Refined Donaldson-Thomas theory and Nekrasov's formula

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Geometric engineering

Geometric engineering starts with a product $X \times \mathbb{C}^2$, where X is a (local) Calabi–Yau threefold.

- For appropriate X, integrating out the X-directions results in a gauge theory on $\mathbb{C}^2$, with gauge group $G = G(X)$. The partition function $Z_{\mathbb{C}^2}$ is (a version of) Nekrasov's partition function
- Integrating out the $\mathbb{C}^2$-directions results in an SU(1) gauge theory on the threefold X. The partition function Z_X is (a version of) the topological string partition function of X

The aim of the talk is to discuss the precise relationship

$$Z_{\mathbb{C}^2} \sim Z_X$$

in the simplest example, and the relationship between the underlying vector spaces.

Gauge-theoretic moduli spaces on $\mathbb{C}^2$

For X a family of ALE spaces of type A_{n-1} over $\mathbb{P}^1$, $G(X) = \mathrm{SU}(n)$.

We have the moduli space of framed finite-action instantons on $\mathbb{C}^2$ of rank n

$$\mathcal{M}_{n,k} = \left\{ (\mathcal{E}, \phi) \text{ framed rank-}n\text{ torsion-free sheaf on } \mathbb{P}^2, c_2(\mathcal{E}) = k \right\}.$$

This is noncompact, but nonsingular and complete in a natural hyperkähler metric.

The SU(n) Nekrasov partition function is

$$Z_{\mathbb{C}^2}^{\mathrm{SU}(n)}(\Lambda) = \sum_{k \geq 0} \Lambda^k \int_{\mathcal{M}_{n,k}} 1.$$

This is an ill-defined expression.

Symmetries of the gauge-theoretic moduli spaces

Note that all $\mathcal{M}_{n,k}$ carry an action of the torus $T = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^{n-1}$.

- The first component acts on $\mathcal{E}$ via its action on $\mathbb{C}^2 \subset \mathbb{P}^2$.
- The second component acts on the framing.

Crucial fact: the fixed point set $\mathcal{M}_{n,k}^T$ is a finite set for all n, k.

Thus T-equivariant integrals make sense on the moduli space $\mathcal{M}_{n,k}$.

The equivariant index and the K-theoretic partition function

We will be particularly interested in a K-theoretic interpretation of the partition function (M-theory).

- The integrand 1 is interpreted as the (K-theory class of) the trivial line bundle $\mathcal{O}_{\mathcal{M}_{n,k}}$.
- Integration gets replaced by cohomology (pushforward to the point).

Thus Nekrasov defines (following Losev–Moore–Nekrasov–Shatashvili)

$$Z_{\mathbb{C}^2}^{\mathrm{SU}(n)}(\Lambda, q_1, q_2, a_1, \ldots, a_{n-1}) = \sum_{k \geq 0} \Lambda^k \operatorname{char}_T H^*(\mathcal{M}_{n,k}, \mathcal{O}_{\mathcal{M}_{n,k}}).$$

Here, for a representation V of T, $\operatorname{char}_T V \in \mathbb{Z}[q_i, a_j]$ denotes its T-character.

Nekrasov's partition function for SU(1)

Assume now that X is the resolved conifold, so in fact $G = \mathrm{SU}(1)$.

There is only one line bundle on $\mathbb{P}^2$ with trivial determinant, so

$$\mathcal{M}_{1,k} \cong \mathrm{Hilb}^k(\mathbb{C}^2)$$

the Hilbert scheme of k points on the plane.

It can be shown that there is no higher cohomology

$$H^i\left(\mathrm{Hilb}^k(\mathbb{C}^2), \mathcal{O}\right) = 0 \text{ for } i > 0$$

and that

$$H^0\left(\mathrm{Hilb}^k(\mathbb{C}^2), \mathcal{O}\right) \cong H^0\left(S^k(\mathbb{C}^2), \mathcal{O}\right)$$

where $S^k(\mathbb{C}^2)$ is the k-th symmetric power of $\mathbb{C}^2$.

