June 11, 2023

THE SECOND VARIATION OF THE HODGE NORM AND HIGHER PRYM REPRESENTATIONS

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Abstract. Let $\chi \in H^1(\Sigma_h, \mathbb{Q})$ denote a rational cohomology class, and let H_χ denote its Hodge norm. We recover the result that H_χ is a plurisubharmonic function on the Teichmüller space $\mathcal{T}_h$, and characterize complex directions along which the complex Hessian of H_χ vanishes. Moreover, we find examples of $\chi \in H^1(\Sigma_h, \mathbb{Q})$ such that H_χ is not strictly plurisubharmonic. As part of this construction, we find an unbranched covering $\pi : \Sigma_h \to \Sigma_2$ such that the subgroup of $H_1(\Sigma_h, \mathbb{Q})$ generated by lifts of simple curves from Σ_2 is strictly contained in $H_1(\Sigma_h, \mathbb{Q})$. Finally, combining the characterization theorem with the Riemann-Roch, and the Li-Yau [17] gonality estimate, we show that geometrically uniform covers of Σ_g satisfy the Putman-Wieland Conjecture about the induced Higher Prym representations.

1. Introduction

1.1. **Second variation of the Hodge norm.** Let Σ_h denote a closed surface of genus $h \geq 2$, and $\mathcal{T}_h$ the corresponding Teichmüller space. Fix $\chi \in H^1(\Sigma_h, \mathbb{Q})$. By Hodge theory, on every Riemann surface $S \in \mathcal{T}_h$ there exists a unique Abelian differential ϕ such that the real-valued harmonic 1-form $\mathrm{Re}(\phi)$ represents χ. The Hodge norm of χ is the function $\mathrm{H}_\chi : \mathcal{T}_h \to \mathbb{R}$ defined as

$$\mathrm{H}_\chi(S) = \frac{i}{2} \int_S \phi \wedge \overline{\phi}. \tag{1}$$

The function H_χ is smooth (and real analytic in fact). The reader can consult the paper by McMullen [13] for the background regarding the analytical properties of the Hodge norm.

In this paper we prove several theorems, the main results being Theorem 1.3, Theorem 1.4, and Theorem 1.9.

Theorem 1.1. *The Hodge norm* H_χ *is a plurisubharmonic function on* $\mathcal{T}_h$.

Remark 1. This result is well known. For example, it was proved by Forni in [4] (it also follows from a more general result by Toledo [16]). The reason we include the proof here is because it is necessary for the proof of Theorem 1.4 below.

Theorem 1.2. *For some* $h \geq 2$*, there exists* $\chi \in H^1(\Sigma_h, \mathbb{Z})$ *such that* H_χ *is not strictly plurisubharmonic.*

2000 *Mathematics Subject Classification.* Primary 20H10.

This work was supported by the *Simons Investigator Award* 409735 from the Simons Foundation

Remark 2. The examples of cohomology classes whose Hodge norm is not strictly subharmonic are known (an interesting example is contained in the work by Forni-Matheus-Zorich [5]).

In the construction underpinning the proof of the previous theorem, we exhibit an unbranched covering $\pi : \Sigma_h \to \Sigma_2$ with the following property.

Theorem 1.3. *The subgroup of $H_1(\Sigma_h, \mathbb{Q})$ generated by lifts of simple closed curves from Σ_2 is strictly contained in $H_1(\Sigma_h, \mathbb{Q})$.*

Remark 3. Malestein-Putman [11] showed the existence of branched covers of $\Sigma_h \to \Sigma_g$ such that the subgroup of $H_1(\Sigma_h, \mathbb{Q})$ generated by lifts of simple closed curves from Σ_g is strictly contained in $H_1(\Sigma_h, \mathbb{Q})$. Theorem 1.3 resolves the open problem which asks whether the same can hold for an unbranched cover.

Next, we characterize the direction along which the complex Hessian of H_χ vanishes.

Theorem 1.4. *The complex Hessian of H_χ vanishes in the infinitesimal direction corresponding to a Beltrami differential μ on $S \in \mathcal{T}_h$ if and only if there exists a smooth function $g : S \to \mathbb{C}$ such that*

$$\mu\phi = \overline{\partial} g, \tag{2}$$

where ϕ is the Abelian differential on S corresponding to χ.

Remark 4. It suffices to prove Theorem 1.1, and Theorem 1.4, assuming $\chi \in H^1(\Sigma_h, \mathbb{Z})$.

Corollary 1.5. *If the complex Hessian of H_χ vanishes in the infinitesimal direction corresponding to a Beltrami differential μ on $S \in \mathcal{T}_h$, then for any Abelian differential ψ on S we have*

$$\int_S \mu\phi \wedge \psi = 0.$$

Proof. From Theorem 1.4, we get the equality $\mu\phi \wedge \psi = \overline{\partial}(g\psi)$ (here we use $\overline{\partial}\psi = 0$). The corollary now follows from the Stokes' Theorem. □

1.2. **Higher Prym Representation.** Let Σ_g^n denote a surface of genus $g \geq 2$ with $n \geq 0$ marked points, and denote by Mod_g^n the corresponding pure mapping class group, i.e. the group of self homeomorphisms of Σ_g^n that fix the punctures pointwise, considered up to isotopy. By thinking of some fixed basepoint on Σ_g^n as another puncture, we obtain a standard action of Mod_g^{n+1} on $\pi_1(\Sigma_g^n)$.

Suppose $K < \pi_1(\Sigma_g^n)$ is a normal finite index subgroup, and let $\pi_K : \Sigma_h^m \to \Sigma_g^n$ be the covering corresponding to K. Let $\Gamma_K < \mathrm{Mod}_g^{n+1}$ be the set of mapping classes that preserve K (as a set). Thus Γ_K is a finite index subgroup consisting of mapping classes which naturally lift to Σ_h^m via the covering π_K.

Remark 5. Note that $\pi_1(\Sigma_g^n) < \Gamma_K$. Furthermore, if K is a characteristic subgroup then $\Gamma_K = \mathrm{Mod}_g^{n+1}$.

The group Γ_K naturally acts on $H^1(\Sigma_h, \mathbb{Q})$ inducing the linear representation $\Gamma_K \to \mathrm{Aut}(H^1(\Sigma_h, \mathbb{Q}))$ called the Higher Prym representation (see [15]).

Definition 1.6. *We say that K has the Putman-Wieland property if for each nonzero vector $v \in H^1(\Sigma_h, \mathbb{Q})$, the Γ_K-orbit of v is infinite.*

In [15] Putman and Wieland made the following conjecture (see Conjecture 1.2. in [15]).

Conjecture 1.7. *Let $g \geq 2$, and $n \geq 0$. Then every finite index subgroup $K < \pi_1(\Sigma_g^n)$ has the Putman-Wieland property.*

Remark 6. It is assumed in [15] that K is a characteristic subgroup. However, it is easy to see that Conjecture 1.7 is equivalent to Conjecture 1.2. in [15]. They also state the homological version of the conjecture, rather than the cohomological Conjecture 1.7 stated above. However the symplectic intersection form on $H_1(\Sigma_h, \mathbb{Q})$ provides a Mod_h-equivariant isomorphism between $H^1(\Sigma_h, \mathbb{Q})$ and $H_1(\Sigma_h, \mathbb{Q})$, so the two versions of the conjecture are equivalent. It was shown in [12] that the conjecture does not hold in the case $g = 2$.

The importance of Conjecture 1.7 stems from its close connections with the Ivanov conjectures about the virtual cohomology of mapping class groups. When $n = 0$, Conjecture 1.7 has been verified by Looijenga [10] assuming the covering $\pi_K : \Sigma_h \to \Sigma_g$ is Abelian, and by Grunewald-Larsen-Lubotzky-Malestein [7] provided that π_K is "redundant". We prove the conjecture for geometrically uniform subgroups $K < \pi_1(\Sigma_g^n)$.

Definition 1.8. *Let $\epsilon > 0$, and $K < \pi_1(\Sigma_g^n)$ a normal, finite index subgroup. For $X \in \mathcal{T}_{g,n}$, we let $Z^o \in \mathcal{T}_{h,m}$ be the corresponding covering $\pi_K : Z^o \to X$. By Z we denote the closed Riemann surface underlying Z^o. We say that K is geometrically ϵ-uniform if there exists $X \in \mathcal{T}_{g,n}$ such that $\lambda_1(Z) \geq \epsilon$, where λ_1 is the smallest non-zero eigenvalue of the Laplacian on Z.*

Theorem 1.9. *For every $\epsilon > 0$ and every $g \geq 3, n \geq 0$ the following holds. If $g \geq \frac{1+2\epsilon}{2\epsilon}$, then every geometrically ϵ-uniform normal subgroup $K < \pi_1(\Sigma_g^n)$ has the Putman-Wieland property.*

Observe that K is geometrically ϵ-uniform subgroup if and only if $\pi_1(\Sigma_g)/K$ is an ϵ'-expander group (for a suitable ϵ' depending on ϵ and g). This indicates that Theorem 1.9 yields many new examples of subgroups satisfying the Putman-Wieland property. In the following example we construct concrete examples of subgroups which can be shown to satisfy the Putman-Wieland property using Theorem 1.9.

Example. Let $\{N_j\}_{j\in\mathbb{N}}$ be a decreasing collection of normal subgroups of Σ_g, and suppose that $\pi_1(\Sigma_g)$ has the property (τ) with respect to this collection. That is, we assume that the Cayley graphs of $\pi_1(\Sigma_g)/N_j$ with respect to a fixed generating set form a family of expanders, see [9]. We now fix an arbitrary Riemann surface structure X on Σ_g, and let X_j be the Riemann surface that covers X corresponding to the subgroup $N_j \leq \pi_1(\Sigma_g) \cong \pi_1(X)$. By a standard result of Brooks [2], we have $\lambda_1(X_j) > \epsilon$ for some $\epsilon > 0$. Let j be such that X_j has genus at least $\frac{1+2\epsilon}{2\epsilon}$.

