Planar percolation and the loop O(n) model

Matan Harel

Northeastern University

Joint Work with Alexander Glazman and Nathan Zelesko

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Oxford Discrete Mathematics and Probability Seminar

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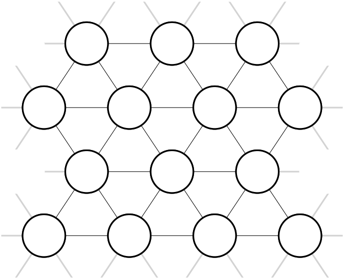

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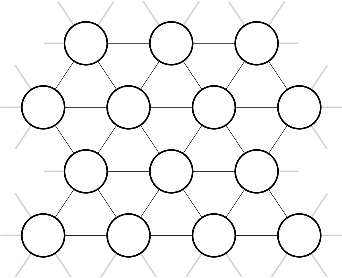

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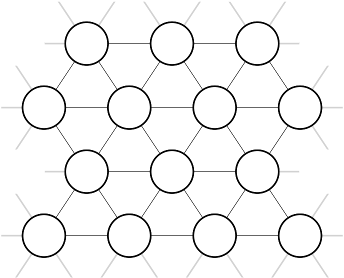

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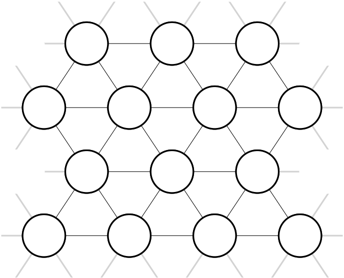

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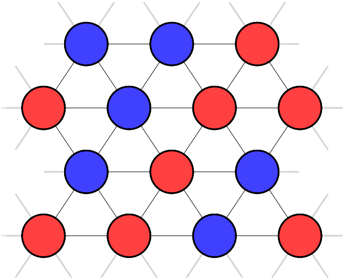

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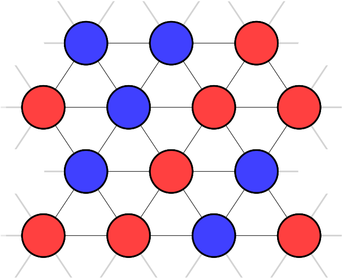

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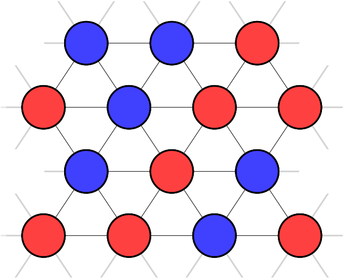

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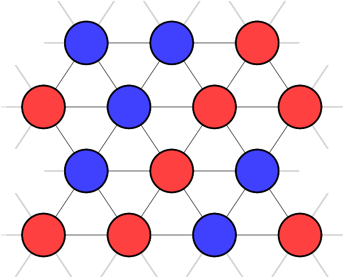

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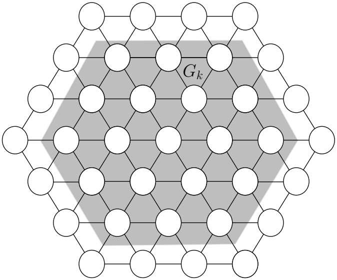

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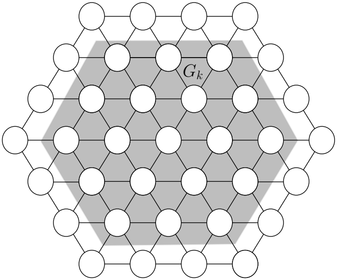

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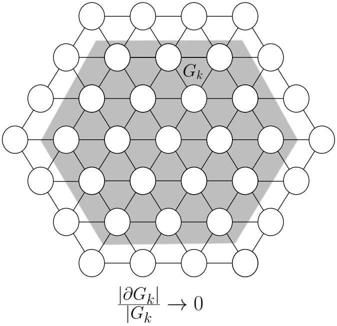

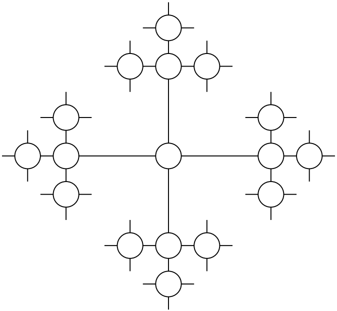

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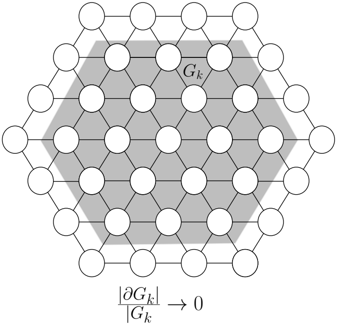

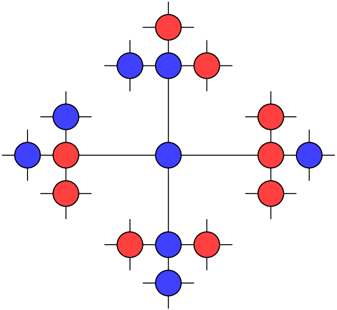

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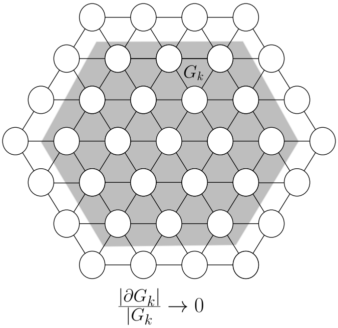

