Locally bipartite subgraphs via multicolor Ramsey numbers

Raphael Steiner

ETH zürich

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A combinatorial classic: Chromatic number and girth

Theorem (Erdős, 1959)

For all k, g, there is a graph G with

$$\chi(G) \geq k \qquad \textit{and} \qquad \mathrm{gir}(G) \geq g.$$

How special are these graphs? Can we find them everywhere?

Conjecture (Erdős–Hajnal, 1969)

For every $g \geq 4$, there is a function f_g such that every graph with $\chi(G) \geq f_g(k)$ has a subgraph G' with

$$\chi(G') \geq k \qquad \text{and} \qquad \mathrm{gir}(G') \geq g.$$

Rödl (1977): True for $g = 4$. Elegant proof, but bound on $f_4(k)$ is tower of height $\Theta(k^2 \log k)$.
Pettie–Tardos–Walczak (2025): If $f_5(k)$ exists, it must grow at least as a tower of height linear in k.

The odd-girth version

$\mathrm{ogir}(G)$ is the length of a shortest odd cycle

Conjecture (Mohar–Wu, 2018)

For every odd $g \geq 5$, there is a function h_g such that every G with $\chi(G) \geq h_g(k)$ has a subgraph G' with

$$\chi(G') \geq k \qquad \text{and} \qquad \mathrm{ogir}(G') \geq g.$$

"Must graphs with huge chromatic number have locally bipartite subgraphs with large chromatic number?"

Conjecture (Erdős-Hajnal, 1975)

For every infinite cardinal $\mathfrak{m}$ and every odd $g \geq 5$, every graph of chromatic number $\mathfrak{m}$ has a subgraph of chromatic number $\mathfrak{m}$ and odd-girth at least g.

Main result: locally bipartite subgraphs

Theorem (S., 2026)

For every fixed $p \geq 1$:

$$\chi(G) \geq \exp^{(p)}(k^{3+o(1)}) \implies \exists G' \subseteq G : \begin{cases} \chi(G') \geq k, \\ \operatorname{ogir}(G') \geq 2p+3. \end{cases}$$

p	excluded cycles	sufficient $\chi(G)$
1	C_3	$\exp(k^{3+o(1)})$
2	C_3, C_5	$\exp(\exp(k^{3+o(1)}))$
3	C_3, C_5, C_7	$\exp(\exp(\exp(k^{3+o(1)})))$

Corollary (Easy using De Bruijn-Erdős)

Every graph with infinite chromatic number has subgraphs of infinite chromatic number with arbitrarily large odd-girth.

A more general meta-theorem

Theorem (S., 2026)

Let f be χ-amenable. For every fixed $p \geq 1$:

$$f(G) \geq \exp^{(p)}(k^{3+o(1)}) \implies \exists G' \subseteq G : \begin{cases} f(G') \geq k, \\ \operatorname{ogir}(G') \geq 2p+3. \end{cases}$$

Many well-studied graph parameters other than χ are χ-amenable:

Theorem (S., 2026)

The following parameters are χ-amenable:

- *The fractional chromatic number χ_f $\Rightarrow$ fractional variant (conjectured by Mohar–Wu 2018, who did triangle-free case)*
- *The Hall-ratio $\rho(G) = \max_{\varnothing \neq H \subseteq G} \frac{|V(H)|}{\alpha(H)}$,*
- *The strict vector chromatic number $\overline{\vartheta}(G) := \vartheta(\overline{G})$ (a.k.a. Lovász Theta function of the complement)*

Multicolor Ramsey numbers of odd cycles

$\mathcal{O}_p$ denotes the set of the first p odd cycles (C_3 through C_{2p+1}).

Definition

$R_q(\mathcal{O}_p)$ is the smallest N such that every q-edge-coloring of K_N contains a monochromatic member of $\mathcal{O}_p$.

Best-studied case: $p = 1$ (multicolor Ramsey number of triangles).
Best bounds for a long time (Ageron et al. 22, Xu-Xi-Chen 02):

