Contractions and monodromy in homological mirror symmetry

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ABSTRACT

This paper discusses the mirror correspondence between contractions and degenerations of Calabi–Yau varieties, originally due to Morrison, in the light of recent developments. In homological mirror symmetry, degenerations lead to symplectomorphisms, whereas contractions give rise to Fourier–Mukai functors. Several explicit examples are treated, many of them conjectural.

Introduction

Kontsevich' Homological Mirror Symmetry conjecture [14] connects holomorphic and symplectic geometry in a deep and surprising way. It relates a pair of Calabi–Yau varieties, one with fixed holomorphic (complex) structure, one with fixed symplectic structure, and predicts an equivalence of two very different kinds of categories. On the symplectic side, "the (derived) Fukaya category" went through many transfigurations over the years, and there is still no unique definition. On the holomorphic side, the category is well known: it is the derived category of coherent sheaves. The conjectured equivalence between these two categories implies that there should be a correspondence between their symmetries. The obvious symmetries are symplectic, respectively holomorphic automorphisms; however, typically there are many more of the former than the latter. The idea of the Fourier–Mukai functor [19] comes to the rescue: the derived category of a Calabi–Yau variety possesses more symmetries than just the obvious ones, by virtue of the very fact that its canonical bundle is globally trivial.

The purpose of this note is to review the dictionary between symplectic isomorphisms and Fourier–Mukai functors in a pedagogical way. Versions of a global correspondence (at least on the cohomological level) are discussed for example in [3, 10, 30]; a comprehensive account will presumably be attempted in [7]. My aim here is rather more limited, summarizing and slightly extending the range of constructions that are local to singularities of Calabi–Yau varieties. I will discuss surface double points, threefold nodes, isolated threefold singularities and finally curves of singularities on threefolds, proving no new results but posing several open problems. The presentation is at the level of suggestive analogies, relying on the shape of cohomology actions, relations, toric examples and the like.

The real question this note completely disregards is of course what exactly *is* mirror symmetry for singularities. Recent ideas of Kontsevich, Kapustin–Li [13] and Orlov [20] on the one hand, related on the physics side to D-branes in Landau–Ginzburg models, and Seidel [24, 25] on categories defined by vanishing cycles on the other, will provide the tools to ask and eventually answer this question in much more precise detail than attempted here.

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1. Homological mirror symmetry in a nutshell

1.1. I shall make to attempt to give a general overview of mirror symmetry. There are several good sources available; most relevant for this note is Kontsevich' 1994 ICM address [14] and its 2000 ECM update by Manin [16], both with extensive bibliography. As originally formulated in string theory, mirror symmetry relates two Calabi–Yau manifolds (X, ω_X) and (Y, ω_Y), both equipped with complex structures and compatible symplectic (Kähler) forms, and certain structures defined on (families of) them. In fact, at least deep inside the moduli space, mirror symmetry "decouples" the symplectic and the complex structure, and interchanges one with the other. Kontsevich conjectured that eventually all mirror symmetry constructions should be understood as an equivalence of categories depending on symplectic, respectively holomorphic data:

$$m\colon D^b\mathrm{Fuk}(Y,\omega) \longrightarrow D^b(X).$$

Here $D^b\mathrm{Fuk}(Y,\omega)$ is the "derived Fukaya category". It should be constructed purely and functorially in symplectic terms, using Lagrangian submanifolds of (Y,ω) (now simply thought of as a symplectic manifold, with no complex structure), and their Floer homology. The analytic details are highly non-trivial, and in fact to this date not completely settled. On the right hand side, $D^b(X)$ is the bounded derived category of coherent sheaves on the smooth Calabi–Yau variety X. This is a perfectly well-defined and relatively well-known (triangulated) category; see [9] for a detailed introduction.

Irrespective of the details of its definition, functoriality of the Fukaya category should imply that it carries an action of the group of symplectomorphisms $\mathrm{Symp}(Y,\omega)$. Symplectomorphisms symplectically homotopic to the identity (in the C^∞-topology) are expected to act trivially. The mirror symmetry equivalence m should then give rise to a map

$$\mu\colon \pi_0(\mathrm{Symp}(Y,\omega)) \to \mathrm{AutEq}(D^b(X))/\langle[1]\rangle, \tag{1}$$

which can be studied independently of the intricacies of the Fukaya category. The group of self-equivalences $\mathrm{AutEq}(D^b(X))$ of the derived category (together with its triangulated structure) contains [1], the translation functor, which is expected to correspond to the translation functor in the derived Fukaya category. This corresponds to the difference between "ordinary" and "graded" symplectic geometry, as explained in [26, Introduction].

1.2. The previous discussion implies that we need a handy supply of self-equivalences of the derived category. As I already discussed in the introduction, there are two obvious sources: automorphisms of the complex manifold X, as well as line bundles acting by tensor product. However, one soon realizes that this is not enough. For example, a generic quintic has no automorphisms other than the identity, and only a $\mathbb{Z}$ worth of line bundles. Its mirror in turn has many more symplectomorphisms.

