Reflexive Polytopes and Calabi-Yau Threefolds

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Introduction

Calabi-Yau threefolds occur in string theory as target spaces of $N = 2$ nonlinear σ-models. A natural isomorphism on the level of phycics results in an unexpexted symmetry on the level of algebraic geometry called mirror symmetry; this essay intends to exhibit some aspects of this nice and mysterious phenomenon. The author attempted to present some of the physical background to the extent he understands it; however, the essay is mainly concerned with the mathematical theory and constructions involved.

The first chapter is devoted to the origins of the problem and definitions, as well as a specific example which has been very influential since the time of its publication in 1990-91: the quintic-mirror family, its exposition being based on Morrison's paper [20]. The second chapter is completely independent of the first, describes the basics of toric geometry, which will provide a convenient framework for our later investigations. The third chapter contains the main mathematical results: the toric construction of Batyrev [6] using reflexive polytopes,[1] encompassing all previously known examples of mirror pairs. Finally, the last chapter presents some recent work and speculations on families of mirror pairs, based on papers by Aspinwall, Greene and Morrison [2], [3], [4], [21], [22].

Throughout the essay, some acquaintance will be assumed with standard notions and constructions of algebraic geometry such as blowing-ups, proper transforms, sheaf cohomology, Gorenstein and $\mathbb{Q}$-factorial (or quasi-smooth) varieties, nef divisors, canonical and terminal singularities. References are Hartshorne's classic [15] and the paper by Wilson [27].

1 Mirror Symmetry

1.1 The physical motivation

A certain path of recent research in string theory has lead to quantum field theories called $N = 2$ *nonlinear σ-models.* These models are constructed from spaces of maps from real surfaces Σ to a fixed manifold, the *target space.* Imposing a symmetry condition called *space-time supersymmetry* results in certain geometrical restrictions on the target space; translated into the language of algebraic geometry, one obtains the following definition:

Definition 1.1.1 *A* Calabi-Yau variety X *is an n-dimensional normal complex projective variety with only Gorenstein canonical singularities, which has trivial canonical bundle and satisfies* $H^i(X, \mathcal{O}_X) = 0$ *for* $0 < i < n$.[2]

The physics of the system is determined by a 'Langrangian' functional associated to the spaces of maps mentioned above, which depends on the choice of data (g_{ij}, B), where g_{ij} is a Riemannian metric on X with holonomy (contained in) $SU(n)$,[3] and B is a cohomology class in $H^2(X, \mathbb{R})$. The exact shape of the Lagrangian dictates certain identifications between these pairs; two such

[1] Throughout the essay, we will use the words 'polyhedron' and 'polytope' interchangeably, meaning the convex hull of a finite set of points in Euclidean space.

[2] We do not assume smoothness in our definition, this will be convenient in our later work; we will refer to smooth C-Y varieties as C-Y manifolds. Also, the definition given here is more restrictive than the physicists' one for dimension greater than three, but we will mostly be concerned with threefolds anyway.

[3] This condition on the holonomy ensures that the metric is Ricci-flat [7]. The holonomy is the whole of $SU(n)$ in the 'generic' case, we will assume this for the rest of the essay; see Section 5 of [22] for some comments on the more general case.

pairs are equivalent, $(g_{ij}, B) \sim (g'_{ij}, B')$, if there is a diffeomorphism $\phi : X \to X$ with $\phi^*(g_{ij}) = g'_{ij}$ and if the difference $\phi^*(B) - B'$ is an integral cohomology class. Finally, one usually includes in this description the complex structure t with respect to which g_{ij} is Kähler; the existence of such a complex structure comes from the fact that the holonomy is contained in $U(n)$. So, following Morrison [21], we define the moduli space of $N = 2$ nonlinear σ-models or the $N = 2$ *moduli space* for short as

$$\mathcal{M}_{N=2} = \{(g_{ij}, B, t)\}/\sim .$$

There is no guarantee that all choices will correspond to well-behaved physical theories, but still, properties of this space should reflect the properties of the moduli space of physical theories.

At a more abstract level, one can consider *superconformal field theories* associated to the quantum field theories above; these theories can be treated mathematically as unitary representations of a certain algebra, called the *Virasoro algebra*. In the relevant cases, this algebra can be enlarged by a copy of a $U(1)$-subalgebra; in fact, one gets a unitary representation of *two* commuting copies of this algebra, so an induced representation of $U(1) \times U(1)$, these subalgebras corresponding to moving charges in the physical theory. If we change the signs in the representation of *one* copy of $U(1)$, we get an isomorphic field theory; however, if one follows the definitions backwards, one sees that going back to the corresponding σ-model, one obtains a topologically different Calabi-Yau manifold as the target space. So the notion of mirror symmetry emerged in the following form [13]:

Definition 1.1.2 *Two Calabi-Yau varieties X and Y are said to constitute a* mirror pair*, if they correspond to the same field theory and the association of geometrical objects on the two manifolds to fields in the field theory differs by a reversal of the sign of the left-moving $U(1)$-charges.*

(For the geometrical objects involved, see the Section 1.3.) There is a certain ambiguity involved in this definition. It turns out that as opposed to point particle theories requiring a smooth space-time, the quantum physics may be well-behaved in the presence of certain singularities on the varieties; however, some of the geometrical properties relevant here are properties of some desingularization of the varieties. So we will refer to both the singular and the desingularized varieties as the mirror pair. The resolution is also ambiguous; we will return to this fact later.

The first concrete examples of mirror pairs appeared in the paper [13] by Greene and Plesser, where they used arguments from physics to show the following:

Theorem 1.1.3 *Suppose that X is a Calabi-Yau hypersurface of Fermat type in weighted projective four-space (i.e. its equation in the homogeneous coordinates x_i is given by $\sum x_i^{a_i} = 0$ for suitable integers a_i). As the canonical bundle of X is trivial, there is a unique global three-form on X; let G be the finite group of coordinate-scaling symmetries which preserve this three-form. Then X possesses a mirror $Y = X/G$.*

Here X/G is a singular space, it admits a desingularization to a smooth Calabi-Yau threefold, we can also take that as the dual of X. For a specific example of this construction, see the quintic-mirror family discussed in Sections 1.4 and 3.3.

However, for other more general Calabi-Yau manifolds the dual turned out to be much harder to find. In fact, the manifolds which appear in the above statement and their deformations are the only ones for which the mirrors are known to exist in the sense of Definition 1.1.2. So the strategy is to analyse the geometry of the manifolds and extract some implications of the mirror property which can be taken as a less restrictive definition.

1.2 Background: the mathematical motivation

Having now presented the physical origins of mirror symmetry, the question arises: why are mathematicians interested in all that? The answer lies in the very definition of a Calabi-Yau manifold. After the (relatively easy) classification of complex projective curves, and the (highly nontrivial) birational classification of surfaces, attention was focused around the classification of threefolds. Mori's theory and the use of holomorphic forms provided some answers in most cases; it is precisely the class of Calabi-Yau threefolds, where the triviality of the canonical bundle and non-existence of holomorphic 1- and 2-forms prevents the application of these methods.

On the other hand, mirror symmetry establishes nontrivial connections between individual Calabi-Yau threefolds and families, and it is hoped that it will reveal a deeper structure and may eventually bring closer to a complete classification. The author intended to present some results in this direction as well, for lack of space however they are omitted; the reader is referred to the paper [8] as a typical example.

1.3 Local geometry of the moduli spaces

For a given conformal field theory, there is a special subset of the allowed operators called *truly marginal operators*, which have the property that they can be used to deform the original theory so that the resulting theory is still conformal. There are two ways to deform a given field theory by means of truly marginal operators; one of the two possible ways can be realized on the target space X as deformation of the complex structure. The other way is connected to the Kähler structure of X, we will investigate this 'extra structure' in more detail below. For $N = 2$ superconformal theories these deformations exhaust all the truly marginal operators, and they are independent of each other, so the moduli space of field theories can – at least locally – be thought of as a product of the complex structure and the 'extra structure' moduli spaces. Further, these truly marginal operators are endowed with an additional quantum number, their charge under the $U(1)$ algebra, which can be 1 or -1. Reversing this sign yields an isomorphic field theory, in which the role of the complex structure and the extra structure moduli are exchanged.

We now investigate these two kinds of moduli for an n-dimensional Calabi-Yau manifold. By Kodaira-Spencer theory, the tangent space to complex moduli is $H^1(X, \Theta_X)$, where Θ_X is the holomorphic tangent bundle of X; choosing a nowhere vanishing section of the canonical bundle, we see that this space is isomorphic to $H^1(X, \Omega_X^{n-1}) = H^{n-1,1}(X)$, where Ω_X denotes the cotangent bundle. Although the obstruction space $H^2(X, \Theta_X)$ is in general nonzero, we know from results of Bogomorov, Tian and Todorov (see e.g. [26]) that the complex moduli problem is unobstructed and the complex structure moduli space is smooth of dimension $h^{n-1,1}(X)$; we use the notation $h^{p,q}(X) = \dim H^{p,q}(X) = \dim H^q(X, \Theta^p)$.