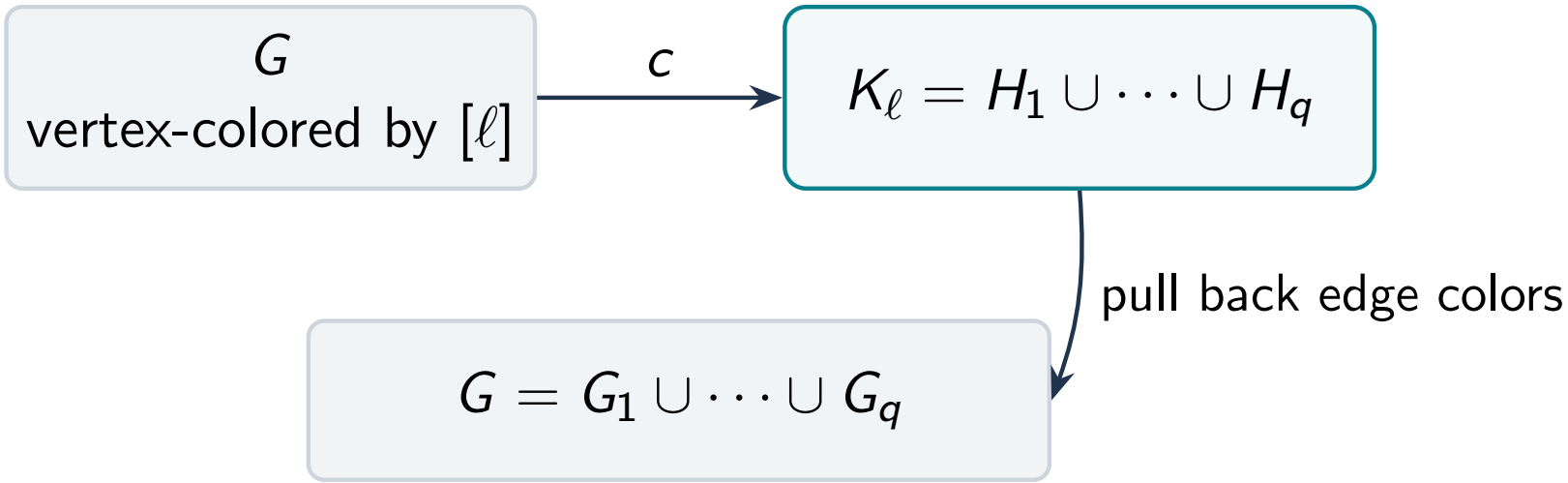
$$3.28^q \leq R_q(C_3) \leq 2.55q!$$

Fix $p \geq 2$, best bounds (Day-Johnson 17, Axenovich et al. 26):

$$(2+\delta_p)^q \leq R_q(\mathcal{O}_p) \leq q^{q/p+o(q)}.$$

A surprising connection

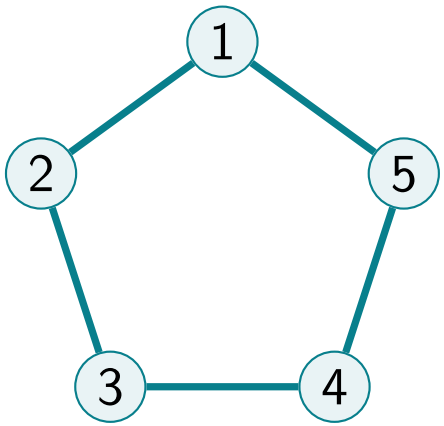
Let $\ell = \chi(G)$ and fix a proper vertex-coloring $c : V(G) \to [\ell]$.
Suppose K_ℓ has a q-edge-coloring with color classes $H_1, \ldots, H_q$, each of odd-girth at least $2p + 3$.



Odd-girth survives the pullback

$$uv \in E(G_i) \iff uv \in E(G) \text{ and } c(u)c(v) \in E(H_i).$$

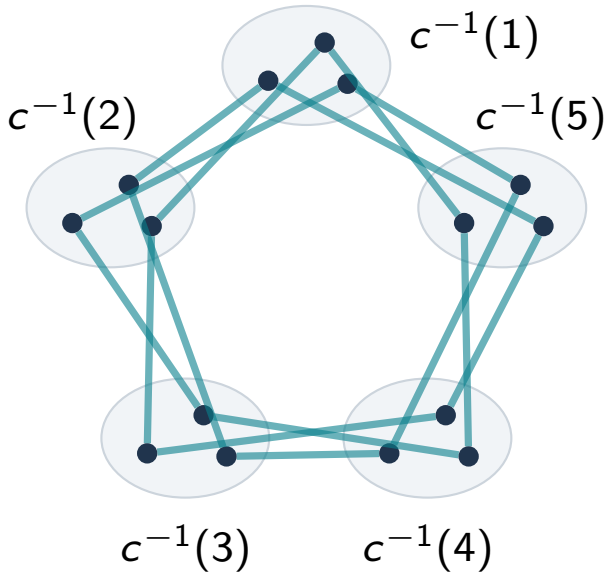
H_i on the color labels



c

a graph homomorphism

G_i on the original vertices



An odd cycle in G_i maps to a closed odd walk in H_i. A closed odd walk of length at most g yields an odd cycle of length at most g.

One part must keep high chromatic number

For graphs on the same vertex set, product coloring gives

$$\chi(G_1 \cup \cdots \cup G_q) \leq \prod_{i=1}^{q} \chi(G_i).$$

Therefore our decomposition satisfies

$$\ell = \chi(G) \leq \prod_{i=1}^{q} \chi(G_i), \qquad \boxed{\max_i \chi(G_i) \geq \ell^{1/q}.}$$

Ramsey construction useful if it efficiently decomposes/colors a clique without monochromatic short odd cycles

Superexponential growth is what matters

Choose q *minimal* such that $R_q(\mathcal{O}_p) > \ell$. Then

$$R_{q-1}(\mathcal{O}_p) \leq \ell < R_q(\mathcal{O}_p).$$

If every G_i had $\chi(G_i) < k$, the product inequality would give

$$R_{q-1}(\mathcal{O}_p) \leq \ell < k^q.$$

Corollary

For every fixed $p \geq 1$, if $R_q(\mathcal{O}_p)$ grows superexponentially in q, then every sufficiently high-chromatic graph G has an $\mathcal{O}_p$-free k-chromatic subgraph.

I got to this point more than a year ago. I was excited and asked some Ramsey experts, but they told me: Nah, everyone thinks $R_k(C_3)$ grows exponentially. And so I mentally discarded the idea as proving something interesting about a hypothetical dream world.

Surprise: We live in a crazy dream world!

Theorem (OpenAI, 2026)

There is an absolute constant $c > 0$ such that

$$R_q(C_3) \geq \left(\frac{c\, q^{1/3}}{\log q}\right)^q = q^{q/3 - o(q)}.$$

I will soon explain the nice proof of this breakthrough.

Consequence: Triangle-free subgraphs

Recall: Given a graph G of chromatic number ℓ, we chose q minimal such that $\ell < R_q(C_3)$ and obtained a decomposition into q triangle-free subgraphs. If all triangle-free subgraphs had chromatic number below k, then $R_{q-1}(C_3) \leq \ell < k^q$, and so

$$q^{1/3-o(1)} \leq R_{q-1}(C_3)^{1/q} < k, \qquad \text{so} \qquad q \leq k^{3+o(1)}.$$

The factorial upper bound gives

$$\ell < R_q(C_3) \leq 3q! \leq \exp(k^{3+o(1)}).$$

$\implies \chi(G) \geq \exp(k^{3+o(1)})$ forces a triangle-free subgraph with chromatic number at least k.

