The unbearable lightness of deformation theory

A tutorial introduction

Balázs Szendrői

Draft, November 1999

Introduction

The purpose of this note is to introduce some ideas in deformation theory, leading up to the Theorem on Formality and Unobstructedness of Calabi–Yau manifolds. The latter is a more-or-less random choice, obviously influenced by my particular taste; however, it may be reassuring to know that already a limited amount of theory has rather interesting consequences. The material is completely standard and well-known to several circles of experts; I claim absolutely no originality at all. The aim is to motivate the constructions via accessible examples, in particular to introduce the language of differential graded Lie algebras and deformation functors in a reasonably lightweight manner. The message I am trying to get across is that this framework is completely *natural* in the context and it is *easy*, despite its appearance and the several hours of sleep one is likely to loose over it; hence the title.

In the note, several statements have the word (Exercise) attached to them. These are all elementary exercises; however I believe that this material can only be understood if the reader takes an active part in the process by working out the solutions on a sheet of paper.

Some words of attribution: the idea of DGLAs in deformation theory appears to be due to Deligne; the idea of formality is due to Sullivan, cf. [4]. Unobstructedness for Calabi–Yaus was proved by Bogomolov, Tian [9] and Todorov [10]; it was re-interpreted as a formality result by Goldman and Millson [5] following a suggestion of Bogomolov. The theory presented here appears as a basic ingredient in the solution of the deformation quantization problem by Kontsevich [8] and in the recent work, some of it in progress, of Kontsevich–Barannikov on mirror symmetry and extended deformations [3], [1], [2]; later I will comment on these aspects briefly.

I learned a large part of this material from Kontsevich' lecture notes [7]. There are several paragraphs taken almost verbatim from these notes; I will make no further fuss about this. I should thank Ian Grojnowski for encouraging me to learn some of this material and teaching me about formality, and Sergei Barannikov for not letting me drown in the Sea of Galilee and answering my questions on [1], [2].

1 General blurb

1.1 I work throughout this note over a base field k, algebraically closed of characteristic zero.

1.2 Pick an (algebraic) structure A, for example an associative k-algebra, a smooth projective variety over k or (for $k = \mathbb{C}$) a complex manifold. The aim of deformation theory is the following: we want to understand structures 'similar' but not 'isomorphic' to A inside the category A naturally lives in. 'Similar' means 'a member of a family containing A' (perhaps 'close' to A); this presupposes that we know what families of objects in our category are. In the above categories, the notion of 'family' is fairly clear; e.g. in the case of complex manifolds, families are complex analytic submersions over complex base spaces.

In the deformation problem associated to a fixed A, one distinguishes several different aspects:

One *First-order theory*: this is the 'tangent space' to the problem; here one works over the ring $k(\varepsilon)/(\varepsilon^2)$ of dual numbers, or the corresponding base space consisting of a point and a (direction-less) vector sticking out of it.

Two *Higher-order theory*: here come 'higher order tangent spaces' or 'Taylor coefficients'; the spiel now happens over rings like $k(\varepsilon)/(\varepsilon^n)$ or more general 'Artinian local rings'. This is where obstructions start to crop up.

Three *Formal theory*: this is where we think of 'power series'; the rings typically look like $k[[\varepsilon]]$ or more complicated relatives coming under the heading 'complete Artinian local rings'.

Four *Algebraization/analytic theory*: here the power series become 'convergent' or the rings become completions of local rings of some base space.

For completeness, I also mention the last topic,

Five *Global theory* or *moduli theory*: here one attempts to glue the base spaces together, by defining various equivalence relations; then the aim is to show that the obtained big space has a pleasant structure like that of an algebraic space, quasi-projective variety, etc.

This note will concern itself mainly with the first two aspects. The formal theory is a reasonably harmless generalization, and I will only make one or two remarks about it. The last two aspects require completely new methods and so lie way outside my current scope.

1.3 Whereas the above story is laying out the algebraic program in detail, it is instructive to keep the geometric picture in mind. Corresponding to 'all conceivable structures' of the kind we are interested in, there corresponds to a big, usually infinite dimensional vector space V. This space will contain a subspace S, given by some 'equations' (often quadratic) that specify further requirements about our structure. Finally, there is a gauge group G acting on V, preserving S. $M = S/G$ is the 'moduli space' of the problem: all possible structures up to equivalence.

