

ON THE AMPLE CONE OF AN AMPLE HYPERSURFACE

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For a smooth ample hypersurface D of dimension at least 3 in a smooth projective variety X , the restriction map $r : \text{Pic}(X) \rightarrow \text{Pic}(D)$ is an isomorphism by weak Lefschetz. In a recent paper [3], Hassett, Lin and Wang gave an example of a pair $D \subset X$ with $\dim D = 3$ such that the cone of ample divisor classes of D is strictly larger than the image under r of the cone of ample divisor classes of X ; in other words, the weak Lefschetz principle can fail for ample cones.

The purpose of this note is to point out that not only is the ample cone of a hypersurface D often larger than the ample cone of the ambient space X , in general it cannot even be computed from birational models of X . This contrasts with the conjecture of [2] (already disproved for semi-ample hypersurfaces in [5]), which expected that such a description is possible. Note that in the special case studied in [3], the ample cone is the union of ample cones of birational models of X (Remark 2.3). However, a slight generalization of the example gives hypersurfaces with ample cone larger than the union of ample cones of all birational models of the ambient space (Corollary 3.1).

For a projective variety Y , denote by $\mathcal{N}(Y) \subset \text{Pic}_{\mathbb{R}}(Y)$ the cone of nef divisor classes; this is the closure of the ample cone and also the dual of the Mori cone by Kleiman's criterion. Facets (codimension-one faces) of $\mathcal{N}(Y)$ correspond in favourable cases to primitive (non-factorizable) contraction morphisms on Y . To avoid heavy notation, $j = 3 - i$ for $i = 1, 2$ denotes the other possible index wherever appropriate.

1. THE AMBIENT SPACE

I use the notations of [3]. Let $\varphi : X \rightarrow \mathbb{P}^4$ be the blowup of two distinct points $p_1, p_2 \in \mathbb{P}^4$. The line $p_1 p_2 = l' \subset \mathbb{P}^4$ has proper transform $l \subset X$. The Picard group $\text{Pic}(X) = \langle H, E_1, E_2 \rangle$ is generated by the classes of exceptional divisors E_1, E_2 of φ and the proper transform H of a general hyperplane section of \mathbb{P}^4 through p_1, p_2 . Let \mathcal{B}, \mathcal{C} be the cones in $\text{Pic}_{\mathbb{R}}(X)$ spanned by the rays $H + E_1, H + E_2, H + E_1 + E_2$ and $H, H + E_1, H + E_2$ respectively.

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Proposition 1.1.

- (i) *The nef cone of X is $\mathcal{N}(X) = \mathcal{B}$; the three facets of the cone correspond to the contractions of E_i , and to a small contraction $\psi : X \rightarrow \bar{X}$ contracting the single extremal curve $l \cong \mathbb{P}^1$ with $K_X \cdot l = 1$.*
- (ii) *The antiflip $\psi^+ : X^+ \rightarrow \bar{X}$ of ψ exists. The proper transform E_i^+ of E_i is isomorphic to \mathbb{P}^3 blown up in a point; $E_1^+ \cap E_2^+ \cong \mathbb{P}^2$ is the exceptional divisor of both blowups and also the small exceptional locus of ψ^+ .*
- (iii) *Let $\alpha : X^+ \dashrightarrow X$ be the flip and identify the Picard groups of X and X^+ by the isomorphism $\alpha_* : \text{Pic}(X^+) \rightarrow \text{Pic}(X)$ induced by proper transform of divisors. Then $\mathcal{N}(X^+) = \mathcal{C}$. The three facets of the cone correspond to the contraction ψ^+ and two different \mathbb{P}^1 -bundle structures $\pi_i : X^+ \rightarrow Z$. The image Z is the blowup of \mathbb{P}^3 in one point. The fibres l_i of π_i are generically the lines of the original \mathbb{P}^4 through the point p_i . On E_i^+ , π_i is the identity; on E_j^+ it is the unique projection to \mathbb{P}^2 .*

Proof. Once the statement is formulated, the proof becomes an elementary exercise; my personal favourite is toric geometry, which provides a clean and uniform proof. In the lattice $N = \mathbb{Z}^4$, the fan spanned by

$$v_0 = (-1, -1, -1, -1), \quad v_3 = (1, 0, 0, 0), \dots, \quad v_6 = (0, 0, 0, 1)$$

describes \mathbb{P}^4 . Barycentric subdivision of the cones $v_0v_3v_4v_5$ and $v_0v_3v_4v_6$ by the vertices

$$v_1 = (0, 0, 0, -1), \quad v_2 = (0, 0, -1, 0)$$

leads to X . In this language, E_i are the torus-invariant divisors corresponding to the rays v_i and l is the torus orbit corresponding to the facet $v_0v_3v_4$. A short check using toric Mori theory [4] shows that there are three primitive contractions on X : the contractions of E_i and a contraction ψ with one-dimensional exceptional locus l . The extremal curves define three inequalities on ample classes which gives $\mathcal{N}(X) = \mathcal{B}$.

