A short proof of the Erdős–Sós Conjecture

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Abstract

The Erdős–Sós Conjecture was recently proved by GPT-6 Astra, using a very ingenious and surprising argument. In this note, we present a simplified version of this argument in an (arguably) more natural form. We also determine the extremal graphs for the Erdős–Sós Conjecture, and prove a related conjecture of Addario-Berry, Havet, Linhares Sales, Reed and Thomassé, again determining the extremal graphs.

In the 1960s, Erdős and Sós made the following beautiful conjecture concerning the extremal numbers of arbitrary trees:

Conjecture 1. *(Erdős–Sós Conjecture [5]) Let G be a graph with average degree greater than $k-2$. Then G contains a copy of every tree T on k vertices.*

The bound is best possible for any T, as shown by a vertex-disjoint union of complete graphs K_{k-1}. Over the last 60 years, the Erdős–Sós Conjecture attracted considerable attention. Special cases were proved by, for example, Brandt and Dobson [4], McLennan [8], and Pokrovskiy [10]; for a more detailed history see the survey by Stein [12]. The conjecture was proved in a certain asymptotic sense by Rohzoň [9], and independently by Besomi, Pavez-Signé and Stein [2].

In a startling development, the Erdős–Sós conjecture was recently proved in full by GPT-6 Astra [6], with a short and ingenious argument; there is also a Lean verification [7]. An exposition of the proof was posted by Thomas Bloom [3].

The aim of this note is to present a simplified and hopefully more intuitive version of GPT-6 Astra's proof. In addition, we determine the extremal graphs, that is the graphs that have average degree $|T|-2$ but do not contain a copy of T.

Theorem 2. *Let T be a tree with k vertices, and let G be a graph with average degree $k-2$. If G does not contain a copy of T then:*

- *If $T = K_{1,k-1}$, then G is $(k-2)$-regular.*
- *If $T \neq K_{1,k-1}$, then G is a vertex-disjoint union of copies of K_{k-1}.*

We further show that with very minor modifications, the proof of the Erdős–Sós Conjecture can be used to prove a related conjecture for digraphs. An orientation of a graph is *antidirected* if it does not contain an oriented path with two edges; in other words, every non-isolated vertex is either a source or a sink. We prove the following conjecture of Addario-Berry, Havet, Linhares Sales, Reed and Thomassé.[1]

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[1] As we were writing this, we discovered that the same result was proved independently and a few days earlier by Santos, Stein and Williams [11] (see their Section 6), also by modifying the ChatGPT proof.

Conjecture 3. *(Addario-Berry, Havet, Linhares Sales, Reed and Thomassé [1]) Let D be a digraph with average outdegree greater than $k-2$. Then D contains a copy of every antidirected tree with k vertices.*

Note that this is a generalization of the Erdős–Sós Conjecture, as given a graph G we can consider the digraph formed by replacing every edge xy with a pair of directed edges, one in each direction. The antidirected condition is necessary: we can avoid any tree that is not antidirected by taking D to be a complete bipartite graph, with two classes of size $k-1$ and all edges directed from the first to the second class. For recent progress on this conjecture, see, for example, [14, 13, 11].

Finally, we determine the extremal digraphs.

Theorem 4. *Let $\overrightarrow{T}$ be an antidirected tree with k vertices, and let D be a digraph with average outdegree $k-2$. If D does not contain a copy of $\overrightarrow{T}$ then:*

- *If $\overrightarrow{T}$ is a star and the centre of the star is a source, then any $(k-2)$-out-regular digraph is extremal; if the centre is a sink then any $(k-2)$-in-regular digraph is extremal.*
- *If $\overrightarrow{T}$ is not a star, then D is a vertex-disjoint union of copies of complete digraphs on $k-1$ vertices, with every pair of vertices joined by edges in both directions.*

We give the (very short) proofs of Conjecture 1 and Conjecture 3 in sections 1 and 2. Theorems 5 and 6 are proved in sections 1 and 4 respectively.

1 Graphs

Let T be a rooted tree. Given a graph G with an ordered vertex set $v_1 < \cdots < v_n$, we say that an edge v_1v_i is *T-jumping* if G contains a copy of T rooted at v_1, with all its vertices in $\{v_1, \ldots, v_{i-1}\}$.

We will prove the following theorem.