A given structure A corresponds to a point $m \in M$. Correspondingly, I can pick an element $\tilde{m} \in V$ mapping to m under the quotient map. The orbit $G\tilde{m}$ is a smooth manifold, and there exists a small transversal slice T to it in V. The intersection $S_A = S \cap T$ is then a representative of 'all' (small) deformations of my original structure A; of course, it may be reducible, singular, or non-reduced. S_A is called a *miniversal space* of A; it is unique up to isomorphism.

Notice that here I actually realized Part Four of the above program: S_A is a 'geometric' space. In practice, one usually goes the other way, working one's way up from Part One.

2 Associative algebras

2.1 After all this nonsense, it's time to see some of this in action. The most harmless example, where nevertheless several key features of the theory can be seen in a completely explicit and computable fashion, is that of finite dimensional associative k-algebras with unit. So let's fix one of these guys A. A has the structure of a finite-dimensional vector space over k, and the extra data corresponds to an associative multiplication $A \otimes A \to A$. The geometric approach is completely transparent here: $V \cong \mathrm{Hom}(A \otimes A, A)$ is the space of all multiplications (finite dimensional this time); S is given by requiring associativity, which, as we'll see below, is a set of quadratic equations in the entries of a matrix representing a multiplication, and $G = GL(A)$. However, this is too general to be really useful.

2.2 The systematic approach begins with the investigation of the 'tangent space' of the problem, i.e. the first-order theory. Suppose that A does have an associative multiplication, and try 'perturbing it' a little bit. What this means in practice is to define a new multiplication

$$a \star b = ab + \varepsilon f(a, b)$$

from the old one (simply denoted by juxtaposition) where $f \in \mathrm{Hom}(A \otimes A, A)$. Writing out the condition

$$(a \star b) \star c = a \star (b \star c)$$

for this to be an associative multiplication (Exercise; remember we are supposed to be working in $k(\varepsilon)/(\varepsilon^2)$, so powers of ε of degree higher than one are automatically set to zero), one obtains a condition for f:

$$f(ab, c) + f(a, b)c = f(a, bc) + af(b, c) \tag{1}$$

This is the 'linearized' defining condition for S. One still has to work out the linearized action of G; to this end, consider an automorphism

$$T(a) = a + \varepsilon g(a)$$

of A that reduces to the identity for $\varepsilon = 0$; here $g \in \mathrm{Hom}(A, A)$. One can 'pull back' the multiplication $\star$ to a new multiplication $\star_T$ given by

$$a \star_T b = T(T^{-1}(a) \star T^{-1}(b)).$$

This changes f to f_T; writing this out explicitly (Exercise) one obtains

$$f_T(a, b) = f(a, b) - g(a)b - ag(b) + g(ab) \tag{2}$$

This is then the answer: up to first order, the solutions of the problem are given by maps $f \in \mathrm{Hom}(A \otimes A, A)$ satisfying (1), under the equivalence relation (2) defined by $g \in \mathrm{Hom}(A, A)$.

2.3 Die-hard homological algebraists will start shouting 'homology' at this point. Indeed, the above stuff has a rather elegant reformulation in terms of the following complex:

$$\mathrm{Hom}(k, A) \xrightarrow{d} \mathrm{Hom}(A, A) \xrightarrow{d} \mathrm{Hom}(A \otimes A, A) \xrightarrow{d} \mathrm{Hom}(A \otimes A \otimes A, A) \xrightarrow{d} \ldots$$

where

$$d : \mathrm{Hom}(A^{\otimes(n-1)}, A) \to \mathrm{Hom}(A^{\otimes n}, A)$$

is defined by

$$(df)(a_1, \ldots, a_n) = a_1 f(a_2, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n f(a_1, \ldots, a_{n-1}) a_n.$$

You should convince yourself (Exercise) that I have really not done anything but translated formulae (1) and (2) into a general definition. Also, you should prove (Exercise) that $d^2 = 0$ i.e. we really have a complex here. Incidentally, the indexing of this complex is taken to be somewhat unusual in this note: by definition, it starts in degree -1. This is only in order to be in agreement with later conventions.