OpenAI's lower bound

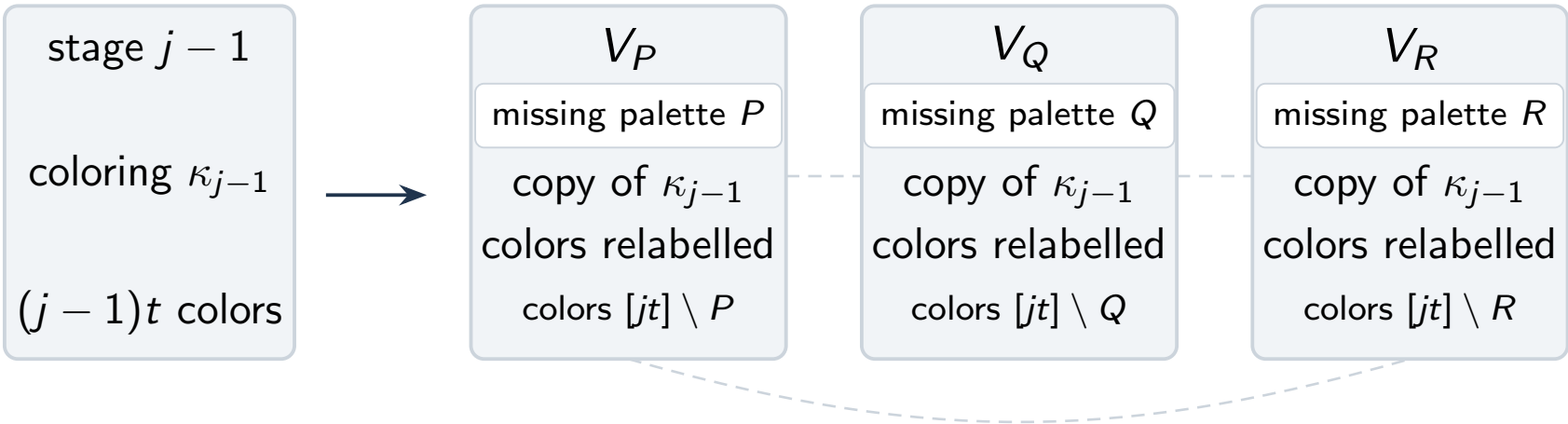
Goal: Construct a triangle-free edge-coloring of huge complete graph with few colors.

The construction works recursively through several stages indexed by $j = 0, 1, \ldots, H$. At stage j, use colors from $[jt]$.

At each stage, we multiply the size of the complete graph we color by roughly $j^{(1-o(1))t} \Rightarrow$ superexponential.

For stage j, repeat the vertex-set of the complete graph from stage $j-1$ disjointly $j^{(1-o(1))t}$ times (blocks).

Different blocks exclude different sets of t colors from $[jt]$.



Choosing color-palettes with significant differences

Choose a maximum size family $\mathcal{P}_j \subseteq \binom{[jt]}{t}$ of color palettes subject to

$$|P \setminus Q| = |Q \setminus P| \geq s \quad (P \neq Q).$$

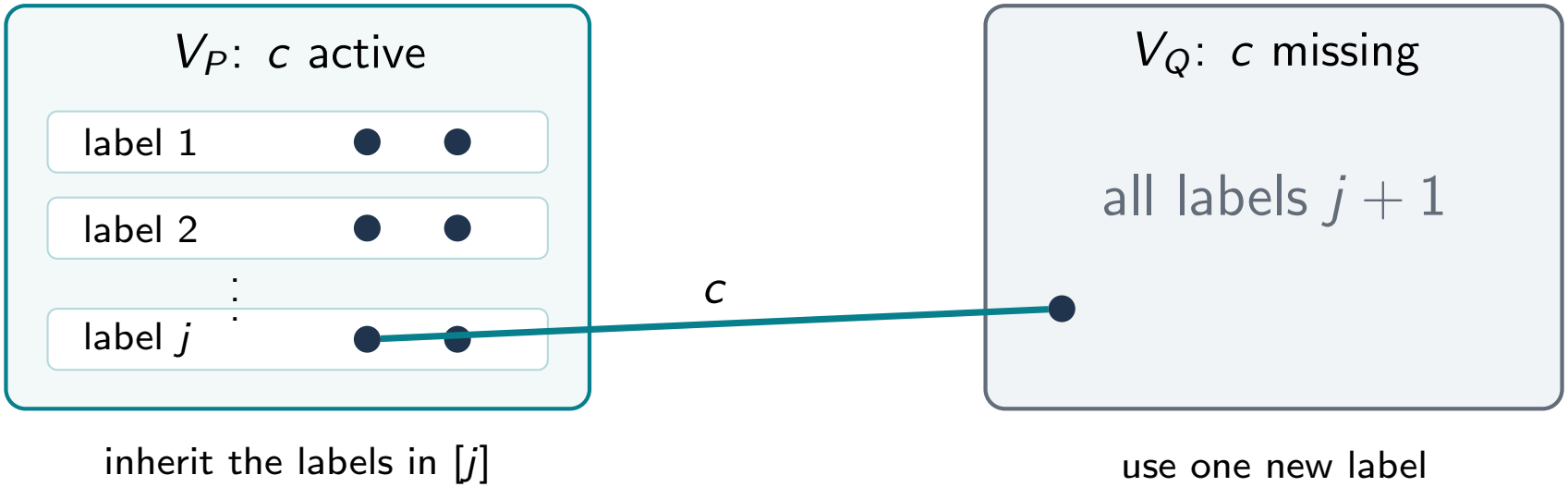
Here think of $s \ll t$ but only by a log factor $s = t/\log H$. A simple greedy packing/volume argument shows that we can take

$$|\mathcal{P}_j| = j^{t-o(t)}.$$

We'll use $P \Delta Q$ to color cross-edges between blocks V_P and V_Q.