Then by Theorem 1.9, the subgroups $N_k \leq \pi_1(X_j) = N_j$ for $k \geq j$ all have the Putman-Wieland property.

1.3. **The organization of the paper.** To prove Theorem 1.1 we need information on the second derivative of the Hodge norm H_χ. The Hodge norm H_χ is plurisubharmonic in the direction defined by a Beltrami form μ if and only if

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{H}_\chi(S^{t\mu}) + \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{H}_\chi(S^{it\mu}) \geq 0 \tag{3}$$

The key step is contained in Lemma 2.3 where we compute second variation of H_χ. This lemma is proved in Section 3 and Section 4. The proof of (3) (given in Section 2) then follows from a Cauchy-Schwartz type inequality. To characterize direction along which H_χ is not strictly plurisubharmonic we need to analyze the case when the equality holds in (3). Straightforward analysis then yields the proof of Theorem 1.4 (given in Section 2).

In Proposition 5.2 in Section 5, we show that if $n \geq 1$, and $K < \pi_1(\Sigma_g^n)$ does not have the Putman-Wieland property, then there exists a finite index subgroup $\Gamma < \mathrm{Mod}_g^{n+1}$ which admits a point pushing epimorphism $\rho : \Gamma \to \mathbb{Z}$, such that $\Gamma \cap \pi_1(\Sigma_g^{n+1})$ is geometrically ϵ-uniform. In Proposition 5.3 we show that if $K < \pi_1(\Sigma_g)$ does not have the Putman-Wieland property, then there exists $\Gamma < \mathrm{Mod}_g^2$ admitting a point pushing epimorphism onto $\mathbb{Z}$, and such that $\Gamma \cap \pi_1(\Sigma_g^1)$ is geometrically ϵ-uniform. In Section 6 we derive some elementary properties of point pushing epimorphisms, while in Section 7 we establish the connection with harmonic maps and the Hodge norm. In Section 8 we prove Lemma 5.4 which rules out the existence of a point pushing epimorphism for such Γ. In Section 9 we prove Theorem 1.2 and Theorem 1.3.

2. The Hodge Norm

We first state Lemma 2.3 which computes the second variation of H_χ (compare with Liu-Su [8] who computed the second variation of the extremal length function on $\mathcal{T}_h$, and Toledo [16] who did the same for the energy function corresponding to harmonic maps from surfaces into negatively curved targets). Using this computation we first prove the inequality (3), which in turn proves Theorem 1.1. To characterize directions along which H_χ is not strictly plurisubharmonic we need to analyze the case when the equality holds in (3). This analysis is also straightforward, and after carrying it out we prove Theorem 1.4.

2.1. **The Hodge Norm and harmonic functions.** Fix a non-zero cohomology class $\chi \in H^1(\Sigma_h, \mathbb{Z})$ (without loss of generality we may assume that χ is a primitive element). By a slight abuse of notation, we let $\chi : \pi_1(\Sigma_h) \to \mathbb{Z}$ denote the corresponding epimorphism.

Proposition 2.1. *There exists a unique (up to homotopy) continuous mapping $f_\chi : \Sigma_h \to \mathbb{R}/\mathbb{Z}$ such that the identity $\chi = (f_\chi)_*$ holds on $\pi_1(\Sigma_h)$.*

Proof. The proof follows from the fact that both Σ_h and the circle $\mathbb{R}/\mathbb{Z}$ have contractible universal covers. □

For $S \in \mathcal{T}_h$, we let $\omega : S \to \mathbb{R}/\mathbb{Z}$ be a harmonic map homotopic to f_χ. The map ω is unique up to post-composition with the rotation of $\mathbb{R}/\mathbb{Z}$. Recall that $\partial\omega = \phi$ is the Abelian differential on S corresponding to the cohomology class χ (via the Hodge correspondence). From (1), we have

$$\mathrm{H}_\chi(S) = \frac{i}{2}\int_S \partial\omega \wedge \overline{\partial\omega} = \int_S |\omega_z|^2\, dxdy, \tag{4}$$

where $z = x + iy$ is a local complex parameter on S. Denote $\mathcal{V}_h$ the universal curve over $\mathcal{T}_h$. The next proposition shows that we can choose harmonic maps $\omega : S \to \mathbb{R}/\mathbb{Z}$ to vary smoothly over $\mathcal{V}_h$.

Proposition 2.2. *There exists a smooth map $W : \mathcal{V}_h \to \mathbb{R}/\mathbb{Z}$ such that for a fixed $S \in \mathcal{T}_h$, the map $W(S, \cdot) : S \to \mathbb{R}/\mathbb{Z}$ is harmonic and homotopic to f_χ.*

Proof. Since $\mathcal{T}_h$ is contractible, there exists a smooth section $\mathbf{s} : \mathcal{T}_h \to \mathcal{V}_h$ of the universal curve $\mathcal{V}_h \to \mathcal{T}_h$. We now let $W : \mathcal{V}_h \to \mathbb{R}/\mathbb{Z}$ be defined on the fibre $S \in \mathcal{T}_h$ to be the unique harmonic map $S \to \mathbb{R}/\mathbb{Z}$ homotopic to f_χ, such that $W(S, \mathbf{s}(S)) = 0$. It remains to show that W is smooth, which will follow from classical results of Eells-Lemaire [6].

We now fix a topological surface Σ_h, and let $(\mathbf{g}_S : S \in \mathcal{T}_h)$ be a smoothly varying family of hyperbolic metrics on Σ_h, such that $\mathbf{g}_S$ is in (the isotopy class of) the conformal class $S \in \mathcal{T}_h$. It now follows from Proposition 5.3 in [6] that the map

$$\tilde{W} : \mathcal{T}_h \to \frac{C^\infty(\Sigma_h, \mathbb{R}/\mathbb{Z})}{\mathbb{R}}$$

that sends $S \in \mathcal{T}_h$ to the harmonic map $(\Sigma_h, \mathbf{g}_S) \to \mathbb{R}/\mathbb{Z}$ homotopic to f_χ. Here $\mathbb{R}$ acts on $\mathbb{R}/\mathbb{Z}$ by translations, and thus on $C^\infty(\Sigma_h, \mathbb{R}/\mathbb{Z})$. Since the map

$$\begin{aligned} \frac{C^\infty(\Sigma_h, \mathbb{R}/\mathbb{Z})}{\mathbb{R}} \times \Sigma_h &\longrightarrow C^\infty(\Sigma_h, \mathbb{R}/\mathbb{Z}) \\ ([f], z) &\longrightarrow f - f(z) \end{aligned}$$

is smooth, it follows that W is smooth. □

Let μ be a smooth Beltrami differential on S, that is, μ is a $(-1, 1)$ form. Let $t \in \mathbb{R}$ be such that $||t\mu||_\infty < 1$. Then we solve the Beltrami equation and find the diffeomorphism $f^{t\mu} : S \to S^{t\mu}$ such that

$$\mu = \left(\frac{f^{t\mu}_{\bar{z}}}{f^{t\mu}_z}\right)\frac{d\bar{z}}{dz},$$

where $S^{t\mu}$ is the resulting Riemann surface. Define the function $\dot{\omega}^\mu : S \to \mathbb{R}$ by

$$\dot{\omega}^\mu(p) = \lim_{t\to 0} \frac{W(S^{t\mu}, p) - W(S, p)}{t}. \tag{5}$$

Although W takes values in $\mathbb{R}/\mathbb{Z}$, the function $\dot{\omega}^{\mu}$ is real valued. From a differential geometric point of view, we are simply taking a unit vector field on the circle and using it to trivialize its tangent bundle.

In the next section we prove the following lemma.

Lemma 2.3. *Fix a Riemann surface $S \in \mathcal{T}_g$, and a smooth Beltrami form μ on S. Then*

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{H}_{\chi}(S^{t\mu}) = 4\int_S |\mu|^2 |\omega_z|^2 \,\mathrm{d}x\mathrm{d}y - 2\int_S |\dot{\omega}^{\mu}_z|^2 \,\mathrm{d}x\mathrm{d}y, \tag{6}$$

and

$$\int_S |\dot{\omega}^{\mu}_z|^2 \,\mathrm{d}x\mathrm{d}y = \int_S \bar{\mu}\omega_{\bar{z}}\dot{\omega}^{\mu}_{\bar{z}}\mathrm{d}x\mathrm{d}y + \int_S \mu\omega_z\dot{\omega}^{\mu}_z\mathrm{d}x\mathrm{d}y. \tag{7}$$

2.2. **Proof of Theorem 1.1.** The Hodge norm H_{χ} is plurisubharmonic in the direction defined by a Beltrami form μ if and only if

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{H}_{\chi}(S^{t\mu}) + \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{H}_{\chi}(S^{it\mu}) \geq 0 \tag{8}$$

We use Lemma 2.3 to verify this inequality. From (7) we get

$$\begin{aligned}\int_S |\dot{\omega}^{\mu}_z|^2 \,\mathrm{d}x\mathrm{d}y + \int_S \left|\dot{\omega}^{i\mu}_z\right|^2 \mathrm{d}x\mathrm{d}y &= \int_S \mu\omega_z(\dot{\omega}^{\mu}_z + i\dot{\omega}^{i\mu}_z)\mathrm{d}x\mathrm{d}y + \int_S \bar{\mu}\omega_{\bar{z}}(\dot{\omega}^{\mu}_{\bar{z}} - i\dot{\omega}^{i\mu}_{\bar{z}})\mathrm{d}x\mathrm{d}y \\ &\leq \int_S |\mu\omega_z|\left|\dot{\omega}^{\mu}_z + i\dot{\omega}^{i\mu}_z\right| \mathrm{d}x\mathrm{d}y + \int_S |\bar{\mu}\omega_{\bar{z}}|\left|\dot{\omega}^{\mu}_{\bar{z}} - i\dot{\omega}^{i\mu}_{\bar{z}}\right| \mathrm{d}x\mathrm{d}y.\end{aligned} \tag{9}$$