A fundamental constrcution of Mukai [19] comes to the rescue. By analogy with the theory of classical correspondences, an object $\mathcal{F} \in D^b(X \times X)$ defines a functor

$$\Phi^{\mathcal{F}}\colon D^b(X) \to D^b(X)$$

by

$$\Phi^{\mathcal{F}}(-) = \mathbf{R}p_{1*}(\mathcal{F} \overset{\mathbf{L}}{\otimes} p_2^*(-)),$$

where $p_i\colon X \times X \to X$ are the two projections, and $\overset{\mathbf{L}}{\otimes}$, $\mathbf{R}p_{1*}$ denote operations in the derived category. It often happens that, under appropriate conditions, $\Phi^{\mathcal{F}}$ is a self-equivalence of the triangulated category $D^b(X)$, and then it is called a *Fourier–Mukai functor* [19].

As a special case, this definition includes the example of a self-equivalence defined by an automorphism; here $\mathcal{F}$ is (the structure sheaf of) the graph of the automorphism. The self-equivalence defined by tensoring with a line bundle $\mathcal{L}$ is realized by $\mathcal{F} = \mathcal{O}_\Delta(\mathcal{L})$ where $\Delta \in X \times X$ is the diagonal. However, the main virtue of this framework is that in the Calabi–Yau context, there are many more Fourier–Mukai functors. Several examples will be given below.

1.3. The mirror symmetry map m is expected to be compatible with a cohomology isomorphism

$$\bar{m}\colon H^{\mathrm{middle}}(Y,\mathbb{Q}) \longrightarrow H^{\mathrm{even}}(X,\mathbb{Q})$$

under, on the syplectic side, the map taking a Lagrangian submanifold to its cohomology class, and on the complex side, the map taking a sheaf or complex to its K-theory class and then via the Chern character to cohomology. For technical reasons, one uses a slight modification of the Chern character, the Mukai map $\mathrm{ch}(-)\sqrt{\mathrm{Td_X}}$, to pass from K-theory to cohomology, which makes the map compatible with natural bilinear pairings. This plays no role in this note.

In fact, here I would like to invoke the "symmetry" of mirror symmetry, which says that actually the complex geometry of Y is also expected to be equivalent to the symplectic geometry of (X, ω_X) in the sense of a categorical equivalence. For the examples studied here (K3s, Calabi–Yau threefolds, but also for elliptic curves) this means that one actually expects a full isomorphism

$$\bar{m}\colon H^*(Y, \mathbb{Q}) \longrightarrow H^*(X, \mathbb{Q}) \tag{2}$$

between $\mathbb{Q}$-vector spaces.

A symplectomorphism of (Y, ω) acts on homology by pushforward. In the context of cohomology actions, Y will always be a compact manifold; hence I can use Poincaré duality to think of cohomology as covariant for symplectomorphisms. A Fourier–Mukai functor $\Phi^{\mathcal{F}}\colon D^b(X) \to D^b(X)$ acts on the cohomology of X via the cohomological correspondence induced by the Chern class of the complex $\mathcal{F}$. One expects the correspondence μ of (1) between symplectomorphisms and Fourier–Mukai functors to be compatible with cohomology actions via the isomorphism $\bar{m}$ of (2).

On specific issue in the Calabi–Yau contect is that cupping with the holomorphic n-form induces an isomorphism, well-defined up to constant, $H^1(X, T_X) \cong H^1(X, \Omega_X^{n-1})$. Via Hodge theory, the latter is a direct summand of cohomology. The map $\varphi^{\mathcal{F}}$ preserves Hodge structures, so one obtains an action of a Fourier–Mukai functor on $H^1(X, T_X)$. This is the tangent space to deformations of X, and the interpretation of this action is that the Fourier–Mukai self-equivalence only deforms as an self-equivalence along deformation directions fixed by its cohomology action. This issue is spelled out in [31, Theorem 2.1].

2. Degenerations and contractions

2.1. The specific issue considered here is the relation of symplectomorphisms and derived equivalences arising from mirror symmetry between degenerations and contractions [17]. Fix some projective ambient space $\mathbb{P}$, and consider a family $\mathcal{Y} \subset T \times \mathbb{P}$ with projection $\pi\colon \mathcal{Y} \to T$ over a smooth base T, having smooth total space $\mathcal{Y}$ and smooth Calabi–Yau fibres over an open set $T^0 \subset T$ whose complement is a divisor with normal crossings. Assume also that $\mathbb{P}$ is equipped with a symplectic form Ω, compatible with its complex structure, whose restriction ω_s to every smooth fibre Y_s of π is still non-degenerate. The map π then becomes a Lefschetz fibration [23]. In particular, there is a notion of symplectic parallel transport over T^0. Fixing a base point $t \in T^0$ with fibre $(Y, \omega) = (Y_t, \omega_t)$, there is a map

$$\sigma : \pi_1(T^0) \to \pi_0(\mathrm{Symp}(Y, \omega)),$$

the *symplectic monodromy* of the family π.

As always, if the fibre (Y_s, ω_s) is smooth for all $s \in T$, then the symplectic monodromy is trivial. The interesting case is when Y_s is singular for $s \in T \backslash T^0$, in other words when π represents a degeneration of the complex structure on the Calabi–Yau manifold Y_t, and $T \setminus T^0$ lies in the boundary of the space of complex structures. There is a vast literature on degenerations of Calabi–Yau manifolds; I will describe some specific situations below.