The 'extra' structure arises from the Kähler structure on X and the chosen real cohomology class B. A Kähler metric $\sum g_{ij} dz^i \otimes d\bar{z}^j$ gives rise to the real Kähler form $\omega = \sum i g_{ij} dz^i \wedge d\bar{z}^j$, which determines a $(1,1)$-cohomology class J, a real element of $H^{1,1}(X)$ i.e. an element of $H^2(X, \mathbb{R})$. Conversely, Yau's theorem [30] implies that in every cohomology class $J \in H^2(X, \mathbb{R})$ which contains Kähler forms, there is a unique form ω giving rise to a Kähler metric with $SU(n)$-holonomy. By a deformation of the Kähler structure one means a deformation of the metric which cannot be realized by coordinate changes on X; such a deformation corresponds to choosing a new form ω' in a different cohomology class. Not all choices lead to new Kähler metrics however; the form must give positive volume for nontrivial algebraic cycles, so ω' has to satisfy a certain set of inequalities.

Definition 1.3.1 *The open cone of cohomology classes in $H^2(X, \mathbb{R})$ containing Kähler forms is called the* Kähler cone *of X and is denoted by $\mathcal{K}$.*

We remark here, that the closure of this cone is well known from algebraic geometry. Recall that the group of Cartier divisors modulo linear equivalence, $\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$. If X is Calabi-Yau, the exponential exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

gives an isomorphism $H^1(X, \mathcal{O}_X^*) \cong H^2(X, \mathbb{Z})$. So the cohomology group $H^2(X, \mathbb{Z})$ is the group of linearly equivalent Cartier divisors; it is then a standard fact that the closure of $\mathcal{K}$ is the $\mathbb{R}$-span of the numerically effective divisors, the so-called *nef cone*.

The Kähler cone of a Calabi-Yau threefold is investigated in great detail by Wilson [28]. In particular, he shows that for most Calabi-Yau threefolds, including all those occuring in the present essay, it is locally independent of the complex structure, so at least locally the moduli space can indeed be thought of as a product. It fails to be independent only under very special circumstances, which are well understood.

The quantum theory in this case is however more complicated than this mathematical description. First, the Kähler structure deformation is only a first approximation to the marginal operators, which must be modified by adding on 'quantum corrections' which are difficult to compute explicitly. (Incidentally, we note here that the complex stucture deformation turns out to be an exact desription of the corresponding set of operators, no quantum corrections are needed.) Second, conformal field theory gives that we must combine our real form ω with a real 2-form b lying in the cohomology class B into a complex form $b + i\omega$, and further, as we saw above, the conformal field theory is invariant under shifts of B by integral classes, i.e. elements of $H^2(X, \mathbb{Z})$. So it is more natural to consider the coordinates $e^{2\pi i(B_k + iJ_k)}$ where B and J are expressed in terms of a basis of $H^2(X, \mathbb{Z})$. The local deformations of the structure will correspond to elements of $H^{1,1}(X)$.

1.4 Triple products, the weak definition and the quintic-mirror family

We now restrict attention to the case $n = 3$, which is most directly relevant in the physical setup. In this case we have two further ingredients coming from field theory, the so-called 'quantum triple products'. These are approximated to first order by triple products coming from cup product in the cohomology ring. Specifically, the form

$$I_X^{2,1} : H^{2,1}(X) \times H^{2,1}(X) \times H^{2,1}(X) \to \mathbb{C}$$

is the so-called *normalized Yukawa coupling* which arises as follows ([20]): since taking the cup product is compatible with the Hodge decomposition, for every n-dimensional Kähler manifold we have a natural adjoint map

$$H^{0,n}(X) \cong H^{n,0}(X)^*.$$

Further, the cup product also determines a natural map

$$\text{Sym}^n\, H^1(X, \Theta_X) \to \text{Hom}\,(H^{0,n}(X), H^{n,0}(X)).$$

Using the isomorphism between Hom and tensor product and also the adjoint isomorphism, we get a natural map

$$\operatorname{Sym}^n H^1(X,\Theta_X) \to (H^{0,n}(X)^*)^{\otimes 2}.$$

This is called the *unnormalized Yukawa coupling.* If we compose this map with evaluation on an element of $(H^{0,n}(X))^{\otimes 2}$ and also identify $H^1(X,\Theta_X)$ with $H^1(X,\Omega_X^{n-1})$, we get the required product.

The product on $H^{1,1}$ is simply given by the intersection form

$$I_X^{1,1} : H^{1,1}(X) \times H^{1,1}(X) \times H^{1,1}(X) \to \mathbb{C}.$$

The quantum triple product $I_{X,Q}^{2,1}$ turns out to agree with the mathematical approximation, but the intersection product needs to be modified using 'instanton corrections' which arise from the presence of rational curves ('worldsheet instantons') on X and can be expressed in the following form, see [4] (2.6):

$$I_{X,Q}^{1,1}(b^{(1)},b^{(2)},b^{(3)}) = \int_X b^{(1)}\wedge b^{(2)}\wedge b^{(3)} + \sum_{m,\{u\}} \left(\int_{\mathbb{P}^1} u^* b^{(1)} \int_{\mathbb{P}^1} u^* b^{(2)} \int_{\mathbb{P}^1} u^* b^{(3)} \right) \exp\left[2\pi i \int_{\mathbb{P}^1} u_m^* K \right]$$

where the $b^{(i)}$ are $(1,1)$-forms on X, $u : \mathbb{P}^1 \to \Gamma$ is a holomorphic map to a rational curve Γ in X, $\pi_m : \mathbb{P}^1 \to \mathbb{P}^1$ is an m-fold cover, $u_m = u \circ \pi_m$ and $K = B + iJ$ is the complexified Kähler class. This sum as it stands is not quite meaningful; one has to ensure that there are finitely many terms, and that the integrals converge. It is at this point that simplectic topology and more specifically Gromov-Witten invariants enter the scene; for more on this, see [21].

In 1.1.2 we insisted that the change in the sign of the $U(1)$-charges should relate the geometric properties of the manifolds involved. Therefore the following weaker version of the original definition offers itself:

Definition 1.4.1 *Two Calabi-Yau threefolds X and Y, equipped with choices of data (g_{ij}, B, t), are said to be a* weakly mirror pair *if the following two conditions hold:*

(i) the Hodge numbers of X and Y satisfy

$$h^{p,1}(X) = h^{3-p,1}(Y) \quad (p = 1,2),$$

(ii) the spaces $H^{p,1}(X)$ and $H^{3-p,1}(Y)$ are isomorphic under an isomorphism which preserves the quantum triple product.

(i) is a purely topological statement, whereas (ii) depends also on the metric and the complex structure. (ii) certainly implies (i); however, as we see later, in many cases only (i) can be proved.

One of the most spectacular predictions obtained by mirror symmetry methods use these triple products and property (ii) in the following way. From the above expression for $I_{X,Q}^{1,1}$, we obtain an expansion of the form

$$I_{X,Q}^{1,1} = \sum_{m=0}^{\infty} \sigma_m e^{-mR}.$$

Here the parameter R depends on the 'extra structure' of X and the leading coefficent σ_0 is the intersection product. The limit $\operatorname{Re} R \to \infty$ can be interpreted physically (see Section 4.2), it is referred to as the 'large radius limit'. At this limit point, all instanton corrections vanish: the cubic form on $H^{1,1}(X)$ agrees with the topological form.

The paper [9] by Candelas, de la Ossa, Green and Parkes takes a specific example for which the result of Greene and Plesser applies, the so-called quintic-mirror family, which arises as follows: consider the smooth quintic Fermat hypersurface

$$X = \left\{ \sum_{i=0}^{4} x_i^5 = 0 \right\} \subset \mathbb{P}^4$$

together with the family $\mathcal{X}$ obtained by varying the complex structure of X; it is the family of smooth quintic threefolds in $\mathbb{P}^4$. These are Calabi-Yau threefolds, toric methods (see later) allow us to calculate $h^{1,1}(X) = 1, h^{2,1}(X) = 101$, which hold for any member of the family; so we expect the mirror family to have a one-dimensional complex moduli space. Let μ_5 be the multiplicative group of fifth roots of unity, consider the group

$$\tilde{G} = \left\{ (\alpha_0, \ldots, \alpha_4) \in (\mu_5)^5 \mid \prod_{i=0}^{4} \alpha_i = 1 \right\}.$$

This group acts on $\mathbb{P}^4$ by coordinatewise multiplication; there is a subgroup of order 5 whose action is trivial, let G be the quotient. The group G also acts on the threefold

$$U_\beta = \left\{ \sum_{i=0}^{4} x_i^5 - 5\beta \prod_{i=0}^{4} x_i = 0 \right\}$$

so we can consider the quotients U_β/G. These varieties will have canonical singularities which can be resolved using toric geometry again, one obtains smooth threefolds Y_β . For any $\alpha \in \mu_5$ multiplying the first coordinate by α^{-1} gives an isomorphism between $U_{\alpha\beta}$ and U_β, so between the quotients; the resolution of singularities can be performed in such a way that the isomorphism can be extended to $Y_{\alpha\beta} \cong Y_\beta$. So we use the more natural parameter $\gamma = \beta^5$:

Definition 1.4.2 *The* quintic-mirror family *is defined to be*

$$\mathcal{Y} = \{Y_{\sqrt[5]{\gamma}}\} \to \{\gamma\} \cong \mathbb{C}.$$

Proposition 1.4.3 *The dual of X is Y_0; more generally, the dual of (a part of) the moduli space $\mathcal{X}$ of smooth quintic threefolds is the family $\mathcal{Y}$.*

Proof (sketch) It is easy to show that G is the group of coordinate scaling symmetries of X fixing the three-form, so by Theorem 1.1.3 the dual of X is (the desingularization of) X/G. Perturbation arguments then extend the mirror property to other parts of the moduli space.

