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UNRAMIFIED CORRESPONDENCES AND VIRTUAL PROPERTIES OF MAPPING CLASS GROUPS

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Abstract. We establish a connection between the conjecture of Bogomolov-Tschinkel about unramified correspondences and the Ivanov conjecture about the virtual homology of mapping class groups. Given $g \geq 2$, we show that every genus g Riemann surface X virtually dominates a fixed Riemann surface Y of genus at least two if and only if there exists a finite index subgroup $\Gamma < \mathrm{Mod}_g^1$ which allows a point pushing epimorphism onto a free group of rank two. As a consequence of this result we show that the Putman-Wieland conjecture about the Higher Prym representations does not hold when $g = 2$.

1. Introduction

Let Σ_g^n denote a surface of genus g with n marked points, and Mod_g^n the corresponding (pure) mapping class group. The group Mod_g^n has been thoroughly studied and our knowledge about it is extensive. But if one replaces it with one of its finite index subgroups $\Gamma < \mathrm{Mod}_g^n$ (of which there are plenty) the situation changes dramatically. One reason behind this stark difference is that Γ may not contain any torsion while Mod_g^n is rich in torsion.

The natural next step is to understand virtual properties of Mod_g^n (i.e. properties of its finite index subgroups). For example, it is was proved by Mumford [15] and Powell [16] that $\mathrm{H}^1(\mathrm{Mod}_g^n, \mathbb{Q}) = 0$ when $g \geq 2$ (another consequence of intricate relations that exist in Mod_g^n). A well known conjecture in the field (Ivanov's Conjecture [12]) is that the same holds for every finite index subgroup $\Gamma < \mathrm{Mod}_g^n$ when $g \geq 3$. A closely related question is whether Mod_g^n is large. In other words, is there a finite index subgroup $\Gamma < \mathrm{Mod}_g^n$ which admits an epimorphism onto F_2 (the free group of rank two)?

Remark. It follows from the work of McCarthy [14] that each Mod_2^n is large.

1.1. **Point pushing epimorphism.**

Fix a marked point $* \in \Sigma_g^{n+1}$. Forgetting $*$ yields the Birman exact sequence (see Chapter 4 in the book by Farb-Margalit [11])

$$1 \to \pi_1(\Sigma_g^n) \to \mathrm{Mod}_g^{n+1} \to \mathrm{Mod}_g^n \to 1. \tag{1}$$

The subgroup $\pi_1(\Sigma_g^n) < \mathrm{Mod}_g^{n+1}$ is the point pushing subgroup of Mod_g^{n+1} corresponding to $*$.

Definition 1.1. *Suppose $\Gamma < \mathrm{Mod}_g^{n+1}$ and $\rho : \Gamma \to G$ an epimorphism onto some group G. We say that ρ is a point pushing epimorphism if there exists a marked point $*$ such that*

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ρ does not annihilate the group $\Gamma \cap \pi_1(\Sigma_g^n)$, where $\pi_1(\Sigma_g^n)$ is the point pushing subgroup corresponding to $*$.

1.2. **Unramified correspondences.** Bogomolov-Tschinkel [7] initiated the theory of unramified correspondences between Riemann surfaces (algebraic curves). Recall that a Riemann surface Z dominates a Riemann surface Y if there exists a holomorphic surjection from Z onto Y. Moreover, we say that Z f-dominates Y if this surjective holomorphic map is homotopic to f.

Definition 1.2. *Let X and Y be two closed Riemann surfaces. We say that X virtually dominates Y, and write $X \implies Y$, if X has an unbranched cover Z which dominates Y.*

The following is one of their main results (see [7] and Theorem 1.5 in [18]).

Theorem 1.1. *Let Y_0 be the closed Riemann surface of genus two given by the equation $y^6 = x(x-1)$. Then every closed hyperelliptic Riemann surface of genus at least two virtually dominates Y_0.*

The focus of [7] is on algebraic curves defined over number fields. However, Theorem 1.1 holds for all hyperelliptic algebraic curves defined over $\mathbb{C}$ (in other words, for all hyperelliptic Riemann surfaces). The reader can consult the paper by Poonen [18] where this is clearly stated.

In fact, Bogomolov-Tschinkel show that every hyperelliptic Riemann surface X has an unbranched cover Z of degree 648 which holomorphically surjects onto Y_0. Based on Theorem 1.1, Bogomolov-Tschinkel made several conjectures. The following is a version of one of their conjectures.

Conjecture 1.1. *Fix $g \geq 2$. There exists a closed Riemann surface Y of genus at least two which is virtually dominated by every closed Riemann surface of genus g.*

Clearly, Theorem 1.1 implies Conjecture 1.1 when $g = 2$ (since every Riemann surface of genus two is hyperelliptic).

1.3. **The main results.** Our main result is the equivalence between Conjecture 1.1 and the existence of virtual point pushing epimorphism onto F_2.

Theorem 1.2. *Conjecture 1.1 holds for some $g \geq 2$ if and only if there exists a finite index subgroup $\Gamma < \mathrm{Mod}_g^1$ which allows a point pushing epimorphism $\rho : \Gamma \to \mathrm{F}_2$.*

In [20] Putman-Wieland introduced the Higher Prym Representation of the mapping class group Mod_g^{n+1}. Suppose $K < \pi_1(\Sigma_g^n)$ is a characteristic finite index subgroup of the point pushing group $\pi_1(\Sigma_g^n) < \mathrm{Mod}_g^{n+1}$. Let $V_K = \mathrm{H}_1(K, \mathbb{Q})/B$, where B is the boundary subspace of $\mathrm{H}_1(K, \mathbb{Q})$ spanned by the homology classes of the loops freely homotopic into the punctures of the surface which is the covering of Σ_g^n corresponding to the subgroup K. Then Mod_g^{n+1} naturally acts on V_K inducing the linear representation $\mathrm{Mod}_g^{n+1} \to \mathrm{Aut}(V_K)$ called the Higher Prym representation. They made the following conjecture (see Conjecture 1.2. in [20]).