Furthermore, for any $a, b \in \mathbb{R}$, we have $(a - \frac{1}{2}b)^2 \geq 0$, so $ab \leq a^2 + \frac{1}{4}b^2$. Applying this inequality pointwise to the integrands in (9), we have

$$\begin{aligned}\int_S |\dot{\omega}^{\mu}_z|^2 \,\mathrm{d}x\mathrm{d}y + \int_S \left|\dot{\omega}^{i\mu}_z\right|^2 \mathrm{d}x\mathrm{d}y &\leq \int_S \left(|\mu\omega_z|^2 + \frac{1}{4}\left|\dot{\omega}^{\mu}_z + i\dot{\omega}^{i\mu}_z\right|^2\right) \mathrm{d}x\mathrm{d}y \\ &\quad + \int_S \left(|\bar{\mu}\omega_{\bar{z}}|^2 + \frac{1}{4}\left|\dot{\omega}^{\mu}_{\bar{z}} - i\dot{\omega}^{i\mu}_{\bar{z}}\right|^2\right) \mathrm{d}x\mathrm{d}y.\end{aligned} \tag{10}$$

Since ω is (locally) real-valued, we have

$$\overline{(\omega_z)} = \omega_{\bar{z}}. \tag{11}$$

Likewise, $\dot{\omega}^{\mu}, \dot{\omega}^{i\mu}$ are real-valued, so we have

$$\overline{\left(\dot{\omega}^{\mu}_z\right)} = \dot{\omega}^{\mu}_{\bar{z}}, \qquad \text{and} \qquad \overline{\left(\dot{\omega}^{i\mu}_z\right)} = \dot{\omega}^{i\mu}_{\bar{z}}. \tag{12}$$

Therefore after expanding and collecting terms on the right hand side of (10), we obtain

$$\int_S \left|\dot{\omega}_z^{\mu}\right|^2 \mathrm{d}x\mathrm{d}y + \int_S \left|\dot{\omega}_z^{i\mu}\right|^2 \mathrm{d}x\mathrm{d}y \leq 2\int_S \left|\mu\right|^2 \left|\omega_z\right|^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_S \left(\left|\dot{\omega}_z^{\mu}\right|^2 + \left|\dot{\omega}_z^{i\mu}\right|^2\right) \mathrm{d}x\mathrm{d}y + C,$$

where

$$\begin{aligned}
C &= \frac{i}{2}\int_S \left(\dot{\omega}_{\bar{z}}^{\mu}\dot{\omega}_z^{i\mu} - \dot{\omega}_z^{\mu}\dot{\omega}_{\bar{z}}^{i\mu}\right) \mathrm{d}x\mathrm{d}y \\
&= -\frac{1}{4}\int_S \partial\dot{\omega}^{i\mu} \wedge \bar{\partial}\dot{\omega}^{\mu} + \bar{\partial}\dot{\omega}^{i\mu} \wedge \partial\dot{\omega}^{\mu} \\
&= -\frac{1}{4}\int_S \left(\partial\dot{\omega}^{i\mu} \wedge \bar{\partial}\dot{\omega}^{\mu} + \dot{\omega}^{i\mu}\partial\bar{\partial}\dot{\omega}^{\mu}\right) + \left(\bar{\partial}\dot{\omega}^{i\mu} \wedge \partial\dot{\omega}^{\mu} + \dot{\omega}^{i\mu}\bar{\partial}\partial\dot{\omega}^{\mu}\right) \\
&= -\frac{1}{4}\int_S \partial(\dot{\omega}^{i\mu}\bar{\partial}\dot{\omega}^{\mu}) + \bar{\partial}(\dot{\omega}^{i\mu}\partial\dot{\omega}^{\mu}) \\
&= -\frac{1}{4}\int_S \mathrm{d}(\dot{\omega}^{i\mu}\bar{\partial}\dot{\omega}^{\mu} + \dot{\omega}^{i\mu}\partial\dot{\omega}^{\mu}) = 0.
\end{aligned}$$

In the calculation above, we used Stokes' theorem in the last line and the fact that $\mathrm{d}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0$ in going from the second to the third line. Hence

$$\int_S \left|\dot{\omega}_z^{\mu}\right|^2 \mathrm{d}x\mathrm{d}y + \int_S \left|\dot{\omega}_z^{i\mu}\right|^2 \mathrm{d}x\mathrm{d}y \leq 4\int_S \left|\mu\right|^2 \left|\omega_z\right|^2 \mathrm{d}x\mathrm{d}y.$$

The above inequality is exactly equivalent to (8) by equations (6) and (7). This proves Theorem 1.1.

2.3. **Proof of Theorem 1.4: Part 1.** If H_{χ} is not strictly plurisubharmonic in the complex direction defined by μ, then the equality must hold in (8) for μ. Hence equality must hold in each inequality we derived above. In particular, the equality in (9) yields the equation

$$\mu\omega_z(\dot{\omega}_z^{\mu} + i\dot{\omega}_z^{i\mu}) \geq 0,$$

while the equality in (10) yields

$$2\left|\mu\omega_z\right| = \left|\dot{\omega}_z^{\mu} + i\dot{\omega}_z^{i\mu}\right|.$$

Putting these two together we obtain the identity

$$\overline{\dot{\omega}_z^{\mu} + i\dot{\omega}_z^{i\mu}} = 2\mu\omega_z. \tag{13}$$

As we already observed, for $\tau \in \{\mu, i\mu\}$ we have $\overline{(\dot{\omega}_z^{\tau})} = \left(\overline{\dot{\omega}^{\tau}}\right)_{\bar{z}} = \dot{\omega}_{\bar{z}}^{\tau}$. Combining (13) and (12) implies the identity

$$2\mu\omega_z = \left(\dot{\omega}^{\mu} - i\dot{\omega}^{i\mu}\right)_{\bar{z}}.$$

Set $g = \frac{1}{2}\left(\dot{\omega}^{\mu} - i\dot{\omega}^{i\mu}\right)$. Then $\bar{\partial}g = \mu\omega_z\mathrm{d}\bar{z} = \mu\partial\omega$, and one direction of the equivalence in Theorem 1.4 is proved.

2.4. **Proof of Theorem 1.4: Part 2.** Let us prove the other direction of the equivalence. Suppose there exists a smooth map $g : S \to \mathbb{C}$ so that $\mu\phi = \bar{\partial} g$. We show that the complex Hessian of H_χ in the direction defined by μ vanishes. We first establish the following claim.

Claim 2.4. *If $\mu\phi = \bar{\partial} g$, then*

$$2\mu\omega_z = \dot{\omega}^{\mu}_{\bar{z}} - i\dot{\omega}^{i\mu}_{\bar{z}}.$$

Proof. It is shown in formula (24) below that $\dot{\omega}^{\mu}_{z\bar{z}} = \left(\bar{\mu}\omega_{\bar{z}}\right)_{\bar{z}} + \left(\mu\omega_z\right)_z$. Note that here $\phi = \omega_z \mathrm{d}z$, so

$$\mu\omega_z = g_{\bar{z}}. \tag{14}$$

Hence, using (14) and (12) in the equality (24), we get

$$\begin{aligned}\dot{\omega}^{\mu}_{z\bar{z}} &= \left(\overline{\mu\omega_z}\right)_{\bar{z}} + \left(\mu\omega_z\right)_z = \left(\overline{g_{\bar{z}}}\right)_{\bar{z}} + g_{z\bar{z}} \\ &= (g + \bar{g})_{z\bar{z}} = 2\left(\mathrm{Re}(g)\right)_{z\bar{z}}.\end{aligned}$$

From (14) we also get $i\mu\omega_z = (ig)_{\bar{z}}$. Therefore the same argument applies and we have

$$\begin{aligned}\left(\dot{\omega}^{\mu} - 2\,\mathrm{Re}(g)\right)_{z\bar{z}} &= 0 \\ \left(\dot{\omega}^{i\mu} - 2\,\mathrm{Re}(ig)\right)_{z\bar{z}} &= 0.\end{aligned}$$

Since S is compact, harmonic functions $S \to \mathbb{C}$ are constant. So, we have

$$\begin{aligned}\dot{\omega}^{\mu} &= 2\,\mathrm{Re}(g) + c_1, \\ \dot{\omega}^{i\mu} &= -2\,\mathrm{Im}(g) + c_2,\end{aligned}$$

for some constants $c_1, c_2 \in \mathbb{C}$. Thus,

$$g = \frac{1}{2}\left(\dot{\omega}^{\mu} - i\dot{\omega}^{i\mu}\right) + c,$$

for some constant $c \in \mathbb{C}$. Hence

$$2\mu\omega_z = \dot{\omega}^{\mu}_{\bar{z}} - i\dot{\omega}^{i\mu}_{\bar{z}},$$

and the claim proved. □

This is enough to see that the the equality holds in (8). Explicitly, from the claim we have (taking conjugates of the equality above)

$$\begin{aligned}2\mu\omega_z &= \dot{\omega}^{\mu}_{\bar{z}} - i\dot{\omega}^{i\mu}_{\bar{z}}, \\ 2\bar{\mu}\omega_{\bar{z}} &= \dot{\omega}^{\mu}_{z} + i\dot{\omega}^{i\mu}_{z},\end{aligned}$$

and hence

$$\begin{aligned}\int_S \mu\omega_z(\dot{\omega}^{\mu}_z + i\dot{\omega}^{i\mu}_z)\mathrm{d}x\mathrm{d}y &= 2\int_S |\mu|^2\,\omega_z\omega_{\bar{z}}\mathrm{d}x\mathrm{d}y, \\ \int_S \bar{\mu}\omega_{\bar{z}}(\dot{\omega}^{\mu}_{\bar{z}} - i\dot{\omega}^{i\mu}_{\bar{z}})\mathrm{d}x\mathrm{d}y &= 2\int_S |\mu|^2\,\omega_z\omega_{\bar{z}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

From (11) we conclude $\omega_z\omega_{\bar{z}} = \left|\omega_z\right|^2$, and it follows that

$$\int_S \mu\omega_z(\dot{\omega}_z^{\mu} + i\dot{\omega}_z^{i\mu})\mathrm{d}x\mathrm{d}y + \int_S \bar{\mu}\omega_{\bar{z}}(\dot{\omega}_{\bar{z}}^{\mu} - i\dot{\omega}_{\bar{z}}^{i\mu})\mathrm{d}x\mathrm{d}y = 4\int_S \left|\mu\right|^2 |\omega_z|^2\mathrm{d}x\mathrm{d}y.$$

Combining this with Lemma 2.3 implies that the equality holds in (8) and we are finished.