Nekrasov's partition function for SU(1): the computation

Now we can finish the computation:

$$\begin{aligned}
Z_{\mathbb{C}^2}^{\mathrm{SU}(1)}(\Lambda, q_1, q_2) &= \sum_{k\geq 0} \Lambda^k \,\mathrm{char}_T\, H^*\left(\mathrm{Hilb}^k(\mathbb{C}^2), \mathcal{O}\right) \\
&= \sum_{k\geq 0} \Lambda^k \,\mathrm{char}_T\, H^0(S^k(\mathbb{C}^2), \mathcal{O}) \\
&= \sum_{k\geq 0} \Lambda^k \,\mathrm{char}_T\, S^k\mathbb{C}[x, y] \\
&= \mathrm{char}_{T\times\mathbb{C}^*}\, S^*\mathbb{C}[x, y] \\
&= \prod_{i,j\geq 0} (1 - q_1^i q_2^j \Lambda)^{-1}.
\end{aligned}$$

Relation to the conifold

We have

$$Z_{\mathbb{C}^2}^{\mathrm{SU}(1)}(\Lambda, q_1, q_2) = \prod_{i,j\geq 0} (1 - q_1^i q_2^j \Lambda)^{-1}.$$

Setting $q_1 = q_2 = q$ and $T = q^{-1}\Lambda$,

$$Z_{\mathbb{C}^2}^{\mathrm{SU}(1)}(\Lambda = qT, q = q_1 = q_2)^{-1} = \prod_{k\geq 1}(1 - q^k T)^k$$

the reduced topological string partition function of the resolved conifold X.

Gauge-theoretic interpretation on X

The reduced topological string partition function

$$\prod_{k\geq 1}(1-q^kT)^k = \sum_{n,l} q^n T^l P_{n,l}$$

corresponds to a version of SU(1) gauge theory on the resolved conifold X: Pandharipande-Thomas stable pairs theory (Toda, Bridgeland, Nagao-Nakajima).

The coefficients $P_{n,l}$ are the (virtual) numbers of SU(1)-sheaves on X of a certain kind, associated to highly singular gauge-theoretic moduli spaces $\mathcal{N}_{n,l}$ on X.

More about gauge-theoretic moduli spaces on X

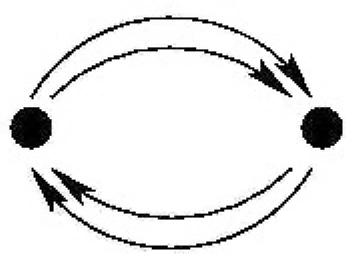
Theorem (Sz., Nagao-Nakajima) The spaces $\mathcal{N}_{n,l}$ are global critical loci

$$\mathcal{N}_{n,l} = \mathrm{Zeros}\,(df_{n,l})$$

of smooth functions $f_{n,l}$ on smooth manifolds $N_{n,l}$.

This uses the quiver description of the conifold and the Klebanov-Witten superpotential;

$$f_{n,l} = \mathrm{Tr}(W).$$



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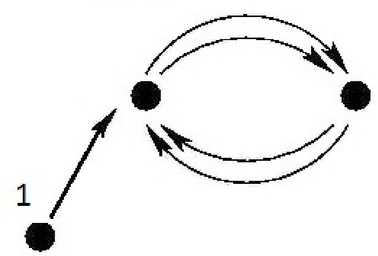


Refining the numerical gauge theoretic invariants of X

Using the critical locus interpretation, one gets a topological coefficient system $\phi_{n,l}$ on the singular moduli spaces $\mathcal{N}_{n,l}$, and a corresponding cohomology theory with mixed Hodge structure

$$H^*(\mathcal{N}_{n,l}, \phi_{n,l}).$$

This gives a weight polynomial $W_{\mathcal{N}_{n,l}}(t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$. This has the property that

$$W_{\mathcal{N}_{n,l}}\left(t^{\frac{1}{2}} = 1\right) = P_{n,l},$$

the numerical coefficient in the reduced partition function.

This weight polynomial refinement of the Euler characteristic is equivalent to the motivic refinement introduced by Kontsevich-Soibelman and studied by Behrend-Bryan-Sz., Dimofte-Gukov and others.

Interpretation of the full partition function on X

Let

$$Z_X(q,t,T) = \sum_{n,l} q^n T^l W_{\mathcal{N}_{n,l}}(t)$$

be the refined partition function of the conifold.