Vertex labels for every edge color

Alongside the edge-coloring κ_j, maintain labelings $L_{j,c} : V_j \to [j+1]$ for each $c \in [jt]$. $L_{j,c}$ properly colors c-edges.

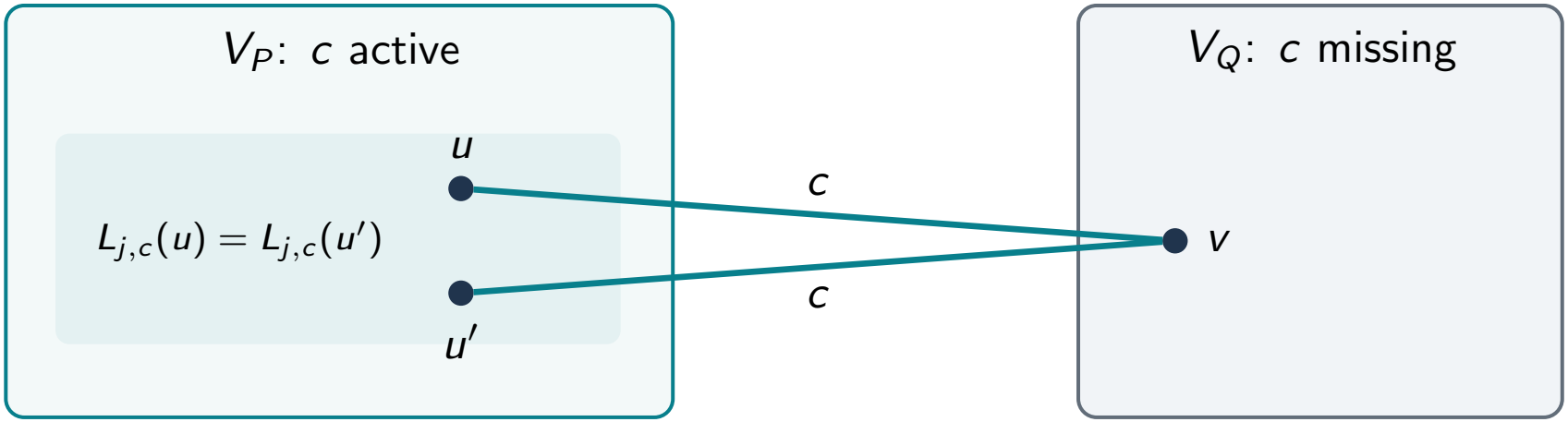


$$L_{j,c}(u) = \begin{cases} \text{the inherited label in } [j], & c \notin P, \\ j+1, & c \in P, \end{cases} \qquad u \in V_P.$$

The crucial property: Coloring crossing edges

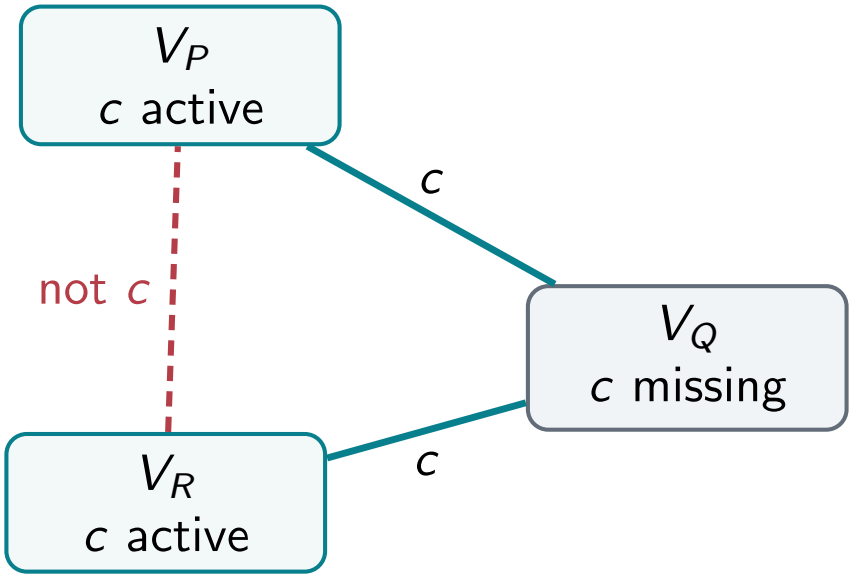
Recall: We will color edges between two blocks V_P and V_Q only with colors from $P \Delta Q$.

But the key property we want is that for every color $c \in P \Delta Q$:



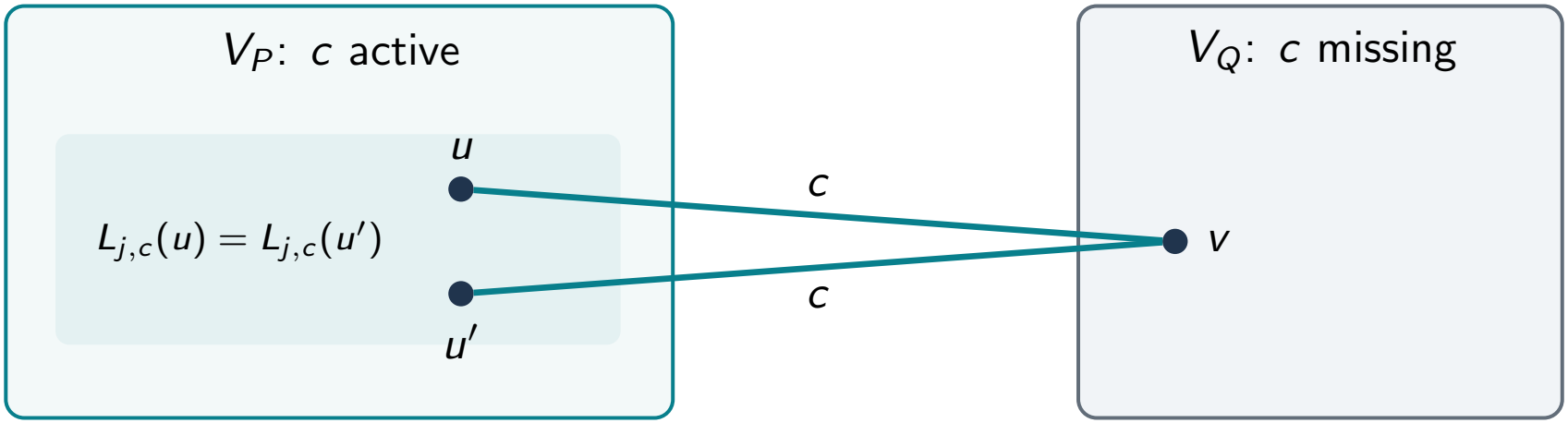
Triangles intersecting one or three parts

Inside blocks, there are no monochromatic triangles by induction.There also cannot be monochromatic triangles (say in color c) visiting 3 blocks, since color c-cross-edges only go between blocks on which c is active and not active (bipartite).



Triangles intersecting two parts

Suppose $u, u' \in V_P$, $v \in V_Q$ and both cross-edges have color c.



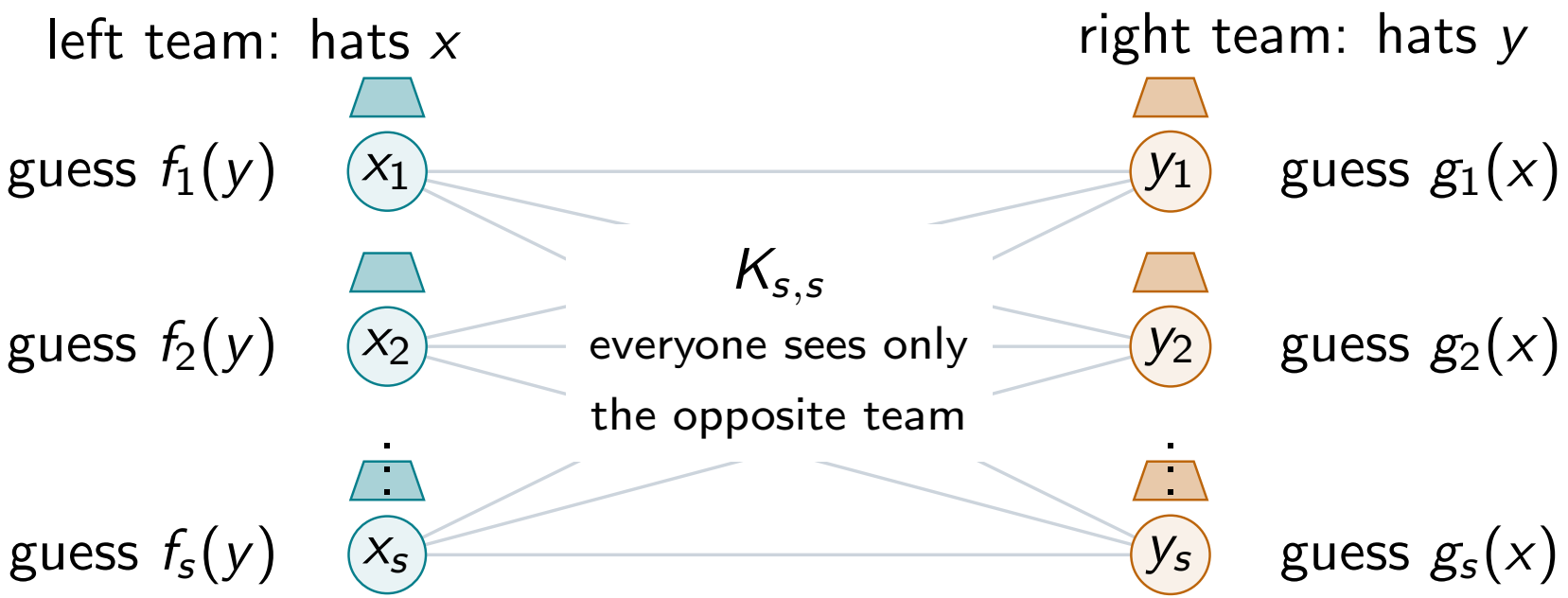
If c is missing on V_P:

there is no internal c-edge uu'.

If c is active on V_P:

Key property forces $L_{j,c}(u) = L_{j,c}(u')$, and since $L_{j,c}$ properly colors c-edges, uu' cannot be a c-edge.

Achieving the key property: Hat guessing on complete bipartite graphs



Each player wears a hat in $[H]$ and makes one simultaneous guess. The team wins if at least one guess is correct, for every hat assignment.

$$\mathrm{HG}(K_{s,s}) \geq H \quad \Longleftrightarrow \quad \exists f, g : [H]^s \to [H]^s \text{ with this guarantee.}$$

Complete bipartite graphs can win with many colors

Theorem (Alon, Ben-Eliezer, Shangguan, Tamo, 2020)

