Dimer models and local non-commutative algebraic geometry

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LMS lectures, Imperial College London, April 2008

Dimer model

$\Gamma \subset T^2$: a bipartite graph embedded in the torus

- Fix a partition of vertices of Γ into black and white
- Assume Γ admits perfect matchings (and other conditions...)

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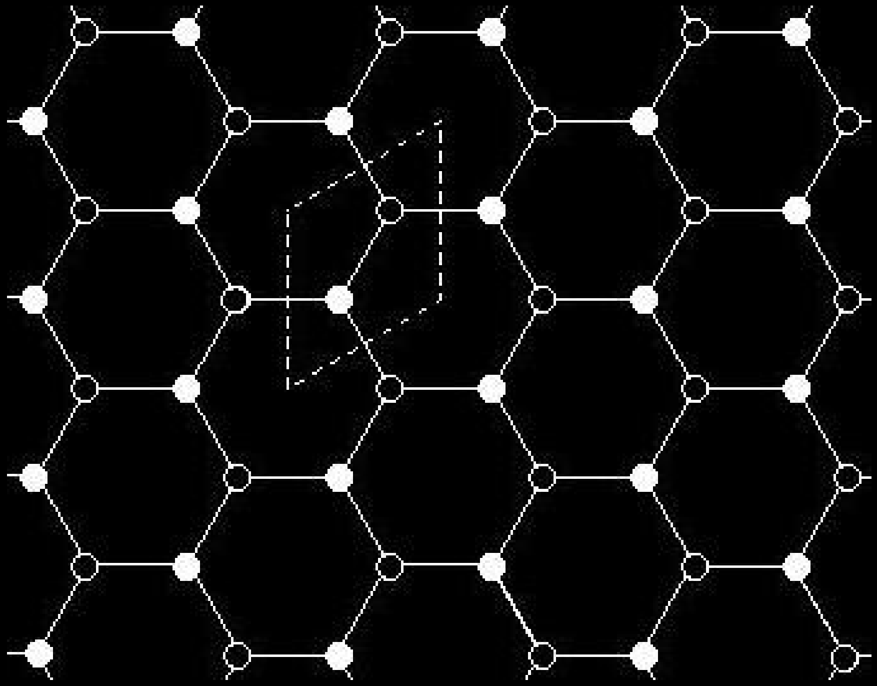
$\widetilde{\Gamma} \subset \mathbb{R}^2$: the pullback to the universal cover

Dimer configuration Δ: one-regular subgraph of $\tilde{\Gamma}$.

Hexagonal lattice

Example 1. $\tilde{\Gamma}$ is the hexagonal lattice.



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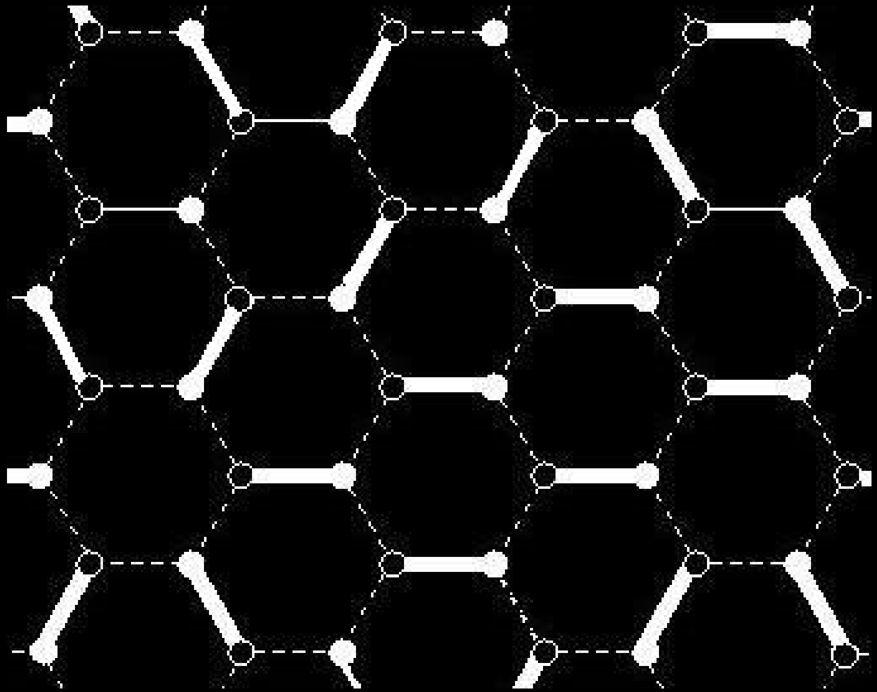

