SOME FINITENESS RESULTS FOR CALABI–YAU THREEFOLDS

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Abstract

The moduli theory of Calabi–Yau threefolds is investigated, and using Griffiths' work on the period map, some finiteness results are derived. In particular, a prediction of Morrison's cone conjecture is confirmed.

Introduction

If X is a smooth complex projective n-fold, Hodge–Lefschetz theory provides a filtration on the primitive cohomology $H_0^n(X, \mathbb{C})$ by complex subspaces, satisfying certain compatibility conditions with a bilinear form Q on cohomology. This gives a map called the *period map*, from a suitably defined moduli space containing X to a complex analytic space $\mathscr{D}/\Gamma$, the study of which was initiated by Griffiths. He showed in particular that if X has trivial canonical bundle, then this map is locally injective on the Kuranishi family of X; further, if the global moduli theory is well behaved, then the map can be extended to a proper map and so finiteness results can be derived.

This paper considers *Calabi–Yau threefolds*. A complex projective manifold X is Calabi–Yau if it has trivial canonical bundle and satisfies $H^i(X, \mathscr{O}_X) = 0$ for $0 < i < \dim(X)$. In Section 1 we recall a theorem about their Hilbert schemes, in Section 2 we investigate the moduli theory. Then we specialize to threefolds, and recall some of Griffiths' results in Section 3, which will enable us to deduce the crucial finiteness statement Theorem 4.2: the period point determines the threefolds up to finitely many choices among those with bounded polarization. This will imply Corollaries 4.3–4.5, which constitute the main results of this paper. In particular, we confirm the following consequence of Morrison's cone conjecture.

Corollary. *Let X be a smooth Calabi–Yau threefold, and fix a positive integer κ. Up to the action of* $\mathrm{Aut}(X)$*, there are finitely many ample divisor classes L on X with $L^3 \leqslant \kappa$. In particular, if the automorphism group is finite, there are finitely many such classes.*

Conventions. All schemes and varieties are assumed to be defined over $\mathbb{C}$, points of schemes are $\mathbb{C}$-valued points. By a polarized variety (X, L) we mean a projective variety with a choice of an ample invertible sheaf, $(X, L) \cong (X', L')$ if there is an isomorphism $\phi: X \xrightarrow{\sim} X'$ with $\phi^*(L') \sim L$. The highest self-intersection of L is

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denoted by L^n. A family of polarized varieties is a flat proper morphism or holomorphic map $\mathcal{X} \longrightarrow S$, with an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ whose restriction to every fibre is ample.

1. *The Hilbert scheme of Calabi–Yau manifolds*

First we recall the unobstructedness theorem for manifolds with trivial canonical bundle ($\mathcal{T}_X$ is the holomorphic tangent bundle of X).

THEOREM 1.1 (Bogomolov, Tian [**22**], Todorov [**23**] in the complex case, Ran [**19**], Kawamata [**10**] in the algebraic case). *Let X be a smooth projective n-fold with trivial canonical bundle, then it has a versal deformation space $\mathcal{X} \longrightarrow S$ over a complex germ or spectrum of a complete Noetherian local $\mathbb{C}$-algebra S with $0 \in S$, $\mathcal{X}_0 \cong X$, and S smooth. If $H^0(X, \mathcal{T}_X) = 0$, this deformation is universal. The tangent space of S at 0 is canonically isomorphic to $H^1(X, \mathcal{T}_X)$.*

Using standard arguments, for example [**7**, Appendix A], one obtains the following proposition.

PROPOSITION 1.2. *Let $n \geqslant 3$, X be as above and assume further that $H^2(X, \mathcal{O}_X) = 0$. The family $\mathcal{X} \longrightarrow S$ is also a versal family for invertible sheaves on X, that is given any sheaf L on X, there is a sheaf $\mathcal{L}$ on $\mathcal{X}$ restricting to L on the central fibre, with the obvious versal property.*

Let now (X, L) be a polarized Calabi–Yau n-fold with Hilbert polynomial p. Matsusaka's big theorem [**15**] gives us an integer m with the following property: for any polarized algebraic manifold (X_1, L_1) with Hilbert polynomial p, the sheaf $L_1^{\otimes m}$ is very ample and has no higher cohomology. Put $N = h^0(L_1^{\otimes m}) - 1 = p(m) - 1$, then we have embeddings $\phi_{|L_1^{\otimes m}|} \colon X_1 \longrightarrow \mathbb{P}^N$ and in particular embeddings $\phi_{|L^{\otimes m}|} \colon X \longrightarrow \mathbb{P}^N$, depending on the choice of a basis of $H^0(X, L^{\otimes m})$. Thus for a fixed choice of basis, we get a point $x \in \mathrm{Hilb}^p_{\mathbb{P}^N}$, where $\mathrm{Hilb}^p_{\mathbb{P}^N}$ is the fine projective moduli scheme representing the Hilbert functor of $\mathbb{P}^N$ with polynomial p.