3. Proof of Lemma 2.3: Part 1

In this section we prove the identity (6) from Lemma 2.3. Throughout this proof, we consider Riemann surfaces $S^{t\mu}$ as different almost complex structures on a fixed Riemann surface S. In formula (15) we express the integrand in the expression for H_χ in terms of this almost complex structure on S. We begin with the following (general) observation. Let X be a Riemann surface. Given a field of endomorphisms $A \in \mathrm{End}(\mathbb{C}\otimes TX)$, and a (complex valued) 1-form α, i.e. a section of $\mathbb{C}\otimes T^*X$, denote by $A\alpha$ the 1-form obtained by fibrewise composition $\alpha \circ A$.

Claim 3.1. *Let $J \in \mathrm{End}(\mathbb{C}\otimes TX)$ be the complex structure on X, and $f : X \to \mathbb{R}/\mathbb{Z}$ a smooth map. Then*

$$\partial f \wedge \bar{\partial} f = \frac{i}{2}\mathrm{d}f \wedge J\mathrm{d}f.$$

Proof. We have $\partial f = \frac{1}{2}\left(\mathrm{id}_{TX} - iJ\right)\mathrm{d}f$ and $\bar{\partial} f = \frac{1}{2}\left(\mathrm{id}_{TX} + iJ\right)\mathrm{d}f$. Therefore

$$\partial f \wedge \bar{\partial} f = \frac{1}{4}(\mathrm{d}f - iJ\mathrm{d}f) \wedge (\mathrm{d}f + iJ\mathrm{d}f) = \frac{i}{2}\mathrm{d}f \wedge J\mathrm{d}f.$$

□

Consider the smooth path $S^{t\mu} \in \mathcal{T}_h$. Since W (locally) takes values in $\mathbb{R}$, we have

$$\mathrm{H}_\chi(S^{t\mu}) = \frac{i}{2}\int\limits_{S^{t\mu}} \partial W \wedge \overline{\partial W} = \frac{i}{2}\int\limits_{S^{t\mu}} \partial W \wedge \bar{\partial} W.$$

Combining this with the previous claim, we express $\mathrm{H}_\chi(S^{t\mu})$ as follows:

$$\begin{aligned}\mathrm{H}_\chi(S^{t\mu}) &= \frac{i}{2}\int\limits_{S^{t\mu}} \partial W \wedge \bar{\partial} W = -\frac{1}{4}\int\limits_{S^{t\mu}} \mathrm{d}W \wedge J\mathrm{d}W \\ &= -\frac{1}{4}\int\limits_{S} \mathrm{d}(W \circ f^{t\mu}) \wedge J^{t\mu}\mathrm{d}(W \circ f^{t\mu}),\end{aligned} \tag{15}$$

where $J^{t\mu}$ denotes the almost complex structure on S obtained as the pullback of the complex structure J from $S^{t\mu}$ by $f^{t\mu}$. Define

$$F(s,t) = -\frac{1}{4}\int\limits_{S} \mathrm{d}(W \circ f^{t\mu}) \wedge J^{s\mu}\mathrm{d}(W \circ f^{t\mu}).$$

From (15) we conclude

$$F(t,t) = \mathrm{H}_\chi(S^{t\mu}). \tag{16}$$

Note that W is harmonic on $S^{t\mu}$ and hence $W \circ f^{t\mu}$ is harmonic on $(S, J^{t\mu})$. Since the energy of a harmonic map is stationary with respect to variation of the map, we have

$$\frac{\partial F}{\partial t}(t,t) = 0. \tag{17}$$

Differentiating both sides yields

$$\frac{\partial^2 F}{\partial t^2}(t,t) + \frac{\partial^2 F}{\partial s \partial t}(t,t) = 0. \tag{18}$$

We now have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathrm{H}_\chi(S^{t\mu}) &= \frac{\partial^2 F}{\partial t^2}(t,t) + 2\frac{\partial^2 F}{\partial t \partial s}(t,t) + \frac{\partial^2 F}{\partial s^2}(t,t)\\ &= \frac{\partial^2 F}{\partial s^2}(t,t) - \frac{\partial^2 F}{\partial t^2}(t,t),\end{aligned}$$

where the second equality follows from (18). In particular, we get

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathrm{H}_\chi(S^{t\mu})\right|_{t=0} = \frac{\partial^2 F}{\partial s^2}(0,0) - \frac{\partial^2 F}{\partial t^2}(0,0). \tag{19}$$

It remains to compute $\frac{\partial^2 F}{\partial s^2}(0,0)$ and $\frac{\partial^2 F}{\partial t^2}(0,0)$. This is done in the next two subsections.

3.1. **Computing $\frac{\partial^2 F}{\partial t^2}(0,0)$.** Define the function $\ddot{\omega} : S \to \mathbb{R}$ by

$$\ddot{\omega} = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} W \circ f^{t\mu}.$$

Using Claim 3.1, for small t we get

$$\begin{aligned}F(0,t) &= -\frac{1}{4}\int_S \mathrm{d}(W \circ f^{t\mu}) \wedge J\mathrm{d}(W \circ f^{t\mu}) = \frac{i}{2}\int_S \partial(W \circ f^{t\mu}) \wedge \bar{\partial}(W \circ f^{t\mu})\\ &= \frac{i}{2}\int_S \left(\partial\omega + t\partial\dot{\omega} + \frac{t^2}{2}\partial\ddot{\omega}\right) \wedge \left(\bar{\partial}\omega + t\bar{\partial}\dot{\omega} + \frac{t^2}{2}\bar{\partial}\ddot{\omega}\right) + o(t^2),\end{aligned}$$

which yields

$$F(0,t) = \mathrm{H}_\chi(S) + \frac{it}{2}\int_S \partial\omega \wedge \bar{\partial}\dot{\omega} + \partial\dot{\omega} \wedge \bar{\partial}\omega + \frac{it^2}{2}\int_S \left(\partial\dot{\omega} \wedge \bar{\partial}\dot{\omega} + \frac{\partial\omega \wedge \bar{\partial}\ddot{\omega} + \partial\ddot{\omega} \wedge \bar{\partial}\omega}{2}\right) + o(t^2).$$

Thus,

$$\frac{\partial^2 F}{\partial t^2}(0,0) = i\int_S \partial\dot{\omega} \wedge \bar{\partial}\dot{\omega} + i\int_S \frac{\partial\omega \wedge \bar{\partial}\ddot{\omega} + \partial\ddot{\omega} \wedge \bar{\partial}\omega}{2}. \tag{20}$$

Note that $\bar{\partial}(\ddot{\omega}\partial\omega) = -\partial\omega \wedge \bar{\partial}\ddot{\omega}$ because ω is harmonic, i.e. $\bar{\partial}\partial\omega = 0$. Similarly, $\partial(\ddot{\omega}\bar{\partial}\omega) = \partial\ddot{\omega} \wedge \bar{\partial}\omega$. Using this, we get

$$\mathrm{d}(\ddot{\omega}\partial\omega) = \partial(\ddot{\omega}\partial\omega) + \bar{\partial}(\ddot{\omega}\partial\omega) = \partial\ddot{\omega} \wedge \partial\omega + \ddot{\omega}\partial^2\omega - \partial\omega \wedge \bar{\partial}\ddot{\omega}.$$

Observe that $\partial\ddot{\omega}\wedge\partial\omega = 0$ because $\mathcal{A}^{(2,0)}(S) = 0$, and $\partial^2\omega = 0$ since $\partial^2 \equiv 0$. Thus, $\mathrm{d}(\ddot{\omega}\partial\omega) = -\partial\omega\wedge\bar{\partial}\ddot{\omega}$. Likewise $\mathrm{d}(\ddot{\omega}\bar{\partial}\omega) = \partial\ddot{\omega}\wedge\bar{\partial}\omega$. So, by Stokes' theorem the second term on the right hand side in (20) vanishes. Hence

$$\frac{\partial^2 F}{\partial t^2}(0,0) = i\int_S \partial\dot{\omega}\wedge\bar{\partial}\dot{\omega} = 2\int_S \left|\dot{\omega}_z\right|^2 \mathrm{d}x\mathrm{d}y. \tag{21}$$

3.2. **Computing $\frac{\partial^2 F}{\partial s^2}(0,0)$.** It will be convenient to have an expression for J^μ in terms of μ.