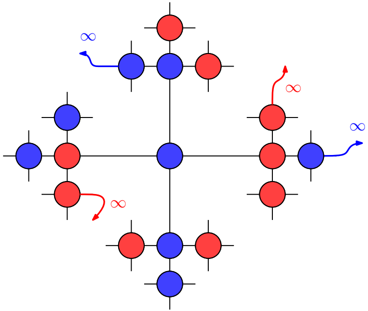

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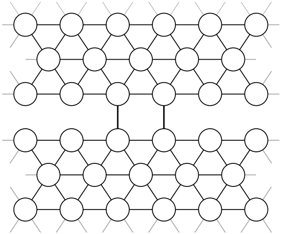

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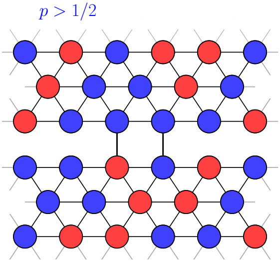

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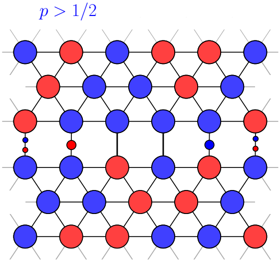

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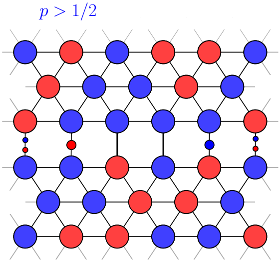

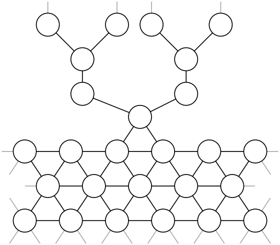

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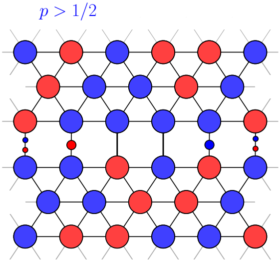

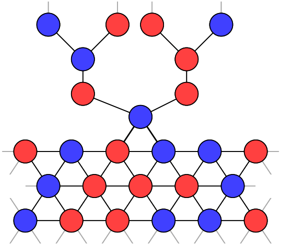

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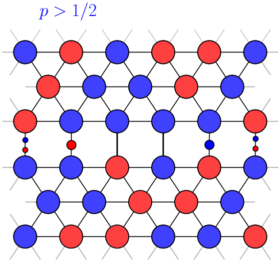

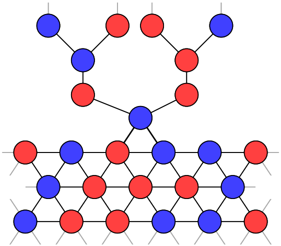

Conjecture 8 (Benjamini–Schramm, 1996)

Let G be *any* locally finite, connected planar graph. Then, for Bernoulli percolation of parameter $1/2$, $N_\infty(\{\sigma = 1\}) \in \{0, \infty\}$.

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In particular, the result holds for Bernoulli percolation of parameter $p \leq 1/2$.

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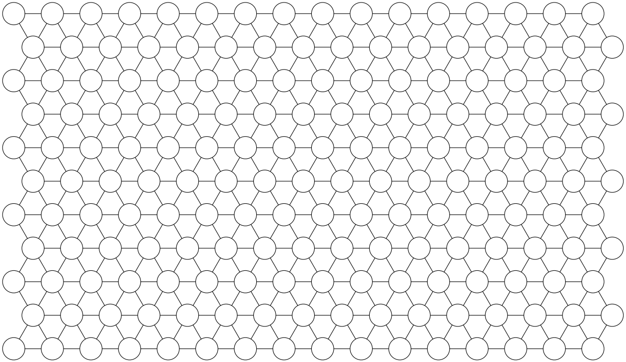
Peled (2020) showed that $\exists p^* > 0$ such that, $p < p^*$ implies the circle packing of Bernoulli percolation exhibits exponential decay in *geometric* distance.

Application: Divide and Color Models

Corollary (Glazman– H. – Zelesko, 2025+)

Consider a translation-invariant partition of $\mathbb{T}$ into *finite* classes.



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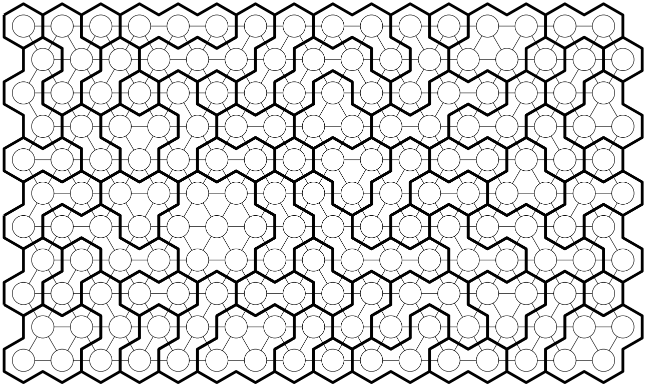

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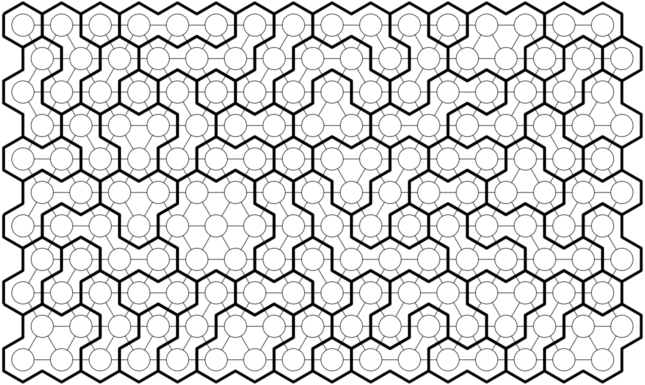