Conjecture 1.2. *Fix $g \geq 2$, and $n \geq 0$. Let $K < \pi_1(\Sigma_g^n)$ be a finite index characteristic subgroup. Then for all nonzero vectors $v \in V_K$, the Mod_g^{n+1}-orbit of v is infinite.*

Combining Theorem 1.2 and Theorem 1.1 we show that this conjecture does not hold when $g = 2$.

Theorem 1.3. *For every $n \geq 0$, there exist a finite index characteristic subgroup $K < \pi_1(\Sigma_2^n)$, and a non-zero vector $v \in V_K$, such that the Mod_2^{n+1}-orbit of v is finite.*

Remark. Actually, Putman-Wieland define the Higher Prym Representation of the mapping class group $\mathrm{Mod}_{g,b}^{n+1}$ of the surface $\Sigma_{g,b}^n$ of genus g, with n marked points, and b discs removed. The theorem extends to this case, that is, for each $n, b \geq 0$ there exists $K < \pi_1(\Sigma_{2,b}^n)$, and a non-zero vector $v \in V_K$, such that the $\mathrm{Mod}_{2,b}^{n+1}$-orbit is finite.

Proof. It suffices to prove the theorem for $n = 0$ (see [20]). From Theorem 1.2 and Theorem 1.1 we conclude that there exists $\Gamma < \mathrm{Mod}_2^1$ which allows a point pushing epimorphism $\rho : \Gamma \to \mathrm{F}_2$. In turn this yields a point pushing epimorphism $\rho_1 : \Gamma \to \mathbb{Z}$. Replacing Γ by one of its subgroups if necessary, we may assume that $K = \Gamma \cap \pi_1(\Sigma_2)$ is a characteristic subgroup of $\pi_1(\Sigma_2)$. By $\sigma : K \to \mathbb{Z}$ we denote the restriction of ρ_1 to $K < \Gamma$, and let $u \in H^1(K, \mathbb{Q})$ be the corresponding element induced by σ.

Denote by $L : \mathrm{Mod}_2^1 \to \mathrm{Aut}(V_K)$ the corresponding Higher Prym representation. We first establish the following invariance of u with respect to L.

Claim 1.1. *Let $f \in \Gamma$. Then $u = u \circ L(f)$.*

Proof. Let $\Gamma' < \mathrm{Mod}_2$ be the image of Γ under the homomorphism $\mathrm{Mod}_2^1 \to \mathrm{Mod}_2$ from the Birman exact sequence (1). Consider the induced exact sequence

$$1 \to \pi_1(\Sigma_h) \to \Gamma \to \Gamma' \to 1,$$

where

$$K = \pi_1(\Sigma_h) = \Gamma \cap \pi(\Sigma_2).$$

If we regard the marked point $* \in \Sigma_2^1$ as the base point, we obtain the action of the group Mod_2^1 on $\pi_1(\Sigma_2, *)$. Since $K < \pi_1(\Sigma_2, *)$ is a characteristic subgroup, it follows that Mod_2^1 acts on K, and we have the induced homomorphism $\mathrm{Mod}_2^1 \to \mathrm{Mod}_h^1$. Composing it with the homomorphism $\mathrm{Mod}_h^1 \to \mathrm{Mod}_h$, we obtain the desired homomorphism $\mathrm{Mod}_2^1 \to \mathrm{Mod}_h$. If $f \in \mathrm{Mod}_2^1$, we let $\widetilde{f}$ denote the corresponding image in Mod_h (the reader can consult [11] for the proofs of the above facts).

As observed by Putman-Wieland on page 3 in [20], the lift $f \to \widetilde{f}$ induces the Higher Prym representation $L : \mathrm{Mod}_2^1 \to \mathrm{Aut}(V_K)$ by

$$L(f)(\omega) = \widetilde{f}(\omega), \tag{2}$$

where $\omega \in \mathrm{H}_1(\Sigma_h, \mathbb{Q})$, and $f \in \mathrm{Mod}_2^1$. Since $\mathbb{Z}$ is an Abelian group, we have

$$\sigma(g) = \sigma(f^{-1} \circ g \circ f) \qquad \forall f \in \Gamma, \ \forall g \in K \tag{3}$$

(note that $f^{-1} \circ g \circ f \in K$ because $K < \Gamma$ is a normal subgroup). The geometric interpretation of this is as follows. Let $\gamma \subset \Sigma_h$ be the closed curves representing the conjugacy class $g \in K = \pi_1(\Sigma_h)$. Let $\widetilde{f}(\gamma)$ be the closed curve which is the image of γ under $\widetilde{f}$ (here we consider $\widetilde{f}$ as the corresponding isotopy class of homeomorphisms of Σ_h). Denote by $[\gamma], [\widetilde{f}(\gamma)] \in \mathrm{H}_1(\Sigma_h)$ the corresponding homology classes. Then (3) implies the equality

$$u([\gamma]) = u(\widetilde{f}([\gamma])),$$

(recall that $u \in \mathrm{H}^1(K, \mathbb{Q})$ is induced by $\sigma : K \to \mathbb{Z}$). But this holds for every closed curve γ because (3) holds for every $g \in K = \pi_1(\Sigma_h)$. We conclude that

$$u(\omega) = u(\widetilde{f}(\omega)), \qquad \forall \omega \in \mathrm{H}_1(\Sigma_h, \mathbb{Q}), \quad \forall f \in \Gamma.$$