As mirror symmetry interchanges complex and Kähler moduli, the construction mirror to degenerations of the complex structure should be a degeneration of the Kähler structure of the mirror X of Y. The way this can be thought of as monodromy in an actual moduli space around some boundary should not concern us here; compare [18] and the much more advanced approach [5] for details. I will take the old-fashioned view [17] which says that in many cases, Kähler degeneration means that a collection of Kähler classes $[\omega_t]$ on the (complex) Calabi–Yau manifold X tends to some limit class $[\omega_0]$ on the boundary of the *Kähler cone* [33] of the complex Calabi–Yau manifold X, the cone of Kähler classes on X in the vector space $H^2(X, \mathbb{R})$. Such boundary points, in favorable cases, correspond to *contractions* on the complex manifold X: algebraic morphisms $X \to \bar{X}$ satisfying certain conditions [33].

We then get our basic correspondence: the map μ arising from mirror symmetry considerations should in certain situations relate symplectic monodromy around boundary points of the complex moduli space of Y to Fourier–Mukai transforms defined from contractions on the complex Calabi–Yau manifold X. I will further restrict attention to cases where the contraction $X \to \bar{X}$ is *birational*, contracting a locus $E \subset X$ to a locus $C \subset \bar{X}$ and inducing an isomorphism between non-empty open subsets $X \setminus E \cong \bar{X} \setminus C$. As such contractions are more easily classified, at least into broad classes, I will always start from the contraction and discuss the corresponding degeneration. I will attempt to be rigorous in notation: X will be a Calabi–Yau threefold (with fixed complex structure) and $X \to \bar{X}$ a birational contraction. On the mirror side, Y_0 will always be the degenerate complex manifold (possibly local) with a symplectic smoothing (Y, ω) and, if needed, a resolution $\widehat{Y_0} \to Y_0$.

2.2. Note that the above discussion assumed a number of details about the specific form of the degeneration of the complex structure of Y. Just because the Calabi–Yau variety Y is embedded in some ambient space $\mathbb{P}$ (such as a weighted projective space or a toric variety), it is by no means certain that all deformations of Y can also be embedded in $\mathbb{P}$ or that all symplectic forms on Y arise as restrictions of some form Ω from $\mathbb{P}$. Hence for a general discussion of the relation between diffeomorphisms and Fourier–Mukai functors, one needs to treat both symplectic structure and complex structure in families; compare [30]. However, for my present purposes the above considerations suffice.

3. Surface double points

In dimension two, there is only one class of birational contractions on Calabi–Yau varieties: the contraction of a (-2)-curve on a K3 surface, or more generally the contraction of a tree of such curves. I begin by discussing the case of a single curve.

3.1. Let X be a smooth K3 surface containing a rational curve $\mathbb{P}^1 \cong E \subset X$ of square (-2). There is a contraction $X \to \bar{X}$ contracting this curve to a point $P \in \bar{X}$, which is locally analytically isomorphic to the surface node (simple double point)

$$\{x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbb{A}^3.$$

The mirror degeneration to this contraction is known to be a degeneration to the same type of singularity: this means a one-dimensional family of K3 surfaces $\widetilde{\pi}\colon \widetilde{\mathcal{Y}} \to \Delta$ with smooth fibres over the punctured disc Δ^* and a singular fibre Y_0 containing a single double point. Locally analytically the family $\widetilde{\mathcal{Y}}$ is isomorphic to the family

$$\{x_1^2 + x_2^2 + x_3^2 = t\} \subset \mathbb{A}^4$$

where t is the coordinate in the base Δ and the symplectic form ω_t on an open set of Y_t is given by the restriction of the standard form on $\mathbb{C}^3$.

Fix $t \neq 0$ as the base point and assume for simplicity that t is positive real. Then $Y = Y_t$ contains a two-sphere $S^2 \simeq S \subset Y_t$ given by

$$S = \{x_1^2 + x_2^2 + x_3^2 = t \,|\, x_i \in \mathbb{R}\}.$$

In fact this sphere is Lagrangian in (Y, ω), and it represents the *vanishing cycle*: its homology class generates the subspace of homology disappearing under passage from Y to Y_0. The symplectic monodromy of the family $\pi\colon \mathcal{Y} \to \Delta^*$ was constructed explicitly with the help of the vanishing cycle S by Arnold [2] as follows.

Consider first a model situation: the cotangent bundle T^*S^2 with its canonical symplectic form η. By means of the standard metric, identify T^*S^2 with the tangent bundle TS^2. The latter has a circle action σ, defined by the normalized geodesic flow, transporting a tangent vector ξ with unit speed along the geodesic emanating from it, irrespective of its length; regard this as a circle action on T^*S^2. Consider also an auxiliary smooth real function ψ such that $\psi(t)+\psi(-t) = 2\pi$ for all $t \in \mathbb{R}$, and $\psi(t) = 0$ for $t >> 0$. Now define $\tau\colon T^*S^2 \to T^*S^2$ by

$$\tau(\xi) = \begin{cases} \sigma(e^{i\psi(|\xi|)})(\xi) & \xi \in Y \setminus S^2 \\ A(\xi) & \xi \in S^2 \end{cases}$$

where A is the antipodal map on the sphere. It is easy to check that τ is continuous, and acts trivially away from a small neighbourhood of the zero-section. A short argument also shows that it is a symplectomorphism of (T^*S^2, η) which, up to symplectic isotopy, is independent of the choice of ψ. τ is the *model Dehn twist* of T^*S^2 with respect to its zero section.