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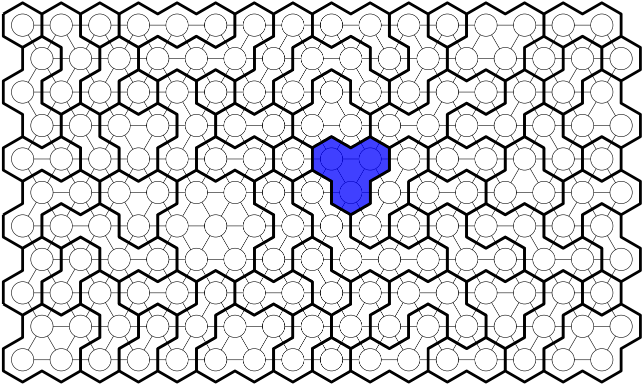

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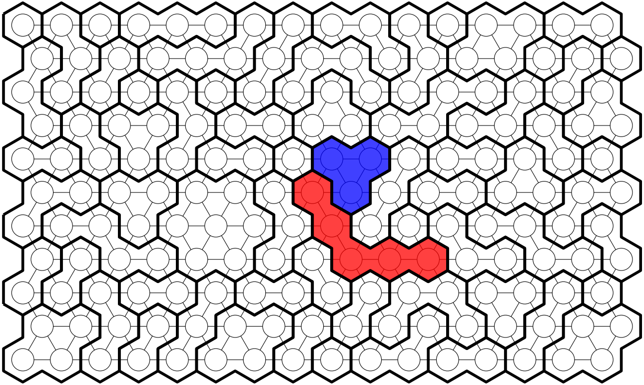

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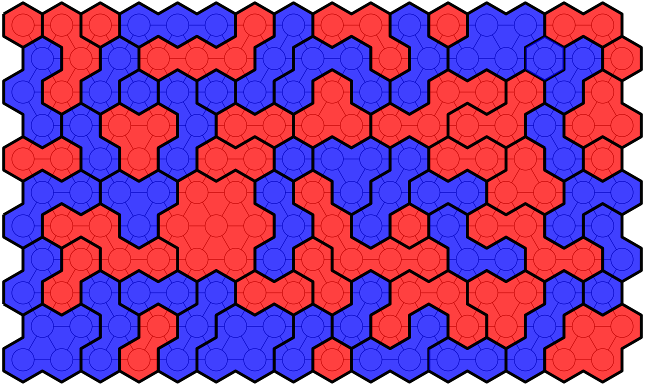

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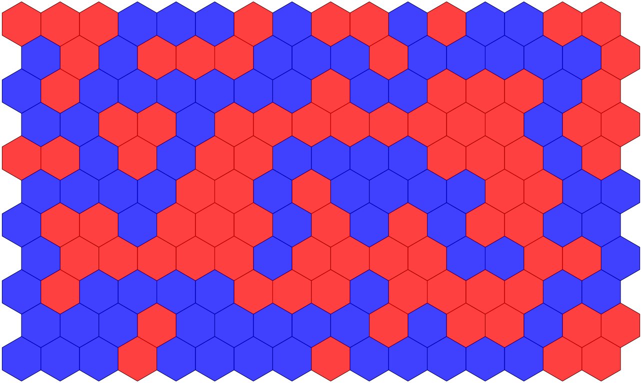

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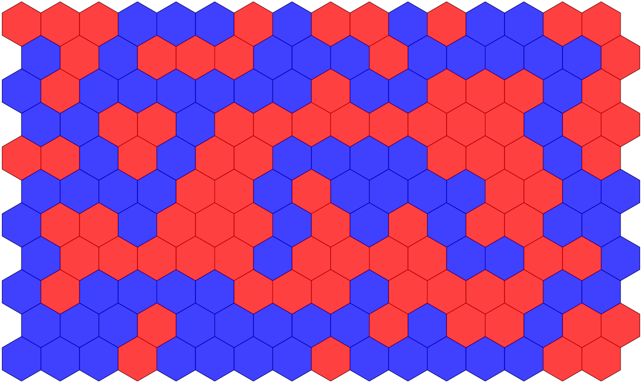
Consider a translation-invariant partition of $\mathbb{T}$ into *finite* classes. Color each class red or blue, uniformly and independently. Then the resulting 2-coloring is translation invariant and has finite connected components.



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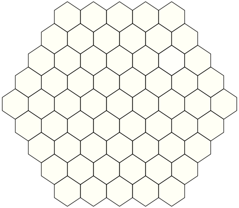

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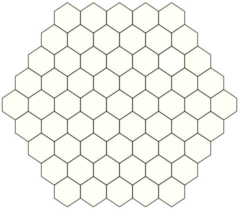
We say ω, a subgraph of $\mathcal{D}$, is a *loop configuration* if every vertex in $\mathcal{D}$ has even degree in ω.



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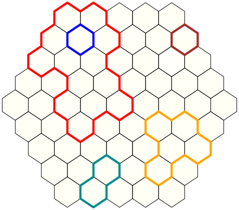
Let $\mathcal{D}$ be a finite domain in the hexagonal lattice $\mathbb{H}$.

We say ω, a subgraph of $\mathcal{D}$, is a *loop configuration* if every vertex in $\mathcal{D}$ has even degree in ω.