From (2) we obtain the equality

$$u = u \circ L(f), \qquad \forall f \in \Gamma$$

which proves the claim. □

Let $v \in \mathrm{H}_1(K, \mathbb{Q})$ be the vector dual of u. From the claim we conclude that $\widetilde{f}(v) = v$ (in $\mathrm{H}_1(\Sigma_h, \mathbb{Q})$) for every $f \in \Gamma$. Since Γ has finite index in Mod_2^1, it follows that the orbit $\mathrm{Mod}_2^1(v)$ is finite. □

1.4. **Brief outline.** We split the proof of Theorem 1.2 into two parts. In the next section we prove that if Conjecture 1.1 holds for some $g \geq 2$ then there exists a finite index subgroup $\Gamma < \mathrm{Mod}_g^1$ which allows a point pushing epimorphism $\rho : \Gamma \to \mathrm{F}_2$. The assumption that Conjecture 1.1 holds enables us to construct a continuous map $f : \mathcal{M}_{g,1}^{\Gamma} \to Y$, where $\mathcal{M}_{g,1}^{\Gamma}$ is a covering of the moduli space $\mathcal{M}_{g,1}$ corresponding to Γ, and Y a closed Riemann surface of genus at least two. Then the induced homomorphism $f_* : \Gamma \to \pi_1(Y)$ is point pushing. Composing f_* with any epimorphism $\pi_1(Y) \to \mathrm{F}_2$ yields the claim.

We then prove the other direction in Theorem 1.2. Using the Siu-Beauville theorem [3], [23], we show that the existence of a point pushing epimorphism $\rho : \Gamma \to \mathbf{F}_2$ yields a holomorphic surjection $f : \overline{\mathcal{M}_{g,1}^{\Gamma}} \to Y$ which does not factor through the forgetful map F_Γ. On the other hand, the fibres of F_Γ are biholomorphic to the unramified coverings $\widetilde{X}_\pi \to X$ corresponding to the subgroup $\Gamma \cap \pi_1(\Sigma_g)$, where $\pi_1(\Sigma_g)$ is the point pushing subgroup of Mod_g^1. Restricting f to these fibres produces the required holomorphic surjections $\widetilde{X}_\pi \to Y$.

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2. Constructing a point pushing epimorphism onto F_2

In this section we prove one direction of the equivalence stated in Theorem 1.2. This is the content of Lemma 2.2 below. As usual, $\mathcal{T}_g$ denotes the Teichmüller space of Riemann surfaces marked by Σ_g. We recall that $\mathcal{T}_g$ is the space of equivalence classes of pairs (X, α), where X is a Riemann surface and $\alpha : \Sigma_g \to X\alpha$ a homeomorphism. Two such pairs (X_1, α_1) and (X_2, α_2) are equivalent (and thus give the same point in $\mathcal{T}_g$) if the map $\alpha_2 \circ \alpha_1^{-1} : X_1 \to X_2$ is homotopic to a biholomorphic map.

Remark. When we write $X \in \mathcal{T}_g$, we mean that X is a marked Riemann surface equipped with a marking $\alpha : \Sigma_g \to X$ which is only well defined up to a post-composition with an isomorphism of X onto itself.

Fix an unbranched covering $\pi : \Sigma_h \to \Sigma_g$, and let $X \in \mathcal{T}_g$ be a marked Riemann surface. Then there exists a unique $\widetilde{X}_\pi \in \mathcal{T}_h$, and a holomorphic unbranched covering $\pi_X : \widetilde{X}_\pi \to X$, such that the following diagram commutes

$$\begin{tikzcd} \widetilde{X}_\pi \arrow[d, "\pi_X"'] & \Sigma_h \arrow[l, "\beta"'] \arrow[d, "\pi"] \\ X & \Sigma_g \arrow[l, "\alpha"] \end{tikzcd}$$

where $\beta : \Sigma_h \to \widetilde{X}_\pi$, and $\alpha : \Sigma_g \to X$, denote the corresponding markings. We call $\widetilde{X}_\pi$ the π-covering of X.

Remark. Observe that the correspondence $X \to \widetilde{X}_\pi$ defines the standard holomorphic embedding $\mathcal{T}_g \to \mathcal{T}_h$.

Definition 2.1. *Suppose Y is a closed Riemann surface of genus at least two, and $\phi : \Sigma_h \to Y$ a continuous map. Let $\pi : \Sigma_h \to \Sigma_g$ denote an unbranched covering. For $X \in \mathcal{T}_g$, we write $X \overset{\pi,\phi}{\Longrightarrow} Y$ if $\widetilde{X}_\pi$ $(\phi \circ \beta^{-1})$-dominates Y (see Definition 1.2) where $\beta : \Sigma_h \to \widetilde{X}_\pi$ is a marking of $\widetilde{X}_\pi$. By $S(\pi, \phi) \subset \mathcal{T}_g$ we denote the set of marked Riemann surfaces X such that $X \overset{\pi,\phi}{\Longrightarrow} Y$.*

Remark. Let $\beta_1, \beta_2 : \Sigma_h \to \widetilde{X}_\pi$ be two equivalent markings. Then $\widetilde{X}_\pi$ $(\phi \circ \beta_1^{-1})$-dominates Y if an only if $\widetilde{X}_\pi$ $(\phi \circ \beta_2^{-1})$-dominates Y.

