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DERIVATIVE OF THE RIEMANN-HILBERT MAP

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Abstract. Given a pair (X, ∇), consisting of a closed Riemann surface X and a holomorphic connection ∇ on the trivial principal bundle $X \times \mathrm{SL}_2(\mathbb{C}) \to X$, the Riemann–Hilbert map sends (X, ∇) to its monodromy representation. We compute the derivative of this map, and provide a simple description of the locus where it is injective, recovering in the process several previously obtained results.

1. Introduction

By Σ_g we denote a closed topological surface of genus g, and by $\mathcal{T}_g$ the Teichmüller space of marked closed Riemann surfaces of genus g. Given $X \in \mathcal{T}_g$, we let $\Omega^1(X)$ denote the space of Abelian differentials on X. By $\mathfrak{sl}_2(\Omega^1(X))$ we denote the space of traceless 2×2 matrices of holomorphic 1-forms on X.

Fix $A \in \mathfrak{sl}_2(\Omega^1(X))$, and consider the trivial holomorphic principal bundle $\underline{\mathrm{SL}_2(\mathbb{C})} := X \times \mathrm{SL}_2(\mathbb{C}) \to X$ equipped with the connection $\nabla = d + A$. This is a flat holomorphic connection over a marked Riemann surface X inducing the monodromy representation

$$\rho_A : \pi_1(\Sigma_g) \to \mathrm{SL}_2(\mathbb{C}).$$

The space of $\mathfrak{sl}_2$-systems over Σ_g is defined as

$$\mathrm{Syst}_g = \{(X, A) : X \in \mathcal{T}_g,\ A \in \mathfrak{sl}_2(\Omega^1(X)),\ \rho_A \text{ is irreducible}\} /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

where we identify pairs (X, A) and (X, B) if A and B are conjugated by and element of $\mathrm{SL}_2(\mathbb{C})$. The Riemann–Hilbert map

$$\mathbf{M} : \mathrm{Syst}_g \to \mathrm{Rep}_g$$

is defined by $\mathbf{M}(X, A) = \rho_A$. Here

$$\mathrm{Rep}_g = \{\rho \in \mathrm{Hom}\big(\pi_1(\Sigma_g) \to \mathrm{SL}_2(\mathbb{C})\big) : \rho \text{ is irreducible}\} /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

is the character variety of irreducible representations of the surface group $\pi_1(\Sigma_g)$.

It is well known that $\mathbf{M}$ is a holomorphic map between connected complex manifold of complex dimension $6g - 6$ (see [1] and [2]). Our main result is Theorem 2 below is the explicit computation of the derivative $D\mathbf{M}$. But first, we state its main corollary characterising the pairs (X, A) where the

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derivative of $\mathbf{M}$ is injective (and hence bijective), answering a well known question (see [2], and also [4] for the broader picture).

Denote by $\mathrm{QD}(X)$ the space of holomorphic quadratic differentials on X. Given $\alpha, \beta, \gamma \in \Omega^1(X)$, we define the subspace $\mathrm{QD}(\alpha,\beta,\gamma) \leq \mathrm{QD}(X)$ by

$$\mathrm{QD}(\alpha,\beta,\gamma) = \{\alpha\varphi_1 + \beta\varphi_2 + \gamma\varphi_3 : \varphi_1, \varphi_2, \varphi_3 \in \Omega^1(X)\}.$$

Theorem 1. *Let $(X, A) \in \mathrm{Syst}_g$, and write*

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

for $\alpha, \beta, \gamma \in \Omega^1(X)$. The derivative $D_{(X,A)}\mathbf{M}$ is injective if and only if

$$\mathrm{QD}(\alpha,\beta,\gamma) = \mathrm{QD}(X). \tag{1}$$

Remark 1. This theorem has a number of applications which we prove in the last section. In particular, we show how it can be used to recover theorems by Biswas-Dumitrescu [1], Calsamiglia-Deroin-Heu-Loray [2]. In Proposition 3.1 in [1], Biswas-Dumitrescu give a sufficient condition on the pair (X, A) for the derivative $D_{(X,A)}\mathbf{M}$ to be injective. This condition can be viewed as a cohomological version of the equality (1).

Before stating Theorem 2 we describe our parameterisation of the tangent spaces to Syst_g and Rep_g.

1.1. Beltrami differentials and tamed matrix forms.

Let $\mathrm{Belt}(X)$ denote the vector space of smooth Beltrami differentials on X. Two differentials $\mu, \nu \in \mathrm{Belt}(X)$ are equivalent if

$$\int_X \mu\phi = \int_X \nu\phi$$

for every holomorphic quadratic differential $\phi \in \mathrm{QD}(X)$. This is a linear equivalence relation, that we denote $\sim$, and the quotient vector space $\mathrm{Belt}(X)/\sim$ is naturally isomorphic to the tangent space of the Teichmüller space $T_X\mathcal{T}_g$.

