The University of Oxford

MSc (Mathematics and Foundations of Computer Science)

Applications of commutative harmonic analysis

Trinity Term 2012

The steps of (each) miniproject are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the miniproject, but should make this assumption clear in your presentation.

Please write or print on one side of the paper only.

The aim of this project is to investigate Roth-type results for solutions to certain systems of equations. Our focus will be on groups of bounded exponent: a group G has exponent r if every element of G has order at most r.

Throughout the project G is a finite Abelian group of exponent r endowed with Haar probability measure. At various points we shall be interested in specific examples of such G and when that is the case we shall say so.

1. Show that for every $\epsilon \in (0,1]$ and $A \subset G$ of density α there is a subgroup $V \leqslant G$ with

$$|V| \geqslant r^{-O(\epsilon^{-1}\log\alpha^{-1})}|G|,$$

and some $x \in G$ such that $A' := V \cap (x + A)$ has

$$\mathbb{P}_V(A') \geqslant \alpha \text{ and } \sup_{\gamma \neq 0_{\widehat{V}}} |\widehat{1_{A'}}(\gamma)| \leqslant \epsilon \mathbb{P}_V(A').$$

(The Fourier transform $\widehat{1_{A'}}(\gamma)$ denotes the Fourier transform of $1_{A'}$ considered as an element of $L^1(V)$.)

Recall that a triple $(x, y, z) \in G^3$ is a three-term arithmetic progression if $x + z = 2y$. We say that a three-term arithmetic progression (x, y, z) is *trivial* if at least two of x, y and z are the same; we say that it is *really trivial* if all of x, y, z are the same. (In particular, every really trivial progression is trivial.) Trivial and really trivial are different notions if and only if the group has 2-torsion.

2. Show that if $A \subset G := (\mathbb{Z}/4\mathbb{Z})^n$ has density α and contains no progressions that are not really trivial then $\alpha = O(|G|^{-1/2})$.

3. Show that if $A \subset G := (\mathbb{Z}/4\mathbb{Z})^n$ has density α and $\sup_{\gamma\neq 0_{\widehat{G}}} |\widehat{1_A}(\gamma)| \leqslant \epsilon\alpha$ then the number of three-term progressions in A is at least $(\alpha - \epsilon)|A|^2$.

4. Hence show that if $A \subset G := (\mathbb{Z}/4\mathbb{Z})^n$ has density α and contains no progressions that are not trivial then $\alpha = O(\log\log|G|/\log|G|)$. Can you improve this bound slightly?

An *arithmetic triangle* in G is a sextuple $(x, y, z, (x+y)/2, (x+z)/2, (y+z)/2) \in G^6$; such a triangle is *trivial* if at least two of its entries are the same.

5. Show that if $A \subset G := (\mathbb{Z}/3\mathbb{Z})^n$ has density α and contains no non-trivial arithmetic triangles then $\alpha = O(\log^{-\Omega(1)}|G|)$. [*Hint: this part is harder; it may be helpful to count the number of arithmetic triangles in A using expressions of the form $\langle(1_{2A} - \alpha) * f_z, f_z\rangle$ where $f_z(x) = 1_A(x)1_A((x+z)/2)$*]

Note that x is a three-term arithmetic progressions in G^3 if and only if $(1, -2, 1)\cdot x = 0$; we think of a three-term progression as a solution to the equation

$$\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \end{pmatrix}.$$

Similarly, $x \in G^6$ is an arithmetic triangle if and only if

$$\begin{pmatrix} 1 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Four-term arithmetic progressions can also be expressed in terms of solutions to systems of linear equations: $x \in G^4$ is a four-term arithmetic progression if

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A four-term progression is *trivial* if at least two of its entries are the same. While it is true that if $A \subset G$ has density α and contains no non-trivial four-term progressions then $\alpha = o(1)$, it is much harder to prove and cannot be shown by the above methods as we shall now see.

6. Let $G := (\mathbb{Z}/5\mathbb{Z})^n$ and write

$$A := \{x \in G : x \cdot x = 0\},$$

where $x \cdot y = x_1y_1 + \cdots + x_ny_n$ for all $x, y \in G$. Show that $\mathbb{P}_G(A) = 1/5 + o(1)$ and $\sup_{\gamma \neq 0_{\widehat{G}}} |\widehat{1_A}(\gamma)| = o(1)$.

7. Show that the set A above contains $|G|^2(\mathbb{P}_G(A)^3 + o(1))$ four-term arithmetic progressions. [*Hint: it may be useful to note that $x \cdot x - 3(x+d)\cdot(x+d) + 3(x+2d)\cdot(x+2d) = (x+3d)\cdot(x+3d)$.*]

8. Explain why the set A above presents an obstacle to extending the method of proof used to establish the Roth-Meshulam Theorem, to a four-term version in $G := (\mathbb{Z}/5\mathbb{Z})^n$.

9. What can you say in general about which systems of equations the Roth-Meshulam method can be used to tackle?