Note that $S(\pi, \phi) = S(\pi, \psi)$ if the maps $\phi, \psi : \Sigma_h \to Y$ are homotopic to each other.

Proposition 2.1. *Each $S(\pi, \phi)$ is a closed subset of $\mathcal{T}_g$.*

Proof. Since π is fixed, to simplify the notation in this proof we write $\widetilde{X} = \widetilde{X}_\pi$. Suppose $X_n \to X$ in $\mathcal{T}_g$. Then we can choose markings $\beta_n : \Sigma_h \to \widetilde{X_n}$, and $\beta : \Sigma_h \to \widetilde{X}$, such that the map $\beta_n \circ \beta^{-1} : \widetilde{X} \to \widetilde{X}_n$ is L_n-bilipschitz, and K_n-quasiconformal, homeomorphism, where $L_n, K_n \to 1$ when $n \to \infty$. Here we assume that X and X_n are equipped with the respective hyperbolic metrics.

Since $X_n \in S(\pi, \phi)$, there exists surjective holomorphic maps $g_n : \widetilde{X_n} \to Y$ homotopic to $\phi \circ \beta_n^{-1}$. Let $h_n = g_n \circ \beta_n \circ \beta^{-1}$. Then each $h_n : \widetilde{X} \to Y$ is homotopic to $\phi \circ \beta^{-1}$. Moreover, each h_n is L_n-lipschitz because $\beta_n \circ \beta^{-1} : \widetilde{X} \to \widetilde{X_n}$ is L_n-bilipschitz, and $g_n : \widetilde{X_n} \to Y$ is 1-lipschitz (by the Schwartz lemma) considering the hyperbolic metric on Y. We also note that h_n is K_n-quasiregular.

Since the family of maps $h_n : \widetilde{X} \to Y$ is uniformly lipschitz, and $\widetilde{X}$ and Y are closed, after passing onto a subsequence if necessary, it follows that h_n converges to a continous map $h : \widetilde{X} \to Y$. Moreover, $d(h(p), h_n(p)) \to 0$, $n \to \infty$, uniformly in $p \in \widetilde{X}$. This implies that h is homotopic to $\phi \circ \beta^{-1}$ (since each h_n is homotopic to $\phi \circ \beta^{-1}$). Also, h is 1-quasiregular, which is the same as saying it is holomorphic. We have constructed a holomorphic map $h : \widetilde{X} \to Y$, homotopic to $\phi \circ \beta^{-1}$. Thus, $X \in S(\pi, \phi)$.

Remark. The above argument crucially depends on the assumption that Y is hyperbolic. The key point is the use of the Schwartz lemma which we used to conclude that the maps $h_n : \widetilde{X} \to Y$ are uniformly lipschitz. For example, if Y is a Riemann sphere instead, and $h_n : \widetilde{X} \to Y$ a sequence of surjective holomorphic maps, then we can not conclude that $d(h(p), h_n(p)) \to 0$, $n \to \infty$, uniformly in $p \in \widetilde{X}$ with respect to any Riemann metric on the sphere. In fact, it map happen that the limiting map $h : \widetilde{X} \to Y$ is constant.

□

Proposition 2.2. *Suppose that Conjecture 1.1 holds for some $g \geq 2$. Then there exists an unbranched covering $\pi : \Sigma_h \to \Sigma_g$, a closed Riemann surface Y of genus at least two, and a continuous map $\phi : \Sigma_h \to Y$, such that $S(\pi, \phi) = \mathcal{T}_g$.*

Proof. Since the assumption is that Conjecture 1.1 holds, there exists a fixed Riemann surface Y of genus at least two, such that each $X \in \mathcal{T}_g$ has an unbranched cover which dominates Y. We have

$$\mathcal{T}_g = \bigcup_{\pi, \phi} S(\pi, \phi),$$

where the union goes over all possible unbranched coverings $\pi : \Sigma_h \to \Sigma_g$, and all continuous maps $\phi : \Sigma_h \to Y$. Up to homotopy, there are countably many such pairs (π, ϕ). Since each $S(\pi, \phi)$ is closed the Baire's Category Theorem implies that at least one set $S(\pi, \phi)$ has a non-empty interior. Let $S(\pi, \phi) \subset \mathcal{T}_g$ be such a set. In the rest of the proof we show $S(\pi, \phi) = \mathcal{T}_g$.

For each $W \in \mathcal{T}_h$, we let $\omega : W \to Y$ be the harmonic map homotopic to $\phi : W \to Y$ (the map ω is harmonic with respect to the hyperbolic metric on Y). Since $S(\pi, \phi)$ is a non-empty set, the map $\phi : \Sigma_h \to Y$ is homotopic to a branched covering. This implies that the harmonic map ω is surjective, and therefore unique in its homotopy class. Thus, the energy function

$$\mathrm{E}_\phi : \mathcal{T}_h \to \mathbb{R},$$

given by

$$\mathrm{E}_\phi(W) = \int_W |\omega_z|^2 \, dxdy,$$

is well defined (here $z = x + iy$ denotes a local complex parameter on W). Define the energy function $\mathrm{F}_\phi : \mathcal{T}_g \to \mathbb{R}$ by $\mathrm{F}_\phi(X) = \mathrm{E}_\phi(\widetilde{X}_\pi)$.