Theorem (Morrison-Mozgovoy-Nagao-Sz.) Under the change of variables

$$q_1 = qt^{\frac{1}{2}}, q_2 = qt^{-\frac{1}{2}}, \Lambda = qT,$$

we have

$$Z_X(q,t,T) = \prod_{i,j\geq 0}(1 - q_1^i q_2^j \Lambda) = Z_{\mathbb{C}^2}^{\mathrm{SU}(1)}(\Lambda, q_1, q_2)^{-1}.$$

Thus we obtain a cohomological interpretation on X of the full Nekrasov partition function in this case.

Inverting the partition function

Recall from the computation that

$$Z_{\mathbb{C}^2}^{\mathrm{SU}(1)}(\Lambda, q_1, q_2) = \mathrm{char}_{T\times\mathbb{C}^*}\, S^*\mathbb{C}[x,y],$$

just the Hilbert series of a triply-graded vector space, the symmetric space of the space $\mathbb{C}[x,y]$ of functions on $\mathbb{C}^2$.

Then,

$$Z_{\mathbb{C}^2}^{\mathrm{SU}(1)}(\Lambda, q_1, q_2)^{-1} = \mathrm{char}_{T\times\mathbb{C}^*}\, \Lambda^*\mathbb{C}[x,y],$$

the corresponding exterior space!

The vector space underlying the conifold partition function

Recall

$$Z_X(q,t,T) = \mathrm{char}_{T\times\mathbb{C}^*}\,\Lambda^*\mathbb{C}[x,y]$$

is the generating series of weight polynomials of the mixed Hodge structures on

$$H^*(\mathcal{N}_{n,l},\phi_{n,l}).$$

Theorem There exists a GL(2)-equivariant isomorphism

$$\bigoplus_{n,l} H^*(\mathcal{N}_{n,l},\phi_{n,l}) \cong \Lambda^*\mathbb{C}[x,y].$$

This depends on two additional ingredients (Davison–Maulik–Schürmann–Sz.):

- the mixed Hodge structure on the cohomology above is in fact pure, and
- it admits a Lefschetz sl(2)-action.

In particular, this proves a version of GMN's "no exotics" or "strong positivity" conjecture in this example (and many others).

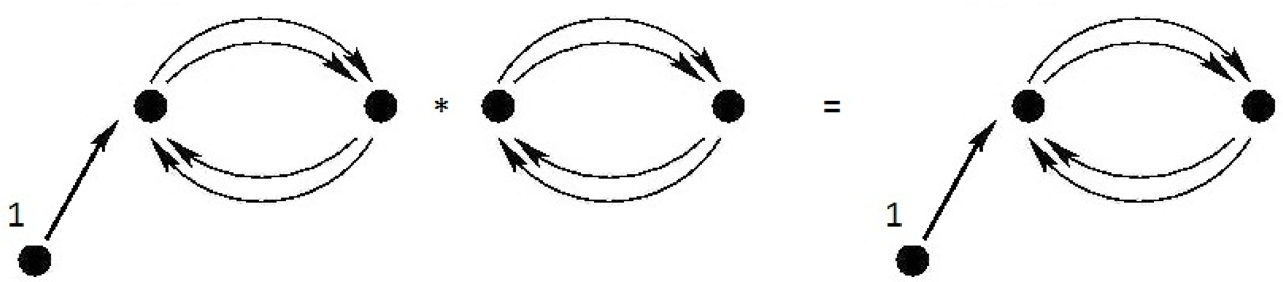
A module for the conifold COHA

Given a quiver Q with superpotential W, Kontsevich–Soibelman define an associative algebra $\mathcal{A}_{Q,W}$, the COHA (cohomological Hall algebra), a realization of the Harvey–Moore BPS state algebra.

The space

$$\bigoplus_{n,l} H^*(\mathcal{N}_{n,l}, \phi_{n,l})$$

is a module for the algebra $\mathcal{A}_{Q,W}$ associated to the unframed conifold quiver.



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The isomorphism

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should be an isomorphism of $\mathcal{A}_{Q,W}$-modules. Perhaps this can be used to learn more about the unknown algebra $\mathcal{A}_{Q,W}$!


The End