Theorem 5. *Let G be a graph with average degree d, and let T be a rooted tree. Then the expected number of T-jumping edges in a uniformly random ordering of the vertices of G is at least $d-|T|+1$.*

To deduce the Conjecture 1 for an arbitrary tree T, apply Theorem 5 to a tree T' obtained by removing a leaf from T, and rooting the resulting tree at the neighbour of the removed leaf. If G contains no copy of T, then however its vertices are ordered, there are no T'-jumping edges, so the average degree is at most $|T'|-1 = |T|-2$.

Proof of Theorem 5. We proceed by induction on $|T|$. For $|T| = 1$ the statement is immediate: every edge incident with v_1 is T-jumping, and the expected number of such edges is d.

Now suppose that T is a tree with root r, and we have proved the result for smaller values of $|T|$. There are two cases, depending on the degree of r.

Case 1: r has degree 1. In this case let s be the neighbour of r in T, and let A be the tree obtained from T by deleting r and declaring s to be the new root. By induction, the expected number of A-jumping edges is at least $d-|A|+1 = d-|T|+2$.

Call an edge of G *A-good* if it joins v_1 to some vertex v_i, and there is a copy of A rooted at v_i with all its vertices contained in $\{v_2, \ldots, v_i\}$. A given edge xy is A-jumping in an ordering π if and only if it is A-good in the ordering π' obtained by swapping the positions of x and y. It follows that the probabilities (in a random ordering) that e is A-good and that e is A-jumping are equal. Summing over e, the expected number of A-good edges is thus at least $d-|T|+2$.

For a fixed ordering, if $v_i < v_j$ and the edges v_1v_i and v_1v_j are both A-good then the edge v_1v_j is T-jumping, since v_1v_i and the copy of A guaranteed by the goodness of v_1v_i give a copy of T rooted at v_1 (see Fig 1). Thus, in every ordering, all but at most one A-good edge is T-jumping. It

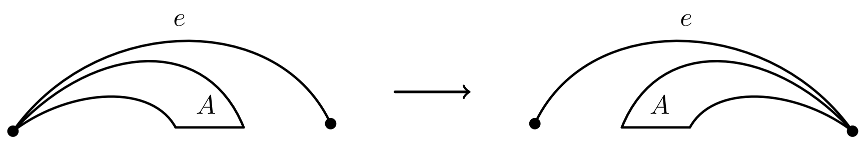

Figure 1: A rooted copy of A with a jumping edge becomes a rooted copy of T when the ends of e are exchanged

follows that the expected number of T-jumping edges in a random ordering is at least $d - |T| + 1$, as required.

Case 2: r has degree at least 2. We can divide T into two trees A and B, each with at least two vertices, overlapping only at the root r. Given an ordering π of the vertices of G, call an edge of G *B^*-jumping* if it is B-jumping in the ordering obtained from π by reversing the order of $v_2, \ldots, v_n$ (we can think of this as looking to the left of v_1 in the cyclic ordering). Let L_π be the set of A-jumping edges in π, let R_π be the set of B^*-jumping edges, and let $I_\pi = L_\pi \cap R_\pi$ be the set of edges that are jumping in both directions. Since L_π and R_π are subsets of a set of size $d_G(v_1)$, we have $|I_\pi| \geqslant |L_\pi| + |R_\pi| - d_G(v_1)$, so for a uniformly random permutation π it follows by induction that

$$\begin{aligned}\mathbb{E}|I_\pi| &\geqslant \mathbb{E}|L_\pi| + \mathbb{E}|R_\pi| - \mathbb{E}d_G(v_1)\\ &\geqslant (d - |A| + 1) + (d - |B| + 1) - d\\ &= d - |T| + 1.\end{aligned}$$

To convert this into a bound on the number of T-jumping edges, we apply an involution to the set of orderings. Given an ordering $\pi : v_1 < \cdots < v_n$, let i be maximal such that G contains a copy of B rooted at v_1 and with all other vertices in $\{v_i, \ldots, v_n\}$; if no such i exists then set $i = 2$. Let π' be the ordering obtained from π by concatenating v_1 and $v_n, v_{n-1}, \ldots, v_i$ and $v_2, \ldots, v_{i-1}$ (thus $v_i, \ldots, v_n$ reverse order and are inserted between v_1 and v_2). This is a bijection: for the inverse map, let i be minimal such that G contains a copy of B rooted at v_1 and with all other vertices in $\{v_2, \ldots, v_i\}$, or take $i = n$ if there is no such i; we then reverse the order of $\{v_2, \ldots, v_i\}$ and move them to the end. Thus π' is also uniformly random.