Claim 3.2. *Let μ be a smooth Beltrami differential on S. Then with respect to the basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$, we have*

$$J^\mu = \frac{i}{1-|\mu|^2}\begin{pmatrix} 1+|\mu|^2 & -2\bar{\mu} \\ 2\mu & -1-|\mu|^2 \end{pmatrix}. \tag{22}$$

Proof. Denote by z some local coordinate on S and by ζ a local coordinate on S^μ so that $\zeta_{\bar{z}} = \mu\zeta_z$. Note that

$$\begin{aligned}
\frac{\partial}{\partial z} &= \zeta_z\frac{\partial}{\partial\zeta} + \overline{\zeta_{\bar{z}}}\frac{\partial}{\partial\bar{\zeta}} = \zeta_z\frac{\partial}{\partial\zeta} + \bar{\mu}\overline{\zeta_z}\frac{\partial}{\partial\bar{\zeta}} \\
\frac{\partial}{\partial \bar{z}} &= \zeta_{\bar{z}}\frac{\partial}{\partial\zeta} + \overline{\zeta_z}\frac{\partial}{\partial\bar{\zeta}} = \mu\zeta_z\frac{\partial}{\partial\zeta} + \overline{\zeta_z}\frac{\partial}{\partial\bar{\zeta}},
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{\partial}{\partial\zeta} &= \frac{1}{\zeta_z\left(1-|\mu|^2\right)}\left(\frac{\partial}{\partial z} - \bar{\mu}\frac{\partial}{\partial\bar{z}}\right) \\
\frac{\partial}{\partial\bar{\zeta}} &= \frac{1}{\overline{\zeta_z}\left(1-|\mu|^2\right)}\left(-\mu\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{z}}\right).
\end{aligned}$$

Hence note that

$$\begin{aligned}
J^\mu\left(\frac{\partial}{\partial z}\right) &= \zeta_z i\frac{\partial}{\partial\zeta} - \bar{\mu}\overline{\zeta_z}i\frac{\partial}{\partial\bar{\zeta}} \\
&= \frac{i}{1-|\mu|^2}\left(\left(1+|\mu|^2\right)\frac{\partial}{\partial z} - 2\bar{\mu}\frac{\partial}{\partial\bar{z}}\right),
\end{aligned}$$

and

$$\begin{aligned}
J^\mu\left(\frac{\partial}{\partial\bar{z}}\right) &= \mu\zeta_z i\frac{\partial}{\partial\zeta} - \overline{\zeta_z}i\frac{\partial}{\partial\bar{\zeta}} \\
&= \frac{i}{1-|\mu|^2}\left(2\mu\frac{\partial}{\partial z} - \left(1+|\mu|^2\right)\frac{\partial}{\partial\bar{z}}\right).
\end{aligned}$$

The result is shown. □

Using Claim 3.2 we see that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} J^{s\mu} = 2i|\mu|^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i|\mu|^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 4|\mu|^2 J,$$

and

$$\begin{aligned} \frac{\partial^2 F}{\partial s^2}(0,0) &= -\frac{1}{4}\int_S \mathrm{d}W \wedge \left(\left.\frac{d^2}{ds^2}\right|_{s=0} J^{s\mu} \right) \mathrm{d}W \\ &= -\frac{1}{4}\int_S 4|\mu|^2 \,\mathrm{d}W \wedge J\mathrm{d}W \\ &= 2i\int_S |\mu|^2 \,\partial\omega \wedge \bar{\partial}\omega \\ &= 4\int_S |\mu|^2 |\omega_z|^2 \,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{23}$$

The result now follows by combining (19), (21), and (23).

4. Proof of Lemma 2.3: Part 2

In this section we prove the identity (7) from Lemma 2.3. For the ease of notation, in this proof we write $\dot{\omega} = \dot{\omega}^{\mu}$. First note that $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J^{t\mu} = 2i \begin{pmatrix} 0 & -\bar{\mu} \\ \mu & 0 \end{pmatrix}$. Let $\partial^{t\mu}$ denote the pullback of the ∂ derivative from the Riemann surface $S^{t\mu}$. We have

$$\begin{aligned} \partial^{t\mu}(W \circ f^{t\mu}) &= \frac{1}{2}(\mathrm{id}_{TS} - iJ^{t\mu})\mathrm{d}(W \circ f^{t\mu}) \\ &= \frac{1}{2}\left(\mathrm{id}_{TS} - iJ + 2t \begin{pmatrix} 0 & -\bar{\mu} \\ \mu & 0 \end{pmatrix} + o(t) \right) (\mathrm{d}\omega + t\mathrm{d}\dot{\omega} + o(t)) \\ &= \partial\omega + t\left(\frac{1}{2}(\mathrm{id}_{TS} - iJ)\mathrm{d}\dot{\omega} + \begin{pmatrix} 0 & -\bar{\mu} \\ \mu & 0 \end{pmatrix} \mathrm{d}\omega \right) + o(t) \\ &= \partial\omega + t\left(\partial\dot{\omega} - \bar{\mu}\omega_{\bar{z}}\mathrm{d}z + \mu\omega_z \mathrm{d}\bar{z} \right) + o(t). \end{aligned}$$

Since $W \circ f^{t\mu}$ is harmonic on $(S, J^{t\mu})$, the form $\partial^{t\mu}(W \circ f^{t\mu})$ is closed for all t in a neighbourhood of 0. Hence $\partial\dot{\omega} - \bar{\mu}\omega_{\bar{z}}\mathrm{d}z + \mu\omega_z\mathrm{d}\bar{z}$ is also closed. Hence

$$\begin{aligned} 0 &= \mathrm{d}\left(\partial\dot{\omega} - \bar{\mu}\omega_{\bar{z}}\mathrm{d}z + \mu\omega_z\mathrm{d}\bar{z} \right) \\ &= \dot{\omega}_{z\bar{z}}\mathrm{d}\bar{z} \wedge \mathrm{d}z - (\bar{\mu}\omega_{\bar{z}})_{\bar{z}}\mathrm{d}\bar{z} \wedge \mathrm{d}z + (\mu\omega_z)_z \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \left(\dot{\omega}_{z\bar{z}} - (\bar{\mu}\omega_{\bar{z}})_{\bar{z}} - (\mu\omega_z)_z \right)\mathrm{d}\bar{z} \wedge \mathrm{d}z. \end{aligned}$$

Therefore

$$\dot{\omega}_{z\bar{z}} = (\bar{\mu}\omega_{\bar{z}})_{\bar{z}} + (\mu\omega_z)_z, \tag{24}$$

or in the coordinate-free notation

$$\bar{\partial}\partial\dot{\omega} = \bar{\partial}(\bar{\mu}\bar{\partial}\omega) - \partial(\mu\partial\omega).$$

From (11) we obtain $\left|\dot{\omega}_z\right|^2 \mathrm{d}x\mathrm{d}y = \dot{\omega}_z\dot{\omega}_{\bar{z}}\mathrm{d}x\mathrm{d}y = \frac{i}{2}\partial\dot{\omega}\wedge\bar{\partial}\dot{\omega}$. We now have

$$\begin{aligned}
\int_S \left|\dot{\omega}_z\right|^2 \mathrm{d}x\mathrm{d}y &= \frac{i}{2}\int_S \partial\dot{\omega}\wedge\bar{\partial}\dot{\omega} = -\frac{i}{2}\int_S \bar{\partial}(\dot{\omega}\partial\dot{\omega}) - \dot{\omega}\bar{\partial}\partial\dot{\omega} \\
&= \frac{i}{2}\int_S \dot{\omega}\bar{\partial}\partial\dot{\omega} = \frac{i}{2}\int_S \dot{\omega}\bar{\partial}(\bar{\mu}\bar{\partial}\omega) - \frac{i}{2}\int_S \dot{\omega}\partial(\mu\partial\omega) \\
&= -\frac{i}{2}\int_S \bar{\partial}\dot{\omega}\wedge\left(\bar{\mu}\bar{\partial}\omega\right) + \frac{i}{2}\int_S \partial\dot{\omega}\wedge\left(\mu\partial\omega\right) \\
&= \int_S \bar{\mu}\dot{\omega}_{\bar{z}}\omega_{\bar{z}}\mathrm{d}x\mathrm{d}y + \int_S \mu\omega_z\dot{\omega}_z\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

In passing from second to third line we used integration by parts on both terms.

5. Point Pushing epimorphisms

In this section we first observe the connection between the Putman-Wieland property and point pushing epimorphisms. Then we state Lemma 5.4 which rules out such epimorphisms for subgroups $\Gamma < \mathrm{Mod}_g^{n+1}$ assuming that $\Gamma \cap \pi_1(\Sigma_g^n)$ is geometrically uniform. Fix a marked point $* \in \Sigma_g^n$. Forgetting $*$ yields the Birman exact sequence

$$1 \to \pi_1(\Sigma_g^n) \to \mathrm{Mod}_g^{n+1} \to \mathrm{Mod}_g^n \to 1. \tag{25}$$

The subgroup $\pi_1(\Sigma_g^n) < \mathrm{Mod}_g^{n+1}$ is the point pushing subgroup of Mod_g^{n+1} corresponding to $*$.

Definition 5.1. *Suppose $\Gamma < \mathrm{Mod}_g^{n+1}$, and $\rho : \Gamma \to G$ an epimorphism onto some group G. We say that ρ is a point pushing epimorphism if ρ does not annihilate the group $\Gamma \cap \pi_1(\Sigma_g^n)$, where $\pi_1(\Sigma_g^n)$ is the point pushing subgroup corresponding to $*$.*

Remark 7. Suppose $K' < \pi_1(\Sigma_g^n)$ is a normal subgroup which does not have the Putman-Wieland property. Let $K < \pi_1(\Sigma_g^{n+1})$ be the pre-image of K' under the homomorphism $\pi_1(\Sigma_g^{n+1}) \to \pi_1(\Sigma_g^n)$ induced by the inclusion $\Sigma_g^{n+1} \to \Sigma_g^n$. Then K is a normal subgroup which does not have the Putman-Wieland property. Moreover, if K' is geometrically ϵ-uniform, then so is K.

The following proposition establishes the connection between the Putman-Wieland property and point pushing epimorphisms assuming $n \geq 1$.