If now (Y, ω) is a symplectic manifold containing a Lagrangian two-sphere S, then by a theorem of Weinstein a neighbourhood of this two-sphere can be identified with a neighbourhood of the zero-section in (T^*S^2, η). Since the model Dehn twist of T^*S^2 acts trivially outside a neighbourhood of the zero-section which can be made arbitrarily small, there is a symplectomorphism τ_S of Y defined by the model Dehn twist in a suitable neighbourhood of S.

Proposition 3.1. (Arnold) *The symplectic monodromy of the family $\mathcal{Y} \to \Delta$ deforming the surface node is generated by the Dehn twist $[\tau_S] \in \pi_0(\mathrm{Symp}(Y, \omega))$ in the vanishing cycle S. The cohomology action of $[\tau_S]$ is the map $(\tau_S)_* \colon H^*(Y, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ given by*

$$(\tau_S)_*(\alpha) = \alpha + ([S] \cdot \alpha)[S]$$

which is a reflection since $[S]^2 = -2$.

What is then the mirror of this symplectic monodromy transformation under the map μ of (1)? According to our basic principle, it should be a Fourier–Mukai functor associated to the contraction of E in X.

Proposition 3.2. (after Seidel–Thomas) *Consider the structure sheaf $\mathcal{F} = \mathcal{O}_{X \times_{\bar{X}} X}$ of the correspondence $X \times_{\bar{X}} X \subset X \times X$. This sheaf defines a Fourier–Mukai equivalence*

$$\Phi^{\mathcal{F}} \colon D^b(X) \to D^b(X)$$

with cohomology action $\varphi^{\mathcal{F}} \colon H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ given by

$$\varphi^{\mathcal{F}}(\alpha) = \alpha + ([E] \cdot \alpha)[E]$$

which is a reflection since $[E]^2 = -2$.

A plausible guess, which under a suitable choice of mirror map can indeed be made precise, is that the mirror $\mu([\tau_S])$ of the symplectic monodromy $[\tau_S]$ is the functor $\Phi^{\mathcal{F}}$. Their cohomology actions are indeed easy to match. I wrote the sheaf $\mathcal{F}$ in the above form to show its explicit dependence on the contraction, but in fact it can be shown that the above definition is equivalent to the original definition of [26] as a twist functor. Without going into the details, which can be found in [26], I note that this particular transform is the (inverse) twist on X defined by the spherical sheaf $\mathcal{O}_E(-1)$.

3.2. The story for simple nodes can be generalized to other double point singularities. In the local situation, mirror symmetry relates deformations and smoothings of the arbitrary A_n surface singularity

$$\{x_1^2 + x_2^2 + x_3^{n+1} = 0\} \subset \mathbb{A}^3.$$

As shown in [22], a smoothing Y of this singularity contains a collection $S_1, \ldots, S_n$ of Lagrangian spheres, with a single transversal intersection point between S_i and S_{i+1} and no other intersections. There are Dehn twists in all these spheres, which satisfy the relations of the braid group on $n+1$ strings [22]. On the contraction side, X contains holomorphic spheres (rational curves) $E_1, \ldots, E_n$, and has a corresponding collection of derived self-equivalences which act by the braid group. The actions on cohomology are given on both sides by reflections generating the symmetric (Weyl) group. For details, consult [26].

4. Isolated threefold singularities

I now turn to birational contractions of three-dimensional Calabi–Yau varieties. There are now three classes of possibilities, depending on the dimension of the exceptional locus $E \subset X$ and its image. In this section I consider cases where this image is a point.

4.1. The first possibility to consider is when E is one-dimensional, and hence necessarily rational. The simplest case is when $E \cong \mathbb{P}^1$ with normal bundle $N_{E/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1,-1)$, the $(-1,-1)$-curve. The contraction of such a curve leads to a threefold $\bar{X}$ with an ordinary double point singularity

$$\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\} \subset \mathbb{A}^4.$$

This contraction is the first half of the conifold transition in physics, which is considered to be self-mirror [17]. Hence its degeneration mirror should be the other half of the conifold transition, the smoothing $\mathcal{Y} \to \Delta^*$ of the node locally given by

$$\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = t\} \subset \mathbb{A}^5.$$

For fixed t positive real, this local geometry contains a Lagrangian three-sphere given by $x_i \in \mathbb{R}$, which is again the vanishing cycle and is Lagrangian. Globally we have a symplectic Calabi–Yau threefold (Y, ω) containing a Lagrangian S^3, and the situation is analogous to the surface case:

Proposition 4.1. (Arnold) *The symplectic monodromy of the smoothing of the threefold node is given by a Dehn twist $[\tau_S]$ in the vanishing cycle $S \simeq S^3$ defined exactly as in the surface case. Its cohomology action is*

$$(\tau_S)_*(\alpha) = \alpha + ([S] \cdot \alpha)[S];$$

as $[S]^2 = 0$, this is a map of infinite order.

One plausible mirror of this symplectic monodromy map is given by

Proposition 4.2. (Seidel–Thomas) *Consider a $(-1,-1)$-curve E in a Calabi–Yau manifold X. Then its structure sheaf $\mathcal{O}_E$ defines a spherical sheaf, and there is a corresponding twist functor*

$$T_{\mathcal{O}_E} \in \mathrm{AutEq}(D^b(X))$$

with cohomology action given by

$$\alpha \mapsto \alpha + (\mathrm{ch}(\mathcal{O}_E) \cdot \alpha)\mathrm{ch}(\mathcal{O}_E).$$

One notable feature of this pair of symmetries is their cohomology action. As the formulae show, $(\tau_S)_*$ acts trivially on even cohomology as $[S]$ is in degree three; mirror to this, $T_{\mathcal{O}_E}$ acts trivially on odd cohomology as $\mathrm{ch}(\mathcal{O}_E)$ lives in even degree. As mentioned in Section 1, this implies that $T_{\mathcal{O}_E}$ acts trivially on the deformation space of X. Thus this Fourier–Mukai functor must exist on all (small) deformations of X; indeed, contractions of $(-1,-1)$-curves are known to have this property [33].