Finally, we note that if e belongs to I_π, then e is a T-jumping edge in π'. Indeed, if $e \in I_\pi$ joins v_1 to w, then in π there are copies A_0, B_0 of A, B rooted at v_1 such that the rest of A_0 lies to the left of w and the rest of B_0 lies to the right. Choosing the 'right-most' such copy B_0, in π' all vertices of $A_0 \cup B_0$ lie to the left of w (see Fig 2). Thus the expected number of T-jumping edges in a uniformly random ordering of the vertices of G is at least $d - |T| + 1$. □

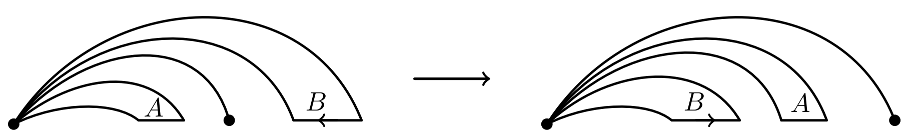

Figure 2: An element of I_π becomes part of a copy of T with a jumping edge (B reverses order)

The proof above is inspired by Thomas Bloom's presentation of GPT-6 Astra's argument. However, it is written in (arguably) more natural language, and Case 2 in the proof is simplified, allowing us to use a simple involution on permutations.

2 Digraphs

An orientation of a graph is *antidirected* if it does not contain an oriented path with two edges; in other words, every non-isolated vertex is either a source or a sink. We think of a single-vertex tree as having two antidirected orientations, one in which the vertex is a source, and one where it is a sink.

Let $\overrightarrow{T}$ be a rooted tree with an antidirected orientation. Given a digraph D with an ordered vertex set $v_1 < \cdots < v_n$, we say that an edge e joining v_1 and v_i is $\overrightarrow{T}$*-jumping* if

- G contains a copy of $\overrightarrow{T}$ rooted at v_1, with all its vertices in $\{v_1, \ldots, v_{i-1}\}$; and
- the orientation of e is such that adding e to $\overrightarrow{T}$ gives an antidirected tree.

We allow the case $|\overrightarrow{T}| = 1$, in which case we declare the vertex to be a source or a sink, which determines the orientation for jumping edges.

We will prove the following theorem.

Theorem 6. *Let D be a digraph with average outdegree d, and let $\overrightarrow{T}$ be a rooted tree with an antidirected orientation. Then the expected number of $\overrightarrow{T}$-jumping edges in a uniformly random ordering of the vertices of D is at least $d - |\overrightarrow{T}| + 1$.*

The proof is essentially the same as before: for the base case, if $|\overrightarrow{T}| = 1$ and its vertex is declared to be a source then a $\overrightarrow{T}$-jumping edge in an ordering is just an outedge from the first vertex, and the expected number of these is the average outdegree d; similarly, if $|\overrightarrow{T}|$ is a sink then the expected number of jumping edges is the average indegree d.

The two cases now work as in the proof of Theorem 5 (noting that in Case 2, the quantity $d_G(v_1)$ is replaced by the outdegree or indegree of v_1 as appropriate): the tree constructed in both cases remains antidirected.

To deduce the Conjecture 3 for an arbitrary tree $\overrightarrow{T}$ with an antidirected orientation, apply Theorem 6 to a tree $\overrightarrow{T}'$ obtained by removing a leaf from $\overrightarrow{T}$, and rooting the resulting tree at the neighbour of the removed leaf. If D contains no copy of $\overrightarrow{T}$, then however its vertices are ordered, there are no $\overrightarrow{T}'$-jumping edges, so the average outdegree is at most $|\overrightarrow{T}'| - 1 = |\overrightarrow{T}| - 2$.

3 Extremal graphs for Erdős–Sós

Proof of Theorem 2. Suppose that $e(G) = (k-2)|G|/2$ and T is a tree with k vertices. If G does not contain a copy of T then there are two cases:

- If $T = K_{1,k-1}$ then G has no vertex with degree more than $k-2$. It follows that G is $(k-2)$-regular; and any $(k-2)$-regular graph is extremal.
- If T is not a star, then any vertex-disjoint union of copies of K_{k-1} is extremal. We will show that every extremal graph has this form.