□

This family has a natural compactification $\bar{\mathcal{Y}} \to \mathbb{P}^1$, the point ∞ corresponding to the large radius limit. Candelas et al now compare the forms $I^{1,1}_{X_s,Q}$ with the forms $I^{2,1}_{Y_\beta,Q}$ on the mirror manifolds Y near the large radius limit, obtaining explicit expressions for the expansion coefficents. Making some plausible assumptions about which quantities should correspond to which other quantities under the mirror correspondence enables them to give predictions for the number n_k of rational curves of degree k on the general quintic threefold. The predictions

$$n_1 = 2875, n_2 = 609250, n_3 = 317206375$$

are in exact agreement with the values obtained by mathematicians using algebraic geometric methods. In fact, their predictions go further than that, the next value being

$$n_4 = 242467530000$$

but no one has calculated these new numbers using exact mathematical methods to check the predictions.

This example indicates the power of the mirror property, so we would like to construct many other mirror pairs or even better, mirror moduli spaces. The ideal state of affairs would be

Conjecture 1.4.4 *For every Calabi-Yau threefold X with a choice of data (g_{ij}, B, t), there exists another such threefold Y, such that they form a (weakly) mirror pair.*

One problem with this conjecture is that from the mathematical point of view, the normalization of the Yukawa coupling needs to be clarified further. Another problem is that there exist Calabi-Yau threefolds X with $h^{2,1}(X) = 0$, whose mirror should satisfy $h^{1,1}(Y) = 0$ which is impossible if Y is Kähler. So instead of proving the conjecture in this generality, one either investigates the moduli spaces of the known pairs to a greater detail, or constructs new applicants for mirror pairs which satisfy at least some of the requirements. In the next two sections we construct many pairs of families with the property that the numbers $h^{1,1}$ and $h^{n-1,1}$ are interchanged, whereas the final section is devoted to the moduli space problem.

2 Toric Geometry

2.1 Generalities

In this section we summarize the basics of toric geometry, mainly in order to fix notation. The material presented here can be found in great detail in [10] or [12].

A toric variety is determined by a certain set Σ of rational polyhedral cones σ having a common vertex at the origin of a real space $N_{\mathbb{R}} = N \otimes \mathbb{R}$, where N is an n-dimensional integral lattice. The set Σ is called a *fan* if the cones σ satisfy the usual compatibility conditions: any face τ of a cone σ is also in Σ and any two cones $\sigma_1, \sigma_2 \in \Sigma$ meet in a common face. We denote by $\Sigma^{(i)}$ the set of i-dimensional cones in Σ, and by $\Sigma^{[i]}$ the set of cones of dimension at most i. Clearly, the latter is also a fan in $N_{\mathbb{R}}$.

Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice of N. For any cone $\sigma \in N_{\mathbb{R}}$ we can consider the dual cone $\check{\sigma} \in M_{\mathbb{R}}$ and the affine algebraic variety $\mathbb{A}_{\sigma,N} = \mathrm{Spec}\,\mathbb{C}[M \cap \check{\sigma}]$, also denoted by $\mathbb{A}_\sigma$ if the lattice is question is unambiguous. If σ is of dimension r and we denote the r-dimensional sublattice containing it by $N(\sigma)$ then this variety can be written as $\mathbb{A}_{\sigma,N} = \mathbb{A}_{\sigma,N(\sigma)} \times (\mathbb{C}^*)^{n-r}$, where $\mathbb{C}^* = \mathrm{Spec}\,\mathbb{C}[x]_x$ is considered as an affine variety. If the cones of Σ cover the whole of $N_{\mathbb{R}}$ then we say the fan Σ is *complete*; a fan is complete if and only if the corresponding variety is complete.

If $\tau = \sigma_1 \cap \sigma_2$ then $\mathbb{A}_\tau$ sits inside $\mathbb{A}_{\sigma_i}$ as an open subvariety, and the conditions for a fan ensure that the $\mathbb{A}_\sigma$ can be glued together along these open subvarieties.

Definition 2.1.1 *The n-dimensional variety obtained by glueing the above affine varieties is called a* toric variety *and is denoted by $\mathbb{X}_{\Sigma,N}$ or simply $\mathbb{X}_\Sigma$. $\mathbb{X}_\Sigma^{[i]}$ denotes the open subvariety corresponding to the fan $\Sigma^{[i]}$.*

Definition-Proposition 2.1.2 *The open subvariety $\mathbb{X}^{[0]} \cong (\mathbb{C}^*)^n$ will be denoted by $\mathbb{T}$; it is an n-dimensional algebraic torus. There is an algebraic group action of $\mathbb{T}$ on itself, which extends to the whole variety $\mathbb{X}_\Sigma$. This action divides the variety $\mathbb{X}_\Sigma$ into orbits $\mathbb{T}_\sigma$, one for each cone $\sigma \in \Sigma$, which are themselves tori $\mathbb{T}_\sigma \cong (\mathbb{C}^*)^{n-\dim\sigma}$ and satisfy*

$$\mathbb{X}_\Sigma^{[i]} = \bigcup_{\dim\sigma\leq i} \mathbb{T}_\sigma.$$

In fact, $\mathbb{X}_\Sigma$ can be thought of as a (partial) compactification of $\mathbb{T}$. The orbits $\mathbb{T}_\sigma$ define a convenient stratification of $\mathbb{X}_\Sigma$ which we will make great use of later.

There is another way to define toric varieties, which yields only a subclass of the varieties defined above; however, it is also of great interest. One starts with an n-dimensional convex integral polyhedron Δ, and considers the variety $\mathbb{P}_{\Delta,M} = \operatorname{Proj}\mathbb{C}[M \cap c\Delta]$ for a sufficently large integer c, also denoted by $\mathbb{P}_\Delta$. This variety, as opposed to $\mathbb{X}_\Sigma$, is therefore automatically projective, comes with an ample invertible sheaf $\mathcal{O}_\Delta(1)$. $\mathbb{P}_\Delta$ also contains a big torus $\mathbb{T}$; if Θ is an arbitrary l-dimensional polyhedral face of Δ then the corresponding orbit of the action will be denoted $\mathbb{T}_\Theta$; $\mathbb{P}_\Theta$ will denote its closure in $\mathbb{P}_\Delta$. It is easy to show that to every polyhedron Δ in M we can associate a fan $\Sigma(\Delta)$ in the dual lattice N such that the variety defined by this fan is isomophic to $\mathbb{P}_\Delta$; moreover, there will be a $1-1$ inclusion-reversing correspondence between polyhedral faces of Δ and cones in Σ. The other direction however is more interesting. Suppose Σ is a complete fan; consider funcions $h : N \to \mathbb{Q}$ that are linear on each cone σ called *support funcions.* It is known that such support functions correspond to $\mathbb{T}$-invariant $\mathbb{Q}$-Cartier divisors D_h on $\mathbb{X}_\Sigma$. If h is *integral,* i.e. $h(N) \subset \mathbb{Z}$, then the corresponding divisor is Cartier. If we further assume that h is upper convex on N, then D_h is generated by sections; finally, if the restrictions $h_\sigma, h_{\sigma'}$ corresponding to any pair of n-dimensional cones are different, then h is called *strictly upper convex* and the divisor is ample. The variety $\mathbb{X}_\Sigma$ is projective precisely when there exists a strictly convex support function h on N; in the case when such a function exists, we have the following result:

Proposition 2.1.3 *Treating the maps h_σ as elements of M, define the convex polyhedron*

$$\Delta = \Delta(\Sigma, h) = \bigcap_{\sigma\in\Sigma^{(n)}} (-h_\sigma + \check{\sigma}).$$

Then $\mathbb{X}_\Sigma \cong \mathbb{P}_\Delta$ and $\mathcal{O}(D_h) \cong \mathcal{O}_\Delta(1)$.

2.2 Morphisms and singularities

In this section we define morphisms in the toric category and recall some statements, which describe arising classes of singularities in toric varieties together with their combinatorial properties. At the end of the section we quote a theorem concerning suitable partial desingularizations of our varieties, which will be very important in the sequel.

Suppose $\tilde{\phi} : N' \to N$ is a homomorphism of lattices, Σ', Σ fans in $N'_\mathbb{Q}, N_\mathbb{Q}$ respectively such that for every cone $\sigma' \in \Sigma'$ we can find a cone $\sigma \in \Sigma$ satisfying $\tilde{\phi}(\sigma') \subset \sigma$. Then there is a morphism of toric varieties

$$\phi : \mathbb{X}_{\Sigma',N'} \to \mathbb{X}_{\Sigma,N},$$

defined on affine pieces in an obvious way.