The above complex is called the *Hochschild complex* of A; its cohomology is called *(lower) Hochschild cohomology* $H^*(A, A)$ of A (with coefficients in A). By the above discussion,

$$H^1(A, A) = \text{equivalence-classes of first-order deformations of } A$$

You can check (Exercise) that in fact

$$H^{-1}(A, A) = \text{center of } A$$

$$H^0(A, A) = \text{derivations of } A \text{ modulo inner derivations}$$

and finally, we will have occasion to meet

$$H^2(A, A) = \text{obstructions to deformations of } A.$$

Example 2.4 I promised concrete computations, so here is one. Take $A = k[\mathbb{Z}/2\mathbb{Z}]$, the group algebra of the cyclic group of order two. As a vector space, it is based by $\{1, x\}$ where x is the nontrivial group element. Writing out the cocycle condition (1) for every pair of basis elements (Exercise), one obtains two conditions for f:

$$\begin{aligned} f(1, x) &= f(x, 1) \\ f(1, x) &= x f(1, 1) \end{aligned}$$

with no restriction on $f(x, x)$. Now it follows that in the given basis,

$$\begin{aligned} f(1, 1) &= \lambda_1 + \lambda_2 x \\ f(1, x) &= \lambda_2 + \lambda_1 x \\ f(x, 1) &= \lambda_2 + \lambda_1 x \\ f(x, x) &= \lambda_3 + \lambda_4 x \end{aligned}$$

for a quadruple of complex numbers (λ_i). On the other hand, writing

$$\begin{aligned} g(1) &= \mu_1 + \mu_2 x \\ g(x) &= \mu_3 + \mu_4 x, \end{aligned}$$

the coboundary corresponding to g is

$$\begin{aligned} (dg)(1, 1) &= \mu_1 + \mu_2 x \\ (dg)(1, x) &= \mu_2 + \mu_1 x \\ (dg)(x, 1) &= \mu_2 + \mu_1 x \\ (dg)(x, x) &= (2\mu_4 - \mu_1) + (2\mu_3 - \mu_2)x \end{aligned}$$

The meaning of this computation is that as you can easily see, all cocyles are in fact coboundaries. Thus $H^1(A, A) = 0$: there are no deformations at all. In the newspeak of deformation theory, we say that A is *rigid*. This of course matches our intuition that finite groups are pretty rigid, 'unbendable' objects, and in fact their group rings are like that too.

Example 2.5 To give an example of nontrivial deformation theory is not hard either. Consider the nilpotent cousin $A = k(x)/(x^2)$ of the previous example (the ring of dual numbers, now studied as an associative algebra). In fact, the cocycle conditions turn out to be identical to the previous one, giving the general cocycle as

$$\begin{aligned} f(1,1) &= \lambda_1 + \lambda_2 x \\ f(1,x) &= \lambda_1 x \\ f(x,1) &= \lambda_1 x \\ f(x,x) &= \lambda_3 + \lambda_4 x \end{aligned}$$

The coboundary defined by g is

$$\begin{aligned} (dg)(1,1) &= \mu_1 + \mu_2 x \\ (dg)(1,x) &= \mu_1 x \\ (dg)(x,1) &= \mu_1 x \\ (dg)(x,x) &= \mu_3 x \end{aligned}$$

This is pretty good! In this case, $H^1(A,A)$ is one-dimensional: a nontrivial deformation of the multiplication is given by

$$f(x,x) = \lambda_3$$

with all other terms zero. Thus, A deforms back (at least up to first order) to something isomorphic to the group ring, as more-or-less expected.

2.6 I now push on with the formal theory. As it turns out, the setup is simple enough so that we do not have to work our way through the higher-order framework, we can just write down the answer. So think once again about the associativity condition. For a general map $m : A \otimes A \to A$ it reads

$$m(m(a,b),c) = m(a,m(b,c)). \tag{3}$$

Keeping in mind the interpretation of $H^0(A,A)$ from above, I write down the condition for $h : A \to A$ to give a derivation with respect to the multiplication given by m ('Leibniz rule'):

$$h(m(a,b)) = m(h(a),b) + m(a,h(b)). \tag{4}$$

With a certain amount of ingenuity or (in the case of the present author) divine grace, one can realize that the last two formulae are special cases of vanishing of a *bracket* on the Hochschild complex: if $f \in \mathrm{Hom}(A^{\otimes n}, A)$ and $g \in \mathrm{Hom}(A^{\otimes m}, A)$ then we can define $[f,g] \in \mathrm{Hom}(A^{\otimes(n+m-1)}, A)$ by

$$\begin{aligned} &[f,g](a_1, \ldots, a_{n+m-1}) = \\ &\sum_{i=1}^{n} (-1)^{im} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+m-1}), a_{i+m}, \ldots, a_{n+m-1}) + \\ &\sum_{j=1}^{m} (-1)^{jn} g(a_1, \ldots, a_{j-1}, f(a_j, \ldots, a_{j+n-1}), a_{j+n}, \ldots, a_{n+m-1}). \end{aligned}$$

It is easy to see (Exercise) that (3) and (4) are equivalent to

$$[m,m] = 0$$

and

$$[h, m] = 0$$

respectively. (Note that the bracket is that of a *graded* Lie algebra, see the next Section, so it is only graded antisymmetric. Hence things like $[m, m] = 0$ do not happen automatically.) Further (Exercise) the differential d defined in the complex, with respect to the multiplication m, can be written as

$$df = [m, f]$$

for $f \in \mathrm{Hom}(A^{\otimes n}, A)$.