If $X \in S(\pi, \phi)$, then $\omega : \widetilde{X}_\pi \to Y$ is the corresponding holomorphic map homotopic to ϕ. This yields the equality

$$\mathrm{F}_\phi(X) = \deg(\phi)\mathrm{Area}(Y), \qquad \forall X \in S(\pi, \phi). \tag{4}$$

On the other hand, it was shown by Slegers (see Proposition 3.3 in [22]) that the function E_ϕ (and thus F_ϕ) is real analytic. This implies that (4) holds for every $X \in \mathcal{T}_g$ because it holds on an open subset of $\mathcal{T}_g$. But this implies that total energy and the Jacobian of ω are equal, which is only possible if the anti-holomorphic part of the energy density $\overline{\partial}\omega$ is identically zero (see Section 1 in the book [21] by Schoen-Yau). Thus, we have shown that $\omega : \widetilde{X}_\pi \to Y$ is holomorphic for each $X \in \mathcal{T}_g$. So, every X belongs to $S(\pi, \phi)$. The proposition is proved. $\square$

Proposition 2.3. *Suppose that Conjecture 1.1 holds for $g \geq 2$, and let $\pi : \Sigma_h \to \Sigma_g$, and $\phi : \Sigma_h \to Y$, be such that $S(\pi, \phi) = \mathcal{T}_g$. Then there exist a finite index subgroup $\Gamma < \mathrm{Mod}_g^1$ so that the maps ϕ and $\phi \circ \widetilde{f}$ are homotopic to each other for every $f \in \Gamma$, where $\widetilde{f}$ is the lift of f to Σ_h.*

Proof. Recall that de Franchis-Severi theorem (originally proved by de Franchis in [10]) says that for two closed Riemann surfaces C_1 and C_2 of genus at least two, there are at most finitely many holomorphic surjections $C_1 \to C_2$. Let Φ be the set of homotopy classes of all continuous maps $\psi : \Sigma_h \to Y$ such that $S(\pi, \psi) = \mathcal{T}_g$. From de Franchis-Severi theorem we conclude that the set Φ is finite.

Furthermore, we may assume π is a characteristic covering (we can always replace π by a larger covering). Thus, the lift $f \to \widetilde{f}$, where $f \in \mathrm{Mod}_g^1$, and $\widetilde{f} \in \mathrm{Mod}_h$, is well defined (see the proof of Theorem 1.3 above). We observe the equality

$$S(\pi, \phi \circ \widetilde{f}) = f^{-1}\left(S(\pi, \phi)\right), \qquad \forall f \in \mathrm{Mod}_g^1. \tag{5}$$

But $S(\pi, \phi) = \mathcal{T}_g$, which together with (5) implies that $S(\pi, \phi \circ \widetilde{f}) = \mathcal{T}_g$ as well. This means that the homotopy class of $\phi \circ \widetilde{f}$ belongs to the set Φ for every $f \in \mathrm{Mod}_g^1$. Let $\Gamma \subset \mathrm{Mod}_g^1$ denote the subset consisting of elements for which $\phi \circ \widetilde{f}$ is homotopic to ϕ. Firstly, Γ is a subgroup. Secondly, Φ is a finite set which implies that Γ has finite index. The proposition is proved.

$\square$

Proposition 2.4. *Suppose that Conjecture 1.1 holds for $g \geq 2$, and let $\pi : \Sigma_h \to \Sigma_g$, and $\phi : \Sigma_h \to Y$, be such that $S(\pi, \phi) = \mathcal{T}_g$. Then there exist a finite index subgroup $\Gamma < \mathrm{Mod}_g^1$ with the following properties:*

(1) The maps ϕ and $\phi \circ \widetilde{f}$ are homotopic to each other for every $f \in \Gamma$, where $\widetilde{f}$ is the lift of f to Σ_h.

(2) $\Gamma \cap \pi_1(\Sigma_g) = \pi_1(\Sigma_h)$.

Proof. By Proposition 2.3, there exists a finite index subgroup $\Gamma_1 < \mathrm{Mod}_g^1$ which satisfies the condition (1) for some covering $\pi_1 : \Sigma_{h_1} \to \Sigma_g$, and a continuous map $\phi_1 : \Sigma_{h_1} \to Y$. On the other hand, in Lemma 2.1 (proved in the next subsection) we show that there exists a finite index subgroup $\Gamma_2 < \mathrm{Mod}_g^1$, such that $\Gamma_2 \cap \pi_1(\Sigma_g) \subset \pi_1(\Sigma_{h_1})$. Let $\Gamma = \Gamma_1 \cap \Gamma_2$, and $\Sigma_h = \Gamma_2 \cap \pi_1(\Sigma_g)$. Then Γ satisfies (1) for the induced covering $\pi : \Sigma_h \to \Sigma_g$, and the induced map $\phi : \Sigma_h \to Y$. Moreover, by definition $\Gamma \cap \pi_1(\Sigma_g) = \Sigma_h$, and thus Γ satisfies the property (2) as well. The proposition is proved.

□

2.1. **The point-pushing subgroup lemma.** In this subsection we prove that one can find a finite index subgroup of Mod_g^1 whose point pushing part is contained in a given finite index subgroup of the point pushing group $\pi_1(\Sigma_g) < \mathrm{Mod}_g^1$.

Lemma 2.1. *Let $G < \pi_1(\Sigma_g)$ be a finite index subgroup. Then there exists a finite index subgroup $\Gamma < \mathrm{Mod}_g^1$, such that $\Gamma \cap \pi_1(\Sigma_g) \subset G$.*

Proof. Replacing G with a deeper finite index subgroup, we can assume it is characteristic, and thus is a normal subgroup of Mod_g^1. Let $F = \pi_1(\Sigma_g)/G$, and $\overline{\mathrm{Mod}}_g^1 = \mathrm{Mod}_g^1/G$. Thus F is a finite group, and we have a short exact sequence

$$1 \to F \to \overline{\mathrm{Mod}}_g^1 \to \mathrm{Mod}_g \to 1.$$

The lemma is equivalent to the assertion that we can find a finite index subgroup $\overline{\Gamma} < \overline{\mathrm{Mod}}_g^1$, such that $\overline{\Gamma} \cap F = 1$. Since F is finite, it is enough to prove that $\overline{\mathrm{Mod}}_g^1$ is residually finite. In the remainder of the proof we do this by showing that $\overline{\mathrm{Mod}}_g^1$ is a subgroup of a different mapping class group.