There are two important particular cases. The first one arises if $\tilde{\phi}$ is injective, $[N : \tilde{\phi}(N')] < \infty$ and $\phi(\Sigma') = \Sigma$, then the morphism ϕ is finite. The other special case is when $\tilde{\phi}$ is an isomorphism and Σ' is a subdivision of Σ, then ϕ is a proper birational morphism, the most important tool to desingularise toric varieties.

We now investigate points of the toric variety $\mathbb{X}_{\Sigma,N}$. A typical affine piece corresponding to an s-dimensional cone σ can be written as

$$\mathbb{A}_{\sigma,n} = \mathbb{A}_{\sigma,N(\sigma)} \times (\mathbb{C}^*)^{n-s}$$

and further, there is a unique $\mathbb{T}/\mathbb{T}_\sigma$-invariant point p_σ on the affine variety $\mathbb{A}_{\sigma,N(\sigma)}$. Any point on the orbit $\mathbb{T}_\sigma$ will have an open analytical neighbourhood isomorphic to a neighbourhood of p_σ, so we concentrate on the properties of the point p_σ.

Proposition 2.2.1 *Suppose that $n_1, \dots, n_r$ are primitive generators of the 1-dimensional faces of σ. Then*

(i) the point $p_\sigma \in \mathbb{A}_{\sigma,N(\sigma)}$ is $\mathbb{Q}$-factorial (quasismooth) if and only if the cone is simplicial, *i.e. $r = s$. It is smooth, if the vectors $n_1, \dots, n_s$ can be extended to a basis of N, in which case we will refer to the simplex spanned by them as* regular.

(ii) The point p_σ is $\mathbb{Q}$-Gorenstein if and only if the elements $n_1, \dots, n_r$ are contained in an affine hyperplane

$$H_\sigma = \{y \in N_\mathbb{Q} \mid \langle k_\sigma, y \rangle = 1\}$$

for some $k_\sigma \in M_\mathbb{Q}$. $\mathbb{A}_{\sigma,N(\sigma)}$ is Gorenstein if and only if $k_\sigma \in M$. (Note that if the k_σ exist, they define the support function h_K corresponding to the anticanonical divisor K on $\mathbb{X}_\Sigma$.)

Now assume that $\mathbb{X}_\Sigma$ is $\mathbb{Q}$-Gorenstein. Then

(iii) p_σ is at most a terminal singularity if and only if

$$N \cap \sigma \cap \{y \in N_\mathbb{Q} \mid \langle k_\sigma, y \rangle \leq 1\} = \{0, n_1, \dots, n_r\}.$$

If $s = r$, we call the simplex spanned by the n_i elementary; *in this case, the point p_σ is at most a Gorenstein $\mathbb{Q}$-factorial terminal singularity.*

(iv) p_σ is at most a canonical singularity if and only if

$$N \cap \sigma \cap \{y \in N_\mathbb{Q} \mid \langle k_\sigma, y \rangle < 1\} = \{0\}.$$

Proof (i) and (ii) are standard facts about toric varieties. (iii) and (iv) can be found in [23], p.294; compare also [24], pp.400-401.

□

The remark about resolving singularities is now clear: it is sufficent to subdivide 'bad' cones into simplicial and then regular ones – this can indeed be done in a finite number of steps, resulting in a smooth toric variety birationally equivalent to the original one.

Proposition 2.2.2 *Suppose $\mathbb{X}_\Sigma$ is a toric variety. Then*

(i) $\mathbb{X}_\Sigma^{[1]}$ *is smooth.*

(ii) If $\mathbb{X}_\Sigma$ *has only terminal singularities,* $\mathbb{X}_\Sigma^{[2]}$ *is smooth.*

(iii) If $\mathbb{X}_\Sigma$ *has only Gorenstein* $\mathbb{Q}$*-factorial terminal singularities,* $\mathbb{X}_\Sigma^{[3]}$ *is smooth.*

Proof (i) follows from definitions. As any 2-dimensional elementary simplex is regular, we have (ii). For (iii), suppose that σ is a 3-dimensional cone spanned by a triple of primitive vectors $\{n_1, n_2, n_3\}$ lying in the hyperplane $\langle k_\sigma, y\rangle = 1$. Then for any lattice point y_0 lying in the parallelepiped spanned by the n_i, we must have $\langle k_\sigma, y_0\rangle = 0, 1, 2$ or 3. So if the simplex is elementary, i.e. it does not contain any extra lattice points, then the parallelepiped does not contain any either, so the simplex is regular.

□

As a consequence of definitions, we also obtain

Corollary 2.2.3 *Any Gorenstein toric singularity is canonical.*

Definition-Proposition 2.2.4 *A proper birational morphism of* $\mathbb{Q}$*-Gorenstein toric varieties*

$$\phi : \mathbb{X}'_\Sigma \to \mathbb{X}_\Sigma$$

is crepant, *i.e. satisfies* $\phi^* K = K'$ *(where* K, K' *are respective canonical divisors), if and only if for every* n*-dimensional cone* $\sigma \in \Sigma$*, all 1-dimensional cones* $\sigma' \in \Sigma'$ *contained in* σ *are generated by primitive integral vectors lying in the hyperplane spanned by the generators of* σ*.*

Proof This follows easily from the description of the support function corresponding to the anticanonical divisor in Proposition 2.2.1, (ii).

□

We now quote the following very important theorem:

Theorem 2.2.5 *Let* $\mathbb{X}_\Sigma$ *be a complete toric Fano variety (the anticanonical divisor is ample; equivalently, the anticanonical support function is integral and strictly upper convex). Then it admits a* maximal projective crepant partial desingularization *(MCPC-desingularization)*

$$\phi : \mathbb{X}_{\Sigma'} \to \mathbb{X}_\Sigma,$$

by which we mean that ϕ *is crepant and* $\mathbb{X}_{\Sigma'}$ *has only* $\mathbb{Q}$*-factorial terminal singularities.*

Proof The proof can be found in [6], pp.502-504. It is based on a combinatorial investigation of the polyhedron $\Delta(\Sigma, h_K)$ and its dual (cf. Proposition 2.1.3). One proves the existence of a *maximal projective triangulation* $\mathcal{T}$ of Δ^* i.e. a triangulation admitting a strictly convex support function, and considers the spanned fan $\Sigma' = \Sigma(\mathcal{T})$.

□

2.3 Hypersurfaces

The principal objects of our study will not be toric varieties themselves, but rather hypersurfaces in them defined by *Laurent polynomials*

$$f(X) = \sum_{m \in M} c_m X^m.$$

The *Newton polyhedron* $\Delta(f)$ of f is defined to be the convex hull of the points $m \in M$ with $c_m \neq 0$ in $M_{\mathbb{Q}}$. The linear space generated by Laurent polynomials with a fixed Newton polyhedron Δ is denoted by $L(\Delta)$; it is not hard to see that this space can be identified with the space of global sections of the invertible sheaf $\mathcal{O}_\Delta(1)$ on $\mathbb{P}_\Delta$. $f \in L(\Delta)$ defines the hypersurface

$$Z_f = Z_{f,\Delta} = \{X \in \mathbb{T} \mid f(X) = 0\},$$

and if we denote its closure in $\mathbb{P}_\Delta$ by $\bar{Z}_{f,\Delta}$ then we get hypersurfaces $Z_{f,\Theta} = \bar{Z}_{f,\Delta} \cap \mathbb{T}_\Theta$ for any face Θ of Δ.

Definition 2.3.1 *A Laurent polynomial $f \in L(\Delta)$ and the corresponding hypersurfaces defined above are said to be Δ-regular, if for any face Θ of Δ, the hypersurface $Z_{f,\Theta}$ is empty or is a smooth subvariety of codimension one in $\mathbb{T}_\Theta$.*

There are several equivalent geometric and homological characterizations of Δ-regularity, see [5], in particular Proposition 4.3 and Theorem 4.8 on pp.362-363. From the above identification of $L(\Delta)$ with sections of an invertible sheaf if follows by Bertini's theorem that the set of Δ-regular hypersurfaces is a Zariski-open subset of $\mathbb{P}(L(\Delta))$.

Similarly one can define the hypersurfaces $Z_{f,\Sigma} = Z_f \subset \mathbb{T}$, $\bar{Z}_{f,\Sigma} \subset \mathbb{X}_\Sigma$, $Z_{f,\sigma} \subset \mathbb{T}_\sigma$ for a fan Σ and cones σ and the definition of Σ-regularity is the same. We define open subvarieties of $\bar{Z}_{f,\Sigma}$ as follows:

$$Z^{[i]}_{f,\Sigma} = \bar{Z}_{f,\Sigma} \cap \mathbb{X}^{[i]}_\Sigma.$$

Theorem 2.3.2 *(i) Suppose σ is an s-dimensional cone in Σ. A point on the affine stratum $Z_{f,\sigma} \in \bar{Z}_{f,\Sigma}$ has a small analytical neighbourhood isomorphic to the product of an $(n-s-1)$-dimensional open ball and an analytical neighbourhood of the point p_σ on the affine toric variety $\mathbb{A}_{\sigma,N(\sigma)}$.*

(ii) The open set $Z^{[1]}_{f,\Sigma}$ consists of smooth points. If $\mathbb{X}_\Sigma$ has only terminal singularities, $Z^{[2]}_{f,\Sigma}$ is smooth. Moreover, if $\mathbb{X}_\Sigma$ has only $\mathbb{Q}$-factorial Gorenstein terminal singularities, then $Z^{[3]}_{f,\Sigma}$ is also smooth.