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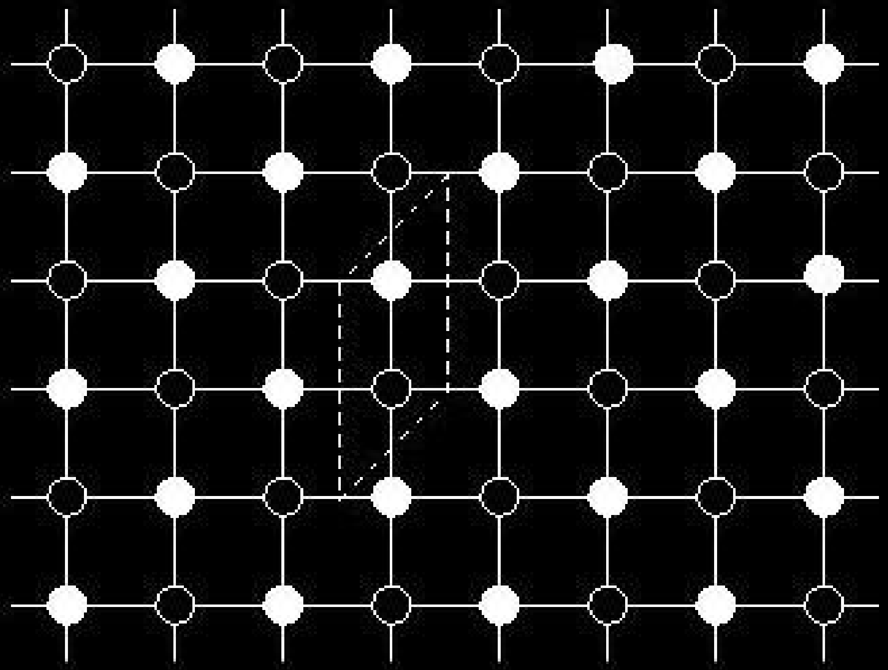
Example 2. $\tilde{\Gamma}$ is the square lattice.



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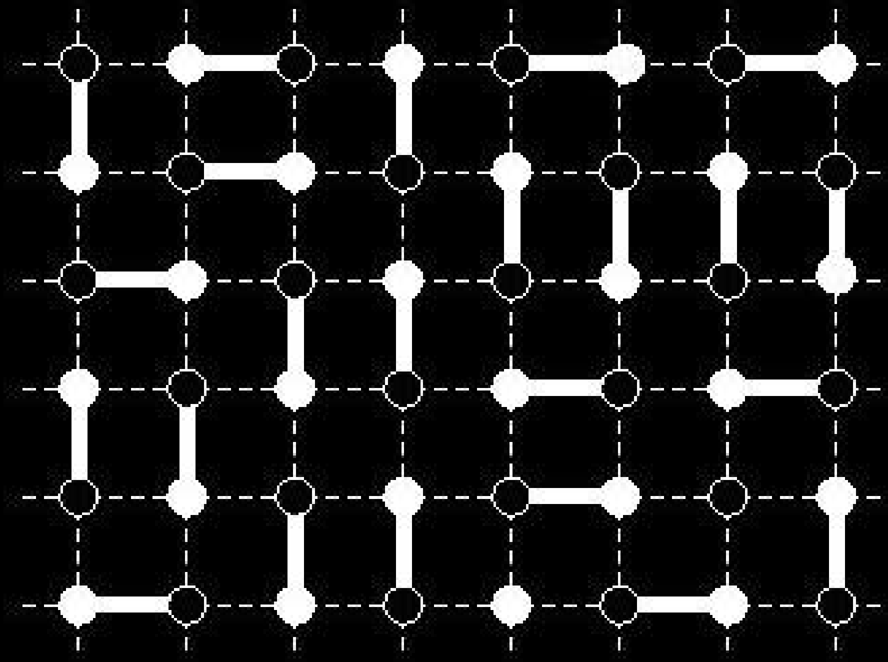