THEOREM 1.3 (cf. [**8**]). *Assume that $n \geqslant 3$. If (X, L) is a polarized Calabi–Yau n-fold, then the scheme $\mathrm{Hilb}^p_{\mathbb{P}^N}$ is smooth at x.*

Proof. Using the Euler sequence of $\mathbb{P}^N$ restricted to X and Kodaira vanishing, one obtains $H^1(X, \mathcal{T}_{\mathbb{P}^N}|_X) = 0$. The normal bundle sequence now gives that $H^1(X, \mathcal{N}_{X/\mathbb{P}^N}) \longrightarrow H^2(X, \mathcal{T}_X)$ is injective, whereas $H^0(X, \mathcal{N}_{X/\mathbb{P}^N}) \longrightarrow H^1(X, \mathcal{T}_X)$ is surjective. Hence by unobstructedness the deformations of $X \subset \mathbb{P}^N$ are also unobstructed, and all deformations of X can be realized in $\mathbb{P}^N$. The theorem follows. □

The following lemma is also standard.

LEMMA 1.4. *Let $\mathcal{X} \longrightarrow S$ be a flat family of projective varieties with smooth fibres over the base S. If for $0 \in S$ the fibre $\mathcal{X}_0$ is Calabi–Yau, then all fibres are Calabi–Yau.*

Let $\mathscr{X}_{\mathrm{Hilb}} \longrightarrow \mathrm{Hilb}^{p}_{\mathbb{P}^N}$ be a universal family over the Hilbert scheme with the universal relatively ample invertible sheaf $\mathscr{L}_{\mathrm{Hilb}}$ over $\mathscr{X}_{\mathrm{Hilb}}$, H' be the open subset of $\mathrm{Hilb}^{p}_{\mathbb{P}^N}$ over which this family has smooth fibres. The quasi-projective scheme H' has several irreducible components, fix one component H which contains a point corresponding to a smooth polarized Calabi–Yau fibre (X, L). By Lemma 1.4, all fibres of the pullback family $\mathscr{X}_H \longrightarrow H$ are polarized Calabi–Yau manifolds, so H is a smooth quasi-projective variety.

Now let $G = \mathrm{SL}(N+1, \mathbb{C})$. As usual, there is an action of G on $\mathrm{Hilb}^{p}_{\mathbb{P}^N}$. From the definition of H and connectedness of G, it follows that there is an induced action $\sigma: G \times H \longrightarrow H$. By the universal property, the action extends to an action of G on $\mathscr{X}_H$. The action σ is proper; this follows from 'separatedness of the moduli problem': since fibres are never ruled by Lemma 1.4, an isomorphism of polarized families over the generic point of the spectrum of a discrete valuation ring specializes to an isomorphism over the special fibre (Matsusaka and Mumford [**16**]). Any $h \in H$ has reduced finite automorphism group.

2. *The moduli space*

Proposition 2.1. *The quotient H/G is a separated algebraic space of finite type over $\mathbb{C}$.*

Proof. This follows from [**18**, II, Theorem 1.4]. For an algebraic proof, see [**12**]. □

The aim of this section is to prove the following theorem.

Theorem 2.2. *There exists a quasiprojective scheme Z with the following properties:*

(i) *There exists a family $\mathscr{X}_Z \longrightarrow Z$ of smooth Calabi–Yau varieties over Z, polarized by an invertible sheaf $\mathscr{L}_Z$ on $\mathscr{X}_Z$.*

(ii) *The classifying map of the family $\mathscr{X}_Z \longrightarrow Z$ is a finite surjective map of algebraic spaces $Z \longrightarrow H/G$.*

(iii) For $t \in Z$, *let $\mathscr{X}_t$ be the fibre of the family. Then the spectrum of the completion of the local ring $\mathscr{O}_{Z,t}$ together with the induced family is the* (*algebraic*) *versal family of $\mathscr{X}_t$. In particular, by unobstructedness, Z is smooth.*