Let $(\Sigma_h, \overline{*}) \to (\Sigma_g, *)$ be the based cover corresponding to the subgroup $G < \pi_1(\Sigma_g, *)$. Regard $*$ as the marked point of Mod_g^1, and $\overline{*}$ as the marked point of Mod_h^1. Since G is a characteristic subgroup, it is preserved by the action of Mod_g^1, so we can lift elements of Mod_g^1 to elements of Mod_h^1. This gives a homomorphism $f : \mathrm{Mod}_g^1 \to \mathrm{Mod}_h^1$. The homomorphism f is injective. Indeed, elements of its kernel act trivially on G and thus (since elements of $\pi_1(\Sigma_g, *)$ have unique roots) trivially on $\pi_1(\Sigma_g, *)$, and hence are isotopic to the identity by the Dehn-Nielsen-Baer theorem.

We have Birman exact sequences

$$1 \to \pi_1(\Sigma_g) \to \mathrm{Mod}_g^1 \to \mathrm{Mod}_g \to 1,$$

and

$$1 \to G \to \mathrm{Mod}_h^1 \to \mathrm{Mod}_h \to 1.$$

We have $G < \pi_1(\Sigma_g)$, and from our constructions it is clear that $f(G) = G$, where $G < \mathrm{Mod}_h^1$ is as in our Birman exact sequence. It follows that we can identify $\overline{\mathrm{Mod}}_g^1 = \mathrm{Mod}_g^1/G$ with a subgroup of $\mathcal{M}_h$, as desired.

□

2.2. **The Moduli space.** The moduli space of Riemann surfaces of genus g with n marked points is denoted by $\mathcal{M}_{g,n}$. The (orbifold) fundamental group of $\mathcal{M}_{g,n}$ is Mod_g^n. By $\mathcal{M}_{g,n}^{\Gamma}$ we denote the finite covering of $\mathcal{M}_{g,n}$ corresponding to the subgroup $\Gamma < \mathrm{Mod}_g^n$. That is,

$$\pi_1\left(\mathcal{M}_{g,n}^{\Gamma}\right) = \Gamma. \tag{6}$$

The Birman sequence (1) induces the "forgetful" map

$$F : \mathcal{M}_{g,1} \to \mathcal{M}_g$$

between the corresponding moduli spaces.

Assume now that Conjecture 1.1 holds, and let $\Gamma < \mathrm{Mod}_g^1$ be the finite index subgroup from Proposition 2.4. Set $\Gamma' = F(\Gamma)$. Then F lifts to the forgetful map

$$F_\Gamma : \mathcal{M}_{g,1}^{\Gamma} \to \mathcal{M}_g^{\Gamma'}.$$

The map F is a holomorphic fibration. We let $\mathrm{Fib}(X)$ denote the fiber of F above $X \in \mathcal{M}_g$. Then $\mathrm{Fib}(X)$ is a closed Riemann orbi-surface biholomorphic to X (meaning that the closed Riemann surface underlying the fiber $\mathrm{Fib}(X)$ is biholomorphic to X). Likewise, the fiber $\mathrm{Fib}_\Gamma(X)$ of F_Γ is biholomorphic to $\widetilde{X}_\pi \in \mathcal{M}_h$, where $\widetilde{X}_\pi$ is the π-covering of X. Moreover,

$$\pi_1\left(\mathrm{Fib}_\Gamma(X)\right) = \Gamma \cap \pi_1(\Sigma_g). \tag{7}$$

2.3. **Constructing the epimorphism.** We state and prove the lemma which proves one direction in Theorem 1.2.

Lemma 2.2. *Suppose that Conjecture 1.1 holds for some $g \geq 2$. Then there exists a finite index subgroup $\Gamma < \mathrm{Mod}_g^1$ which allows a point pushing epimorphism $\rho : \Gamma \to \mathrm{F}_2$.*

Proof. By Proposition 2.2 there exist π and ϕ such that $S(\pi, \phi) = \mathcal{T}_g$. Let Γ be the group from Proposition 2.3. We define the map $f : \mathcal{M}_{g,1}^{\Gamma} \to Y$ as follows. On the fiber $\mathrm{Fib}_\Gamma(X) \subset \mathcal{M}_{g,1}^{\Gamma}$, we let f be equal to the holomorphic map $\omega : \widetilde{X}_\pi \to Y$ which is homotopic to ϕ. Proposition 2.3 implies that f is a well defined continuous map on $\mathcal{M}_{g,1}^{\Gamma}$. The induced homomorphism

$$f_* : \pi_1\left(\mathcal{M}_{g,1}^{\Gamma}\right) \to \pi_1(Y),$$

together with (6), yields the homomorphism $f_* : \Gamma \to \pi_1(Y)$. Since the restriction of f to each fiber $\mathrm{Fib}_\Gamma(X)$ is a holomorphic surjection it follows from (7) that f_* is point pushing homomorphism. Composing f_* with some epimorphism $\pi_1(Y) \to \mathrm{F}_2$ produces the required point pushing homomorphism $\rho : \Gamma \to \mathrm{F}_2$.