From the dimer model to the quiver

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- The vertex set V of Q is the set of connected components of $T^2 \setminus \Gamma$
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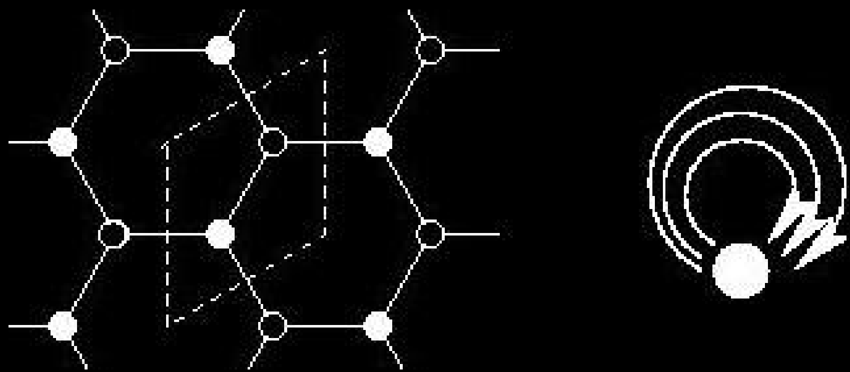

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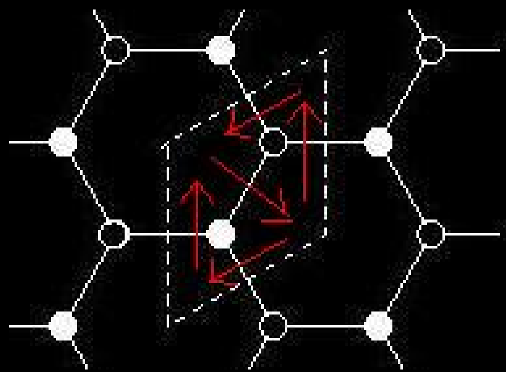

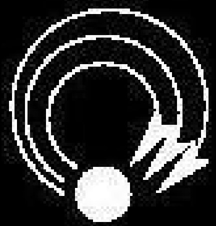

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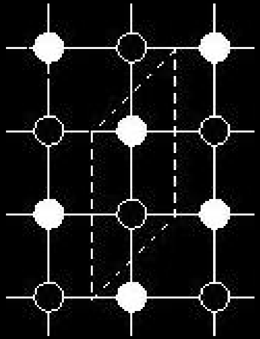

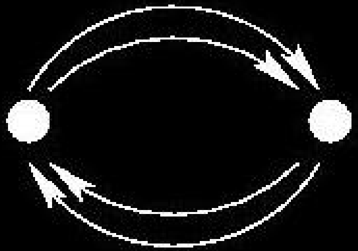

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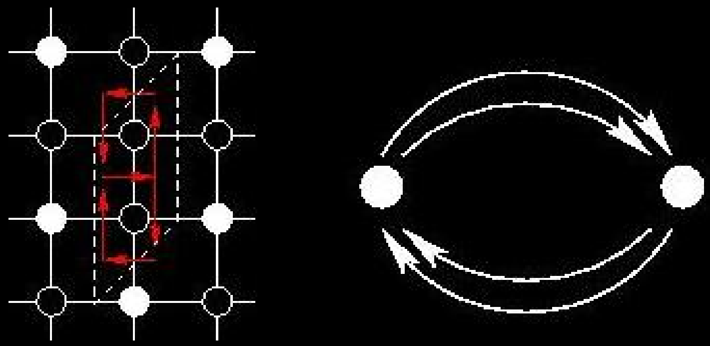

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Following [Hanany et al], define the superpotential

$$W = \sum_{\substack{i \in V(\Gamma) \\ i \text{ black}}} W_i - \sum_{\substack{i \in V(\Gamma) \\ i \text{ white}}} W_i$$

The non-commutative algebra

Let $\mathbb{C}Q$ be the quiver algebra of the quiver

- $\mathbb{C}$-algebra generated by oriented paths
- Product is given by concatenation of oriented paths (or zero)
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Define the non-commutative quotient algebra

$$A_{Q,W} = \mathbb{C}Q / \langle\langle \partial_e W : e \in E(Q) \rangle\rangle$$

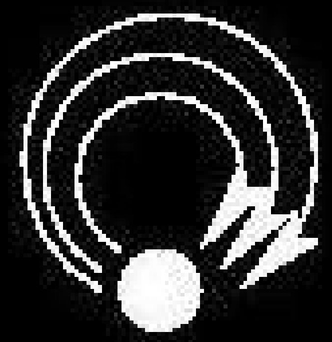
The algebra in Example 1

Take the quiver of Example 1. We have a free non-commutative algebra

$$\mathbb{C}Q = \mathbb{C}\langle x, y, z\rangle,$$

with superpotential

$$W = xyz - xzy.$$



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hence

$$A_{Q,W} = \mathbb{C}[x, y, z]$$

commutative!