This theorem can be proved in two different ways. The proof we give below consists of two steps: first one builds an étale cover $H^{\mathrm{et}} \longrightarrow H$ directly, with a free G-action, using a rigidification construction; then Z exists as an algebraic space, and property (i) of the theorem together with results of Viehweg [**24**] imply that Z is quasi-projective. An alternative way was pointed out to the author by Alessio Corti: H/G is the coarse moduli space representing the stack $\mathscr{Z}$ whose category of sections over a $\mathbb{C}$-scheme S is the set of polarized families of Calabi–Yau n-folds over S as objects, with isomorphisms over S as morphisms. $\mathscr{Z}$ is in fact a Deligne–Mumford stack, and one can show the existence of Z as a finite union of affine schemes satisfying (i)–(iii) using Artin's method as follows: consider algebraizations of formal versal families of individual varieties [**2**], and use openness of versality [**3**] to show that a finite union of them covers H/G and (iii) is satisfied. The author decided to give the proof below because he feels that it is natural in the context and it is more concrete.

PROPOSITION 2.3. *Condition* (iii) *of Theorem 2.2 follows from*
(iii)′ $Z \cong H^{\mathrm{et}}/G$ *where* $H^{\mathrm{et}} \longrightarrow H$ *is a finite étale cover and* G *acts freely on* H^{et}.

Proof. Let $\mathcal{Y} \longrightarrow S$ be the versal deformation space of $\mathcal{X}_t$ over the spectrum of a complete local $\mathbb{C}$-algebra, with $\mathcal{Y}_0 \cong \mathcal{X}_t$. The variety $\mathcal{X}_t$ comes equipped with a distinguished ample sheaf $\mathcal{L}_t$ over it. By Proposition 1.2, there is a relatively ample sheaf $\mathcal{L}$ over $\mathcal{Y}$ extending $\mathcal{L}_t$, and we can think of S as the base space of versal deformations of $(\mathcal{X}_t, \mathcal{L}_t)$.

Let $U = \operatorname{Spec}(\widehat{\mathcal{O}}_{Z,t})$ with closed point still denoted by t; then the pullback family $\mathcal{X}_U \longrightarrow U$ is a flat polarized deformation of $\mathcal{X}_t$. By the definition of the versal family, there is a morphism $\epsilon: U \longrightarrow S$ mapping t to 0 such that the family over U is a pullback by ϵ.

On the other hand, we may assume that $\mathcal{L}$ can be trivialized by $N+1$ sections over S. From the universal property of the Hilbert scheme, this determines a morphism $S \longrightarrow \operatorname{Spec}(\widehat{\mathcal{O}}_{H,h})$, so a morphism $S \longrightarrow \operatorname{Spec}(\widehat{\mathcal{O}}_{H^{\mathrm{et}},h'})$, where h' is chosen so that the composition with the morphism coming from $H^{\mathrm{et}} \longrightarrow Z$ gives a map $\tau: S \longrightarrow U$, mapping 0 to t.

The composite $\tau \circ \epsilon: U \longrightarrow U$ fixes t and pulls back the family over U to itself, so as the action of G on H^{et} is free, it is the identity. Similarly, $\epsilon \circ \tau: S \longrightarrow S$ fixes 0 and pulls back the polarized family over S to itself, so by universality it has finite order (it can be nontrivial, giving an automorphism of the polarized central fibre). Hence ϵ and τ are isomorphisms, and (iii) follows. □

Proof of Theorem 2.2. First we construct H^{et}, using a method which is best known in the case of curves, and was applied in the higher dimensional case by Popp [**18**, I], followed by a direct construction.

Let us consider a smooth polarized family $\mathcal{Y} \longrightarrow S$ of complex projective n-folds such that the automorphisms of fibres are finite, let (X, L) be a fixed fibre. Denote $H_{\mathbb{Z}}(\mathcal{Y}_s) = H^n(\mathcal{Y}_s, \mathbb{Z})/\text{torsion}$, a free $\mathbb{Z}$-module with a pairing Q_s; consider the map

$$\theta_s: \operatorname{Aut}(\mathcal{Y}_s, \mathcal{L}_s) \longrightarrow \operatorname{Aut}(H_{\mathbb{Z}}(\mathcal{Y}_s), Q_s), \tag{1}$$

with image Θ_s. For any $s \in S$, $H_{\mathbb{Z}}(\mathcal{Y}_s) \cong H_{\mathbb{Z}}(X)$, as the family is locally topologically trivial. Let $\mathrm{I}(s)$ be the set of minimal orthonormal or symplectic generating systems of $H_{\mathbb{Z}}(\mathcal{Y}_s)$; then $\Lambda = \operatorname{Aut}(H_{\mathbb{Z}}(\mathcal{Y}_s), Q_s)$, a group of matrices over $\mathbb{Z}$, acts transitively on $\mathrm{I}(s)$. Consider the disjoint union $\tilde{S} = \bigcup_{s \in S} \mathrm{I}(s)$; then there is a map $\gamma: \tilde{S} \longrightarrow S$ which allows one to put a topology on $\tilde{S}$ in a standard way, such that γ is a topological covering with covering group Λ. Now recall the following result.