The first bullet is clear; all that remains is to prove that if T is not a star, then any vertex-disjoint union of copies of K_{k-1} is extremal.

Let T' be the tree obtained by deleting the leaves of T: since T is not a star, T' must have at least two vertices. Let x be a leaf of T', and let y be the neighbour of x in T'. Let S be the set of leaves of T adjacent to x, and let A be the rooted tree obtained from T by deleting x and all leaves of T adjacent to it, and declaring y to be the root (see Fig 3). By construction we have $s := |S| \geqslant 1$ and $|A| = k - s - 1 \geqslant 2$ (the latter since y and at least one leaf of T belong to A).

We break the proof into three steps.

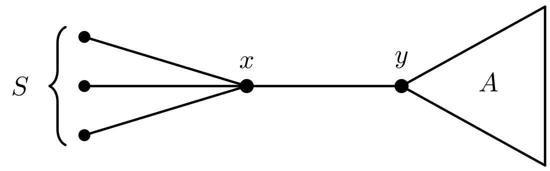

Figure 3: The tree T

(1) Every ordering has exactly s edges that are A-good.

By Theorem 5, the expected number of A-jumping edges in a random ordering of the vertices of G is at least s. It follows (as in the proof of Case 1) that the expected number of A-good edges is at least s. If there is an ordering with at least $s+1$ A-good edges, then we obtain a copy of T: take any $s+1$ such edges, and add the copy of A guaranteed by the earliest edge. It follows that every ordering has exactly s edges that are A-good.

(2) G is $(k-2)$-regular.

For any vertex v, consider any ordering with v as first vertex, followed by its neighbours and then the remaining vertices: since s edges are A-good then the copy of A guaranteed by the earliest of these edges is contained in the neighbourhood of v and is disjoint from the other A-good edges, so v has at least $|A|+s-1=k-2$ neighbours. It follows that G is $(k-2)$-regular.

(3) Every component of G is a complete graph.

Suppose not. We will show for a contradiction that G contains a copy of T, by constructing a suitable embedding $\phi : V(T) \to V(G)$.

Since G has a non-complete component it contains vertices x, y and z such that xy, $yz \in E(G)$ but $xz \notin E(G)$. Let $v_1 v_2 \cdots$ be a longest path in T, which has at least 4 vertices, since T is not a star. Set $\phi(v_2)=x$, $\phi(v_3)=y$ and $\phi(v_4)=z$. Now continue greedily step-by-step, each time embedding a vertex u of T with a (necessarily unique) already embedded neighbour v. At each step, the only constraint is that $\phi(u)$ should be an unused neighbour of $\phi(v)$. Since $\phi(v)$ has $k-2$ neighbours this is always possible, unless u is the last vertex to be embedded, and all vertices other than u, v are mapped to neighbours of $\phi(v)$. To avoid this, we simply embed v_1 last (which is possible since it's a leaf), noting that $\phi(v_4)=z$ is not a neighbour of $\phi(v_2)=x$. $\square$

4 Extremal digraphs

Proof of Theorem 4. Now suppose that D is a digraph with n vertices and $(k-2)n$ edges, and $\overrightarrow{T}$ is an antidirected tree with k vertices. If D does not contain a copy of $\overrightarrow{T}$ then there are two cases:

- If $\overrightarrow{T}$ is a star then the antidirected condition implies that the centre of the star is either a source or a sink. In the first case, any $(k-2)$-out-regular digraph is extremal; and in the second, any $(k-2)$-in-regular digraph is extremal. It is clear that these are the only extremal digraphs.
- If $\overrightarrow{T}$ is not a star, then any vertex-disjoint union of copies of complete digraphs on $k-1$ vertices, with all edges directed in both directions, is extremal. We will show that every extremal digraph has this form.

In the graph case, the key step was to show that G is regular. In the digraph case, we need biregularity. We start by showing that this is sufficient.

(1) It suffices to show that D is both in- and out-regular.