Proof (i) follows from the standard decomposition

$$\mathbb{A}_{\sigma,N} = \mathbb{A}_{\sigma,N(\sigma)} \times (\mathbb{C}^*)^{n-s},$$

the usual criterion for smoothness and the implicit function theorem. (ii) is a direct consequence of (i) and Proposition 2.2.2.

□

We end this section with a proposition concerning morphisms between hypersurfaces:

Proposition 2.3.3 *(i) Suppose $\bar{Z}_f$ is Σ-regular, $\tilde{\phi} : \Sigma' \to \Sigma$ is a subdivision of the fan Σ. Then the hypersurface $\bar{Z}_{\phi^* f} \subset \mathbb{X}_{\Sigma'}$ is Σ'-regular.*

(ii) Assume $\mathbb{X}_\Sigma$ is projective with Gorenstein singularities, and we have is an MCPC-desingularisation

$$\phi : \mathbb{X}_{\Sigma'} \to \mathbb{X}_\Sigma.$$

Then $\bar{Z}_{\phi^* f}$ is an MCPC-desingularization of $\bar{Z}_f$.

Proof (i) follows from the fact that if $\sigma' \in \Sigma'$ and $\phi(\sigma') \subset \sigma \in \Sigma$ then

$$Z_{\phi^* f, \sigma'} \cong Z_{f,\sigma} \times (\mathbb{C}^*)^{\dim \sigma - \dim \sigma'}.$$

By Proposition 2.2.1 and Theorem 2.3.2.(i), $\bar{Z}_f$ has only Gorenstein singularities and $\bar{Z}_{\phi^* f}$ has at most $\mathbb{Q}$-factorial terminal singularities. So the latter satisfies the conditions for a suitable desingularization and the adjunction formula gives that the restriction $\phi : \bar{Z}_{\phi^* f} \to \bar{Z}_f$ is crepant. □

2.4 Hodge Theory

As we saw in Chapter 1, the statement of the Mirror Symmetry Conjecture involves the equality of certain Hodge numbers of Calabi-Yau varieties. Although for smooth projective varieties these numbers can be defined in terms of sheaf cohomology and also in terms of the usual Hodge decomposition of the complex cohomology groups, during the general calculation one encounters noncomplete (affine) or singular varieties as well, for which one has to generalize Hodge theory, as Hodge decompositions only exist on certain weight filtrations of the cohomology groups. This theory is due to Deligne; we do not have time to introduce it here, a summary can be found in Section 2 of [11]. We only set up notation and quote the statements important for our later calculation. As in the paper referred to above, we shall also use complex cohomology with compact supports, denoted by $H^i_c(.)$; the dimension of the (p,q)-part of the i-th compact cohomology group will be denoted by $h^{p,q}(H^i_c(.))$. These numbers vanish if $p+q > i, p < 0$ or $q < 0$.

Definition-Proposition 2.4.1 *For any algebraic variety X, define the (p,q)-*Euler characteristic *$e^{p,q}_c(X)$ as the alternating sum of the Hodge-Deligne numbers*

$$e^{p,q}(X) = \sum_{i \geq 0} h^{p,q}(H^i_c(X)).$$

This has the following properties:

(i) If $X = X_1 \times X_2$ is a product of varieties then

$$e^{p,q}(X) = \sum_{(p_1,q_1)+(p_2,q_2)=(p,q)} e^{p_1,q_1}(X_1) e^{p_2,q_2}(X_2)$$

(ii) If X is a disjoint union of a finite number of strata (locally closed subvarieties) X_i such that the closure of any stratum is a union of strata then

$$e^{p,q}(X) = \sum_i e^{p,q}(X_i).$$

(iii) If the cohomology groups $H_c^i(X)$ have the pure Hodge structure of weight i then

$$e^{p,q}(X) = (-1)^{p+q} h^{p,q}(X).$$

PROOF (i) is proved in [11] 1.4, (ii) there in 1.8. (iii) simply follows from definitions.

□

We see that if X is smooth and projective then (iii) applies. More generally, if X is only assumed to be compact and quasi-smooth, then the cohomology groups still have the pure Hodge structure of weight i, so (iii) still applies (see [10], Corollary 14.4). The following theorem is another important statement, which shows that the Hodge-Deligne theory is indeed a useful genaralization:

Theorem 2.4.2 *Suppose that X is a quasismooth irreducible variety of dimension n. Then there exists a Poincaré pairing*

$$H_c^i(X) \otimes H^{2n-i}(X) \to H_c^{2n}(X) \cong \mathbb{C}$$

which is compatible with Hodge structures. Here $H_c^{2n}(X)$ is assumed to have Hodge type (n,n).

We now list the important properties of the quantities definied above for Δ-regular $(n-1)$-dimensional affine hypersurfaces $Z_f \in \mathbb{T}$ (the reader is reminded that Δ-regularity in this case simply means that Z_f is a smooth subvariety of codimension 1 in $\mathbb{T}$). For this, we need

Definition 2.4.3 *Suppose that S is a compact subset of $M_{\mathbb{Q}}$. We denote by $l(S), l^*(S)$ the number of lattice points in S and the number of lattice points in the interior $\dot{S}$ respectively.*

Theorem 2.4.4 *The Hodge-Deligne numbers of Z_f satisfy*

(i) $h^{p,q}(H_c^k(Z_f)) = 0$ *for* $k < n-1$,

(ii) $h^{p,q}(H_c^k(Z_f)) = 0$ *for* $p \neq q$ *and* $k > n-1$,

(iii) $h^{p,q}(H_c^{n-1}(Z_f)) = 0$ *for* $p+q > n-1$.

Moreover, if $n = \dim \Delta \geq 4$ then the (p,q)-Euler characteristic satisfies

(iv) $e^{p,0}(Z_f) = (-1)^{n-1} \sum_{\dim \Theta = p+1} l^*(\Theta)$,

(v) $e^{n-2,0}(Z_f) + e^{n-2,1}(Z_f) = (-1)^{n-1}(l^*(2\Delta) - (n+1)l^*(\Delta))$.

PROOF (i) follows from the general fact that for smooth affine varieties $H_c^i(Z) = 0$ if $i < \dim(Z)$. The other statements are proved in [11]: (ii) is a direct consequence of 3.11 and 1.10.e, for (iii) see 1.4.d, (iv) and (v) come from 5.6 and 5.8.

□

3 Batyrev's Construction

3.1 Calabi-Yau threefolds as hypersurfaces in toric varieties

In this section we define certain families of Calabi-Yau threefolds, such that there is a natural involution on the set of all such families, which is conjecured to correspond to mirror symmetry. We first need a couple of combinatorial definitions:

Definition 3.1.1 *Suppose that M is an n-dimensional integral lattice with dual lattice N,*

$$H = \{y \in M_{\mathbb{Q}} \mid \langle y, l \rangle = c\}$$

is a hyperplane in $M_{\mathbb{Q}}$, where $c \in \mathbb{Z}$ and l is a primitive integral vector in N. Then for any point $p \in M$, the value $|c - \langle y, l \rangle|$ is called the integral distance *of p from H.*

Definition 3.1.2 *Suppose that Δ is a convex integral polytope in $M_{\mathbb{Q}}$ containing the origin. Then the pair (Δ, M) is called* reflexive *if the integral distance between 0 and the affine hyperplane generated by any of the $(n-1)$-dimensional faces of Δ equals 1.*[4]

Notice in particular that a reflexive polytope contains exactly one lattice point in its interior. The following simple observation is crucial:

Theorem 3.1.4 *Suppose (Δ, M) is a reflexive pair. Then (Δ^*, N) is also a reflexive pair, where Δ^* denotes the dual polytope of Δ.*

PROOF Let $p_1, \ldots, p_m$ denote the vertices, $\Theta_1, \ldots, \Theta_l$ the $n-1$-dimensional faces of Δ which lie in the hyperplanes $H_1, \ldots, H_l$ respectively. Then by the reflexivity property there exist primitive integral elements $q_1, \ldots, q_l$ of N such that $H_i = \{x \in M_{\mathbb{Q}} \mid \langle x, q_i \rangle = -1\}$. By definition, Δ^* is the convex hull of the q_i, so it is an integral polyhedron. Its $(n-1)$-dimensional faces are given by $\{y \in \Delta^* \mid \langle p_i, y \rangle = -1\}$, so the integral distance between $0 \in N$ and the hyperplane generated by any of these faces is 1.