I leave the world of the simple node, though of course there would be much more to say regarding the issue of multiple nodes and flops. I leave these to more able hands; [17, 26] and [27] respectively have all the details.

4.2. The mirror pairs considered sofar can be encoded in toric geometry, which suggests generalizations. Namely, consider a convex polytope Π with vertices in a lattice $\mathbb{Z}^{n-1}$. Embed $\mathbb{Z}^{n-1} \subset \mathbb{Z}^n$ as the affine hyperplane with last coordinate 1. Let N be the sublattice of $\mathbb{Z}^n$ spanned by the origin and the vertices of Π, and let τ be the cone over Π. Then the toric variety $\bar{X} = \mathbb{X}_{N,\tau}$ (in the covariant description [8]) is an affine n-fold with a canonical Gorenstein singularity at the origin, which in dimension at most three has toric crepant resolution(s) $X \to \bar{X}$.

On the other hand, the polytope Π can be thought of as the Newton polytope of a polynomial f in variables x_i, x_i^{-1} for $i = 1, \ldots, n-1$. Adding two auxiliary variables u, v, one obtains a family of affine n-folds

$$\mathcal{Y} = \{u^2 + v^2 + f(x_i, x_i^{-1}) = 0\} \subset (\mathbb{A} \setminus \{0\})^{n-1} \times \mathbb{A}^2 \times T$$

over the base T defined by the moduli of the polynomial f. The complex structure of this variety will degenerate for particular f, giving rise to a boundary locus $T \setminus T^0$ and symplectic monodromy. Toric geometry suggests that this family is the mirror of the contraction $X \to \bar{X} = \mathbb{X}_{N,\tau}$.

It is a simple matter to see that the surface A_n singularity is the case when $P = [0, n+1]$ is 1-dimensional, whereas the threefold node arises from the two-dimensional polytope $P = [0,1] \times [0,1]$.

4.3. Examples of birational maps contracting a surface $E \subset X$ to a point also arise in this way. For example, consider the polytope in $\mathbb{R}^2$ with vertices at $(-1,-1), (1,0)$ and $(0,1)$. The contraction $X \to \bar{X} = \mathbb{X}_{N,\tau}$ contracts a projective plane $\mathbb{P}^2 \subset X$ to a Gorenstein singularity $P \in \bar{X}$. The mirror of this singularity is the family of threefolds

$$\mathcal{Y} = \{a_0 + a_1x_1 + a_2x_2 + a_3x_1^{-1}x_2^{-1} + u^2 + v^2 = 0\} \subset (\mathbb{A} \setminus \{0\})^2 \times \mathbb{A}^2 \times T$$

over a base $T \subset \mathbb{A}^4$ with coordinates a_i. The symplectic geometry of this family is studied in detail in [25, Proposition 3.2].

4.4. The list can be continued with various two-dimensional polytopes. One interesting example is the following: consider a Calabi–Yau threefold $\widehat{Y}_0$ containing a contractible surface $E \subset \widehat{Y}_0$ abstractly isomorphic to $\mathbb{P}^2$ blown up in three points. Then $E^3 = (K_E)^2 = 6$ and after contraction, Y_0 contains a toric Gorenstein singularity $(P \in Y_0)$ corresponding to the polytope Π with vertices at $(-1,-1)$, $(0,-1)$, $(1,0)$, $(1,1)$, $(0,1)$ and $(-1,0)$. The mirror to this singularity is the family

$$\mathcal{X} = \{u^2 + v^2 + f(x_i, x_i^{-1}) = 0\} \subset \mathbb{A}^2 \times (\mathbb{A} \setminus \{0\})^2 \times T$$

defined by polynomials f with Newton polytope Π. It might be interesting to study the symplectic geometry of degenerations in this family in more detail.

On the other hand, the deformation theory of $(P \in Y_0)$ is quite interesting: by [1, 2.1 and 8.4], the local first-order deformation space is three-dimensional, but only a one- and a two-dimensional subspace can be realized as actual deformations, the two components corresponding to two essentially different ways in which Π decomposes into the Minkowski sum of two polytopes Π_1 and Π_2. There are therefore two different (not even diffeomorphic) symplectic smoothings $Y_s^{(j)}$ here. On the other side, as Mark Gross points out, for every decomposition $\Pi = \Pi_1 + \Pi_2$ into Minkowski summands, there are degenerate members

$$\bar{X}^{(j)} = \{u^2 + v^2 + f_1(x_i, x_i^{-1}) f_2(x_i, x_i^{-1}) = 0\} \subset \mathbb{A}^2 \times (\mathbb{A} \setminus \{0\})^2,$$

in the family $\mathcal{X}$, where f_i has Newton polytope Π_i. These varieties are singular, with nodes at the points where $u = v = f_1 = f_2 = 0$, and have small resolutions $X^{(j)} \to \bar{X}^{(j)}$ mirroring the two symplectic smoothings $Y_s^{(j)}$.