The algebra in Example 2

Take the quiver of Example 2. We have the quiver algebra

$$\mathbb{C}Q = (\mathbb{C} \oplus \mathbb{C})\langle a, b, c, d\rangle,$$

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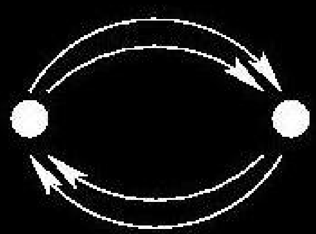

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which gives a non-commutative algebra $A_{Q,W}$. Its center

$$Z(A_{Q,W}) = \mathbb{C}[x, y, z, t]/(xy - zt)$$

with

$$x = ac + ca, \ldots$$

is the coordinate ring of the ordinary double point singularity.

Properties of $A_{Q,W}$

Under suitable conditions (King/Broomhead) the following hold:

- $A_{Q,W}$ is a smooth non-commutative 3-Calabi–Yau algebra
  - vanishing of Ext's on A-mod above degree 3
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- The algebra $A_{Q,W}$ is Van Den Bergh's non-commutative crepant resolution of $X = \mathrm{Spec}(Z(A))$

Moduli spaces of A-modules

Fix dimension vector $\mathbf{n} \in \mathbb{N}^V$ and a vertex $i \in V$.

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- Cyclicity: stability condition
- In Example 1:

$$\mathcal{M}_n \cong (\mathbb{C}^3)^{[n]},$$

  the Hilbert scheme of n points on $\mathbb{C}^3$.

Partition function

Proposition $\mathcal{M}_{\mathbf{n},i}$ is cut out in a smooth variety by zeros of one-form $d\mathrm{Tr}W$. Hence

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Fixing the vertex $i \in V$, get partition function of cyclic $A_{Q,W}$-modules based at vertex i:

$$Z(\mathbf{q}) = \sum_{\mathbf{n}\in\mathbb{N}^V} e_{\mathrm{vir}}(\mathcal{M}_{\mathbf{n},i})\mathbf{q}^{\mathbf{n}}$$

for a set of auxiliary variables $\mathbf{q} = \{q_1, \ldots\}$.

Torus action

Recall: the center of $A_{Q,W}$ defines a toric Calabi–Yau X.

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Proposition

- The torus T acts as outer automorphisms of the algebra $A_{Q,W}$.
- This defines an action of T on all moduli spaces $\mathcal{M}_{\mathbf{v},i}$.
- The T-fixed points on $\mathcal{M}_{\mathbf{v},i}$ are isolated.

Hence the partition function can be computed by localizing to the fixed points.

Torus localization: dimer configurations

For simplicity, restrict now to Examples 1-2.

Proposition The torus-fixed points on the moduli spaces $\mathcal{M}_{\mathbf{v},i}$ are in one-to-one correspondence with dimer configurations Δ on the corresponding graph $\tilde{\Gamma}$, asymptotic to a fixed dimer configuration Δ_0.

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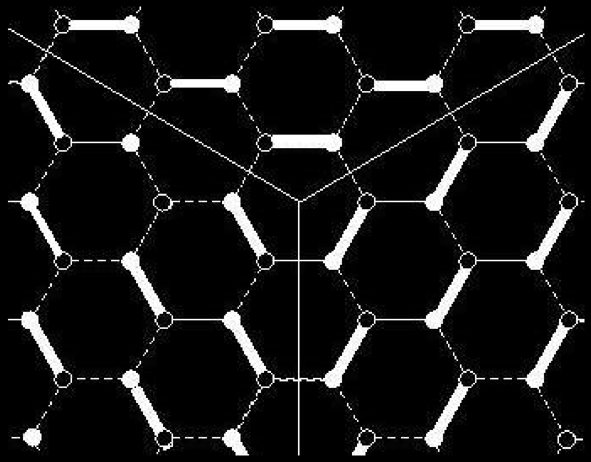

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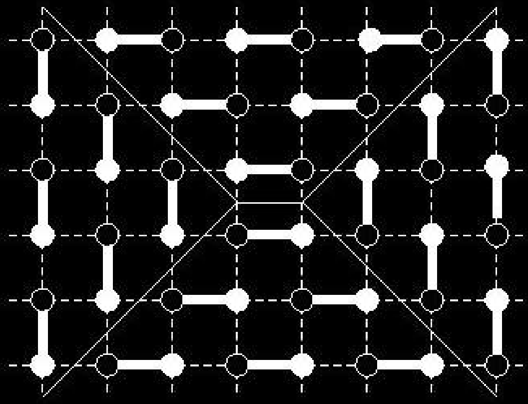
Example 2: the "empty room" dimer configuration Δ_0 on the square lattice.



