# 4-acceleration

Back to a particle in general motion:

**Definition 1** *The 4-vector* $A = \mathrm{d}V/\mathrm{d}\tau$ *with components* $(c\ddot{t}, \ddot{x}, \ddot{y}, \ddot{z})$, *where* $\cdot = \mathrm{d}/\mathrm{d}\tau$, *is called the* 4-acceleration.

Recall that $\mathrm{d}t/\mathrm{d}\tau = \gamma$. So

$$A = \gamma \frac{\mathrm{d}}{\mathrm{d}t} \gamma \frac{\mathrm{d}}{\mathrm{d}t} X \, .$$

To make this explicit, we use:

**Lemma 1** *Put* $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$ *then*

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\gamma^3 v}{c^2} \frac{\mathrm{d}v}{\mathrm{d}t}.$$

**Proof:**

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{v}{c^2} \frac{\mathrm{d}v}{\mathrm{d}t} \frac{1}{(1 - v^2/c^2)^{3/2}} \, .$$

□

Thus

$$\begin{aligned} (c\ddot{t}, \ddot{x}, \ddot{y}, \ddot{z}) &= \frac{\mathrm{d}}{\mathrm{d}\tau} (\gamma(v)(c, \mathbf{v})) \\ &= \gamma(v) \frac{\mathrm{d}}{\mathrm{d}t} (\gamma(c, \mathbf{v})) \\ &= \gamma^4 v \frac{\mathrm{d}v}{\mathrm{d}t} (\gamma(c, \mathbf{v})) + \gamma^2 (0, \mathbf{a}) \end{aligned}$$

where $\mathbf{a} = \mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ (components of acceleration measured in the ICS).

**Definition 2** *The instantaneous rest frame (IRF) is the frame in which* $v = 0$.

In the IRF $v = 0$ so that $\gamma = 1$ and $V$ and $A$ have components $(c, 0, 0, 0)$ and $(0, \mathbf{a})$. We deduce:

(1) $g(A, A) = -a^2$ where $a$ is the acceleration measured in the IRF (i.e. acceleration felt).

(2) $g(A, V) = 0$ (this can also be seen by differentiating $g(V, V) = c^2$.

## Constant Acceleration

Consider a particle moving along the $x$-axis with constant acceleration $a$ measured in its rest frame at each point.

$V$ has components $(c\dot{t}, \dot{x}, 0, 0)$, and $A$ has components $(c\ddot{t}, \ddot{x}, 0, 0)$ where $\cdot = \mathrm{d}/\mathrm{d}\tau$. We have

$$c^2 = g(V,V) = c^2\dot{t}^2 - \dot{x}^2\,, \quad -a^2 = g(A,A) = c^2\ddot{t}^2 - \ddot{x}^2$$

Differentiate the first equation to get $c^2\dot{t}\ddot{t} = \dot{x}\ddot{x}$ and eliminate $\ddot{x}$ and $\dot{x}$ from 2nd:

$$c^4\dot{t}^2\ddot{t}^2 = c^2(\dot{t}^2 - 1)(c^2\ddot{t}^2 + a^2)$$

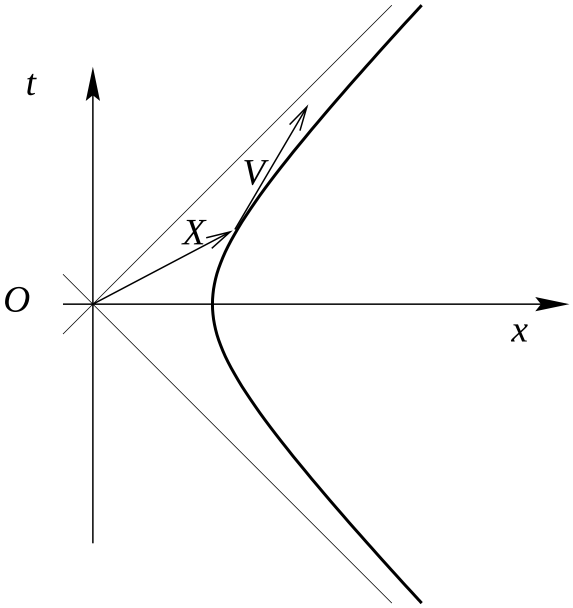
so $\ddot{t} = \frac{a}{c}(\dot{t}^2 - 1)^{1/2}$ which integrates to give

$$\dot{t} = \cosh(a\tau/c) \quad \text{by choosing 0 for } \tau\text{, and}$$

$$t = \frac{c}{a}\sinh(a\tau/c) \quad \text{by choosing 0 for } t\text{, and}$$

$$x = \frac{c^2}{a}\cosh(a\tau/c) \quad \text{by choosing 0 for } x.$$

Worldline is the hyperbola $c^2t^2 - x^2 = -c^4/a^2$.