Proposition 5.2. *Let $n \geq 1$. Suppose $K < \pi_1(\Sigma_g^n)$ is a normal subgroup. If K does not have the Putman-Wieland property, then there exists a finite index subgroup $\Gamma < \mathrm{Mod}_g^{n+1}$ which admits a point pushing epimorphism $\rho : \Gamma \to \mathbb{Z}$, such that $\Gamma \cap \pi_1(\Sigma_g^n) = K$.*

Proof. We essentially dualize the argument from Section 4 of [15]. Let $\Gamma'_K < \mathrm{Mod}_g^n$ be the image of Γ_K under the forgetful map $\mathrm{Mod}_g^{n+1} \to \mathrm{Mod}_g^n$, and recall that $\pi_1(\Sigma_g^n) < \Gamma_K$. The exact sequence (25) induces the new exact sequence

$$1 \to \pi_1(\Sigma_g^n) \to \Gamma_K \to \Gamma'_K \to 1. \tag{26}$$

Since we assume $n \geq 1$, this sequence splits, and we have $\Gamma_K \cong \pi_1(\Sigma_g^n) \rtimes \Gamma'_K$. Since K does not have the Putman-Wieland property, let $\nu \in H^1(K, \mathbb{Q})$ be a non-zero cohomology class with a finite Γ'_K orbit. Define $\Gamma = K \rtimes \mathrm{Stab}_{\Gamma'_K}(\nu) < \Gamma_K$. Then clearly $\Gamma \cap \pi_1(\Sigma_g^n) = K$, and we define

$$\rho : \Gamma \to \mathbb{Q}$$

by $\rho|_K = \nu$ and $\rho|_{\mathrm{Stab}_{\Gamma'_K}(\nu)} = 0$. Since Γ is finitely generated, we may rescale ρ to get a point-pushing epimorphism $\Gamma \to \mathbb{Z}$. □

The next proposition deals with the case $n = 0$.

Proposition 5.3. *Let $\epsilon > 0$, and suppose $K < \pi_1(\Sigma_g)$ is a normal, geometrically ϵ-uniform subgroup. If K does not have the Putman-Wieland property, then there exists a finite index subgroup $\Gamma < \mathrm{Mod}_g^2$ which admits a point pushing epimorphism $\rho : \Gamma \to \mathbb{Z}$, and such that $\Gamma \cap \pi_1(\Sigma_g^1)$ is a normal, geometrically ϵ-uniform.*

Proof. Let $K' < \pi_1(\Sigma_g^1)$ be the pre-image of K under the homomorphism $\pi_1(\Sigma_g^1) \to \pi_1(\Sigma_g)$ induced by the inclusion $\Sigma_g^1 \to \Sigma_g$. Then K' does not have the Putman-Wieland property by definition. Moreover, since K is geometrically ϵ-uniform, so is K', again by definition. We now apply the previous Proposition 5.2 to K'. □

In the following Sections 6, 7, and 8 of the paper we prove the following lemma.

Lemma 5.4. *Let $\epsilon > 0$, and $g \geq 3, n \geq 0$. If $g \geq \frac{1+2\epsilon}{2\epsilon}$, and $\Gamma < \mathrm{Mod}_g^{n+1}$ is a finite index subgroup such that $\Gamma \cap \pi_1(\Sigma_g^n)$ is a normal, geometrically ϵ-uniform subgroup, then there is no point pushing epimorphism $\rho : \Gamma \to \mathbb{Z}$.*

5.1. **Proof of Theorem 1.9.** If $n \geq 1$, the proof follows by combining Lemma 5.4 and Proposition 5.2. If $n = 0$, the proof follows by combining Lemma 5.4 and Proposition 5.3

6. Geometric properties of point pushing epimorphisms

In this and the following two sections we take $g \geq 3$, and we make the following assumptions:

(1) $\Gamma < \mathrm{Mod}_g^{n+1}$ is a finite index subgroup.
(2) Let $K = \pi_1(\Sigma_g^n) \cap \Gamma$. Then K is a normal subgroup of the point pushing group.
(3) There exists a point pushing epimorphism $\rho : \Gamma \to \mathbb{Z}$.

We may also assume that $\Gamma < \Gamma_K$ (after replacing Γ by $\Gamma \cap \Gamma_K$ if necessary). Recall that $\pi_K : \Sigma_h^m \to \Sigma_g^n$ is the covering corresponding to the subgroup K. In this section we establish basic geometric and homotopy invariance properties of a map $f_\rho : \Sigma_h \to \mathbb{R}/\mathbb{Z}$ which induces ρ.

Proposition 6.1. *Let $A \in \pi_1(\Sigma_h^m)$, and denote by $\alpha \subset \Sigma_h^m$ a closed curve which represents the conjugacy class of A in $\pi_1(\Sigma_h^m)$. If α is a lift of a simple closed curve on Σ_g^n (by the covering $\pi_K : \Sigma_h^m \to \Sigma_g^n$), then A is in the kernel of ρ_*.*

Remark 8. Proposition 6.1 holds when $g \geq 3$ (this is in fact the only place we use this assumption). When $g = 2$, we do not know if this proposition holds for every ρ. However, in Section 9 we construct such ρ for which the conclusion of the proposition holds.

Proof. Since α is a lift a simple closed curve from Σ_g^n we know that $A \in \mathrm{Mod}_g^{n+1}$ is a product of powers of Dehn twists. Bridson [1] and Putman [14] proved that $\rho(A) = 0$ for such A. □

By $\iota : \Sigma_h^m \to \Sigma_h$ we denote the inclusion map, and by $\iota_* : \pi_1(\Sigma_h^m) \to \pi_1(\Sigma_h)$ the induced map between fundamental groups.

Proposition 6.2. *There exists a unique (up to homotopy) continuous mapping $f_\rho : \Sigma_h \to \mathbb{R}/\mathbb{Z}$ such that the identity $\rho = (f_\rho)_* \circ \iota_*$ holds on $\pi_1(\Sigma_h^m)$.*

Proof. Let $g_\rho : \Sigma_h^m \to \mathbb{R}/\mathbb{Z}$ be a continuous map such that

$$(g_\rho)_* = \rho, \qquad \text{on } \pi_1(\Sigma_h^m). \tag{27}$$

The previous proposition implies that g_ρ can be continuously extended over the m punctures on Σ_h^m. Denote this extension by $f_\rho : \Sigma_h \to \mathbb{R}/\mathbb{Z}$. Then $(f_\rho)_* \circ \iota_* = (g_\rho)_*$, which together with (27) implies the equality $\rho = (f_\rho)_* \circ \iota_*$ on $\pi_1(\Sigma_h^m)$. Furthermore, the surjectivity of ι_* implies that $(f_\rho)_*$ is completely determined by ρ. In turn this shows the uniqueness of f_ρ (up to homotopy). □

6.1. **Lifting the mapping classes from Σ_g^{n+1} to Σ_h.** By $\mathcal{L} : \Gamma_K \to \mathrm{Mod}_h^{m+1}$ we denote the corresponding lift. Furthermore, we let $\mathcal{F} : \mathrm{Mod}_h^{m+1} \to \mathrm{Mod}_h$ be the forgetful map induced by the inclusion $\iota : \Sigma_h^{m+1} \to \Sigma_h$. Define $\mathcal{B} : \Gamma_K \to \mathrm{Mod}_h$ by letting $\mathcal{B} = \mathcal{F} \circ \mathcal{L}$. Since $\Gamma < \Gamma_K$, the lift

$$\mathcal{B} : \Gamma \to \mathrm{Mod}_h$$

is well defined on Γ. For simplicity of the notation we let $\mathcal{B}(C) = C_{\mathcal{B}}$ for every $C \in \Gamma_K$.

Proposition 6.3. *The map $f_\rho : \Sigma_h \to \mathbb{R}/\mathbb{Z}$ is homotopic to the map $(f_\rho \circ C_{\mathcal{B}}) : \Sigma_h \to \mathbb{R}/\mathbb{Z}$ for every $C \in \Gamma$.*

Proof. Fix $C \in \Gamma$, and let $A \in \pi_1(\Sigma_h^m)$. Note that $C \circ A \circ C^{-1} \in \pi_1(\Sigma_h^m)$ because $\pi_1(\Sigma_h^m) < \mathrm{Mod}_h^{m+1}$ is a normal subgroup. Furthermore, the equality

$$\rho(A) = \rho(C \circ A \circ C^{-1}) \tag{28}$$

stands because $\mathbb{Z}$ is an Abelian group. On the other hand, $C \circ A \circ C^{-1}$ and $(\mathcal{L}(C))_*(A)$ are conjugate to each other in $\pi_1(\Sigma_h^m)$. Thus,

$$\rho(C \circ A \circ C^{-1}) = \rho\big((\mathcal{L}(C))_*(A)\big) = (\rho \circ (\mathcal{L}(C))_*)(A),$$

because $\mathbb{Z}$ is Abelian. Putting this together with (28) yields the identity

$$\rho(A) = (\rho \circ (\mathcal{L}(C)_*))(A), \qquad \forall A \in \pi_1(\Sigma_h^m).$$

Replacing ρ by $(f_\rho)_* \circ \iota_*$ yields

$$(f_\rho)_*(\iota_*(A)) = (f_\rho \circ C_{\mathcal{B}})_*(\iota_*(A)), \qquad \forall A \in \pi_1(\Sigma_h^m).$$

Therefore, the induced maps $(f_\rho)_*$ and $(f_\rho \circ C_{\mathcal{B}})_*$ agree on $\pi_1(\Sigma_h)$, and we conclude that f_ρ and $f_\rho \circ C_{\mathcal{B}}$ are homotopic to each other for every $C \in \Gamma$.