Next, we define the notion of a tame matrix valued form.

Definition 1. Let $A \in \mathfrak{sl}_2(\Omega^1(X))$, and $\mu \in \mathrm{Belt}(X)$. We say that $\dot{A}$ is (μ, A)-tamed form if $\dot{A}$ a closed $\mathfrak{sl}_2(\mathbb{C})$-valued 1-form whose $(0,1)$ part satisfies the equality $\dot{A}^{0,1} = \mu A$.

The following lemma is elementary and its proof is left to the reader.

Lemma 1. *Fix $A \in \mathfrak{sl}_2(\Omega^1(X))$. Let $\mu, \nu \in \mathrm{Belt}(X)$, and suppose $\dot{A}$ and $\dot{B}$ are (μ, A) and (ν, A) tamed respectively. If $\mu \sim \nu$ then there exists a smooth function $T : X \to \mathfrak{sl}_2(\mathbb{C})$ such that*

$$\dot{A} - \dot{B} - dT \in \mathfrak{sl}_2(\Omega^1(X)).$$

1.2. **Parametrisation of the tangent space to** Syst_g**.** Fix $(X, A) \in \mathrm{Syst}_g$, and define the vector space

$$Z(X, A) = \{(\mu, \dot{A}) : \mu \in \mathrm{Belt}(X),\ \dot{A} \text{ is } (\mu, A)\text{-tamed}\},$$

That is, $Z(X, A)$ is a set of pairs $(\mu, \dot{A})$ where μ is a smooth Beltrami form on X, and $\dot{A}$ a closed $\mathfrak{sl}_2(\mathbb{C})$-valued 1-form satisfying the equality $\dot{A}^{0,1} = \mu A$. We define the linear equivalence relation on $Z(X, A)$ by letting $(\mu, \dot{A}) \sim (\nu, \dot{B})$ if the following two conditions are satisfied

(1) $\mu \sim \nu$,
(2) $\dot{A}^{1,0} - \dot{B}^{1,0} - \partial T \in [A, \mathfrak{sl}_2(\mathbb{C})]$, where $T : X \to \mathfrak{sl}_2(\mathbb{C})$ is the function from Lemma 1.

Remark 2. Whenever $\mu \sim \nu$, we have $\dot{A}^{1,0} - \dot{B}^{1,0} - \partial T \in \mathfrak{sl}_2(\Omega^1(X))$. For the pairs $(\mu, \dot{A})$ and $(\nu, \dot{B})$ to be equivalent we require this element to live in $[A, \mathfrak{sl}_2(\mathbb{C})] \leq \mathfrak{sl}_2(\Omega^1(X))$.

It will be shown in Lemma 5 that the vector space $Z(X, A)/\sim$ is naturally isomorphic to $T_{(X,A)}\mathrm{Syst}_g$.

1.3. **Derivative of the monodromy map M.** The tangent space $T_\rho\mathrm{Rep}_g$ is well-known to coincide with $H^1(\mathrm{Ad}_\rho)$, where Ad_ρ denotes the $\pi_1(\Sigma_g)$-module with underlying vector space $\mathfrak{sl}_2(\mathbb{C})$ and the action $(\gamma, T) \to \rho(\gamma)T\rho(\gamma)^{-1}$. Since Ad_ρ is precisely the monodromy of the flat connection

$$d_A := d + \mathrm{ad}(A)$$

on the bundle $\underline{\mathfrak{sl}_2(\mathbb{C})} := X \times \mathfrak{sl}_2(\mathbb{C}) \to X$, there is a de Rham isomorphism

$$\iota : H^1(X, \mathcal{E}) \to H^1(\mathrm{Ad}_\rho). \tag{2}$$

Here $\mathcal{E}$ denotes the sheaf of d_A-flat sections of the bundle $\underline{\mathfrak{sl}_2(\mathbb{C})}$. We are now ready to state our main result.

Theorem 2. *Let* $(X, A) \in \mathrm{Syst}_g$ *and* $(\mu, \dot{A}) \in Z(X, A)$. *Then* $\dot{A}$ *is a* d_A*-closed* $\mathfrak{sl}_2(\mathbb{C})$*-valued 1-form, and hence defines a cohomology class* $\chi \in H^1(X, \mathcal{E})$. *Moreover,* $D\mathbf{M}\big([(\mu, \dot{A})]\big) = -\iota(\chi)$, *where* $[(\mu, \dot{A})] \in Z(X, A)/\sim$ *denotes the corresponding tangent vector under the identification* $T_{(X,A)}\mathrm{Syst}_g \approx Z(X, A)/\sim$.