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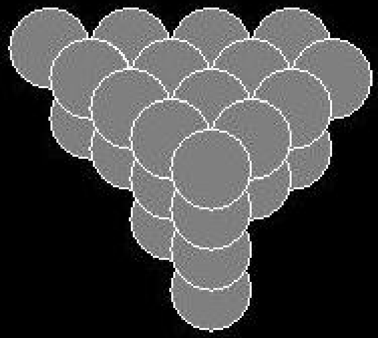
Example 1: the crystal is the triangular pyramid, its suitable subsets are

finite 3-dimensional partitions



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Equivalent combinatorics for Examples 1-2

Proposition The torus-fixed points on the moduli spaces $\mathcal{M}_{\mathbf{v},i}$ are in one-to-one correspondence with suitable finite subsets of a crystal configuration.

Example 2: the crystal is the square-based pyramid, its suitable subsets are

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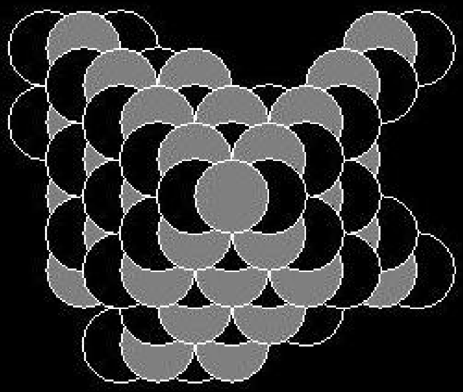
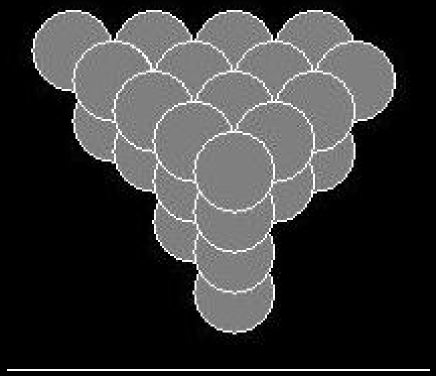


Exact results for the partition function

In Example 1, the set of T-fixed points on $\mathcal{M}_{\mathbf{v}}$ is the set $\mathcal{P}$ of 3-dimensional partitions. Hence

$$Z(q) = \sum_{\alpha \in \mathcal{P}} (-q)^{|\alpha|}$$

where $|\alpha|$ is the volume of the partition.



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Theorem (MacMahon, 19th c.)

$$Z(q) = \prod_{n \geq 1} (1 - (-q)^n)^{-n}$$

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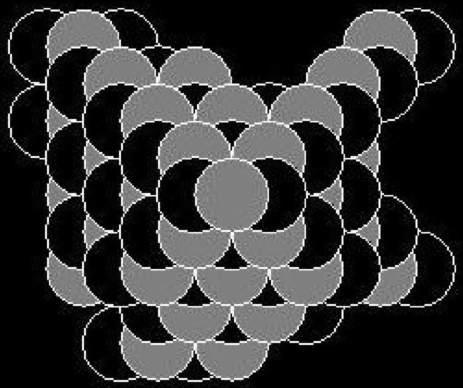
In Example 2, the set of T-fixed points on $\mathcal{M}_{\mathbf{v}}$ is the set $\widetilde{\mathcal{P}}$ of "pyramid partitions". Hence

$$Z(q_1, q_2) = \sum_{\pi \in \widetilde{\mathcal{P}}} q_1^{|\pi|_1} (-q_2)^{|\pi|_2},$$

where $|\pi|_j$ are the number of stones in the pyramid partition π of colour j.



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Theorem (Sz.-Young, 2007/8)

$$Z(q_1, -q_2) = \prod_{n \geq 1} (1 - q_1^n q_2^{n-1})^n (1 - q_1^n q_2^{n+1})^n (1 - q_1^n q_2^n)^{-2n}.$$

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$$Z_Y(q,t) = \prod_n (1 - q^n t)^n (1 - q^n)^{-2n}.$$

The other is the non-commutative crepant resolution, the quiver algebra A.

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$$Z_A(q, t) = \prod_n (1 - q^n t)^n (1 - q^n t^{-1})^n (1 - q^n)^{-2n}$$

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An example of Kontsevich–Soibelman/Denef–Moore wall crossing!