EXAMPLES SHEET, ANALYSIS OF BOOLEAN FUNCTIONS

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Throughout $G := \mathbb{F}_2^n$ and X is a finite set unless otherwise stated. Answers and comments on some of the questions appear at the end.

1. Prove the nesting of the $L^p(X)$-norms and $\ell^p(X)$-norms. Show that in the first case equality holds if and only if the function is constant, and in the latter if and only if the function is a δ-function, meaning that it is supported on exactly one point of the domain.

2. Prove Chebychev's inequality that

$$\mu_X(\{x : |f(x)| \geqslant \epsilon\}) \leqslant \epsilon^{-2}\|f\|_{L^2(X)}^2.$$

Prove an L^p analogue and ℓ^p analogue.

3. Prove the instance $\|f*g\|_{L^1(G)} \leqslant \|f\|_{L^1(G)}\|g\|_{L^1(G)}$ of Young's inequality via the triangle inequality.

4. Prove the instance $\|f * g\|_{L^\infty(G)} \leqslant \|f\|_{L^p(G)}\|g\|_{L^q(G)}$ of Young's inequality via the Hölder's inequality.

5. Prove the general instance of Young's inequality via interpolation if you are familiar with it. If not look up Riesz-Thorin interpolation on wikipedia and try to use it in conjunction with the previous inequalities.

6. Prove that if V and W are (vector) subspaces of G then

$$\dim V + W = \dim V + \dim W - \dim V \cap W.$$

7. Compute the convolution $1_W * 1_{W'}$ if W and W' are affine subspaces.

8. Compute the convolution $1_A * 1_V$ if $A \subset V$ has density α and V is a vector subspace of G.

9. Show that if $S \subset G$ then the map $\pi : \ell^2(G) \to \ell^2(G); f \mapsto f|_S$ is a projection in the sense that it is a linear map with $\pi^2 = \pi$. Show directly that if V is a subspace then the map $P_V : L^2(G) \to L^2(G); f \mapsto f * \mu_V$ is also a projection. Can you show this using the first part?

10. Make sure you believe that $f * g = g * f$ and $f * (g * h) = (f * g) * h$.

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11. Find an upper estimate for $\mu_G(\{x : 1_A * 1_A(x) > c\alpha\})$. Is there an interesting lower estimate? What if $c < \alpha$? Suppose that you know $\|1_A * 1_A\|_{L^2(G)}^2 \geqslant \eta\alpha^3$. Does that help? What if $\eta > 2c$?

12. Prove that if G is a finite group of exponent 2, that is a finite group in which every element has order 2, then G is abelian. Hence, or otherwise, show that G is isomorphic to the additive group of $\mathbb{F}_2^n$ for some $n \in \mathbb{N}_0$.

13. We say that $f \in L^1(G)$ is idempotent if $f * f = f$. Show that if f is idempotent then $\|f\|_{L^1(G)} = 0$ or else $\|f\|_{L^1(G)} \geqslant 1$. Show that if $\|f\|_{L^1(G)} = 1$ then $f = z\mu_W$ where $|z| = 1$, W is an affine subspace of G and μ_W is, as usual, the unique probability measure supported on W.

14. Recall that if B is a (real, finite dimensional) Banach space then B^* denotes its dual space, that is the space of continuous linear functionals $B \to \mathbb{R}$. Prove the Riesz representation theorem that if $\phi \in L^p(X)^*$ then there is some $g \in L^q(X)$ (where $p^{-1}+q^{-1} = 1$) such that $\phi(f) = \langle g, f\rangle_{L^2(X)}$ for all $f \in L^p(X)$. What is $\|g\|_{L^q(X)}$?

15. Suppose that X_1, X_2 are finite sets, (p_1, q_1) and (p_2, q_2) are conjugate pairs of indices. Suppose that T is a linear operator $L^{p_1}(X_1) \to L^{p_2}(X_2)$. The adjoint operator $T^* : L^{q_1}(X_2) \to L^{q_2}(X_1)$ is defined by $\langle Tf, g\rangle_{L^2(X_2)} = \langle f, T^*g\rangle_{L^2(X_1)}$ for all $f \in L^{p_1}(X_1)$ and $g \in L^{q_2}(X_2)$. Check that this produces a well-defined linear operator and compute $\|T^*\|$ in terms of $\|T\|$. Note that the same arguments give the same results for ℓ^p-spaces, and for maps between L^{p_1} and ℓ^{p_2} spaces.

16. Which functions f are idempotent (see the previous question for a definition) and have $\|f\|_{L^p(G)} \leqslant 1$ for some $p > 1$? Prove $\|f\|_{L^\infty(G)} \leqslant 1$ using Young's inequality.