1.4. **Organisation.** Theorem 1 is proved in Section 2. In Section 3 we describe the tangent space to Syst_g and prove that $T_{(X,A)}\mathrm{Syst}_g$ is isomorphic to $Z(X, A)/\sim$. In Section 4 we recall the description of the tangent space of Rep_g and construct the de Rham isomorphism (2). In Section 5 we prove Theorem 2. Applications of Theorem 1 are derived in Section 6.

2. The kernel of $D\mathbf{M}$

Fix $(X, A) \in \mathrm{Syst}_g$, and consider a tangent vector $(\mu, \dot{A}) \in Z(X, A)$. The following three lemmas easily follow from Theorem 2, and the isomorphism $T_{(X,A)}\mathrm{Syst}_g \approx Z(X, A)/\sim$. We then use these lemmas to prove Theorem 1.

Lemma 2. *Let $(\mu, \dot{A}) \in Z(X, A)$. The derivative $D\mathbf{M}(\mu, \dot{A})$ vanishes if and only if there exists a smooth function $\dot{F} : X \to \mathfrak{sl}_2(\mathbb{C})$ such that*

$$\mu A = \bar{\partial}\dot{F} \quad \textit{and} \quad \dot{A} = \partial\dot{F} + [A, \dot{F}] + \mu A. \tag{3}$$

Proof. By Theorem 2 the derivative $D\mathbf{M}(\mu, \dot{A})$ vanishes if and only if $\chi = [\dot{A}] \in H^1(X, \mathcal{E})$ vanishes, that is, if $\dot{A}$ is d_A-exact. This means that $\dot{A} = d_A\dot{F} = d\dot{F} + [A, \dot{F}]$, for some smooth function $\dot{F} : X \to \mathfrak{sl}_2(\mathbb{C})$. Splitting this equality into $(0, 1)$ and $(1, 0)$ parts, and using that $\dot{A}^{0,1} = \mu A$, we derive (3).

□

Lemma 3. *Suppose μ is a Beltrami differential on X. Then there exists $\dot{A}$ such that $D\mathbf{M}(\mu, \dot{A}) = 0$ if and only if the 1-forms $\mu\alpha, \mu\beta, \mu\gamma$ are all $\bar{\partial}$-exact, where*

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

Proof. If such $\dot{A}$ exists then by Lemma 2 there exists $\dot{F} : X \to \mathfrak{sl}_2(\mathbb{C})$ satisfying the system (3). In particular, the equality $\mu A = \bar{\partial}\dot{F}$ holds implying that the 1-forms $\mu\alpha, \mu\beta, \mu\gamma$ are all $\bar{\partial}$-exact.

Suppose now that $\mu\alpha, \mu\beta, \mu\gamma$ are all $\bar{\partial}$-exact. Then we can construct $\dot{F} : X \to \mathfrak{sl}_2(\mathbb{C})$ such that $\mu A = \bar{\partial}\dot{F}$. Set $\dot{A} = \mu A + \partial\dot{F} + [A, \dot{F}]$. One can now verify that $(\mu, \dot{A})$ solves the system (3). □

Lemma 4. *Suppose μ is zero in $T_X\mathcal{T}_g$. Then $D\mathbf{M}(\mu, \dot{A}) = 0$ if and only if the pair $(\mu, \dot{A})$ represents the zero vector in $T_{(X,A)}\mathrm{Syst}_g$.*

Proof. Suppose that $D\mathbf{M}(\mu, \dot{A}) = 0$. We need to prove that $(\mu, \dot{A})$ represents the zero vector in $T_{(X,A)}\mathrm{Syst}_g$. Since μ is zero in $T_X\mathcal{T}_g$, we may assume $\mu = 0$. Let $\dot{F} : X \to \mathfrak{sl}_2(\mathbb{C})$ be the function from Lemma 2. Then $\bar{\partial}\dot{F} = 0$, and hence $\dot{F} : X \to \mathfrak{sl}_2(\mathbb{C})$ is holomorphic. Therefore $\dot{F}$ is constant, and by the second equality in (3) we obtain $\dot{A} = \dot{A}^{1,0} = [A, \dot{F}]$. But then $\dot{A} \in [A, \mathfrak{sl}_2(\mathbb{C})]$, and so $(0, \dot{A})$ vanishes in $T_{(X,A)}\mathrm{Syst}_g$. □

2.1. **Proof of Theorem 1.** We are ready to complete the proof of Theorem 1. Let μ denote a Beltrami differential on X. Then by Lemma 3 and Lemma 4 there exists $\dot{A}$ such that $(\mu, \dot{A})$ is a non-zero vector with $D\mathbf{M}(\mu, \dot{A}) = 0$ if and only if μ is not equal to zero in $T_X\mathcal{T}_g$, and the 1-forms $\mu\alpha, \mu\beta, \mu\gamma$ are all $\bar{\partial}$-exact. By Serre duality, the last condition is equivalent to

$$\int_X \mu\phi = 0$$

for every $\phi \in \alpha\Omega^1(X)+\beta\Omega^1(X)+\gamma\Omega^1(X) \leq \mathrm{QD}(X)$. Thus, $D\mathbf{M}$ is injective if and only if the equality (1) holds.