Displacement 4-vector from the origin is

$$X = \frac{c^2}{a}(\sinh(a\tau/c), \cosh(a\tau/c), 0, 0)$$

and 4-velocity is $V$ (note $|u| < c \ \forall \tau$)

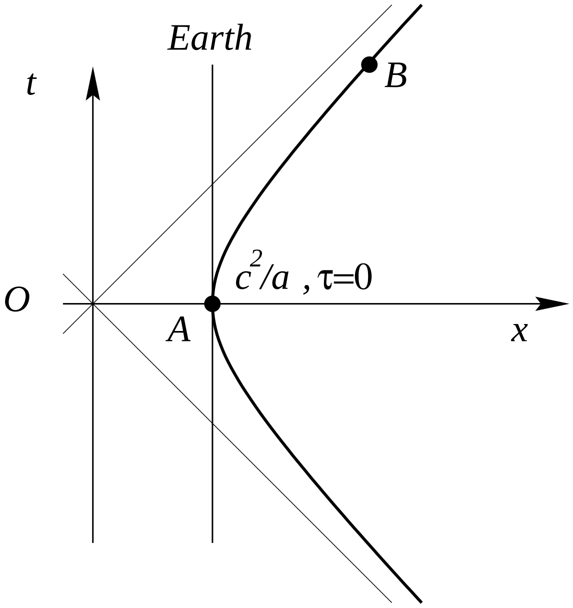
$$V = c(\cosh(a\tau/c), \sinh(a\tau/c), 0, 0)\,.$$

Note $g(X, X) = -c^4/a^2$ and $g(X, V) = 0$. Thus for all $\tau$, an observer on the spaceship at $X(\tau)$ reckons that the event $O$ is simultaneous with $X$, and distance $c^2/a$ away. However, for $a \simeq g$ (the accel. due to gravity), $c^2/a$ is about a light-year.

## The twin paradox revisited

Consider the earth at rest in the $(x, t)$ system at $x = c^2/a$. At $A$, the space-ship is at rest (relative to $(x, t)$ coordinates) at $x = c^2/a$.

At $B$, the space-ship is at $x = \frac{c^2}{a}\cosh(a\tau/c)$.



The time elapsed on earth is $t = \frac{c}{a}\sinh(a\tau/c)$ where $\tau$ is the time elapsed on the space-ship. Suppose $a \simeq g$, (acceleration due to gravity), $\tau \simeq 10$ years. Then

$c/a \simeq 1$ year and

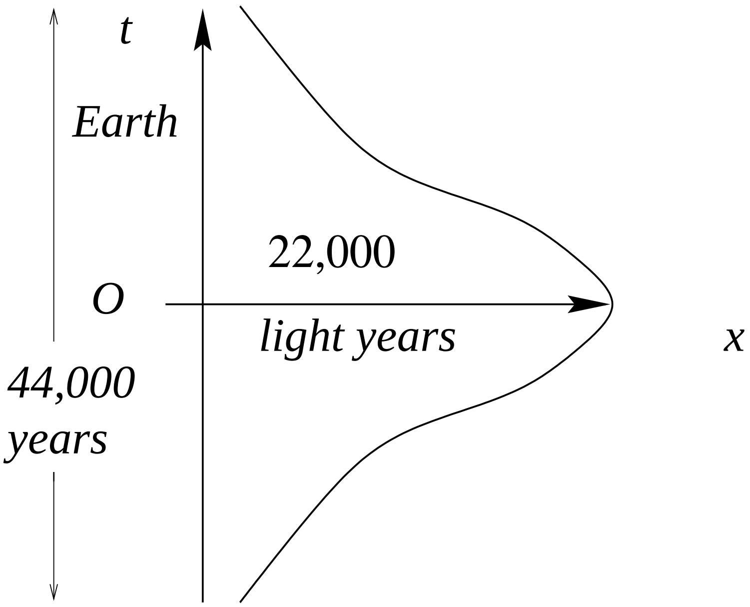
$$t \simeq \frac{1}{2}e^{10} \simeq 11,000 \text{ years.}$$