17. Write $\delta_x : G \to \mathbb{C}$ for the map taking x to $\sqrt{|G|}$ and all other elements to 0. Prove that $(\delta_x)_{x\in G}$ forms a basis for $L^2(G)$. This is the physical space basis of δ-functions.

18. Characterise the homomorphisms $G \to \{-1, 1\}$, where $\{-1, 1\}$ is a group under multiplication.

19. Suppose that $f \in L^1(G)$. Check that you believe that $g \mapsto f * g$ is a linear operator $L^2(G) \to L^2(G)$. Such operators care called convolution operators. What is the operator norm? What is its determinant? What is its characteristic polynomial? What is its minimal polynomial? Why is the operator diagonalisable?

20. Characterise those bases of $L^2(G)$ that simultaneously diagonalise all convolution operators, that is identify all bases $\{e_1, \ldots, e_{|G|}\}$ of $L^2(G)$ such that $f * e_i = \lambda(f, i)e_i$ for all $i \in \{1, \ldots, |G|\}$.

21. Prove the special case $\|\widehat{f}\|_{\ell^\infty(G)} \leqslant \|f\|_{L^1(G)}$ of the Hausdorff-Young inequality.

22. Prove Plancherel's theorem that $\langle f,g\rangle_{L^2(G)} = \langle \widehat{f},\widehat{g}\rangle_{\ell^2(\widehat{G})}$ for all $f,g \in L^2(G)$ using the Fourier inversion formula.

23. Check that the map $L^2(G) \to \ell^2(\widehat{G}); f \mapsto \widehat{f}$ is an isometric isomorphism. What is its adjoint? What is its inverse?

24. Deduce Plancherel's theorem from Parseval's theorem. Unless your proof was very exotic, what you have done is called de-polarisation.

25. Prove the general Hausdorff-Young inequality that $\|\widehat{f}\|_{\ell^p(\widehat{G})} \leqslant \|f\|_{L^{p'}(G)}$ for all $p \in [2,\infty]$ using interpolation or otherwise. Prove the dual version that $\|f\|_{L^p(G)} \leqslant \|\widehat{f}\|_{\ell^{p'}(\widehat{G})}$ for the same range of p.

26. Suppose that G is any finite group of exponent 2. We know that there is some n such that G is isomorphic to $\mathbb{F}_2^n$ and in lectures we defined $\widehat{G}$ through this isomorphism. A much better was is to let $\widehat{G}$ be the set of homomorphisms $G \to \{-1,1\}$ where $\{-1,1\}$ is considered to be a group under multiplication. Show that if $\phi : G \to \mathbb{F}_2^n$ is an isomorphism then

$$\{\gamma \circ \phi : \gamma : G \to \{-1,1\} \text{ is a homomorphism.}\}$$

is equal to the set $\widehat{G}$ as we defined it in lectures. We shall typically use the definitions interchangeably.

27. Suppose that X is a finite set and $\mathcal{A} \subset \mathcal{P}(X)$ is intersection closed and contains X. Then we say that $S \subset X$ generates $A \in \mathcal{A}$ if $S \subset A$ and for all $A' \in \mathcal{A}$ with $S \subset A'$ we have $A \subset A'$. If $\emptyset \neq S \subset G$, how large is the affine space generated by S compared with the vector space?

28. If $A \subset \widehat{G}$ has size $\delta_G(A) = k$, how large and small can $\mu_G(A^\perp)$ possibly be in terms of k?

29. Prove that $\|f\|_{PM(G)} := \|\widehat{f}\|_{\ell^\infty(\widehat{G})}$ is a norm; it is the spectral radius – that is size of the largest eigenvalue – of the convolution operator $g \mapsto f * g$. Prove that $\|f\|_{A(G)} := \sup\{|\langle f,g\rangle_{L^2(G)}| : \|g\|_{PM(G)} \leqslant 1\}$ is an algebra norm, that is, it is a norm such that $\|fg\|_{A(G)} \leqslant \|f\|_{A(G)}\|g\|_{A(G)}$. Show that $\|f\|_{A(G)} = \|\widehat{f}\|_{\ell^1(\widehat{G})}$.

30. Prove the spectral radius formula, that is $\|f^{(n)}\|_{L^2(G)}^{1/n} \to \|\widehat{f}\|_{\ell^\infty(\widehat{G})}$, where $f \in L^1(G)$ and $f^{(n)}$ denotes the n-fold convolution of f with itself. How rapidly does it converge? For which values of $p \in [1,\infty]$ can you replace $L^2(G)$ with $L^p(G)$?

31. Prove that $\|\widehat{f}\|_{\ell^\infty(\widehat{G})} \geqslant \|f\|_{L^1(G)}/\sqrt{|G|}$. Can you do any better?

32. Show that if H is a finite dimensional Hilbert space then H is isometrically isomorphic to $\ell^2(X)$ for some finite set X. On the other hand show that there are finite dimensional Banach spaces B such that B is not isometrically isomorphic to $\ell^p(X)$ for any $p \in [1,\infty]$ and finite set X.