3. The space of $\mathfrak{sl}_2$-systems

Recall that

$$\mathrm{Syst}_g = \{(X, A) : X \in \mathcal{T}_g, A \in \mathfrak{sl}_2(\Omega^1(X)), \rho_A \text{ is irreducible}\} /\!\!/ \mathrm{SL}_2(\mathbb{C}).$$

Note that the quotient here is by the conjugation action of $\mathrm{SL}_2(\mathbb{C})$, in the sense of geometric invariant theory. Since we have restricted to the irreducible locus, this coincides with the quotient in the sense of topology. It is standard that Syst_g is a smooth complex manifold.

Our aim in this section is to describe the tangent space $T_{(X,A)}\mathrm{Syst}_g = Z(X, A)/\sim$, as described in Section 1. Given a smooth Beltrami form $\mu \in \mathrm{Belt}(X)$ with $\|\mu\|_{L^\infty(X)} < 1$, denote by $f^\mu : X \to X^\mu$ the smooth map to a Riemann surface X^μ such that

$$\bar{\partial} f^\mu = \mu \partial f^\mu.$$

Lemma 5. *Let $(X, A) \in \mathrm{Syst}_g$ and $(\mu, \dot{A}) \in Z(X, A)$. Then there exists a path of pairs $(X^{t\mu}, A_t)$ based at (X, A), such that*

$$\left.\frac{d}{dt}\right|_{t=0} \left(f^{t\mu}\right)^* A_t = \dot{A}. \tag{4}$$

Moreover, the map $\tau : Z(X, A) \to T_{(X,A)}\mathrm{Syst}_g$, defined by taking the tangent vector to this path at $t = 0$, is surjective and has kernel $B(X, A)$ consisting of pairs $(\mu, \dot{A}) \in Z(X, A)$ such that

- *μ vanishes in $T_X\mathcal{T}_g$, and*
- *$\dot{A} - dT \in [A, \mathfrak{sl}_2(\mathbb{C})]$ for some function $T : X \to \mathfrak{sl}_2(\mathbb{C})$.*

We split the proof of Lemma 5 into two parts. We construct the path with property (4) in §3.1, and then we describe the kernel of τ in §3.2.

3.1. Constructing the path.

The path we construct will in fact be in

$$\{(X, A) : X \in \mathcal{T}_g, A \in \mathfrak{sl}_2(\Omega^1(X))\}$$

rather than Syst_g. Note that this space is isomorphic to the total space of $\mathcal{H}_g \otimes \mathfrak{sl}_2(\mathbb{C})$, where $\mathcal{H}_g \to \mathcal{T}_g$ is the Hodge bundle of Abelian differentials over a varying Riemann surface, i.e. the fibre of $\mathcal{H}_g$ over $X \in \mathcal{T}_g$ is $\Omega^1(X)$.

Let A_t be a path of $\mathfrak{sl}_2(\mathbb{C})$ valued holomorphic 1-forms on $X^{t\mu}$, chosen so that $[\mathrm{Re}(A_t)] \in H^1(\Sigma_g, \mathfrak{sl}_2(\mathbb{R}))$ depends smoothly on t, and has $\left.\frac{d}{dt}\right|_{t=0} [\mathrm{Re}(A_t)] = [\mathrm{Re}(\dot{A})]$. By the smoothness of the Gauss–Manin connection on $\mathcal{H}_g$, it follows that $\left(f^{t\mu}\right)^* A_t$ depend smoothly on t, and hence we can define

$$B = \left.\frac{d}{dt}\right|_{t=0} \left(f^{t\mu}\right)^* A_t.$$

Since the 1-forms A_t are holomorphic, they are also closed, and hence so is B. Moreover, $\mathrm{Re}(\dot{A}-B)$ is exact by construction.

Claim 1. We have $B^{0,1}=\mu A$.

Proof. We work in a local coordinate z on X, and abuse notation slightly to denote the Beltrami form by $\mu\frac{d\bar{z}}{dz}$ in this coordinate. Then $\tilde{A}_t := \left(f^{t\mu}\right)^* A_t$ is proportional to $f_z^{t\mu}\,(dz+t\mu d\bar{z})$. In particular,

$$\tilde{A}_t^{0,1}=t\mu\tilde{A}_t^{1,0}.$$

Differentiating at $t=0$, we see that $B^{0,1}=\mu A$, as desired. □

In particular, it follows that $B-\dot{A}$ is a closed form of type $(1,0)$, and is therefore holomorphic. Since $B-\dot{A}$ has exact real part, it follows by Hodge theory that $B-\dot{A}=0$. We have thus constructed a path with (4).

