^3 + \alpha_{2101}(t_1,t_2)x_1^2x_2 + \alpha_{1201}(t_1,t_2)x_1x_2^2 + \alpha_{0301}(t_1,t_2)x_2^3 \in \\ & |L_{1,3}|_{\mathbb{F}(0,0)}|; \\ \left.\frac{\partial f}{\partial x_3}\right|_{x_3=y=0} =& \sum_{k=0}^{4} \alpha_{(4-k)k10}(t_1,t_2)x_1^{4-k}x_2^k \in |L_{0,4}|_{\mathbb{F}(0,0)}|. \end{aligned}$$

Therefore, along $\{x_3 = y = 0\}$ we have 4 isolated singularities

$$\begin{aligned} Sing(\widetilde{X}) \cap \{x_3 = y = 0\} =& \mathbb{V}\left(\frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial y}, x_3, y\right) \\ =& |L_{1,3}|_B| \cap |L_{0,4}|_B| \subset \mathbb{P}^1_{[t_i]} \times \mathbb{P}^1_{[x_1:x_2]}. \end{aligned}$$

Along the curve $Sing(\mathbb{F}) = \{x_1 = x_2 = x_3 = 0\}$ the singularity of X is defined by general, nonzero forms in $\mathbb{C}[t_1,t_2]$ of different degrees $3, 3, 4$

$$\begin{aligned} Sing(\widetilde{X}) \cap \mathbb{V}(x_1,x_2,x_3) =& \mathbb{V}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, x_1, x_2, x_3\right) \\ =& \mathbb{V}(\alpha_{1002}(t_i), \alpha_{0102}(t_i), \alpha_{0012}(t_i)) = \emptyset. \end{aligned}$$

The threefold X is therefore quasismooth along $Sing(\mathbb{F})$ of $\frac{1}{2}(1,1,1,0)$ curve singularity on $\mathbb{F}$. This family makes it to the list.

2. For $\mathbb{F} = \mathbb{F}(0,1,2,0|1^3,2)$ we have $X \in |-K_{\mathbb{F}}| = |-L+5M|$. A general section $f(t_1,t_2,x_1,x_2,y_1,y_2)$ of $|-L+5M|$ has coefficients

$$\deg(\alpha_{q_j}(t_i)) = -1 + q_2 + 2q_3$$

as listed in Figure (5.7). A general section of $|-K_{\mathbb{F}(0,1,2,0|1^3,2)}|$ is therefore divisible

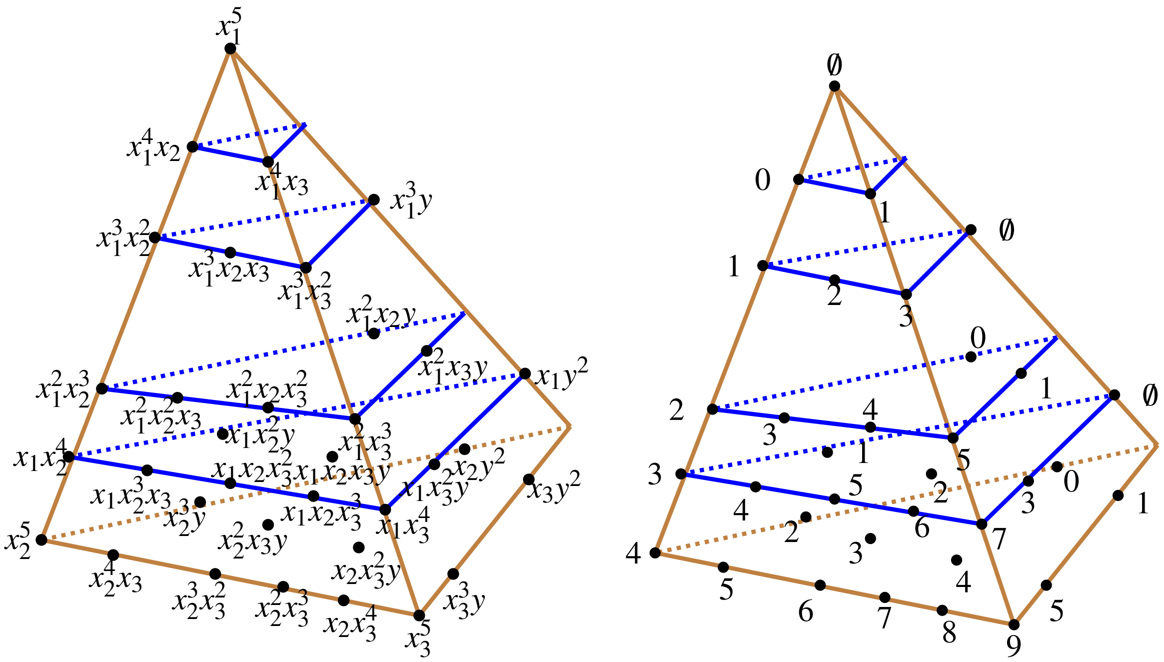


Figure 5.7: Degrees of coefficients of a section of $|-K_{\mathbb{F}(0,1,2,0|1^3,2)}|$

by either x_2 and x_3. Consequently, along the base locus

$$B = Bs(|-L+5M|) = \{x_2 = x_3 = 0\} \cong \mathbb{P}^1_{[t_i]} \times \mathbb{P}^1_{[x_1^2:y]}$$

we have that

$$\left.\frac{\partial f}{\partial x_2}\right|_{x_2=x_3=0} = \alpha_{4100}(t_1,t_2)x_1^4 + \alpha_{2101}(t_1,t_2)x_1^2y + \alpha_{0102}(t_1,t_2)y^2 \in |L_{0,2}|_B|,$$

$$\left.\frac{\partial f}{\partial x_3}\right|_{x_2=x_3=0} = \alpha_{4010}(t_1,t_2)x_1^4 + \alpha_{2011}(t_1,t_2)x_1^2y + \alpha_{0012}(t_1,t_2)y^2 \in |L_{1,2}|_B|.$$

Along B, we have 2 isolated singularities $|L_{0,2}|_B| \cap |L_{1,2}|_B|$. The family X is quasismooth along $Sing(\mathbb{F}) = \{x_1 = x_2 = x_3 = 0\} \subset B$. This family makes it to the list.

The Table (5.2) has two new singular families with with at most isolated singularities along the respective base locus. The new singular families with worse than isolated singularities as well as the quasismooth families classified in [Mul06] are not included in the list. $\square$

5.3 Calabi–Yau threefolds with Sextic fibres

5.3.1 Fibres: $X_6 \subset \mathbb{P}^3[1,1,1,3]$

In this section, we classify sextic K3

$$X_6 \subset \mathbb{P}[b_j] = \mathbb{P}^3[1^3,3]$$

fibred Calabi–Yau threefolds $X|-K_{\mathbb{F}(a_j|1^3,3)}|$ with quotient singularities and finite isolated singular points along $Bs(|-K_{\mathbb{F}(a_j|1^3,3)}|)$.

This is achieved by relaxing the quasismooth requirement on $X \subset \mathbb{F} = \mathbb{F}(a_j|1^3,3)$ so that, other than $\frac{1}{3}(1,1,1)$ singularities on X, there are at worst isolated singularities on the base locus $Bs(|-K_{\mathbb{F}}|)$. The classification theorem proved in this section extends the (1113) list in the appendix A of [Mul06] by 2 new singular families in Table (5.3).

No.	$\mathbb{F} = \mathbb{F}(a_j|1^3,3)$	General $X \in |-K_{\mathbb{F}}|$
11	$\mathbb{F}(0,0,1,2|1^3,3)$	General $X \in |-K_{\mathbb{F}}|$ has 5 isolated ODP singularities on $Bs(|-K_{\mathbb{F}}|)$
12	$\mathbb{F}(0,0,2,1|1^3,3)$	General $X \in |-K_{\mathbb{F}}|$ has 3 isolated ODP singularities on $Bs(|-K_{\mathbb{F}}|)$

Table 5.3: Sextic fibred Calabi–Yau threefolds $X \subset \mathbb{F} = \mathbb{F}(a_j|1^3,3)$ with at worst isolated singularities along $Bs(|-K_{\mathbb{F}}|)$

Lemma 5.3.1. *The unique normal form of the $\mathbb{P}[1,1,1,3]$-bundle over $\mathbb{P}^1$ is the weighted 4-fold scroll*

$$\mathbb{F}(a,b,c,d|1,1,1,3) \cong \mathbb{F}(0,a_2,a_3,a_4|1,1,1,3)$$

with integers $a_3 \geq a_2 \geq 0$ and $a_4 \in \mathbb{Z}$.

`Proof.` Note that

$$\begin{aligned}\mathbb{F}(a_1,a_2,a_3,d|1,1,1,3) =& \mathbb{F}\begin{bmatrix}1 & 1 & -a_{\sigma(1)}+k & -a_{\sigma(2)}+k & -a_{\sigma(3)}+k & -d+3k \\ 0 & 0 & 1 & 1 & 1 & 3\end{bmatrix}, \ \sigma \in S_3 \\ \cong& \mathbb{F}\begin{bmatrix}1 & 1 & 0 & -a_2 & -a_3 & -a_4 \\ 0 & 0 & 1 & 1 & 1 & 3\end{bmatrix}.\end{aligned}$$

Hence, uniquely, we have that

$$\mathbb{F}(a_1,a_2,a_3,d|1^3,3) \cong \mathbb{F}(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)},d|1^3,3) \cong \mathbb{F}(0,a_2,a_3,a_4|1^3,3)$$

with integers $a_3 \geq a_2 \geq 0$ and $a_4 \in \mathbb{Z}$. □

The anticanonical divisor of $\mathbb{F}_A = \mathbb{F}(0,a_2,a_3,a_4|1^3,3)$ is given by

$$-K_{\mathbb{F}_A} = \left(2-\sum_j a_j\right)L + \left(\sum_j b_j\right)M = L_{2-a_2-a_3-a_4,6}.$$

We then have that

$$\mathbf{Sym}^{2+a_2(q_2-1)+a_3(q_3-1)+a_4(q_4-1)}(t_i)\prod_j x_j^{q_j} \subset H^0(\mathbb{F}_A, L_{2-a_2-a_3-a_4,6})$$

with a general 3-fold section $X \in |L_{2-(a_2+a_3+a_4),6}|$ is given by

$$X = \mathbb{V}\left(\sum_{(q_1,q_2,q_3,q_4)_{(1^3,3)}\vdash 6} \alpha_{(q_j)}(t_1,t_2)x_1^{q_1}x_2^{q_2}x_3^{q_3}y^{q_4}\right) \subset \mathbb{F}(0,a_2,a_3,a_4|1^3,3)$$

where $x_1^{q_1}x_2^{q_2}x_3^{q_3}y^{q_4}$ from the Newton tetrahedron in Figure (5.8).

The degrees of coefficients

$$\deg \alpha_{(q_j)_{(1,1,1,3)}\vdash 6}(t_i) = 2+(q_2-1)a_2+(q_3-1)a_3+(q_4-1)a_4$$

of the monomials $x_1^{q_1}x_2^{q_2}x_3^{q_3}y^{q_4}$ increase down the x_1-x_2 edge of the tetrahedron by a_2, increase down the x_1-x_3 edge of the tetrahedron by a_3; increase, decrease or stay constant down the x_1-y edge by $a_4 \in \mathbb{Z}$. They increase from edge x_1-x_2 to edge x_1-x_2 to edge x_1-x_3 by a_2-a_1 and a_3-a_2 respectively. Increase, decrease or stay constant by a_4-a_3 from x_1-y edge to x_1-x_3 edge.

Lemma 5.3.2. *On the fibres of $\pi : \mathbb{F}(0,a_2,a_3,a_4|1^3,3) \to \mathbb{P}^1$ we have*

$$Sing\mathbb{P}^3_{x_j,y}[1^3,3] = \{[0:0:0:1]\}$$

whereas on $\mathbb{F} = \mathbb{F}(0,a_2,a_3,a_4|1^3,3)$ there are $\frac{1}{3}(1,1,1,0)$ singularities

$$Sing(\mathbb{F}) = \mathbb{V}(x_1,x_2,x_3) \cong \mathbb{P}^1_{[t_i]}.$$

Theorem 5.3.3. *Let $\mathbb{F} = \mathbb{F}(0,a_2,a_3,a_4|1,1,1,3)$ be a $\mathbb{P}[1^3,3]$ bundle over $\mathbb{P}^1$ with sextic fibres. There are* 26 *sextic K3 surface fibred Calabi–Yau* 3*-folds with quotient singularities and finite isolated singularities listed in the Tables* (5.4),(5.5).

`Proof.` A general section of $-K_{\mathbb{F}} = L_{2-a_2-a_3-a_4,6}$ having x_1^6, x_2^6, x_3^6 and y^2 terms in $f(t_i,x_j,y)$ implies base point freeness (bpf) of $-K_{\mathbb{F}}$ which in turn implies quasismoothness of a general $X \in |-K_{\mathbb{F}}|$. Equivalently, bpf is implied by the degrees of the coefficients

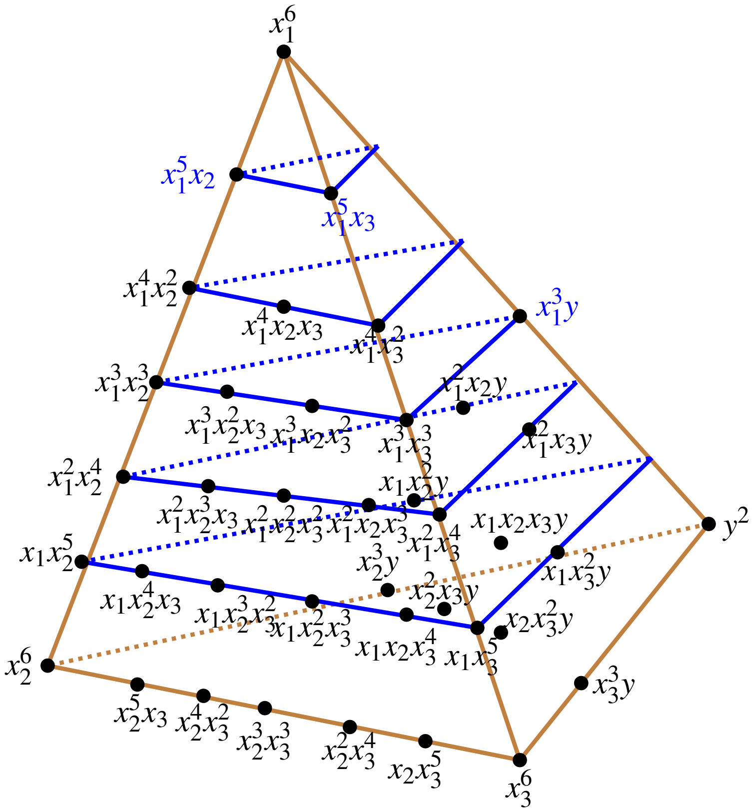


Figure 5.8: The 39 monomial generators of $\mathbb{C}[\mathbb{P}^3_{x_j,y}[1^3,3]]_6$.

$\alpha_{(q_j)_{(1,1,1,3)}\vdash 6}(t_i)$ of monomials $x_1^{q_1}x_1^{q_2}x_3^{q_3}y^{q_4}$ at the vertices of Newton triangle (5.8) being non-negative.

$$\begin{aligned}
2-a_2-a_3+a_4 &\geq 0, \qquad (5.18)\\
2-a_2+5a_3-a_4 &\geq 0,\\
2+5a_2-a_3-a_4 &\geq 0,\\
2-a_2-a_3-a_4 &\geq 0.
\end{aligned}$$

The Inequality (4.7) together with Lemma (5.3.1) corresponds to a list of 10 quasismooth Calabi–Yau three-folds $X \in |-K_{\mathbb{F}(0,a_2,a_3,a_3|1,1,1,3)}|$ in Table (5.4) .

A weaker condition than bpf results in these further cases

1. The threefold $X \in |-K_{\mathbb{F}}|$ is irreducible hence $\dim(Bs(|-K_{\mathbb{F}}|)) \neq 3$. Equivalently, $x_j, y \not| f(t_i, x_j, y)$; that is, the largest corresponding degree of coefficients of the monomials on each face of the Newton tetrahedron (5.8) is nonnegative. The equivalent condition is that the larger of the three corresponding degrees of coefficients of the

No.	$\mathbb{F}=\mathbb{F}(a_j\|1^3,3)$	General $X\in\|-K_{\mathbb{F}}\|$
1	$\mathbb{F}(0,0,0,-2\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is nonsingular
2	$\mathbb{F}(0,0,0,-1\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{3}(1,1,1)$ singularity
3	$\mathbb{F}(0,0,0,0\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{3}(1,1,1)$ singularities
4	$\mathbb{F}(0,0,0,1\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $3\times\frac{1}{3}(1,1,1)$ singularities
5	$\mathbb{F}(0,0,0,2\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $4\times\frac{1}{3}(1,1,1)$ singularities
6	$\mathbb{F}(0,0,1,-1\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{3}(1,1,1)$ singularities
7	$\mathbb{F}(0,0,1,0\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $\frac{1}{3}(1,1,1)$ singularity
8	$\mathbb{F}(0,0,1,1\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{3}(1,1,1)$ singularities
9	$\mathbb{F}(0,0,2,0\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is nonsingular
10	$\mathbb{F}(0,1,1,0\|1^3,3)$	$-K_{\mathbb{F}}$ base point free, general $X\in\|-K_{\mathbb{F}}\|$ is nonsingular
11	$\mathbb{F}(0,0,1,2\|1^3,3)$	General $X\in\|-K_{\mathbb{F}}\|$ has 5 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and $3\times\frac{1}{3}(1,1,1)$ singularities
12	$\mathbb{F}(0,0,2,1\|1^3,3)$	General $X\in\|-K_{\mathbb{F}}\|$ has 3 isolated ODP singularities on $Bs(\|-K_{\mathbb{F}}\|)$ and $\frac{1}{3}(1,1,1)$ singularity
13	$\mathbb{F}(0,0,2,2\|1^3,3)$	General $X\in\|-K_{\mathbb{F}}\|$ is quasismooth with $2\times\frac{1}{3}(1,1,1)$ singularities

Table 5.4: $\mathbb{F}=\mathbb{F}(a_j|1^3,3)$ for which a general $X\in|-K_{\mathbb{F}}|$ with at worst isolated singularities along $Bs(|-K_{\mathbb{F}}|)$

monomials on the x_1-x_3-y face or $x_1-x_2-x_3$ face is nonnegative

$$\begin{cases} 2-a_2+5a_3-a_4\geq 0, a_4-3a_3\geq 0, a_4-3a_2\geq 0 & \text{if } \deg\alpha_{(0002)} \text{ is largest} \\ \begin{cases} 2+5a_2-a_3-a_4\geq 0, a_4-3a_3<0, a_4-3a_2<0 \text{ or} \\ 2-a_2-a_3+a_4\geq 0, a_4-3a_3<0, a_4-3a_2\geq 0 \end{cases} & \text{if } \deg\alpha_{(0060)} \text{ is largest.} \end{cases} \tag{5.19}$$

2. Suppose $\dim(Bs(|L_{2-a_2-a_3-a_4,6}|))=2$. The inequalities (5.19) imply, depending on the value of $a_4\in\mathbb{Z}$, that the equation of a general section of $|L_{2-a_2-a_3-a_4,6}|$ simultaneously has nonzero term involving the pairs $\{x_3^3,y^2\}$ for (5.19)(1), $\{x_2^6,x_3^2\}$ for (5.19)(2) or $\{x_3^3,y^2\}$ for (5.19)(3).