4.5. Beyond toric cases, one example discussed in the literature [28] concerns, on the symplectic side, the degeneration to the simple threefold triple point

$$\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\} \subset \mathbb{A}^4.$$

Denote as usual by (Y, ω) a symplectic smoothing.

Proposition 4.3. (Smith–Thomas) *The symplectic manifold* (Y, ω) *contains a collection of 16 Lagrangian three-spheres* $S_1, \dots, S_{16}$ *meeting in an intricate configuration depicted on* [28, Figure 2]*. In particular, there are 16 Dehn corresponding twists*

$$\tau_{S_i} \in \pi_0(\mathrm{Symp}(Y, \omega)).$$

As far as I know, the mirror of this singularity has not been discussed in the literature. It should have 16 Fourier–Mukai transforms corresponding to the Dehn twists of Proposition 4.3.

5. Non-isolated threefold singularities

The final case left out so far is that of a contraction $(E \subset X) \to (C \subset \bar{X})$ with $\dim C = 1$. Assuming that X is smooth, it follows that the curve C is also smooth [33].

5.1. Start with the simplest case: a projective Calabi–Yau variety $\bar{X}$, smooth outside of a smooth curve $C \subset \bar{X}$ of genus g, locally analytically isomorphic to

$$\{x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbb{A}^4$$

along C (which is locally the line $\mathbb{A}^1_{x_4}$). Blowing up the ideal of C gives a Calabi–Yau resolution $X \to \bar{X}$ containing an exceptional divisor E geometrically ruled over C.

Proposition 5.1. (Horja, Szendrői) *The structure sheaf $\mathcal{F} = \mathcal{O}_{X \times_{\bar{X}} X}$ on the product $X \times X$ gives a Fourier–Mukai self-equivalence*

$$\Phi^{\mathcal{F}} \colon D^b(X) \to D^b(X).$$

The corresponding cohomology action maps

i. a class $\alpha \in H^2(X, \mathbb{Q})$ as

$$\varphi^{\mathcal{F}}(\alpha) = \alpha + ([l] \cdot \alpha)[E] \mod H^4(X, \mathbb{Q})$$

with $[l] \in H^4(X, \mathbb{Q})$ the class of the ruling of E (this is a reflection since $[l] \cdot [E] = -2$);

ii. on third cohomology as an involution with a codimension $2g$ fixed locus and (-1)-eigenspace given by the image of the cylinder homomorphism $H^1(C, \mathbb{Q}) \to H^3(X, \mathbb{Q})$ (note that there is no odd cohomology in different degrees; see [29, Proposition 4.6] *for more details).*

The problem is then clear: find the symplectic mirror of this contraction, together with its symplectic monodromy corresponding to $\Phi^{\mathcal{F}}$.

5.2. Before I move on, I discuss a slight generalization, which will be just as easy (or difficult) to study: suppose that $\bar{X}$ contains a curve of A_n-singularities $C \subset \bar{X}$ locally of the form

$$\{x_1^2 + x_2^2 + x_3^{n+1} = 0\} \subset \mathbb{A}^4.$$

Assume also that repeated blowup leads eventually to a smooth Calabi–Yau resolution X containing a collection $E_1, \ldots, E_n$ of smooth ruled surfaces. Then there is a Fourier–Mukai equivalence $\Phi^{\mathcal{F}_i} \in \mathrm{AutEq}(D^b(X))$ for each of the surfaces, which as a matter of fact satisfy the relations of the braid group [32] just as in the surface case.

5.3. To approach the problem of finding the mirror, I recall the discussion of the above setup in the toric context from [15, Section 3.2]. Assume therefore that $\bar{X}$ is in fact a hypersurface in a toric variety $\mathbb{P}_\Delta$ given by a reflexive polytope Δ spanned by some lattice points in a lattice $M \cong \mathbb{Z}^4$; compare [4]. The singularities of $\bar{X}$ are most easily studied in terms of the normal fan Σ consisting of cones spanned by vertices of the dual polytope Δ° in the dual $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. The specific singularity along C then arises from an edge $\tau = \langle \mathbf{v}_0, \mathbf{v}_{n+1} \rangle$ of Δ° containing n interior lattice points $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The genus g can also be read off from the toric data: it is the number of interior lattice points in the dual two-dimensional face $\tau^\circ \subset \Delta \subset M$.

The mirror Y of X is given, by Batyrev's mirror duality [4], as a hypersurface in a partial toric resolution $\widehat{\mathbb{P}}_{\Delta^\circ}$ of the toric variety $\mathbb{P}_{\Delta^\circ}$. The parameter space of the hypersurface Y is the vector space

$$H^0\left(\widehat{\mathbb{P}}_{\Delta^\circ}, -K_{\widehat{\mathbb{P}}_{\Delta^\circ}}\right) \cong \bigoplus_{\mathbf{m} \in \Delta^\circ \cap N} \mathbb{C}\, \mathbf{x}^{\mathbf{m}}.$$

Here $\mathbf{x}^{\mathbf{m}}$ represents a monomial $\prod x_i^{m_i}$, where x_i are to be thought of as affine variables in the affine torus $(\mathbb{C}^*)^4 \subset \widehat{\mathbb{P}}_{\Delta^\circ}$, and Y is the closure in $\widehat{\mathbb{P}}_{\Delta^\circ}$ of the subvariety of the torus given by the chosen Laurent polynomial. By [6], this vector space can also be viewed dually as the appropriately graded piece of the homogeneous polynomial ring S of the abstract toric variety $\widehat{\mathbb{P}}_{\Delta^\circ}$, with one generating variable y_η for every lattice points $\eta \in N$ on the boundary of the polytope Δ.