We also have

$$x \simeq \frac{1}{2}e^{10} \simeq 11,000 \text{ light-years} \;,$$

By piecing together trajectories such as these, you can travel as far as you like in as short a time as you like (time in space-ship, distance measured on earth) if you are allowed sufficient acceleration.

For twin paradox, piece together 4 such trajectories:



Time lapse on space-ship is 40 years, whereas that on earth is 44,000 years.

## 4-force and relativistic Newton's law

**Definition 3** *The relativistic Newton law is*

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = F\,,$$

*where $P$ is the 4-momentum of the particle and $F$ is the 4-force.*

This is only half the story; we need a law for what $F$ should be. Later we will find an example in the 4-force that arises from the electromagnetic Lorentz force law.

We can also determine a 4-force from consideration of conservation of momentum:

**Example:** A photon rocket propels itself along the $x$-axis by firing $n$ photons per unit (proper) time $\tau$ with frequency $\nu$ in the rocket's rest frame along the negative $x$-axis. Show that, if the initial mass is $M_0$ the magnitude of the acceleration is

$$a = \frac{n\hbar\nu}{M_0 - \frac{n\hbar\nu}{c^2}\tau}$$

**Solution:** Let the mass of the rocket be $M(\tau)$ and 4-velocity be $U(\tau)$. Conservation of momentum over time interval $\delta\tau$ yields

$$0 = \delta(MU) + nN\delta\tau = U\delta M + M\delta U + nN\delta\tau$$

where $N$ is the momentum of the photon. The given frequency implies $g(U, N) = \hbar\nu$. Thus, contracting the equation with $U$ yields

$$c^2\frac{\mathrm{d}M}{\mathrm{d}\tau} = -n\hbar\nu\,, \qquad \text{so} \qquad M = M_0 - \frac{n\hbar\nu}{c^2}\tau$$

using $\dot{U} = A$, $g(U, A) = 0$. However, squaring the conservation of momentum equation yields

$$c^2\dot{M}^2 - M^2a^2 = 0\,,$$

so, taking the positive solutions for $a$:

$$a = -\frac{c^2\mathrm{d}M}{M\mathrm{d}\tau} = \frac{n\hbar\nu}{M_0 - \frac{n\hbar\nu}{c^2}\tau}$$

# Maxwell's equations

We saw in lecture 2 that Maxwell's equations are

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho/\varepsilon_0 && (1)\\
\nabla \cdot \mathbf{B} &= 0 && (2)\\
\nabla \wedge \mathbf{B} - \varepsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{J} && (3)\\
\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 && (4)\\
\mathbf{f} &= e(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) && (5)
\end{aligned}$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field and $\rho$ the charge density, and $\mathbf{J}$ the current density and $\mathbf{f}$ is the force experienced by a particle of charge $e$ due to fields $\mathbf{E}$ and $\mathbf{B}$ (this serves as an experimental definition of $\mathbf{E}$ and $\mathbf{B}$).

The first 4 equations are relativistically invariant (and indeed motivated relativity) but the last is not, only true for $|v| << c$.

## Charge and current density

The charge and current density are defined as:

$$\text{Charge density: } = \lim_{\delta V \to 0} \frac{\sum e}{\delta V} = \rho$$

$$\text{Current density: } = \lim_{\delta V \to 0} \frac{\sum e\mathbf{v}}{\delta V} = \mathbf{J}$$

so $\rho$, $\mathbf{J}$ are functions of $(t, x, y, z)$ and the volumes $V$ are small spheres centred at $\mathbf{r} = (x, y, z)$. They are related by the *continuity equation*, which expresses the fact that charges are neither created or destroyed.

For a fixed volume $V$ bounded by a surface $S$,

(a) $\int_V \rho \mathrm{d}V$ = total charge in $V$.