   (i) From the pair $\{x_3^3,y^2\}$, we deduce that the two dimensional base locus is

$$Bs(|L_{2-a_2-a_3-a_4,6}|)=\mathbb{V}(x_3,y)=\mathbb{F}(0,a_2).$$

No.	$\mathbb{F} = \mathbb{F}(a_j \vert 1^3, 3)$	General $X \in \vert -K_{\mathbb{F}} \vert$
14	$\mathbb{F}(0,1,1,1 \vert 1^3, 3)$	General $X \in \vert -K_{\mathbb{F}} \vert$ is quasismooth with $\frac{1}{3}(1,1,1)$ singularity
15	$\mathbb{F}(0,1,1,2 \vert 1^3, 3)$	General $X \in \vert -K_{\mathbb{F}} \vert$ is quasismooth with $2 \times \frac{1}{3}(1,1,1)$ singularities
16	$\mathbb{F}(0,1,1,3 \vert 1^3, 3)$	General $X \in \vert -K_{\mathbb{F}} \vert$ is quasismooth with $3 \times \frac{1}{3}(1,1,1)$ singularities
17	$\mathbb{F}(0,1,1,4 \vert 1^3, 3)$	$-K_{\mathbb{F}}$ base point free, general $X \in \vert -K_{\mathbb{F}} \vert$ is quasismooth with $4 \times \frac{1}{3}(1,1,1)$ singularities
18	$\mathbb{F}(0,1,1,5 \vert 1^3, 3)$	$-K_{\mathbb{F}}$ base point free, general $X \in \vert -K_{\mathbb{F}} \vert$ is quasismooth with $5 \times \frac{1}{3}(1,1,1)$ singularities
19	$\mathbb{F}(0,1,1,6 \vert 1^3, 3)$	$-K_{\mathbb{F}}$ base point free, general $X \in \vert -K_{\mathbb{F}} \vert$ is quasismooth with $6 \times \frac{1}{3}(1,1,1)$ singularities
20	$\mathbb{F}(0,1,2,1 \vert 1^3, 3)$	General $X \in \vert -K_{\mathbb{F}} \vert$ is nonsingular

Table 5.5: $\mathbb{F} = \mathbb{F}(a_j|1^3,3)$ for which a general $X \in |-K_{\mathbb{F}}|$ with at worst isolated singularities along $Bs(|-K_{\mathbb{F}}|)$

This is equivalent to there being no terms on the $x_1 - x_2$-edge of the Newton tetrahedron (5.8) in $f(t_i, x_j, y)$. Equivalently, there is no x_2^6 term in $f(t_i, x_j, y)$; That is

$$2 + 5a_2 - a_3 - a_4 < 0. \tag{5.20}$$

Further, in the cone

$$\widetilde{X} = q^{-1}(X) \subset \mathbb{C}^2_{t_1,t_2} \setminus \{(0,0)\} \times \mathbb{C}^4_{x_j,y} \setminus \{(0,0,0,0)\} \xrightarrow{q} \mathbb{F}(0, a_2, a_3, a_4|1^3, 3),$$

we observe that

$$\begin{aligned} Sing(\widetilde{X}) \cap \widetilde{\mathbb{V}(x_3,y)} &= \mathbb{V}\left(\frac{\partial f(t_i,x_j,y)}{\partial x_3}, \frac{\partial f(t_i,x_j,y)}{\partial y}, x_3, y\right) \\ &= \mathbb{V}\left(\sum_{j=0}^{5} \alpha_{(5-j,j,1,0)}(t_1,t_2) x_1^{5-j} x_2^j, \sum_{k=0}^{3} \alpha_{(3-k,k,0,1)}(t_1,t_2) x_1^{3-k} x_2^k\right) \\ &= \mathbb{V}(f_1, f_2) \subset \mathbb{F}(0, a_2) \end{aligned}$$

where $f_1 \in |L_{2-a_2-a_4,5}|$ and $f_2 \in |L_{2-a_2-a_3,3}|$ and $\widetilde{\mathbb{V}(x_3,y)} = q^{-1}(F_{a_2})$.

Therefore for X to have isolated singularities, we must have $|L_{2-a_2-a_4,5}|$ and $|L_{2-a_2-a_3,3}|$

both non-empty and without a common component. Equivalently,

$$\begin{aligned}
&\deg \alpha_{(0510)}(t_i) = 2+4a_2-a_4 \geq 0, \\
&\deg \alpha_{(0301)}(t_i) = 2+2a_2-a_3 \geq 0 \text{ and} \\
&\mathrm{Max}(\deg \alpha_{(5010)}(t_i), \deg \alpha_{(3001)}(t_i)) = \begin{cases} 2-a_2-a_4 \geq 0 & \text{for (5.19)(1) or} \\ 2-a_2-a_3 \geq 0 & \text{for (5.19)(3)} \end{cases}.
\end{aligned} \tag{5.21}$$

Therefore, from Lemma (5.3.1), Inequalities (5.19)(1), (5.20) and (5.21) OR from Lemma (5.3.1), Inequalities (5.19)(3), (5.20) and (5.21); we get the lattice point (a_2,a_3,a_4) corresponding to the ambient weighted 4-fold scrolls

$\mathbb{F}(0,a_2,a_3,a_4|1^3,3) = \mathbb{F}(0,0,1,2|1^3,3), \mathbb{F}(0,0,2,1|1^3,3)$ and $\mathbb{F}(0,0,2,2|1^3,3)$;

for 3 Calabi–Yau threefold families $X \in |-K_{\mathbb{F}(a_j|1^3,3)}|$.

For $\mathbb{F} = \mathbb{F}(0,0,1,2|1^3,3)$, let

$$([\gamma_{1i}:\gamma_{2i}],[\beta_{1i}:\beta_{2i}]) \in \mathbb{P}^1_{[t_i]} \times \mathbb{P}^1_{[x_j]} = Bs(|-K_{\mathbb{F}}|).$$

That is, $[\beta_{1i}:\beta_{2i}]$ is one of the 5 roots of a general homogeneous quintic in x_1,x_2 whereas $[\gamma_{1i}:\gamma_{2i}]$ satisfies a linear equation in t_1,t_2 for a fixed $[\beta_{1i}:\beta_{2i}]$. Therefore, for the family $X \in |-K_{\mathbb{F}(0,0,1,2|1^3,3)}|$, we have that

$$\begin{aligned}
&Sing(X) \cap \mathbb{F}(0,0) = \mathbb{V}\left(\sum_{j=0}^{5} c_j x_1^{5-j} x_2^j,\ \sum_{k=0}^{3} \alpha_1(t_1,t_2) x_1^{3-k} x_2^k\right) \\
&= \{e_i = [\gamma_{1i}:\gamma_{2i};\beta_{1i}:\beta_{2i}:0:0] : i = 1,\ldots,5 = (0,5).(1,3)\} \subset \mathbb{P}^1_{[t_i]} \times \mathbb{P}^1_{[x_j]} \subset \\
&X \subset \mathbb{F}(0,0,1,2|1^3,3).
\end{aligned}$$

Locally on the chart $\mathbb{A}^4_{t_2,x_1,x_3,y}$, a change of coordinates

$$t_2 = t_2' + \gamma_{i2},\ x_1 = x_1' + \beta_{1i}$$

results in

$$\begin{aligned}
X \cap \mathbb{A}^4_{t_2,x_1,x_3,y} =& \mathbb{V}\left(x_3 \sum_{j=0}^{5} c_j (x_1'+\beta_{1i})^{5-j} + y \sum_{k=0}^{3} (x_1'+\beta_{1i})^{3-k} \alpha_1(t_2'+\gamma_{i2}) + \text{ H.O.T}\right) \\
=& \mathbb{V}(c_4 x_1' x_3 + y(\delta_2 x_1' + \delta_3 t_2' x_3) + \text{ H.O.T}),\ \text{let } (y')^2 := x_1' y, (t_2'')^2 := t_2' x_3 y \\
=& \mathbb{V}(x_1' x_3 + (y')^2 + (t_2'')^2 + \text{ H.O.T}),\ \text{let } T := y' + i t_2'', Y := y' - i t_2'' \\
=& \mathbb{V}(x_1' x_3 + YT + \text{ H.O.T}) \\
=& \mathbb{V}(q_2(T,x_1',x_3,Y) + \text{ H.O.T }) \subset \mathbb{A}^4_{(T,x_1',x_3,Y)} \subset \mathbb{F}(0,0,1,2|1^3,3),
\end{aligned}$$

where $q_2 = x_1' x_3 + YT$ is a full rank quadratic expression. Therefore, the Calabi–Yau threefold X has 5 isolated threefold Ordinary Double Point (ODP) singularities $\{e_i\}$.

A similar analysis shows that the Calabi–Yau threefold $X \in |-K_{\mathbb{F}(0,0,2,1|1^3,3)}|$ has $3 = (1,5).(0,3)$ isolated threefold ODP singularities $\{f_k\}$.

For the threefold $X \subset \mathbb{F}(0,0,2,2|1^3,3)$, there are no isolated singularities along the base locus $\mathbb{P}^1_{[t_i]} \times \mathbb{P}^1_{[x_j]}$, hence a quasismooth CY3-fold family.

(ii) From the second pair $\{x_2^6, x_3^6\}$, we deduce that the two dimensional base locus is

$$Bs(|L_{2-a_2-a_3-a_4,6}|) = \mathbb{V}(x_2,x_3) = \mathbb{F}(0,a_4|1,3).$$

Equivalently there is no term on the $x_1 - y$-edge of the Newton tetrahedron (5.8) in $f(t_i,x_j,y)$. Equivalently,

$$\mathrm{Max}(2-a_2-a_3-a_4, 2-a_2-a_3+a_4) < 0.$$

That is to say

$$\begin{cases} 2-a_2-a_3-a_4 < 0, 3a_3 \geq 3a_2 \geq 0 > a_4 & \text{or} \\ 2-a_2-a_3+a_4 < 0, 3a_3 \geq 3a_2 > a_4 \geq 0 \end{cases} \tag{5.22}$$

We then have, along $\mathbb{F}(0,a_4|1,3)$, that

$$\begin{aligned} Sing(\widetilde{X}) \cap \widetilde{\mathbb{F}(0,a_4|1,3)} &= \mathbb{V}\left(\frac{\partial f(t_i,x_j,y)}{\partial x_2}, \frac{\partial f(t_i,x_j,y)}{\partial x_3}, x_2, x_3\right) \\ &= \mathbb{V}\left(\alpha_{(5100)}(t_i)x_1^5 + \alpha_{(2101)}x_1^2 y, \alpha_{(5010)}(t_i)x_1^5 + \alpha_{(2011)}x_1^2 y\right) \\ &= \mathbb{V}(g_1,g_2) \subset \mathbb{F}(0,a_4|1,3) \end{aligned}$$

where $g_1 \in |L_{2-a_3-a_4,5}|$ and $g_2 \in |L_{2-a_2-a_4,5}|$.

We observe that $x_1 | g_1, g_2$ hence Sing$X$ is reducible and contains the component $\mathbb{V}(x_1) = \mathbb{F}(0,a_3-a_2,a_4-3a_2|1,1,3)$ along the base locus $Bs(-K_{\mathbb{F}}) = \mathbb{F}(0,a_4|1,3)$. Therefore in this case, there are no new examples under the assumption of the classification problem; at most isolated singularities on X along $Bs(|-K_{\mathbb{F}(a_j|1^3,3)}|)$.

(iii) Let $f = f(t_i,x_j,y)$ be a section of $|-K_{\mathbb{F}(a_j|1^3,3)}| = |L_{2-a_2-a_3-a_4,6}|$ and suppose that $\dim(Bs(|L_{2-a_2-a_3-a_4,6}|)) = 1$. We either have that

$$x_1^6 \notin f \text{ and } x_2^6, x_3^6, y^2 \in f$$

or

$$y^2 \notin f \text{ and } x_1^6, x_2^6, x_3^6 \in f.$$

Equivalently

$$\begin{cases} 2+5a_2-a_3-a_4<0, 2-a_2-a_3+a_4\geq 0 \text{ and } \begin{cases} 2-a_2-a_3\geq 0 & \text{or} \\ 2-a_2-a_4\geq 0 \end{cases} \\ 2-a_2-a_3+a_4<0, 2-a_2-a_3-a_4\geq 0 \end{cases}$$

The one-dimensional base locus is therefore

$$X=\mathbb{V}(f(t_i,x_j,y))=Bs(|L_{2-a_2-a_3-a_4,6}|)=\begin{cases} \mathbb{V}(x_2,x_3,y)=\mathbb{F}(a_1)=\mathbb{F}(0)\cong\mathbb{P}^1 & \text{or} \\ \mathbb{V}(x_1,x_2,x_3)\cong\mathbb{F}(a_4|3). \end{cases}$$

Along $\mathbb{F}(0)$ and on the chart $U_{11}=\{t_1=x_1=1\}=\mathbb{A}^4_{t_2,x_2,x_3,y}$ we have that

$$Sing(\widetilde{X})\cap\widetilde{\mathbb{F}(0)}=\mathbb{V}\left(\frac{\partial f(t_i,x_j,y)}{\partial x_2},\frac{\partial f(t_i,x_j,y)}{\partial x_3},\frac{\partial f(t_i,x_j,y)}{\partial y},x_2,x_3,y\right)$$
$$=\mathbb{V}(\alpha_{(5100)}(t_2),\alpha_{(5010)}(t_2),\alpha_{(3001)}(t_2))\subset\mathbb{C}^2_{t_1,t_2}\setminus\{(0,0)\}\times\mathbb{C}^4_{x_j,y}\setminus\{(0,0,0,0)\}.$$

Along $\mathbb{F}(a_4|3)$ and on the chart $U_{1y}=\{t_1=y=1\}=\mathbb{A}^4_{t_2,x_1,x_2,x_3}$ we have that

$$Sing(\widetilde{X})\cap\widetilde{\mathbb{F}(a_4|3)}=\mathbb{V}\left(\frac{\partial f(t_i,x_j,y)}{\partial x_1},\frac{\partial f(t_i,x_j,y)}{\partial x_2},\frac{\partial f(t_i,x_j,y)}{\partial x_3},x_1,x_2,x_3\right)$$
$$=\emptyset\subset\mathbb{C}^2_{t_1,t_2}\setminus\{(0,0)\}\times\mathbb{C}^4_{x_j,y}\setminus\{(0,0,0,0)\}.$$

Hence, Lemma (5.3.1) and Inequalities (2iii) result in 7 families of quasismooth Calabi–Yau threefolds embedded in fourfold scrolls

$$\begin{aligned}\mathbb{F}(0,a_2,a_3,a_4|1^3,3)=&\mathbb{F}(0,1,2,1|1^3,3),\mathbb{F}(0,1,1,1|1^3,3),\ \mathbb{F}(0,1,1,2|1^3,3)\\ &\mathbb{F}(0,1,1,3|1^3,3),\mathbb{F}(0,1,1,4|1^3,3),\ \mathbb{F}(0,1,1,5|1^3,3)\\ &\text{and } \mathbb{F}(0,1,1,6|1^3,3).\end{aligned}$$

Note that the isolated singularities of the respective Calabi–Yau threefolds families $X\in|-K_{\mathbb{F}(a_j|1^3,3)}|$ do not meet the respective one dimensional base locus $Bs(|-K_{\mathbb{F}(a_j|1^3,3)}|)$ hence are either quasismooth or nonsingular.

From all cases above, $y^2\in f(t_i,x_j,y)$, the isolated singularities of the Calabi–Yau threefold

$$X=\mathbb{V}(f(t_i,x_j,y))\subset\mathbb{F}(0,a_2,a_3,a_4|1^3,3)$$

are therefore

$$X\cap Sing(\mathbb{F}(0,a_2,a_3,a_4|1,1,1,3))=\mathbb{V}(\alpha_{(0002)}(t_2))\times\{[0:0:0:1]\}=$$
$$\{p_k=[\beta_{1k}:\beta_{2k};0:0:0:1]:1\leq k\leq\deg\alpha_{(0002)}\}\subset Sing\,X.$$

So, near p_k, we have that

$$X \cap \{y = t_2 = 1\} = \mathbb{V}(f(t_1, x_j) = \{\alpha_{(0002)}(t_1) + \text{Other Terms} = 0\} \subset \mathbb{C}^4_{(t_1, x_1, x_2, x_3)}$$

from which we deduce that there are $(2 - a_2 - a_3 + a_4) \times \frac{1}{3}(1,1,1)$ quotient singularities at p_k for every k.

In conclusion, the Tables (5.4) and (5.5) summarise with reason the models of 4-fold straight scrolls $\mathbb{F} = \mathbb{F}(0, a_2, a_3, a_4 | 1^3, 3)$ in which the anticanonical threefolds have at most codimension two singularities on base locus of $Bs(|-K_{\mathbb{F}(a_j|1^3,3)}|)$. $\square$

5.3.2 Fibres: $X_6 \subset \mathbb{P}^3[1,1,2,2]$

Theorem 5.3.4. *Let $\mathbb{F} = \mathbb{F}(0,a_2,a_3,a_4|1,1,2,2)$ be a $\mathbb{P}[1^2,2^2]$ bundle over $\mathbb{P}^1$ with $X_6 \subset \mathbb{P}^3[1^2,2^2]$ fibres. There are 43 sextic K3 surface fibred Calabi–Yau 3-folds with at worst $\mathbb{A}^1 \times \frac{1}{2}(1,1)$ singularities along the base locus of $|-K_{\mathbb{F}}|$. The 3 families listed in the Table (5.6) are the families with quotient singularities and finite isolated singular points along the base locus. The quasismooth ones are already, up to isomorphism, in the (1122)-case in Appendix A of [Mul06].*

No.	$\mathbb{F} = \mathbb{F}(a_j\|1^2,2^2)$	General $X \in \|-K_{\mathbb{F}}\|$
1	$\mathbb{F}(0,0,1,2\|1^2,2^2)$	Has 4 ODP singularities along $Bs(\|-K_{\mathbb{F}}\|) = \mathbb{V}(y_1,y_2)$ and a genus 5 curve of $\frac{1}{2}(1,1,0)$ singularity along $\mathbb{V}(x_1,x_2)$
2	$\mathbb{F}(0,2,0,1\|1^2,2^2)$	Has 2 ODP singularities along $Bs(\|-K_{\mathbb{F}}\|) = \mathbb{V}(x_2,y_2)$ with $\mathbb{P}^1$ and a genus 0 curves of $\frac{1}{2}(1,0,1)$ singularity along $\mathbb{V}(x_1,x_2)$

Table 5.6: $\mathbb{F} = \mathbb{F}(a_j|1^2,2^2)$ for which a general $X \in |-K_{\mathbb{F}}|$ with quotient singularities and finite isolated singular points along $Bs(|-K_{\mathbb{F}}|)$

Proof. We have, by a standard isomorphism, that

$$\begin{aligned}\mathbb{F}(a_1,a_2,d_1,d_2|1,1,2,2) =& \mathbb{F}\begin{bmatrix}1 & 1 & -a_{\sigma(1)}+k & -a_{\sigma(2)}+k & -d_{\rho(1)}+2k & -d_{\rho(2)}+2k\\ 0 & 0 & 1 & 1 & 1 & 3\end{bmatrix}, \ \sigma,\rho \in S_2\\ \cong& \mathbb{F}\begin{bmatrix}1 & 1 & 0 & -a_2 & -a_3 & -a_4\\ 0 & 0 & 1 & 1 & 2 & 2\end{bmatrix}.\end{aligned}$$

Hence, uniquely, we have that

$$\mathbb{F}(a_1,a_2,d_1,d_2|1^2,2^2) \cong \mathbb{F}(a_{\sigma(1)},a_{\sigma(2)},d_{\rho(1)},d_{\rho(2)}|1^2,2^2) \cong \mathbb{F}(0,a_2,a_3,a_4|1^2,2^2)$$

with integers $a_2 \geq 0$ and $a_4 - a_3 \geq 0$.