33. Establish the log-convexity of the L^p-norms. That is to say show that

$$\|f\|_{L^p(G)} \leqslant \|f\|_{L^q(G)}^{\theta}\|f\|_{L^r(G)}^{1-\theta} \text{ whenever } \frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r} \text{ and } \theta \in [0,1].$$

34. Show that if V and W are linear subspaces of G then $V \cap W$ is a linear subspace and

$$\operatorname{cod} V \cap W \leqslant \operatorname{cod} V + \operatorname{cod} W;$$

when does equality occur?

35. Prove the following law of large numbers. Suppose that $A \subset G$ has density α, W is the affine subspace of G generated by A, and V is W's vector subspace. If $x_1, \dots, x_{2k}$ are elements of A chosen independently and uniformly at random and $S \subset G$ has density σ then

$$\mathbb{P}(x_1 + \dots + x_{2k} \in S) = \mu_V(S)(1 + o_{\alpha,\sigma;k\rightarrow\infty}(1)).$$

36. Can the Parseval bound, that $|\operatorname{Spec}_\epsilon(f)| \leqslant \epsilon^{-2}\|f\|_{L^2(G)}^2\|f\|_{L^1(G)}^{-2}$ for all $f \in L^2(G)$, be improved?

37. Show that if $A \subset G$ has density $\alpha > 0$ then $A + A + A$ contains an affine subspace of co-dimension at most $O_\alpha(1)$.

38. Suppose that $A \subset G$ is such that there is $A' \subset A$ with

$$\{(x,y,z,w) \in (A')^4 : x + y + z + w = 0_G \text{ and } x \neq y, x \neq z, z \neq w\} = \emptyset$$

and $\mu_G(A \setminus A') \leqslant \epsilon$. Show that $\mu_G(A) \leqslant \epsilon + o(1)$, where $o(1) \rightarrow 0$ as $|G| \rightarrow \infty$. Can you get a reasonable bound for the $o(1)$ term?

39. Prove that the set of vectors in G having at least $n - d$ ones intersects every affine subspace $x + V$ where V is a linear subspace of co-dimension at most d.

40. Show that for n sufficiently large in terms of K there is a set $A \subset G = \mathbb{F}_2^n$ with $\mu_G(A) \geqslant 1/3$ such that for any $X \subset G$ with $|X| \leqslant K$ we have $A + X \neq G$.

41. Suppose that A is an independent subset of G. How large is nA in terms of n and the size of A?

42. (*Khintchine's inequality*) Suppose that $\Lambda \subset \widehat{G}$ is independent and $p \in [1,\infty)$. Prove that

$$\Omega(\|f\|_{\ell^2(\Lambda)}) = \|\sum_{\gamma\in\Lambda} f(\gamma)\gamma\|_{L^p(G)} = O(\sqrt{p}\|f\|_{\ell^2(\Lambda)}) \text{ for all } f \in \ell^2(\Lambda).$$

[Note that some of the inequalities and range of values follow immediately from Rudin; some require an additional argument.]

43. Deduce Rudin's inequality from Beckner's inequality.

44. Show that if $|A+A| \leqslant K|A|$ then $A+A+A$ contains an affine subspace of dimension at least $\log_2|A| - O_K(1)$.

45. Show that if $A \subset G$ has density α and $\epsilon \in (0,1]$. Then $\mathrm{Sym}_{\alpha/2}(A)$ contains $1-\epsilon$ of a subspace V of co-dimension $O_{\epsilon,\alpha}(1)$.

46. Show that if Λ is a set of independent characters and $A \subset G$ has density α then

$$\sum_{S\subset\Lambda, |S|=r} |\widehat{1_A}(\sum_{\lambda\in S}\lambda)|^2 \leqslant O(\log\alpha^{-1})^r\alpha^2.$$

47. Show that if $p(x) = \sum_{i<j} x_i x_j$ then $\langle(-1)^p, (-1)^l\rangle_{L^2(G)} = o(1)$ for all linear polynomials l. That is to say, the conclusion of the U^3-inverse theorem cannot be qualitatively strengthened.

48. Prove directly that if $S \subset G$ and $\phi : G \to G$ is such that

$$\mu_{G^2}(\{(x,y) \in G^2 : \phi(x) + \phi(y) = \phi(x+y), x, y, x+y \in S\}) \geqslant \epsilon,$$

then there is a morphism θ such that $\mu_G(\{x \in S : \phi(x) = \theta(x)\}) = \Omega_\epsilon(1)$.