For sufficiently large s,

$$\mathrm{HG}(K_{s,s}) = \Omega\left(\frac{\sqrt{s}}{\log s}\right).$$

In the construction we will fix s such that $HG(K_{s,s}) \geq H$, by the above we can take

$$s = \Theta(H^2 \log^2 H).$$

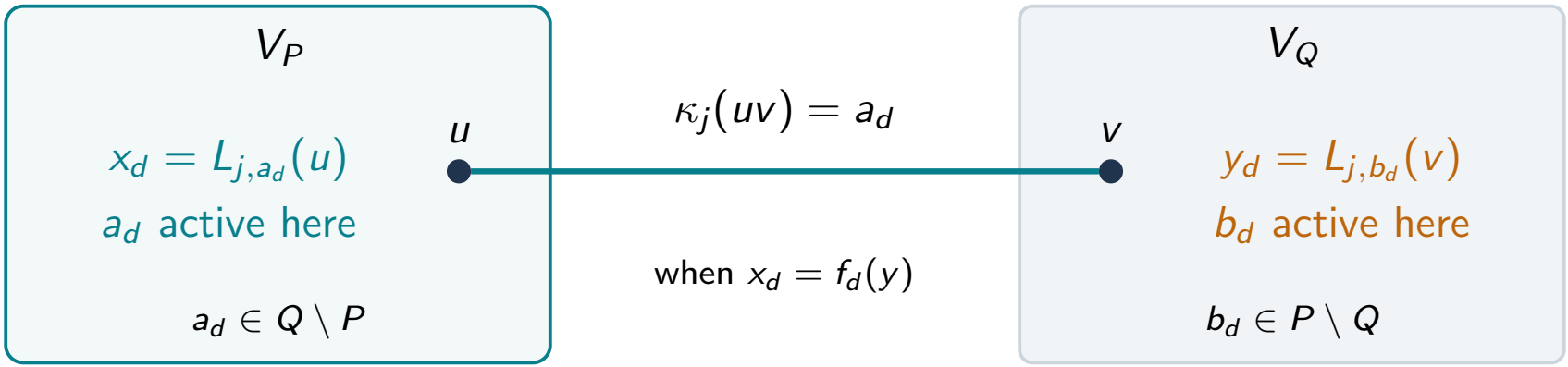
Then there are fixed $f, g : [H]^s \to [H]^s$ such that

$$\forall x, y \in [H]^s \ \exists d \in [s] : \quad x_d = f_d(y) \text{ or } y_d = g_d(x).$$

Coloring cross-edges via hat guessing

For $P \neq Q \in \mathcal{P}_j$, we fix once and for all edge-colors $a_1, ..., a_s \in Q \setminus P$ and $b_1, ...b_s \in P \setminus Q$.
For $u \in V_P$, $v \in V_Q$, form the following left and right hat-color sequences for $K_{s,s}$:

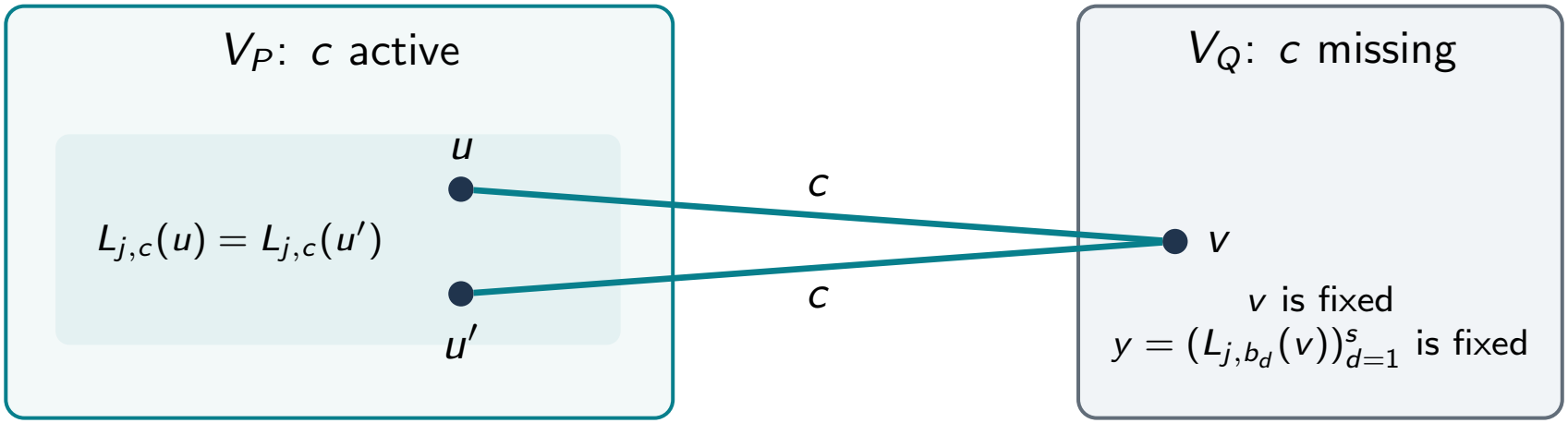
$$x = (L_{j,a_d}(u))_{d=1}^{s}, \qquad y = (L_{j,b_d}(v))_{d=1}^{s} \qquad \in [j]^s \subseteq [H]^s.$$



$$\kappa_j(uv) = \begin{cases} a_d, & \text{for the first } d \text{ with } x_d = f_d(y), \\ b_d, & \text{otherwise, for the first } d \text{ with } y_d = g_d(x). \end{cases}$$

Verifying the key property

Fix V_P, V_Q, a color c active on V_P, and $v \in V_Q$ where c is missing.



Since c is active on V_P, we have $c \in \{a_1, \ldots, a_s\}$, let d be unique s.t. $a_d = c$.Then

$$\kappa_j(uv) = \kappa_j(u'v) = c = a_d \quad \Longrightarrow \quad \begin{array}{l} L_{j,c}(u) = x_d = f_d(y(v)), \\ L_{j,c}(u') = x'_d = f_d(y(v)). \end{array}$$

When c is active on V_Q and missing on V_P, we use $\{b_1, \ldots, b_s\}$ and g instead.