3.2. **Surjectivity and the kernel of τ.** We first show the surjectivity of τ. Given any tangent vector $V\in T_{(X,A)}(\mathcal{H}_g\otimes\mathfrak{sl}_2(\mathbb{C}))$, let $(X^{t\mu},A_t)$ be a path in $\mathcal{H}_g\otimes\mathfrak{sl}_2(\mathbb{C})$ tangent to V at (X,A). Note that each A_t is closed, so is $\left(f^{t\mu}\right)^* A_t$. Thus $\dot{A}:=\frac{d}{dt}\Big|_{t=0}\left(f^{t\mu}\right)^* A_t$ is closed, and by Claim 1 has $(0,1)$-part equal to μA. Thus $\dot{A}$ is (μ,A)-tamed and hence $V=\tau(\mu,\dot{A})$. Thus τ is surjective.

We now show that $\ker\tau = B(X,A)$. Since the paths constructed in §3.1 lie in $\mathcal{H}_g\otimes\mathfrak{sl}_2(\mathbb{C})$, τ factors as

$$Z(X,A)\xrightarrow{\tilde{\tau}} T_{(X,A)}(\mathcal{H}_g\otimes\mathfrak{sl}_2(\mathbb{C}))\to T_{(X,A)}\mathrm{Syst}_g.$$

Since ρ_A is irreducible, the point (X,A) is stable for the $\mathrm{SL}_2(\mathbb{C})$-action on $\mathcal{H}_g\otimes\mathfrak{sl}_2(\mathbb{C})$. Thus $\ker\tau=\tilde{\tau}^{-1}\left((0,[A,\mathfrak{sl}_2(\mathbb{C})])\right)=(0,[A,\mathfrak{sl}_2(\mathbb{C})])+\ker\tilde{\tau}$ by standard geometric invariant theory. In the rest of this subsection, we show that

$$\ker\tilde{\tau}=\{(\mu,\dot{A})\in Z(X,A):\mu \text{ vanishes in } T_X\mathcal{T}_g \text{ and } \dot{A} \text{ is exact}\},$$

which implies $B(X,A)=\ker\tau$ immediately.

Suppose first that $(\mu,\dot{A})\in\ker\tilde{\tau}$. Then μ corresponds to the zero tangent vector in $T_X\mathcal{T}_g$. By the construction of the path in §3.1, it follows that $\mathrm{Re}(\dot{A})$ is exact. Since the period matrix of $X^{t\mu}=X$ remains constant up to terms of order $o(t)$ (for μ is zero in the $T\mathcal{T}_g$), we see that $\mathrm{Im}(\dot{A})$ is exact as well.

Conversely, if μ vanishes in $T_X\mathcal{T}_g$ and $\dot{A}=dT$, then the cohomology class $[\dot{A}]\in H^1(\Sigma_g,\mathfrak{sl}_2(\mathbb{C}))$ vanishes. Thus the path in §3.1 represents a path of holomorphic 1-forms on a fixed Riemann surface, with the same cohomology. Thus the path is constant, and $(\mu,\dot{A})\in\ker\tilde{\tau}$.

4. The representation variety and de Rham isomorphism

In this section, we explain the tangent space to the representation variety Rep_g and the de Rham isomorphism for local systems.

4.1. **Tangent space to** Rep_g**.** The description of $T\mathrm{Rep}_g$ is much more standard than $T\mathrm{Syst}_g$, and can be found for instance in [6]. We summarise the results here for the sake of completeness.

Given an irreducible representation $\rho : \pi_1(\Sigma_g) \to \mathrm{SL}_2(\mathbb{C})$, the tangent space $T_\rho\mathrm{Rep}_g$ can be described as $H^1(\mathrm{Ad}_\rho) := Z(\mathrm{Ad}_\rho)/B(\mathrm{Ad}_\rho)$, where

$$Z(\mathrm{Ad}_\rho) = \{\eta : \pi_1(\Sigma_g) \to \mathfrak{sl}_2(\mathbb{C}) \text{ with } \eta(\gamma_1\gamma_2) = \eta(\gamma_1) + \rho(\gamma_1)\eta(\gamma_2)\rho(\gamma_1)^{-1}\},$$

and

$$B(\mathrm{Ad}_\rho) = \{\rho T \rho^{-1} - T \text{ for } T \in \mathfrak{sl}_2(\mathbb{C})\}.$$

Given an element $\eta \in Z(\mathrm{Ad}_\rho)$, there exists a smooth path $\rho_t : \pi_1(\Sigma_g) \to \mathrm{SL}_2(\mathbb{C})$ such that

$$\left.\frac{d}{dt}\right|_{t=0} \rho_t(\gamma) = \eta(\gamma)\rho(\gamma),$$

and this path is tangent to $\eta + B(\mathrm{Ad}_\rho) \in H^1(\mathrm{Ad}_\rho)$.