□

The argument has the following natural extension:

Proposition 3.1.5 *Suppose (Δ, M) is a reflexive pair, Θ is a face of Δ with vertices $p_1, \ldots, p_m$. Define $\Theta^* = \{y \in \Delta^* \mid \langle p_i, y \rangle = -1 \ \forall\ i\}$. Then Θ^* is a face of Δ^* and the operation * gives the 1-1 inclusion-reversing correspondence between faces of Δ and faces of Δ^*.*

The next theorem sets up the promised correspondence between hypersurfaces in toric varieties and Calabi-Yau varieties:

---

[4]We note the following interesting fact:

Theorem 3.1.3 *For any dimension n, up to unimodular transformations of the n-dimensional lattice M there exist only finitely many reflexive pairs (Δ, M). There exist an algorithm to construct all these pairs for any fixed dimension n; for $n = 2$ the number of these pairs is 16.*

PROOF The finiteness of the number of pairs follows from various bounds on the number of lattice points in a reflexive polyhedron developed by means of lattice geometry, see e.g. [16]. The algorithm appears in the preprint [19] where the case $n = 2$ is derived explicitly; for an improved algorithm, see [25]. □

Theorem 3.1.6 *Suppose that Δ is an n-dimensional integral polyhedron in $M_{\mathbb{Q}}$, $\mathcal{F}(\Delta)$ is the family of Δ-regular hypersurfaces in the toric variety $\mathbb{P}_\Delta$. Then the following are equivalent:*

(i) The family $\mathcal{F}(\Delta)$ consists of Calabi-Yau varieties with Gorenstein singularities;

(ii) the ample invertible sheaf $\mathcal{O}_\Delta(1)$ on $\mathbb{P}_\Delta$ is anticanonical, i.e. $\mathbb{P}_\Delta$ is a toric Fano variety with Gorenstein singularities;

(iii) Δ contains an integral point m_0 in its interior and $(\Delta - m_0, M)$ is a reflexive pair.

Proof The equivalence of (ii) and (iii) follows from the description of the anticanonical divisor in 2.2.1.(ii), the isomorphism 2.1.3, and 3.1.4 above.

To connect (i) and (ii)-(iii), we use the adjunction formula. Recall that elements of $L(\Delta)$ can be identified with global sections of the ample invertible sheaf $\mathcal{O}_\Delta(1)$, hence under the usual sheaf-divisor correspondence, $\mathcal{O}_\Delta(1)$ corresponds to any of the divisors $\bar{Z}_f$. So the adjunction formula takes the form

$$K_{\bar{Z}_f} = (K_{\mathbb{P}_\Delta} \otimes \mathcal{O}_\Delta(1))|_{\bar{Z}_f}.$$

If now (i) is satisfied, we obtain (ii). Conversely, assume (ii) and (iii). Then from the formula, all Z_f have trivial canonical bundle. Further,

$$\mathcal{O}_{\bar{Z}_f} \cong \mathcal{O}_{\mathbb{P}_\Delta}/\mathcal{O}_\Delta(1)^{-1}$$

so we have an exact sequence of sheaves

$$0 \to \mathcal{O}_\Delta(1)^{-1} \to \mathcal{O}_{\mathbb{P}_\Delta} \to \mathcal{O}_{\bar{Z}_f} \to 0.$$

On the other hand, from the vanishing of higher cohomologies of sheaves generated by their sections ([10], Corollary 7.3) we obtain

$$H^i(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}) = 0$$

for $0 < i < n$. Using the long exact sequence of cohomology, together with the Kodaira vanishing theorem for varieties with mild singularities, yields

$$H^i(\bar{Z}_f, \mathcal{O}_{\bar{Z}_f}) = 0$$

for $0 < i < n-1$. Finally, the singularities of $\mathbb{P}_\Delta$, and hence those of $\bar{Z}_f$, are Gorenstein, therefore canonical (2.2.3), so all the conditions are satisfied, the varieties $\bar{Z}_f$ are Calabi-Yau.

□

Next we apply the results of Section 2.2 in the case of Calabi-Yau hypersurfaces.

Theorem 3.1.7 *Suppose $\mathbb{P}_\Delta$ is a toric variety coming from a reflexive pair (Δ, M) and $\bar{Z}_f$ is a Δ-regular Calabi-Yau hypersurface. Then there exists an MCPC-desingularization*

$$\phi : \widehat{Z}_f \to \bar{Z}_f.$$

The codimension of singularities of the Calabi-Yau variety $\widehat{Z}_f$ is at least 4. In particular, Calabi-Yau hypersurfaces in toric varieties associated to reflexive pairs of dimension at most 4 have a smooth Calabi-Yau desingularization.

Proof By Theorem 2.2.5 we have an MCPC-desingularization $\phi : \mathbb{X}_{\Sigma(\mathcal{T})} \to \mathbb{P}_\Delta$ using a maximal projective triangulation $\mathcal{T}$ of Δ^*. By Proposition 2.3.3, $\widehat{Z}_f = \phi^{-1}(\bar{Z}_f)$ is an MCPC-desingularization of $\bar{Z}_f$. In particular, the canonical bundle of $\widehat{Z}_f$ is still trivial and $\widehat{Z}_f$ is Calabi-Yau. The remark on singularities now follows from Theorem 2.3.2.(ii).

□

We analyse this morphism in a bit more detail. The trianguation $\mathcal{T}$ gives a subdivision $\Sigma(\mathcal{T}, \Theta)$ of the cone supporting Θ^* for any face of Δ^*, giving rise to a toric morphism

$$\phi_{\Theta^*} : \mathbb{X}_{\Sigma(\mathcal{T},\Theta)} \to \mathbb{A}_{\sigma,N(\sigma)}.$$

Proposition 3.1.8 *Suppose that Θ is an l-dimensional face of Δ, $p \in Z_{f,\Theta}$ is a closed point of $\bar{Z}_f$. Then the fibre $\phi^{-1}(p)$ is isomorphic to the fibre $\phi^{-1}_{\Theta^*}(p_\sigma)$ of the projective toric morphism ϕ_{Θ^*}. The number of irreducible components in this fibre equals $l^*(\Theta^*)$, where Θ^* is the dual face of Θ (cf. Proposition 3.1.5).*

Proof The map $\phi : \mathbb{X}_{\Sigma(\mathcal{T})} \to \mathbb{P}_\Delta$ is a birational toric morphism, the action of $\mathbb{T}$ induces isomorphisms of the fibers corresponding to a $\mathbb{T}$-orbit $\mathbb{T}_\Theta$, so over closed points of $Z_{f,\Theta}$. $\mathbb{T}_\Theta$ is contained in the $\mathbb{T}$-invariant set $\mathbb{A}_{\sigma,n} = \mathbb{A}_{\sigma,N(\sigma)} \times \mathbb{T}_\Theta$; we have the commutative diagram

$$\begin{array}{ccc} \phi^{-1}(\mathbb{A}_{\sigma,N}) & \longrightarrow & \mathbb{A}_{\sigma,N} \\ \downarrow & & \downarrow \\ \mathbb{X}_{\Sigma(\mathcal{T},\Theta)} & \longrightarrow & \mathbb{A}_{\sigma,N(\sigma)}. \end{array}$$

Here the vertical maps are quotient maps with respect to the action of $\mathbb{T}_\Theta$. So the statement about the isomorphism of fibres follows; also the number of irreducible components is the number of irreducible divisors of $\mathbb{X}_{\Sigma(\mathcal{T},\Theta)}$ over p_σ which is just the number of integral points in the interior of Θ^*.

□

Finally, all the work in this section was done in order to be able to formulate

Conjecture 3.1.9 *Let (Δ, M) be a reflexive pair of dimension 4, let $\mathcal{Z}$ denote the family of Δ-regular hypersurfaces in $\mathbb{P}_\Delta$ and $\widehat{\mathcal{Z}}$ the desingularized family. Let (Δ^*, N) be the dual polytope, $\mathcal{Z}'$ the family of Δ^*-regular hypersurfaces, $\widehat{\mathcal{Z}}'$ the desingularized family. Then the two families $\widehat{\mathcal{Z}}$ and $\widehat{\mathcal{Z}}'$ are weakly mirror families of smooth Calabi-Yau threefolds.*

This conjecture presents several problems again. Most importantly, this is a statement about families and does not give a 1-1 correspondence between manifolds, which was required by the original definition. Secondly, whereas a local isomorphism of the moduli spaces will be made plausible by the calculation of the Hodge numbers in the next chapter, the global isomorphism of the moduli spaces is an extremely delicate question - we will return to this later. Finally, there is no proof as yet of the isomorphism of cohomology groups that respects triple products, except in very special cases.

3.2 Calculation of the Hodge numbers

We first apply the results of Danilov and Khovanskii, quoted in Section 2.4, to calculate the Hodge number $h^{n-2,1}(\widehat{Z}_f)$ for a hypersurface in $\mathbb{P}_\Delta$ with Δ reflexive, $\dim \Delta = n \geq 4$.