Suppose that D is bi-regular, so every vertex has in-degree and out-degree equal to $k-2$. Let $v_1v_2v_3v_4$ be a path in $\overrightarrow{T}$ with v_1 a leaf, and without loss of generality suppose the edges are directed $v_1 \leftarrow v_2 \rightarrow v_3 \leftarrow v_4$. Suppose first that D contains a vertex y whose in-neighbourhood is not bi-complete, so y has in-neighbours x and z with $x \not\rightarrow z$. Set $\phi(v_2) = x$, $\phi(v_3) = y$ and $\phi(v_4) = z$, and embed greedily as before. When embedding u with a unique already-embedded neighbour v, we need either an unused in- or unused out-neighbour of $\phi(v)$. The only possible problem is at the last step, but we take v_1 last, and since $\phi(v_4)$ is not an out-neighbour of $\phi(v_2)$, there is at least one unused out-neighbour of $\phi(v_2)$ to map v_1 to.

Thus we may assume that every in-neighbourhood is bi-complete. But this easily implies that each component of D is bi-complete. Let S be the in-neighbourhood of some vertex v, so $|S| = k-2$. Then every vertex in S has in- and out- degree $k-3$ within S, so exactly one in-edge from outside S. If all such edges come from v, then the component containing v is bicomplete. Otherwise, there is an edge $x \rightarrow s$ for some $s \in S$ and $x \notin S \cup \{v\}$. There is at least one other vertex $t \in S$ (since $k \geqslant 4$ since $\overrightarrow{T}$ is not a star), and now by bi-completeness of the in-neighbourhood of s, the edge $t \rightarrow x$ is present. But then the out-degree of t is at least $k-1$, a contradiction.

Label the vertices of $\overrightarrow{T}$ as in Fig 3: without loss of generality, we assume that x is a source, with $s \geqslant 1$ adjacent leaves, and that $\overrightarrow{A}$ is an oriented tree with at least two vertices. Note that the edge xy is directed from x to y, and that the leaves adjacent to x are sinks (see Fig 4). We now proceed in several steps.

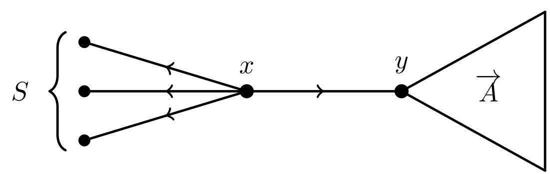

Figure 4: The tree $\overrightarrow{T}$

(2) Every ordering has exactly s edges that are $\overrightarrow{A}$-good.

Consider first the subtree $\overrightarrow{A}$ rooted at y, noting that y is a sink. As in the graph case, the expected number of $\overrightarrow{A}$-jumping edges is at least s, and so the expected number of $\overrightarrow{A}$-good edges is at least s; if there are at least $s+1$ $\overrightarrow{A}$-good edges then we obtain a (correctly oriented) copy of $\overrightarrow{T}$. It follows that every ordering has exactly s $\overrightarrow{A}$-good edges.

(3) D is $(k-2)$-out-regular.

For any vertex v, consider any ordering with v as first vertex, followed by its out-neighbours and then the remaining vertices: since s edges are $\overrightarrow{A}$-good the copy of $\overrightarrow{A}$ guaranteed by the earliest of these edges is contained in the out-neighbourhood $\Gamma^+(v)$ of v and is disjoint from the other $\overrightarrow{A}$-good edges, so v has at least $|\overrightarrow{A}| + s - 1 = k-2$ out-neighbours. Since this holds for every vertex, it follows that D is $(k-2)$-out-regular.

(4) We may suppose that $\overrightarrow{A}$ contains vertices a, b such that a is a source and b is a leaf of $\overrightarrow{T}$ adjacent to a.

As in the undirected case, consider the tree $\overrightarrow{T}'$ obtained from $\overrightarrow{T}$ by deleting all its leaves. Then $\overrightarrow{T}'$ has at least two leaves, one of which is x; let a be another leaf. If a is a sink, then using a in place of x, we get a decomposition of $\overrightarrow{T}$ as in Fig 4, except with a sink at a in place of the source at x. Arguing as above, we get that D is $(k-2)$-in-regular; by (1) and (3), this is enough. Otherwise,

a is a source. Since a is a vertex of $\overrightarrow{T}'$, there is some leaf b of $\overrightarrow{T}$ adjacent to a, and a, b both belong to $\overrightarrow{A}$.