Now return to our concrete situation, restricting attention to $g = 1$ the reasons for which will appear presently; let $\eta_1 \in M$ be the unique lattice point in the interior of the face $\tau^\circ \subset \Delta$ with corresponding homogeneous coordinate y_1. The hypersurface Y is given by a homogeneous equation

$$\{y_1 f(y_j) + g(y_j) = 0\} \subset \widehat{\mathbb{P}}_{\Delta^\circ}$$

where I have separated out the monomials not involving the variable y_1. It can however be checked, using the correspondence between Laurent and homogeneous polynomials [6], that the only terms not involving the monomial y_1 correspond to Laurent monomials $\mathbf{x}^{\mathbf{m}}$ with $\mathbf{m} \in \tau$, one of the $n+2$ lattice points responsible for the singularity of $\bar{X}$. The linear relations between these lattice points translate to multiplicative relations between the Laurent monomials, which implies that the equation of Y can be written on a suitable affine piece of $\widehat{\mathbb{P}}_{\Delta^\circ}$ as

$$y_1 f + \sum_{j=0}^{n+1} a_j x^j = 0$$

where x is an auxiliary affine variable. Moreover, the contraction of X back to $\bar{X}$ corresponds to the degeneration to the hypersurface with equation

$$y_1 f + (b_1 x + b_0)^{n+1} = 0$$

which (assuming appropriate regularity of f) is a threefold exactly of the studied type: singular along the curve $\{y_1 = f = b_1 x + b_0 = 0\}$, with a curve of A_{n+1} singularities.

This is then the correspondence suggested by toric geometry: the mirror to a contraction of a collection of ruled surfaces to an elliptic curve of A_n singularities should be a degeneration to a single curve of A_n singularities.

5.4. Before moving on to more theory, it might be illustrative to give an example, appearing as [12, Example II]. Let

$$\widetilde{\Delta} = \{\mathbf{m} \in M \cong \mathbb{Z}^4 \colon m_i \geq -1, 1 \geq m_1 + 2m_2 + 2m_3 + 2m_4\}$$

with dual polytope $\widetilde{\Delta}^\circ$ spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ and $\mathbf{v}_0 = -\mathbf{e}_1 - 2\mathbf{e}_2 - 2\mathbf{e}_3 - 2\mathbf{e}_4$ in $\widetilde{N} = \oplus \mathbb{Z}\mathbf{e}_i$. The toric variety $\mathbb{P}_{\widetilde{\Delta}}$ is simply weighted projective space $\mathbb{P}^4[1^2, 2^3]$, containing the family of octic Calabi–Yau hypersurfaces. The edge $\langle \mathbf{v}_0, \mathbf{e}_1 \rangle$ contains one interior lattice point, giving rise to a curve of A_1 singularities in the general octic. This curve has genus 3.

Now consider a quotient of this family; let

$$N = \widetilde{N} + \left(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}\right)\mathbb{Z} + \left(0, 0, \frac{1}{4}, \frac{3}{4}\right)\mathbb{Z}$$

with $\Delta^\circ = \widetilde{\Delta}^\circ$ now thought of as a polytope in the lattice N. It can be checked that Δ° is still reflexive, and now $\langle \mathbf{v}_0, \mathbf{e}_1 \rangle$ contains seven interior lattice points, indicating singularities of type A_7 along a curve. The dual to (N, Δ°) is the pair (M, Δ), where

$$M = \widetilde{M} + \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)\mathbb{Z}$$

and $\Delta = \widetilde{\Delta}$ as a polytope with vertices in M. The mirror two-dimensional face to $\langle \mathbf{v}_0, \mathbf{e}_1 \rangle$ looks like the toric diagram for a $\frac{1}{4}(1,1,2)$ singularity, with one interior lattice point. Hence the generic threefold $\bar{X} \subset \mathbb{P}_\Delta$ has a genus-1 curve of A_7 singularities, so falls under the remit of the above discussion.

5.5. I now consider the symplectic geometry of the mirror (Y, ω) smoothing the hypersurface Y_0. The monodromy of the degeneration found above has in fact been studied already by Seidel [22].

Definition/Proposition 5.2. (Seidel) *Let (M^{2n}, ω) be a symplectic manifold. A* set of construction data for a generalized Dehn twist on *M consists of a compact symplectic manifold (M', ω') of dimension $2n - 2r$, a fibre bundle $p\colon S \to M'$ with fibre S^r and structure group O_{r+1}, and an embedding $i\colon S \hookrightarrow M$ such that $i^*(\omega) = p^*(\omega')$. Given a set of such data, there is a well-defined* generalized Dehn twist

$$[\tau_S] \in \pi_0(\mathrm{Symp}(M, \omega)).$$

Dehn twists as discussed in previous sections correspond to the case $M' =$ point. The general idea should now be clear:

Conjecture 5.3. *Let $X \to \bar{X}$ be a contraction of ruled surfaces $E_1, \ldots, E_n$ to an elliptic curve $C \subset \bar{X}$ of A_n-singularities on a Calabi–Yau threefold. Then the symplectic mirror (Y, ω) of X contains a collection $p_i\colon S_i \to \Sigma$ of S^2-bundles over a symplectic Riemann surface (Σ, ω'), providing a set of data for generalized Dehn twists*

$$[\tau_{S_i}] \in \pi_0(\mathrm{Symp}(Y, \omega)).$$

These symplectomorphisms satisfy the relations of the braid group on $n+1$ strings in the obvious way. Their cohomology actions are given for $\alpha \in H^2(Y, \mathbb{Q})$ by

$$(\tau_{S_i})_*(\alpha) = \alpha + ([l] \cdot \alpha)[S_i]$$

with $[l] \in H^4(Y, \mathbb{Q})$ the class of the fibre of p_i; $(\tau_{S_i})_*$ is a reflection since $[l] \cdot [S_i] = -2$. These symplectic automorphisms mirror the action of the Fourier–Mukai transforms of Proposition 2.

Note that the cohomology action is formally identical to that of Proposition 2.*i*. However, remembering that we are in the threefold case, the more important point is the relation to the action in Proposition 2.*ii*, which brings me back to the $g = 1$ issue. Note that in Proposition 2.*i*, the fixed locus is of codimension $2g$. Hence for a simple formula like that in Conjecture 5.3 to hold, with a codimension-one fixed locus, we need to be in the case $g = 1$ (the other half of the fixed locus in the symplectic case is in H^4 by Poincaré duality).

Geometrically, it is abundantly clear where the fibred manifolds S_i come from, if the sketched degeneration picture is correct. The base Σ of the fibration is the singular locus of the degenerate variety Y_0. The S^2 sphere fibres are the vanishing cycles in the local A_n degeneration transverse to that curve.

5.6. In the toric argument, the $g = 1$ assumption manifested itself in the presence of a single coordinate y_1 which can be pulled out of the equation of Y. In the case of higher genus, there are several such coordinates, and the picture suggested by [15] is that of a degeneration to a reducible curve of A_n singularities. These curves will however begin to intersect, necessarily producing singularities worse than A_n, and the geometric picture is not so clear any more. The discussion of the cohomology action also suggests a more complicated symplectic automorphism. It might be worthwhile to study this case in more detail.

Incidentally, the $g = 0$ case is also of some interest. In that case the cohomology action of the Fourier–Mukai functors of Proposition 2 in odd degree is trivial. Mirror to that, I expect a symplectomorphism induced by a submanifold $S \subset (Y, \omega)$ generically fibred in spheres by $p\colon S \to \Sigma$, so that the spheres collapse over special points $P \in \Sigma$ making their cohomology class trivial.

5.7. Return to the example of 5.4. This is discussed in [12] as Example II, where a pair of (special) Lagrangian cycles $S^1 \times S^2 \simeq N_i \subset (Y, \omega)$ is constructed as the fixed locus of a real involution. If Conjecture 5.3 holds, the natural guess is that in fact $N_i = p_4^{-1}(B_i)$ for a pair of Lagrangian circles $S^1 \simeq B_i \subset (\Sigma, \omega')$; remember that the singularity is of type A_7 and the construction using real variables in the complex A_7 equation gives the middle vanishing 2-cycle. The one-dimensional local moduli space of N_i as a special Lagrangian cycle is geometrically realized then as coming simply from moving the circle B_i locally in Σ. Conjecture 5.3 would imply that there is a host of other Lagrangian cycles around, though their realization as special Lagrangians is bound to run into the usual problem of finding sLag representatives of vanishing cycles.

As for the other two examples of [12], Example I involves the original octic as X, with a contraction to a $g = 3$ curve. Its mirror contains a complicated (special) Lagrangian cycle N with $b_2(N) = 5$. As I discussed above, I expect this case to be quite complicated. [12, Example III] involves on the complex side a contraction to a $g = 0$ curve, and again $S^1 \times S^2$ in the mirror; this would arise naturally from the $g = 0$ speculation at the end of 5.6.

5.8. To conclude, I want to return to one point which was swept under the carpet above. Namely, just because a threefold has a curve of A_n singularities, it does not follow that in its resolution one finds n irreducible surfaces all ruled over the same curve. This is an issue of monodromy (in a different sense now, over the curve C), which is discussed in detail in [31, 32]. I only want to point out that [31, Example 4.3] constructs an example of a threefold $\bar{X}$ with an elliptic curve of A_3 singularities, where in the resolution there are only two irreducible surfaces $E_1, E_2 \subset X$. There are two corresponding Fourier–Mukai functors $\Phi_1, \Phi_2 \in \mathrm{AutEq}(D^b(X))$, which satisfy the braid relation

$$\Phi_1 \circ \Phi_2 \circ \Phi_1 \circ \Phi_2 = \Phi_2 \circ \Phi_1 \circ \Phi_2 \circ \Phi_1$$

of the C_3 braid group. The analogue of Conjecture 5.3 would suggest that the mirror (Y, ω) of X should contain a pair of S^2-fibred submanifolds together with the necessary symplectic data, giving rise to a pair of Dehn twists $[\tau_i] \in \pi_0(\mathrm{Symp}(Y, \omega))$ which satisfy the same relation

$$\tau_1 \circ \tau_2 \circ \tau_1 \circ \tau_2 = \tau_2 \circ \tau_1 \circ \tau_2 \circ \tau_1$$

up to symplectic isotopy. I leave the problem of filling in details as a final challenge for you, my Dear Reader.

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