The upshot of all this is that fixing an initial associative algebra structure on A, one can define

$$\Gamma = \bigoplus_{n=-1}^{\infty} \Gamma^n$$

where

$$\Gamma^n = \mathrm{Hom}(A^{\otimes(n+1)}, A).$$

By the foregoing discussion, the beast Γ has an awful lot of algebraic structure on it, coming from its bracket and differential. This structure will be formalized in the next Section, we will just use it for the time being.

2.7 An initial associative algebra structure on A corresponds in the above language to an element $m \in \mathrm{Hom}(A^{\otimes 2}, A)$. A *deformed* multiplication corresponds to $(m + f) \in \mathrm{Hom}(A^{\otimes 2}, A)$, and the associativity condition reads

$$[m + f, m + f] = 0.$$

Using the obvious linearity of the bracket and taking care of the signs, this becomes

$$[m, m] + 2[m, f] + [f, f] = 0.$$

Recalling however that m is associative, and looking back on the definition of d, this becomes

$$\boxed{df + \tfrac{1}{2}[f, f] = 0}$$

This is a very important equation, so important that it has a name: it is called the *Maurer–Cartan equation*, being an analogue of the equation bearing the same name in differential geometry.

Finally, we have to remember that coboundaries coming from Γ^0 have to be taken into account. For every $g \in \mathrm{Hom}(A, A)$, we can define $T = \exp(g) \in G$, an invertible linear transformation of A. You can check (Exercise) that the action

$$(T \circ (m + f))(a, b) = T(m + f)(T^{-1}a, T^{-1}b)$$

corresponds, under our conventions, to the action

$$g \circ f = \exp(g)\, f \exp(-g) + exp(g)\, d\exp(-g). \tag{5}$$

where the product maps here are appropriate compositions. For further reference, I note that this action is simply the exponential of the Lie algebra action

$$\dot{f} = dg + [g, f]$$

as you can check easily by writing out (5) to first order (Exercise).

Thus the final answer is

equivalence classes of deformations of (A, m)	$\cong$	$\{f \in \Gamma^1 \mid df + \frac{1}{2}[f, f] = 0\} / \Gamma^0$
	$=$	'solutions to Maurer–Cartan modulo gauge equivalence'

2.8 Before I move on, let me remark that the 'linearized' problem may be recovered from this description by noting that after linearization one obtains '$df = 0$ modulo linearized gauge equivalence' and this is just one-cocyles modulo coboundaries.

2.9 Finally, this is the right time to introduce *obstructions*. Let $[f] \in H^1(A, A)$ be a vector in the tangent space to the moduli problem, represented by $f \in \Gamma^1$. We say that the direction given by the vector f is *unobstructed*, if there is a deformation

$$\tilde{f} = \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \dots$$

where $f_1 = f$, the f_i are elements of Γ^1 and $\tilde{f}$ represents a (formal) point of the moduli space, i.e.

$$d\tilde{f} + \frac{1}{2}[\tilde{f}, \tilde{f}] = 0 \tag{6}$$

formally in $\Gamma[[\varepsilon]]$. One says that *the deformation problem is unobstructed* if this can be done for all $[f]$. Comparing powers of ε, (6) is equivalent to (Exercise) the set of equations

$$df_n + \frac{1}{2}\sum_{i=1}^{n-1}[f_i, f_{n-i}] = 0. \tag{7}$$

Let s_n denote the big sum. Clearly $s_n \in \Gamma^2$ and one checks (Exercise in handling big sums)

$$ds_n = 0.$$

Thus if $H^2(A, A) = 0$, then s_n, being a cocycle, is *automatically* a coboundary and so (7) can be solved step-by-step. More generally, the non-vanishing of $H^2(A, A)$ means exactly that the equation $df_n = s_n$ may *not* be solvable. It is in this sense that the space $H^2(A, A)$ is *the space of obstructions*. If $H^2(A, A) = 0$, we say *obstructions vanish* and so the problem is unobstructed. Note that the vanishing of H^2 is a sufficient but not necessary condition for unobstructedness.