(5) For every vertex v, every set W of $|\overrightarrow{A}| = k - s - 1$ out-neighbours of v, and every z in W, there is a copy of $\overrightarrow{A}$ rooted at z and with vertex set W.

As in (3), consider an ordering with first vertex v, followed by its out-neighbours $w_1, \ldots, w_{k-2}$ and then the remaining vertices. If vw_i is $\overrightarrow{A}$-good then $i \geqslant |\overrightarrow{A}| = k-s-1$ (as we need to fit in a copy of $\overrightarrow{A}$); and since there are exactly s $\overrightarrow{A}$-good edges, it follows that vw_i is $\overrightarrow{A}$-good for all $i \geqslant k-s-1$. In particular, there is a copy of $\overrightarrow{A}$ rooted at w_{k-s-1} and with vertex set $\{w_1, \ldots, w_{k-s-1}\}$. Since we can take the w_i in any order, the claim follows.

(6) Suppose that u, v have a common out-neighbour z, and u and v are not bi-adjacent. Then u and v are non-adjacent, and $\Gamma^+(u) = \Gamma^+(v)$.

We may suppose without loss of generality that $u \not\to v$. By (5) there is a copy $\overrightarrow{A}_0$ of $\overrightarrow{A}$ contained in $\Gamma^+(u)$ and rooted at z. By assumption, $\overrightarrow{A}_0$ does not contain v. If $\Gamma^+(v) \neq \Gamma^+(u)$ then by out-regularity there is some $w \in \Gamma^+(v) \setminus \Gamma^+(u)$ (we may have $w = u$, but this is no problem). Then adding v and w to $\overrightarrow{A}_0$ gives a copy of $\overrightarrow{T}$, a contradiction. Thus $\Gamma^+(u) = \Gamma^+(v)$, and in particular they are non-adjacent.

(7) For every vertex u there is $v \in \Gamma^+(u)$ such that $u \leftrightarrow v$ and their out-neighbourhoods are equal (apart from each other).

By (5) there is a copy $\overrightarrow{A}_0$ of $\overrightarrow{A}$ in $\Gamma^+(u)$. Now by (4), $\overrightarrow{A}$ contains a source a and a leaf b of $\overrightarrow{T}$ adjacent to a (so b is not the root of $\overrightarrow{A}$). Let v, w be the images of a and b in $\overrightarrow{A}_0$. Suppose v has an out-neighbour w' outside $\Gamma^+(u) \cup \{u\}$. Then we may replace w by w' to obtain a new copy of $\overrightarrow{A}$, and then add u and w to make a copy of $\overrightarrow{T}$. Hence $\Gamma^+(v) \subset \Gamma^+(u) \cup \{u\} \setminus \{v\}$, a set of size $k-2$. Thus $\Gamma^+(v)$ is equal to this set.

(8) D is in-regular.

If not, then since the average in-degree is $k-2$, there is some vertex v whose in-neighbourhood I has $|I| \geqslant k-1$. If I is an independent set, then by (6) all vertices in I have the same set J of out-neighbours which is necessarily disjoint from I. We thus have a complete bipartite graph with both vertex classes of size at least $k-2$ and all edges directed from the first class to the second class. As $\overrightarrow{T}$ is not a star, this embeds $\overrightarrow{T}$. We may thus suppose that I is not independent.

By (6) each pair in I is either non-adjacent or bi-adjacent. Moreover, if u, v, and v, w are non-adjacent, then (applying (6) to u, v, which shows that $w \notin \Gamma^+(u)$), so are u, w. Hence the subgraph induced by I is complete multipartite with all edges bidirectional. Since I is not independent, there are at least two vertex classes.

Now by (7), every vertex is in a bi-directed edge, so there is $u \in I$ with $v \to u$. Suppose w is in a different class within I, and that $v \not\to w$. Since $u \leftrightarrow w$, v and w have a common out-neighbour u, so we obtain a contradiction to (6). Hence v is joined to every vertex not in the same class as u. But applying this once more with w in place of u, v is joined to every vertex in I, and so has out-degree at least $k-1$, a contradiction.

We have shown that D is both in- and out-regular, and so we are done by (1). $\square$

AI declaration: ChatGPT was used to help draw pictures.

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