We assume that the anticanonical divisor class $|-K_{\mathbb{F}}| = |L_{2-a_2-a_3-a_4,6}| \neq \emptyset$. The quasismooth Calabi–Yau threefold hypersurface $X \in |-K_{\mathbb{F}}|$ inherits its singularities from

$$\begin{aligned}Sing(\mathbb{F}(0,a_2,a_3,a_4|1^2,2^2)) &= \{x_1 = x_2 = 0\}\\ &= \mathbb{F}\begin{bmatrix}1 & 1 & -a_3 & -a_4\\ 0 & 0 & 2 & 2\end{bmatrix} \cong \mathbb{F}\begin{bmatrix}1 & 1 & -a_3 & -a_4\\ 0 & 0 & 1 & 1\end{bmatrix} = \mathbb{F}(a_3,a_4).\end{aligned}$$

We then have $\frac{1}{2}(1,1,0)$ quotient singularities on X along $X \cap \mathbb{F}(a_3,a_4)$

$$\begin{aligned}Sing\ X \supseteq& X \cap \mathbb{F}(a_3,a_4)\\ =& \mathbb{V}(\alpha_{(0030)}(t_i) + \alpha_{(0210)}(t_i) + \alpha_{(0120)}(t_i) + \alpha_{(0003)}(t_i)) \times \{[0:0:1:1]\}\\ =& \{r_k = [\beta_{1k} : \beta_{2k}; 0:0:1:1] : 1 \leq k \leq \mathrm{Max}(\deg \alpha_{(0030)}, \deg \alpha_{(0003)})\}.\end{aligned}$$

If

$$\deg \alpha_{(0030)} = 2 - a_2 + 2a_3 - a_4 \geq 2 - a_2 - a_3 + 2a_4 = \deg \alpha_{(0003)},$$

then near r_k we have that

$$X \cap \{y_1 = t_2 = 1\} = \mathbb{V}(f(t_1, x_j, y_2)) = \{\alpha_{(0030)}(t_1) + \text{Other Terms} = 0\}.$$

With β_{1k} a simple root of $f(t_1, x_j, y_2)$, we have that $\frac{\partial f(t_1, x_j, y_2)}{\partial t_1} \neq 0$. Therefore, up to the stabilizer $Stab_{(\mathbb{C}^*)^2}(\mathbb{F}(0, a_2, a_3, a_4|1^2, 2^2))$, we have that (x_1, x_2, y_2) is a set of local coordinates on $X \cap \mathbb{C}^4_{x_1, x_2, y_2}$ satisfying

$$(x_1, x_2, y_2) \mapsto (-x_1, -x_2, y_2) \text{ hence } (2 - a_2 + 2a_3 - a_4) \times \frac{1}{2}(1, 1, 0) \tag{5.23}$$

isolated quotient singularities at r_k for every k. Otherwise, we have

$$(x_1, x_2, y_1) \mapsto (-x_1, -x_2, y_1) \text{ hence } (2 - a_2 - a_3 + 2a_4) \times \frac{1}{2}(1, 1, 0) \tag{5.24}$$

isolated quotient singularities at r_k on $X \cap \mathbb{C}^4_{t_1, x_1, x_2, y_1}$ for every k.

We will refer to the Newton tetrahedon in Figure (5.9) with similar arguments as those used for elliptically fibred K3s in $\mathbb{F}(0, a_2, a_3|1, 2, 3)$.

Base point freeness of $-K_{\mathbb{F}} = L_{2-a_2-a_3-a_4, 6}$ is implied by the Inequalities

$$\begin{aligned} 2 - a_2 - a_3 + 2a_4 &\geq 0, \\ 2 - a_2 + 2a_3 - a_4 &\geq 0, \\ 2 + 5a_2 - a_3 - a_4 &\geq 0 \\ 2 - a_2 - a_3 - a_4 &\geq 0 \end{aligned} \tag{5.25}$$

whose integral solutions under the assumptions that $a_2 \geq 0, a_4 - a_3 \geq 0$ correspond to the following list of 11 quasismooth Calabi–Yau threefolds

$$\begin{aligned} \mathbb{F}(a_1, a_2, a_3, a_4|1^2, 2^2) = &\mathbb{F}(0, 0, -2, -2|1^2, 2^2),\ \mathbb{F}(0, 0, -1, -1|1^2, 2^2),\ \mathbb{F}(0, 0, -1, 0|1^2, 2^2) \\ &\mathbb{F}(0, 0, 0, 0|1^2, 2^2),\ \mathbb{F}(0, 0, 0, 1|1^2, 2^2),\ \mathbb{F}(0, 0, 0, 2|1^2, 2^2), \\ &\mathbb{F}(0, 0, 1, 1|1^2, 2^2),\ \mathbb{F}(0, 1, -1, -1|1^2, 2^2),\ \mathbb{F}(0, 1, 0, 0|1^2, 2^2), \\ &\mathbb{F}(0, 1, 0, 1|1^2, 2^2) \text{ and } \mathbb{F}(0, 2, 0, 0|1^2, 2^2). \end{aligned}$$

For $X = \mathbb{V}(f(t_i, x_j, y_j)) \in |-K_{\mathbb{F}(0, a_2, a_3, a_4|1^2, 2^2)}| = |L_{2-a_2-a_3-a_4, 6}|$ to be irreducible, the second largest of the corresponding degrees of coefficients of the monomials at the vertices

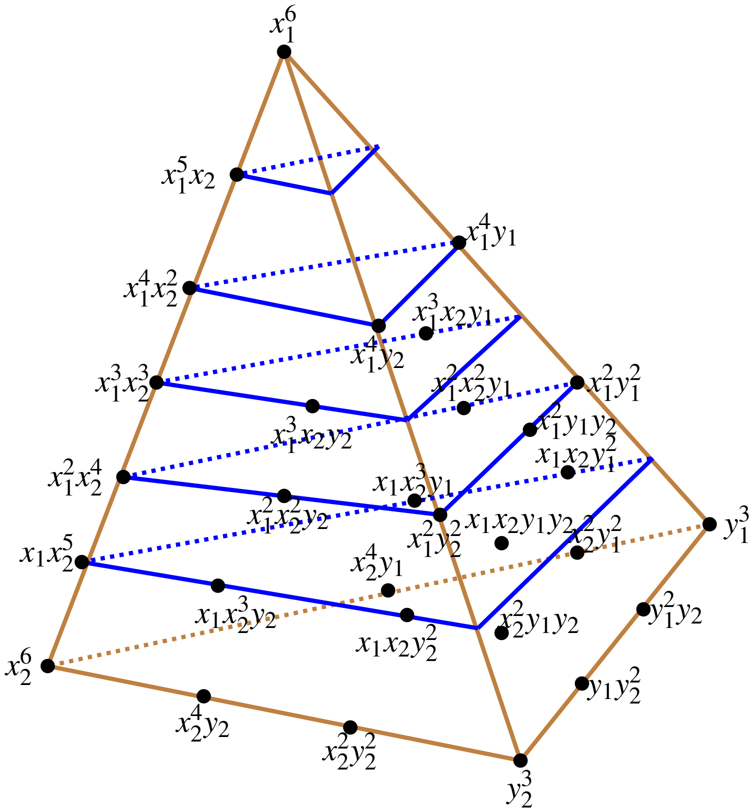


Figure 5.9: The 30 monomial generators of $\mathbb{C}[\mathbb{P}^3_{x_i,y_j}[1^2,2^2]]_6$.

of the Newton tetrahedron (5.9) is nonnegative so that at least one vertex of every face exists

$$\begin{cases} 2-a_2+2a_3-a_4 \geq 0, -2a_2+a_3 \geq 0, -2a_2+a_4 \geq 0, a_3 \geq 0, a_4 \geq 0 & \text{if } x_1^6 \leq x_2^6 \leq y_1^3 \leq y_2^3 \\ 2+5a_2-a_3-a_4 \geq 0, -2a_2+a_3 < 0, -2a_2+a_4 \geq 0, a_3 \geq 0, a_4 \geq 0 & \text{if } x_1^6 \leq y_1^3 \leq x_2^6 \leq y_2^3 \\ 2+5a_2-a_3-a_4 \geq 0, -2a_2+a_3 < 0, -2a_2+a_4 \geq 0, a_3 < 0, a_4 \geq 0 & \text{if } y_1^3 \leq x_1^6 \leq x_2^6 \leq y_2^3 \\ 2-a_2-a_3+2a_4 \geq 0, -2a_2+a_3 < 0, -2a_2+a_4 < 0, a_3 \geq 0, a_4 \geq 0 & \text{if } x_1^6 \leq y_1^3 \leq y_2^3 \leq x_2^6 \\ 2-a_2-a_3+2a_4 \geq 0, -2a_2+a_3 < 0, -2a_2+a_4 < 0, a_3 < 0, a_4 \geq 0 & \text{if } y_1^3 \leq x_1^6 \leq y_2^3 \leq x_2^6 \\ 2-a_2-a_3-a_4 \geq 0, -2a_2+a_3 < 0, -2a_2+a_4 < 0, a_3 < 0, a_4 < 0 & \text{if } y_1^3 \leq y_2^3 \leq x_1^6 \leq x_2^6. \end{cases} \tag{5.26}$$

Inequalities (5.26) guarantee that $\dim(Bs(|L_{2-a_2-a_3-a_4,6}|)) \neq 3$. If

$$\dim(Bs(|L_{2-a_2-a_3-a_4,6}|)) = 1,$$

we must have that

$$\begin{cases} 2-a_2-a_3-a_4<0,\ 2+5a_2-a_3-a_4\geq 0 & \text{from (5.26)}(1);\\ 2-a_2-a_3-a_4<0,\ 2-a_2+2a_3-a_4\geq 0 & \text{from (5.26)}(2);\\ 2-a_2-a_3-a_4\geq 0,\ 2-a_2+2a_3-a_4<0 & \text{from (5.26)}(3);\\ 2-a_2-a_3-a_4<0,\ 2-a_2+2a_3-a_4\geq 0 & \text{from (5.26)}(4);\\ 2-a_2-a_3-a_4\geq 0,\ 2-a_2+2a_3-a_4<0 & \text{from (5.26)}(5);\\ 2-a_2+2a_3-a_4<0, 2-a_2-a_3+2a_4\geq 0 & \text{from (5.26)}(6). \end{cases} \tag{5.27}$$

The one-dimensional base locus is the curve $B'=Bs(|L_{2-a_2-a_3-a_4,6}|)$ given by

$$\begin{cases} B_1=\mathbb{V}(x_2,y_1,y_2)=\mathbb{F}(a_2|1)\cong\mathbb{P}^1 & \text{for (5.27)(1), (5.27)(2), (5.27)(4),}\\ B_2=\mathbb{V}(x_1,x_2,y_2)\cong\mathbb{F}(a_3|2)\cong\mathbb{P}^1 & \text{for (5.27)(3), (5.27)(5), (5.27)(6).} \end{cases}$$

Along the curve B' we have that

$$\begin{aligned} Sing(\widetilde{X})\cap\widetilde{B}' &= \begin{cases} \mathbb{V}\left(\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial y_1},\frac{\partial f}{\partial y_2}\right)\cap B_1\\ \mathbb{V}\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial y_2}\right)\cap B_2; \end{cases}\\ &= \begin{cases} \mathbb{V}\left(\alpha_{5100}(t_i)x_1^5,\alpha_{4010}(t_i)x_1^4,\alpha_{4001}(t_i)x_1^4\right) & x_1\neq 0,\\ \mathbb{V}\left(\emptyset,\emptyset,\alpha_{0021}(t_i)y_1^2\right) & y_1\neq 0. \end{cases} \end{aligned}$$

We then require of the equation above that the largest degree of coefficient is nonnegative to have isolated singularities

$$\begin{cases} 2-a_2-a_3\geq 0 & \text{if } x_1^6\leq x_2^6\leq y_1^3\leq y_2^3;\\ 2-a_2-a_3\geq 0 & \text{if } x_1^6\leq y_1^3\leq x_2^6\leq y_2^3;\\ 2-a_2+a_3\geq 0 & \text{if } y_1^3\leq x_1^6\leq x_2^6\leq y_2^3;\\ \begin{cases} -a_2+a_4\geq 0,\ 2-a_2-a_3\geq 0\\ -a_2+a_4<0,\ 2-a_3-a_4\geq 0 \end{cases} & \text{if } x_1^6\leq y_1^3\leq y_2^3\leq x_2^6;\\ 2-a_2+a_3\geq 0 & \text{if } y_1^3\leq x_1^6\leq y_2^3\leq x_2^6;\\ 2-a_2+a_3\geq 0 & \text{if } y_1^3\leq y_2^3\leq x_1^6\leq x_2^6. \end{cases} \tag{5.28}$$

With the condition $a_2\geq 0,\ a_4-a_3\geq 0$ and Inequalities (5.28) and (5.27), we get other

Calabi–Yau threefolds X embedded in the weighted scrolls $\mathbb{F}(0,a_2,a_3,a_4|1^2,2^2)$

$$\begin{cases} \mathbb{F}(0,1,1,2|1^2,2^2),\ \mathbb{F}(0,1,1,3|1^2,2^2),\ \mathbb{F}(0,1,1,1|1^2,2^2), \\ \mathbb{F}(0,2,1,1|1^2,2^2) \text{ and } \mathbb{F}(0,3,1,1|1^2,2^2) \\ \text{for } (5.27)(1),\ (5.27)(2),\ (5.27)(4); \\ \mathbb{F}(0,1,-1,0|1^2,2^2),\ \mathbb{F}(0,1,-1,1|1^2,2^2),\ \mathbb{F}(0,0,-2,0|1^2,2^2), \\ \mathbb{F}(0,0,-2,1|1^2,2^2),\ \mathbb{F}(0,0,-2,2|1^2,2^2),\ \mathbb{F}(0,0,-2,3|1^2,2^2), \\ \mathbb{F}(0,0,-2,4|1^2,2^2),\ \mathbb{F}(0,0,-1,1|1^2,2^2),\ \mathbb{F}(0,0,-1,2|1^2,2^2), \\ \mathbb{F}(0,0,-1,3|1^2,2^2),\ \mathbb{F}(0,1,-1,2|1^2,2^2),\ \mathbb{F}(0,1,-1,0|1^2,2^2), \\ \mathbb{F}(0,1,-1,1|1^2,2^2),\ \mathbb{F}(0,0,-6,-4|1^2,2^2),\ \mathbb{F}(0,0,-5,-3|1^2,2^2), \\ \mathbb{F}(0,0,-4,-3|1^2,2^2),\ \mathbb{F}(0,0,-4,-2|1^2,2^2),\ \mathbb{F}(0,0,-3,-2|1^2,2^2), \\ \mathbb{F}(0,0,-3,-1|1^2,2^2),\ \mathbb{F}(0,0,-2,-1|1^2,2^2),\ \mathbb{F}(0,1,-3,-2|1^2,2^2) \\ \text{and } \mathbb{F}(0,1,-2,-1|1^2,2^2) \text{ for } (5.27)(3),\ (5.27)(5),\ (5.27)(6). \end{cases}$$

The new families are colour coded blue and have either isolated or more than isolated singularities along the base locu. In Table (5.6), we omit families with more than isolated singularities as well as the quasismooth (non-colour coded) families which had been classified in [Mul06].

Finally, if $\dim(Bs(|L_{2-a_2-a_3-a_4,6}|)) = 2$, we must have that

$$\begin{cases} 2+5a_2-a_3-a_4<0,\ 2-a_2+2a_3-a_4\geq 0 & \text{from (5.26)(1);} \\ 2-a_2+2a_3-a_4<0,\ 2+5a_2-a_3-a_4\geq 0 & \text{from (5.26)(2);} \\ 2-a_2-a_3-a_4<0,\ 2+5a_2-a_3-a_4\geq 0 & \text{from (5.26)(3);} \\ 2-a_2+2a_3-a_4<0,\ 2-a_2-a_3+2a_4\geq 0 & \text{from (5.26)(4);} \\ 2-a_2-a_3-a_4<0,\ 2-a_2-a_3+2a_4\geq 0, & \text{from (5.26)(5);} \\ 2-a_2-a_3+2a_4<0, 2-a_2-a_3-a_4\geq 0 & \text{from (5.26)(6).} \end{cases} \tag{5.29}$$

Consequently, the two dimensional base loci $B = Bs(|L_{2-a_2-a_3-a_4,6}|)$ are

$$\begin{cases} \mathbb{V}(y_1,y_2) = \mathbb{F}(0,a_2|1,1)_{t_i,x_1,x_2} & \text{for (5.29)(1);} \\ \mathbb{V}(x_2,y_2) \cong \mathbb{F}(0,a_3|1,2) \cong \mathbb{F}(0,a_3|1,1)_{t_i,x_1^2,y_1} & \text{for (5.29)(2)-(5.29)(5);} \\ \mathbb{V}(x_1,x_2) \cong \mathbb{F}(a_3,a_4|2,2) \cong \mathbb{F}(0,a_4-a_3|1,1)_{t_i,y_1,y_2} & \text{for (5.29)(6).} \end{cases}$$

For X to have nice singularities of codimension at least two along the base loci $\mathbb{F}(0,a_2|1,1)$ or $\mathbb{F}(0,a_3|1,3)$, we must have that

$$x_1,x_2 \nmid \frac{\partial f(t_i,x_i,y_j)}{\partial y_1},\ \frac{\partial f(t_i,x_i,y_j)}{\partial y_2} \left(\text{or } x_1,y_1 \nmid \frac{\partial f(t_i,x_i,y_j)}{\partial x_2},\ \frac{\partial f(t_i,x_i,y_j)}{\partial y_2} \right).$$

Here,

$$Sing(\widetilde{X})\cap\widetilde{B}=\begin{cases}\mathbb{V}\left(\frac{\partial f}{\partial y_1},\frac{\partial f}{\partial y_2},y_1,y_2\right) & \text{if } x_1^6\le x_2^6\le y_1^3\le y_2^3\\ \mathbb{V}\left(\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial y_2}x_2,y_2\right) & \text{otherwise}\\ \mathbb{V}\left(\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}x_1,x_2\right) & \text{if } y_1^3\le y_2^3\le x_1^6\le x_2^6.\end{cases}$$

where $f=f(t_i,x_i,y_j)$ and

$$\left.\frac{\partial f}{\partial y_2}\right|_{y_1=y_2=0}=\alpha_{4001}(t_i)x_1^4+\alpha_{3101}(t_i)x_1^3x_2+\alpha_{2201}(t_i)x_1^2x_2^2+\alpha_{1301}(t_i)x_1x_2^3+\alpha_{0401}(t_i)x_2^4$$

$$\left.\frac{\partial f}{\partial y_1}\right|_{y_1=y_2=0}=\alpha_{4010}(t_i)x_1^4+\alpha_{3110}(t_i)x_1^3x_2+\alpha_{2210}(t_i)x_1^2x_2^2+\alpha_{1310}(t_i)x_1x_2^3+\alpha_{0410}(t_i)x_2^4,$$

$$\left.\frac{\partial f}{\partial y_2}\right|_{x_2=y_2=0}=\alpha_{4001}(t_i)x_1^4+\alpha_{2011}(t_i)x_1^2y_1+\alpha_{0021}(t_i)y_1^2,$$

$$\left.\frac{\partial f}{\partial x_2}\right|_{x_2=y_2=0}=\alpha_{5100}(t_i)x_1^5+\alpha_{3110}(t_i)x_1^3y_1+\alpha_{1120}(t_i)x_1y_1^2,$$

$$\left.\frac{\partial f}{\partial x_2}\right|_{x_1=x_2=0}=\left.\frac{\partial f}{\partial x_1}\right|_{x_1=x_2=0}\equiv 0.$$

Therefore for X to have isolated singularities we must have that $\frac{\partial f}{\partial y_1}$ and $\frac{\partial f}{\partial y_2}$ $\left(\text{or that } \frac{\partial f}{\partial x_2} \text{ and } \frac{\partial f}{\partial y_2}\right)$ are not both identically zero on B and have no common factor. This is guaranteed by the constraints,

$$\begin{cases}2-a_2-a_4\ge 0,\ 2+3a_2-a_3\ge 0 & \text{if } x_1^6\le x_2^6\le y_1^3\le y_2^3\\ 2-a_3-a_4\ge 0,\ 2-a_2+a_3\ge 0 & \text{if } x_1^6\le y_1^3\le x_2^6\le y_2^3\\ 2-a_3-a_4\ge 0,\ 2-a_2+a_3\ge 0 & \text{if } y_1^3\le x_1^6\le x_2^6\le y_2^3\\ \begin{cases}-a_2+a_4\ge 0,\ 2-a_3-a_4\ge 0,\ 2-a_2+a_3\ge 0\\ -a_2+a_4<0,\ 2-a_2-a_3\ge 0, 2+a_3-a_4\ge 0\end{cases} & \text{if } x_1^6\le y_1^3\le y_2^3\le x_2^6\\ 2-a_2-a_3\ge 0, 2+a_3-a_4\ge 0 & \text{if } y_1^3\le x_1^6\le y_2^3\le x_2^6.\end{cases}\tag{5.30}$$

However, if the family X for which $y_1^3\le y_2^3\le x_1^6\le x_2^6$ exists, it would be singular on the entire surface $\mathbb{F}(0,a_4-a_3|1,1)$ on which both $\frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_1}$ are identically zero. We are not interested in such families and can ignore this case already.