4.2. **De Rham isomorphism for local systems.** There is a standard correspondence between flat vector bundles E of rank n over a manifold M, and representations $\rho : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$ of its fundamental group. Taking the derivative of this correspondence gives an isomorphism $H^1(M, \mathcal{E}) \cong H^1(\pi_1(M), \mathrm{Ad}_\rho)$, where $\mathcal{E}$ is the sheaf of parallel sections of E. In this section, we describe this isomorphism explicitly. We omit the proofs as they are standard application of homological algebra. A much more general result can be found in Proposition 6.3 in [7].

We first describe how $H^1(M, \mathcal{E})$ and $H^1(\mathrm{Ad}_\rho)$ can be understood explicitly. Consider the de Rham resolution of $\mathcal{E}$

$$\mathcal{E} \to E \xrightarrow{d^\nabla} E \otimes T^*M \xrightarrow{d^\nabla} E \otimes \bigwedge^2 T^*M.$$

This is a soft resolution, and its sections can hence be used to compute the cohomology

$$H^1(\Sigma_g, \mathcal{E}) = \frac{\{A \in C^\infty(E \otimes T^*\Sigma_g) : d^\nabla A = 0\}}{\{d^\nabla B : B \in C^\infty(E)\}}.$$

Also recall that $H^1(\mathrm{Ad}_\rho) = Z(\mathrm{Ad}_\rho)/B(\mathrm{Ad}_\rho)$, where

$$Z(\mathrm{Ad}_\rho) = \{\eta : \pi_1(\Sigma_g) \to \mathbb{C}^n \text{ such that } \eta(\gamma_1\gamma_2) = \eta(\gamma_1) + \rho(\gamma_1)\eta(\gamma_2)\rho(\gamma_1)^{-1}\},$$

and

$$B(\mathrm{Ad}_\rho) = \{\rho(\gamma)\eta\rho(\gamma)^{-1} - \eta : \eta \in \mathbb{C}^n\}.$$

We now describe the de Rham isomorphism $\iota : H^1(M, \mathcal{E}) \to H^1(\mathrm{Ad}_\rho)$. Let $p : \tilde{M} \to M$ be the universal cover, and $\tilde{E} = p^*E$ be the pullback of E. Note that the deck group action of $\pi_1(M)$ on $\tilde{M}$ naturally extends to an action on $\tilde{E}$. Fix an arbitrary basepoint $x \in \tilde{M}$.

Given a d^∇-closed E-valued 1-form A, the class $\iota([A])$ can be computed as follows. Since $\tilde{E}$ is the trivial flat bundle, it admits a section T such that

$dT = p^*A$. It is easily seen that $\gamma T - T$ is parallel for any $\gamma \in \pi_1(M)$, and hence

$$\gamma \longrightarrow (\gamma T - T)(x)$$

defines a cycle in $Z(\mathrm{Ad}_\rho)$. The image of this cycle in $H^1(\mathrm{Ad}_\rho)$ is $\iota([\chi])$.

5. Proof of Theorem 2

5.1. **Trivializing the flat $\mathrm{SL}_2(\mathbb{C})$-bundle.** We now describe the setup in which we will prove Theorem 2.

Let X be a Riemann surface and $A \in \mathfrak{sl}_2(\Omega^1(X))$. Let $\rho : \pi_1(X) \to \mathrm{SL}_2(\mathbb{C})$ be the monodromy of $d + A$. The following lemma shows how to trivialize $(\underline{\mathrm{SL}_2(\mathbb{C})}, d + A)$. Denote by $\tilde{X}$ the universal cover of X. This lemma is entirely contained in the literature (see e.g. the introduction to [2]), but we include a proof for completeness.