(b) $\int_{\partial V} \mathbf{J} \cdot \mathrm{d}\mathbf{S}$ = total charge leaving $V$ per unit time (negative charges entering count as positive charges leaving).

where $\mathrm{d}\mathbf{S} = \mathbf{n}\mathrm{d}S$ and $\mathbf{n}$ is the outward normal.

Consider $N$ particles/unit volume each with charge $e$, velocity $\mathbf{v}$ at time $t$. Those that cross $\mathrm{d}\mathbf{S}$ between $t$ and $t + \delta t$ occupy at time $t + \delta t$ a volume $\mathrm{d}S\mathbf{n} \cdot \mathbf{v}\delta t$, so they contribute

$$eN\mathbf{v} \cdot \mathrm{d}\mathbf{S}\delta t$$

to the outflow. Sum over $e\mathbf{v}$, to get total outflow through $\mathrm{d}\mathbf{S}$ of $\mathbf{J} \cdot \mathrm{d}\mathbf{S}\delta t$. Then charge conservation implies

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathrm{d}V + \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{s} = \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}\right) \mathrm{d}V\,.$$

This holds for any $V$. Hence we obtain the continuity equation, true for all physical sources:

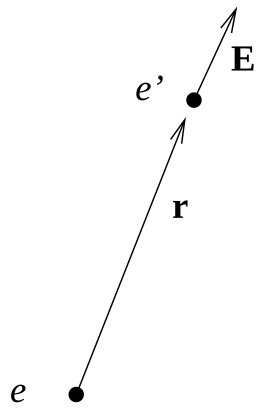
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0\,.$$

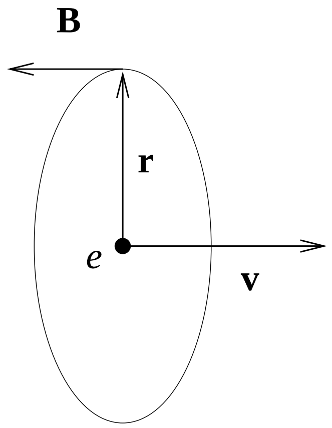
## The origin of Maxwell's equations

All but the 2nd term (the *displacement current*) in equation (6) were known before Maxwell, but in an integrated form. Without that, Maxwell's equations are differential forms of 4 key laws.

In terms of point charges, one has the (non-relativistic) laws that a particle with velocity $\mathbf{v}$, ($|\mathbf{v}| << c$) feels a force $\mathbf{f} = e(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$, (5), (defines $\mathbf{E}$ and $\mathbf{B}$) and generates fields

1. $\mathbf{E} = e\mathbf{r}/4\pi\varepsilon_0 r^3$, *(Coulomb's law)* and
2. $\mathbf{B} = \mu_0 e\mathbf{v} \wedge \mathbf{r}/4\pi r^3$ *(Biot and Savart's law).*
3. Faraday's law of induction (described later).





- Both $\mathbf{E}$ and $\mathbf{B}$ fall off like $1/r^2$.
- The electric force law is like gravity but force between like charges is repulsive.
- They are inconsistent with relativity (e.g. (2) cannot hold for lare $\mathbf{v}$ or $r$ as changes cannot propagate with $\infty$ speed). Maxwell's extra term result in relativistic equations.

Maxwell put these laws into differential form; laws 1 and 2 are equivalent to Maxwells equations without $\partial\mathbf{E}/\partial t$ and $\partial/\partial\mathbf{B}$ (the laws of electrostatics and magnetostatics):

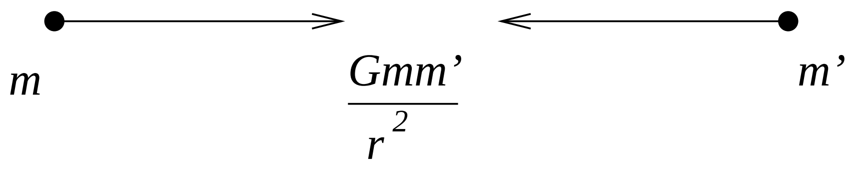
$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\varepsilon_0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \wedge \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \wedge \mathbf{E} &= 0 \end{aligned}$$

The procedure is analogous to that for gravity as discussed in mods and so we recall it next.