What do we get in the end?

$$\begin{aligned} s &= \Theta(H^2 \log^2 H) && \text{hat strategy} \\ t = s \log H &= \Theta(H^2 \log^3 H) && \text{palettes} \\ q = Ht &= \Theta(H^3 \log^3 H) && \text{total number colors} \end{aligned}$$

Number of vertices after all H rounds:

$$\prod_{j=2}^{H} |\mathcal{P}_j| = \prod_{j=2}^{H} j^{(1-o(1))t} = (H!)^{(1-o(1))t} = H^{(1-o(1))Ht} = H^{(1-o(1))q}.$$

Thus

$$R_q(C_3) \geq q^{q/3-o(q)}.$$

From triangles to all short odd cycles

Theorem (ChatGPT 5.6, presented by S., 2026)

For every fixed $p \geq 1$,

$$R_q(\mathcal{O}_p) \geq \left(\log^{(p-1)} q\right)^{q/3-o(q)}.$$

forbidden monochromatic cycles	lower bound
C_3	$q^{q/3-o(q)}$
C_3, C_5	$(\log q)^{q/3-o(q)}$
C_3, C_5, C_7	$(\log\log q)^{q/3-o(q)}$

Combined with the transfer argument: $h_{2p+3}(k) \leq \exp^{(p)}(k^{3+\varepsilon})$.

A generalization: χ-amenable graph parameters

Definition: χ-amenability

A nonnegative graph parameter f is χ-amenable if:

Product inequality: for graphs on the same vertex set,

$$f(G_1 \cup G_2) \leq f(G_1)f(G_2).$$

A bounded chromatic subgraph: for a constant $C \geq 1$, every G has $H \subseteq G$ with

$$f(H) \geq f(G)/C, \qquad \chi(H) \leq Cf(H).$$

$f(\cdot)$ χ-amenable if f is submultiplicative and every graph with high f has a subgraph with comparable f and chromatic number $O(f)$.

χ-amenability of χ_f

$\chi_f(G)$ is the optimal value of the following primal-dual pair of LPs:

$$\min \sum_{I \in \mathcal{I}(G)} x_I \tag{P}$$

$$\text{s.t.} \sum_{I \in \mathcal{I}(G): v \in I} x_I \geq 1 \quad (\forall v \in V(G)),$$

$$x_I \geq 0 \quad (\forall I \in \mathcal{I}(G)).$$

$$\max \sum_{v \in V(G)} y_v \tag{D}$$

$$\text{s.t.} \sum_{v \in I} y_v \leq 1 \quad (\forall I \in \mathcal{I}(G)),$$

$$y_v \geq 0 \quad (\forall v \in V(G)).$$

χ-amenability of χ_f

Let $z = \chi_f(G)$. We can reformulate (P) as:

A distribution on independent sets

There exists a distribution $\mathcal{D}$ on independent sets of G such that

$$\mathbb{P}_{I\sim\mathcal{D}}(v \in I) \geq 1/z \qquad \text{for every vertex } v.$$

and (D) as:

Dual vertex weights

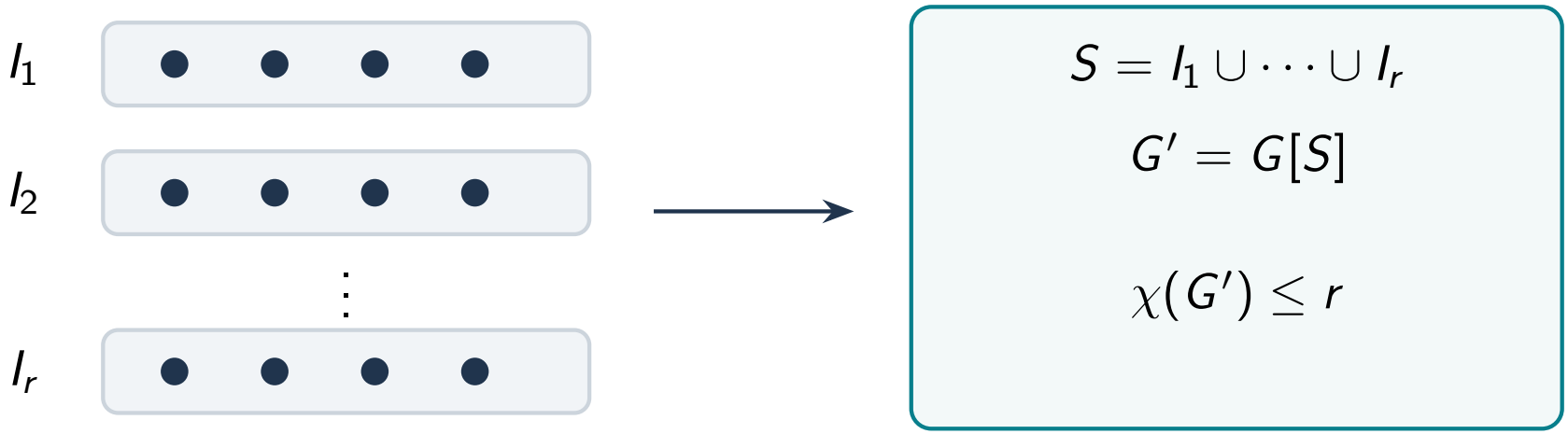
There are nonnegative weights y_v with

$$\sum_v y_v = z, \qquad \sum_{v\in I} y_v \leq 1 \quad \text{for every independent set } I.$$

It is a simple exercise to show that $\chi_f(G_1 \cup G_2) \leq \chi_f(G_1)\chi_f(G_2)$ for G_1, G_2 on the same vertex-set.

Bounded-chromatic subgraph of comparable χ_f

Take $r = \lceil z \rceil$ independent samples $I_1, \ldots, I_r$.