Therefore, with the condition $a_2\ge 0$ and $a_4-a_3\ge 0$, Inequalities (5.29) and (5.30) we get other Calabi–Yau threefolds X embedded the weighted scrolls $\mathbb{F}(0,a_2,a_3,a_4|1^2,2^2)$

$$\begin{cases}\mathbb{F}(0,0,1,2|1^2,2^2) \text{ and } \mathbb{F}(0,0,2,2|1^2,2^2) \text{ from Inequalities}\\ (5.29)(1)+(5.30)(1);\\ \mathbb{F}(0,1,0,2|1^2,2^2),\ \mathbb{F}(0,1,-1,3|1^2,2^2),\ \mathbb{F}(0,2,0,2|1^2,2^2),\\ \mathbb{F}(0,2,0,1|1^2,2^2),\ \text{and } \mathbb{F}(0,3,-1,1|1^2,2^2)\\ \text{from } (5.29)(2)+(5.30)(2),\dots,(5.29)(5)+(5.30)(5).\end{cases}$$

The new families are colour coded blue and we list, in Table (5.6),those with at worst isolated singularities along the respective base locus. The non-color coded families are quasismooth.

We now analyze the singularities in the new families. We are only interested in the families with at worst isolated singularities along the base locus.

1. For $\mathbb{F} = \mathbb{F}(0,0,1,2|1^2,2^2)$ in which case $X \in |-K_{\mathbb{F}}| = |-L+6M|$, we have that

$$Sing(\mathbb{F}) = \{x_1 = x_2 = 0\} \cong \mathbb{F}(1,2|2,2) \cong \mathbb{F}(0,1|1,1) = F_1.$$

   A general section $f(t_1,t_2,x_1,x_2,y_1,y_2)$ of $|-L+6M|$ has coefficients

$$\deg(\alpha_{q_j}(t_i)) = -1 + q_3 + 2q_4$$

   as listed in Figure (5.10). We then get the curve

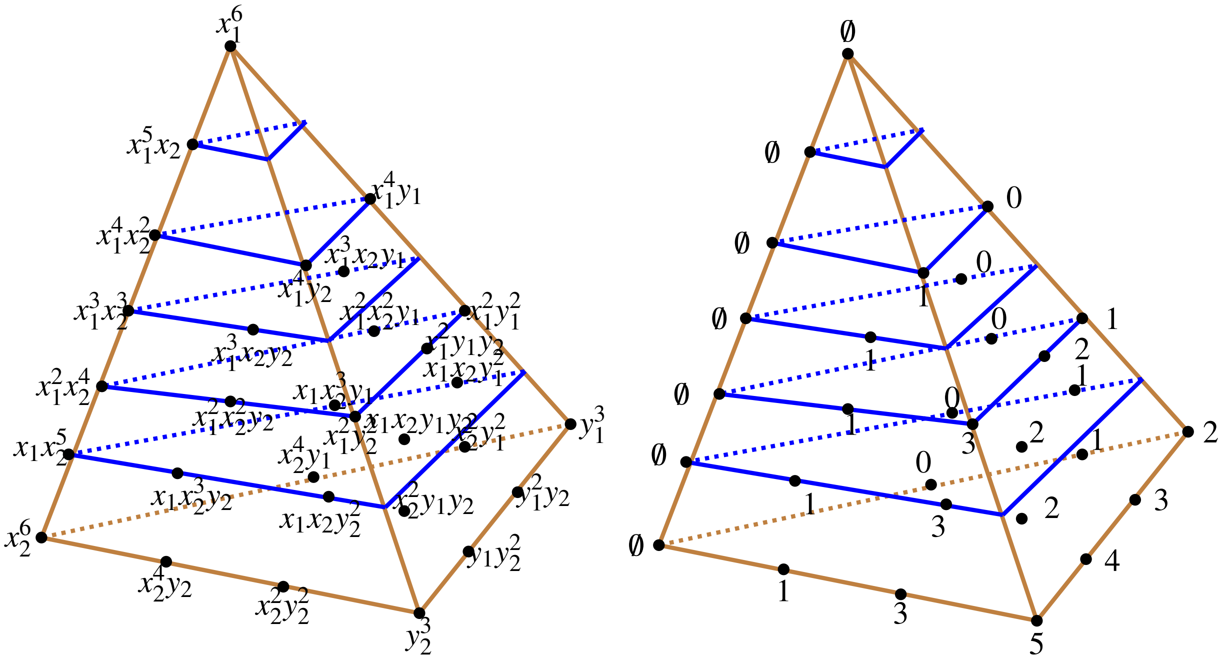


Figure 5.10: Degrees of coefficients of a section of $|-K_{\mathbb{F}(0,0,1,2|1^2,2^2)}|$

$$\begin{aligned} C = X \cap Sing(\mathbb{F}) =& \mathbb{V}(f(t_1,t_2,0,0,y_1,y_2)) \subset F_1 =: F_a \\ =& \mathbb{V}(g_5(t_i)y_2^3 + g_4(t_i)y_1y_2^2 + g_3(t_i)y_1^2y_2 + g_2(t_i)y_1^3) \in \\ & |2L|_{F_1} + 3M|_{F_1}| =: |mL|_{F_1} + nM|_{F_1}| \end{aligned}$$

of $\frac{1}{2}(1,1,0)$ singularities on the surface scroll F_1 whose genus is

$$\begin{aligned} g(C) =& a\binom{n}{2} + (m-1)(n-1) \\ =& 1\times\binom{3}{2} + (2-1)(3-1) = 5. \end{aligned}$$

Since f was general, we have that along the base locus

$$B = Bs(|-L+6M|) = \{y_1 = y_2 = 0\} \cong \mathbb{P}^1_{[t_i]} \times \mathbb{P}^1_{[x_j]}$$

that

$$\begin{aligned} \frac{\partial f}{\partial y_1}\Big|_{y_1=y_2=0} =& \alpha_{4010}(t_i)x_1^4 + \alpha_{3110}(t_i)x_1^3x_2 + \alpha_{2210}(t_i)x_1^2x_2^2 + \alpha_{1310}(t_i)x_1x_2^3 + \\ & \alpha_{0410}(t_i)x_2^4 \in |L_{0,4}|_B| \\ \frac{\partial f}{\partial y_2}\Big|_{y_1=y_2=0} =& \alpha_{4001}(t_i)x_1^4 + \alpha_{3101}(t_i)x_1^3x_2 + \alpha_{2201}(t_i)x_1^2x_2^2 + \alpha_{1301}(t_i)x_1x_2^3 + \\ & \alpha_{0401}(t_i)x_2^4 \in |L_{1,4}|_B|. \end{aligned}$$

Therefore, along the base locus B, we have 4 isolated singularities

$$\begin{aligned} Sing(\widetilde{X}) \cap \widetilde{B} =& \mathbb{V}\left(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, y_1, y_2\right) \\ =& |L_{0,4}|_B| \cap |L_{1,4}|_B| \subset \mathbb{P}^1_{[t_i]} \times \mathbb{P}^1_{[x_j]}. \end{aligned}$$

We also note that $B \cap Sing(\mathbb{F}) = \emptyset$ so that these 4 singular points $p_i = [\gamma_{1i} : \gamma_{2i}; \beta_{1i} : \beta_{2i} : 0 : 0]$ are the worst there is along B.

Now, locally on the chart $U_{12} = \{t_1 = x_2 = 1\} \cong \mathbb{A}^4_{t_2,x_1,y_1,y_2}/\mathbb{Z}_2$, a local change of coordinates $t_2 = t_2' + \gamma_{2i}$, $x_1 = x_1' + \beta_{1i}$ results in

$$\begin{aligned} X \cap U_{12} =& \mathbb{V}\left(y_1 \sum_{k=0}^{4} c_k(x_1' + \beta_{1i})^{4-k} + y_2 \sum_{k=0}^{4} \ell_1^{[k]}(t_2' + \gamma_{2i})(x_1' + \beta_{1i})^{4-k} + \text{ H.O.T}\right) \\ =& \mathbb{V}(y_1 {x_1'}^2 + y_2 g_1(t_2', {x_1'}^2) + \text{ H.O.T}), \text{ let } u_1 =: {x_1'}^2,\ t_2'' := g_1(t_2', u_1) \\ =& \mathbb{V}(y_1u_1 + y_2t_2'' + \text{ H.O.T}) \subset \mathbb{A}^4_{(t_2'', u_1, y_1, y_2)} \subset \mathbb{F}(0,0,1,2|1^2,2^2), \end{aligned}$$

a quadric threefold cone with a full rank quadratic part $q_2 = y_1u_1 + y_2t_2''$. Therefore, up to an analytic $\mathbb{Z}_2$ change of coordinates, the Calabi–Yau threefold X has 4 isolated threefold Ordinary Double Point (ODP) singularities $\{p_i\}$. It is known, see [Ati58] , that blowing up along $B \subset X \subset \mathbb{F}(0,0,1,2|1^2,2^2)$ gives a small (crepant) resolution of the 4 isolated ODP singularities $\{p_i\} \subset B$.

2. For $\mathbb{F} = \mathbb{F}(0,2,0,1|1^2,2^2)$ where $X \in |-K_{\mathbb{F}}| = |-L+6M|$, we have that

$$Sing(\mathbb{F}) = \{x_1 = x_2 = 0\} \cong \mathbb{F}(0,1|2,2) \cong \mathbb{F}(0,1|1,1) = F_1.$$

A general section $f(t_1,t_2,x_1,x_2,y_1,y_2)$ of $|-L+6M|$ has coefficients

$$\deg(\alpha_{q_j}(t_i)) = -1+2q_2+q_4$$

as listed in Figure (5.11). We then get a union of $\{x_1 = x_2 = y_2 = 0\}$ and genus 0 curve in $|L_{0,1}|_{F_2}|$

$$\begin{aligned} C = X \cap Sing(\mathbb{F}) =& \mathbb{V}(f(t_1,t_2,0,0,y_1,y_2)) \\ =& \{x_1 = x_2 = y_2 = 0\} \cup \mathbb{V}(g_2(t_i)y_2^2 + g_1(t_i)y_1y_2 + g_0(t_i)y_1^2) \subset F_1 \end{aligned}$$

on the surface scroll F_1. Both curves are of $\frac{1}{2}(1,1,0)$ singularity. Further along the

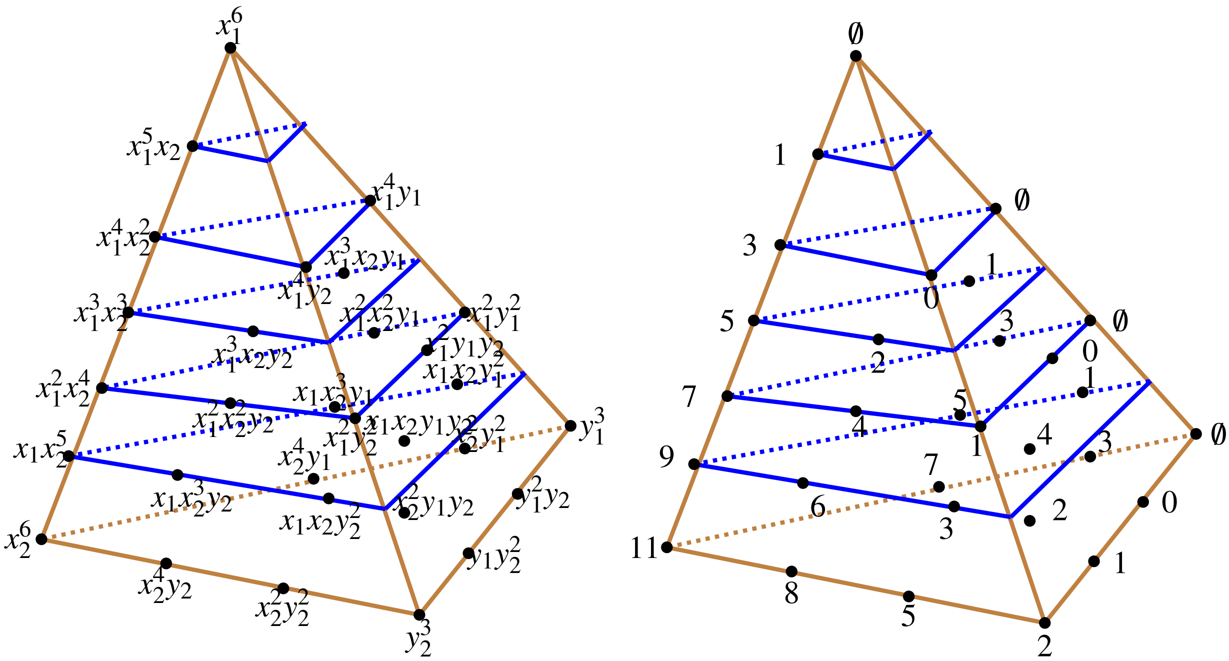


Figure 5.11: Degrees of coefficients of a section of $|-K_{\mathbb{F}(0,2,0,1|1^2,2^2)}|$

base locus

$$B = Bs(|-L+6M|) = \{x_2 = y_2 = 0\} \cong \mathbb{P}^1_{[t_i]} \times \mathbb{P}^1_{[x_1^2:y_1]},$$

we have that

$$\left.\frac{\partial f}{\partial y_2}\right|_{x_2=y_2=0} = \alpha_{4001}(t_i)x_1^4 + \alpha_{2011}(t_i)x_1^2y_1 + \alpha_{0021}(t_i)y_1^2 \in |L_{0,2}|,$$

$$\left.\frac{\partial f}{\partial x_2}\right|_{x_2=y_2=0} = \alpha_{5100}(t_i)x_1^5 + \alpha_{3110}(t_i)x_1^3y_1 + \alpha_{1120}(t_i)x_1y_1^2 \in x_1|L_{1,2}|.$$

There, along the base locus B, we have 2 isolated singularities

$$\begin{aligned}Sing(\widetilde{X})\cap\widetilde{B} =& \mathbb{V}\left(\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial y_2},x_2,y_2\right)\\ =& |L_{0,2}|_B|\cap|L_{1,2}|_B|\subset\mathbb{P}^1_{[t_i]}\times\mathbb{P}^1_{[x_j]}.\end{aligned}$$

We note that the 2 isolated singular points are not in

$$B\cap Sing(\mathbb{F}) = \{x_1 = x_2 = y_2 = 0\}\cong\mathbb{P}^1_{[t_i]}\times\{[0:0:1:0]\}.$$

We therefore have that the family X has 2 isolated singular points along the base locus of $|-K_{\mathbb{F}}|$ and a $\mathbb{P}^1$ curve of $\frac{1}{2}(1,1,0)$ singularities. This family makes it to our list.

In conclusion, the Table (5.6) summarizes with reason the models of 4-fold weighted scrolls $\mathbb{F}=\mathbb{F}(0,a_2,a_3,a_4|1^2,2^2)$ in which the anticanonical threefolds have at most isolated singularities along the base locus of $|-K_{\mathbb{F}}|$. $\square$

Chapter 6

Codimension 2 Calabi–Yau Varieties in Weighted Scrolls

In this chapter, we first construct K3 surfaces over $\mathbb{P}^1$ by complete intersections of two quadrics $E_{2,2} \subset \mathbb{P}^3$ from Table (2.2). This is intended to introduce the classification of Complete Intersection Calabi–Yau threefolds [CICY3] in weighted scrolls fibred by

$$X_{2,3} \subset \mathbb{P}^4$$

which are either nonsingualr or has isolated singular points.

Lemma 6.0.1. *Let $|D_1|$ and $|D_2|$ be base point free linear systems on the scroll $\mathbb{F}_A$. The codimension two surface $X = \mathbb{V}(f_1, f_2)$ with equations f_i of general sections of $|D_i|$ is quasismooth.*

`Proof.` By Bertini's Theorem, since $|D_1|$ is base point free, a general section $X_1 := \mathbb{V}(f_1) \in |D_1|$ is a quasismooth hypersurface in $\mathbb{F}_A$. Since $|D_2|$ is base point free, the restriction $D_2|X_1$ is basepoint-free with at least as many sections as $|D_2|$. Therefore, with f_2 being the equation of a general section of $|D_2|$, the general section $X = \mathbb{V}(f_1, f_2)$ of $D_2|X_1$ is also quasismooth. □

6.1 Elliptic fibrations by $X_{2,2} \subset \mathbb{P}^3$ in Scrolls

Definition 6.1.1. [Codim 2 K3 Surface in scroll] Let $\mathbb{F} = \mathbb{F}(0, a_2, a_3, a_4)$ be a 4-fold straight scroll with weakly increasing twisting data

$$0 \leq a_2 \leq a_3 \leq a_4. \tag{6.1}$$

Let $D_1, D_2 \in Cl(\mathbb{F})$ and suppose that $D_1 + D_2 + K_{\mathbb{F}} = \mathscr{O}_X$. Now, with f_i the equation of a general section of $|D_i|$, we have that

$$X = \mathbb{V}(f_1, f_2) \subset \mathbb{F}(0, a_2, a_3, a_4)$$

is a codimension two surface.

Now since $-K_{\mathbb{F}} = L_{2-a_2-a_3-a_4,4}$, let $D_1 = L_{p_1,2}$ so that $D_2 = L_{2-p_1-a_2-a_3-a_4,2}$. In the $\mathbb{Z}^2-$graded coordinate ring $S = \mathbb{C}[t_i, x_j]$ of the scroll $\mathbb{F}(0, a_2, a_3, a_4)$, we take equations of general section

$$\begin{aligned} f_1(t_i, x_j) =& \alpha_{(2000)}(t_i)x_1^2 + \alpha_{(1100)}(t_i)x_1x_2 + \alpha_{(0200)}(t_i)x_2^2 + \alpha_{(1010)}(t_i)x_1x_3 + \alpha_{(0020)}(t_i)x_3^2 + \\ & \alpha_{(1001)}(t_i)x_1x_4 + \alpha_{(0110)}(t_i)x_2x_3 + \alpha_{(0101)}(t_i)x_2x_4 + \alpha_{(0011)}(t_i)x_3x_4 + \alpha_{(0002)}(t_i)x_4^2 \in |L_{p_1,2}| \end{aligned}$$

and

$$\begin{aligned} f_2(t_i, x_j) =& \beta_{(2000)}(t_i)x_1^2 + \beta_{(1100)}(t_i)x_1x_2 + \beta_{(0200)}(t_i)x_2^2 + \beta_{(1010)}(t_i)x_1x_3 + \beta_{(0020)}(t_i)x_3^2 + \\ & \beta_{(1001)}(t_i)x_1x_4 + \beta_{(0110)}(t_i)x_2x_3 + \beta_{(0101)}(t_i)x_2x_4 + \beta_{(0011)}(t_i)x_3x_4 + \beta_{(0002)}(t_i)x_4^2 \in \\ & |L_{2-p_1-a_2-a_3-a_4,2}|. \end{aligned}$$

Here, $\alpha_{(q_j)}, \beta_{(q_j)} \in \mathbb{C}[t_1, t_2]$ of degrees

$$\begin{aligned} \deg \alpha_{(q_j)}(t_i) &= p_1 + q_2a_2 + q_3a_3 + q_4a_4, \\ \deg \beta_{(q_j)}(t_i) &= 2 - p_1 + (q_2-1)a_2 + (q_3-1)a_3 + (q_4-1)a_4 \end{aligned}$$

are the coefficients of x^q where $q = (q_1, \ldots, q_4) \vdash 2$.