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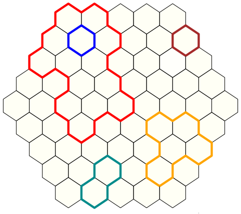

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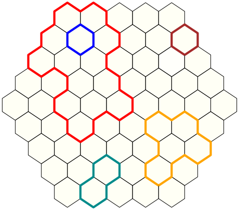

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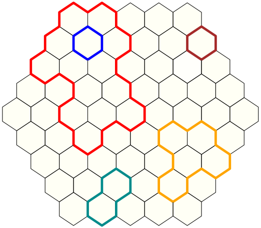

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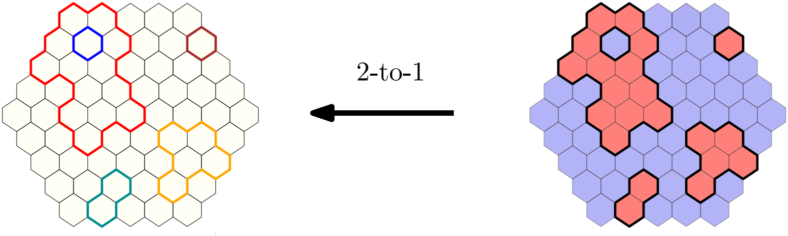

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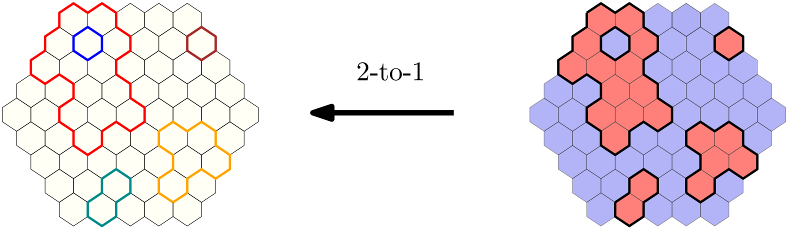

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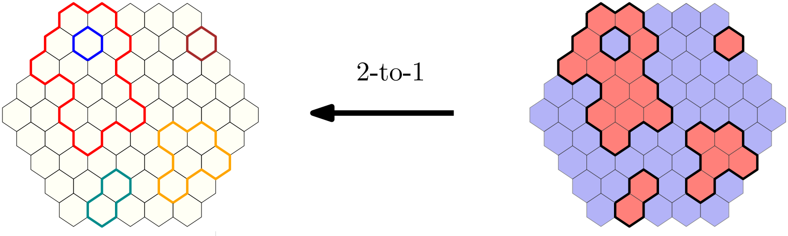

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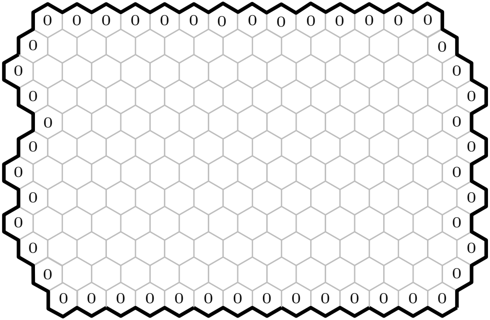

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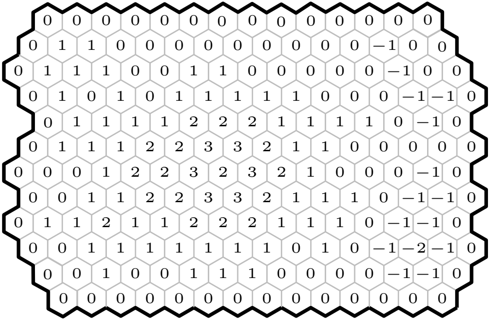

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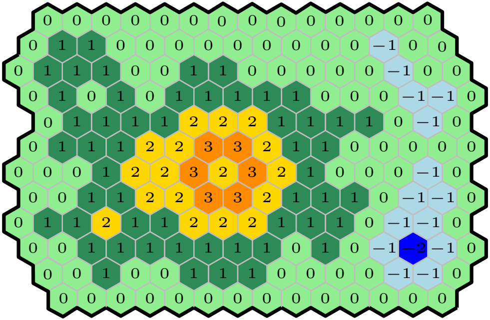

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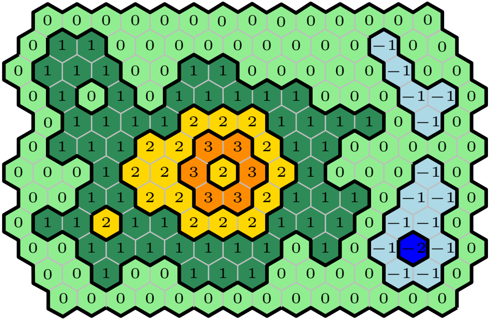

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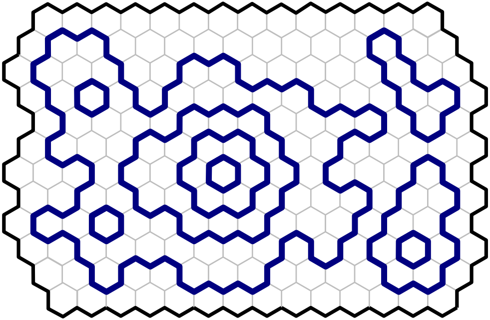

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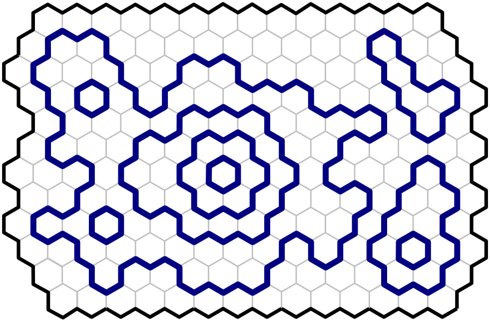

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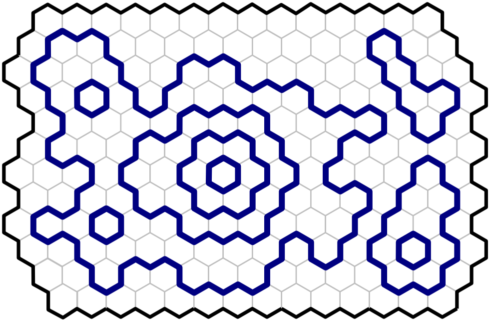

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Therefore, loop $O(2)$ model $\leftrightarrow$ integer valued Lipschitz functions.