Every vertex is covered with probability at least $1 - e^{-1}$. Thus some outcome has

$$\chi_f(G') \ge \sum_{v \in S} y_v \ge (1 - e^{-1})z, \qquad \chi(G') \le \lceil z \rceil \le 2z.$$

χ_f is χ-amenable with $C = 2/(1 - e^{-1})$.

The bittersweet aftermath of the paper

Theorem (KohImeyer–Kruer (AI-assisted), September 2026)

There are graphs of arbitrarily large chromatic number whose C_4-free subgraphs are all 6-colorable.

Theorem (Nguyen–Walczak, September/October 2026)

The same statement holds with 3-colorable in place of 6-colorable.

So the **Erdős–Hajnal conjecture is false** for $g \geq 5$: Short even cycles are sometimes necessary ro retain high χ. The construction actually relies on an earlier construction of Janzer, Sudakov, S. (but the AI drafts did not mention that, NW noticed it).

The odd-girth result is best-possible

Definition

Call F χ-avoidable if, for every k, every graph with sufficiently large chromatic number has an F-free subgraph H with $\chi(H) \geq k$.

I suggested to NW that in their proof one can replace C_4 by $K_{s,t}$.

Theorem (Nguyen–Walczak, 2026)

For $s \geq t \geq 2$, there are G with arbitrarily large $\chi(G)$ such that

$$H \subseteq G, \quad H \text{ is } K_{s,t}\text{-free} \quad \Longrightarrow \quad \chi(H) \leq t+1.$$

Theorem (Nguyen, Walczak, 2026)

$$F \text{ is } \chi\text{-avoidable} \quad \Longleftrightarrow \quad F \text{ is non-bipartite.}$$

So the odd-girth result is best possible in a strong sense: odd cycles (and thus their supergraphs), but nothing else, can be avoided.

The End

Thank you for your attention!

χ-amenability of $\overline{\vartheta}$ (strict vector chromatic number)

A first proof was found by GPT 5.6 Sol with input from S., and later simplified by S.
Recall $\overline{\vartheta}(G) = \vartheta(\overline{G})$. Similar to how χ_f has a primal-dual description in terms of LPs, $\overline{\vartheta}$ has a primal-dual description in terms of SDPs. The dual translates to:

$$\overline{\vartheta}(G) = \max\left\{ \|M\| : \begin{array}{l} M \succeq 0, \quad M_{vv} = 1, \\ M_{uv} = 0 \text{ if } u \neq v \text{ and } uv \notin E(G) \end{array} \right\}.$$

Let M be optimal and $z = \|M\| = \overline{\vartheta}(G)$.

The plan

Partition the vertices into $q = \lceil z \rceil$ classes. Delete the edges within classes (i.e., insert zeros in the corresponding blocks of the matrix), without dropping norm too much.

$\overline{\vartheta}(G_1 \cup G_2) \leq \overline{\vartheta}(G_1)\overline{\vartheta}(G_2)$ is a neat exercise (tensor products).

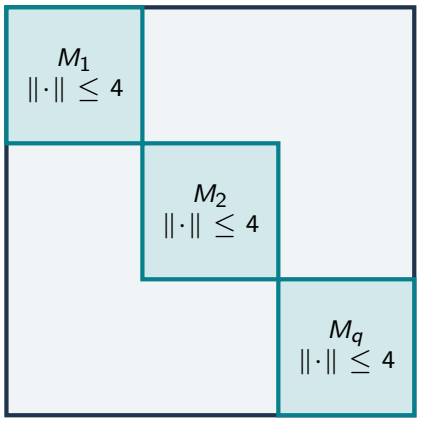
An amazing relation to the Kadison-Singer problem

Amazingly, a partition into blocks with small spectral norm is exactly the problem that Marcus, Spielman, Srivastava achieved in their 2013 resolution of the Kadison-Singer problem, earning a Polya prize.

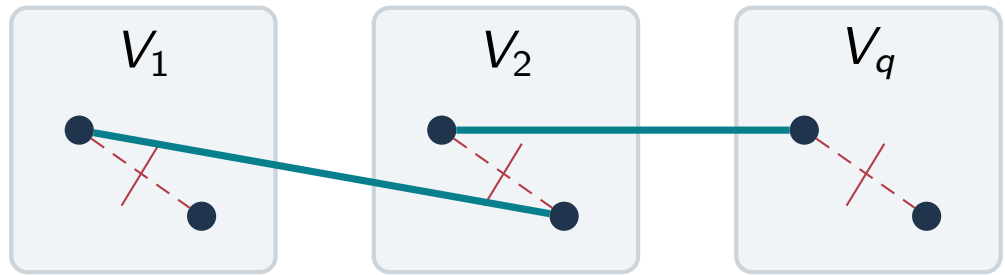
Marcus–Spielman–Srivastava (consequence of "paving theorem")

For $q = \lceil z \rceil = \lceil \|M\| \rceil$, there is a partition $V = V_1 \sqcup \cdots \sqcup V_q$ with

$$\|M[V_i]\| \leq 4.$$



$M_i = M[V_i]$



delete the edges inside the q parts

$\chi(G') \leq q$

Deleting blocks yields a feasible matrix for the subgraph

Let D be the block-diagonal matrix with blocks $M[V_1], \ldots, M[V_q]$. Since $0 \preceq D \preceq 4I$, set

$$M' = I + \tfrac{1}{4}(M - D) \succeq \tfrac{1}{4}M \succeq 0.$$

M' has diagonal 1 and zeros on the nonedges of G'. Therefore

$$\overline{\vartheta}(G') \geq \left\|M'\right\| \geq z/4, \qquad \chi(G') \leq q \leq 2z \leq 8\overline{\vartheta}(G').$$

$\vartheta(\overline{G})$ is χ-amenable with $C = 8$.