Denote a general

$$X = \mathbb{V}(f_1, f_2) = \mathbb{V}\left(\sum_{(q_j)\vdash 2} \alpha_{(q_j)}(t_1,t_2)x^{q_j}, \sum_{(q_j)\vdash 2} \beta_{(q_j)}(t_1,t_2)x^{q_j} \right)$$

by

$$\begin{bmatrix} p_1 & 2-p_1-a_2-a_3-a_4 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0, a_2, a_3, a_4).$$

We also note, by symmetry that

$$\begin{bmatrix} p_1 & 2-p_1-a_2-a_3-a_4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2-p_1-a_2-a_3-a_4 & p_1 \\ 2 & 2 \end{bmatrix}$$

and therefore without loss of generality we may assume that $p_1 \geq 2 - p_1 - \sum a_j$ or

$$-2 + 2p_1 + a_2 + a_3 + a_4 \geq 0. \tag{6.2}$$

We get a fibration $f : X \to \mathbb{P}^1$ by genus 1 curves $E_{2,2} \subset \mathbb{P}^3$ induced from the canonical $\mathbb{P}^3$-fibred scroll map

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{F}(0, a_2, a_3, a_4) \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \pi} \\ & & \mathbb{P}^1. \end{array}$$

The following theorem classifies codimension two surfaces with at most isolated singularities fibred by intersection of two quadrics and living in $\mathbb{P}^3$-bundles over $\mathbb{P}^1$.

Theorem 6.1.2. *Let $\mathbb{F} = \mathbb{F}(0, a_2, a_3, a_4)$ be a fourfold straight scroll over $\mathbb{P}^1$. There are 18 families of mildly singular codimension two surfaces $X = \mathbb{V}(f_1, f_2)$ fibred by $E_{2,2} \subset \mathbb{P}^3$ a complete intersection of two quadrics with at most isolated singularities along either of the base loci $Bs(|L_{p_1,2}|), Bs(|L_{2-p_1-a_2-a_3-a_4,2}|)$. The Tables* (6.1) *and* (6.2) *summarise the classification.*

No.	$X \subset \mathbb{F}(0,a_2,a_3,a_4) = \mathbb{F}$	General $X = \begin{bmatrix} p_1 & 2-p_1-a_2-a_3-a_4 \\ 2 & 2 \end{bmatrix}$
1	$\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,0,2)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular $K3$ surface.
2	$\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,1,1)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular $K3$ surface.
3	$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,0,0)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular $K3$ surface.
4	$\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,0,1)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular $K3$ surface.
5	$\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,0,0)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular $K3$ surface.
6	$\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,1,2)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3, x_4)$ General X has 4 isolated singular points along $Bs(\lvert D_2\rvert)$
7	$\begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,1,3)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3, x_4)$ General X has 4 isolated singular points along $Bs(\lvert D_2\rvert)$
8	$\begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,2,2)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3, x_4)$ General X has 4 isolated singular points along $Bs(\lvert D_2\rvert)$
9	$\begin{bmatrix} 0 & -3 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,2,3)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3, x_4)$ General X has 4 isolated singular points along $Bs(\lvert D_2\rvert)$

Table 6.1: The data (p_1, a_j) for which a general codimension two K3 in $|L_{p_1,2}| \cap |L_{2-p_1-\sum a_j,2}| \subset \mathbb{F}$ has at worst isolated singularities.

No.	$X \subset \mathbb{F}(0,a_2,a_3,a_4) = \mathbb{F}$	General $X = \begin{bmatrix} p_1 & 2-p_1-a_2-a_3-a_4 \\ 2 & 2 \end{bmatrix}$
10	$\begin{bmatrix} 0 & -4 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,2,4)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3,x_4)$ General X has 4 isolated singular points along $Bs(\lvert D_2\rvert)$
11	$\begin{bmatrix} 0 & -3 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,1,2,2)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3,x_4)$ General X has 6 isolated singular points along $Bs(\lvert D_2\rvert)$
12	$\begin{bmatrix} 0 & -3 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,1,2,3)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3,x_4)$ General X has 6 isolated singular points along $Bs(\lvert D_2\rvert)$
13	$\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,1,1)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3,x_4)$ General X has 4 isolated singular points along $Bs(\lvert D_2\rvert)$
14	$\begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,1,2)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_3,x_4)$ General X has 4 isolated singular points along $Bs(\lvert D_2\rvert)$
15	$\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,1,1,1)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_2,x_3,x_4)$ General X is nonsingular along $Bs(\lvert D_2\rvert)$
16	$\begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,1,1,2)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_2,x_3,x_4)$ General X has 1 isolated singular point along $Bs(\lvert D_2\rvert)$
17	$\begin{bmatrix} 0 & -4 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,2,2,2)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_2,x_3,x_4)$ General X is nonsingular along $Bs(\lvert D_2\rvert)$
18	$\begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,1,1,1)$	$\lvert D_1\rvert$ is base-point-free and $Bs(\lvert D_2\rvert) = \mathbb{V}(x_2,x_3,x_4)$ General X is nonsingular along $Bs(\lvert D_2\rvert)$

Table 6.2: The data (p_1, a_j) for which a general codimension two K3 in $|L_{p_1,2}| \cap |L_{2-p_1-\sum a_j,2}| \subset \mathbb{F}$ has at worst isolated singularities.

Proof. Let us first consider that case when $|D_1| = |L_{p_1,2}|$ and $|D_2| = |L_{2-p_1-a_2-a_3-a_4,2}|$ are both base point free on $\mathbb{F}$. By Proposition (3.5.4), base point freeness is implied by 2 Inequalities

$$p_1 \geq 0 \text{ and} \tag{6.3}$$
$$2 - p_1 - a_2 - a_3 - a_4 \geq 0.$$

From (6.1), (6.2) and (6.3) we deduce that

$$(p_1, a_2, a_3, a_4) = (0,0,0,2), (0,0,1,1), (1,0,0,0), (1,0,0,1) \text{ or } (2,0,0,0);$$

the data on 4-fold scrolls where X is embedded in $\mathbb{F}$ and fibred over $\mathbb{P}^1$ by complete intersection $E_{2,2} \subset \mathbb{P}^3$. These families are nonsingular by Lemma (6.0.1).

We further have, by irreducibility of X, that both

$$B_1 = Bs(|L_{p_1,2}|), B_2 = Bs(|L_{2-p_1-a_2-a_3-a_4,2}|)) \subset \mathbb{F}$$

are each of dimension at most two. Otherwise, the general section is reducible; there is a fixed divisor in each member of the linear system whose base locus is 3-dimensional. The surface X is irreducible whenever $\alpha_{(0020)}(t_i)x_3^2$ appears in f_1 and $\beta_{(0020)}(t_i)x_3^2$ appears in f_2

$$\begin{aligned} p_1 + 2a_3 \geq 0 \text{ and} \\ 2 - p_1 - a_2 + a_3 - a_4 \geq 0. \end{aligned} \tag{6.4}$$

Now, define $\text{Sing}(X) = \mathbb{V}(f_1, f_2, \Delta)$ with the ideal

$$\Delta(t_i, x_j) = \left(\text{rank}\begin{bmatrix}\nabla f_1 \\ \nabla f_2\end{bmatrix} < 2\right) = \left(\bigwedge^2 \begin{bmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \end{bmatrix}\right).$$

We would like to find data (p_1, a_j) for which X has at most isolated singularities along B_1 and B_2. The case when $\dim(B_1) = \dim(B_2) = 2$ is such that the surface X contains (hence is exactly) the surface D_{34}; such X is therefore reducible. We omit this case as well as the cases when $\dim(B_1) = \dim(B_2) = 1$ and the case when $\dim(B_1) = 1$ and $\dim(B_2) = 2$ where there no new data. We now check the remaining cases.

1. Consider the case when $|L_{p_1,2}|$ is base point free and $\dim(B_2) = 2$. These assumptions imply that $\alpha_{(2000)}(t_i)x_1^2$ appears in f_1 and $\beta_{(0200)}(t_i)x_2^2$ does not appear in $f_2, \beta_{(0020)}(t_i)x_3^2$ apears in f_2. Equivalently

$$\begin{aligned} p_1 \geq 0, \\ 2 - p_1 + a_2 - a_3 - a_4 < 0 \text{ and} \\ 2 - p_1 - a_2 + a_3 - a_4 \geq 0. \end{aligned} \tag{6.5}$$

   Moreover, the surface X will have isolated singularities whenever

$$f_1|_{x_3=x_4=0} = \alpha_{(2000)}(t_i)x_1^2 + \alpha_{(1100)}(t_i)x_1x_2 + \alpha_{(0200)}(t_i)x_2^2 \text{ and } \Delta(t_i, x_1, x_2, 0, 0)$$

   have no common factor. Equivalently, we want

$$\Delta(t_i, x_1, x_2, 0, 0) = \det\begin{bmatrix} \alpha_{1010}(t_i)x_1 + \alpha_{(0110)}(t_i)x)_2 & \alpha_{(1001)}(t_i)x_1 + \alpha_{(0101)}(t_i)x_2 \\ \beta_{1010}(t_i)x_1 + \beta_{(0110)}(t_i)x)_2 & \beta_{(1001)}(t_i)x_1 + \beta_{(0101)}(t_i)x_2 \end{bmatrix}$$

not to be identically zero which is guaranteed by

$$\deg(\alpha_{(0110)}) = p_1 + a_2 + a_3 \geq 0 \text{ and} \qquad (6.6)$$
$$\deg(\beta_{(0101)}) = 2 - p_1 - a_3 \geq 0.$$

From the Inequalities $a_4 \geq a_3 \geq a_2 \geq 0$, (6.2), (6.5) and (6.6) we get the following data on K3 families

$$\begin{aligned}(p_1, a_2, a_3, a_4) =& (0,0,1,2), (0,0,1,3), (0,0,2,2), (0,0,2,3), (0,0,2,4), (0,1,2,2),\\ & (0,1,2,3), (1,0,1,1) \text{ or } (1,0,1,2).\end{aligned}$$

We now find the singularities along $B_2 = \mathbb{V}(x_3, x_4)$ in each example. Let the respective subscripts of polynomials coefficients from $\mathbb{C}[t_1, t_2]$ defining X be the degree. From the list above, we get that

(i) $X_1 = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,1,2)$ with

$$\begin{aligned} Sing(X_1) \cap \widetilde{B_2} &= \mathbb{V}(\alpha_{(2000)}(t_i)x_1^2 + \alpha_{(1100)}(t_i)x_1x_2 + \alpha_{(0200)}(t_i)x_2^2,\ \Delta(t_i, x_1, x_2, 0, 0)) \\ &= \mathbb{V}(a_0x_1^2 + b_0x_1x_2 + c_0x_2^2,\ A_2(t_i)x_1^2 + B_2(t_i)x_1x_2 + C_2(t_i)x_2^2). \end{aligned}$$

These are 4 isolated singularities along $\mathbb{F}(0,0)$.

$$Sing(X_1) = \{[\gamma_{1i} : 1; 0 : 1 : 0 : 0], [1 : \gamma_{2i}; 1 : 0 : 0 : 0],\ C_2(\gamma_{1i}) = 0 = A_2(\gamma_{2i})\}.$$

Similarly,

$$\begin{aligned} X_2 &= \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,1,3),\ X_3 = \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,2,2), \\ X_4 &= \begin{bmatrix} 0 & -3 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,2,3),\ X_5 = \begin{bmatrix} 0 & -4 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,0,2,4) \end{aligned}$$

each have 4 isolated singularities.

Now, let us have a close look at these 4 isolated singularities on X_1. We first make a change of coordinates

$$\begin{aligned} f_1 =& x_1x_2 + p_1(t_i)x_1x_3 + \widetilde{p_1}(t_i)x_2x_3 + p_2(t_i)x_3^2 + \widetilde{p_2}(t_i)x_1x_4 + q_2(t_i)x_2x_4 + \\ & \widetilde{q_2}(t_i)x_3x_4 + q_4(t_i)x_4^2 \in |L_{0,2}| \text{ and} \\ f_2 =& g_0x_1x_3 + \widetilde{g_0}x_2x_3 + g_1(t_i)x_3^2 + \widetilde{g_1}(t_i)x_1x_4 + h_1(t_i)x_2x_4 + h_2(t_i)x_3x_4 + h_3(t_i)x_4^2 \in |L_{-1,2}|. \end{aligned}$$

Now, near the singular point $[1:\gamma_{2i};1:0:0:0]$, an analytic change of coordinates gives

$$\begin{aligned} X_1\cap\{x_1\neq 0\} =& \mathbb{V}(f_1|_{x_1=1},f_2|_{x_1=1})\subset \mathbb{P}^1_{[t_i]}\times\mathbb{A}^3_{x_2,x_3,x_4} \\ =& \mathbb{V}(p(t_2,x_3,x_4)x_2+q(t_2,x_3,x_4),f_2|_{x_1=1}(t_2,x_2,x_3,x_4))\subset \mathbb{P}^1_{[t_i]}\times\mathbb{A}^3_{x_2,x_3,x_4} \\ \cong& \mathbb{V}(f_2|_{x_1=1}(t_2,r(t_2,x_3,x_4),x_3,x_4))\subset \mathbb{P}^1_{[t_i]}\times\mathbb{A}^2_{x_3,x_4}. \end{aligned}$$

This is a hypersurface singularity.

(ii) $X_6=\begin{bmatrix}0 & -3\\ 2 & 2\end{bmatrix}\subset\mathbb{F}(0,1,2,2)$ with

$$\begin{aligned} Sing(X_6)\cap\widetilde{B_2} &= \mathbb{V}(\alpha_{(2000)}(t_i)x_1^2+\alpha_{(1100)}(t_i)x_1x_2+\alpha_{(0200)}(t_i)x_2^2,\ \Delta(t_i,x_1,x_2,0,0)) \\ &= \mathbb{V}(a_0x_1^2+b_1(t_i)x_1x_2+c_2(t_i)x_2^2,\ (\alpha_{(1010)}(t_i)x_1+\alpha_{(0110)}(t_i)x_2)\beta_{(0101)}x_2) \\ &= \mathbb{V}(a_0x_1^2+b_1(t_i)x_1x_2+c_2(t_i)x_2^2,\ (q_2(t_i)x_1+q_3(t_i)x_2)p_0x_2) \\ &= |L_{0,2}|\cap|L_{1,2}|\subset\mathbb{F}(0,1). \end{aligned}$$

Hence a general X_6 has 6 isolated singular points along $\mathbb{F}(0,1)$. Similarly, a general

$$X_7=\begin{bmatrix}0 & -4\\ 2 & 2\end{bmatrix}\subset\mathbb{F}(0,1,2,3)$$

has 6 isolated singular points along $\mathbb{F}(0,1)$.

(iii) $X_8=\begin{bmatrix}1 & -1\\ 2 & 2\end{bmatrix}\subset\mathbb{F}(0,0,1,1)$ with

$$\begin{aligned} Sing(X_8)\cap\widetilde{B_2} &= \mathbb{V}(\alpha_{(2000)}(t_i)x_1^2+\alpha_{(1100)}(t_i)x_1x_2+\alpha_{(0200)}(t_i)x_2^2,\ \Delta(t_i,x_1,x_2,0,0)) \\ &= \mathbb{V}(a_1(t_i)x_1^2+b_1(t_i)x_1x_2+c_1(t_i)x_2^2,\ A_2(t_i)x_1^2+B_2(t_i)x_1x_2+C_2(t_i)x_2^2) \\ &= |L_{1,2}|\cap|L_{2,2}|\subset\mathbb{P}^1\times\mathbb{P}^1 \end{aligned}$$

Hence a general X_8 has 4 isolated singular points along $\mathbb{F}(0,0)\cong\mathbb{P}^1\times\mathbb{P}^1$. Similarly, a general

$$X_9=\begin{bmatrix}1 & -2\\ 2 & 2\end{bmatrix}\subset\mathbb{F}(0,0,1,2)$$

has 4 isolated singular points along $\mathbb{F}(0,0)$.

2. Finally, consider the case when $|L_{p_1,2}|$ is base point free and $\dim(B_2)=1$. Inequality (6.4) implies that $\alpha_{(2000)}(t_i)x_1^2$ appears in f_1 and $\beta_{(2000)}(t_i)x_1^2$ does not appear in f_2, $\beta_{(0200)}(t_i)x_2^2$ appears in f_2. Equivalently

$$\begin{aligned} p_1&\geq 0, \\ 2-p_1-a_2-a_3-a_4&<0 \text{ and} \\ 2-p_1+a_2-a_3-a_4&\geq 0. \end{aligned} \tag{6.7}$$

Moreover, the K3 surface X will have isolated singularities

$$\begin{aligned} Sing(X) =& \mathbb{V}(\alpha_{(2000)}(t_i)x_1^2,\ \Delta(t_i,x_1,0,0,0)) \subset \mathbb{P}^1 \\ =& \mathbb{V}\left(\alpha_{(2000)}(t_i)x_1^2, \bigwedge^2 \begin{bmatrix} \alpha_{(1100)}(t_i)x_1 & \alpha_{(1010)}(t_i)x_1 & \alpha_{(1001)}(t_i)x_1 \\ \beta_{(1100)}(t_i)x_1 & \beta_{(1010)}(t_i)x_1 & \beta_{(1001)}(t_i)x_1 \end{bmatrix}\right) \end{aligned}$$

along $B_2 = \mathbb{V}(x_2,x_3,x_4)$. We then get, from Inequalities $a_4 \geq a_3 \geq a_2 \geq 0$, (6.2) and (6.7), the K3 families

$$(p_1,a_2,a_3,a_4) = (0,1,1,1), (0,1,1,2), (0,2,2,2) \text{ or } (1,1,1,1).$$

The general members of the families

$$X_1 = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,1,1,1), X_2 = \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,1,1,2) \text{ and } X_3 = \begin{bmatrix} 0 & -4 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,2,2,2)$$

are, respectively, such that

$$Sing(X_i) \cap \widetilde{B_2} = \emptyset,\ \{[t_1:t_2;1:0:0:0] | \alpha_{(1010)}(t_1,t_2) = \ell_1(t_1,t_2) = 0\} \text{ and } \emptyset.$$

The family $X_4 = \begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix} \subset \mathbb{F}(0,1,1,1)$ is also such that $Sing(X) \cap \widetilde{B_2} = \mathbb{V}(a_0x_1^2) = \emptyset$. There are therefore 18 families of codimension 2 K3s fibred by a complete intersection of two quadrics as Listed in Table (6.1).

□

6.2 K3 fibrations by $X_{2,3} \subset \mathbb{P}^4$ in Scrolls

Definition 6.2.1. [Codim 2 Calabi–Yau threefold in scroll] Let $\mathbb{F} = \mathbb{F}(0,a_2,a_3,a_4,a_5)$ be a 5-fold straight scroll with weakly increasing twisting data

$$0 \leq a_2 \leq a_3 \leq a_4 \leq a_5.$$

With $D_1, D_2 \in Cl(\mathbb{F})$ such that $D_1 + D_2 + K_{\mathbb{F}} = \mathscr{O}_X$, we have that

$$X \in |D_1| \cap |D_2| \subset \mathbb{F}(0,a_2,a_3,a_4,a_5)$$

is a potential codimension 2 Calabi–Yau threefold.