Geometry of the loop $O(n)$ model

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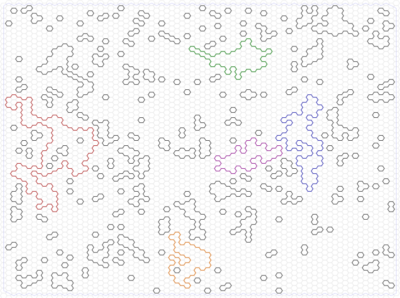

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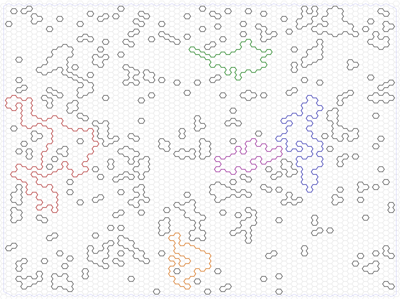

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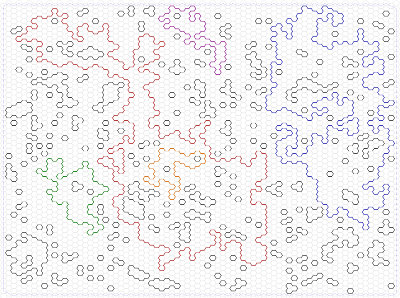

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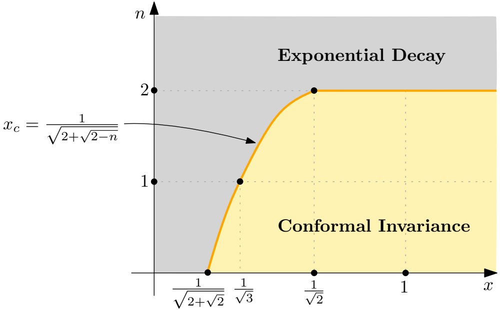

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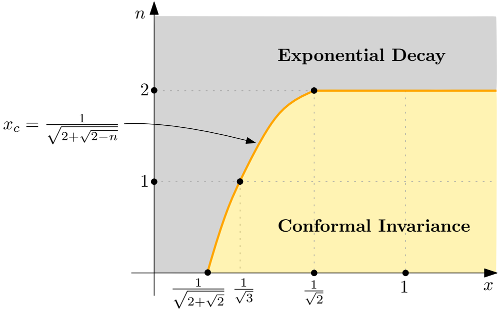

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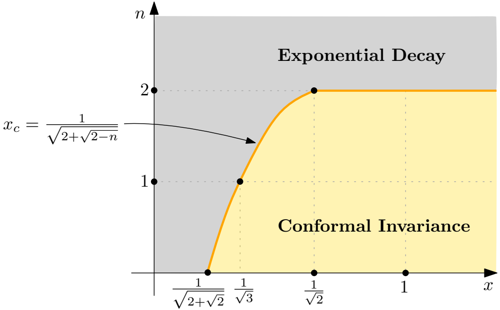

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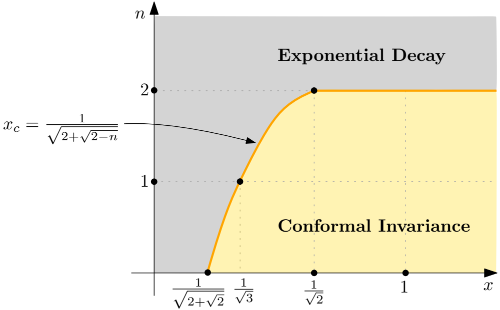

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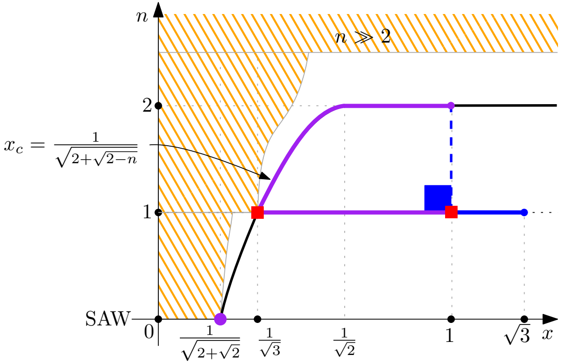

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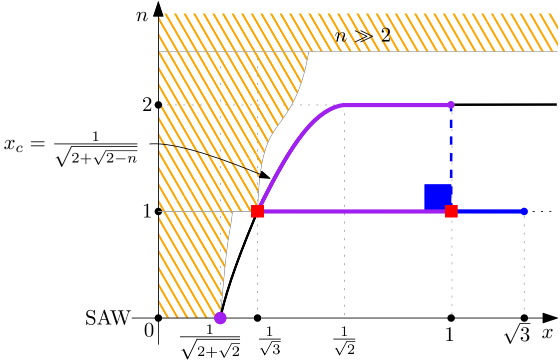
Theorem (Glazman– H. – Zelesko, 2025+)

If $1 \leq n \leq 2$ and $1/\sqrt{2} \leq x \leq 1$, any translation-invariant Gibbs measure has long loops.