Example 2.10 Returning to the simple cases discussed before, I leave $A = k[\mathbb{Z}/2\mathbb{Z}]$ to the reader (recall $H^1(A, A) = 0$ anyway). In the other case $A = k(x)/(x^2)$, we have found one nontrivial (first-order) deformation, corresponding to the one-cocycle $f(x, x) = \lambda$, all other terms zero. This *is* an associative product rule (though without unit) so satisfies $[f, f] = 0$. (This *need not* be the case in more complicated, e.g. non-commutative examples.) Thus f satisfies Maurer–Cartan on the nose, there is no need to modify it by higher-order terms. Obviously the deformation is given by the one-parameter family of algebras $k(x)/(x^2 - \lambda)$.

As a final tortuous Exercise, the reader can calculate that for a map $h \in \mathrm{Hom}(A \otimes A \otimes A, A)$ to be a two-cocycle, it must act on basis elements as

$$\begin{aligned}
h(1,1,1) &= 0 \\
h(1,1,x) &= a + bx \\
h(1,x,1) &= 0 \\
h(1,x,x) &= -ax \\
h(x,1,1) &= c + dx \\
h(x,1,x) &= (a + c)x \\
h(x,x,1) &= -cx \\
h(x,x,x) &= e + fx
\end{aligned}$$

whereas the coboundary given by $f \in \mathrm{Hom}(A \otimes A, A)$ is

$$\begin{aligned}
(df)(1,1,1) &= 0 \\
(df)(1,1,x) &= C + (D - A)x \\
(df)(1,x,1) &= 0 \\
(df)(1,x,x) &= -Cx \\
(df)(x,1,1) &= -E + (A - F)x \\
(df)(x,1,x) &= (C - E)x \\
(df)(x,x,1) &= Ex \\
(df)(x,x,x) &= 0
\end{aligned}$$

for $a, \ldots, F \in k$. The interpretation of this horrendous set of formulae is of course that $H^2(A, A)$ is *two-dimensional*, represented by the two-cocycle $h(x, x, x) = e + fx$, all other images zero. Thus already this innocent-looking algebra provides us with an example of *unobstructed* deformation theory, where however the obstruction space *does not vanish*.

3 DGLAs and deformation functors

3.1 The definitions below are rather technical, however there is no point in postponing them any further. So here we go. A structure

$$\left(\Gamma = \bigoplus_{n=-\infty}^{\infty} \Gamma^n, [\,,\,], d \right)$$

is called a *differential graded Lie algebra* (DGLA) if d is a graded derivation of degree one, $[\,,\,]$ is a bracket

$$[\,,\,] : \Gamma^n \times \Gamma^n \to \Gamma^{n+m}$$

and furthermore the following compatibilities (graded Leibniz, graded antisymmetry, graded Jacobi) hold for homogeneous elements $g_i \in \Gamma^{m_i}$:

$$d[g_1, g_2] = [dg_1, g_2] + (-1)^{m_1}[g_1, dg_2],$$

$$[g_1, g_2] = -(-1)^{m_1 m_2}[g_2, g_1],$$

$$[g_1, [g_2, g_3]] + (-1)^{m_3(m_1+m_2)}[g_3, [g_1, g_2]] + (-1)^{m_1(m_2+m_3)}[g_2, [g_3, g_1]] = 0$$

3.2 Now we are going to construct a deformation functor for a DGLA as follows. Let $\underline{Sets}$ be the category of sets, and let $\underline{Art_k}$ be the category of Artinian local rings with residue field k. (Such a ring is of the form $R = k \oplus \mathbf{m}$ where $\mathbf{m}$ is a finite-dimensional nilpotent algebra. Even more concretely, R is isomorphic to a finitely generated ring $k(x_i)/I$ where I contains some power of the maximal ideal of the origin. So R is like a several variables version of $k(t)/(t^n)$.)