With $-K_{\mathbb{F}} = L_{2-\sum a_j,5}$, let $D_1 = L_{p_1,2}$ so that $D_2 = L_{2-p_1-\sum a_j,3}$ and let X be denoted by

$$X = \begin{bmatrix} p_1 & 2-p_1-\sum a_j \\ 2 & 3 \end{bmatrix}.$$

We get a K3 fibration $f : X \to \mathbb{P}^1$ by codimension 2 K3 surfaces $X_{2,3} \subset \mathbb{P}^4$ and embedded in the scroll map $\pi : \mathbb{F} \to \mathbb{P}^1$. The equations of general sections

$$f_1(t_i, x_j) = \sum_{(q_j)\vdash 2} \alpha_{(q_j)}(t_1,t_2)x^{q_j} \in S_{(p_1,2)},\ f_2(t_i,x_j) = \sum_{(q_j)\vdash 3} \beta_{(q_j)}(t_1,t_2)x^{q_j} \in S_{(2-p_1-\sum a_j,3)}$$

are such that the degrees of coefficients of monomial $x^{q_j} = \prod_{j=1}^{5} x_j^{q_j}$ are

$$\deg \alpha_{(q_j)}(t_i) = p_1 + \sum_{j=2}^{5} q_j a_j,$$
$$\deg \beta_{(q_j)}(t_i) = 2 - p_1 + \sum_{j=2}^{5} (q_j - 1) a_j.$$

The following result classifies codimension 2 Calabi–Yau threefolds fibred by codimension 2 K3 surfaces $X_{2,3} \subset \mathbb{P}^4$ over a projective line and having at most isolated singularities.

Theorem 6.2.2. *Let $\mathbb{F} = \mathbb{F}(0, a_2, a_3, a_4, a_5)$ be a fivefold straight scroll over $\mathbb{P}^1$. There are 12 families of codimension two Calabi–Yau threefolds $X = \mathbb{V}(f_1, f_2)$ fibred by $X_{2,3} \subset \mathbb{P}^4$ a complete intersection of a quadric and cubic, which are nonsingular or has isolated singular points along either of the base loci $Bs(|L_{p_1,2}|), Bs(|L_{2-p_1-a_2-a_3-a_4-a_5,3}|)$. The Table (6.3) summarizes the classification.*

Proof. Consider the case when $|D_1| = |L_{p_1,2}|$ and $|D_2| = |L_{2-p_1-\sum a_j,3}|$ are both base point free on $\mathbb{F}$ which is implied by 2 Inequalities

$$p_1 \geq 0 \text{ and} \qquad (6.8)$$
$$2 - p_1 - \sum a_j \geq 0.$$

From Inequalities (6.8) and $a_5 \geq a_4 \geq a_3 \geq a_2 \geq 0$, we get the following data of nonsingular Calabi–Yau threefold $X \in |D_1| \cap |D_2|$ embedded in 5-fold scrolls

$$(p_1, a_2, a_3, a_4, a_5) = (0,0,0,0,0), (0,0,0,0,1), (0,0,0,0,2), (0,0,0,1,1), (1,0,0,0,0)$$
$$(1,0,0,0,1) \text{ or } (2,0,0,0,0).$$

By irreducibility of X, both $B_1 = Bs(|L_{p_1,2}|), B_2 = Bs(|L_{2-p_1-\sum_j a_j,3}|) \subset \mathbb{F}$ are each of dimension at most three. Equivalently $\alpha_{(00020)}(t_1,t_2)x_4^2$ appears in f_1 and $\beta_{(00030)}(t_1,t_2)x_4^3$ appears in f_2

$$p_1 + 2a_4 \geq 0 \text{ and} \qquad (6.9)$$
$$2 - p_1 - a_2 - a_3 + 2a_4 - a_5 \geq 0.$$

Now, depending on whether $|D_1|$ and $|D_2|$ are base point free or have a base locus of dimension at most 3, there are 15 more cases to consider for potential data $(p_1, a_2, \ldots, a_5)$ for which X has at most isolated singularities. We omit the cases when $1 \leq \dim(B_1) = \dim(B_2) \leq 3$ since for these cases, there is at least a whole curve of singularity along the base loci.

1. For the case when $\dim(B_1) = 1$ and $B_2 = \emptyset$, the Inequality (6.9) implies that $\alpha_{(20000)}(t_1,t_2)x_1^2$ does not appear in f_1, $\alpha_{(02000)}(t_1,t_2)x_2^2$ appears in f_1 and $\beta_{(3000)}(t_1,t_2)x_1^3$ appears in f_2. Equivalently

$$\begin{aligned} p_1 &< 0, \\ p_1 + 2a_2 &\geq 0 \text{ and} \\ 2 - p_1 - a_2 - a_3 - a_4 - a_5 &\geq 0. \end{aligned} \tag{6.10}$$

We define

$$\Delta(t_i, x_1, 0, 0, 0, 0) = \bigwedge^2 \begin{bmatrix} p_{12} & p_{13} & p_{14} & p_{15} \\ p_{22} & p_{23} & p_{24} & p_{25} \end{bmatrix}$$

where $p_{ij} = \frac{\partial f_i}{\partial x_j}$:

$$p_{12} = \alpha_{(11000)}(t_i)x_1, p_{13} = \alpha_{(10100)}(t_i)x_1, p_{14} = \alpha_{(10010)}(t_i)x_1, p_{15} = \alpha_{(10001)}(t_i)x_1;$$
$$p_{22} = \beta_{(21000)}(t_i)x_1^2, p_{23} = \beta_{(20100)}(t_i)x_1^2, p_{24} = \beta_{(20010)}(t_i)x_1^2, p_{25} = \beta_{(20001)}(t_i)x_1^2.$$

We have isolated singularities along the curve $\{x_2 = x_3 = x_4 = x_5 = 0\}$ whenever

$$\begin{aligned} \deg(\alpha_{(10010)}) = p_1 + a_4 &\geq 0 \text{ and} \\ \deg(\beta_{(20001)}) = 2 - p_1 - a_2 - a_3 - a_4 &\geq 0. \end{aligned} \tag{6.11}$$

From Inequalities $a_5 \geq a_4 \geq a_3 \geq a_2 \geq 0$, (6.10) and (6.11) we get no new Complete Intersection Calabi–Yau threefolds

2. For the case when $\dim(B_1) = 2$ and $B_2 = \emptyset$, the Inequality (6.9) implies that $\alpha_{(02000)}(t_1,t_2)x_2^2$ does not appear in f_1, $\alpha_{(00200)}(t_1,t_2)x_3^2$ appears in f_1 and $\beta_{(3000)}(t_i)x_1^3$ appears in f_2. Equivalently

$$\begin{aligned} p_1 + 2a_2 &< 0, \\ p_1 + 2a_3 &\geq 0 \text{ and} \\ 2 - p_1 - a_2 - a_3 - a_4 - a_5 &\geq 0. \end{aligned} \tag{6.12}$$

Define

$$\Delta(t_i, x_1, x_2, 0, 0, 0) := \bigwedge^2 \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \end{bmatrix}$$

where

$$n_{11} = \left.\frac{\partial f_1}{\partial x_3}\right|_{\{x_3=x_4=x_5=0\}} = \alpha_{(10100)}(t_i)x_1 + \alpha_{(01100)}(t_i)x_2,$$
$$n_{12} = \left.\frac{\partial f_1}{\partial x_4}\right|_{\{x_3=x_4=x_5=0\}} = \alpha_{(10010)}(t_i)x_1 + \alpha_{(01010)}(t_i)x_2,$$
$$n_{13} = \left.\frac{\partial f_1}{\partial x_5}\right|_{\{x_3=x_4=x_5=0\}} = \alpha_{(10001)}(t_i)x_1 + \alpha_{(01001)}(t_i)x_2,$$
$$n_{21} = \left.\frac{\partial f_2}{\partial x_3}\right|_{\{x_3=x_4=x_5=0\}} = \beta_{(20100)}(t_i)x_1^2 + \beta_{(11100)}(t_i)x_1x_2 + \beta_{(20100)}(t_i)x_2^2,$$
$$n_{22} = \left.\frac{\partial f_2}{\partial x_4}\right|_{\{x_3=x_4=x_5=0\}} = \beta_{(20010)}(t_i)x_1^2 + \beta_{(11010)}(t_i)x_1x_2 + \beta_{(20010)}(t_i)x_2^2,$$
$$n_{23} = \left.\frac{\partial f_2}{\partial x_5}\right|_{\{x_3=x_4=x_5=0\}} = \beta_{(20001)}(t_i)x_1^2 + \beta_{(11001)}(t_i)x_1x_2 + \beta_{(20001)}(t_i)x_2^2.$$

Note that with

$$\begin{aligned} \deg(\alpha_{(01010)}) &= p_1 + a_2 + a_4 \geq 0 \text{ and} \\ \deg(\beta_{(20001)}) &= 2 - p_1 - a_2 - a_3 - a_4 \geq 0, \end{aligned} \tag{6.13}$$

the threefold X has isolated singularities along $\{x_3 = x_4 = x_5 = 0\}$. These are

$$\begin{aligned} \mathrm{Sing}(\widetilde{X}) \cap \widetilde{\mathbb{F}(0,a_2)} &= \mathbb{V}(f_1, f_2, \Delta(t_i, x_1, x_2, 0, 0, 0), x_3, x_4, x_5) \\ &= \mathbb{V}(f_2(t_i, x_1, x_2, 0, 0, 0),\ \Delta(t_i, x_1, x_2, 0, 0, 0), x_3, x_4, x_5). \end{aligned}$$

We then get, from Inequalities $a_5 \geq a_4 \geq a_3 \geq a_2 \geq 0$, (6.12) and (6.13) the Complete Intersection Calabi–Yau threefold

$$X = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,1,1,1)$$

with 2 isolated singular points

$$\begin{aligned} Sing(X_2) \cap \mathbb{F}(0,0) &= \mathbb{V}(\beta_{(30000)}(t_i)x_1^3 + \ldots + \beta_{(03000)}(t_i)x_2^3,\ \Delta(t_i, x_1, x_2, 0, 0, 0)) \\ &= \mathbb{V}(|L_{0,2}|_{\mathbb{F}(0,0)}|, |L_{1,3}|_{\mathbb{F}(0,0)}|) \subset \mathbb{P}^1 \times \mathbb{P}^1. \end{aligned}$$

3. Consider the case when $\dim(B_1) = 3$ and $B_2 = \emptyset$, the Inequality (6.9) implies that $\alpha_{(00200)}(t_i)x_3^2$ does not appear in f_1, $\alpha_{(00020)}(t_i)x_4^2$ appears in f_1 and $\beta_{(3000)}(t_i)x_1^3$ appears in f_2. Equivalently

$$\begin{aligned} p_1 + 2a_3 &< 0, \\ p_1 + 2a_4 &\geq 0 \text{ and} \\ 2 - p_1 - a_2 - a_3 - a_4 - a_5 &\geq 0. \end{aligned} \tag{6.14}$$

Define $Sing(X) = \mathbb{V}(f_1, f_2, \Delta(t_i, x_j))$ with

$$\Delta(t_i,x_j) = \left(\text{rank}\begin{bmatrix}\nabla f_1\\ \nabla f_2\end{bmatrix} < 2\right) = \left(\bigwedge^2 \begin{bmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \frac{\partial f_1}{\partial x_5} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} & \frac{\partial f_2}{\partial x_5}\end{bmatrix}\right).$$

Now, along $\{x_4 = x_5 = 0\}$, the threefold X has singular locus

$$\begin{aligned}\text{Sing}(\widetilde{X}) \cap \mathbb{F}(\widetilde{0,a_2,a_3}) &= \mathbb{V}(f_1, f_2, \Delta(t_i, x_1, x_2, x_3, 0, 0), x_4, x_5)\\ &= \mathbb{V}(f_2(t_i, x_1, x_2, x_3, 0, 0),\ \Delta(t_i, x_1, x_2, x_3, 0, 0), x_4, x_5)\end{aligned}$$

where

$$\mathbb{F}(\widetilde{0,a_2,a_3}) = q^{-1}(\mathbb{V}(x_4,x_5)) \subset \mathbb{C}^2_{t_i} \setminus \{0\} \times \mathbb{C}^5_{x_j} \setminus \{0\} \xrightarrow{q} \mathbb{F}.$$

The locus $\text{Sing}(\widetilde{X}) \cap \mathbb{F}(\widetilde{0,a_2,a_3})$ are isolated singularities if it is defined by

$$f_2|_{\{x_4=x_5=0\}} \not\equiv 0$$

and an irreducible non-constant quadric

$$\Delta(t_i, x_1, x_2, x_3, 0, 0) = \det\left(\begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\right) \not\equiv 0.$$

Here

$$\begin{aligned}
m_{11} = \left.\frac{\partial f_1}{\partial x_4}\right|_{\{x_4=x_5=0\}} &= \alpha_{(10010)}(t_i)x_1 + \alpha_{(01010)}(t_i)x_2 + \alpha_{(00110)}(t_i)x_3,\\
m_{12} = \left.\frac{\partial f_1}{\partial x_5}\right|_{\{x_4=x_5=0\}} &= \alpha_{(10001)}(t_i)x_1 + \alpha_{(01001)}(t_i)x_2 + \alpha_{(00101)}(t_i)x_3,\\
m_{21} = \left.\frac{\partial f_2}{\partial x_4}\right|_{\{x_4=x_5=0\}} &= \beta_{(20010)}(t_i)x_1^2 + \beta_{(11010)}(t_i)x_1x_2 + \beta_{(10110)}(t_i)x_1x_3 +\\
&\quad \beta_{(02010)}(t_i)x_2^2 + \beta_{(01110)}(t_i)x_2x_3 + \beta_{(00210)}(t_i)x_3^2,\\
m_{22} = \left.\frac{\partial f_2}{\partial x_5}\right|_{\{x_4=x_5=0\}} &= \beta_{(20001)}(t_i)x_1^2 + \beta_{(11001)}(t_i)x_1x_2 + \beta_{(10101)}(t_i)x_1x_3 +\\
&\quad \beta_{(02001)}(t_i)x_2^2 + \beta_{(01101)}(t_i)x_2x_3 + \beta_{(00201)}(t_i)x_3^2.
\end{aligned}$$

Equivalently

$$\begin{aligned}\deg(\alpha_{(00110)}) = p_1 + a_3 + a_4 \geq 0 \text{ and}\\ \deg(\beta_{(00210)}) = 2 - p_1 - a_2 + a_3 - a_5 \geq 0.\end{aligned} \tag{6.15}$$

We then get, from Inequalities $a_5 \geq a_4 \geq a_3 \geq a_2 \geq 0$, (6.15) and (6.14), the following data on Complete Intersection Calabi–Yau threefold in 5-fold scrolls

$$(p_1, a_2, a_3, a_4, a_5) = (-3,0,1,2,2), (-2,0,0,2,2), (-1,0,0,1,1) \text{ and } (-1,0,0,1,2).$$

From the list above, we get

(i) $X_1 = \begin{bmatrix} -3 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,1,2,2)$ with

$$\begin{aligned} Sing(X_1) \cap \mathbb{F}(0,0,1) &= \mathbb{V}(\beta_{(30000)}(t_i)x_1^3 + \ldots + \beta_{(00300)}(t_i)x_3^3,\ \Delta(t_i,x_1,x_2,x_3,0,0)) \\ &= \mathbb{V}(|L_{0,3}|_{\mathbb{F}(0,0,1)}|, |L_{4,3}|_{\mathbb{F}(0,0,1)}|) = \emptyset \subset \mathbb{F}(0,0,1). \end{aligned}$$

(ii) $X_2 = \begin{bmatrix} -2 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,2,2)$ with

$$\begin{aligned} Sing(X_2) \cap \mathbb{F}(0,0,0) &= \mathbb{V}(\beta_{(30000)}(t_i)x_1^3 + \ldots + \beta_{(00300)}(t_i)x_3^3,\ \Delta(t_i,x_1,x_2,x_3,0,0)) \\ &= \mathbb{V}(q_3(x_1,x_2,x_3)) \cap |L_{2,3}|_{\mathbb{F}(0,0,0)}| \subset \mathbb{P}^1_{[t_1:t_2]} \times \mathbb{P}^2_{[x_1:x_2:x_3]} \end{aligned}$$

where $q_3 = \beta_{(30000)}(t_i)x_1^3 + \ldots + \beta_{(00300)}(t_i)x_3^3$.
Here we are intersecting a general $(0,3)$- and a general $(2,3)$-form on $\mathbb{F}(0,0,0) = \mathbb{P}^1 \times \mathbb{P}^2$. Now, with at most 2 vertical rulings and at most 3 horizontal rulings on $\mathbb{F}(0,0,0)$, we concluded that there are $6 = 2 \times 3$ isolated singular points $\{[\gamma_{1i} : \gamma_{2i}; k_{1i} : k_{2i} : k_{3i} : 0 : 0] : i = 1,2,3\}$ on X.

(iii) $X_3 = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,1,1)$ with

$$\begin{aligned} Sing(X_3) \cap \mathbb{F}(0,0,0) &= \mathbb{V}(\beta_{(30000)}(t_i)x_1^3 + \ldots + \beta_{(00300)}(t_i)x_3^3,\ \Delta(t_i,x_1,x_2,x_3,0,0)) \\ &= \mathbb{V}(|L_{1,3}|_{\mathbb{F}(0,0,0)}|, |L_{2,3}|_{\mathbb{F}(0,0,0)}) = \emptyset \subset \mathbb{P}^1 \times \mathbb{P}^2. \end{aligned}$$

(iv) $X_4 = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,1,2)$ with 6 isolated singular points

$$\begin{aligned} Sing(X_2) \cap \mathbb{F}(0,0,0) &= \mathbb{V}(\beta_{(30000)}(t_i)x_1^3 + \ldots + \beta_{(00300)}(t_i)x_3^3,\ \Delta(t_i,x_1,x_2,x_3,0,0)) \\ &= \mathbb{V}(r_3(x_1,x_2,x_3), |L_{2,3}|_{\mathbb{F}(0,0,0)}|) \subset \mathbb{P}^1 \times \mathbb{P}^2 \end{aligned}$$

where $r_3 = \beta_{(30000)}(t_i)x_1^3 + \ldots + \beta_{(00300)}(t_i)x_3^3$.

4. Consider the case when $\dim(B_1) = 3$ and $\dim(B_2) = 1$, the Inequality (6.9) implies that $\alpha_{(00200)}(t_1,t_2)x_3^2$ does not appear in f_1, $\alpha_{(00020)}(t_1,t_2)x_4^2$ appears in f_1 and $\beta_{(3000)}(t_1,t_2)x_1^3$ does not appear in f_2, $\beta_{(0300)}(t_i)x_2^3$ appears in f_2. Equivalently

$$\begin{aligned} p_1 + 2a_3 &< 0, \\ p_1 + 2a_4 &\geq 0, \\ 2 - p_1 - a_2 - a_3 - a_4 - a_5 &< 0, \text{ and} \\ 2 - p_1 + 2a_2 - a_3 - a_4 - a_5 &\geq 0. \end{aligned} \tag{6.16}$$

We then have that

$$B_1 = D_{45} = \{x_4 = x_5 = 0\} \text{ and } B_2 = D_{2345} = \{x_2 = x_3 = x_4 = x_5 = 0\}.$$

Now, along D_{45}, we get isolated singularities whenever Inequalities (6.15) are satisfied. That is

$$p_1 + a_3 + a_4 \geq 0,\ 2 - p_1 - a_2 + a_3 - a_5 \geq 0$$

are satisfied together with Inequalities (6.16). This, however, do not result in new examples. In fact, for all B_1 and B_2 with distinct dimensions

$$1 \leq \dim(B_1), \dim(B_2) \leq 3,$$

the analysis results in no new Calabi–Yau threefolds. □

No.	$X \subset \mathbb{F}(0,a_2,a_3,a_4,a_5) = \mathbb{F}$	General $X = \begin{bmatrix} p_1 & 2-p_1-\sum a_j \\ 2 & 3 \end{bmatrix}$
1	$\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,0,0)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular CY threefold.
2	$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,0,1)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular CY threefold.
3	$\begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,0,2)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular CY threefold.
4	$\begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,1,1)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular CY threefold.
5	$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,0,0)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular CY threefold.
6	$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,0,1)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular CY threefold.
7	$\begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,0,0)$	$\lvert D_1\rvert, \lvert D_2\rvert$ are base-point-free General X is nonsingular CY threefold.
8	$\begin{bmatrix} -3 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,1,2,2)$	$\lvert D_1\rvert$ is base-point-free General X is nonsingular along $Bs(\lvert D_2\rvert) = \mathbb{V}(x_4,x_5)$
9	$\begin{bmatrix} -2 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,2,2)$	$\lvert D_1\rvert$ is base-point-free General X has 6 isolated singular points along $Bs(\lvert D_2\rvert) = \mathbb{V}(x_4,x_5)$
10	$\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,1,1)$	$\lvert D_1\rvert$ is base-point-free General X is nonsingular along $Bs(\lvert D_2\rvert) = \mathbb{V}(x_4,x_5)$
11	$\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,0,1,2)$	$\lvert D_1\rvert$ is base-point-free General X has 6 isolated singular points along $Bs(\lvert D_2\rvert) = \mathbb{V}(x_4,x_5)$
12	$\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \subset \mathbb{F}(0,0,1,1,1)$	$\lvert D_1\rvert$ is base-point-free General X has 2 isolated singular points along $Bs(\lvert D_2\rvert) = \mathbb{P}^1 \times \mathbb{P}^1$

Table 6.3: The data $(p_1, a_2, \dots, a_5|)$ for which a general codimension two Calabi–Yau threefold in $|L_{p_1,2}| \cap |L_{2-p_1-\sum a_j,3}| \subset \mathbb{F}$ has at worst isolated singularities.