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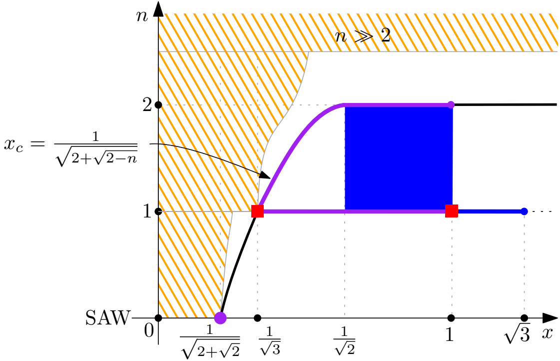

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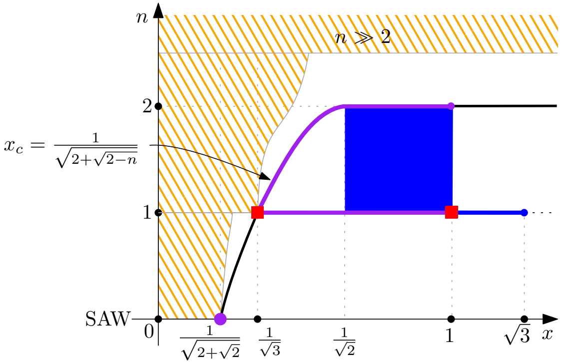

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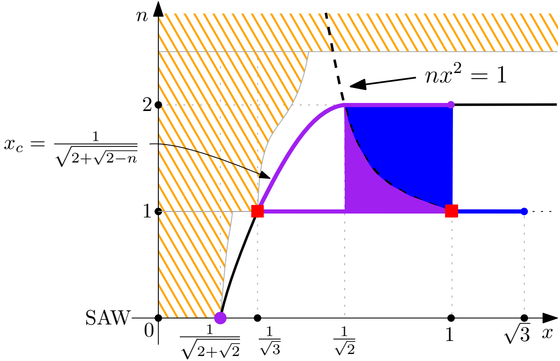

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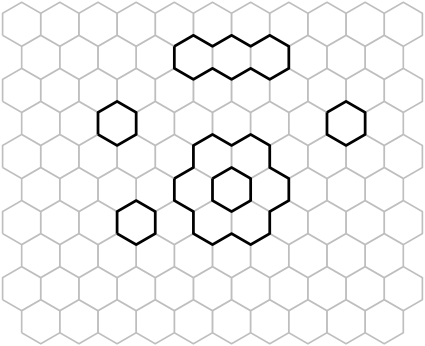

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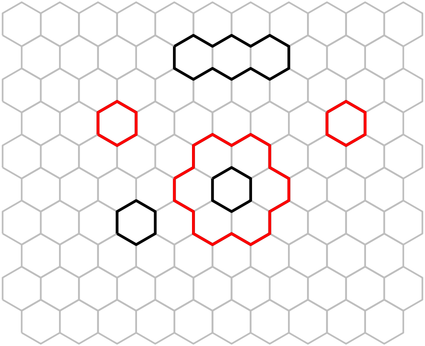

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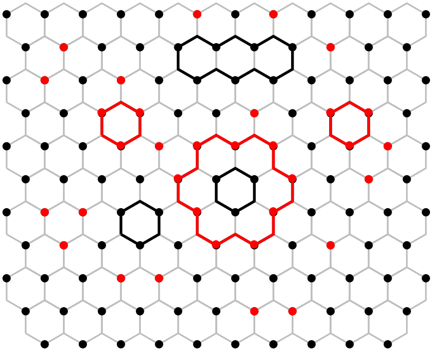

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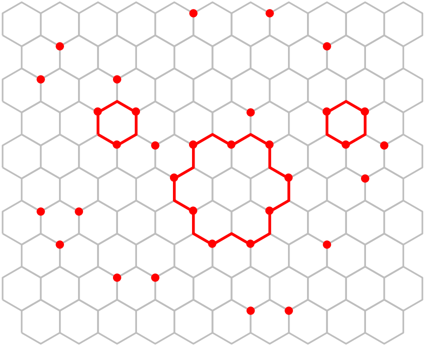

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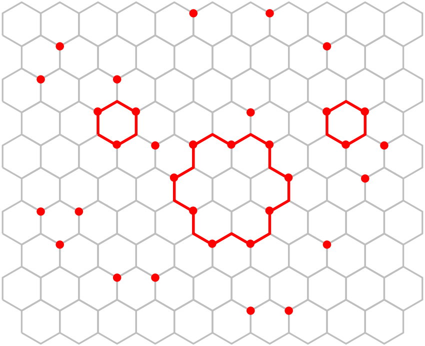

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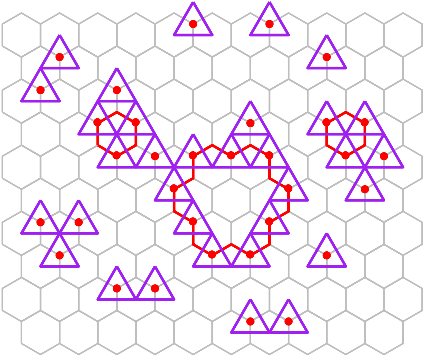

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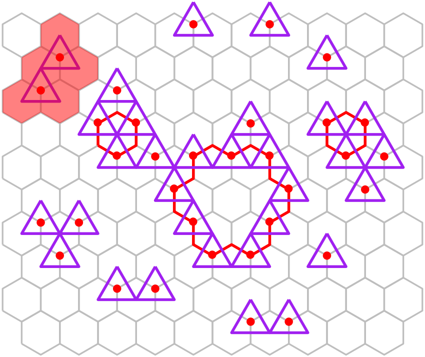

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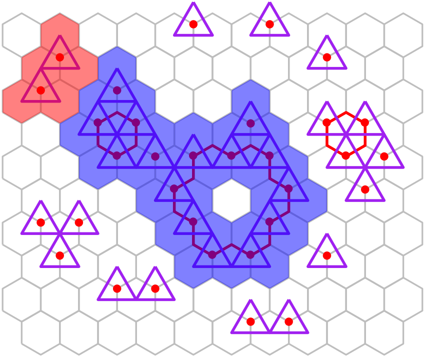