Comments and solutions

38. First note that the number of quadruples $(x,y,z,w) \in A'$ with $x+y+z+w = 0_G$ is

$$\begin{aligned}(1.1)\qquad \sum_{x+y+z+w=0_G} 1_{A'}(x)1_{A'}(y)1_{A'}(z)1_{A'}(w) &= |G|^3 1_{A'} * 1_{A'} * 1_{A'} * 1_{A'}(0_G)\\ &= |G|^3\langle 1_{A'} * 1_{A'}, 1_{A'} * 1_{A'}\rangle_{L^2(G)}.\end{aligned}$$

Now, if A' has

$$\{(x,y,z,w) \in A'^4 : x+y+z+w = 0_G \text{ and } x \neq y, x \neq z, z \neq w\} = \emptyset$$

Then any quadruple $(x,y,z,w) \in A'^4$ with $x+y+z+w = 0_G$ has $x = y$ or $x = z$ or $z = w$. A'ut if $x+y+z+w = 0_G$ and $x = y$ then $z = w$; and similarly if $x = z$ or $z = w$. It follows that there are at most $|G|^2$ such quadruples and hence the left hand side of (1.1) is at most $|G|^2$. Thus

$$|G|^{-1} \geqslant \|1_{A'} * 1_{A'}\|_{L^2(G)}^2 \geqslant \|1_{A'} * 1_{A'}\|_{L^1(G)}^2 = \mu_G(A')^4$$

where the inequality is by Cauchy-Schwarz and $\mu_G(A') \leqslant |G|^{-1/4}$.

Now, since $A' \subset A$ we have

$$\mu_G(A) = \mu_G(A \setminus A') + \mu_G(A') \leqslant \epsilon + \mu_G(A') = \epsilon + o(1).$$

More than this the $o(1)$ bound is rather good and certainly reasonable, satisfying the demand of the second part. The point is that we do *not* just prove a variant of the removal lemma.

46. The point of this question is to highlight the parallels between Beckner's inequality and Chang's theorem. Write

$$q_\epsilon(x) := \prod_{\lambda \in \Lambda}(1 + \epsilon\lambda(x)).$$

Since Λ is independent it is easy to see that if $S \subset \Lambda$ then

$$\widehat{q_\epsilon}(\sum_{\lambda \in S}\lambda) = \epsilon^{|S|}.$$

It follows that

$$\|\widehat{1_A}\widehat{q_\epsilon}\|_{\ell^2(\widehat{G})}^2 \geqslant \sum_{S \subset \Lambda, |S|=r} \epsilon^{2|S|}|\widehat{1_A}(\sum_{\lambda \in S}\lambda)|^2 = \epsilon^{2r}\sum_{S \subset \Lambda, |S|=r}|\widehat{1_A}(\sum_{\lambda \in S}\lambda)|^2.$$

On the other hand, q_ϵ is equal to the Beckner operator p_ϵ (on $G/\bigcap_{\lambda \in \Lambda}\lambda$ with a suitable basis). Hence

$$\|\widehat{1_A}\widehat{q_\epsilon}\|_{\ell^2(\widehat{G})}^2 = \|1_A * q_\epsilon\|_{L^2(G)}^2 \leqslant \|1_A\|_{L^{1+\epsilon^2}(G)}^2 = \alpha^{2/(1+\epsilon^2)}.$$

Optimising as with Chang's theorem we put $\epsilon^2 = 1/(1 + \log\alpha^{-1})$ to get the result of the question. Note that if $r = 1$ we can easily recover Chang's theorem.

48. One way to prove this is to follow the start of the proof of Lemma 7.12 in the notes. First, if $S = \{0_G\}$ then we can take $\theta \equiv 0$ and so we assume not. Define $\tilde{\phi}$ on G to equal ϕ on S and ν on S^c where ν is as in Lemma 7.11 in the notes. Then it is easy to see that

$$\mu_{G^2}(\{(x,y) \in G^2 : \tilde{\phi}(x) + \tilde{\phi}(y) = \tilde{\phi}(x+y), x, y, x+y \in S\}) \geqslant \epsilon.$$

Now we apply the Rough Morphism theorem to get a morphism $\tilde{\theta}$ such that

$$\mu_G(x \in G : \tilde{\theta}(x) = \tilde{\phi}(x)) \geqslant \exp(-O(\epsilon^{-O(1)})).$$

Thus, either $\exp(-O(\epsilon^{-O(1)})) = 2^{-n}.O(n^2)$ or

$$\mu_G(x \in S : \tilde{\theta}(x) = \tilde{\phi}(x)) \geqslant \exp(-O(\epsilon^{-O(1)})).$$

In the first case, let $x' \in S$ have $x' \neq 0_G$ (possible since $S \neq \{0_G\}$) and let θ be a morphism such that $\theta(x') = \phi(x')$ and the result follows; in the second, we let $\theta \equiv \tilde{\theta}$.

The point of the question is that this result could be prove directly following the argument for the Rough Morphism Theorem, rather than by using the above method.

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