Lemma 6. *There exists a map $F : \tilde{X} \to \mathrm{SL}_2(\mathbb{C})$ such that*

$$dF + AF = 0 \text{ and } F(\gamma x) = F(x)\rho(\gamma)^{-1}.$$

Then the map $\Phi : \tilde{X} \times \mathrm{SL}_2(\mathbb{C}) \to \tilde{X} \times \mathrm{SL}_2(\mathbb{C})$ defined by $\Phi(x, T) = (x, F(x)T)$ is an isomorphism of principal bundles with the following two properties

(1) *Φ conjugates the $\pi_1(X)$-action $\gamma \cdot (x, T) = (\gamma x, \rho(\gamma)T)$ to the action $\gamma \cdot (x, T) = (\gamma x, T)$, and*
(2) *$\Phi^*(d + A) = d$.*

Proof. Fix an arbitrary basepoint $\tilde{x}_0 \in \tilde{X}$, and let $F_0 \in \mathrm{SL}_2(\mathbb{C})$ be such that ρ is the monodromy of $d + A$ with basepoint $\tilde{x}_0$ relative to the basis that consists of the columns of F_0. Define $F(x)$ as follows: for $i = 1, 2$, its i-th column is the d_A-parallel transport of the i-th column of F_0 along an arbitrary path from $\tilde{x}_0$ to x. It is immediate that $dF + AF = 0$, since F is obtained through parallel transport. Since the monodromy of $d + A$ is precisely ρ, it follows that $F(x) = F(\gamma x)\rho(\gamma)$.

Moreover, if we let $\sigma : \tilde{X} \to \mathrm{SL}_2(\mathbb{C})$ be a section of the bundle $(\underline{\mathrm{SL}_2(\mathbb{C})}, d_A)$, then

$$\begin{aligned} \nabla(F^{-1}\sigma) &= \sigma^{-1} F d(F^{-1}\sigma) \\ &= \sigma^{-1} F \left(-F^{-1}(dF)F^{-1}\sigma + F^{-1} d\sigma \right) \\ &= \sigma^{-1} d\sigma + \sigma^{-1} A \sigma = d_A \sigma, \end{aligned}$$

and the result is shown. □

5.2. **Computing the derivative.** We now let $(X^{t\mu}, A_t)$ be the path at $(X, A) \in \mathrm{Syst}_g$ tangent to the vector $(\mu, \dot{A})$ with $\dot{A}^{0,1} = \mu A$, as in Lemma 5. Let $F_t : \tilde{X}^{t\mu} \to \mathrm{SL}_2(\mathbb{C})$ be as in Lemma 6, and suppose that the monodromy of $d + A_t$ is $\rho_t : \pi_1(X^{t\mu}) \to \mathrm{SL}_2(\mathbb{C})$.

We use $f^{t\mu}$ to transport A_t, F_t and ρ_t to X. Recall that $\frac{d}{dt}\Big|_{t=0} A_t = \dot{A}$. We have

$$dF_t + A_t F_t = 0 \text{ and } F_t(\gamma x) = F_t(x) \cdot \rho_t(\gamma)^{-1}.$$

Let $F_t = (\mathrm{id} + t\dot{F} + o(t)) \cdot F$ for some map $\dot{F} : \tilde{X} \to \mathfrak{sl}_2(\mathbb{C})$. Then

$$d_A \dot{F} + \dot{A} = d\dot{F} + \dot{A} + [A, \dot{F}] = 0.$$

We thus have $d_A \dot{F} = -\dot{A}$. Similarly, if we set $\rho_t(\gamma) = (\mathrm{id} + t\dot{\rho}(\gamma) + o(t)) \cdot \rho(\gamma)$, we have

$$\dot{\rho}(\gamma) = F(\gamma x)^{-1} \left(\dot{F}(x) - \dot{F}(\gamma x) \right) F(\gamma x).$$

Set $B = F^{-1}\dot{F}F$. Then $\dot{\rho}(\gamma) = \rho(\gamma)B(x)\rho(\gamma)^{-1} - B(\gamma x)$. It is easy to see that $\rho(\gamma)B(x)\rho(\gamma)^{-1} - B(\gamma x)$ does not depend on $x \in \tilde{X}$, so we have

$$\dot{\rho}(\gamma) = \rho(\gamma)B(\gamma^{-1}x)\rho(\gamma)^{-1} - B(x).$$

From the description of the de Rham isomorphism $H^1(X, \mathcal{E}) \to H^1(\mathrm{Ad}_\rho)$ in §4.2, we see that $[\dot{\rho}] = -\iota[\dot{A}]$.

6. Applications

Recall that Theorem 1 states that the derivative $D\mathbf{M}$ is injective on the tangent space $T_{(X,A)}\mathrm{Syst}_g$ if and only if $\mathrm{QD}(\alpha, \beta, \gamma) = \mathrm{QD}(X)$, where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha. \end{pmatrix}$$

In this section we derive several applications of this theorem.

6.1. Preliminaries.

Let X be a Riemann surface of genus at least two, and let $\xi_i \in \Omega^1(X)$, $i = 1, ..., k$. Let $\mathrm{QD}(\xi_1, ..., \xi_k) \leq \mathrm{QD}(X)$ be the vector subspace generated by the products $\xi_i \varphi$, $i = 1, ..., k$, and $\varphi \in \Omega^1(X)$. The following was proved by Rauch (see statement C in [10]).