LEMMA 2.4 (Serre [**20**]). *Let* $l \geqslant 3$ *be an integer, and* $\alpha \in \mathrm{GL}_m(\mathbb{Z})$ *be an invertible matrix of finite order satisfying* $\alpha \equiv I_m \pmod{l}$. *Then* $\alpha = I_m$, *the identity.*

By assumption, Θ_s is a finite subgroup of Λ for any s, so all its elements have finite order. For any integer $l \geqslant 3$, let $\Lambda^{(l)}$ be the *lth congruence subgroup* of Λ. Applying Lemma 2.4, the intersection of $\Lambda^{(l)}$ and any Θ_s will be trivial. Let $\bar{\Lambda}^{(l)}$ be the quotient of Λ by $\Lambda^{(l)}$, let $S^{(l)}$ be the finite unramified covering of S corresponding to this finite group. $S^{(l)}$ is called the *level-l cover* of S. It is naturally a complex analytic space, so by the generalized Riemann existence theorem it has the structure of a complex scheme such that $S^{(l)} \longrightarrow S$ is an étale morphism.

Consider the finite unramified cover $H^{(l)} \longrightarrow H$, together with the family $\mathscr{X}_{H^{(l)}} \longrightarrow H^{(l)}$ of polarized Calabi–Yaus pulled back from the Hilbert family $\mathscr{X}_H \longrightarrow H$.

LEMMA 2.5. *There is a proper action of the group* $G = \mathrm{SL}(N+1, \mathbb{C})$ *on* $H^{(l)}$,

$$\rho : G \times H^{(l)} \longrightarrow H^{(l)}$$

and the map $H^{(l)} \longrightarrow H$ *is a G-morphism. G also acts on* $\mathscr{X}_{H^{(l)}}$.

Proof. See [**18**, I, 2.19]. □

The stabilizers under the action ρ are the kernels of the maps (1), which are of very special type.

LEMMA 2.6. *Let* (X, L) *be a polarized Calabi–Yau,* $\alpha \in \ker(\theta) \subset \mathrm{Aut}(X, L)$*, that is, assume that* α *acts trivially on the nth cohomology. Let* $\mathscr{X} \longrightarrow S$ *be a small polarized deformation of* (X, L)*; then* α *extends to give an automorphism of the family* $\mathscr{X}$ *over S leaving S fixed and also fixing the polarization.*

Proof. Once α extends to $\mathscr{X}$ fixing S, it fixes the polarization also, since it fixes L and the Picard group of a Calabi–Yau is discrete. We may also assume that $\mathscr{X} \longrightarrow S$ is in fact the Kuranishi family. Then by universality, α gives an automorphism $\tilde{\alpha}$ of $\mathscr{X}$ over S. Assume that $\tilde{\alpha}$ acts nontrivially on S. Then it must also act nontrivially on the tangent space to S at X, that is, $H^1(X, \mathscr{T}_X)$. This is however a direct summand of $H^n(X, \mathbb{C})$, so $\tilde{\alpha}$ acts nontrivially on that and hence also on $H_{\mathbb{Z}}(X)$. This is a contradiction. □

LEMMA 2.7. *There exists a cover* $H^{(p)} \longrightarrow H^{(l)}$ *with a finite covering group K, which becomes a finite unramified map when we give* $H^{(p)}$ *the induced scheme structure. There is an induced action of G on* $H^{(p)}$*, which is proper and free. The action extends to an action on the pullback family* $\mathscr{X}_{H^{(p)}} \longrightarrow H^{(p)}$.

Proof. For any $h \in H^{(l)}$, denote by K_h the set of automorphisms of $(\mathscr{X}_h, \mathscr{L}_h)$ that extend to the Kuranishi family fixing the base. For any h, this is isomorphic to the group K of generic isomorphisms of the family. Let $H^{(p)} = \bigcup_{h \in H^{(l)}} K_h$; then, as before, $H^{(p)}$ can be given an induced scheme structure such that we get an unramified covering. It is easy to check that there is a proper action of G on $H^{(p)}$, and Lemma 2.6 will imply that the action is free. □

Now $H^{(p)} \longrightarrow H$ is the cover $H^{\mathrm{et}} \longrightarrow H$ in (iii)′ of Proposition 2.3.