We note that $\widehat{Z}_f$ is compact and quasi-smooth, so the cohomology groups have pure Hodge structure of the appropriate weight. By 2.4.1.(iii) we have

$$h^{n-2,1}(Z_f) = (-1)^{n-1} e^{n-2,1}(Z_f),$$

hence it suffices to calculate this Euler characteristic. To this end, let us define a stratification of $\widehat{Z}_f$. We have a birational morphism $\phi : \widehat{Z}_f \to \bar{Z}_f$, so we can write

$$\widehat{Z}_f = \bigcup_{\Theta \subset \Delta} \phi^{-1}(Z_{f,\Theta}).$$

By Proposition 3.1.8, fibers of ϕ over closed points of $Z_{f,\Theta}$ can be stratified by smooth affine varieties, which are isomorphic to products $Z_{f,\Theta} \times (\mathbb{C}^*)^k$ for nonnegative integers k. Hence we obtain a stratification of $\widehat{Z}_f$ by smooth affine varieties of the same form, for faces Θ of Δ.

Now we use Theorem 2.4.1.(i) and (ii). Noting that the (p,q)-Euler characteristic of tori is nonzero only if $p = q$ ([11] 1.10.e), for a single stratum we obtain

$$e^{n-1,1}(Z_{f,\Theta} \times (\mathbb{C}^*)^k) = e^{n-3,0}(Z_{f,\Theta}) \cdot e^{1,1}((\mathbb{C}^*)^k) + e^{n-2,1}(Z_{f,\Theta}) \cdot e^{0,0}((\mathbb{C}^*)^k).$$

A simple calculation based on Theorem 2.4.4 (i)-(iii) confirms, that $\operatorname{codim} \Theta > 0$ implies the vanishing of $e^{n-2,1}(Z_{f,\Theta})$, if $\operatorname{codim} \Theta > 2$ then also $e^{n-3,0}(Z_{f,\Theta}) = 0$, so we we are left with the following cases:

- $\operatorname{codim} \Theta = 0$, then $k = 0$, $\Theta = \Delta$, and we can apply the formulae from Theorem 2.4.4.(iv)-(v).
- $\operatorname{codim} \Theta = 1$, then $k = 0$ still (as we are in a stratum of positive codimension), so the contribution is zero.
- $\operatorname{codim} \Theta = 2$, then to get nonzero contribution we need $k = 1$. It is clear that $e^{1,1}(\mathbb{C}^*) = 1$ and by Theorem 2.4.4.(iv) we have $e^{n-3,0}(Z_{f,\Theta}) = (-1)^{n-3} l^*(\Theta)$, so we only need to find the number of such strata. For a fixed Θ, strata isomorphic to $Z_{f,\Theta} \times \mathbb{C}^*$ come from fibers of ϕ over the locally closed subvariety $Z_{f,\Theta} \subset \bar{Z}_f$ of codimension two. By Proposition 3.1.8, we know that $\phi^{-1}(Z_{f,\Theta})$ consists of $l^*(\Theta^*)$ irreducible components isomorphic to $Z_{f,\Theta} \times \mathbb{P}^1$, so the number of strata isomorphic to $Z_{f,\Theta} \times \mathbb{C}^*$ is also $l^*(\Theta^*)$.

Putting all this together, and using the reflexivity of Δ i.e. $l^*(\Delta) = 1, l^*(2\Delta) = l(\Delta)$ we obtain

Theorem 3.2.1 *For $n \geq 4$,*

$$h^{n-2,1}(\widehat{Z}_f) = l(\Delta) - n - 1 - \sum_{\operatorname{codim} \Theta = 1} l^*(\Theta) + \sum_{\operatorname{codim} \Theta = 2} l^*(\Theta) \cdot l^*(\Theta^*),$$

where for a face Θ of Δ, Θ^* is the dual face of the dual polytope Δ^*.

The formula for the other Hodge number of interest is provided by

Theorem 3.2.2 *Under the same conditions as before, we have*

$$h^{1,1}(\widehat{Z}_f) = l(\Delta^*) - n - 1 - \sum_{\mathrm{codim}\ \Theta=1} l^*(\Theta^*) + \sum_{\mathrm{codim}\ \Theta=2} l^*(\Theta^*) \cdot l^*(\Theta).$$

Proof As $\widehat{Z}_f$ is quasi-smooth, we can use Poincaré duality, Theorem 2.4.2 to see that $h^{1,1}(\widehat{Z}_f) = h^{n-2,n-2}(\widehat{Z}_f)$. This number can then be easily calculated using long exact sequences, comparing Hodge types and invoking formulae for Hodge numbers of tori.

□

If we fix $n = 4$, comparing the two expressions obtained we see that desigularizations of three-dimensional hypersurfaces in toric varieties corresponding to a dual pair of reflexive polytopes satisfy (i) of Definition 1.4.1, so we obtain a partial justification of Conjecture 3.1.9.

3.3 An explicit example

We now show how the construction enables us to reproduce the quintic-mirror family defined before. Take the n-dimensional polyhedron Δ_n defined by the inequalities

$$x_1 + \cdots + x_n \leq 1, \quad x_i \geq -1 \quad (1 \leq i \leq n).$$

This is a reflexive polytope, the corresponding family is the family of all degree $(n+1)$ hypersurfaces in $\mathbb{P}^n = \mathbb{P}_{\Delta_n}$. The dual polyhedron Δ_n^* has the following vertices:

$$u_1 = (1, 0, \ldots, 0), \ldots, u_n = (0, \ldots, 0, 1),\ u_{n+1} = (-1, \ldots, -1).$$

The variety obtained is $\mathbb{P}_{\Delta_n^*} = \mathbf{H}_n$, the singular hypersurface

$$\prod_{i=1}^{n+1} w_i = w_0^{n+1}$$

in $\mathbb{P}^{n+1}$ with coordinates w_i. The reflexive pairs (N, Δ_n^*), (M, Δ_n) are related via the finite morphism of lattices

$$\tilde{\phi} : (N, \Delta_n^*) \to (M, \Delta_n)$$

where $\tilde{\phi}(u_{n+1}) = (-1, \ldots, -1)$ and $\tilde{\phi}(u_i) = (-1, \ldots, n, \ldots, -1)$ for $1 \leq i \leq n$. It is easy to see that

$$M/\tilde{\phi}(N) \cong (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}.$$

A Calabi-Yau hypersurface $\bar{Z}_f$ in $\mathbf{H}_n$ has equation

$$\sum_{i=0}^{n+1} a_i w_i = 0.$$

If we apply the result of the previous section, we see that $h^{n-1,1} = 1$, so we expect a one-parameter family here. Indeed, $\mathbf{H}_n$ is invariant under the action of the n-dimensional torus

$$\mathbb{T} = \left\{ (t_1, \ldots, t_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid t_1 \cdots t_{n+1} = 1 \right\},$$

and also we can multiply the equation of the hypersurface by a nonzero complex number, so up to isomorphism all hypersurfaces have an equation of the form

$$\sum_{i=1}^{n+1} w_i + a w_o = 0,$$

where the number a is well-defined up to an $(n+1)$-th root of unity. Finally, using the mapping $\phi : \mathbb{P}^n \to \mathbf{H}_n$ induced by $\tilde{\phi}$ and using coordinates v_i on $\mathbb{P}^n$, we can rewrite this equation and obtain

Theorem 3.3.1 *The family given by the reflexive polytope construction as the mirror of the family of degree $(n+1)$ hypersurfaces in $\mathbb{P}_n$ is the one-parameter family of quotients of the hypersurfaces in $\mathbb{P}^n$ given by the equation*

$$\sum_{i=0}^{n} v_i^{n+1} + a \prod_{i=0}^{n} v_i = 0$$

by the group $(\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$, with a^{n+1} being a canonical parameter for this family.

If we put $n = 4$, we arrive at the quintic-mirror family indeed. We also remark here, that similar methods enable one to obtain all the examples covered by the Greene-Plesser construction, see Corollary 5.5.6 in [6].

4 The Global Picture

4.1 Flops

In this section we discuss an interesting geometrical construction, which arises naturally in the present context.

Definition 4.1.1 *Suppose we have a diagram of varieties and maps*

$$\begin{array}{ccc} X_0 & \stackrel{\phi}{\dashrightarrow} & X_1 \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

with the following properties:

(i) the X_i are normal projective varieties with nef $\mathbb{Q}$-Cartier canonical divisors K_{X_i};

(ii) the birational map $\phi : X_0 \dashrightarrow X_1$ is an isomorphism except over subsets of codimension at least 2;

(iii) $\psi_i : X_i \to Y$ are contraction morphisms with exceptional sets E_i of codimension at least 2;

(iv) the restriction of K_{X_i} to the fibres of ψ_i is numerically trivial.