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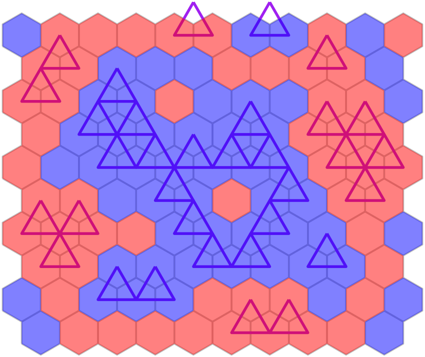

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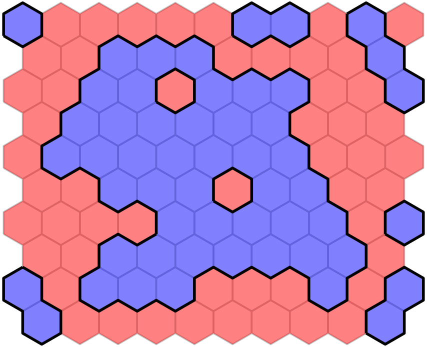

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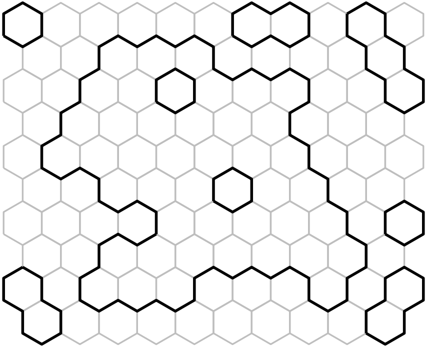

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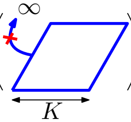
$\varepsilon > \mathbb{P}\Bigg($  $\Bigg)$

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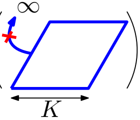
$\varepsilon > \mathbb{P}\Big($  $\Big)$

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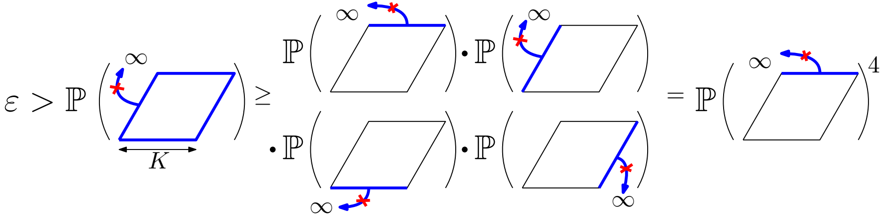

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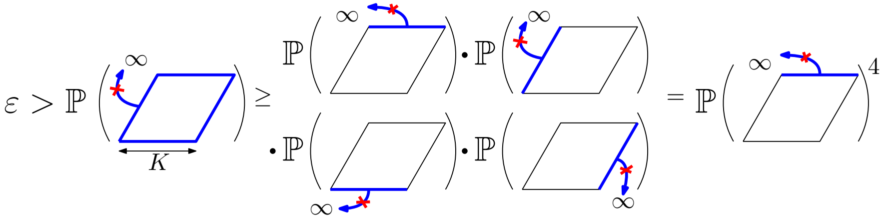

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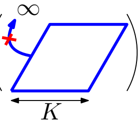
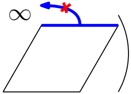
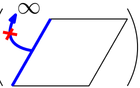
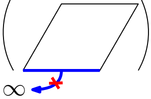
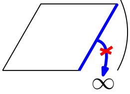
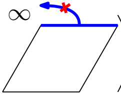
$\varepsilon > \mathbb{P}\Big($ 

$\Big) \geq \mathbb{P}\Big($ 

$\Big) \cdot \mathbb{P}\Big($ 

$\Big) \cdot \mathbb{P}\Big($ 

$\Big) \cdot \mathbb{P}\Big($ 

$\Big) = \mathbb{P}\Big($ 

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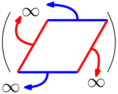
$1 - 4\varepsilon^{1/4} < \mathbb{P}\Big($ 

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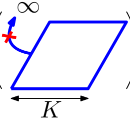
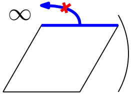
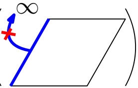
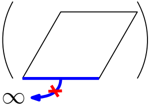
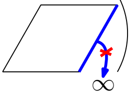
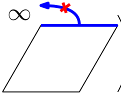
$\varepsilon > \mathbb{P}\Big($ 

$\Big) \geq \mathbb{P}\Big($ 

$\Big) \cdot \mathbb{P}\Big($ 

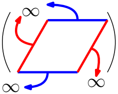
$\Big) \cdot \mathbb{P}\Big($ 

$\Big) \cdot \mathbb{P}\Big($ 

$\Big) = \mathbb{P}\Big($ 

$\Big)^4$

Now, by a union bound and color switching ($p = 1/2$),

$1 - 4\varepsilon^{1/4} < \mathbb{P}\Big($ 

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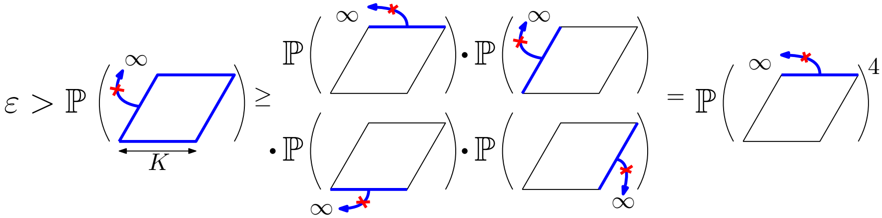

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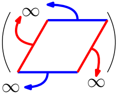
$1 - 4\varepsilon^{1/4} < \mathbb{P}\Big($  $\Big) \leq \mathbb{P}\big(\text{exists} \geq 2 \text{ open or} \geq 2 \text{ closed infinite clusters}\big)$

By ergodicity, $N_\infty(\{\sigma = 1\}) = \infty$, almost surely.