□

3. Admissible subgroups

It remains to prove the other direction of the equivalence in Theorem 1.2. The Deligne-Mumford compactification of $\mathcal{M}_{g,n}$ by noded Riemann surfaces (with marked points) is denoted by $\overline{\mathcal{M}_{g,n}}$. Recall that $\mathcal{M}^{\Gamma}_{g,n}$ is the finite covering of $\mathcal{M}_{g,n}$ which corresponds to Γ. By $\overline{\mathcal{M}^{\Gamma}_{g,n}}$ we denote the normalization of $\mathcal{M}^{\Gamma}_{g,n}$ with respect to $\overline{\mathcal{M}_{g,n}}$.

The forgetful maps F and F_{Γ} defined in the previous section extend to the Deligne-Mumford compactifications

$$F : \overline{\mathcal{M}_{g,1}} \to \overline{\mathcal{M}_{g}}, \qquad F_{\Gamma} : \overline{\mathcal{M}^{\Gamma}_{g,1}} \to \overline{\mathcal{M}^{\Gamma'}_{g}}.$$

Definition 3.1. *Let Y be any set. We say that a map $f : \mathcal{M}^{\Gamma}_{g,1} \to Y$ (or $f : \overline{\mathcal{M}^{\Gamma}_{g,1}} \to Y$) factors through the forgetful map F_{Γ} if there exists a map $g : \mathcal{M}^{\Gamma'}_{g} \to Y$ (or $g : \overline{\mathcal{M}^{\Gamma'}_{g}} \to Y$) such that $f = g \circ F_{\Gamma}$.*

Combining results of Pikaart-de Jong [19], Boggi-Pikaart [6], Bridson [8], and Putman [17], with the classical Siu-Beauville theorem [3], [23], we prove the following.

Lemma 3.1. *Suppose $g \geq 2$, and let $\Gamma < \mathrm{Mod}^1_g$ be a finite index subgroup which admits a point pushing epimorphism $\rho : \Gamma \to \mathbf{F}_2$. Then there exists a finite index subgroup $\Theta < \mathrm{Mod}^1_g$ with the following properties:*

(1) $\Theta < \Gamma$,

(2) there exists a Riemann surface Y of genus at least two, and a surjective holomorphic map $f : \overline{\mathcal{M}^{\Theta}_{g,1}} \to Y$ which does not factor through the forgetful map F_{Θ}.

We prove this lemma in the remainder of this section. But before that we prove the other direction in Theorem 1.2 using Lemma 3.1.

3.1. **Constructing unramified correspondences.** We have the following lemma.

Lemma 3.2. *Suppose that for some $g \geq 2$ there exists a finite index subgroup $\Gamma < \mathrm{Mod}^1_g$ which allows a point pushing epimorphism $\rho : \Gamma \to \mathrm{F}_2$. Then Conjecture 1.1 holds for such g.*

Proof. Let $\Theta < \mathrm{Mod}^1_g$ be the subgroup from Lemma 3.1. After passing onto a subgroup, we may assume that Θ is a characteristic subgroup (clearly, the conclusions of Lemma 3.1 hold for every finite index subgroup of Θ). Set $K = \Theta \cap \pi_1(\Sigma_g)$, and let $\pi : \Sigma_h \to \Sigma_g$ be the corresponding covering such that $K = \pi_1(\Sigma_h)$ (see above). The fiber $\mathrm{Fib}_{\Theta}(X)$ of the forgetful map F_{Θ} is biholomorphic to $\widetilde{X}_{\pi} \in \mathcal{M}_h$, where $\widetilde{X}_{\pi}$ is the π-covering of X. Let $f : \overline{\mathcal{M}^{\Theta}_{g,1}} \to Y$ be the holomorphic surjection from Lemma 3.1 which does not factor through the forgetful map. Since $\mathcal{M}^{\Theta}_{g,1}$ is an open and dense subset of $\overline{\mathcal{M}^{\Theta}_{g,1}}$, it follows that for at least on $X \in \mathcal{M}_g$ the restriction of $f : \mathrm{Fib}_{\Theta}(X) \to Y$ is surjective. This shows that the restriction $f : \mathcal{M}^{\Theta}_{g,1} \to Y$ does not factor either. However, to prove the lemma we need to show that the restriction of f to every fiber $\mathrm{Fib}_{\Theta}(X)$, $X \in \mathcal{M}_g$, is surjective.

Let $U \subset \mathcal{M}_g$ be such that $X \in U$ if the restriction $f : \mathrm{Fib}_\Theta(X) \to Y$ is surjective. We have already observed that U is not empty. Note that if a holomorphic map between closed surfaces is not surjective then it is constant. This implies that U is an open subset of $\mathcal{M}_g$. Moreover, by Proposition 2.1 U is also closed. Thus, U is both open and closed subset of $\mathcal{M}_g$, and therefore $U = \mathcal{M}_g$. This shows that Conjecture 1.1 holds.

□

3.2. **Twists are dead.** We now state the result of Bridson [8], and Putman [17]. Recall that a multicurve γ is a collection of mutually disjoint simple closed curves $\gamma_1, \ldots, \gamma_k$. A multi-twist about γ is the product of powers of Dehn twists about the γ_i's.

Lemma 3.3. *Let $g \geq 3$, and let $\rho : \Gamma \to \mathbb{Z}$ be an epimorphism, where $\Gamma < \mathrm{Mod}_g^n$. Suppose $\gamma \subset \Sigma_g^n$ is a multicurve and let $t \in \mathrm{Mod}_g^n$ denote a multi-twist about γ. If $t \in \Gamma$, then $\rho(t) = 0$.*

Remark. Originally this result was proved when γ is a single curve. It was extended to multicurves by Putman in (see Corollary 2.10 in [20]).