Then ϕ is called a flop *or ψ_1 is called a* flop *of ψ_0.*

Flops occur in the classification theory of higher dimensional varieties, in particular they connect different minimal models of threefolds, because of the following (see [18], Theorem 4.9):

Theorem 4.1.2 *Let X_0 and X_1 be projective threefolds with $\mathbb{Q}$-factorial terminal singularities. Assume also that both K_{X_i} are nef. Then any birational map $g : X_0 \dashrightarrow X_1$ can be written as a composite of flops.*

An important property of flops (or their composites) is the following ([18], Theorem 4.6):

Theorem 4.1.3 *Under the conditions of the previous theorem, the exceptional loci (where g or g^{-1} is not regular) are unions of rational curves.*

Hence intuitively one can think of a flop as a blow-down of a rational curve on a threefold X_0 followed by a blow-up to obtain another threefold X_1. These threefolds will have the same analytical singularities.[5]

4.2 Moduli Spaces and Kawamata's Movable Cone

We now want to investigate some global properties of the moduli space defined in Section 1.1. As we saw before, a quantum field theory on a fixed Calabi-Yau threefold X is determined by the triple (g_{ij}, B, t) where g_{ij} is a metric whose holonomy is (contained in) $SU(n)$, t is a complex structure with respect to which g_{ij} is Kähler and $B \in H^2(X, \mathbb{R})$; we arrived at the moduli space $\mathcal{M}_{N=2}$. There is a related moduli space, the moduli space $\mathcal{M}_{complex}$ of complex structures modulo diffeomorphisms on X; we have a natural forgetful map

$$\mathcal{M}_{N=2} \to \mathcal{M}_{complex}.$$

If we write X_t for X with the complex structure t and $\mathcal{K}_t$ for its Kähler cone, the fibres of this map can be described as follows: giving a quantum field theory for a manifold with a fixed complex structure t amounts to choosing a 'complexified Kähler class', an element $B+iJ \in H^2(X, \mathbb{R})+i\mathcal{K}_t$. The resulting theory is isomorphic for choices differing by an integral cohomology class, and further, under the action of the complex automorphism group of X_t. Hence the fibres take the form $\mathcal{D}_t/\Gamma_t$ where $\mathcal{D}_t = H^2(X, \mathbb{R}) + i\mathcal{K}_t$ and $\Gamma_t = H^2(X, \mathbb{Z}) \times \mathrm{Aut}(X_t)$.

From now on, let us restrict ourselves to threefolds again. As we remarked already, by results of Wilson [28], except for very special cases the moduli space decomposes locally as a product of an open set in $\mathcal{M}_{complex}$ and the fibre, which will be invariant in this open set. The mirror symmetry correspondence is then expected to exchange the roles of these two sets: if Y_s is the mirror of X_t then the open set in $\mathcal{M}_{complex,X}$ should be isomorphic to some open set in the fibre space of Y, and so moving around in the moduli spaces should establish some kind of global isomorphism. This however presents us with the following puzzle: the fibre space is obtained by taking a quotient by some group action of a complexified open cone in a vector space, so it is some partly bounded domain - this can be seen even better in the exponentiated coordinates introduced at the end of Section 1.3: in that coordinate system, the fibre becomes a bounded domain. The complex moduli space however is not bounded, typically it is an open subset (corresponding to nonsingular deformations) of some projective variety. The puzzle is simply: how can these spaces be isomorphic?

One resolution of the puzzle might be that mirror symmetry only holds in a certain bounded part of the complex moduli space, thereby restoring the symmetric structure of the spaces. This

[5] In fact, this picture is slightly misleading: the result of the first blowdown may well be a nonalgebraic variety, and we may need to contract more rational curves to restore algebraicity. This will not cause any problems for us.

however contradicts both the naturality of the constructions and experimental evidence, which shows that in certain cases there does not seem to be any bounded domain singled out by mirror symmetry. A new approach is needed, which emerged in the works of Aspinwall, Greene and Morrison [3], [4] and later, in a more mathematical form, in Morrison's paper [21], which we now describe.

To see the intuitive idea, let us consider once again the Kähler cone of a fixed Calabi-Yau manifold X. It consists of cohomology classes containing real 2-forms J satisfying the inequalities

$$\int_{X'} J^r > 0$$

for all r-dimensional subvarieties X'. In particular, this must hold for all rational curves (if any) $C \subset X$. Now consider a process in which we 'shrink down' a chosen rational curve C to 'zero volume', i.e. we choose new cohomology classes so that in the limiting case the above inequality becomes an equality. We obtain a singular variety Z, on which the image P of C is a single point. Now we can try and blow up this singularity in a different way, so that we may be able to continue our path in the cohomology space; the image of P will be a different rational curve C'. In short, we performed a flop - and in this way, it might be possible to attach Kähler cones of topologically related but distinct manifolds together and restore symmetry with the complex moduli space.

The intuitive picture presented here can indeed be made rigorous, one has the following result (see [28], Fact 1):

Theorem 4.2.1 *Define the cubic cone W^* of X as*

$$W^* = \left\{ D \in H^2(X, \mathbb{R}) \mid D^3 = 0 \right\}.$$

Then the closed Kähler cone $\bar{\mathcal{K}}$ is locally rational polyhedral away from W^*, the codimension one faces correponding to primitive birational contractions of X (i.e contractions that cannot be factored further into birational maps).

There are three kinds of birational contractions to be considered:

- Type I. The birational map contracts a finite number of rational curves.
- Type II. The map contracts a surface down to a point.
- Type III. The map contracts a surface down to a curve.

We have only described the Type I case; however, the others occur in examples as well. The flopping case is the one which is also investigated by Morrison in great detail in [21]; he shows that the Kähler cones can indeed be attached along a flopping wall, one does have a continuous path in the cohomology space across this wall and moreover, the corresponding triple products are also the same on either side of the wall. This is a very important result, since as we saw before, mirror symmetry is also expected to relate triple products of the pair of manifolds. The path across the wall is very interesting from the physical point of view: it connects theories with topologically different target spaces by passing through a singular theory. In the paper [4] it is argued that the physics may be perfectly well-behaved in this situation.

Let us now try to attach several Kähler cones together. Here we recall definitions and results of Kawamata [17]:

Definition 4.2.2 *Let X be a Calabi-Yau variety. We define the (closed)* movable cone *of X, $\mathrm{Mov}(X) \subset H^2(X,\mathbb{R})$ to be the closed cone generated by divisors whose associated linear system has base locus of codimension at least 2.*

This cone certainly contains the nef cone, the closure of the cone of ample divisor classes, so the Kähler cone as well. Further, results in Kawamata ([17], Section 5) imply the following: suppose that X is a $\mathbb{Q}$-factorial Calabi-Yau threefold with terminal singularities, $\{\phi\}$ is the set of birational maps $\phi : X \dashrightarrow X'$ to other such varieties. We know (4.1.3), that the exceptional loci consist of rational curves, in particular all such maps are isomorphisms in codimension one. If we map the Kähler cones of all the X' to $H^2(X,\mathbb{R})$ via the proper transform maps, we get a sectional decomposition of the movable cone $\mathrm{Mov}(X)$, where neighbouring sections correspond to varieties related by primitive contractions followed by blowups. So we managed to attach cones belonging to different manifolds together, thereby enlarging the moduli space. This also explains a phenomenon seen earlier: different desingularizations of the Greene-Plesser construction occupy different regions in this enlarged space, none of them is preferred to the others.

We mention briefly that this construction has an analogue in the toric setup as well, see [4] and [2]. One again considers hypersurfaces of toric varieties $\mathbb{P}_\Delta$, and restricts attention to those monomial deformations and Kähler classes of the hypersurface which come from the large toric variety. One obtains subspaces $H^{1,1}(X)_{toric} \subset H^{1,1}(X)$, $H^{n-1,1}(X)_{toric} \subset H^{n-1,1}(X)$ and if Y is the conjectured mirror of X as constructed from the dual polytope, under certain conditions one has natural isomorphisms

$$\begin{aligned} H^{1,1}(X)_{toric} &\cong H^{n-1,1}(Y)_{toric}, \\ H^{n-1,1}(X)_{toric} &\cong H^{1,1}(Y)_{toric}, \end{aligned}$$

called the *monomial-divisor mirror map* in [2]. It turns out that the toric Kähler cones are also polyhedral and the attached cones give a new (noncomplete) fan. This fan is contained in the so-called *secondary fan*, which is related to the combinatorics of Δ. The 'large structure limit point' can also interpreted in this way: it is some kind of a limit of those theories whose associated points in the moduli space are 'far' from the boundaries of the Kähler cone; there is one such limit point for every cone in the fan. We do not have time to go into details, they can be found in the sources cited above.

The movable cone construction presents us with a very natural picture – however, it still fails to give a satisfactory answer to our puzzle, for we only replaced our Kähler cone with another cone; in the toric setup, we only obtain a subfan of the secondary fan. This means that we should perhaps attach meaning to other parts of the space $H^2(X,\mathbb{R})$ as well; similarly, most of the cones in the secondary fan still require explanation. This is an area of active study – it appears that from the physical point of view, one leaves the realm of σ-models and obtains other physical theories such as Landau-Ginzburg theories[6]; from the mathematical point of view, one obtains other kinds of orbifolds (quotients of smooth varieties by finite groups) and reducible varieties as well. It is clear however, that in this way one also gets far beyond the author's knowledge of the subject, so the essay ends here.

[6]This phenomenon also occurs, although from a different approach, in the work of Witten [29].

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