Proof Outline

Theorem (Glazman– H. – Zelesko, 2025+)

Let G be a locally finite, connected planar graph, and $\sigma : V(G) \to \{0, 1\}$ a site percolation process that satisfies:

- tail triviality,
- positively association (i.e. the FKG inequality), and
- stochastic domination of $\{\sigma = 1\}$ by $\{\sigma = 0\}$.

Then $N_\infty(\{\sigma = 1\}) = 0$ a.s., or $N_\infty(\{\sigma = 1\}) = \infty$ a.s.

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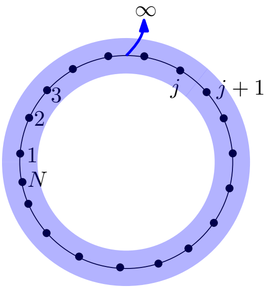

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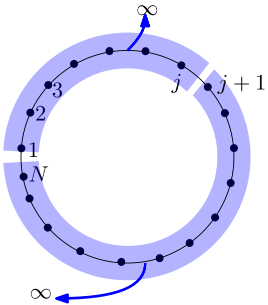

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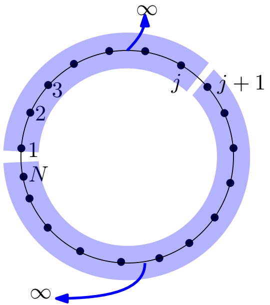
Pick *j* to be the largest index so that

$$\mathbb{P}\left([1,j] \not\leftrightarrow \infty\right) > \sqrt{\varepsilon/2}.$$



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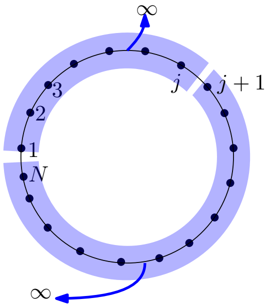

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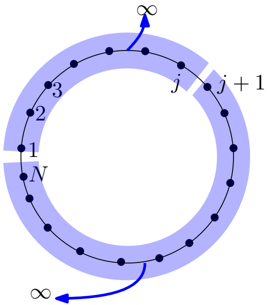

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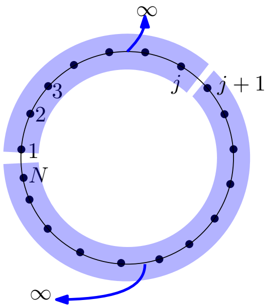

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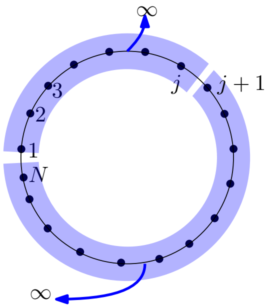

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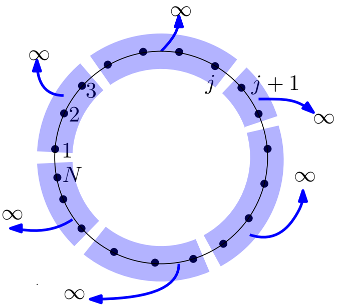

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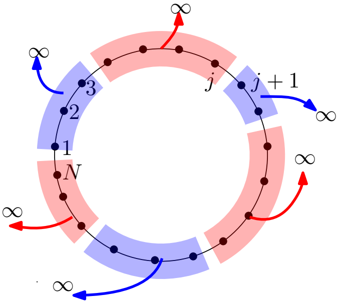

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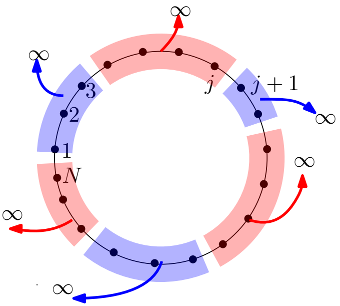

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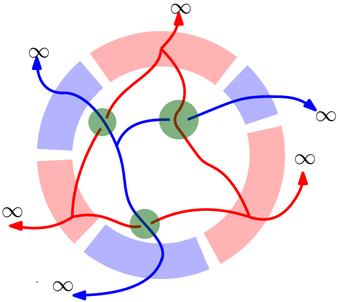

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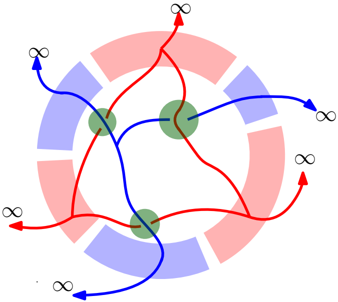

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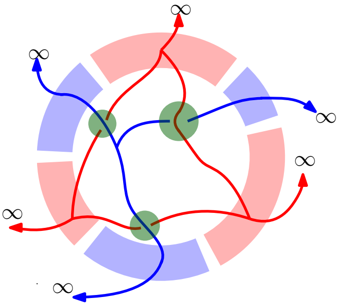

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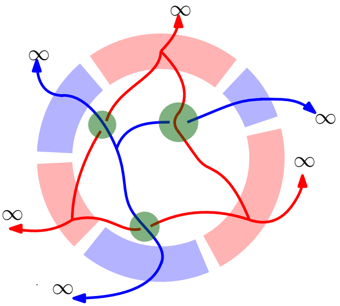

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- BKT phase transitions on more general planar geometries?

Thank you!