3.3. **Admissible groups.** We begin by defining admissible subgroups.

Definition 3.2. *We say that $\Gamma < \mathrm{Mod}_g^n$ is admissible if it is a subgroup of finite index, and if $\overline{\mathcal{M}_{g,n}^{\Gamma}}$ is a complex projective manifold.*

The first examples of admissible subgroups were found by Looijenga [13]. The following was shown by Boggi-Pikaart (Corollary 2.10 in [6]). We sketch the proof of this proposition in the appendix for the sake of completeness.

Proposition 3.1. *Suppose $g \geq 2$, and let $\Gamma < \mathrm{Mod}_g^n$ be a finite index subgroup. Then there exists an admissible group $\Theta < \Gamma$.*

In order to prove Lemma 3.1, we show that if an admissible group Γ admits a point-pushing epimorphism onto $\mathbf{F}_2$ then there exists a non-constant holomorphic map from $\overline{\mathcal{M}_{g,1}^{\Gamma}}$ to a closed Riemann surface Y of genus at least two which does not factor through the forgetful map F_Γ. To carry out this plan we must compute the fundamental group of $\overline{\mathcal{M}_{g,1}^{\Gamma}}$. The following proposition follows from Lemma 16 in the paper [2] by Aramayona-Funar (we recall the proof in the appendix).

Proposition 3.2. *Let $\Gamma < \mathrm{Mod}_g^n$ denote an admissible group. Then*

$$\pi_1\left(\overline{\mathcal{M}_{g,n}^{\Gamma}}\right) \cong \frac{\Gamma}{Q(\Gamma)},$$

where $Q(\Gamma) < T(\Gamma)$, and $T(\Gamma) < \Gamma$ is the group generated by multitwists contained in Γ.

Remark. This proposition holds for any finite index subgroup Γ (not necessarily an admissible one).

3.4. **Holomorphic fibrations of $\overline{\mathcal{M}_{g,n}^{\Gamma}}$.** The following proposition suggests that it is convenient to study epimorphism of admissible groups onto free groups because they lead to non-constant holomorphic maps of $\overline{\mathcal{M}_{g,n}^{\Gamma}}$.

Proposition 3.3. *Suppose $g \geq 2$, and let $\Gamma < \mathrm{Mod}_g^1$ be admissible. Suppose there exists a point pushing epimorphism $\rho : \Gamma \to \mathbf{F}_2$. Then there exist a surjective holomorphic map $f : \overline{\mathcal{M}_{g,1}^{\Gamma}} \to Y$ which does not factor through F_Γ, where Y is a closed Riemann surface of genus at least two.*

Proof. It follows from Lemma 3.3 that $Q(\Gamma) < \mathrm{Ker}(\rho)$. Thus, ρ factors through $\Gamma/Q(\Gamma)$ and induces the epimorphism of $\Gamma/Q(\Gamma)$ to $\mathbf{F}_2$. Putting this together with Proposition 3.2 yields a point pushing epimorphism $\sigma : \pi_1(\overline{\mathcal{M}_{g,1}^{\Gamma}}) \to \mathbf{F}_2$. On the other hand, $\overline{\mathcal{M}_{g,1}^{\Gamma}}$ is a projective complex manifold, thus it is Kähler. The proposition now follows from the classical Siu-Beauville theorem (see [3], [23]), which say that if the fundamental group of a compact Kähler manifold surjects onto $\mathbf{F}_2$, then the Kähler manifold fibers over a closed Riemann surface of genus at least 2.

It remains to explain why the map f does not factor through the forgetful map. Consider the induced map $\sigma^* : \mathrm{H}^1(\mathbf{F}_2) \to \mathrm{H}^1(\overline{\mathcal{M}_{g,1}^{\Gamma}})$. The image $\sigma^*\left(\mathrm{H}^1(\mathbf{F}_2)\right)$ is an isotropic subspace. The Castelnuovo-de Franchis Theorem (see the book [1]; also the same was proved by Catanese in Theorem 1.10 in [9]) shows that

$$\sigma^*\left(\mathrm{H}^1(\mathbf{F}_2)\right) \subset f^*\left(\mathrm{H}^1(Y)\right). \tag{8}$$

Since σ is point pushing, it follows that

$$\sigma^*\left(\mathrm{H}^1(\mathbf{F}_2)\right) \cap \iota\left(\mathrm{H}^1(\mathrm{Fib}_\Theta(X))\right) \neq \emptyset,$$

where $\iota : \mathrm{Fib}_\Theta(X) \to \overline{\mathcal{M}_{g,1}^{\Gamma}}$ is the inclusion map. From (8) we find that f does not factor through F_Γ.

□

3.5. **Proof of Lemma 3.1.** Suppose $\Gamma < \mathrm{Mod}_g^1$ is a finite index subgroup equipped with a point pushing epimorphism $\rho : \Gamma \to \mathbf{F}_2$. By Proposition 3.1 there exists an admissible group $\Theta < \Gamma$. Although the restriction $\rho : \Theta \to \mathbf{F}_2$ may not be an epimorphism, the image $\rho(\Theta)$ is a finite index subgroup of F_2. Thus, we can find an epimorphism $\rho_1 : \Theta \to \mathbf{F}_2$. Then by Proposition 3.3 there exists a non-constant holomorphic map $f : \overline{\mathcal{M}_{g,1}^{\Theta}} \to Y$ which does not factor through the forgetful map. Lemma 3.1 is proved.

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