The fan of X^+ is obtained by replacing the cones $v_0v_1v_3v_4$ and $v_0v_2v_3v_4$ by the cones $v_0v_1v_2v_3$, $v_0v_1v_2v_4$ and $v_1v_2v_3v_4$. The new codimension-two face v_1v_2 represents the small exceptional locus $E_1^+ \cap E_2^+ \cong \mathbb{P}^2$. There are three primitive contractions again; the two \mathbb{P}^1 -bundle structures π_i come from projecting the whole fan along the vector v_i for $i = 1, 2$. The tasks of drawing the pictures and filling in the details are left to the reader. \square

2. THE HYPERSURFACE

Let now $D' \subset \mathbb{P}^4$ be a smooth hypersurface of degree $d > 2$ containing p_1 and p_2 ; assume for simplicity that the line l' is either contained in D' or is at most simply tangent to D' at p_i while meeting it transversally elsewhere. Let $D \subset X$ be the proper transform of D' . A simple computation shows that D is ample in X ; hence by the Lefschetz principle, the Picard groups of X and D can be identified by the restriction isomorphism $r : \text{Pic}(X) \rightarrow \text{Pic}(D)$.

Proposition 2.1.

(i) *If $l' \subset D'$, then*

$$\mathcal{N}(D) = \mathcal{B}.$$

(ii) *If D' does not contain l' but for both $i = 1, 2$, either*

- a. D' contains a line through p_i , or*
- b. the line l' is tangent to D' at p_j , then*

$$\mathcal{N}(D) = \mathcal{B} \cup \mathcal{C}.$$

(iii) *In all other cases, there is a strict inclusion*

$$\mathcal{N}(D) \supset \mathcal{B} \cup \mathcal{C}.$$

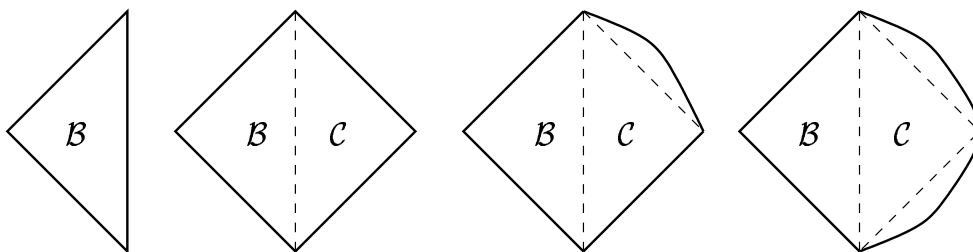


FIGURE 1. Possibilities for the shape of $\mathcal{N}(D)$

Proof. $\mathcal{N}(D) \supset \mathcal{B}$ is clear as a restriction of an ample class from X to D is also ample. Part (i) follows immediately from Proposition 1.1(i), since every face of the nef cone of X corresponds to a contraction with D containing at least one fibre.

To prove (ii), let $\bar{D} \subset \bar{X}$, $D^+ \subset X^+$ be the proper transforms of D in \bar{X} and X^+ . As D meets l in a finite number of points, the map

$\psi|_D : D \rightarrow \bar{D}$ is finite and birational and there is a Cartesian diagram

$$\begin{array}{ccc} & \tilde{D} & \\ & \swarrow & \searrow \\ D & & D^+ \\ & \searrow & \swarrow \\ & \bar{D} & \end{array}$$

By the conditions put on D' , the map $\sigma : \tilde{D} \rightarrow D$ is the blowup of the intersection points of l and D ; the map $\tau : \tilde{D} \rightarrow D^+$ is again finite and birational. Write $\beta : D^+ \dashrightarrow D$ for the composition $\sigma \circ \tau^{-1}$.

Take any ample divisor class A^+ on X^+ . Part (ii) will follow if I can show that the divisor class $r(\alpha_* A^+)$ is ample on D . However, using the fact that τ is finite birational,

$$r(\alpha_* A^+) = \beta_* r^+(A^+) = \sigma_* \tau^* r^+(A^+),$$

where $r^+ : \text{Pic}(Y^+) \rightarrow \text{Pic}(X^+)$ is the restriction map. As A^+ is ample on Y^+ , $r^+(A^+)$ is an ample Cartier divisor class on X^+ and hence $\tau^* r^+(A^+)$ is ample on \tilde{D} . Now the ampleness of $r(\alpha_* A^+)$ on D follows by applying Lemma 2.2 to the map $\sigma : \tilde{D} \rightarrow D$.