Given a DGLA Γ, define the functor

$$\mathrm{Def}_\Gamma : \underline{Art_k} \to \underline{Sets}$$

as follows: the set corresponding to an Artinian local ring $R = k \oplus \mathbf{m}$ is

$$\left\{ g \in \Gamma^1 \otimes \mathbf{m} \mid dg + \frac{1}{2}[g,g] = 0 \right\} \Big/ (\Gamma^0 \otimes \mathbf{m})$$

Here the bracket and the differential extend in the obvious way to $\Gamma^1 \otimes \mathbf{m}$, and the action of $\Gamma^0 \otimes \mathbf{m}$ is via the exponential of the Lie algebra action

$$g \in \Gamma^0 \otimes \mathbf{m} \;\text{ acts by }\; \dot{h} = dg + [g,h].$$

Morphisms between the sets are defined using the morphisms between the respective Artin rings.

Believe it or not, *everything* here is a straightforward formalization of the material of the previous Section: we are considering solutions of Maurer–Cartan up to gauge equivalence. Perhaps the only confusing concept is the last one, about exponentiating the Lie algebra action. The point is that most the DGLAs arising in nature are actually *infinite dimensional.* Taking solutions of Maurer–Cartan in Γ^1 and trying to divide by an infinite dimensional Lie algebra action is not going to lead anywhere. By tensoring with the *nilpotent* algebra $\mathbf{m}$, the exponential does make sense (it is just a finite sum) and so do the resulting sets. We could avoid this in the example of the previous Section because the Lie algebra occuring was finite-dimensional.

The tangent space in this framework is of course obtained by linearizing the above construction: it is the cohomology $H^1(\Gamma)$ of the complex in degree one.

Remark 3.3 It was Deligne who suggested first that in characteristic zero *every* sensitive deformation theory should have an associated DGLA. As we have seen above, the Hochschild complex together with the bracket constructed governs the deformation theory of f.d. associative k-algebras with unit. As Kontsevich says, it is sometimes an 'art' to find the correct DGLA for a given deformation problem. However, if it exists, then the above procedure provides the answer to Parts One and Two of the program outlined in 1.2.

Remark 3.4 By a reasonably standard result of Schlessinger, the functor defined above is pro-represented by a *complete* local k-algebra R_Γ in the sense that the the functors Def_Γ, $\mathrm{Hom}_{k-\mathrm{alg}}(R_\Gamma, -)$ from $\underline{Art_k}$ to $\underline{Sets}$ are isomorphic. The ring R_Γ (or its spectrum) is then the solution to Part Three of 1.2.

Remark 3.5 Of course this definition is open to several generalizations. First of all, according to Grothendieck–Schlessinger, the natural range of the functor Def_Γ is *groupoids* rather than sets. The well-informed reader is invited to satisfy himself with reformulating the definitions accordingly.

Also, one may wonder why, of the enormously big structure Γ, only Γ^1 and Γ^0 (perhaps also Γ^2) appear. In fact, this need not be the case, if one defines the functor taking values on $\mathbb{Z}$-graded Artin local rings. This leads to the *extended deformation functors* of Barannikov–Kontsevich [3], appearing naturally in mirror symmetry and elsewhere.

3.6 Next, I introduce an equivalence relation on DGLAs. A pair Γ, Γ' of DGLAs is said to be *quasi-isomorphic* if there is a chain

$$\Gamma = \Gamma_0 \to \Gamma_1 \leftarrow \Gamma_2 \to \ldots \to \Gamma_m = \Gamma'$$

of differential graded Lie algebra homomorphisms, which induce isomorphisms on the cohomologies of (Γ_i, d_i).

Theorem 3.7 (Main Theorem of deformation theory) *Quasi-isomorphic DGLAs give rise to isomorphic deformation functors.*

This is proved in [7] as a sequence of rather elementary but tricky lemmas in homological algebra.

Remark 3.8 As it turns out, the condition of this Theorem is too strong. There is a much weaker equivalence relation, that of L_∞-equivalence, which leads to the same conclusion. This is one of the crucial points of Kontsevich' work on deformation quantization [8].

3.9 A final definition that will only play a role in the last Section: a DGLA Γ is called *formal*, if it is quasi-isomorphic to its cohomology Lie algebra with zero differential. Formality is a very pleasant property that frequently holds; it implies for example (Exercise) that the (formal) moduli space (the spectrum of the ring R_Γ) is an intersection of homogeneous quadric cones.

4 Deformations of complex structures

4.1 Here of course $k = \mathbb{C}$. The material of this Section is entirely standard [6], and I am going to give the definitions only. A *complex manifold* is of course a real manifold with holomorphic transition functions between charts. One wishes to consider the deformation problem for complex manifolds, i.e. to understand 'nearby complex structures' to a given one.