□

6.2. **The map f_ρ is K-Lipschitz.** The covering $\pi_K : \Sigma_h^m \to \Sigma_g^n$ induces the holomorphic map

$$\sigma_K : \mathcal{T}_{g,n} \to \mathcal{T}_h. \tag{29}$$

The map σ_K is the composition of the holomorphic embedding $\pi_K^* : \mathcal{T}_{g,n} \to \mathcal{T}_{h,m}$ induced by π_K, and the forgetful map $\mathcal{T}_{h,m} \to \mathcal{T}_h$.

Proposition 6.4. *There exists a constant $L > 0$ with the following property. Let $X \in \mathcal{T}_{g,n}$, and set $Z = \sigma_K(X) \in \mathcal{T}_h$. Then there exists a L-Lipschitz map $f : Z \to \mathbb{R}/\mathbb{Z}$ homotopic to $f_\rho : Z \to \mathbb{R}/\mathbb{Z}$.*

Proof. Suppose the statement of the proposition fails. Let $X_1, X_2, ... \in \mathcal{T}_{g,n}$ be a sequence such that, setting $Z_j = \sigma_K(X_j)$, the least possible Lipschitz constant of a map $Z \to \mathbb{R}/\mathbb{Z}$ homotopic to f_ρ diverges.

We first restrict the way in which X_j could diverge in $\mathcal{T}_{g,n}$. We say that a topological decomposition $\mathcal{P}$ into pairs of pants is a b-decomposition if the hyperbolic length of every geodesic cuff of $\mathcal{P}$ on X is at most b. Bers proved that there exists a constant $b_{g,n}$ such that any $X \in \mathcal{T}_{g,n}$ admits a $b_{g,n}$-decomposition (the marked points are cuffs of length zero). Let such a decomposition for X_j be $\mathcal{P}_j$.

Note that Mod_g^n acts transitively on the set of isotopy classes of pants decompositions of Σ_g^n, so since Γ has finite index in Mod_g^{n+1}, we may pass to a subsequence of $(X_j : j \geq 1)$ such that there exists a sequence $(\gamma_j \in \Gamma : j \geq 2)$ with $\gamma_j \mathcal{P}_j = \mathcal{P}_1$. By Proposition 6.3, the action of $\mathcal{B}(\Gamma)$ on any Z_j preserves the homotopy class of f_ρ, so we may replace X_j with $\gamma_j^{-1} \cdot X_j$ for $j \geq 2$. We are now in the following situation:

(1) there is a topological pants decomposition $\mathcal{P} := \mathcal{P}_1$ such that $\mathcal{P}$ is a $b_{g,n}$-decomposition for each X_j, and
(2) the least possible Lipschitz constant of maps $Z_j = \sigma_K(X_j) \to \mathbb{R}/\mathbb{Z}$ in the homotopy class of f_ρ tends to infinity.

We let $Q^1, Q^2, ..., Q^k$ be the connected components of the lifts to Σ_h^m of pairs of pants in $\mathcal{P}$ via π_K. Note that Q^i may have positive genus. Let Z_j^i be the subsurface of Z_j with geodesic boundary isotopic to Q^i.

We observe that, by construction, any closed curve on Z_j^i that is not homotopic into ∂Z_j^i projects to a curve on some pair of pants P that is not homotopic into ∂P. By basic hyperbolic geometry, there exists a universal constant $c_{g,n}$, so that on a hyperbolic pair of pants P with cuff lengths at most $b_{g,n}$, any closed geodesic not

homotopic into ∂P has length at least $c_{g,n}$. It follows that any closed curve on Z_j^i not homotopic into ∂Z_j^i has length at least $c_{g,n}$.

Let the markings $f_j : \Sigma_h^m \to Z_j$ correspond to the point $Z_j \in \mathcal{T}_{g,n}$. We may modify them by an isotopy so that $f_j(Q^i) = Z_j^i$. Denote by g_j the hyperbolic metric on Z_j. From the last claim in the previous paragraph, it follows that, after possibly applying a further isotopy, the metrics $f_j^* g_j$ converge uniformly on compact subsets of $(Q^i)^\circ = Q^i \setminus \partial Q^i$ to some metric g^i. This metric makes $(Q^i)^\circ$ into a hyperbolic surface with geodesic boundary, and possibly with cusps.

We are now ready to finish the proof. We modify $f_\rho : \Sigma_h^m \to \mathbb{R}/\mathbb{Z}$ by a homotopy so that it is locally constant in a small neighbourhood of $\bigcup_i \partial Q^i$. This is possible since connected components of ∂Q^i are precisely lifts of cuffs in $\mathcal{P}$, so by Proposition 6.1, $(f_\rho)_*$ annihilates all curves in $\bigcup_i \partial Q^i$. But then functions $f_\rho : (Q^i, g^i) \to \mathbb{R}/\mathbb{Z}$ have bounded Lipschitz constants by compactness. Since $f_j^* g_j$ converge to g^i uniformly on compact subsets of $(Q^i)^\circ$, it follows that $f_\rho \circ f_j^{-1} : Z_j \to \mathbb{R}/\mathbb{Z}$ have uniformly bounded Lipschitz constants. This is a contradiction. □

7. The Hodge norm of an epimorphism ρ

Set $\chi = (f_\rho)_*$. Then $\chi : \pi_1(\Sigma_h) \to \mathbb{Z}$ is an epimorphism. By $\mathrm{H}_\chi : \mathcal{T}_h \to \mathbb{R}$ we denote the corresponding Hodge norm of χ. The first goal of this section is to prove that the composition $\mathrm{H}_\chi \circ \sigma_K : \mathcal{T}_{g,n} \to \mathbb{R}$ is constant (the holomorphic map σ_K was defined above (29)).

Proposition 7.1. *The function* H_χ *is invariant under the subgroup* $\mathcal{B}(\Gamma) < \mathrm{Mod}_h$ *(here we use the identification* $\mathrm{Mod}_h \approx Aut(\mathcal{T}_h)$*). That is,*

$$\mathrm{H}_\chi = \mathrm{H}_\chi \circ C_{\mathcal{B}}, \qquad \forall C \in \Gamma. \tag{30}$$

Proof. This follows from Proposition 6.3 which implies that $\omega : S \to \mathbb{R}/\mathbb{Z}$ and $\omega \circ C_{\mathcal{B}} : S \to \mathbb{R}/\mathbb{Z}$ are homotopic to each other for every $S \in \mathcal{T}_h$, and every $C \in \Gamma$. Therefore $\mathrm{H}_\chi(S) = \mathrm{H}_\chi(C_{\mathcal{B}}(S))$ for every $S \in \mathcal{T}_h$. □

Observe that the slice $\sigma_K(\mathcal{T}_{g,n})$ is invariant under the action of the group $\mathcal{B}(\Gamma)$. Furthermore, the quotient $\sigma_K(\mathcal{T}_{g,n})/\mathcal{B}(\Gamma)$ is biholomorphic to $\mathcal{M}_{g,n}^{\Gamma'}$, where $\Gamma' < \mathrm{Mod}_g^n$ is the image of Γ under the forgetful map $\mathrm{Mod}_g^{n+1} \to \mathrm{Mod}_g^n$. Here $\mathcal{M}_{g,n}^{\Gamma'}$ is the finite (étale) covering of the Moduli space $\mathcal{M}_{g,n}$ corresponding to the subgroup Γ'.

Proposition 7.2. *The restriction of the function* H_χ *to the slice* $\sigma_K(\mathcal{T}_{g,n}) \subset \mathcal{T}_h$ *is constant.*

Proof. Let $X \in \mathcal{T}_{g,n}$, and set $Z = \sigma_K(X) \in \mathcal{T}_h$. From Proposition 6.4 we know that there exists a L-Lipschitz map $f : Z \to \mathbb{R}/\mathbb{Z}$ homotopic to f_ρ. Thus, the energy of f is bounded above by $(4h-4)\pi L^2$. Since harmonic maps minimize the energy in their homotopy class, we find that

$$\mathrm{H}_\chi(Z) \leq (4h-4)\pi L^2.$$

Therefore, H_χ is bounded on this slice.

On the other hand, the composition $\mathrm{H}_\chi \circ \sigma_K$ is plurisubharmonic function on $\mathcal{T}_{g,n}$ (because precomposing a plurisubharmonic function with a holomorphic function produces a plurisubharmonic function). Invoking the invariance formula (30) from Proposition 7.1 shows that the composition $\mathrm{H}_\chi \circ \sigma_K$ gives a well defined function on $\mathcal{M}_{g,n}^{\Gamma'}$. That is, we have constructed the function

$$\mathrm{H}_\chi \circ \sigma_K : \mathcal{M}_{g,n}^{\Gamma'} \to \mathbb{R}$$

which is plurisubharmonic and bounded on $\mathcal{M}_{g,n}^{\Gamma'}$. Since $\mathcal{M}_{g,n}^{\Gamma'}$ is a quasi-projective variety we conclude that $\mathrm{H}_\chi \circ \sigma_K$ is constant. This proves the proposition. □

Proposition 7.3. *Let $X \in \mathcal{T}_{g,n}$, and let ν be a Beltrami differential on X. Set $\sigma_K(X) = Z \in \mathcal{T}_h$, and denote by μ the lift of ν to Z. Then $\mathrm{H}_\chi : \mathcal{T}_h \to \mathbb{R}$ is not strictly plurisubharmonic at Z in the complex direction determined by μ. In particular, for any Abelian differential ψ on Z we have*

$$\int_Z \mu\phi \wedge \psi = 0. \tag{31}$$

Proof. The complex disc $Z^{\lambda\mu}$ (for $|\lambda|$ small) is contained in the slice $\sigma_K(\mathcal{T}_{g,n})$. Since H_χ is constant on this slice, it follows that H_χ is not strictly plurisubharmonic in the complex direction determined by μ. The second part of the proposition follows from Corollary 1.5.