Chapter 7

Further Directions

7.1 General Theory of Fibrations in Scrolls

Let $X \subset \mathbb{P}^n$ be a projective variety with well-understood resolution structure and such that the cone $C(X) \subset \mathbb{A}^{n+1}$ has a large number of possible torus actions that induces various grading on the coordinate ring $\mathbb{C}[C(X)]$. One can make other interesting varieties $Y \subset X \subset \mathbb{P}^n$ by taking complete intersections in X. If this is the case then we call X a **key variety** of Y. The straight projective variety $\mathbb{P}^n$ is an obvious key variety for the usual weighted complete intersection in $\mathbb{P}^n$. One could also take, as a key variety, products of projective spaces with their respective Segre embedding such as

$$\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8 \text{ or } \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$$

with nice resolution structure and construct interesting varieties in them. For examples, in [BKQ18], codimension 4 Fano 3-folds are constructed from $\mathbb{P}^2 \times \mathbb{P}^2$ as a key variety.

Other examples include more Fano 3-folds of codimension 4 and 5 constructed in [ST20] by taking rank 2 cluster varieties as the key variety. In [CR02], codimension 3 K3's in weighted projective spaces are constructed by taking (quasi) linear sections of weighted Grassmannian $wGr(2,5) \hookrightarrow \mathbb{P}^9[b_j]$; this simplifies the harder unprojection construction in [Alt98] of the same varieties. Generalized Gorenstein resolution formats are used in [BKL19] to construct non-complete intersection Calabi–Yau threefolds.

In this section, we introduce examples of relative key varieties; that is, key varieties in families over a base B (we use $B = \mathbb{P}^1$ for illustration) in which projective fibrations in weighted scrolls may be embedded by taking hypersurface sections of these relative key varieties. We would like to make sense of "construction of interesting fibrations" over the same base in specific cases.

We also start to develop machinery for studying the geometry of such projective fibrations using geometry of bigraded rings.

7.1.1 Bigraded Rings from Projective Fibrations

Let (X,H) and (B,D) be polarised projective varieties where H is an ample $\mathbb{Q}$-Cartier Weil divisor on X, and D is an ample Cartier divisor on the nonsingular base B. Assume that $f_*\mathcal{O}_X = \mathcal{O}_B$ so that the projective morphism $f : X \to B$ has connected polarised fibres $(X_t,H_t)_{t\in B}$ with $H_t = H|_{X_t}$.

Now with $L_{p_1,p_2} := p_1 f^*D + p_2H \in Cl(X)$, define the bigraded ring

$$R := \bigoplus_{(p_1,p_2)\in\mathbb{Z}^2} H^0(X, L_{p_1,p_2}).$$

Since the base B is nonsingular and the fibres of f are connected, we get from the projection formula that

$$f_*f^*\mathcal{O}_B(D) \cong f_*\mathcal{O}_X \otimes \mathcal{O}_B(D) \cong \mathcal{O}_B(D).$$

At the level of cohomology, we have that

$$H^0(X, p_1 f^*\mathcal{O}_B(D)) \cong H^0(B, p_1 f_*f^*\mathcal{O}_B(D)) \cong H^0(B, p_1\mathcal{O}_B(D)). \tag{7.1}$$

We then have a subring

$$R_1 := \bigoplus_{p_1\geq 0} H^0(X, p_1 f^*\mathcal{O}_B(D)) \cong \bigoplus_{p_1\geq 0} H^0(B, p_1\mathcal{O}_B(D)) = \langle u_1,\dots,u_k\rangle_{\mathbb{C}} \subset R$$

where the algebra generators $u_1,\dots,u_k$ of R_1 are of bidegree $(c_i,0)$ for some positive integers c_i.

Remark 7.1.1. *For a hypersurface $X \subset \mathbb{F}$, we have that the restriction map*

$$r : \bigoplus_{(p_1,p_2)\in\mathbb{Z}^2} H^0(\mathbb{F}, \mathcal{O}_{\mathbb{F}}(p_1L + p_2M)) = \mathbb{C}[\mathbb{F}] \to R_X = \bigoplus_{(p_1,p_2)\in\mathbb{Z}^2} H^0(X, \mathcal{O}_X(p_1L|_X + p_2M|_X))$$

is not surjective in general. For example, on F_2 with $L = f^*D$ and $M = H$, we know from Theorem (3.9.2) that $p_1 f^*D + p_2H$ is ample for $p_1, p_2 > 0$ but not anything else. That is, for a hypersurface $X \subset F_2$, unlike in the absolute graded pieces $H^0(X, p_1 f^*D)$ of $R_B = R_1$, the bigraded pieces $H^0(X, p_1 f^*D + p_2H)$ of R can have sections without $p_1 f^*D + p_2H$ necessarily being ample.

Now, consider the finitely generated bigraded ring

$$R_{B,H} = \langle R_B, R_H\rangle \subset R_X$$

where the absolutely graded rings

$$R_B = R_1 := \bigoplus_{p_1 \geq 0} H^0(X, p_1 f^* \mathscr{O}_B(D)) \text{ and } R_H = \bigoplus_{p_2 \geq 0} H^0(X, p_2 H)$$

are finitely generated subrings of R_X. *Even though* $\mathrm{Proj}(R_{B,H}) = X$, *this re-embeds* X *into something bigger than* $\mathbb{F}_A$ *so that we do not have*

$$(f : X \to \mathbb{P}^1) \hookrightarrow (\mathbb{F}_A \to \mathbb{P}^1)$$

as we would like. Therefore, we need a bigraded ring that remembers the base and has just enough generators to recover X *with* $\mathbb{C}[X] \cong \mathbb{C}[\mathbb{F}_A]/(Relations)$.

It is the case that exactly one of $H^0(X, p_1 f^* D + p_2 H)$ *or* $H^0(X, -p_1 f^* D - p_2 H)$ *is nonzero and that* $H^0(X, p_1 f^* D + p_2 H) = 0$ *if* $p_1 \leq 0, p_2 < 0$.

The following result introduces a set-up for constructing projective fibrations

$$(f : X \to B) \hookrightarrow (\mathbb{F} \to \mathbb{P}^{k-1}[c_i]).$$

Proposition 7.1.2. *Let* $f : X \to B$ *be a projective fibration with connected polarised fibres* $(X_s, H|_{X_s} := \mathscr{O}_X(1)|_H)$ *as above. Assume the bigraded ring* R *defined above is finitely generated. Consider algebra generators* $v_1, \ldots, v_n$ *of* R *over* R_1. *We have a surjection*

$$\begin{array}{ccc} R = R_1[v_1, \ldots, v_n] & \xleftarrow[v_j \leftarrow x_j]{} & S = S_1[x_1, \ldots, x_n] \\ \uparrow & & \uparrow \\ R_1 = \mathbb{C}[u_1, \ldots, u_k] & \xleftarrow[u_i \leftarrow t_i]{} & S_1 = \mathbb{C}[t_1, \ldots, t_k]. \end{array}$$

from which we have the diagram below

$$\begin{array}{ccc} X & \xhookrightarrow{\quad i \quad} & \mathrm{Proj}\, S \cong \mathbb{F}_A \\ \Big\downarrow{\scriptstyle f = \pi|_X} & & \Big\downarrow{\scriptstyle \pi} \\ B & \xhookrightarrow{\quad j \quad} & \mathrm{Proj}\, S_1 \cong \mathbb{P}^{k-1}[c_i] \end{array}$$

where

$$A := wt(t_1, \ldots, t_k, x_1, \ldots, x_n) = \begin{bmatrix} c_1 & \ldots & c_k & -a_1 & \ldots & -a_n \\ 0 & \ldots & 0 & b_1 & \ldots & b_n \end{bmatrix}.$$

Proof. We first show positivity of the weights b_j so that we have the embedding $h_s : X_s \hookrightarrow \mathbb{P}[b_1,\dots,b_n]$. It suffices to show that $H^0(X, p_1 f^*D + p_2 H) = 0$ whenever $p_2 < 0$. To do this we note that the divisor H is ample on fibres X_s and that f^*D is nef. In particular, we have that

$$p_1 f^*D|_{X_s} + p_2 H|_{X_s} = p_2 H|_{X_s}$$

is ample. Suppose $p_2 < 0$, from the inclusion $i_s : X_s \hookrightarrow X$, we have that

$$i_s^* : H^0(X, p_1 f^*D + p_2 H) \to H^0(X_s, p_2 H|_{X_s}) = 0.$$

A section $\sigma \in H^0(X, p_1 f^*D + p_2 H)$ on X is therefore such that $i_s^*\sigma = 0$ for all $s \in B$ hence $\sigma = 0$. We also have a nonsingular base B so that all requirements of [Har77][Cor 12.9] are satisfied; hence

$$f_*\mathscr{O}_X(D) \cong H^0(X_s, \mathscr{O}_{X_s}(D|_{X_s})) \cong H^0(X_s, p_2 H|_{X_s}) = 0.$$

It then follows from the Equation (7.1) that

$$H^0(X, p_1 f^*D + p_2 H) = 0.$$

Now let $S_1 = \mathbb{C}[t_1,\dots,t_k]$ and consider free ring S generated over S_1 by $x_1,\dots,x_n$ such that $\deg(t_i) = (c_i, 0), \deg(x_j) = (-a_j, b_j)$ with integers $a_j \in \mathbb{Z}, c_i, b_j > 0$. We then have that

$$S = \mathbb{C}[t_1,\dots,t_k; x_1,\dots,x_n] \cong \mathbb{C}\left[\mathbb{F}\begin{bmatrix} c_1 & \dots & c_k & -a_1 & \dots & -a_n \\ 0 & \dots & 0 & b_1 & \dots & b_n \end{bmatrix}\right].$$

We then note, by surjectivity, that the upper horizontal ring map preserves the grading of S so that S_{p_1,p_2} is mapped onto R_{p_1,p_2}. Let $R = S/J$ and $R_1 = S_1/I$ with bi-homogeneous ideal $J \lhd S$ and homogeneous ideal $I \lhd S_1$.

Taking bigraded Proj, as in Lemma 6.4 of [KU19], at the vertices of the first diagram gives the second diagram with $X = \mathbb{V}(J)$ and $B = \mathbb{V}(I)$. Note that with $R_2 = \bigoplus H^0(X, p_2 H) \hookrightarrow R$, we also have $\mathbb{V}(J) = \text{Proj}(R) \hookrightarrow \text{Proj}(R_2) = X$.

We further have that

$$f^*D = L|_X \text{ and } H = M|_X \text{ with } Cl(\mathbb{F}_A) = \mathbb{Z}[L, M]$$

where

$$L := \pi^*\mathscr{O}_{\mathbb{P}}(1) \text{ and } M|_{\pi^{-1}(t)} \cong \mathscr{O}_{\mathbb{P}^{n-1}[b_j]}(1).$$

Since $H \sim H + f^*D$, we have

$$R_2 \twoheadleftarrow S_2 = \bigoplus_{p_2 \geq 0} H^0(\mathbb{F}_A, p_2 M) \hookrightarrow S \text{ with ample } M$$

so that

$$X = \mathrm{Proj}(R_2) \hookrightarrow \mathbb{P}(S_{(0,1)}) \xleftarrow{\varphi_{|M|}} \mathbb{F}_A.$$

We then have the following diagram

$$\begin{array}{ccc} X & \xhookrightarrow{\quad i \quad} & \mathbb{F}_A \\ \uparrow{\scriptstyle i_s} & & \uparrow{\scriptstyle k_t} \\ X_s & \xhookrightarrow{\quad h_s \quad} & \mathbb{P}^{n-1}[b_j]. \end{array}$$

Here, the inclusion

$$k_t : \pi^{-1}(t) \cong \mathbb{P}^{n-1}[b_j] \hookrightarrow \mathbb{F}_A$$

is such that

$$k_t{}^*M = \mathbb{P}^{n-1}[b_j](1) = M|_{\pi^{-1}(t)}. \tag{7.2}$$

By Equation (7.2) and pullback of H along i_s we obtain

$$\begin{aligned} H|_{X_s} = i_s{}^*H &= i_s{}^*(i^*M|_X) \\ &= i_s{}^*(k_t^*M|_X) \\ &= i_s{}^*(\mathscr{O}_X(1)|_H). \end{aligned}$$

We have therefore exhibited polarised fibres $(X_s, H|_{X_s})$ such that

$$(X_s, H|_{X_s}) \hookrightarrow \left(\mathbb{P}^{n-1}[b_j], \mathscr{O}_{\mathbb{P}^{n-1}[b_j]}(1)\right).$$

Starting with the general hyperplane $H = \sum_{i=1}^{k} a_i t_i$ on $\mathbb{P}$ one gets that

$$f^* i^* = j^* \pi^*$$

so that the diagram above commutes. Hence

$$(f : X \to B) \hookrightarrow (\mathbb{F} \to \mathbb{P}^{k-1}[c_i])$$

with $f := \pi \circ i|_B$. □

The fibration $f : X \to B$ gets embedded in the fibration $\pi : \mathbb{F}_A \to \mathbb{P}^{k-1}[c_i]$ whose fibres are $\mathbb{P}_t := \pi^{-1}(t) = \mathbb{P}^{n-1}[b_j]$. We also have $(B, D) \overset{j}{\hookrightarrow} (\mathbb{P}, \mathscr{O}_{\mathbb{P}}(1))$. If the embedding codimension of the fibres is low and/or j is an isomorphism with $\mathbb{P} = \mathbb{P}^1$ or $\mathbb{P}^2$, we can hope to study f using explicit methods.

In the next example, we show that we can use the same methods of this thesis to classify and study other classes of varieties in Weighted Scrolls over $\mathbb{P}^2$. The example constructs a 3-fold of general type fibred by a family of plane curves of a fixed genus over $\mathbb{P}^2$.

Example 7.1.3. Let $\pi : \mathbb{F} = \mathbb{F}(2,1,0) \to \mathbb{P}^2$ be a 4-fold scroll over $\mathbb{P}^2_{[t_i]}$ with $\mathbb{P}^2_{[x_j]}$ fibres. We also have that

$$Cl(\mathbb{F}(2,1,0)) = \mathbb{Z}[L,M] \text{ with } M = [\mathbb{V}(x_3)], L = [\mathbb{V}(t_i)]$$

Let L_{p_1,p_2} be ample on $\mathbb{F}$ and assume that $K_X = L_{p_1,p_2}|_X$. We get that $X = \mathbb{V}(f_{p_1,p_2}) \subset \mathbb{F}$ is a 3-fold of general type. Here

$$S_{(p_1,p_2)} = \bigoplus_{q_1+q_2+q_3=p_2} \mathbf{Sym}^{p_1+2q_1+q_2}(t_1,t_2,t_3)x_1^{q_1}x_2^{q_2}x_3^{q_3}$$

and

$$f_{p_1,p_2} = \sum_{q_1+q_2+q_3=p_2} \alpha_{p_1+2q_1+q_2}(t_1,t_2,t_3)x_1^{q_1}x_2^{q_2}x_3^{q_3}.$$

For X chosen generally, its fibre is a nonsingular plane curve of degree p_2.

We conclude a general $X \in |L_{p_1,p_2}|$ in $\mathbb{F}$ is a 3-fold of general type fibred by a family of plane curves which are generically nonsingular of genus

$$\frac{1}{2}(p_2-1)(p_2-2).$$

7.2 Projective Bundles over Relative Key Varieties

Consider $u : X \hookrightarrow \Sigma$ with Σ a key variety of X. Let $g : \Sigma \to B$ be a family of Σ over a nonsingular base B. This section will introduce these relative key varieties $g : \Sigma \to B$ whose relative hypersurface sections are fibration $f : X \to B$ fitting into the following diagram

$$\begin{array}{ccccc} X & \overset{u}{\hookrightarrow} & \Sigma & \hookrightarrow & \mathbb{F} \\ & \overset{f}{\searrow} & \downarrow g & & \downarrow \pi \\ & & B & \hookrightarrow & \mathbb{P}^{k-1}[c_i]. \end{array}$$

Construction of these fibrations will form part of future projects.

7.2.1 A relative Key variety for Degree 5 Elliptic fibrations

From the graded ring

$$R(E,5P) = \bigoplus_{n\geq 0} H^0(E, \mathscr{O}_E(5nP)),$$

projective models of Elliptic curves E, with $P \in E$, are given on Table 7.1 .

We want to define a $Gr(2,5)$-bundle over $\mathbb{P}^k[c_i]$ and hence construct Calabi–Yau varieties

Hilbert Series $P_R(t)$	Degree	Ambient Space	Description of E
$\frac{1-5t^2+5t^3-t^5}{(1-t)^5}$	5	$\mathbb{P}^4$	$\mathbb{P}^4$ sections of $Gr(2,5) \subset \mathbb{P}^9$

Table 7.1: Data on elliptic curve $\mathrm{Proj}(R(E,5P))$

X with at worst rational double point singularities and which are fibred over $\mathbb{P}^k[c_i]$ by $\mathbb{P}^4$ sections of $Gr(2,5)$: degree 5 elliptic curves.