LEMMA 2.8. *The quotient Z of* $H^{(p)}$ *by G exists as a quasi-projective scheme, and there is a polarized family* $\mathscr{X}_Z \longrightarrow Z$ *with smooth fibres over it.*

Proof. As before, the quotient Z exists as an algebraic space of finite type, and [**18**, III, 1.4] shows that there is a polarized family $\mathscr{X}_Z \longrightarrow Z$ with smooth fibres. The total space $\mathscr{X}_Z$ is at the moment also an algebraic space only, but it is quasi-projective if Z is. Also, from the construction and Proposition 2.3 we obtain that Z is smooth.

Using smoothness of $H^{(\rho)}$ and the assumptions about the action of G on it, by Seshadri's theorem [**21**, 6.1] we have the following diagram.

$$\begin{array}{ccccc} V & \xrightarrow{p} & H^{(\rho)} & \longrightarrow & H \\ {\scriptstyle q}\downarrow & & \downarrow & & \\ T & \longrightarrow & Z & & \end{array}$$

Here T, V are normal schemes, G acts on V with geometric quotient T, and a finite group F also acts on V with quotient $H^{(\rho)}$ such that the actions of G and F on V commute. In particular, the map $T \longrightarrow Z$ is finite.

Let us pull back the family $\mathscr{X}_Z \longrightarrow Z$ to the family $\tau\colon \mathscr{X}_T \longrightarrow T$; denote $\mathscr{L} = \mathscr{L}_T$, $\omega = \omega_{\mathscr{X}_T/T}$. We will use the deep results due to Viehweg [**24**] to show that the scheme T is quasi-projective; we refer for terminology and results to [**24**]. Let m be an integer such that for $t \in T$, $\mathscr{L}_t^{\otimes m}$ is very ample on $\mathscr{X}_t$ and has no higher cohomology. Choosing an integer $l > (\mathscr{L}_t^{\otimes m})^n + 1$, all the conditions of the weak positivity criterion [**24**, 6.24] are satisfied (the dualizing sheaf is trivial on fibres). In particular $\tau_*(\mathscr{L}^{\otimes m})$ is locally free of rank r on T, and we obtain a weakly positive sheaf

$$\left(\bigotimes^{rm} \tau_*(\mathscr{L}^{\otimes rm} \otimes \omega^{\otimes lrm})\right) \otimes (\det(\tau_* \mathscr{L}^{\otimes m}))^{-rm}$$

over T. Using [**24**, 2.16d] the sheaf

$$\mathscr{A} = \tau_*(\mathscr{L}^{\otimes rm} \otimes \omega^{\otimes lrm}) \otimes (\det(\tau_* \mathscr{L}^{\otimes m}))^{-1}$$

is also weakly positive over T. Then for integers $\mu > 1$, denote

$$\mathscr{Q} = \tau_*(\mathscr{L}^{\otimes rm\mu} \otimes \omega^{\otimes lrm\mu}) \otimes (\det(\tau_* \mathscr{L}^{\otimes m}))^{-\mu}$$

and look at the multiplication map

$$S^{\mu}(\mathscr{A}) \longrightarrow \mathscr{Q}.$$

For μ large enough, exactly as in the argument on [**24**, p. 304], the kernel of this map has maximal variation (it is here that we use the fact that every polarized fibre occurs only finitely many times by construction). The ampleness criterion [**24**, 4.33] therefore applies, so for suitable (large) integers a, b the sheaf

$$\mathscr{B} = \det(\mathscr{A})^{\otimes a} \otimes \det(\mathscr{Q})^{\otimes b}$$

is ample on T. Hence T is quasi-projective.

To finish the proof, we use the following lemma.

LEMMA 2.9. *Assume that $\delta\colon Y' \longrightarrow Y$ is a finite surjective map from a scheme Y' to a normal algebraic space Y, let L be an invertible sheaf on Y. If $\delta^*(L)$ is ample on Y', then L is ample on Y.*

Proof. This follows from [**6**, 2.6.2], noting that the proof given there carries over to the case when Y is an algebraic space. □

If we construct the sheaves $\mathscr{A}_Z, \mathscr{Q}_Z, \mathscr{B}_Z$ using the relative dualizing sheaf and polarization of the family over Z exactly as for T, they pull back to the sheaves $\mathscr{A}, \mathscr{Q}, \mathscr{B}$ on T via the finite surjective map $T \longrightarrow Z$. By Lemma 2.9, $\mathscr{B}_Z$ is ample on Z, so the proof of Lemma 2.8, and therefore also the proof of Theorem 2.2, are complete. □