4.2 A useful way to look at a complex structure is to view it as an integrable almost complex structure. An almost complex structure is given by a sub-bundle of the complexified tangent bundle of the manifold X. A 'nearby' almost complex structure is given by a section of $\mathrm{Hom}(\bar{T}_X, T_X)$ or equivalently, a section of $\Omega_X^{0,1}(T_X)$, where T_X is the holomorphic tangent bundle in the initial complex structure. This (infinite dimensional) space is the space V in the geometric framework of 1.3. The space S is given by the condition that the new almost complex structure should be integrable. The integrability condition for $\gamma \in \Gamma(X, \Omega_X^{0,1}(T_X))$ is

$$\bar{\partial}\gamma + \frac{1}{2}[\gamma, \gamma] = 0,$$

i.e. the Maurer-Cartan equation again. Here the $\bar{\partial}$ operator is given by the complex structure of X, it simply differentiates the $(0,1)$-part; the bracket is the Lie bracket on holomorphic vector fields and cup product on forms.

Equivalences between structures are given by the action of the diffeomorphism group. A one-parameter family of diffeomorphisms is given by a section of the real tangent bundle. This section can be decomposed as $\eta+\bar{\eta}$ for $\eta \in \Gamma(T_X)$. Then the action of the diffeomorphism group on integrable $\gamma \in \Gamma(X, \Omega_X^{0,1}(T_X))$ can be identified with the action of η by

$$\dot{\gamma} = \bar{\partial}\eta + [\eta, \gamma].$$

4.3 This description fits perfectly well into the general framework. The tangent space to the deformation problem is $\ker(\bar{\partial})/\operatorname{im}(\bar{\partial})$, which is just the Dolbeault representation of the cohomology group $H^1(X, T_X)$. The governing DGLA is the direct sum of

$$\Gamma^q = \Gamma(X, \Omega_X^{0,q}(T_X))$$

for $q \geq 0$, with differential $\bar{\partial}$ and bracket as described above.

Remark 4.4 This DGLA sits naturally in a larger (bigraded) one

$$\tilde{\Gamma} = \bigoplus_{p,q} \Gamma^{p,q}$$

the direct sum of

$$\Gamma^{p,q} = \Gamma\left(X, \Omega_X^{0,q}\left(\bigwedge^p T_X\right)\right).$$

The bracket extends to the Schouten bracket on polyvector fields. This DGLA governs the 'extended deformations of complex structure', cf. [3], [1]. One is supposed to think of these deformations as 'non-commutative' (perhaps 'non-associative') deformations of X, more precisely its derived category of coherent sheaves. We are promised more details in [2].

5 Formality for Calabi–Yau manifolds

5.1 A compact complex Kähler manifold X of dimension N for the purposes of this note is called *Calabi–Yau* if it possesses a nowhere vanishing holomorphic N-form $\Omega \in \Omega_X^{N,0}$. Contraction with the form Ω gives an isomorphism

$$i_q : \Gamma\left(X, \Omega_X^{0,q}\left(\bigwedge^p T_X\right)\right) \longrightarrow \Gamma\left(X, \Omega_X^{N-p,q}\right).$$

The $\bar{\partial}$-operator acts on both sides, and as Ω is holomorphic, the actions commute with the i-s. The ∂-operator on the right hand side gives, under the isomorphisms i, a map

$$\Delta : \Gamma\left(X, \Omega_X^{0,q}\left(\bigwedge^p T_X\right)\right) \to \Gamma\left(X, \Omega_X^{0,q-1}\left(\bigwedge^p T_X\right)\right).$$

The two operators Δ and $\bar{\partial}$ act on the algebra $\tilde{\Gamma}$ introduced in 4.4 and satisfy $\Delta \circ \bar{\partial} + \bar{\partial} \circ \Delta = 0$. The following is a direct consequence of the famous $\partial\bar{\partial}$-lemma of Kähler geometry:

Lemma 5.2 *One has*

$$\operatorname{im}(\bar{\partial}) \cap \ker(\Delta) = \ker(\bar{\partial}) \cap \operatorname{im}(\Delta) = \operatorname{im}(\Delta \circ \bar{\partial}).$$

The following is known as the Tian–Todorov lemma [9], [10]:

Lemma 5.3 *For* $\gamma_i \in \Gamma^{p_i,q_i}$,

$$[\gamma_1, \gamma_2] = (-1)^{p_1+q_1+1} \left(\Delta(\gamma_1 \wedge \gamma_2) - \Delta\gamma_1 \wedge \gamma_2 - (-1)^{p_1+q_1}\gamma_1 \wedge \Delta\gamma_2\right).$$