Theorem 3. *Let X be a hyperelliptic Riemann surface of genus at least two, and suppose $\xi_1, ..., \xi_g$ is a basis of $\Omega^1(X)$. Then dim$\big(\mathrm{QD}(\xi_1, ..., \xi_g)\big) = 2g - 1$.*

The next theorem is a classical result.

Theorem 4. *Suppose $g \geq 3$. There exists a dense open subset $U \subset \mathcal{H}_g^{\oplus 3}$ such that the equality $\mathrm{QD}(\alpha, \beta, \gamma) = \mathrm{QD}(X)$ holds for each $(\alpha, \beta, \gamma) \in U$.*

Remark 3. Let $p : \mathcal{H}_g^{\oplus 3} \to \mathcal{T}_g$ be the natural projection. According to the result of [5], the set U from Theorem 4 can in fact be taken such that $U \cap p^{-1}(X)$ is dense in $p^{-1}(X)$ for any non-hyperelliptic $X \in \mathcal{T}_g$. The same result can be recovered from the well known Max Noether theorem (see III.11.20 in [3]).

Proof. Let X be a non-hyperelliptic Riemann surface. By Theorem 1.1 in [5], the set of triples $(\alpha,\beta,\gamma)\in\big(\Omega^1(X)\big)^3$ such that $\mathrm{QD}(\alpha,\beta,\gamma)=\mathrm{QD}(X)$ is an open and dense subset of $\big(\Omega^1(X)\big)^3$. Taking the union of these dense open subsets over all non-hyperelliptic Riemann surface in $\mathcal{T}_g$ yields the required open and dense subset of $\mathcal{H}_g^{\oplus 3}$. □

6.2. **Two corollaries.** The following is a corollary of Theorem 1 and Theorem 3.

Theorem 5. *Suppose $X\in\mathcal{T}_g$ is hyperelliptic and let $(X,A)\in\mathrm{Syst}_g$. Then $D\mathbf{M}$ is injective on the tangent space $T_{(X,A)}\mathrm{Syst}_g$ if and only if $g=2$.*

Remark 4. That $D\mathbf{M}$ is injective on the tangent space $T_{(X,A)}\mathrm{Syst}_g$ if $g=2$ is a well known theorem by Calsamiglia-Deroin-Heu-Loray [2].

Proof. Assume at first that $g>2$. Then by Theorem 3 the following inequality holds

$$\dim\big(\mathrm{QD}(\alpha,\beta,\gamma)\big)\le 2g-1<3g-3=\dim\big(\mathrm{QD}(X)\big).$$

Thus, the equality (1) can never hold on such X regardless of the choice of Abelian differentials α,β,γ.

We now consider the case $g=2$. Since $(X,A)\in\mathrm{Syst}_g$, we know that the monodromy $\rho_A:\pi_1(\Sigma_g)\to\mathrm{SL}_2(\mathbb{C})$ is irreducible. Then the three Abelian differentials α,β, and γ, span $\Omega^1(X)$. This was observed by Biswas-Dumitrescu (see the proof of Proposition 4.1 in [1]). Since $\dim(\mathrm{QD}(X))=2$, without loss of generality we may assume that α and β span $\Omega^1(X)$. But then by Theorem 3 we know that the three quadratic differentials $\alpha^2,\alpha\beta,\beta^2$ represent a basis for $\mathrm{QD}(X)$. By Theorem 1 the derivative $D\mathbf{M}$ is injective on $T_{(X,A)}\mathrm{Syst}_g$. □

In the remainder of this section we let

$$I:\mathcal{H}_g^{\oplus 3}\to\mathcal{H}_g\otimes\mathfrak{sl}_2(\mathbb{C})$$

denote the identification

$$(\alpha,\beta,\gamma)\ \mapsto\ A=\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix}.$$

Theorem 6. *The derivative $D\mathbf{M}$ is locally injective on an open and dense subset of Syst_g.*

Remark 5. This is a recent theorem by Biswas-Dumitrescu [1].

Proof. Let $U\subset\mathcal{H}_g^{\oplus 3}$ be the open dense set from Theorem 4. Moreover, let V be the open dense subset of $\mathcal{H}_g\otimes\mathfrak{sl}_2(\mathbb{C})$ that corresponds to irreducible flat connections. Then $I(U)\cap V$ is a dense open subset of $\mathcal{H}_g\otimes\mathfrak{sl}_2(\mathbb{C})$. Since the quotient map $q:V\to V/\!\!/\,\mathrm{SL}_2(\mathbb{C})$ is open, it follows that $q(I(U)\cap V)$ is the desired dense open set by Theorem 1. □

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