3. *The period map*

From now on, let us assume that $n = 3$. If X is a Calabi–Yau threefold, Serre duality gives $H^5(X, \mathbb{C}) = 0$, so the whole third cohomology is primitive for topological reasons. Fix a non-negative integer b; let $V_{\mathbb{Z}}$ be the unique $(2b+2)$-dimensional lattice with a unimodular alternating form Q (the fact that this lattice is unique is proved in [**1**, 6.2.36]). Let $V = V_{\mathbb{Z}} \otimes \mathbb{C}$. The period map for Calabi–Yau threefolds X with $H^3(X, \mathbb{C}) \cong V$ takes values in the domain

$$\mathscr{D} = \{\text{flags } V = F^0 \supset F^1 \supset F^2 \supset F^3 \text{ with } \dim F^p = f_p, \text{ satisfying (R)}\},$$

where $f_0 = 2b+2$, $f_1 = 2b+1$, $f_2 = b+1$, $f_3 = 1$ and (R) are the Riemann bilinear relations $Q(F^p, F^{4-p}) = 0$, $(-1)^{p+1} iQ(\xi, \bar{\xi}) > 0$ for nonzero $\xi \in F^p \cap \bar{F}^{3-p}$. The arithmetic monodromy group $\Gamma = \operatorname{Aut}(V_{\mathbb{Z}}, Q)$ acts on $\mathscr{D}$, it is well known [**4**] that the action is proper and discontinuous, and $\mathscr{D}/\Gamma$ is a separated complex analytic space.

Let us define the set

$$\mathscr{C}_b = \{X \mid X \text{ a Calabi–Yau threefold with } b_3(X) = 2b+2\}/\cong.$$

We have isomorphisms $V \cong H^3(X, \mathbb{C})$ for any $X \in \mathscr{C}_b$ well-defined up to elements of Γ, so there is a map (the 'period map')

$$\phi : \mathscr{C}_b \longrightarrow \mathscr{D}/\Gamma$$

mapping X to the filtration on the primitive cohomology.

This is only a map between sets. However, assume that $\pi : \mathscr{X} \longrightarrow S$ is a smooth complex analytic family of Calabi–Yau threefolds with $b_3(\mathscr{X}_s) = 2b+2$ over a smooth contractible complex base. Fixing a point $0 \in S$, the fibre X over 0 and a marking of the cohomology $V_{\mathbb{Z}} \cong H^3(X, \mathbb{Z})$, we can define the map

$$\psi : S \longrightarrow \mathscr{D}$$

using the Leray cohomology sheaf $\mathscr{E} = R^3\pi_* \mathbb{C}$ on S, equipped with the Gauss–Manin connection, and the bilinear form $Q : \mathscr{E} \times \mathscr{E} \longrightarrow \mathscr{O}_S$ defined by integrating over the fibres the wedge product of two 3-forms. Griffiths [**4**] proved that the map ψ is holomorphic, and if $\pi : \mathscr{X} \longrightarrow S$ is the Kuranishi family of X, the derivative of ψ is injective, so it is locally an embedding ('infinitesimal Torelli' holds for X).

To discuss global properties of ψ, assume that the base S is quasi-projective, not necessarily contractible, and $\mathscr{X} \longrightarrow S$ is a smooth polarized algebraic family. There is a smooth compactification

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{i} & \bar{\mathscr{X}} \\ \pi \downarrow & & \downarrow \bar{\pi} \\ S & \xrightarrow[j]{} & \bar{S}, \end{array}$$

where i, j are inclusions, $\bar{\pi} : \bar{\mathscr{X}} \longrightarrow \bar{S}$ is a proper map between smooth projective varieties with connected fibres and the boundary divisor $D = \bar{S} \backslash S$ has simple normal crossings. Using the Gauss–Manin connection again, we can define a map

$$\psi : S \longrightarrow \mathscr{D}/\Gamma.$$

This map is in fact well-defined if one quotients $\mathscr{D}$ by the image Γ_0 of the fundamental group $\pi_1(S)$ under the monodromy representation, but we want a map whose range does not depend on S.

Let X be a fixed fibre; for any irreducible component Δ_i of the boundary divisor D, there is a quasi-unipotent transformation

$$T_i : H^3(X,\mathbb{Z}) \longrightarrow H^3(X,\mathbb{Z}),$$

the Picard–Lefschetz transformation. If $D = \bigcup_i \Delta_i$ is the decomposition into irreducible components, we may assume that for $i = 1,\dots,k$, $T_i \in \Gamma$ is of finite order, and for $i \geqslant k+1$ it is of infinite order.