□

8. Proof of Lemma 5.4

The proof is by contradiction. Assume that there exists a point pushing epimorphism $\rho : \Gamma \to \mathbb{Z}$ (we remind the reader that $K = \Gamma \cap \pi_1(\Sigma_g^n)$ is geometrically ϵ-uniform). We then show that (31) can not hold. This is a contradiction which proves the lemma.

Choose $X \in \mathcal{T}_{g,n}$ so that $\lambda_1(Z) \geq \epsilon$ (we can do this by the assumption that K is geometrically ϵ-uniform). Set $\sigma_K(X) = Z$, and let $\delta(Z)$ denote the gonality of Z (the smallest possible degree of a non-trivial meromorphic function on Z). Li-Yau [17] proved the following inequality

$$2\lambda_1(Z)(h-1) \leq \delta(Z). \tag{32}$$

Next, by d we denote the degree of the map $\pi_K : Z \to X$. For a regular value $p \in X$, let $\pi_K^{-1}(p) = \{q_1, ..., q_d\} \subset Z$. Furthermore, we can find such p so that $\phi(q_d) \neq 0$. Let D, D_1, be the divisors on Z given by

$$D = q_1 + \cdots + q_d, \qquad D_1 = q_1 + \cdots + q_{d-1}.$$

Claim 8.1. *If $g > \frac{1+2\epsilon}{2\epsilon}$, then there exists an Abelian differential ψ on Z such that*

$$\psi(q_j) = 0, \quad 1 \leq j \leq d-1,$$

$$\psi(q_d) \neq 0.$$

Proof. The Riemann-Hurwitz formula yields the estimate $d \leq \frac{h-1}{g-1}$. Combining this with the assumption $g > \frac{1+2\epsilon}{2\epsilon}$, we get $d < 2\epsilon(h-1)$. Together with (32) this gives

$$\delta(Z) > d.$$

The Riemann-Roch theorem gives the equalities

$$\dim L(-D) = d + 1 - h + \dim \Omega(D),$$
$$\dim L(-D_1) = (d-1) + 1 - h + \dim \Omega(D_1),$$

where $L(-D)$ is the space of meromorphic functions that have at most single poles at the points of D, and $\Omega(D)$ the space of Abelian differentials that are equal to zero at every point of D (and likewise for D_1). Because $\delta(Z) > d$, we have $\dim L(-D) = L(-D_1) = 0$, so $\dim \Omega(D_1) = 1 + \dim \Omega(D)$. Thus, there exists $\psi \in \Omega(D_1)$ with the stated properties. □

Let $\Theta : \mathrm{QD}(Z) \to \mathrm{QD}(X)$ be the linear map induced by π_K (here QD denotes the space of holomorphic quadratic differentials). For $\varphi \in \mathrm{QD}(Z)$, the value $\Theta(\varphi)(p)$ is obtained by summing up the values of φ over the points from the set $\pi_K^{-1}(p)$.

Set $\alpha = \Theta(\phi\psi)$. Since $\phi(q_d) \neq 0$, from the previous claim we conclude that $\alpha(p) \neq 0$. Thus, $\alpha \not\equiv 0$. Define the Beltrami dilatation

$$\nu = \frac{\overline{\alpha}}{|\alpha|}.$$

Then

$$\frac{1}{d}\int_Z \mu\phi \wedge \psi = \int_X \nu\Theta(\phi\psi)\, dxdy = \int_X \nu\alpha\, dxdy = \int_X |\alpha|\, dxdy > 0.$$

This contradicts (31) and we are finished.

9. Proof of Theorem 1.2 and Theorem 1.3

Recall that given a finite index subgroup $\Gamma < \mathrm{Mod}_g^1$, by $\mathcal{M}_{g,1}^\Gamma$ we denote the finite (étale) covering of $\mathcal{M}_{g,1}$ corresponding to Γ. Then

$$\Gamma \approx \pi_1\left(\mathcal{M}_{2,1}^\Gamma\right). \tag{33}$$

Denote by $K = \pi_1(\Sigma_g) \cap \Gamma$ the corresponding subgroup of the point pushing subgroup $\pi_1(\Sigma_g)$, and by $\pi_K : \Sigma_h \to \Sigma_g$ the corresponding unbranched covering. The following lemma essentially follows from [12] (see Lemma 2.2 in [12]).

Lemma 9.1. *There exists a characteristic finite index subgroup $\Gamma < \mathrm{Mod}_2^1$, a closed Riemann surface Y of genus two, and a holomorphic map $H : \mathcal{M}_{2,1}^\Gamma \to Y$, with the following properties. Let $H_* : \Gamma \to \pi_1(Y)$ denote the homomorphism which is induced by*

$$H_* : \pi_1\left(\mathcal{M}_{2,1}^\Gamma\right) \to \pi_1(Y)$$

via the identification (33). Then

(1) *The homomorphism $H_* : \Gamma \to \pi_1(Y)$ (induced by the identification (33) is a point pushing epimorphism.*

(2) *Let $A \in \pi_1(\Sigma_h)$, and denote by $\alpha \subset \Sigma_h$ a closed curve which represents the conjugacy class of A in $\pi_1(\Sigma_h)$. If α is a lift of a simple closed curve from Σ_2 (by the covering $\pi_K : \Sigma_h \to \Sigma_2$), then A is in the kernel of H_*.*

Proof. In Section 2 of [12] (see the proof of Lemma 2.2 in [12], and also [3]), we constructed a characteristic subgroup $\Gamma < \mathrm{Mod}_2^1$, a closed Riemann surface Y of genus two, and a holomorphic map $H : \mathcal{M}_{2,1}^{\Gamma} \to Y$, satisfying the first property of the lemma. We show below that any such H must automatically satisfy the second property as well.

Remark 9. In fact, the cover $\pi_K : \Sigma_h \to \Sigma_2$ constructed in [12] is of degree 648 which forces $h = 649$. But this cover may not be characteristic, so we pass to a further cover to arrange this.

Consider the fibration map $F : \mathcal{M}_{2,1}^{\Gamma} \to \mathcal{M}_2$ obtained by composing the covering $\mathcal{M}_{2,1}^{\Gamma} \to \mathcal{M}_{2,1}$, and the forgetful map $\mathcal{M}_{2,1} \to \mathcal{M}_2$. If $\Gamma < \mathrm{Mod}_2^1$ is a characteristic subgroup, then $F^{-1}(X)$ is connected and biholomorphic to the Riemann surface $\widetilde{X}$, where $\widetilde{X} \in \mathcal{M}_h$ denotes the unbranched covering of X induced by $\pi_K : \Sigma_h \to \Sigma_2$. Moreover, any choice of marking of X yields the marking of $\widetilde{X}$ through the covering π_K. In other words, if X and $\widetilde{X}$ are endowed with markings, then $X \in \mathcal{T}_2$, and $\widetilde{X} = \sigma_K(X)$.

Let $\beta \subset \Sigma_2$ denote a simple closed curve and $\alpha \subset \Sigma_h$ a simple closed curve such that $\pi_K(\alpha) = \beta$. Let X be a marked Riemann surface of genus two, and denote by $\beta(X)$ the closed geodesic homotopic to β (using a chosen marking $\Sigma_2 \to X$). Then $\widetilde{X}$ is marked by Σ_h, and we let $\alpha(\widetilde{X}) \subset \widetilde{X}$ denote the closed geodesic homotopic to α (the geodesics are taken with respect to the underlying hyperbolic metrics).

Suppose that the degree of the covering $\pi_K : \Sigma_h \to \Sigma_2$ is equal to D. Then

$$|\alpha(\widetilde{X})| \leq D|\beta(X)|, \tag{34}$$

where the absolute values denote the lengths of the corresponding geodesics. On the other hand, the restriction $H : \widetilde{X} \to Y$ is a holomorphic map and therefore a contraction (with respect to the hyperbolic metrics on $\widetilde{X}$ and Y respectively), and we get

$$|H(\alpha(\widetilde{X}))| \leq |\alpha(\widetilde{X})|. \tag{35}$$

Putting (34) and (35) together yields the estimate

$$|H(\alpha(\widetilde{X}))| \leq D|\beta(X)|. \tag{36}$$

Denote by $\mathrm{sys}(Y)$ the length of the shortest geodesic on Y. Since we can choose X to be any marked Riemann surface of genus two, we choose X so that

$$|\beta(X)| < \frac{\mathrm{sys}(Y)}{D}.$$

Combining this with (36) shows that $H(\alpha(\widetilde{X}))$ is homotopically trivial on Y, which implies that $H_*(A) = \mathrm{Id}$ in $\pi_1(Y)$. The lemma is proved.

□

We are now ready to prove Theorem 1.2 and Theorem 1.3. Composing H_* with some epimorphism $\pi_1(Y) \to \mathbb{Z}$ produces the required point pushing homomorphism $\rho : \Gamma \to \mathbb{Z}$. Let $f_\rho : \Sigma_h \to \mathbb{R}/\mathbb{Z}$ be the continuous map from Proposition 6.2, and set $\chi = (f_\rho)_*$ as before.

Remark 10. By Lemma 9.1, the statement of the Proposition 6.1 holds for such a ρ. Since proving Proposition 6.1 is the only place in Sections 6,7,8 where we used the standing assumption $g \geq 3$, we may now apply those same results to this ρ.

By Proposition 7.2 the function H_χ is constant on the slice $\sigma_K(\mathcal{T}_2)$. Thus, by Proposition 7.3 the function H_χ is not strictly plurisubharmonic at points from $\sigma_K(\mathcal{T}_2) \subset \mathcal{T}_h$. This proves Theorem 1.2.

By Proposition 6.1 if $A \in \pi_1(\Sigma_h) < \pi_1(\Sigma_2)$ represents a simple closed curve, then A lies in the kernel of χ. Thus, the subgroup of $H_1(\Sigma_h, \mathbb{Q})$ generated by lifts of simple curves from Σ_2 lies in the kernel of χ. This proves Theorem 1.3.

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