The proof is a local computation. Now we are ready for

Theorem 5.4 (Formality and Unobstructedness of Calabi–Yau manifolds) *Let* Γ *be the governing DGLA of the deformation theory of a Calabi–Yau* N*-fold* X *introduced in 4.3. Then the maps below are morphisms of DGLAs inducing isomorphisms on cohomology:*

$$(\Gamma, [\,,\,], \bar\partial) \xleftarrow{\psi} (\ker(\Delta), [\,,\,], \bar\partial) \xrightarrow{\varphi} (\ker(\Delta)/\operatorname{im}(\Delta), 0, 0)$$

where ψ *is the inclusion and* ψ *is the projection. In particular, the degree* q *piece of the last complex is isomorphic to* $H^q(X, T_X)$*. Consequently, the local moduli space of* X *can be identified with a smooth disc neighbourhood of the origin in* $H^1(X, T_X)$*.*

Proof It is just as well for this note to contain at least one proof, so this will be it. The proof happens inside the larger complex

$$\tilde\Gamma = \bigoplus_{p,q} \Gamma^{p,q}$$

containing all $(0,q)$-forms with values in the polyvector fields.

First of all, the injection ψ is clearly a morphism of DGLAs. To check that φ is a morphism too, one has to check the compatibility of the differential and the bracket. If $\gamma \in \ker(\Delta)$, then clearly $\bar\partial\gamma$ is Δ-closed and $\bar\partial$-exact. So by Lemma 5.2, it is Δ-exact as well. So $\bar\partial$ vanishes on $\ker(\Delta)/\operatorname{im}(\Delta)$. On the other hand, the bracket of two Δ-closed forms is Δ-exact by Lemma 5.3. So the bracket also vanishes on $\ker(\Delta)/\operatorname{im}(\Delta)$.

The rest of the proof is a completely formal consequence of Lemma 5.2 and the linear algebra machinery of [4, Section 5]: whenever one has a double complex with a pair of maps satisfying the analogue of the $\partial\bar\partial$-lemma, the three complexes of the theorem have isomorphic cohomology under the given maps. Just to illustrate this point, I show that pullback under ψ is onto in cohomology. So take $\alpha \in \Gamma$ with $\bar\partial\alpha = 0$. Consider $\Delta\alpha$: it is $\bar\partial$-closed and Δ-exact, so by Lemma 5.2, $\Delta\alpha = \bar\partial\Delta\beta$. Then $\gamma = \alpha + \bar\partial\beta$ is $\bar\partial$-closed, represents the same cohomology class as α but also $\Delta\gamma = \Delta\alpha + \Delta\bar\partial\beta = 0$. So indeed, γ is an element of the smaller complex $\ker(\Delta)$.

Similar arguments show that ψ^* is injective, and also that φ induces an isomorphism on cohomology. In particular the cohomology of the rightmost complex, $\ker(\Delta)/\operatorname{im}(\Delta)$ with zero differential, is isomorphic to the cohomology of the left hand complex at degree q which by Dolbeault's theorem is isomorphic to $H^q(X, T_X)$.

The final statement of the proof follows directly from Main Theorem 3.7: the right hand DGLA has zero differential and bracket, so *every* element in its degree one part gives an unobstructed solution to Maurer–Cartan and the gauge equivalence relation is trivial. □

References

[1] S. Barannikov, Generalized periods and mirror symmetry in dimensions $n > 3$

[2] S. Barannikov, in preparation

[3] S. Barannikov, M. Kontsevich: Frobenius manifolds and formality of Lie algebras of polyvector fields

[4] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan: The real homotopy theory of Kähler manifolds

[5] W. Goldman, J. Millson: The homotopy invariance of the Kuranishi space

[6] K. Kodaira: Complex manifolds and deformation of complex structures

[7] M. Kontsevich: Topics in algebra-deformation theory, course given at Berkeley, Fall 1994, notes by Alan Weinstein and M.K.

[8] M. Kontsevich: Deformation quantization of Poisson manifolds I

[9] G. Tian: Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Petersson–Weil metric

[10] A. Todorov: The Weil–Petersson geometry of the moduli space of $SU(n \geq 3)$ (Calabi–Yau) manifolds I

Mathematics Institute, University of Warwick
Coventry CV4 7AL, United Kingdom
Fax: +44 24 7652 4182
E-mail address: balazs@maths.warwick.ac.uk