THEOREM 3.1 (Griffiths [**5**]). *The map* ψ *has a holomorphic extension* (*not necessarily locally liftable*)

$$\tilde{\psi} : \bar{S} \setminus \bigcup_{i>k} \Delta_i \longrightarrow \mathscr{D}/\Gamma$$

such that the map $\tilde{\psi}$ *is proper onto its image.*

Proof. In the language of [**5**], the map ψ is holomorphic, locally liftable and horizontal. Hence the statements follow from [**5**, 9.10, 9.11], noting that [**5**, 9.11] remains valid if Γ is not the monodromy group Γ_0, the image of $\pi_1(S)$ under the monodromy representation, but the full arithmetic monodromy group that we use. □

4. *Finiteness results*

Now we can put everything together. Fix the lattice $V_{\mathbb{Z}}$ together with the bilinear form Q, $V = V_{\mathbb{Z}} \otimes \mathbb{C}$ as before. Let $\mathscr{D}/\Gamma$ be the appropriate period domain.

LEMMA 4.1. *For any positive integer* κ*, there is a finite set of polynomials* $p_1,\dots,p_k$ *with the following property: if* (X,L) *is a pair consisting of a Calabi–Yau threefold* X *and an ample* L *on* X *with* $L^3 \leqslant \kappa$*, then there exists* $1 \leqslant i \leqslant k$ *such that the Hilbert polynomial of* (X,L) *equals* p_i.

Proof. By assumption, the leading coefficient of the Hilbert polynomial can only assume finitely many values, and the next coefficient is 0 as $c_1 = 0$. The conclusion now follows from [**14**]. □

Let

$$\mathscr{C}_{b,\kappa} = \{(X,L) \mid X \in \mathscr{C}_b,\ L \text{ an ample invertible sheaf on } X \text{ with } L^3 \leqslant \kappa\}/\cong,$$

where the equivalence relation is now given by polarized isomorphisms, and let ϕ_κ be the restriction of the period map ϕ to $\mathscr{C}_{b,\kappa}$. (The reason for including the ample sheaf here will become clear in Corollary 4.5.)

THEOREM 4.2. *Fix a positive integer* κ *such that the set* $\mathscr{C}_{b,\kappa}$ *is nonempty. The image* $\Phi_\kappa = \phi_\kappa(\mathscr{C}_{b,\kappa})$ *is a locally closed analytic subspace of the complex analytic space* $\mathscr{D}/\Gamma$. *For any point* $x \in \Phi_\kappa$*, there are finitely many* $(X,L) \in \mathscr{C}_{b,\kappa}$ *satisfying* $\phi_\kappa(X,L) = x$.

Proof. Lemma 4.1 gives us polynomials $p_1,\dots,p_k$ as possible Hilbert polynomials. Choose an m such that $L^{\otimes m}$ is very ample and has no higher cohomology for any (X,L) with Hilbert polynomial in the above set, and consider the corresponding Hilbert schemes $\mathrm{Hilb}^{p_i}_{\mathbb{P}^{N_i}}$. Look at the open subsets over which the fibres of the universal families are smooth, and pick those irreducible components which contain Calabi–Yau threefold fibres with $b_3 = 2b+2$. (The Hilbert scheme may contain components where the fibres are Calabi–Yau threefolds with different b_3, but these components are irrelevant for our discussion.) We obtain a finite set of smooth quasi-projective varieties $H_1, \ldots, H_d$ with polarized families $\mathscr{X}_{H_j} \longrightarrow H_j$. A group $\mathrm{SL}(N_j+1, \mathbb{C})$ acts on H_j for every j, and as proved in Section 2, choosing an integer $l \geqslant 3$ and taking finite covers

$$H_j^{(\rho)} \longrightarrow H_j^{(l)} \longrightarrow H_j$$

we obtain a finite number of quotient families $\pi_j : \mathscr{X}_{Z_j} \longrightarrow Z_j$ over smooth quasi-projective bases. By construction, every $(X, L) \in \mathscr{C}_{b,\kappa}$ appears at least once as fibre.

We may assume that each Z_j is embedded in a smooth projective variety $\bar{Z}_j$ as the complement of a normal crossing divisor D_j. Corresponding to the families over Z_j, there are period maps

$$\psi_j : Z_j \longrightarrow \mathscr{D}/\Gamma.$$

As discussed in the previous section, every ψ_j has a proper extension

$$\tilde{\psi}_j : \tilde{Z}_j \longrightarrow \mathscr{D}/\Gamma,$$

where $\tilde{Z}_j = \bar{Z}_j \backslash E_j$, where E_j is a union of some components of D_j. (Notice that all monodromies of $R^3\pi_{j*}\mathbb{C}$ of finite order are trivial; this follows from Serre's lemma and the construction. Therefore the extensions remain locally liftable in this case.)