To prove (iii), take $A = \pi_i^* B$ where B is ample on Z ; A is a nef but non-ample divisor class in the boundary of the cone $\mathcal{N}(X^+)$. If none of the special circumstances of (i)-(ii) happen, then D^+ does not contain any fibre of π_i , so π_i restricts to D^+ as a finite morphism. Hence the restriction of A to D^+ is ample. The argument of (ii) now shows that $r(\alpha_* A)$ is ample on D and as ampleness is an open condition, (iii) follows. \square

Lemma 2.2. *Let Y be a smooth threefold, $f : \tilde{Y} \rightarrow Y$ the blowup of a closed point. If A is an ample divisor class on \tilde{Y} , then its proper transform $B = f_* A$ is ample on Y .*

Proof. Let E be the exceptional divisor; then $f^* B = A + \lambda E$ with a non-negative integer λ . If C is a curve in Y with proper transform \tilde{C} in \tilde{Y} , then

$$B \cdot C = f^* B \cdot \tilde{C} = A \cdot \tilde{C} + \lambda E \cdot \tilde{C} > 0$$

as $A \cdot \tilde{C} > 0$ and $E \cdot \tilde{C} \geq 0$ since \tilde{C} is not contained in E . If S is any surface in X with proper transform \tilde{S} then

$$B^2 \cdot S = (A + \lambda E)^2 \tilde{S} = A \cdot (A + \lambda E) \cdot \tilde{S} \geq A^2 \cdot \tilde{S} > 0.$$

The second equality holds since $(A + \lambda E) \cdot E \cdot \tilde{S} = f^* B \cdot \tilde{S}|_E = 0$ by the projection formula, as E and \tilde{S} intersect in a curve contracted by f ;

the first inequality uses $A \cdot E \cdot \tilde{S} = A \cdot \tilde{S}|_E \geq 0$. Finally in same way

$$B^3 = (A + \lambda E)^3 \geq (A + \lambda E)^2 \cdot A = (A + \lambda E) \cdot A^2 \geq A^3 > 0.$$

Hence by the Nakai–Moishezon criterion, B is ample. \square

Remark 2.3. Note that in the special case of smooth cubics studied in [3], the first condition of Proposition 2.1(ii) is always satisfied: a smooth cubic threefold is covered by lines. Hence either $l' \subset D'$ and $\mathcal{N}(D) = \mathcal{B}$, or $l' \not\subset D'$ and $\mathcal{N}(D) = \mathcal{B} \cup \mathcal{C}$. [3, Assumptions 2.5] fixes the type of the primitive contractions on D induced by π_i : if D' contains finitely many lines through p_i and l' is not tangent to D' at p_i , then (the Stein factorization of) $\pi_i|_D$ is a small flopping contraction ($K_D \cdot l_i = 0$); otherwise it is divisorial.

3. GOOD HYPERSURFACES

In the original example of [3], as well as in all cases where the conditions of Proposition 2.1(i)-(ii) are satisfied, the ample cone of the hypersurface D has an explicit description in terms of nef cones of models of its ambient space; D is good in the following sense:

Good hypersurface: Call a hypersurface $D \subset X$ *good*, if the nef cone $\mathcal{N}(D)$ is a union of cones $r(\alpha_{i*}\mathcal{N}(X_i))$, where $\alpha_i : X_i \dashrightarrow X$ are certain birational models of X isomorphic to X in codimension one.

Anticanonical hypersurfaces in certain toric varieties were conjectured to be good by Cox and Katz [2, Conjecture 6.2.8]. By [5], this conjecture can fail if D is only semi-ample. Alas, an ample but non-anticanonical $D \subset X$ can also fail to be good.

Corollary 3.1. *Take any smooth degree $d > 3$ hypersurface D' in \mathbb{P}^4 with no lines through p_i and with l' not tangent at p_i and let D be its proper transform in X as before. Then $D \subset X$ is not good.*

Proof. By Proposition 2.1(iii), $\mathcal{N}(D)$ is strictly larger than $\mathcal{B} \cup \mathcal{C}$. On the other hand, X has no further birational models which are isomorphic in codimension one. For if $\alpha_i : X_i \dashrightarrow X$ is any such model, then $\alpha_{i*}A$ is a movable divisor class for A ample on X_i . However, from the explicit description of Proposition 1.1 it is clear that the movable cone of X is the interior of $\mathcal{B} \cup \mathcal{C}$. So X_i is isomorphic to X or X^+ . \square

Note that a 3-dimensional ample and anticanonical hypersurface is always good: if $D \in |-K_X|$ is ample, $r(\mathcal{N}(X)) = \mathcal{N}(D)$ by Kollár's result [1].

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