We first note that the Grassmannian

$$Gr(2,5) = \mathbb{P}(\mathrm{Im}\,\eta) \overset{p}{\hookrightarrow} \mathbb{P}(\wedge^2\mathbb{C}^5) \cong \mathbb{P}^9_{[x_{ij}]}$$

is the projectivisation of the variety defined by the image of

$$\eta : \mathbb{C}^5 \times \mathbb{C}^5 \xrightarrow{(u,v)\mapsto u\wedge v} \wedge^2\mathbb{C}^5 \cong \mathbb{C}^{10}.$$

In coordinates, we write

$$\bigwedge^2 \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix} = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ & 0 & x_{23} & x_{24} & x_{25} \\ & & 0 & x_{34} & x_{35} \\ -sym & & & 0 & x_{45} \\ & & & & 0 \end{bmatrix} = (w_{ij}), \text{ with } x_{ij} = \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}.$$

These variables x_{ij} are known to satisfy 5 Plücker quadrics

$$P_5 := Pf_{1234} = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$$

$$P_4 := Pf_{1235} = x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23}$$

$$P_3 := Pf_{1245} = x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24}$$

$$P_2 := Pf_{1345} = x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34}$$

$$P_1 := Pf_{2345} = x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34}$$

from 4×4 Pffafians P_k of the 5×5 skew anti-symmetric matrix (x_{ij}). We have that the nonsingular 6-dimensional Grassmannian is given by

$$Gr(2,5) = \mathrm{Proj}\left(\mathbb{C}[x_{ij}]/\left\langle Pf_{ijkl}\right\rangle\right).$$

The curve $E \subset Gr(2,5)$ is generated by the 5 Plücker quadrics and 5 general linear forms $\ell_j(x_{ij})$ on $\mathbb{P}^9_{[x_{ij}]}$ or 5 non-general quadrics q_k on a general linear subspace $\mathbb{P}^4_{[\ell_j]} \subset \mathbb{P}^9_{[x_{ij}]}$

$$\begin{aligned} \mathbb{I}(E) =& (P_1,\dots,P_5,\ell_1,\dots,\ell_5) \lhd \mathbb{C}[\mathbb{P}^9_{x_{ij}}] \\ =& (q_1,\dots,q_5) \lhd \mathbb{C}[\mathbb{P}^4_{\ell_j}]. \end{aligned}$$

That is

$$\begin{array}{ccc} E = Gr(2,5)\cap \mathbb{P}^4 & \overset{i}{\hookrightarrow} & \mathbb{P}^4 \\ \Big\downarrow i & & \Big\downarrow j \\ Gr(2,5) & \overset{p}{\hookrightarrow} & \mathbb{P}^9. \end{array}$$

Now,

$$K_{Gr(2,5)} = p^*\mathscr{O}_{\mathbb{P}^9}(-5) = \mathscr{O}_{Gr(2,5)}(-5)$$

and since $\mathbb{P}^4$ is a general linear subspace of $\mathbb{P}^9$, the normal bundle $N_{E/Gr(2,5)}$ is given by

$$N_{E/Gr(2,5)}|_E = i^*N_{\mathbb{P}^4/\mathbb{P}^9} = i^*\mathscr{O}_{\mathbb{P}^4}(1)^{\oplus 5}.$$

Indeed, by the adjunction formula, we have that

$$\begin{aligned} K_E =& K_{Gr(2,5)}|_E \otimes \det(N_{E/Gr(2,5)})|_E \\ =& \mathscr{O}_E(-5)\otimes \mathscr{O}_E(5) = \mathscr{O}_E \end{aligned}$$

so that E is an elliptic curve as we want.

We now take the weight matrix on $\mathbb{C}[\mathbb{P}^9_{x_{ij}}]$ to be

$$W = (wt(x_{ij})) = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} & b_{15} \\ & 0 & b_{23} & b_{24} & b_{25} \\ & & 0 & b_{34} & b_{35} \\ -sym & & & 0 & b_{45} \\ & & & & 0 \end{bmatrix}. \tag{7.3}$$

Define the cone $C(Gr(2,5))$ over Grassmannian $Gr(2,5)$ by

$$q^{-1}(\mathbb{V}(P_1,\dots,P_5))\cup\{0\}\subset\mathbb{C}^{10} \text{ where } q:\mathbb{C}^{10}\setminus\{0\}\to\mathbb{P}^9[b_{ij}].$$

We want to single out the (relative key variety Σ) twisting vectors (a_{ij}) where the ordered integers a_{ij} satisfy

$$\begin{aligned} a_{12}+a_{34} &= a_{13}+a_{24} = a_{14}+a_{23}; \qquad (7.4)\\ a_{12}+a_{35} &= a_{13}+a_{25} = a_{15}+a_{23};\\ a_{12}+a_{45} &= a_{14}+a_{25} = a_{15}+a_{24};\\ a_{13}+a_{45} &= a_{14}+a_{35} = a_{15}+a_{34};\\ a_{23}+a_{45} &= a_{24}+a_{35} = a_{25}+a_{34}. \end{aligned}$$

A $Gr(2,5)$-bundle over $\mathbb{P}^k[c_j]$ is then defined by

$$\mathbb{F}_A := \left(\mathbb{C}^k_{t_i}\setminus\{0\}\times C(Gr(2,5))\setminus\{0\}\right)/\mathbb{C}^*\times\mathbb{C}^*,\ A = \begin{bmatrix} c_i & -a_{ij} \\ 1 & b_{ij} \end{bmatrix} \tag{7.5}$$

where $\mathbb{C}^* \times \mathbb{C}^*$ acts by

$$(\lambda, \mu).(t_i, x_{ij}) = (\lambda^{c_i} t_i, \lambda^{-a_{ij}} \mu^{b_{ij}} x_{ij}).$$

The system (7.4) of 10 linear equations in a_{ij} has solutions $a_{ij} = w_i + w_j$ with $w_i, w_j \in \mathbb{Z}\left[\frac{1}{2}\right]$

$$\left\{ (a_{ij}) \in \mathbb{Z}^{10} : \begin{bmatrix} a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{23} \\ a_{24} \\ a_{25} \\ a_{34} \\ a_{35} \\ a_{45} \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + w_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + w_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

We considering $Gr(2,5)$-bundle Σ over $\mathbb{P}^1$ fitting into the following set-up with superscripts denoting respective dimensions:

$$\begin{array}{ccccc} X^2 & \hookrightarrow & \Sigma^7 & \hookrightarrow & \mathbb{F}^{10}(a_{12}, \ldots, a_{45}) \\ & \searrow^{f} & \downarrow \phi & \swarrow_{\pi} & \\ & & \mathbb{P}^1_{[t_i]}. & & \end{array}$$

Here, the fibres of π are $\mathbb{P}^9_{x_{ij}}[b_{ij}]$, the fibres of ϕ are copies of $Gr(2,5)$ and f-fibres are copies of the elliptic curve E.

On $\mathbb{F}$, we have $L := [\mathbb{V}(t_i)]$, $M := [\mathbb{V}(x_{12})] \in Cl(\mathbb{F}) = \mathbb{Z}[L, M]$ with the relations

$$L^2 = 0,\ LM^9 = 1 \text{ and } M^{10} = \sum a_{ij} = 4(w_1 + \ldots + w_5).$$

I will go on to first understand the hyperplane sections of Σ as families over $\mathbb{P}^1$. With this, I plan to characterize 5 relative hyperplanes whose intersection is the surface X fibred by degree 5 elliptic curves.

7.2.2 Weighted Grassmannian Fibrations

Another possible class of relative key variety is families of weighted Grassmannians. In this subsection, we construct fibrations with weighted Grassmannians as fibres. We start by recalling a modified construction of Grassmannians in [CR02].

Definition 7.2.1. Let $w = (w_1, \ldots, w_5) \in \mathbb{Q}^5$ be weights on $\mathbb{C}^5_{x_i}$ such that $w_{ij} + k := w_i + w_j + k$ are positive integers for some rational k and for all i, j. A **weighted Grassmannian** $wG(2,5)$ is defined by

$$wG(2,5) := (aGr(2,5) \setminus 0) / \mathbb{C}^* \subset w\mathbb{P}(\wedge^2 \mathbb{C}^5_{x_i}) = \mathbb{P}^9_{[x_{ij}]}[w_{ij+k}]$$

with action defined by $\lambda \bullet (x_{ij}) \sim \lambda^{w_{ij}+k} x_{ij}$ for $\lambda \in \mathbb{C}^*$. Hence

$$wG(2,5) = \text{Proj}\left(\mathbb{C}[x_{ij}] / \left\langle Pf_{ijkl} \right\rangle\right)$$

with weights $wt_{ij} = w_{ij} + k$ of x_{ij}.

The following result from [CR02] gives numerical data on weighted Grassmannians

Theorem 7.2.2. *Let $d = 2k + \sum_{i=1}^5 w_i$. The numerical data of*

$$wGr(2,5) \subset w\mathbb{P} := \mathbb{P}^9_{[x_{ij}]}[w_{ij} + k]$$

are

(i) The Hilbert numerator

$$\prod_{i,j} (1 - t^{w_{ij}+k}) P(t) = 1 - \sum_{i=1}^{5} t^{d-w_i} + \sum_{j=1}^{5} t^{d+w_j+k} - t^{2d+k}.$$

(ii) The degree

$$\deg wGr = \frac{\sum \binom{d-w_i}{3} - \sum \binom{d+w_i+k}{3} + \binom{2d+k}{3}}{\prod (w_{ij} + k)}.$$

(iii) $K_{w\mathbb{P}} = -\det(\wedge^2 \mathbb{C}^5 \otimes \mathbb{C}) = \mathscr{O}(-\sum wtx_{ij}) = \mathscr{O}(-4d - 2k)$.

(iv) If $wGr(2,5)$ is well-formed, its canonical class is $K_{wGr(2,5)} = \mathscr{O}(-2d - k)$ where

$$wGr(2,5) \subset w\mathbb{P}(\wedge^2 \mathbb{C}^5 \otimes \mathbb{C})$$

with adjunction number

$$2d + k = \deg(\mathbb{C} \otimes \wedge^2 \mathbb{C}^5 \otimes \mathbb{C}^2).$$

The following example shows how to construct families of weighted Grassmannians with an example of a fibration living in them.

Example 7.2.3. Consider a projective fibration $\mathbb{G}_A \xrightarrow{\pi} \mathbb{P}^1$ whose fibres are weighted Grassmannians

$$wGr(2,4) = (aGr(2,4)\setminus 0)/\mathbb{C}^* \subset w\mathbb{P}(\wedge^2\mathbb{C}^4_{x_i}) = \mathbb{P}^5_{[x_{ij}]}[w_{ij+k}]$$

where $aGr(2,4) = \mathbf{Spec}\left(\mathbb{C}[x_{ij}]/\langle Pf_{1234}\rangle\right)$. Let

$$w = (w_1,w_2,w_3,w_4) = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$$

be the weights on $\mathbb{C}^4_{x_i}$ and $k=0$; we get the weight matrix

$$W = \begin{pmatrix} w_{12}+k & b_{13}+k & w_{14}+k \\ & w_{23}+k & w_{24}+k \\ -sym & & w_{34}+k \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ & 1 & 2 \\ -sym & & 2 \end{pmatrix} \tag{7.6}$$

We can then define

$$\mathbb{G}_A := \left(\mathbb{C}^2_{<t_1,t_2>} \times Cone(Gr(2,4)_{x_{ij}})\setminus Z\right)/\mathbb{C}^*\times\mathbb{C}^* \tag{7.7}$$

with Z a closed locus of $Gr(2,4)$ where $\mathbb{C}^*\times\mathbb{C}^*$ acts by the weight matrix

$$\begin{aligned} A = wt(t_1,t_2;x_{ij}) &= \begin{pmatrix} 1 & 1 & b_{12} & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \\ 0 & 0 & w_{12}+k & w_{13}+k & w_{14}+k & w_{23}+k & w_{24}+k & w_{34}+k \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & b_{12} & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix} \end{aligned}$$

with the point $(b_{12},b_{13},b_{14},b_{23},b_{24},b_{34}) \in aGr(2,4) \cong \mathbb{C}^6/(\mathbb{Z}/w_{ij}) \subset \mathbb{P}^5_{[x_{ij}]}[1^3,2^3]$. We then get $\pi : \mathbb{G}_A \to \mathbb{P}^1_{[t_i]}$ a weighted Grassmannian fibration with $\pi^{-1}(p) = wtGr(2,4)$.

$$\begin{array}{ccccc} X & \hookrightarrow & \mathbb{G} & \overset{\varphi_{|L_{d_1,d_2}|}}{\dashrightarrow} & \mathbb{P}(S_{(d_1,d_2)}) \\ & \searrow^{f} & \downarrow \pi & & \\ & & \mathbb{P}^1. & & \end{array}$$

In this example, a general $\mathbb{G}_A \supset X \in |-K_{\mathbb{G}_A}|$ is a Calabi–Yau variety fibred by $K3$ hypersurfaces in $wGr(2,4)$. With this set-up, I plan to extend classification problems in this thesis to include Calabi–Yau varieties fibred by K3 surfaces in weighted Grassmannians.

Bibliography

[ABR02] S. Altinok, G. Brown and M. Reid, *Fano* 3*-folds, K3 surfaces and graded rings.* Contemp. Math. Vol. 314 (2002), 25 – 53. arXiv:math/0202092

[ADHL10] I. Arzhantsev, U. Derenthal, J. Hausen and A. Laface, *Cox Rings.* Cambridge University Press (2014). arXiv.1003.4229

[Alt98] S. Altinok, *Graded rings corresponding to polarised K3 surfaces and* $\mathbb{Q}$*-Fano 3-folds.* Univ. of Warwick PhD thesis (1998).

[AFT05] M. Artin, F.R-Villegas, J. Tate, *On the Jacobians of plane cubics.* Adv. Math. Vol. 198 (2005), 366 – 382.

[Ati58] M. Atiyah, *On Analytic Surfaces with Double Points.* Proceedings of Royal Soc. Lond. A 247 (1958), 237 – 244

[BB96] V. Batyrev and L.A. Borisov, *On Calabi–Yau complete intersections in toric varieties.* Higher-dimensional complex varieties, M. Andreatta and T. Peternell (eds.) (1994), 39 – 65. arXiv:alg-geom/9412017

[BCZ05] G. Brown, A.Corti and F. Zucconi, *Birational geometry of 3-fold Mori fibre spaces.* The Fano Conference, Proceedings, A. Collino, A. Conte, M. Marchisio (eds.), Universitá di Torino (2005), 235 – 275. arXiv:math/0307301

[BE77] D. A. Buchsbaum and D. Eisenbud, *Algebra Structures for Finite Free Resolutions, and Some Structure Theorems for Ideals in Codimension* 3. Amer. J. Math Vol. 99 (1997), 447 – 485.

[BK04] G. Brown and A. Kasprzyk, *Graded Ring Database: Classifications of toric varieties, polarised K3 surfaces, and Fano 3-folds and 4-folds.* http://www.grdb.co.uk/

[BKL19] G. Brown, A. Kasprzyk and L. Zhu, *Gorenstein formats, canonical and Calabi–Yau threefolds.* Exp. Math. Vol 31[1] (2022), 146 – 164. arXiv:1409.4644

[BKQ18] G. Brown, A. Kasprzyk and M. I. Qureshi, *Fano 3-folds in* $\mathbb{P}^2 \times \mathbb{P}^2$ *format, Tom and Jerry* Eur. J. Math. Vol. 4 (2018), 51 – 72. arXiv:1707.00736

[Bun19] D. Bunnet, *Moduli of hypersurfaces in toric orbifolds* . Freien Universität Berlin PhD thesis (2019).

[Cou08] S. Coughlan, *Key varieties for surfaces of general type* . University of Warwick PhD thesis (2008).

[CDLS98] P. Candelas , A. M. Dale, C.A. Lútken and R. Schimmrigk, *Complete intersection Calabi–Yau manifolds.* Nuclear Phys. B Vol. 298 (1988), 493 – 525.

[CDT18] D. Charles, F. David and K. Tyler, *Equivalences of families of stacky toric Calabi–Yau hypersurfaces.* Proc. AMS Vol. 146 (2018), 4633 – 4647.

[CLS11] D.Cox, J.B Little, H. K Schenck, *Toric varieties.* AMS Graduate Studies in Mathematics. 124 (2011).

[CP22] S. Coughlan and R. Pignatelli, *Simple fibrations in (1,2)-surfaces.* (2022) arXiv:2207.06845

[CPR00] A. Corti, A. Pukhlikov and M. Reid, *Birationally rigid Fano hypersurfaces: Explicit birational geometry of* 3*-folds.* Cambrige Univesity, Lecture Notes 281 (2000), 175 – 258.

[CR00] A. Corti and M. Reid, *Foreword: Explicit Birational Geometry for* 3*-folds.* Cambrige Univesity Press, Lecture Notes 281 (2000), 1 – 20. arXiv:math/0007004

[CR02] A. Corti and M. Reid, *Weighted Grassmannians.* Algebraic geometry. A volume in memory of Paolo Francia. M. C. Beltrametti, F. Catanese, C. Ciliberto, A. Lanteri and C. Pedrini (eds.) Walter de Gruyter and Co. Berlin Germany (2002). arXiv:math/0206011

[Deb16] O. Debarre, *Introduction to Mori Theory Cours de M2. 2010–2011.* Université Paris Diderot (2016). M2

[Eis95] D. Eisenbud, *Commutative Algebra : with a View Toward Algebraic Geometry.* Springer New York 1st ed. (1995).

[Fle00] A.R. Fletcher, *Working With Weighted Complete Intersections: Explicit Birational Geometry for* 3*-folds.* Cambrige Univesity Press, Lecture Notes 281 (2000), 101 – 173.

[Ful93] W. Fulton, *Introduction to toric varieties.* Princeton University Press, Annals of Math Studies 131 (1993).

[Har77] Robin Hartshorne, *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer (1977).

[HM70] R. Hartshorne, C. Musili, *Ample subvarieties of algebraic varieties.* Lecture Notes in Mathematics, No. 156. Springer (1970).

[GS22] G. Mboya and B. Szendroi, *On K3 fibred Calabi–Yau threefolds in Weighted Scrolls.* arXiv:2210.10559

[Gro97] M. Gross, *The deformation space of Calabi–Yau n-folds with canonical singularities can be obstructed.* Mirror symmetry, II, Adv. Math (1997), $401-411$.

[GW78] S. Goto and K-I Watanabe, *On graded rings.* The Journal of Mathematical Society of Japan 30 (1978), $179-213$.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties.* Cambridge Tracts in Mathematics, 134, Cambridge University Press (1998).

[KS00] M. Kreuzer, H. Skarke, *Complete classification of reflexive polyhedra in four dimensions* . Adv. Theor. Math. Phys. 4 (2000), $1209-1230$.

[Kuh03] M. Kühnel, *Calabi–Yau threefolds with Picard number* $\rho(X) = 2$ *and their Kähler cone I.* Mathematische Zeitschrift 245[2] (2003), $233-254$. arXiv:math/0110054

[Kuh04] M. Kühnel, *Calabi–Yau-threefolds with Picard number* $\rho(X) = 2$ *and their Kähler cone II.* Pacific Journal of Mathematics 217[1] (2004), $115-137$. arXiv:math/0201295

[KU19] A. Küronya and S. Urbinati, *Geometry of multigraded rings and embeddings of toric varieties.*(2019). arXiv.1912.04374

[Lar20] R. Lazarsfeld, *Positivity in Algebraic Geometry.* Springer. A Series of Modern Survey in Mathematics Vol. 49 (2000).

[Lop89] M. Mendes-Lopes, *The relative canonical algebra for genus three fibrations.* Univ. of Warwick Ph.D. thesis (1989).

[Muk10] S. Mukai, *Curves and symmetric spaces, II.* Ann. of Math. 172[2] (2010), $1539-1558$.

[Mul06] J. Mullet, *On Toric Calabi–Yau hypersurfaces fibred by weighted K3 hypersurfaces.* Comm. Anal. Geom. 17 (2009), $107-138$. arXiv:math/0611338

[ST20] S. Coughlan, T. Ducat, *Constructing Fano 3-folds from cluster varieties of rank 2.* Compos. Math. 156 (2020), $1873-1914$. arXiv:1811.10926

[SP22] S. Coughlan, R. Pignatelli, *Simple Fibrations in* $(1,2)$-*Surfaces.*(2022). arXiv:2207.06845

[Tei99] M. Teicher, *Hirzebruch surfaces: degenerations, related braid monodromy, Galois covers.* Contemp. Math. 241 (1999), $305-325$.

[Tho00] R. Thomas, *A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on K3 fibrations.* J. Differential Geom. 54 (2000), $367-438$.

[Reid79] M. Reid, *Exercises on Graded Rings.* Manuscript, (1979). homework

[Reid89] M. Reid, *Problems on pencils of small genus.* Manuscript, (1989). atoms

[Reid97] M. Reid, *Chapters on Algebraic Surfaces.* Complex Algebraic Geometry IAS/Park City Mathematics Series 3 (1997), 3 – 159. arXiv:alg-geom/9602006

[Reid02] M. Reid, *Graded rings and varieties in weighted projective space.* Manuscript, (2002). Graded Rings

[RT09] J. Ross and R. Thomas, *Weighted projective embeddings, stability of orbifolds and constant scalar curvature Kähler metrics.* J.Differ. Geom. 88 (2011), 109 – 159.

[Ruan96] Y. Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory.* Duke Math. J. 83 (1996), 461 – 500.

[Tho11] A. Thompson, *Models for threefolds fibred by K3 surfaces of degree two.* Univ. of Oxford PhD thesis (2011).