By the proper mapping theorem, $\tilde{\psi}_j(\tilde{Z}_j)$ is a closed analytic subspace of $\mathscr{D}/\Gamma$. $\psi_j(Z_j)$ is relatively open in this set, so it is locally closed in $\mathscr{D}/\Gamma$. Then

$$\Phi_\kappa = \bigcup_{j=1}^{d} \psi_j(Z_j)$$

so it is also locally closed.

Further, since the action of Γ is discontinuous on $\mathscr{D}$, the maps ψ_j do not have positive dimensional fibres by infinitesimal Torelli, and they have proper extensions $\tilde{\psi}_j$ as above. For $x \in \Phi_\kappa$ the sets $\psi_j^{-1}(x) = \tilde{\psi}_j^{-1}(x) \cap Z_j$ are therefore discrete (perhaps empty), and they have only finitely many components from the properness of $\tilde{\psi}_j$. Hence these sets are finite, which implies the finiteness of $\phi_\kappa^{-1}(x)$. □

We now recall a definition. A projective surface E is called an *elliptic quasi-ruled surface* if there is a map $E \longrightarrow C$ exhibiting E either as a smooth $\mathbb{P}^1$-bundle over the smooth elliptic curve C, or a conic bundle over such a C all of whose fibres are line pairs.

COROLLARY 4.3. *Let X be a smooth Calabi–Yau threefold such that no deformation of X contains an elliptic quasi-ruled surface.* (*This holds, for example, if $b_2(X) = 1$.*) *Then the period point determines the manifolds among complex deformations of X up to finitely many possibilities.*

Proof. Let Y be a (large) deformation of X, then by the main result of Wilson [**26**], any ample class L on X deforms to a class M on Y which is ample. Thus any Y possesses an ample class with self-intersection $\kappa = L^3$ and the result follows. □

The recent result of Voisin [**25**] for quintic threefolds in $\mathbb{P}^4$ is of course much stronger than this, namely in that case the period point determines the generic threefold up to automorphisms ('weak global Torelli' holds). No similar result is known for other classes of Calabi–Yau threefolds.

Using results of [**27**], one can formulate various conditions on X which ensure the existence of ample classes with bounded self-intersection in the presence of elliptic quasi-ruled surfaces as well. This is left to the reader.

We can also deduce a corollary for birationally equivalent threefolds.

COROLLARY 4.4. *For any positive integer* κ*, the number of minimal models* (*up to isomorphism*) *of a smooth Calabi–Yau threefold* X*, which possess an ample sheaf* L *with* $L^3 \leqslant \kappa$*, is finite.*

Proof. By Kawamata [**9**], different minimal (that is, $\mathbb{Q}$-factorial terminal) models of X are related by birational maps called *flops*. According to Kollár [**13**], these different models are all smooth and have isomorphic third cohomology, the isomorphisms respecting Hodge structure and polarization (which comes from Poincaré duality). Hence the statement follows from Theorem 4.2. □

We remark here that the unconditional finiteness of the number of minimal models up to isomorphism has recently been proved by Kawamata [**11**] for relative Calabi–Yau models, that is, fibre spaces $X \longrightarrow S$ with relatively (numerically) trivial canonical sheaf K_X, $\dim X = 3$, $\dim S \geqslant 1$. The absolute case of Calabi–Yau threefolds is however unknown.

Finally we would like to point out a connection to Morrison's cone conjecture [**17**], which arose from string theoretic considerations leading to the phenomenon called mirror symmetry.

COROLLARY 4.5. *Let* X *be a smooth Calabi–Yau threefold, and fix a positive integer* κ*. Up to the action of* $\mathrm{Aut}(X)$*, there are finitely many ample divisor classes* L *on* X *with* $L^3 \leqslant \kappa$*. In particular, if the automorphism group is finite, there are finitely many such classes.*

Proof. By construction, every pair (X, L) with $L^3 \leqslant \kappa$ appears as a fibre of some $\mathscr{X}_{Z_j} \longrightarrow Z_j$. On the other hand, the period point does not depend on the choice of the ample sheaf; hence under the period map, pairs $(X, L^{\otimes m})$ map to the same point of $\mathscr{D}/\Gamma$. By Theorem 4.2, there are finitely many such pairs up to the action of $\mathrm{Aut}(X)$. Considering m-torsion as well, we get finitely many pairs (X, L) up to the action of the automorphism group. □

The statement certainly follows from the cone conjecture, but seems to have been unknown otherwise.

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