## Gravity

Recall the two descriptions of gravity:

(I) Particles and action at a distance with force $Gmm'/r^2$:



(II) Particles and fields $\mathbf{g} = -\nabla\phi$ ($\phi$ = gravitational potential and $\mathbf{g}$ = gravitational acceleration) and force $\mathbf{f} = m\mathbf{g}$. The field satisfies the field equations

$$\nabla\cdot\mathbf{g} = -\nabla^2\phi = -4\pi G\rho\,, \quad \nabla\wedge\mathbf{g} = 0$$

where $\rho(\mathbf{r})$ is the mass density at $\mathbf{r}$.

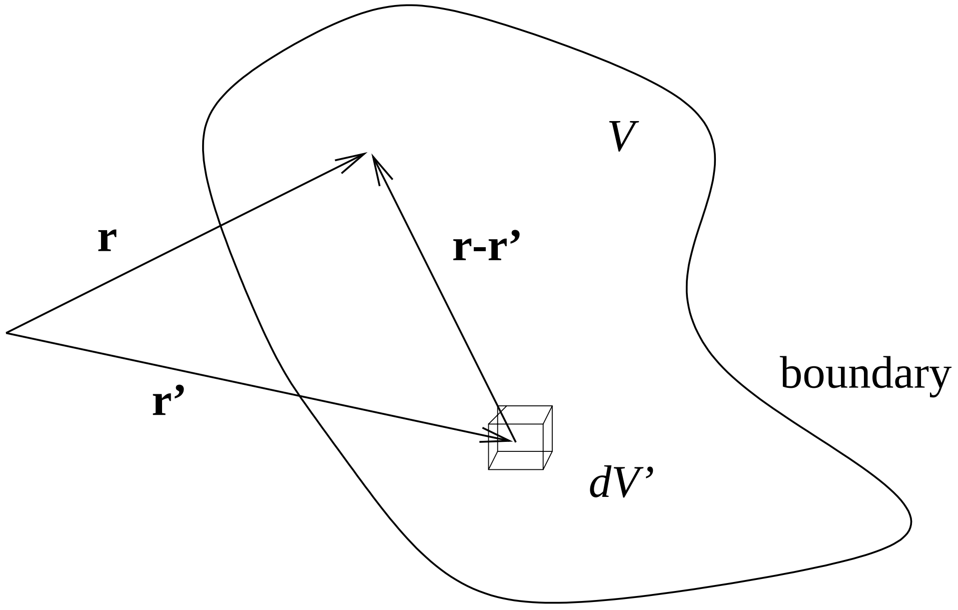
**Proposition 1** *These formulations are equivalent.*

**Proof:** I $\Rightarrow$ II:
For a single particle $M$, setting $\phi = 0$ at $\infty$, we get

$$\mathbf{g} = -\frac{GM\mathbf{r}}{r^3} = -\nabla\phi\,, \quad \text{where} \quad \phi = -\frac{Gm}{r}$$

Sum up over a mass distribution with density $\rho(\mathbf{r})$ to get

$$\phi(\mathbf{r}) = -\int_{V'}\frac{G\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'$$



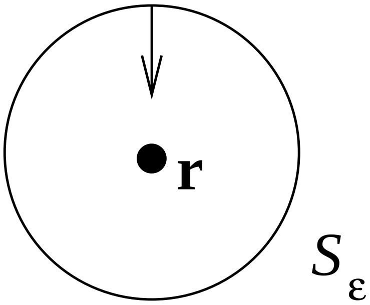
Since $\nabla 1/|\mathbf{r}-\mathbf{r}'| = -\nabla' 1/|\mathbf{r}-\mathbf{r}'|$ where $\nabla'$ is the gradient wrt $\mathbf{r}'$

$$\begin{aligned}\nabla\phi &= -\int_{V'} \nabla\left(\frac{G}{|\mathbf{r}-\mathbf{r}'|}\right)\rho \mathrm{d}V' \\ &= \int_{V'} \nabla'\left(\frac{G}{|\mathbf{r}-\mathbf{r}'|}\right)\rho \mathrm{d}V' , \\ &= -\int_{V'} \frac{G}{|\mathbf{r}-\mathbf{r}'|}\nabla'\rho \mathrm{d}V' ,\end{aligned}$$

