EXAMPLES SHEET, TOPICS IN ANALYTIC NUMBER THEORY

TOM SANDERS

1. Prove that $\tau(n) = n^{o(1)}$.

2. Prove that $\sum_{x>n} \frac{\chi(x)}{x} = O(1/n)$.

3. Prove that $\phi(n) = \Omega(n/\log\log n)$.

4. Prove that.

$$\sum_{p \leqslant n} \frac{1}{p} \sim \log\log n.$$

5. More generally, prove that

$$\sum_{p \leqslant n : p \equiv a \pmod q} \frac{1}{p} \sim \frac{1}{\phi(q)} \log\log n,$$

where $(a, q) = 1$.

6. Suppose that $(a_n)_n$ is a sequence of complex numbers with $\sum_n |a_n| < \infty$. Show that the product

$$\prod_{n=1}^{\infty} (1 - a_n) := \lim_{N \to \infty} \prod_{n \leqslant N} (1 - a_n)$$

converges and is zero if and only if $a_n = 1$ for some n.

7. Show that if $A \subset \mathbb{Z}/p\mathbb{Z}$ has $|A| < \log p$ then

$$\sup_{\gamma \neq 0_{\widehat{G}}} |\widehat{1_A}(\gamma)| = \Omega(|A|).$$

8. Suppose that G is a finite abelian group, $A \subset G$ has density α, and

$$S \subset \{\gamma \in \widehat{G} : |\widehat{1_A}(\gamma)| \geqslant \epsilon\alpha\}$$

for some $\epsilon \in (0, 1]$. Show that

$$|S| \leqslant \epsilon^{-2}\alpha^{-1}.$$

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9. Show that if p is prime and $w \in \mathbb{Z}/p\mathbb{Z}$ then there are elements x, y, z such that $w \equiv x^2 + y^2 + z^2 \pmod{p}$.

10. Show that there is some function $p_0(\alpha)$ such that if $p > p_0(\alpha)$ is prime and $A \subset \mathbb{Z}/p\mathbb{Z}$ has density α, then every $x \in \mathbb{Z}/p\mathbb{Z}$ has $x \equiv u^2 + a_1 + a_2 \pmod{p}$ for some $a_1, a_2 \in A$ and $u \in \mathbb{Z}/p\mathbb{Z}$.

11. Show that there is an absolute constant $C > 0$ such that if $x > C$ is odd and $N > C$ is a natural then $x \equiv u_1 + u_2 + u_3 \pmod{N}$ where u_1, u_2, u_3 are all coprime to N.

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