by parts (i.e. moving $\mathbf{r}$ is equivalent to moving the body). So differentiating under the integral and using $\nabla^2|\mathbf{r}-\mathbf{r}'|^{-1} = 0$:

$$\begin{aligned}\nabla^2\phi &= -\int_{V'} G\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\cdot\nabla'\rho \mathrm{d}V' , \\ &= \int_{V'} G\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\cdot\nabla'\rho \mathrm{d}V' , \\ &= \int_{V'} G\nabla'\cdot\left(\rho\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' , \\ &= \lim_{\varepsilon\to 0}\int_{S_\varepsilon} G\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\cdot\rho \mathrm{d}\mathbf{S}' ,\end{aligned}$$

To obtain the last line we have applied the divergence outside a small sphere radius $\varepsilon$ assuming $\rho \mapsto 0$ at $\infty$.



But (putting $\mathbf{r} = 0$ for simplicity)

$$\nabla'\frac{1}{r'} = \frac{-\mathbf{r}'}{r'^3} , \quad \mathrm{d}\mathbf{S}' = -\frac{\mathbf{r}'}{r'}\mathrm{d}S' , \quad r' = \varepsilon$$

Hence

$$\nabla^2\phi = \int_{S_\varepsilon} \frac{G\rho}{\varepsilon^2}\mathrm{d}S' \longrightarrow 4\pi G\rho(0)$$

since the area of $S_\varepsilon$ is $4\pi\varepsilon^2$. Thus $\nabla^2\phi = 4\pi G\rho$. □

II $\Rightarrow$ I: Consider potential of a uniform sphere. Then by spherical symmetry, $\mathbf{g} = g(r)\hat{\mathbf{r}}$. If the total mass is $M$, $\nabla\cdot\mathbf{g} = -4\pi G\rho$ implies

$$\int_S \mathbf{g}\cdot\mathrm{d}\mathbf{S} = \int_V \nabla\cdot\mathbf{g}\mathrm{d}V = -4\pi G\int\rho\mathrm{d}V = -4\pi GM$$

where $M$ is the mass enclosed by a sphere of radius $r > a$ centre 0. Thus

$$\int \mathbf{g}\cdot\mathrm{d}\mathbf{S} = 4\pi r^2 g(r) = -4\pi GM$$

so that $g = -GM/r^2$. □

## Electrostatics

An identical argument, then, gives the equivalence of Coulomb's law with the differential equations

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0\,, \quad \nabla\wedge\mathbf{E} = 0$$

and the existence of a potential $\phi$ satisfying

$$\mathbf{E} = -\nabla\phi\,, \quad \nabla^2\phi = -\rho/\varepsilon_0$$

where $\phi$ is now the electric potential and $\rho$ is now the charge density. The reversal of sign yields a repulsive force for like charges. The potential is measured in volts 'energy/charge'.

Problems in electrostatics can be solved by mods maths methods, except that we have new boundary conditions:

E.g. an earthed conductor has $\phi = 0$ (so that $\mathbf{E} = \nabla\phi = 0$ tangent to conductor) and $\phi = 0$ at $\infty$.

**Example:** Consider an earthed conducting plate in the $(x, y)$-plane with charge at $(0, 0, a)$. What is the force between the charge and the plate?

**Solution:** We must find $\phi$ s.t.

1. $\nabla^2\phi = 0$ on $z \geq 0$,
2. $\phi = 0$ at $z = 0$,
3. $\phi \to 0$ at $\infty$,
4. $\phi = \frac{e}{4\pi\varepsilon_0(x^2+y^2+(z-a)^2)^{\frac{1}{2}}} + \psi$ where $\psi$ is smooth on $z \geq 0$ including $(0, 0, a)$.

Place an image charge $-e$ at $(0, 0, -a)$ to get

$$\phi = \frac{e}{4\pi\varepsilon_0}\left(\frac{1}{(x^2+y^2+(z-a)^2)^{\frac{1}{2}}} - \frac{1}{(x^2+y^2+(z+a)^2)^{\frac{1}{2}}}\right)$$

and we find that there is an attractive force

$$f = \frac{e^2}{4\pi\varepsilon_0(2a)^2}\,.$$